Invent. math. 134, 335–361 (1998)

Inventiones mathematicae

On a reverse form of the Brascamp-Lieb inequality

Franck Barthe*

Equipe d'Analyse et de Mathématiques Appliquées, Université de Marne-la-Vallée, Cité Descartes, 5 boulevard Descartes, F-77454 Marne-la-Vallée Cedex 2, France (e-mail: barthe@clipper.ens.fr)

Oblatum 12-VI-1997 & 21-XI-1997

Abstract. We prove a reverse form of the multidimensional Brascamp-Lieb inequality. Our method also gives a new way to derive the Brascamp-Lieb inequality and is rather convenient for the study of equality cases.

Introduction

We work on $\mathbb{R}^n$ with its usual Euclidean structure and we denote by $\langle , \rangle$ the canonical scalar product. In [BL], H. J. Brascamp and E. H. Lieb showed that for $m, n \in \mathbb{N}$, $p_1, \dots, p_m > 1$ and $a_1, \dots, a_m \in \mathbb{R}^n$, the norm of the multilinear operator Φ from $L_{p_1}(\mathbb{R}) \times \cdots \times L_{p_m}(\mathbb{R})$ into $\mathbb{R}$ defined by

$$\Phi(f_1, \dots, f_m) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(\langle x, a_i \rangle) dx$$

can be computed as the supremum over centered Gaussian functions $g_1, \dots, g_m$ of

$$\frac{\Phi(g_1, \dots, g_m)}{\prod_{i=1}^{m} \|g_i\|_{p_i}} .$$

* This work will form part of a doctoral thesis under the supervision of professors B. Maurey and A. Pajor. Their advice and encouragement have been invaluable.

In other words, Φ is "saturated" by Gaussian functions. This theorem is a very convenient tool to derive sharp inequalities. Brascamp and Lieb applied it successfully to prove the optimal version of Young's convolution inequality (also established independently by Beckner [Bec]), to rederive Nelson's hypercontractivity. Their proof is based on a rearrangement inequality of Brascamp, Lieb and Luttinger [BLL] and on the fact that radial functions of a large number of variables behave like Gaussians. However, their method left open, except in some special cases, the multidimensional problem. Let $m \geq n, p_1, \ldots, p_m > 1$ and let $n_1, \ldots, n_m$ be integers. For each $i \leq m$ let B_i be a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^{n_i}$. Is the multilinear operator on $L_{p_1}(\mathbb{R}^{n_1}) \times \cdots \times L_{p_m}(\mathbb{R}^{n_m})$ defined by

$$\Psi(f_i, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(B_i x) dx$$

saturated by Gaussian functions?

This question was solved positively by Lieb in his article "Gaussian kernels have only Gaussian maximizers" [Lie]. The key point is that Ψ can be viewed as a limit case of multilinear operators with Gaussian kernels.

In [Bar3], we gave a simple proof, for functions of one real variable, of the Brascamp-Lieb inequality and of a new family of inequalities which can be understood as a reverse form, or as a dual form of the Brascamp-Lieb inequalities. These inequalities can be stated as follows: let $m \geq n$, $p_1, \ldots, p_m > 1$ and $a_1, \ldots, a_m \in \mathbb{R}^n$; the largest constant E such that

$$\int^*_{\mathbb{R}^n} \sup_{x=\sum c_i\theta_i a_i} \prod_{i=1}^{m} f_i(\theta_i) dx \geq E \prod_{i=1}^{m} \|f_i\|_{p_i}$$

holds for all $f_i, \ldots, f_m$ is also the largest constant such that the inequality holds for centered Gaussian functions, where $\int^*$ is the outer integral. Again, Gaussian functions play an extremal role. This new inequality was inspired by convexity theory. The strength of the Brascamp-Lieb inequality for volume estimates of convex bodies was noticed by K. Ball (see [Bal1], [Bal2] and [Bal3]), who also remarked in [Bal3] that a reverse inequality would give dual results. For geometric applications of the reverse Brascamp-Lieb inequality, see [Bar1] and also section III of the present paper.

In the first section, we prove a fully multidimensional version of the reverse Brascamp-Lieb inequality. Our method also gives a new proof of the multidimensional Brascamp-Lieb inequality. It is very similar to the one we used for the one-dimensional case and uses a theorem of Brenier ([Bre1], [Bre2]) refined by McCann ([McC1], [McC2]) on measure preserving mappings deriving from convex potentials. Notice that this result was applied by McCann in [McC1] to prove the Prékopa-Leindler inequality ([Pré], [Lei]), which is a particular case of the reverse Brascamp-Lieb inequalities.

In section II, we focus on the one-dimensional case in order to deal in detail with equality cases. This problem was left open for the Brascamp-Lieb inequality because the previous proofs depended on limit processes. We push further the study of [BL] in the spirit of [Lie] to see when there is a Gaussian maximizer for the Brascamp-Lieb inequality (or a Gaussian minimizer for the reverse form) and whether it is unique.

In section III, we study the particular case of the Brascamp-Lieb inequality which was pointed out by K. Ball [Ball] and which is so useful in convexity. We state the corresponding converse inequality. The equality cases are completely solved, which allows us to find new characteristic properties of simplices and parallelotopes. The multidimensional version of the reverse Brascamp-Lieb inequality implies a Brunn-Minkowski type theorem for sets that are contained in subspaces.

1 Proof of the Brascamp-Lieb inequality and its converse

We first introduce some notation. Let $\mathscr{S}^+(\mathbb{R}^n)$ be the set of $n \times n$ symmetric definite positive matrices. For $A \in S^+(\mathbb{R}^n)$ we denote by G_A the centered Gaussian function on $\mathbb{R}^n$

$$G_A(x) = \exp(-\langle Ax, x\rangle) \ .$$

We also denote by $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, identified with $m \times n$-matrices. If $B \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $B^* \in \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)$ will be its Euclidean adjoint. We work with the set $L_1^+(\mathbb{R}^n)$ of integrable non-negative functions on $\mathbb{R}^n$.

The fully multidimensional version of the Brascamp-Lieb inequality and its converse is as follows:

Theorem 1 *Let m, n be integers. Let $(c_i)_{i=1}^m$ be positive real numbers and $(n_i)_{i=1}^m$ be integers smaller than n such that*

$$\sum_{i=1}^m c_i n_i = n \ .$$

For $i = 1, \dots, m$, let B_i be a linear surjective map from $\mathbb{R}^n$ onto $\mathbb{R}^{n_i}$. Assume that

$$\bigcap_{i \le m} \ker B_i = \{0\} \ .$$

We define two functions I and J on $L_1^+(\mathbb{R}^{n_1}) \times \cdots \times L_1^+(\mathbb{R}^{n_m})$ as follows: if $f_i \in L_1^+(\mathbb{R}^{n_i})$, $i = 1, \dots, m$ then

$$I((f_i)_{i=1}^m) = \int_{\mathbb{R}^n}^{*} \sup\left\{ \prod_{i=1}^m f_i^{c_i}(y_i);\ \sum_{i=1}^m c_i B_i^* y_i = x \text{ and } y_i \in \mathbb{R}^{n_i} \right\} dx \ ,$$

and

$$J((f_i)_{i=1}^m) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(B_i x) dx \ .$$

Let E be the largest constant such that for all $(f_i)_{i=1}^m$,

$$I((f_i)_{i=1}^m) \ge E \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i \right)^{c_i} \ , \tag{RBL}$$

and let F be the smallest one such that for all $(f_i)_{i=1}^m$,

$$J((f_i)_{i=1}^m) \le F \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i \right)^{c_i} \ . \tag{BL}$$

Then E and F can be computed using centered Gaussian functions only, that is

$$E = \inf\left\{ \frac{I((g_i)_{i=1}^m)}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_i\right)^{c_i}};\ g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \dots, m \right\} \ ,$$

and

$$F = \sup\left\{ \frac{J((g_i)_{i=1}^m)}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_i\right)^{c_i}};\ g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \dots, m \right\} \ ,$$

Moreover, if we denote by D the largest real number such that

$$\det\left(\sum_{i=1}^{m} c_i B_i^* A_i B_i\right) \geq D \prod_{i=1}^{m} (\det A_i)^{c_i} \quad ,$$

for all $A_i \in \mathscr{S}^+(\mathbb{R}^{n_i}), i = 1, \ldots, m$, *then*

$$E = \sqrt{D} \qquad \textit{and} \qquad F = \frac{1}{\sqrt{D}} \quad .$$

Remark 1. The hypothesis $\sum_{i=1}^{m} c_i n_i = n$ is just a necessary homogeneity condition for E to be positive and for F to be finite. The condition on $\cap \ker B_i$ ensures that $\sum_{i=1}^{m} c_i B_i^* A_i B_i$ is an isomorphism. Actually, the conclusion of the theorem remains obviously valid without this condition, since $D = 0$ when it is not satisfied.

Remark 2. Notice that the reverse Brascamp-Lieb inequality for $m = 2$, $n_1 = n_2 = n$, $B_1 = B_2 = B_1^* = B_2^* = I_n$ and $c_1 = \alpha = 1 - c_2$, where I_n is the identity map on $\mathbb{R}^n$ and $0 < \alpha < 1$, is the inequality of Prékopa-Leindler [Pré] [Lei]. Indeed the constant D is

$$D = \inf_{A_1, A_2 \in \mathscr{S}^+(\mathbb{R}^n)} \frac{\det(\alpha A_1 + (1-\alpha) A_2)}{(\det A_1)^{\alpha} (\det A_2)^{1-\alpha}} = 1$$

by the arithmetic-geometric inequality. So (RBL) becomes, for all $f, g \in L_1^+(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}^{*} \sup_{x = \alpha u + (1-\alpha) v} f^{\alpha}(u) g^{1-\alpha}(v) dx \geq \left(\int_{\mathbb{R}^n} f\right)^{\alpha} \left(\int_{\mathbb{R}^n} g\right)^{1-\alpha} \quad .$$

It is well-known that this inequality implies the Brunn-Minkowski theorem: for A, B compact non-void subsets of $\mathbb{R}^n$,

$$\mathrm{Vol}^{\frac{1}{n}}(A + B) \geq \mathrm{Vol}^{\frac{1}{n}}(A) + \mathrm{Vol}^{\frac{1}{n}}(B) \quad .$$

The proof of Theorem 1 is divided into lemmas. We first deal with the study of the behavior of I and J with respect to centered Gaussian functions. We set

$$E_g = \inf\left\{\frac{I((g_i)_{i=1}^m)}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_i\right)^{c_i}};\ g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \ldots, m\right\},$$

and

$$F_g = \sup\left\{\frac{J((g_i)_{i=1}^m)}{\prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} g_i\right)^{c_i}};\ g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i=1,\dots,m\right\}.$$

Our aim is to prove that $E = E_g = \sqrt{D}$ and $F = F_g = D^{-1/2}$. We begin with a classical computation, taken from [BL]; it only uses the fact that if $M \in \mathcal{S}^+(\mathbb{R}^k)$, then

$$\int_{\mathbb{R}^k} \exp(-\langle x, Mx\rangle)dx = \sqrt{\frac{\pi^k}{\det M}} \ .$$

Lemma 1 *With the notation of Theorem* 1, *we have*

$$F_g = \frac{1}{\sqrt{D}} \ .$$

Our next lemma links E_g and F_g by means of duality between quadratic forms.

Lemma 2 *With the previous notation*, *we have*

$$E_g \cdot F_g = 1 \ ,$$

and $E_g = 0$ *if and only if* $F_g = +\infty$.

Proof. For $i = 1, \dots, m$, let $A_i \in \mathcal{S}^+(\mathbb{R}^{n_i})$ and let Q be the quadratic form on $\mathbb{R}^n$ defined by

$$Q(y) = \left\langle \sum_{i=1}^m c_i B_i^* A_i B_i y, y \right\rangle \ .$$

Let Q^* be the dual quadratic form of Q. We recall that it is defined on $\mathbb{R}^n$ by

$$Q^*(x) = \sup\left\{ |\langle x, y\rangle|^2;\ Q(y) \le 1 \right\} \ .$$

We also introduce the function R on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$,

$$R(x) = \inf\left\{ \sum_{i=1}^m c_i \langle A_i^{-1} x_i, x_i\rangle;\ x = \sum_{i=1}^m c_i B_i^* x_i \text{ and for all } i, x_i \in \mathbb{R}^{n_i} \right\} \ .$$

We now show that $R = Q^*$. Indeed, assume that $x = \sum_{i=1}^m c_i B_i^* x_i$ with $x_i \in \mathbb{R}^{n_1}$ for $i = 1, \ldots, m$, then

$$|\langle x, y\rangle|^2 = \left|\left\langle \sum_{i=1}^m c_i B_i^* x_i, y \right\rangle\right|^2 = \left|\sum_{i=1}^m \langle \sqrt{c_i} A_i^{-1/2} x_i, \sqrt{c_i} A_i^{1/2} B_i y\rangle\right|^2 \ .$$

By the Cauchy-Schwarz inequality, applied to the quadratic form ϕ on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ defined by $\phi(X_i, \ldots, X_m) = \sum_{i=1}^m \langle X_i, X_i\rangle$, one gets:

$$\begin{aligned} |\langle x, y\rangle|^2 &\leq \left(\sum_{i=1}^m |\sqrt{c_i} A_i^{-1/2} x_i|^2\right)\left(\sum_{i=1}^m |\sqrt{c_i} A_i^{1/2} B_i y|^2\right) \\ &= \left(\sum_{i=1}^m c_i \langle x_i, A_i^{-1} x_i\rangle\right)\left(\left\langle \sum_{i=1}^m c_i B_i^* A_i B_i y, y\right\rangle\right) \ . \end{aligned}$$

In fact, one easily checks that there is equality in the previous argument if one takes

$$y = \left(\sum_{i=1}^m c_i B_i^* A_i B_i\right)^{-1} x$$

and

$$x_i = A_i B_i y \qquad i = 1, \ldots, m \ .$$

It follows that $R = Q^*$.

We apply this result to our integrals of Gaussian functions. Straightforward computations give that

$$\frac{J(G_{A_1}, \ldots, G_{A_m})}{\prod_{i=1}^m \left(\int G_{A_i}\right)^{c_i}} = \sqrt{\frac{\prod_{i=1}^m (\det A_i)^{c_i}}{\det Q}} \ ,$$

and

$$\frac{I(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}})}{\prod_{i=1}^m \left(\int G_{A_i^{-1}}\right)^{c_i}} = \sqrt{\frac{\prod_{i=1}^m (\det A_i)^{-c_i}}{\det R}} \ .$$

Using that $R = Q^*$ and the classical duality relation $\det Q \cdot \det Q^* = 1$, one has

$$\frac{J(G_{A_1}, \ldots, G_{A_m})}{\prod_{i=1}^m \left(\int G_{A_i}\right)^{c_i}} \cdot \frac{I(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}})}{\prod_{i=1}^m \left(\int G_{A_i^{-1}}\right)^{c_i}} = 1 \ ,$$

and therefore $E_g = F_g^{-1}$. □

Remark. Let us emphasize the equivalence for $A_i \in \mathcal{S}^+(\mathbb{R}^n)$, $i = 1, \ldots, m$ of the assertions

- $\det\big(\sum_{i=1}^m c_i B_i^* A_i B_i\big) = D \prod_{i=1}^m (\det A_i)^{c_i}$.
- The m-tuple of centered Gaussians $(G_{A_1}, \ldots, G_{A_m})$ is a maximizer for (BL).
- The m-tuple of centered Gaussians $(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}})$ is a minimizer for (RBL).

Lemma 3 *For $i = 1, \ldots, m$, let f_i and h_i belong to $L_1^+(\mathbb{R}^{n_i})$ and satisfy $\int_{\mathbb{R}^{n_i}} f_i = \int_{\mathbb{R}^{n_i}} h_i = 1$. Then*

$$I(f_1, \ldots, f_m) \geq D \cdot J(h_1, \ldots, h_m) \ .$$

Taking the supremum over $(h_i)_{i=1}^m$ and the infimum over $(f_i)_{i=1}^m$ in the previous inequality yields $E \geq DF$. So Lemmas 1, 2 and 3 imply $\sqrt{D} = E_g \geq E \geq DF \geq DF_g = \sqrt{D}$. Thus the proof of Theorem 1 will be complete as soon as Lemma 3 is established.

In [Bar3], we proved Lemma 3 for functions of one real variable, using measure-preserving mappings. Given two non-negative functions f and h on $\mathbb{R}$ with integral one, there exists a non-decreasing mapping u such that for all $x \in \mathbb{R}$:

$$\int_{-\infty}^{u(x)} f = \int_{-\infty}^{x} h \ .$$

In other words, u maps the probability measure of density h onto the probability measure of density f. Our proof in the general case (i.e. for functions of several variables) is also based on measure-preserving mappings. But, in dimension larger than one, there is a large choice of such mappings between two sufficiently regular probability measures. For our purpose, the Brenier mapping (see [Bre1], [Bre2]) fits perfectly; it has the additional convenient property of deriving from a convex potential. Brenier proved its existence and uniqueness under certain integrability assumptions on the moments of the measures, which where later removed by McCann [McC1], [McC2]. Let us state the result that we need.

Theorem 2 *Let f_1, f_2 be non-negative measurable functions on $\mathbb{R}^n$ with integral one. There exists a convex function ϕ on $\mathbb{R}^n$ such that the map $u = \nabla\phi$ has the following property: for every non-negative Borel function b on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} b(u(x))f_2(x)dx = \int_{\mathbb{R}^n} b(x)f_1(x)dx \quad .$$

The function ϕ given by this theorem represents a generalized solution of the Monge-Ampère equation

$$\det(\nabla^2\phi(x))f_2(\nabla\phi(x)) = f_1(x) \quad .$$

In fact, the gradient of ϕ is unique $f_1\, dx$-almost everywhere. Since it is convenient to work with strong solutions, we recall here a corollary of a theorem of Caffarelli [Caf], who has developed a regularity theory for these convex solutions.

Theorem 3 *For $i = 1, 2$, let Ω_i be bounded domains of $\mathbb{R}^n$ and let f_i be non-negative functions, supported on Ω_i. Assume that f_i and $1/f_i$ are bounded on Ω_i and that Ω_2 is convex. If f_i, $i = 1, 2$ are Lipschitz then the Brenier mapping ϕ is twice continuously differentiable.*

Let $\mathscr{C}_L(\mathbb{R}^n)$ be the set of functions $f \in L_1^+(\mathbb{R}^n)$ which are the restriction to some open Euclidean ball of a positive Lipschitz function on $\mathbb{R}^n$.

Let us remark that it suffices to establish (BL) and (RBL) for functions in $\mathscr{C}_L(\mathbb{R}^{n_i})$. We strongly rely on the monotonicity of the functions I and J. Assume that (RBL) holds for functions in $\mathscr{C}_L(\mathbb{R}^{n_i})$. Then it is clearly true for positive Lipschitz functions. By classical monotone convergence arguments, this yields (RBL) for functions in the classes $\mathscr{D}_L(\mathbb{R}^{n_i})$ of pointwise limits of decreasing sequences of positive Lipschitz functions. Let $f_i \in L_1^+(\mathbb{R}^{n_i}), i = 1, \dots, m$, and let $\varepsilon > 0$. Then there exist m functions $(s_i)_{i=1}^m$, which are positive linear combination of characteristic functions of compact sets and such that

$$f_i \geq s_i \text{ and } \int f_i - \int s_i \leq \varepsilon \quad .$$

Since s_i belongs to $\mathscr{D}_L(\mathbb{R}^{n_i})$, (RBL) holds for $(s_i)_{i=1}^m$. Thus

$$I((f_i)_{i=1}^m) \geq I((s_i)_{i=1}^m) \geq E\prod_{i=1}^m \left(\int s_i\right)^{c_i} \geq E\prod_{i=1}^m \left(-\varepsilon + \int f_i\right)^{c_i} \quad .$$

Hence (RBL) is always true. The same kind of argument is valid for (BL).

Proof of Lemma 3. We assume that $D > 0$. By homogeneity we can also assume that $\int f_i = \int h_i = 1$ for all i. The previous remark allows us to work with functions f_i, h_i belonging to $\mathscr{C}_L(\mathbb{R}^{n_i})$, so that we can use Caffarelli's regularity result and Brenier's theorem. We denote by Ω_{h_i} the domain where h_i is positive. For $i = 1, \dots, m$, we get a differentiable function T_i deriving from a convex potential and such that for all $x \in \Omega_{h_i}$,

$$\det(dT_i(x)) \cdot f_i(T_i(x)) = h_i(x) \ .$$

Since T_i derives from a convex potential, its differential is symmetric semi-definite positive and because of the previous equation and of the non-vanishing property of h_i, we know that for all $x \in \Omega_{h_i}$, $dT_i(x) \in \mathscr{S}^+(\mathbb{R}^{n_i})$.

We define a function Θ from $\bigcap_{i=1}^m B_i^{-1}(\Omega_{h_i}) \subset \mathbb{R}^n$ into $\mathbb{R}^n$ by

$$\Theta(y) = \sum_{i=1}^m c_i B_i^*(T_i(B_i y)) \ .$$

Its differential is symmetric semi-definite positive

$$d\Theta(y) = \sum_{i=1}^m c_i B_i^* dT_i(B_i y) B_i \ ,$$

and it is actually definite positive because:

$$\det\left(\sum_{i=1}^m c_i B_i^* dT_i(B_i y) B_i\right) \geq D \prod_{i=1}^m (\det dT_i(B_i y))^{c_i} > 0$$

In particular for all $v \neq 0$ in $\mathbb{R}^n$,

$$\langle d\Theta(y) \cdot v, v \rangle > 0$$

so Θ is injective. Denoting $S = \bigcap_{i=1}^m B_i^{-1}(\Omega_{h_i})$, we can write

$$\begin{aligned}\int_{\mathbb{R}^n} \prod_{i=1}^m h_i^{c_i}(B_i y)\, dy &= \int_S \prod_{i=1}^m h_i^{c_i}(B_i y)\, dy \\ &= \int_S \prod_{i=1}^m \big(f_i(T_i(B_i y)) \det dT_i(B_i y)\big)^{c_i}\, dy\end{aligned}$$

$$\leq \frac{1}{D}\int_S \prod_{i=1}^m f_i(T_i(B_i y))^{c_i} \det\left(\sum_{i=1}^m c_i B_i^* dT_i(B_i y) B_i\right) dy$$

$$\leq \frac{1}{D}\int_S \sup_{\Theta(y)=\sum_{i=1}^m c_i B_i^* x_i} \left(\prod_{i=1}^m f_i(x_i)^{c_i}\right) \det(d\Theta(y))\, dy$$

$$\leq \frac{1}{D}\int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^m c_i B_i^* x_i} \left(\prod_{i=1}^m f_i(x_i)^{c_i}\right) dx$$

which concludes the proof of Lemma 3. □

2 Equality cases

In this section, we restrict to functions of one real variable. With the notation of Theorem 1 there are vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$ such that $\mathrm{span}((v_i)_{i=1}^m) = \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$B_i(x) = \langle x, v_i \rangle, \quad B_i^*(t) = t v_i \quad \text{and} \quad B_i^* B_i = v_i \otimes v_i \ .$$

We are going to study the best constant in inequalities (BL) and (RBL) and to characterize equality cases. We call maximizers the non-zero functions that give equality in (BL) and minimizers those that provide equality in (RBL).

2.1 The geometric structure of the problem

Given $(v_i)_{i=1}^m$ which span $\mathbb{R}^n$, we introduce some notation. For a subset K of $\{1, \ldots, m\}$, we denote by E_K the linear span in $\mathbb{R}^n$ of the vectors $(v_k)_{k \in K}$. We call *adapted partition* a partition S of $\{1, \ldots, m\}$ such that:

$$\mathbb{R}^n = \bigoplus_{K \in S} E_K \ .$$

These partitions are useful because this splitting of the space $\mathbb{R}^n$ yields a splitting of the Brascamp-Lieb inequality and of its converse, so that one can work separately on each piece. We shall first show that there exists a best adapted partition.

Proposition 1 *Let $\bowtie$ be the relation on $N_m = \{1, \ldots, m\}$ defined by as follows: $i \bowtie j$ if and only if there exists a subset K of N_m of cardinality*

$n-1$ such that both $(v_i, (v_k)_{k\in K})$ and $(v_j, (v_k)_{k\in K})$ are bases of $\mathbb{R}^n$. Let $\sim$ be the transitive completion of $\bowtie$ ($i \sim j$ means that there exists a path between i and j in which two consecutive elements are in relation for $\bowtie$).

Then $\sim$ is an equivalence relation and the subdivision C of N_m into equivalence classes for $\sim$ is the finest adapted partition.

Proof. We first establish that C is more accurate than any adapted partition S. Let $I, J \in S$, $I \neq J$ and let $i \in I$, $j \in J$. It suffices to show that $i \bowtie j$ is impossible.

Assume precisely that $i \bowtie j$. Then there exists $K \subset N_m$ such that

$$\mathbf{e}_i = (v_i, (v_k)_{k\in K}) \text{ and } \mathbf{e}_j = (v_j, (v_k)_{k\in K})$$

are bases of $\mathbb{R}^n$. As S is adapted, we have $\mathbb{R}^n = \bigoplus_{H\in S} E_H$, each of them being spanned by some v_i's. So, every basis of $\mathbb{R}^n$ with elements taken among the v_i's must contain $\dim(E_H)$ elements in E_H. But our bases $\mathbf{e}_i$ and $\mathbf{e}_j$ do not have the same number of vectors in E_I because $v_i \in E_I$ and $v_j \in E_J$. Thus we have a contradiction.

We now show that the partition C is adapted to our geometric setting. Let I be an equivalence class for $\sim$ and let E_I be the corresponding space. Since the vectors $(v_i)_{i=1}^m$ span $\mathbb{R}^n$, we find a permutation of indices such that $\mathbf{b} = (v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$ and $(v_1, \ldots, v_r)$ is a basis of E_I for some $r \leq n$. Let us denote by F the span of $v_{r+1}, \ldots, v_n$.

Let $i \in N_m$; the vector v_i can be decomposed in the basis $\mathbf{b}$:

$$v_i = \sum_{k=1}^{n} \alpha_k v_k \quad .$$

For any $j \leq n$, we notice that

$$\det_{\mathbf{b}}(v_1, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n) = \alpha_j \quad ,$$

hence $\alpha_j \neq 0$ implies that v_i and v_j belong to neighbor bases, that is $i \sim j$. So, if $i \in I$, as i can be in relation for $\bowtie$ only with elements of I we have $\alpha_{r+1}, \ldots, \alpha_n = 0$. Thus $i \in I$ implies $v_i \in E_I$. By a similar argument, if $i \notin I$, $\alpha_1, \ldots, \alpha_r = 0$ and v_i belongs to F. We have proved that $\mathbb{R}^n = \mathrm{span}\{v_i, i \in I\} \bigoplus \mathrm{span}\{v_i, i \notin I\}$, this is the first step of the decomposition. The result follows by induction, noticing that the relation $\sim$ can be restricted to F. □

As a consequence of Proposition 1, let us observe that it is sufficient to study the case when the relation $\bowtie$ has only one equivalence class. In this case we say that $(\mathbb{R}^n, (v_i)_{i=1}^m)$ is *irreducible*.

2.2 The Gaussian case

Let $v_1, \dots, v_m$ be vectors of $\mathbb{R}^n$ such that $\mathbb{R}^n = \operatorname{span}((v_i)_{i=1}^m)$. For $I \subset \{1, \dots, m\}$ of cardinality $|I| = n$, we denote

$$d_I = \det((v_i)_{i \in I})^2 \ .$$

For each m-tuple $c = (c_i)_{i=1}^m$ of positive numbers, we study the constant D_c defined by

$$D_c = \inf\left\{ \frac{\det(\sum_{i=1}^m \lambda_i v_i \otimes v_i)}{\prod_{i=1}^m \lambda_i^{c_i}} ; \lambda_i > 0, i = 1 \dots m \right\} \ .$$

We wish to know when D_c is positive and when the infimum is achieved. We will sometimes call minimizers the m-tuples $(\lambda_i)_{i=1}^m$ for which D_c is achieved.

The computation of the previous determinant is made possible by the Cauchy-Binet formula which we recall:

Proposition 2 *Let $m \geq n$ be integers; let A be a $n \times m$ matrix and let B be a $m \times n$ matrix. For $I \subset N_m$* of *cardinality n we denote by A_I the square matrix obtained from A by keeping only the columns with indices in I; we denote by B^I the square matrix obtained from B by keeping the rows with indices in I. Then we have the formula*

$$\det(AB) = \sum_{|I|=n} \det(A_I) \det(B^I)$$

where the sum is over all subsets of cardinality n of N_m.

The relevance of this formula is clear from

Corollary 1 *Let $m \geq n$ and let $(v_1, \dots, v_m)$ be vectors in $\mathbb{R}^n$. Then*

$$\det\left(\sum_{i=1}^m \lambda_i v_i \otimes v_i \right) = \sum_{|I|=n} \lambda_I \big(\det((v_i)_{i \in I})\big)^2 = \sum_{|I|=n} \lambda_I d_I \ ,$$

where for $I \subset N_m$, we have set $\lambda_I = \prod_{i \in I} \lambda_i$.

The condition for D_c to be non-zero has a rather nice geometric description which requires some notation. For $I \subset \{1, \dots, m\}$, let 1_I be the vector of $\mathbb{R}^m$ of coordinates $(1_I)_i = \delta_{i \in I}$ (it is the characteristic function of I). One has the following result:

Proposition 3 *The infimum D_c is positive if and only if the vector $c \in \mathbb{R}^m$ belongs to the convex hull of the characteristic vectors 1_I of the subsets I of cardinality n such that the vectors $(v_i)_{i \in I}$ form a basis of $\mathbb{R}^n$.*

Proof. We shall first show that the condition is sufficient. Assume that we have a family of non-negative real numbers $(t_I)_{|I|=n}$ indexed by the subsets of cardinality n of $\{1, \dots, m\}$, such that

$$t_I = 0 \text{ whenever } d_I = 0 \ ,$$
$$c_i = \sum_{|I|=n, i \in I} t_I, \text{ for all } i \ .$$

Let λ_i, $i = 1, \dots, m$ be positive. By the Cauchy-Binet formula and the arithmetic mean-geometric mean inequality with coefficients t_I (their sum is indeed one), we have:

$$\det\left(\sum_{i=1}^m \lambda_i v_i \otimes v_i\right) = \sum_{|I|=n} \lambda_I d_I = \sum_{t_I \neq 0} t_I \left(\frac{\lambda_I}{t_I} d_I\right) + \sum_{t_I = 0} \lambda_I d_I \geq \prod_{t_I \neq 0} \left(\frac{\lambda_I d_I}{t_I}\right)^{t_I}.$$

Each λ_i appears with the total exponent

$$\sum_{i \in I, t_I \neq 0} t_I \ ,$$

equal to c_i by hypothesis. Thus we have

$$\det\left(\sum_{i=1}^m \lambda_i v_i \otimes v_i\right) \geq \prod_{t_I \neq 0} \left(\frac{d_I}{t_I}\right)^{t_I} \prod_{i=1}^m \lambda_i^{c_i} \ .$$

Since $t_I \neq 0$ implies $d_I \neq 0$, the constant D_c is positive.

Let us now prove that the condition is necessary. For all $\lambda_i > 0$, $i = 1, \dots, m$, let

$$\Delta(\lambda_1, \dots, \lambda_m) := \frac{\sum_{|I|=n} d_I \lambda_I}{\prod_{i=1}^m \lambda_i^{c_i}} \ .$$

Let $(x_i)_{i=1}^m \in \mathbb{R}^n$ and let $N > 0$. There always exists a positive a such that the quantity $\Delta((e^{-N x_i})_{i=1}^m)$ is equivalent when N tends to infinity to

$$a \cdot \exp N\left(\sum_{i=1}^m x_i c_i + \max\left\{-\sum_{i \in I} x_i;\ |I| = n \textit{ and } d_I \neq 0\right\}\right) \ .$$

If we assume that D_c is positive, then $\Delta((e^{-Nx_i})_{i=1}^m) \geq D_c > 0$ cannot tend to zero. Necessarily, for all $(x_i)_{i=1}^m \in \mathbb{R}^m$, one has:

$$\sum_{i=1}^m x_i c_i \geq \min\left\{\sum_{i\in I} x_i;\ |I| = n \text{ and } d_I \neq 0\right\} .$$

This can be reformulated in terms of convex cones as $\cap_{d_I\neq 0}\mathscr{C}_{1_I} \subset \mathscr{C}_c$, where, for $y \in \mathbb{R}^m$, $\mathscr{C}_y = \{x \in \mathbb{R}^m;\ \langle x, y\rangle \geq 0\}$. By duality of convex cones, this implies that the vector c belongs to the convex cone generated by the vectors 1_I such that $d_I \neq 0$. Thus there exist non-negative real numbers $(t_I)_{I,d_I\neq 0}$ such that for all $i \leq m$,

$$c_i = \sum_{|I|=n \text{ and } i\in I} t_I \ .$$

Summing over i yields $\sum_{d_I\neq 0} t_I = (\sum_{i=1}^m c_i)/n$. But the hypothesis $D_c > 0$ implies that the numerator and the denominator of Δ have the same homogeneity degree in the variables. So $\sum_{i=1}^m c_i = n$ and we have shown that c belongs to the convex hull of the 1_I such that $d_I \neq 0$. □

Remark. Let $K = \{x \in [0,1]^m;\ \sum_{i=1}^m x_i = n\}$, it is the convex hull of the vectors $(1_I)_{|I|=n}$. By the previous result, D_c is non-zero only if c is in K. If the vectors (v_i) are in generic position, $D_c \neq 0$ if and only if $c \in K$. But as the 1_I are clearly the only extremal points of K, any geometrical degeneracy (i.e. any d_I equal to zero) will imply a reduction of the domain where c must be.

We know that D_c is positive if and only if c can be written as a convex combination of certain vectors. The next proposition states that D_c is achieved if and only if there exists a convex combination with some additional property.

Proposition 4 *The constant D_c is achieved if and only if there exist positive numbers $(t_I)_{|I|=n}$ and $(\lambda_i)_{i=1}^m$ such that*

$$c = \sum_{|I|=n} t_I 1_I$$

and for all I

$$t_I = d_I \prod_{i\in I} \lambda_i \ .$$

Notice that $d_I = 0$ implies $t_I = 0$, so the result is coherent with the previous one.

Proof. The *if* part comes from a precise study of the first half of the proof of Proposition 3. All inequalities stated in this proof become equalities for the particular m-tuple $(\lambda_i)_{i=1}^m$. The arithmetic-geometric inequality is an equality because for all I, $\lambda_I d_I / t_I = 1$. Moreover, the term $\sum_{I, t_I = 0} \lambda_I d_I$ is zero.

The *only if* part is obvious by differentiation. □

We are going to rewrite our problem in the setting of Fenchel duality for convex functions in order to use the following result (see [Roc] p. 264):

Proposition 5 *Let ϕ be a l.s.c. convex function on $\mathbb{R}^m$ and let ϕ^* be its Fenchel conjugate, defined for $x \in \mathbb{R}^m$ by*

$$\phi^*(x) = \sup_{y \in \mathbb{R}^m} \langle x, y \rangle - \phi(y) \ .$$

Then $\phi^*(x)$, which is a supremum, is achieved if and only if ϕ^* is subdifferentiable at the point x. In particular, it is achieved when x belongs to the relative interior of $\mathrm{dom}(\phi^*) = \{y \in \mathbb{R}^m;\ \phi^*(y) < +\infty\}$.

Let us define the function ϕ on $\mathbb{R}^m$ by

$$\phi(x_1, \ldots, x_m) = \log\det\left(\sum_{i=1}^m e^{t_i} v_i \otimes v_i\right) \ .$$

The next proposition links our problem on D_c with the study of the Fenchel conjugate of ϕ.

Proposition 6 1. *The function ϕ is convex.*

2. *The constant D_c is equal to* $\exp(-\phi^*(c))$.
3. *D_c is positive if and only if $c \in$* $\mathrm{dom}(\phi^*)$.
4. *D_c is achieved if and only if $\phi^*(c)$ is.*
5. $\mathrm{dom}\,(\phi^*)$ *is equal to $K =$* $\mathrm{conv}\{1_I;\ d_I \neq 0\}$.
6. *The constant D_c is achieved when c belongs to the relative interior of K.*

Proof. The convexity of ϕ is a consequence of the Cauchy-Schwarz inequality: let $s, t \in \mathbb{R}^m$,

$$\phi\left(\frac{t+s}{2}\right) = \log\left(\sum_{|I|=n}\left\{d_I \exp\left(\sum_{i\in I} t_i\right)\right\}^{\frac{1}{2}}\left\{d_I \exp\left(\sum_{i\in I} s_i\right)\right\}^{\frac{1}{2}}\right)$$
$$\leq \log\left(\left\{\sum_{i\in I} d_I \exp\left(\sum_{i\in I} t_i\right)\right\}^{\frac{1}{2}}\left\{\sum_{i\in I} d_I \exp\left(\sum_{i\in I} s_i\right)\right\}^{\frac{1}{2}}\right)$$
$$= \frac{\phi(t)+\phi(s)}{2} \quad .$$

The other assertions are also very simple. □

The last statement of the previous proposition allows us to recover a result of [BL].

Corollary 2 *If for all $I \subset N_m$ of cardinality n, $d_I = \det((v_i)_{i\in I})$ is not zero, then for all $c = (c_i)_{i=1}^m$ such that:*

$$\sum_{i=1}^m c_i = n \quad \textit{and} \quad 0 < c_i < 1 \textit{ for all } i \ ,$$

the constant D_c is achieved for a certain $(\lambda)_{i=1}^m$.

The following result shows that the reciprocal statement is almost true.

Proposition 7 *If $(\mathbb{R}^n, (v_i)_{i=1}^m)$ is irreducible and if $c_1 = 1$, then D_c is achieved only when $m = n = 1$.*

We come to unicity results: if D_c is achieved, there is a unique minimizer, up to scalar multiplication.

Proposition 8 *Assume that $(\mathbb{R}^n, (v_i)_{i=1}^m)$ has the irreducibility property. If $(\lambda_i)_{i=1}^m$ and $(\mu_i)_{i=1}^m$ are two minimizers, then there exists $r \in \mathbb{R}$ such that for all i, $\lambda_i = r\mu_i$.*

Proof. Let $t = ((t_i)_{i=1}^m)$ and $s = ((s_i)_{i=1}^m)$ such that for all i, $\lambda_i = e^{t_i}$ and $\mu_i = e^{s_i}$. Let ψ be the function on $\mathbb{R}^m$ defined for all $((x_i)_{i=1}^m)$ by

$$\psi((x_i)) = \phi((x_i)) - \sum_{i=1}^m c_i x_i \ .$$

Then ψ reaches its minimum at the points t, s and also at $(t+s)/2$ because it is convex. So we have

$$\frac{\phi(t)+\phi(s)}{2}=\phi\Big(\frac{t+s}{2}\Big) \ ,$$

and there must be equality in the Cauchy-Schwarz inequality in the proof of Proposition 6. Hence, there exists $a\in\mathbb{R}$ such that for all I, $|I|=n$,

$$d_I\exp\left(\sum_{i\in I}t_i\right)=a\cdot d_I\exp\left(\sum_{i\in I}s_i\right) \ .$$

In particular, if $d_I\neq 0$, one has

$$\prod_{i\in I}\left(\frac{\lambda_i}{\mu_i}\right)=a \ .$$

Let $i,j\in N_m$ such that $i\bowtie j$. By definition, there exists $K\subset N_m$ of cardinality $n-1$, such that $d_{\{i\}\cup K}$ and $d_{\{j\}\cup K}$ are both non-zero. So, we have

$$\prod_{l\in\{i\}\cup K}\left(\frac{\lambda_l}{\mu_l}\right)=\prod_{l\in\{j\}\cup K}\left(\frac{\lambda_l}{\mu_l}\right) \ ,$$

and after simplification

$$\frac{\lambda_i}{\mu_i}=\frac{\lambda_j}{\mu_j} \ .$$

By the irreducibility property (see Proposition 1), this implies $\frac{\lambda_1}{\mu_1}=\cdots=\frac{\lambda_m}{\mu_m}$. □

2.3 The general case

We studied existence and uniqueness of centered Gaussian maximizers for (BL) and minimizers for (RBL). Now, we turn to the general study. As explained before, we may assume that $(\mathbb{R}^n,(v_i)_{i=1}^m)$ is irreducible. The behavior of extremal functions is very different for $n=1$ and for $n\geq 2$.

2.3.1 The case $n=1$

If $n=1$, then $n_i=1$ for all $i\leq m$, the condition on $(c_i)_{i=1}^m$ is just $\sum_{i=1}^m c_i=1$, and the v_i's are just real numbers. The inequality (BL) is nothing else than Hölder's inequality for the functions $x\mapsto f_i(v_ix)$, whereas (RBL) is the Prékopa-Leindler inequality for $x\mapsto f_i(x/v_i)$.

The equality cases can be settled from our proof; we will not do it because they are well-known: if $\sum_{i=1}^{m} c_i = 1$, and $f_i \in L_1^+(\mathbb{R})$, $i = 1, \dots, m$ are non identically zero, then

$$\int\limits_{\mathbb{R}} \prod_{i=1}^{m} f_i^{c_i}(x)dx = \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i \right)^{c_i}$$

holds if and only if

$$\frac{f_1}{\int_{\mathbb{R}} f_1} = \cdots = \frac{f_m}{\int_{\mathbb{R}} f_m} \quad .$$

Under the same assumptions,

$$\int\limits_{\mathbb{R}} \sup_{\sum c_i x_i = x} \prod_{i=1}^{m} f_i^{c_i}(x_i)dx = \prod_{i=1}^{m} \left(\int\limits_{\mathbb{R}} f_i \right)^{c_i}$$

holds if and only if there exists $(y_i)_{i=1}^{m} \in \mathbb{R}^m$ such that

$$\frac{f_1(\cdot - y_1)}{\int_{\mathbb{R}} f_1} = \cdots = \frac{f_m(\cdot - y_m)}{\int_{\mathbb{R}} f_m} \text{is a log-concave function} \quad .$$

2.3.2 The case $n \geq 2$

We prove that if there is a centered Gaussian extremizer, then up to dilatation and scalar multiplication, it is the only extremizer.

Theorem 4 *Let $n \geq 2$ and let $(\mathbb{R}^n, (v_i)_{i=1}^{m})$ be irreducible. Let $(c_i)_{i=1}^{m}$ and $(\lambda_i)_{i=1}^{m}$ be positive numbers such that D_c is achieved for $(\lambda_i)_{i=1}^{m}$:*

$$\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i \right) = D_c \prod_{i=1}^{m} \lambda_i^{c_i} \quad .$$

Then $(h_i)_{i=1}^{m}$ is a maximizer for (BL) if and only if there exist $a > 0$, $(\alpha_i)_{i=1}^{m}$ positive and $y \in \mathbb{R}^n$ such that for all i and for all $t \in \mathbb{R}$,

$$h_i(t) = \alpha_i \exp(-\lambda_i(at - \langle y, v_i \rangle)^2) \quad . \tag{1}$$

The m-tuple $(h_i)_{i=1}^{m}$ is a minimizer for (RBL) if and only if there exist $b > 0$, $(\beta_i)_{i=1}^{m}$ positive and $(t_i)_{i=1}^{m}$ real such that for all i and for all $t \in \mathbb{R}$,

$$h_i(t) = \beta_i \exp(-(bt - t_i)^2/\lambda_i) \quad .$$

Proof. By Lemma 1 and by the proof of Lemma 2, we know that $(G_{\lambda_i})_{i=1}^m$ is a maximizer for (BL) and $(G_{\lambda_i^{-1}})_{i=1}^m$ a minimizer for (RBL), so by simple changes of variables in $\mathbb{R}^n$, one can check that the previous functions are extremizers.

Let $(h_i)_{i=1}^m$ be a maximizer for (BL) and $(f_i)_{i=1}^m$ be a minimizer for (RBL). We may assume that f_i, h_i are positive and continuous for all i. Indeed by the following lemma (which was communicated to us by K. Ball) we know that $(h_i * G_{\lambda_i})_{i=1}^m$ is a positive and continuous maximizer for (BL). If we know that it is Gaussian, then so is $(h_i)_{i=1}^m$ by the properties of the Fourier transform. The same argument is relevant for (RBL).

Lemma 4 *If* $(f_i)_{i=1}^m$ *and* $(g_i)_{i=1}^m$ *are maximizers for* (*BL*), *then so is* $(f_i * g_i)_{i=1}^m$. *If* $(f_i)_{i=1}^m$ *and* $(g_i)_{i=1}^m$ *are minimizers for* (*RBL*), *then so is* $(f_i * g_i)_{i=1}^m$.

A proof of the first part of this lemma appears in [Bar2], the proof of the second part is similar. Notice that this lemma is valid for the multidimensional version of the inequalities.

Let us now prove that positive continuous maximizers for (BL) are of the form (1); the proof for (RBL) is analogous and a bit simpler. Let $(f_i)_{i=1}^m$ such that $f_i(x) = \exp\left(-x^2/\lambda_i\right)$; it is a minimizer for (RBL). Let $(h_i)_{i=1}^m$ be a positive continuous maximimizer for (BL). We study precisely the proof of Lemma 3 applied with $(h_i)_{i=1}^m$ and $(f_i)_{i=1}^m$. Since our functions are positive, the changes of variables T_i's are increasing differentiable bijections of $\mathbb{R}$, such that for all $t \in \mathbb{R}$,

$$T_i'(t) \cdot f_i(T_i(t)) = h_i(t) \quad .$$

There must be equality in every step of the proof. In particular, for all $y \in \mathbb{R}^n$, one has

$$\det\left(\sum_{i=1}^m T_i'(\langle y, v_i\rangle) v_i \otimes v_i\right) = D_c \prod_{i=1}^m \left(T_i'(\langle y, v_i\rangle)\right)^{c_i} \quad .$$

By irreducibility and Proposition 8, one gets for all $y \in \mathbb{R}^n$,

$$\frac{T_1'(\langle y, v_i\rangle)}{\lambda_1} = \cdots = \frac{T_m'(\langle y, v_i\rangle)}{\lambda_m} \quad .$$

Since $n \geq 2$, for all $i \leq m$ there exists $j \leq m$ such that v_i and v_j are not collinear; so there exists $z \in \mathbb{R}$ such that $\langle z, v_i\rangle = 1$ and $\langle z, v_i\rangle = 0$. The previous relation for $y = tz$ means that for all $t \in \mathbb{R}$

$$\frac{T_i'(t)}{\lambda_i} = \frac{T_j'(0)}{\lambda_j} \quad .$$

Consequently, there exist $a > 0$ and real numbers $(s_i)_{i=1}^m$ such that for all i and for all $t \in \mathbb{R}$,

$$T_i(t) = a\lambda_i t + s_i$$

and by the change of variable formula between h_i and f_i we get

$$h_i(t) = T_i'(t)\exp\left(-T_i^2(t)/\lambda_i\right) = \mu_i \exp(-\lambda_i(at - t_i)^2) \ ,$$

for some positive (μ_i) and some real (t_i).

It remains to find which translates of a centered Gaussian maximizer are still maximizers. Let $(g_i)_{i=1}^m$ be a maximizer,

$$g_i(t) = \exp(-\lambda_i t^2) \ ,$$

and let $x = (x_i)_{i=1}^m \in \mathbb{R}^m$ and for $i \le m$, $h_i(t) = g_i(t - x_i)$. Let us consider $\mathbb{R}^m$ with the Euclidean metric given by

$$N^2(w) = \sum_{i=1}^m c_i \lambda_i w_i^2 \ ,$$

and the subspace

$$K = \left\{ (\langle y, v_i \rangle)_{i=1}^m ; \ y \in \mathbb{R}^n \right\} \ .$$

Let s be the orthogonal projection of x onto A. Then there exists $z \in \mathbb{R}^n$ satisfying $s_i = \langle z, v_i \rangle$ for all i; moreover, by the Pythagore Theorem

$$N\left((\langle y, v_i \rangle - x_i)_{i=1}^m\right) \ge N\left((\langle y, v_i \rangle - \langle z, v_i \rangle)_{i=1}^m\right)$$

with equality only if x belongs to A, that is $x = s$. Thus $J((h_i)_{i=1}^m) \le J((g_i)_{i=1}^m)$, with equality only if $x_i = \langle z, v_i \rangle$ for all i. □

Remark. There are, for $n \ge 2$, some remaining questions. If there is no centered Gaussian maximizer, is there any maximizer at all? The answer seems to be no: if $(f_i)_{i=1}^m$ is a maximizer then by the Brascamp-Lieb-Luttinger inequality [BLL] so is $(f_i^*)_{i=1}^m$, where f^* is the symmetric rearrangement. As K. Ball noticed it, for every integer k

$$\Big(\sqrt{k}\underbrace{f_i^* * \cdots * f_i^*}_{k \text{ times}}(\sqrt{k}\cdot)\Big)_{i=1}^m$$

is also a maximizer. Moreover, under some integrability assumptions, it converges to a centered Gaussian m-tuple by the Central Limit Theorem.

Notice that our method gives the answer when there are positive continuous maximizers for (BL) and positive continuous minimizers for the corresponding (RBL). The study of the equality case of Lemma 3 shows that the constant D must be achieved, so there is a centered Gaussian maximizer.

3 Applications to convex geometry

3.1. Dimension one

K. Ball noticed that an additional geometrical hypothesis, which is often available in convexity, supports an easy computation of the optimal constant in the Brascamp-Lieb inequality. His version of (BL) gives sharp upper estimates for the volume of sections of the unit ball of $\ell_p^m = (\mathbb{R}^m, \|\cdot\|_p)$ (see [Bal1] and Proposition 8 of [Bal3]). Using (BL), K. Ball also proved that simplices have maximal volume ratio [Bal3], and that among symmetric convex bodies, parallelotopes do have [Bal1]. We recall that the volume ratio $\mathrm{vr}(K)$ of a convex body $K \subset \mathbb{R}^n$ is the ratio of the volume of K to the volume of the maximal volume ellipsoid contained in K (called the John ellipsoid, see [Joh]).

However, the question of equality cases in Ball's version of (BL) and in applications was open. Moreover, one needed a corresponding tool for dual problems such as minorizing the volume of n-dimensional projections of the unit ball of ℓ_p^m. In this section, we point out a version of (RBL) which applies to such dual problems and we characterize equality cases in (BL), (RBL) and in the applications.

Ball's version of the Brascamp-Lieb inequality and our reverse version are as follows:

Theorem 5 *Let $m \geq n$, let $(u_i)_{i=1}^m$ be unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be positive real numbers such that*

$$\sum_{i=1}^m c_i u_i \otimes u_i = I_n \quad .$$

Then for all $f_i \in L_1^+(\mathbb{R})$, $i = 1, \dots, m$ one has

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(\langle x, u_i \rangle) dx \leq \prod_{i=1}^m \left(\int f_i \right)^{c_i} \quad ,$$

and

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum c_i\theta_i u_i} \prod_{i=1}^{m} f_i^{c_i}(\theta_i)dx \geq \prod_{i=1}^{m}\left(\int f_i\right)^{c_i} .$$

This result follows from Theorem 1. The optimal constant (equal to 1) is provided by Ball's observation:

Proposition 9 *Let $m \geq n$, let $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^n$ such that $\sum_{i=1}^{m} v_i \otimes v_i = I_n$, where I_n stands for the identity map. Then for every m-tuple $(\lambda_i)_{i=1}^{m}$ of positive numbers*

$$\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i\right) \geq \prod_{i=1}^{m} \lambda_i^{|v_i|^2} .$$

There is equality when $\lambda_1 = \cdots = \lambda_m$.

Proof. By the Cauchy-Binet formula, we have

$$1 = \det I_n = \det\left(\sum_{i=1}^{m} v_i \otimes v_i\right) = \sum_{|I|=n} d_I .$$

Hence we can use the arithmetic-geometric inequality with coefficients d_I:

$$\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i\right) = \sum_{|I|=n} \lambda_I d_I \geq \prod_{|I|=n} \lambda_I^{d_I} .$$

Each λ_i appears with the total exponent $\sum_{I,i\in I} d_I$. Applying Corollary 1 to the m-tuple $(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_m)$, we get:

$$\begin{aligned}\sum_{I,i\in I} d_I &= \sum_{I} d_I - \sum_{I,i\notin I} d_I \\ &= 1 - \det(v_1 \otimes v_1 + \cdots + v_{i-1} \otimes v_{i-1} + v_{i+1} \\ &\qquad \otimes v_{i+1} + \cdots + v_m \otimes v_m) \\ &= 1 - \det(I_n - v_i \otimes v_i) = |v_i|^2 .\end{aligned}$$ □

The equality cases in Theorem 5 are completely settled by Proposition 1 and Theorem 4. The space $\mathbb{R}^n$ is the direct sum of irreducible subspaces. The additional hypothesis on $(u_i)_{i=1}^{m}$ clearly implies that the sum is orthogonal. On irreducible subspaces of dimension one there is equality for (BL) if and only if the functions are equal up to scalar multiplication and, for (RBL), if and only if all the functions are equal, up to multiplication and translation, to a common log-concave function. On irreducible spaces of dimension ≥ 2, there is equality if and only if the functions are (up to scalar multiplications, up to translations and only coherent translations in the direct form) equal to a common centered Gaussian function. In particular, we get

Corollary 3 *Let $m \geq n$, let $(u_i)_{i=1}^m$ be m distinct unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be positive real numbers such that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n \quad .$$

If $(f_i)_{i=1}^m$ are non-identically-zero functions in $L_1^+(\mathbb{R})$ such that none of them is a Gaussian and

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i^{c_i}(\langle x, u_i \rangle) dx = \prod_{i=1}^{m} \left(\int f_i \right)^{c_i} \quad ,$$

then $m = n$ and $(u_i)_{i=1}^m$ is an orthonormal basis of $\mathbb{R}^n$.

This result allows us to settle equality cases in Ball's volume ratio estimates. We denote by Q_n the unit cube and by Δ_n the regular simplex.

Proposition 10 *Let $K \subset \mathbb{R}^n$ a convex body.*

- If *K is symmetric and* $\mathrm{vr}(K) = \mathrm{vr}(Q_n)$ *then K is a parallelotope.*
- *If* $\mathrm{vr}(K) = \mathrm{vr}(\Delta_n)$ *then K is a simplex.*

As mentioned by Ball in [Bal3], one gets, as a consequence of (RBL), the sharp lower bounds of the outer volume ratio, which involves the minimal volume ellipsoid containing a body K. A proof of such lower estimates relies on the following dual version of Proposition 8 of [Bal3].

Proposition 11 *Let $m \geq n$, let $(u_i)_{i=1}^m$ be unit vectors in $\mathbb{R}^n$ and $(c_i)_{i=1}^m$ positive numbers such that $\sum_{i=1}^m c_i u_i \otimes u_i = I_n$. Let $(\alpha_i)_{i=1}^m$ be positive numbers and let $p \geq 1$. If K is the unit ball of $\mathbb{R}^n$ with the norm*

$$\|x\| = \sup_{x=\sum_{i=1}^m c_i \theta_i u_i} \left(\sum_{i=1}^{m} \alpha_i |\theta_i|^p \right)^{\frac{1}{p}} \quad ,$$

then

$$\mathrm{Vol}(K) \geq \frac{2^n \Gamma(1+1/p)^n}{\Gamma(1+n/p)} \prod_{i=1}^{m} \left(\frac{c_i}{\alpha_i}\right)^{\frac{c_i}{p}} .$$

Proof. We apply (RBL) in the form of Theorem 5 to the functions f_i such that $f_i(t) = \exp(-\alpha_i |t|^p / c_i)$ for all $t \in \mathbb{R}$. We get

$$\begin{aligned}
\mathrm{Vol}(K) &= \frac{1}{\Gamma(1+n/p)} \int_{\mathbb{R}^n} e^{-\|x\|^p} dx \\
&= \frac{1}{\Gamma(1+n/p)} \int_{\mathbb{R}^n} \sup_{x=\sum c_i \theta_i u_i} \prod_{i=1}^{m} f_i^{c_i}(\theta_i) dx \\
&\geq \frac{1}{\Gamma(1+n/p)} \prod_{i=1}^{m} \left(2 \left(\frac{c_i}{\alpha_i}\right)^{\frac{1}{p}} \Gamma(1+1/p) \right)^{c_i} . \qquad \square
\end{aligned}$$

Remark. Applying (RBL) with the functions $f_i = \mathbf{1}_{[-\beta_i/c_i, \beta_i/c_i]}$ yields Ball's lower estimate for the volume of zonoids [Bal2]: under the same assumptions on $(u_i)_{i=1}^m$ and $(c_i)_{i=1}^m$, one has

$$\mathrm{Vol}\left(\sum_{i=1}^{m} \beta_i [-u_i, u_i] \right) \geq 2^n \prod_{i=1}^{m} \left(\frac{\beta_i}{c_i}\right)^{c_i} .$$

3.2 Larger dimensions

We obtain a multidimensional generalization of Ball's version of the Brascamp-Lieb inequality and its converse. It follows from Theorem 1. The computation of the optimal constant, here equal to 1, comes from a simple extension of Proposition 9.

Theorem 6 *Let m, n be integers. For $i = 1, \ldots, m$ let E_i be a subspace of $\mathbb{R}^n$ of dimension n_i and let P_i be the orthogonal projection onto E_i (on each E_i there is a Lebesgue measure compatible with the induced Euclidean structure). Assume that there exist positive numbers $(c_i)_{i=1}^m$ satisfying*

$$\sum_{i=1}^{m} c_i P_i = I_n .$$

If for $i = 1, \ldots, m$, f_i is a non-negative integrable function on E_i, then one has

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(P_i x)\, d^n x \leq \prod_{i=1}^m \left(\int_{E_i} f_i \right)^{c_i} ,$$

and

$$\int^{*}_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^m c_i x_i,\, x_i \in E_i} \prod_{i=1}^m f_i^{c_i}(x_i)\, d^n x \geq \prod_{i=1}^m \left(\int_{E_i} f_i \right)^{c_i} .$$

Remark. When the f_i's are characteristic functions of sets, the reverse inequality provides a Brunn-Minkowski type result for compact subsets of subspaces: if $K_i \subset E_i$ then

$$\mathrm{Vol}_n \left(\sum_{i=1}^m c_i K_i \right) \geq \prod_{i=1}^m (\mathrm{Vol}_{E_i}(K_i))^{c_i} .$$

References

[Bal1] K.M. Ball. Volumes of sections of cubes and related problems. In J. Lindenstrauss and V.D. Milman, editors, Israel seminar on Geometric Aspects of Functional Analysis, number 1376 in Lectures Notes in Mathematics. Springer-Verlag, 1989

[Bal2] K.M. Ball. Shadows of convex bodies. Transactions of the American Mathematical Society, **327(2)**, 891–901 (1991)

[Bal3] K.M. Ball. Volume ratio and a reverse isoperimetric inequality. Journal of the London Math. Soc., **44(2)**, 351–359 (1991)

[Bar1] F. Barthe. An extremal property of the mean width of the simplex. To appear in Math. Annalen

[Bar2] F. Barthe. Optimal Young's inequality and its converse: a simple proof. Geom. funct. anal., **8**, 234–242 (1998)

[Bar3] F. Barthe. Inégalités de Brascamp-Lieb et convexité. C. R. Acad. Sci. Paris, **324**, 885–888 (1997)

[Bec] W. Beckner. Inequalities in Fourier analysis. Annals of Math., **102**, 159–182 (1975)

[BL] H.J. Brascamp, E.H. Lieb. Best constants in Young's inequality, its converse and its generalization to more than three functions. Adv. Math., **20**, 151–173 (1976)

[BLL] H.J. Brascamp, E.H. Lieb, and J.M. Luttinger. A general rearrangement inequality for multiple integrals. J. Funct. Anal., **17**, 227–237 (1974)

[Bre1] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. C. R. Acad. Sci. Paris Sér. I Math., **305**, 805–808 (1987)

[Bre2] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. Comm. Pure Appl. Math, **44**, 375–417 (1991)

[Caf] L. Caffarelli. The regularity of mappings with a convex potential. J. Amer. Math. Soc., **4**, 99–104 (1992)

[Joh] F.John. Extremum problems with inequalities as subsidiary conditions. In Courant Anniversary Volume, pages 187–204, New York, 1948. Interscience

[Lei] L. Leindler. On a certain converse of Hölder's inequality. II. Acta Sci. Math. Szeged, **33**, 217–223 (1972)

[Lie] E.H. Lieb. Gaussian kernels have only Gaussian maximizers. Inventiones Mathematicae, **102**, 179–208 (1990)

[McC1] R.J. McCann. A Convexity Theory for Interacting Gases and Equilibrium Crystals. PhD thesis, Princeton University, 1994

[McC2] R.J. McCann. Existence and uniqueness of monotone measure-preserving maps. Duke Math. Journal, **80(2)**, 309–323 (1995)

[Pré] A. Prékopa. On logarithmic concave measures and functions. Acta Scient. Math., **34**, 335–343 (1973)

[Roc] R.T. Rockafellar. Convex Analysis. Princeton University Press, Princeton, 1972