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Borel-Weil-Bott theory on the moduli stack of *G*-bundles over a curve

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Abstract. Let G be a semi-simple group and $\mathfrak{M}$ the moduli stack of G-bundles over a smooth, complex, projective curve. Using representation-theoretic methods, I prove the pure-dimensionality of sheaf cohomology for certain "evaluation vector bundles" over $\mathfrak{M}$, twisted by powers of the fundamental line bundle. This result is used to prove a Borel-Weil-Bott theorem, conjectured by G. Segal, for certain generalized flag varieties of loop groups. Along the way, the homotopy type of the group of algebraic maps from an affine curve to G, and the homotopy type, the Hodge theory and the Picard group of $\mathfrak{M}$ are described. One auxiliary result, in Appendix A, is the Alexander cohomology theorem conjectured in [Gro2]. A self-contained account of the "uniformization theorem" of [LS] for the stack $\mathfrak{M}$ is given, including a proof of a key result of Drinfeld and Simpson (in characteristic 0). A basic survey of the simplicial theory of stacks is outlined in Appendix B.

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0. Introduction

Loop groups are known to admit a class of representations (those of positive energy) that are similar to those of semi-simple groups. Furthermore, their subgroups of loops that extend holomorphically to some compact Riemann surface are analogues of maximal parabolic subgroups. (This point of view I owe to Graeme Segal.) Accordingly, this paper, which is a sequel to [T2], presents some theorems of Borel-Weil-Bott type. It can be read independently of the prior article, granting its main result (which is recalled in Sect. I).

I shall work mostly in an algebraic framework. Let G be a complex, simple[1], simply connected Lie group, Σ a complex affine curve with compactification Σ^c, smooth, unless otherwise indicated. The group G^Σ (also denoted by $G[\Sigma]$) of algebraic G-valued functions on Σ is contained in the product $\hat{L}G$ of its formal completions $\hat{L}_iG$ at the points at ∞. (The $\hat{L}_iG$ are abstractly isomorphic to the group of G-valued formal Laurent series, an algebraic substitute for the smooth loop group of G). It therefore acts on a completed tensor product $\mathbf{H}$ of irreducible, positive energy representations of the $\hat{L}_iG$. These shall all be chosen at the same level h. Further, given distinct points $z_1, \ldots, z_m$ on Σ, labeled by irreducible representations ("irreps") V_k of G, the space $\mathbf{V} := \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_m$ becomes a representation of G^Σ, if maps $g : \Sigma \to G$ act on the factor V_k by means of their value at z_k.

Several related cohomologies are associated to these data, and it is my purpose in this paper to describe them.

(0.1) The cohomology of the space G^Σ, topologized as a union of closed subvarieties.

(0.2) The cohomology of the Lie algebra $\mathfrak{g}^\Sigma$ of $\mathfrak{g}$-valued algebraic functions on Σ, with coefficients in $\mathbf{H} \otimes \mathbf{V}$. (This, we know from [T2].)

(0.3) The algebraic group cohomology $H^*_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V})$ of G^Σ, with the same coefficients.

(0.4) The (coherent sheaf) cohomology of the vector bundle $\mathscr{L}^{\otimes h} \otimes \mathscr{V}$ over $\mathfrak{M}$, where:

- $\mathfrak{M}$ is the moduli stack of G-bundles over Σ^c (cf. Appendix B);
- $\mathscr{L}$ is the generator of $\mathrm{Pic}(\mathfrak{M})$ (cf. [LS], or Sect. V in this paper);
- $\mathscr{V}$ is the tensor product of the V_k-bundles associated to the "evaluation bundles" $\mathscr{G}(z_k)$,

[1] The corresponding statements for semi-simple, simply connected Lie groups are clear. The changes needed for arbitrary semi-simple groups are discussed in Sect. V

• $\mathscr{G}(z_k)$ is the restriction to $\mathfrak{M} \cong \mathfrak{M} \times \{z_k\}$ of the universal G bundle $\mathscr{G}$ over $\mathfrak{M} \times \Sigma^c$.

(0.5) The cohomology of the bundle $\mathscr{L}^{\otimes h} \otimes \mathscr{V}$ over the generalized flag variety $\hat{L}G/G^\Sigma$. Here, $\mathscr{V}$ is associated to the representation $\mathbf{V}$ of G^Σ, and $\mathscr{L}$ is the product of the basic line bundles (Sect. I) over the $\hat{L}_iG$. This abuse of notation, relative to (0.4), will be justified later; but, except in the context of the variety $\hat{L}G/G^\Sigma$, $\mathscr{L}$ and $\mathscr{V}$ will carry their meanings of (0.4).
The fifth item was added to the list during revision of the paper. It was conjecturally described by Graeme Segal (cf. Theorem 4), and its knowledge is, on the surface, stronger than that of (0.4); but I realized that the two were equivalent after a tip from Shrawan Kumar.

Some background for these problems is outlined in Remark (0.6) below. Meanwhile, the main results are summarized as follows.

Concerning (0.1), we shall see that G^Σ is homotopy equivalent to the group $C^\infty(\Sigma; G)$ of smooth maps from Σ to G. The latter, in turn, is equivalent to a product of G and a number (N) of copies of the based loop space ΩG, so we have

$$(0.1') \qquad H^*(G[\Sigma]) \cong H^*(G) \otimes H^*\left(\Omega G^{\times N}\right) .$$

Next, the first three items are related by means of the *van Est* spectral sequence

$$E_2^{p,q} = H^p_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V}) \otimes H^q(G[\Sigma]) \Rightarrow H^*(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \ ,$$

and the Lie algebra cohomology (0.2) was already determined in [T2]:

$$(0.2') \qquad H^{*+\ell}(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \cong H^\ell(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \otimes H^*(G) \otimes H^*(\Omega G^{\times N}) \ ,$$

for a certain degree ℓ, depending only on $\mathbf{V}$ and on the level h. When Σ is smooth, “N” is the same as in (0.1′); the two results will imply the collapse at E_2 of the spectral sequence, and the *pure-dimensionality* of the group cohomology (0.3):

$$H^\ell_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V}) \cong H^\ell(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}), \quad \text{whereas } H^q_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V}) = 0 \quad \text{if } q \neq \ell \ . \qquad (0.3')$$

The van Est spectral sequence also collapses when Σ is singular, but we no longer obtain the purity result (0.3′). For a nodal curve, we shall find that group cohomology acquires a factor of $H^*(\Omega G)$ for each node.

The cohomology of the vector bundles over $\mathfrak{M}$ follows, it turns out, from the knowledge of group cohomology. As will be shown in Sect. I, there is a natural isomorphism

$$(0.4') \qquad H^*(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{V}) \cong H^*_{G[\Sigma]}\left(\mathbf{H}_{0,h} \otimes \mathbf{V}\right) \ ,$$

where $\mathbf{H}_{0,h}$ is the product of the *vacuum* representations. The isomorphism applies, whether or not Σ is smooth; it follows, by standard arguments, from the "uniformization theorem" of [LS], and a "Borel-Weil-Bott" theorem for loop groups, due to Kumar [K] and Mathieu [M].

Finally, there is a natural morphism from $\hat{L}G/G^{\Sigma}$ to $\mathfrak{M}$, under which the $\mathscr{L}$ and $\mathscr{V}$ of (0.4) and (0.5) correspond (whence the abusive notation). One can "ascend" from $\mathfrak{M}$ back to $\hat{L}G/G^{\Sigma}$, and identify the cohomology of the pull-back bundle as follows:

$$(0.5') \qquad H^*(LG/G^{\Sigma}; \mathscr{L}^{\otimes h} \otimes \mathscr{V}) \cong \bigoplus\nolimits_{\check{\mathbf{H}}} \check{\mathbf{H}} \otimes H^*\left(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{V} \otimes \mathscr{U}^t\right) ,$$

where the sum goes over the *highest-weight representations* $\check{\mathbf{H}}$ of $\hat{L}G$ at level h, and $\mathscr{U}$ is the evaluation bundle over $\mathfrak{M}$, obtained by attaching, to the points at infinity of Σ^c, the *highest-energy spaces* of the tensor factors of $\check{\mathbf{H}}$. The delicate step in going from (0.4′) to (0.5′) is showing that the left-hand side is a sum of highest-weight representations of $\hat{L}G$ (Prop. 8.3).

There is an analytic version of the results (0.3′) and (0.4′), if one is careful with the statements. We may replace $\mathfrak{M}$ by the stack of holomorphic G-bundles over Σ^c, and the algebraic ind-varieties used in the paper (such as G^{Σ}) by their underlying analytic objects. Using analytic sheaf cohomology, the proofs apply, *mutatis mutandis*. On the other hand, one might try to replace G^{Σ} by the group of holomorphic maps to G. In a sense, (0.4′) is unaffected (cf. Props. 3.17 and 3.18); but, while it seems likely that some version of (0.3′) and (0.5′) should continue to hold, one runs into difficulties when topologizing the representations, and when trying to prove vanishing theorems for the cohomology of honestly infinite-dimensional varieties; and I do not know any tools suited to the task.

Remarks (*0.6*) (i) The results (0.1′) and (0.3′) were announced in [T2]. The connection with the stack was already "obvious" then, but a proper treatment had to wait for the clarifications in [DS] and [LS].

(ii) The homotopy equivalence $G^{\Sigma} \subset C^{\infty}(\Sigma; G)$ had been independently conjectured by Kumar; see [Ham] for some partial results. Connectedness of G^{Σ}, for simply connected G, was also proved in [LS] (the argument is attributed to V. Drinfeld). The case of the Laurent polynomial group was well-known (essentially [PS], Prop. 8.6.6).

(iii) The spaces of invariants in (0.2′) or (0.3′) were worked out in [TUY], who proved the conjectured "Verlinde factorization formula" for their dimensions. An alternative proof was given in [F] (and, later, [T2]). The connection with the space of regular sections over the stack of G-bundles, and over the GIT moduli space, was developed in [BL1], [F], [KNR], [LS].

(iv) The vanishing of the higher cohomology of $\mathscr{L}^{\otimes h}$ on the GIT moduli space is proved in [KN]. This would be closely related to the theorem for the stack, if we knew the vanishing of the cohomology with supports on the unstable strata (cf. Sect. IX). This seems to be a reasonable conjecture, but, to my knowledge, it has not been verified.

The title of the paper derives from the following epitome for the cohomological purity results. Let $P \subset G$ be a parabolic subgroup, E an irrep of G, F one of P. Knowledge of the group cohomology $H^*_P(\mathsf{E} \otimes \mathsf{F})$ of P with coefficients in all the $\mathsf{E} \otimes \mathsf{F}$ is equivalent to the Borel-Weil-Bott theorem ("BWB") for the flag variety G/P. The argument (due to Bott) is usually given in Lie algebra terms, so let me rephrase it. If $\mathscr{F}$ is the sheaf of sections of the algebraic vector bundle $G \times_P \mathsf{F}$ over G/P, one has a spectral sequence of "cohomological descent from G to G/P", with

(0.7) $$E_2^{p,q} = H^p_P(H^q(G; \mathscr{O}_G \otimes \mathsf{F})) \Rightarrow H^*(G/P; \mathscr{F}) \ .$$

On the left-hand side, P acts simultaneously on F and (by right translation) on $\mathscr{O}_G$. As G is affine, $H^q(G; \ldots) = 0$ for $q > 0$, so the spectral sequence must collapse at E_2. Further, the Peter-Weyl theorem implies that

(0.8) $$H^0(G; \mathscr{O}_G \otimes \mathsf{F}) = \bigoplus\nolimits_{\mathsf{E}} \mathsf{E}^t \otimes \mathsf{E} \otimes \mathsf{F} \ ,$$

and a correct tracking of the P-action turns (0.7) into the isomorphism

(0.9) $$H^*(G/P; \mathscr{F}) \cong \bigoplus\nolimits_{\mathsf{E}} \mathsf{E}^t \otimes H^*_P(\mathsf{E} \otimes \mathsf{F}) \ .$$

BWB says that the left-hand side lives in a single degree, where it gives an irreducible representation of G. This amounts to the vanishing of $H^*_P(\mathsf{E} \otimes \mathsf{F})$, except possibly in a single degree (and, given F, also for a single choice of E, when the non-zero cohomology is one-dimensional). Pure-dimensionality of $H^*_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V})$ is the corresponding loop group statement, because $\mathbf{H}$ is, in a suitable sense, an irreducible representation of $\hat{L}G$, while $\mathbf{V}$ is an irrep of the parabolic subgroup G^Σ. Note, however, that the remaining BWB assertions fail, in general; for instance, the non-trivial cohomology group is a space of conformal blocks in the WZW model, whose dimension, usually not 0 or 1, is given by a "factorization formula" as in [TUY].

The main results of the paper are direct consequences of the Lie algebra theorem in [T2], except for the difficulty that G^Σ is not a Lie group, but an algebraic ind-group. It is formally smooth, but its only known description is as an increasing union of *singular* varieties. A priori, what appears in the van Est spectral sequence is the Lie algebra cohomology of the space of algebraic functions on G^Σ, rather than its singular cohomology; to relate (0.1) and (0.2), we must verify the equality of the two[2]. For Lie groups, this follows from the algebraic de Rham theorem [Gro1]; the argument in Sect. VI extends this to G^Σ.

I have endeavored to render the paper accessible to readers who are not stack-initiated. While there is nothing difficult about the concept, the state of the literature makes its intelligible use problematic, and constant attempts to relate to the original definitions (inherited from the 1960's) tend to produce convoluted output. Also, to the usual category-theoretic approach

[2] In other words, we must check the "crystalline cohomology theorem" for $G[\Sigma]$

(probably best exposed in [LM]), I prefer the homotopy-theory point of view, in which stacks are represented by simplicial schemes. This way, sheaf cohomology comes almost for free. Whenever possible, I discuss stacks as if they were ordinary schemes: the statements are literally true, yet a reader willing to accept "obvious" properties of stacks could follow most arguments without knowing the definition. (Knowledgeable readers will want to consult Appendix B, where the basics of the simplicial theory of stacks are outlined). Of course, explicit simplicial constructions are at times needed. Also, the notions of a Grothendieck site – in particular, the site of complex schemes in the étale topology – sheaves over such, and their cohomology, were not always avoidable; a quick introduction to the topic is [G].

The Appendices offer some background material on Alexander cohomology (used in Sect. VII) and a proof of the "uniformization theorem" over $\mathbb{C}$, in simplicial language. A direct proof of a key result of Drinfeld and Simpson (in characteristic 0) is included in Sect. III.

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I. Definitions and statements

1. General background

Several kinds of cohomology will be used, sometimes in the same context. Sheaf cohomology over algebraic or analytic objects will be denoted by H^*, or H^*_{et} when the étale topology needs to be emphasized; H^*_s will indicate ordinary (singular) cohomology. The notations for group and Lie algebra cohomology present no ambiguities.

Recall some loop group-related definitions [T2]. Having fixed an integral level $h \geq 0$, an irrep of G is called either **regular** or **singular**, according to the affine weight $(\lambda + \rho, h + c)$ (λ is its highest weight, ρ the half-sum of the positive roots, and c the dual Coxeter number of $\mathfrak{g}$). To a regular representation V, one assigns a non-negative integral **length** $\ell(\mathsf{V})$, and a **ground form**, an irrep whose ρ-shifted highest weight lies inside the positive Weyl alcove at level $h + c$. They are described by the following property: the ρ-shifted highest weight of V and of its ground form are related by an affine Weyl transformation (at level $h + c$) of length $\ell(\mathsf{V})$.

As before, $\mathbf{H}$ denotes a formally completed tensor product of n positive energy irreps of the loop groups $\hat{L}_iG$ at level h, completed with respect to the total energy filtration. We reserve $\mathbf{H}_{0,h}$ for the product of the *vacuum* representations. $\mathbf{H}$ is the algebraic dual of a *highest-weight representation* (HWR) $\check{\mathbf{H}}$ of $\hat{L}G$ (at level -h), which is filtered, by the total energy, as a union of finite-dimensional, $\hat{L}^+G$-invariant subspaces. ($\hat{L}^+G \subset \hat{L}G$ is the product of subgroups of regular power series within the $\hat{L}_iG$). When all the selected representations V_k are regular, call $\ell(\mathbf{V}) = \sum \ell(\mathsf{V}_k)$, let W_k be the ground form of V_k, and $\mathbf{H} \otimes \mathbf{W}$ the corresponding $G[\Sigma]$-representation. The main result of [T2] is the following.

Theorem. [T2, Thm. 2.5] *When all* the V_k *are regular*, *there is a vector space isomorphism*

$$(1.1) \qquad H^*(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \cong H^{\ell(\mathbf{V})}(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \otimes H^{*-\ell(\mathbf{V})}(C^\infty(\Sigma; G)) \ ;$$

further, $\dim H^{\ell(\mathbf{V})}(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) = \dim(\mathbf{H} \otimes \mathbf{W})^{\mathfrak{g}[\Sigma]}$. *All cohomology groups vanish as soon as one of the* V_k *is singular.*

The dimension of $(\mathbf{H} \otimes \mathbf{W})^{\mathfrak{g}[\Sigma]}$ is determined by the "factorization theorem" ([TUY]), Prop. 2.2.6), or by more explicit "Verlinde formulas" [BL1]. Its dependence on Σ is only through the genus.

It was also claimed in [T2] that the isomorphism (1.1) is induced by the cup-product

$$(1.2) \qquad H^\ell(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \otimes H^{*-\ell}(\mathfrak{g}[\Sigma]; \mathbb{C}) \xrightarrow{\wedge} H^*(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V}) \ ;$$

and further, that the level-zero isomorphism $H^*(\mathfrak{g}[\Sigma]; \mathbb{C}) \cong H^*(C^\infty(\Sigma; G))$ is obtained by interpreting Lie algebra cocycles as left-invariant de Rham forms. These naturality properties will emerge from the proofs in Sect. VII.

Call Q_i the standard flag variety of the loop group $\hat{L}_iG$. Identifying the latter with the standard formal loop group $G((z)), Q_i$ is the quotient $G((z))/G[[z]]$. When G is simple and simply connected, it is known ([KN], Lemma 2.2) that the Picard group of Q_i is $\mathbb{Z}$, identified with $H^2(Q; \mathbb{Z})$ by the first Chern class; the positive generator gives the *basic central extension* of $\hat{L}_iG$. Let $\mathscr{L}$ be the product of the basic line bundles over $X := \hat{L}G/\hat{L}^+G$, the product of the Q_i. A special case of the "BWB" theorem of Kumar and Mathieu says that $\Gamma(X; \mathscr{L}^{\otimes h}) = \mathbf{H}_{0,h}$, while all higher cohomology vanishes. (We get all other representations $\mathbf{H}$, if we twist $\mathscr{L}^{\otimes h}$ with vector bundles over X, coming from suitable evaluation representations of $\hat{L}^+G$.) X carries an obvious action of G^Σ, which lifts uniquely to $\mathscr{L}$ (see Sect. V).

2. The ind-group G^Σ

We shall encounter several "ind-schemes", or analytic "ind-varieties". (One such example was X.) These are directed systems of closed embeddings, modulo the cofinality relation. It is best to replace them by the *sheaf* direct limits of the functors represented by the constituent schemes. (We shall generally use the étale topology, on the category of schemes of finite type, and the classical topology on analytic spaces; but see also Remark 3.1.) However, the difference between the ind-objects and their sheaf direct limits is somewhat subtle, and, as it turns out, immaterial for the results of this paper. I shall evade the distinction and call them *spaces*, in statements that support both interpretations.

Starting with an embedding $G \subset \mathrm{GL}_N(\mathbb{C})$, G^{Σ} is defined as a closed subspace of $M_N[\Sigma]$, the direct limit of the affine space $M_N[\Sigma]_d$ of matrix-valued functions of Σ of degree no more than d: $G[\Sigma] = \lim_{\to d} G[\Sigma]_d$, where $G[\Sigma]_d$ is the scheme $G[\Sigma] \cap M_N[\Sigma]_d$, with the reduced induced structure. The resulting space does not depend on the chosen embedding. In topological questions, G^{Σ} will be considered with its direct limit (classical) topology.

Theorem 1. *The inclusion $G^{\Sigma} \subset C^0(\Sigma; G)$ is a homotopy equivalence.* (Σ *may be singular*).

Remark (*1.3*) The result was well-known for the Laurent polynomial loop group $G[z, z^{-1}]$; however, an equivalent form of this special case will be used in the proof.

There is an equivalent formulation of this theorem, whose importance has become increasingly apparent to the author, largely as a result of discussions with Carlos Simpson. (Again, Σ need not be smooth.)

Theorem 1′. *The stack $\mathfrak{M}$ has the homotopy type of* $C^{\infty}(\Sigma^c; BG)$.

Remark (*1.4*) For now, we may interpret the "homotopy type of $\mathfrak{M}$" as the homotopy quotient X/G^{Σ}, using the "uniformaization theorem" (cf. Remark 1.6 below); see Appendix B for the true definition. However, Theorem 1′, which holds for any linear G, is better stated (and proved) independently of the uniformization theorem, and of Theorem 1, which only hold when G is semi-simple. To see the correct statement, note that the universal bundle $\mathcal{G}$ of the Introduction determines a classifying morphism from $\mathfrak{M} \times \Sigma^c$ to BG. Applying geometric realization to their simplicial presentations gives an arrow between their homotopy types. This arrow corresponds to a homotopy class of maps, from the homotopy type of $\mathfrak{M}$ to $C^{\infty}(\Sigma^c; BG)$; this is the asserted equivalence. A proof of Theorem 1′, for all reductive groups, follows from the Atiyah-Bott construction (cf. Cor. 2.11).

3. Regular functions on G^{Σ}

Define[3] the space $\Gamma(G[\Sigma]; \mathcal{O})$ of regular functions on $G[\Sigma]$ as $\lim_{\leftarrow d} \mathbb{C}[G[\Sigma]_d]$. The corresponding definition for $\Gamma(G[\Sigma]^{\times p}; \mathcal{O})$, with the obvious product

[3] The low-brow definitions given here agree with the "correct" ones – in this case, the space of global sections of the structure sheaf $\mathcal{O}$ over the big site of $G[\Sigma]$

filtration on $G[\Sigma]^{\times p}$, produces $\Gamma(G[\Sigma];\mathcal{O})^{\hat{\otimes} p}$, the formal completion of the pth tensor power of $\Gamma(G[\Sigma];\mathcal{O})$. There is, then, associated to the action of G^{Σ} on $\mathbf{H}\otimes\mathbf{V}$, an Eilenberg-MacLane cochain complex (where "$\hat{\otimes}$" denotes completed tensor products with respect to the total filtration)

$$\mathbf{H}\otimes\mathbf{V}\to\Gamma(G[\Sigma];\mathcal{O})\hat{\otimes}\mathbf{H}\otimes\mathbf{V}\to\Gamma(G[\Sigma];\mathcal{O})^{\hat{\otimes}2}\hat{\otimes}\mathbf{H}\otimes\mathbf{V}\to\dots\ , \tag{1.5}$$

whose cohomology we call the **group cohomology** of G^{Σ}, with coefficients in $\mathbf{H}\otimes\mathbf{V}$. One can verify by hand that the standard differentials in (1.5) are well-defined on these completions; but an alternative is given by the following construction.

The *simplicial homotopy quotient* $X_\bullet$ of X by G^{Σ} is the simplicial space $(EG^{\Sigma}_\bullet\times X)/G^{\Sigma}$, where $EG^{\Sigma}_\bullet$ is the *bar construction* of the universal G^{Σ}-bundle. All pull-backs to X_p, by the face maps, of the vector bundle $\mathcal{L}^{\otimes h}\otimes\mathbf{V}$ over X are canonically isomorphic, because the action of G^{Σ} on X has a canonical lifting to this bundle. We take this lifting as part of the definition of $\mathbf{V}$, using the G^{Σ}-action of the Introduction. However, a lifting for $\mathcal{L}$ amounts to a splitting of the projective cocycle of the $\hat{L}G$-representation $\mathbf{H}$, when restricted to G^{Σ} (see Sect. V); group cohomology would not make sense otherwise. $\mathcal{L}^{\otimes h}\otimes\mathbf{V}$, with these compatibility isomorphisms, is said to be a vector bundle over $X_\bullet$. The Eilenberg-MacLane complex (1.5) is associated, in the usual manner, to the co-simplicial complex of global sections of this bundle: the differentials are the alternating sums of the coface morphisms.

Remark (*1.6*) In the simplicial realization of stacks (App. B), $X_\bullet$ represents the moduli stack $\mathfrak{M}$ of algebraic G-bundles over Σ^c; this is the "uniformization theorem" of [LS].

Theorem 2. *Let Σ be smooth. For regular $\vee_k$, the group cohomology $H^*_{G[\Sigma]}(\mathbf{H}\otimes\mathbf{V})$ lives only in degree $\ell(\mathbf{V})$, and has the same dimension as the space of G^{Σ}-invariants in $\mathbf{H}\otimes\mathbf{W}$. It vanishes in all degrees, as soon as one $\vee_k$ is singular.*

Remark (*1.7*) When $\Sigma=\mathbb{P}^1\setminus\{0,\infty\}$ and $\mathbf{H}$ and $\mathbf{V}$ are the trivial representations, this is an algebraic version of the vanishing of higher group cohomology of LG^{cpt} (loops in the compact form of G), with smooth cochains and complex coefficients ([PS], Thm. 14.6.9). That was regarded in *loc. cit.* as a simile between LG^{cpt} and compact groups (thus, between LG and semi-simple groups). However, our interpretation is quite different: the Laurent polynomial loop group is a parabolic subgroup of the product of two formal loop groups. For example, twisting the coefficients by an evaluation Ad-representation shifts the cohomology into degree one.

We can also describe the group cohomology when Σ is deformed to a nodal curve. We do allow the degenerate curve to be reducible, but require that each of its components be affine, and that all the marked points remain smooth.

Theorem 2′. *Under such a deformation,* $H^*_{G[\Sigma]}(\mathbf{H}\otimes\mathbf{V})$ *acquires a tensor factor of* $H^*(\Omega G)$ *for each node.*

The factors $H^*(\Omega G)$ are "attached" to the nodes in a manner that will be clarified in Sect. VII. By contrast, the factors $H^*(\Omega G)$ in the (singular) cohomology of G^Σ come from non-trivial loops in Σ. The loops that get contracted under the degeneration give rise to group cohomology classes, instead of topological ones.

4. Vector bundles over the moduli stack and a conjecture of G. Segal's

The most interesting application of Theorem 2 is the cohomological single-dimensionality of the "evaluation vector bundles" over the stack $\mathfrak{M}$ of G-bundles over Σ^c, for smooth Σ. We shall see, in Sect. VIII, how this implies a Borel-Weil-Bott theorem for certain flag varieties of the loop group. The statement for the stack is simple enough to prove here (accepting, for now, one detail that will be clarified in Sect. III).

Theorem 3. *For any* Σ (*smooth or not*), *there is a natural isomorphism*

$$H^*(\mathfrak{M};\mathscr{L}^{\otimes h}\otimes\mathscr{V})\cong H^*_{G[\Sigma]}(\mathbf{H}_{0,h}\otimes\mathbf{V})\ .$$

Remark (*1.8*) Using this and the smooth case of Theorem 2, one can give a geometric proof of Theorem 2′: $H^*(\mathfrak{M};\mathscr{L}^{\otimes h}\otimes\mathscr{V})$, for a nodal curve, can be computed by fibering $\mathfrak{M}$ over the moduli stack of bundles over the normalized curve (cf. Sect. VII).

Proof of Theorem 3. We shall see in Sect. III.4 that the uniformization theorem identifies the sheaf cohomology $H^*(\mathfrak{M};\mathscr{L}^{\otimes h}\otimes\mathscr{V})$ over $\mathfrak{M}$ with the total cohomology of $\mathscr{L}^{\otimes h}\otimes\mathbf{V}$ over the simplicial space $X_\bullet$. We then have a Leray spectral sequence for the projection $p: X_\bullet\to BG^\Sigma{}_\bullet$, with

$$E_2^{p,q}=H^p(BG^\Sigma{}_\bullet;H^q(X;\mathscr{L}^{\otimes h}\otimes\mathbf{V})\Rightarrow H^{p+q}(\mathfrak{M};\mathscr{L}^{\otimes h}\otimes\mathscr{V})\ . \tag{1.9}$$

Topologists know it from equivariant cohomology, but in the language of [SGA4], this would be the "spectral sequence of descent from X to $\mathfrak{M}$": X is regarded as a principal G^Σ-bundle over $\mathfrak{M}$ (which is true, in a simplicial sense, if X is replaced by $X\times EG^\Sigma$). By the vanishing theorem of Kumar and Mathieu, this sequence collapses at E_2, giving

$$H^p(BG^\Sigma{}_\bullet;p_*(\mathscr{L}^{\otimes h})\otimes\mathbf{V})\cong H^p(\mathfrak{M};\mathscr{L}^{\otimes h}\otimes\mathscr{V})\ . \tag{1.10}$$

Note that $p_*(\mathscr{L}^{\otimes h})$ is not quite a vector bundle with fiber $\mathbf{H}_{0,h}$; rather, on each component $G[\Sigma]^{\times p}$, it is a *product* of copies of $\mathscr{O}$, indexed by a basis of that space. Still, the higher cohomology of this sheaf vanishes, by (A.18), and the spectral sequence for cohomology over $BG^\Sigma{}_\bullet$,

$$(1.11) \quad E_1^{p,q} = H^q(G[\Sigma]^{\times p}; p_*(\mathscr{L}^{\otimes h}) \otimes \mathbf{V}) \Rightarrow H^{p+q}(BG^{\Sigma}_{\bullet}; p_*(\mathscr{L}^{\otimes h}) \otimes \mathbf{V}) \ ,$$

becomes the Eilenberg-MacLane resolution of $H^*_{G[\Sigma]}(\mathbf{H}_{0,h} \otimes \mathbf{V})$. □

The final theorem is the algebraic version of a conjecture by Graeme Segal (1990). It motivated the sequence of papers [T1], [T2], leading to the present one, and it can now be proved. In the original formulation, Σ was a Riemann surface with boundary, and the $\hat{L}_i G$ were smooth loop groups. Algebraically, we shall work with the "thick" generalized flag variety $X_\Sigma := \hat{L}G/G^\Sigma$; this can be shown to be a scheme (of infinite type), but we shall not use this fact now, and regard X_Σ as a sheaf over the category $\mathfrak{Sch}$ of all schemes over $\mathbb{C}$, carrying the $\hat{L}G$-action.

Call $\check{\mathbf{H}}_{\mathsf{U}}$ the HWR of $\hat{L}G$ at level $(-h)$, with zero-energy spaces the G-irreps $\mathsf{U}_1, \dots, \mathsf{U}_n$.

Theorem 4[4]. *$H^*(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V})$ lives only in degree $\ell = \ell(\mathbf{V})$, and is a sum of HWRs at level $(-h)$. The multiplicity space of $\check{\mathbf{H}}_{\mathsf{U}}$ identified with the conformal block $H^\ell(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{V} \otimes \mathscr{U}^t)$. Here, $\mathscr{U}$ is the evaluation bundle over $\mathfrak{M}$ associated to the U_k, attached to the points at infinity on Σ^c.*

In other words, using Theorems 2 and 3, there is a natural isomorphism

$$(1.12) \qquad H^*(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V}) \cong \bigoplus\nolimits_{\check{\mathbf{H}}} \check{\mathbf{H}} \otimes H^*_{G[\Sigma]}(\check{\mathbf{H}}^t \otimes \mathbf{V}) \ ,$$

summing over all HWRs of $\hat{L}G$ at level $(-h)$. This isomorphism holds for any curve, but of course we will not get dimensional purity in general. (1.12) would immediately follow from a "Peter-Weyl theorem" for the space of holomorphic sections of the line bundle $\mathscr{L}^{\otimes h}$ over $\hat{L}G$, in the manner explained in the Introduction (0.7–0.9). However, the Peter-Weyl statement is *false*; nonetheless, Theorem 4 and (1.12) are true.

Remark (*1.13*) Theorem 4 implies all the other BWB theorems by standard methods, as employed throughout this paper. The ascent from $\mathfrak{M}$ to X_Σ, on the other hand, does use some special features of loop groups and of the stack $\mathfrak{M}$ (Lemmata 8.4 and 8.8). The extra information in (1.12), compared to Theorem 3, is that the cohomologies over X_Σ are sums of HWRs. In the analytic setting, this is not known, and I had mistakenly assumed the algebraic analogue to be equally problematic. The simple but effective idea of using filtrations, instead of gradations, to verify the "highest-weight" condition emerged in a conversation with S. Kumar.

5. A finite-dimensional analogue of nodal degeneration

The deformation of $G[z, z^{-1}]$ to $G[L]$, where L is a union of two affine lines meeting at a point, has the following finite-dimensional counterpart. Choose

[4] Added during revision of the paper

a Borel subgroup $B \subset G$, and consider the subgroup $K \subset B \times \overline{B}$ of pairs mapping to the same point in the maximal torus. K is a degeneration of G. To see this on Lie algebras, filter $\mathfrak{g}$ by the norm of the eigenvalue of the ad-action of the highest co-root: the associated graded Lie algebra is $\mathfrak{k}$ (This Lie algebra degeneration was also studied by Kostant.)

The diagonal action of G on the product $\mathsf{U} \otimes \mathsf{V}$ of irreducible representations degenerates to an action of K, restricted from the natural B-action of U and $\overline{B}$-action on V. Assume that $\mathsf{V} = \mathsf{U}^t$ (otherwise, all cohomologies will vanish); we then have a natural isomorphism $H^*(\mathfrak{g}, \mathfrak{t}; \mathsf{U} \otimes \mathsf{V}) \cong H^*(G/T)$. Indeed, the space of invariants in $\mathsf{U} \otimes \mathsf{V}$ is one-dimensional, and the Koszul resolution of $H^*(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ is naturally identified with the complex of left-invariant de Rham forms on G/T. Slightly less obvious is an isomorphism $H^*(\mathfrak{k}, \mathfrak{t}; \mathsf{U} \otimes \mathsf{V}) \cong H^*(G/T)$ – arising, this time, from group cohomology, courtesy of the van Est spectral sequence for the pair (K, T) (because K/T is contractible). Letting $\hat{T} := \operatorname{Hom}(T; \mathbb{C}^\times)$, we have

$$H_K^*(\mathsf{U} \otimes \mathsf{V}) \cong H^*_{B\times\overline{B}}\left(\operatorname{Ind}_K^{B\times\overline{B}}(\mathsf{U} \otimes \mathsf{V})\right) \cong \bigoplus_{\lambda\in\hat{T}} H_B^*(\mathsf{U} \otimes \mathbb{C}_\lambda) \otimes H_{\overline{B}}^*\left(\mathsf{V} \otimes \mathbb{C}_{\overline{\lambda}}\right) \ ,$$

and from BWB, the last sum equals a direct sum of lines, labeled by elements of the Weyl group, in degree equal to twice the length of the label. This is $H^*(G/T)$ (non-canonically). So, the Lie algebra cohomology groups are rigid under the degeneration of $\mathfrak{g}$ to $\mathfrak{k}$, but singular cohomology has turned into group cohomology.

The deformation of stacks, from $\overline{B}\backslash G/B \cong (G/B \times G/\overline{B})/G$ to $(G/B \times G/\overline{B})/K$, corresponds to a "bubbling" deformation from the genus zero moduli stack to the stack of bundles over two crossing copies of $\mathbb{P}^1$. More interestingly, if A is an annulus with parametrized boundary, $LG/\operatorname{Hol}(A; G)$ is the moduli stack of holomorphic G-bundles over the elliptic curve obtained by sewing together the boundary circles of A. Deforming A to two crossing disks degenerates this stack to that of bundles over the self-crossing projective line. This seems to bear some resemblance to the deformation of the quotient stack G/G (conjugation action) to G/K (left and right action).

II. The homotopy type of G^Σ

There is some overlap between this section and the next (esp. in the proofs of Proposition 2.4 and of Proposition 3.18). This is unavoidable, because the homotopy equivalence between algebraic and continuous mapping groups is really a consequence of the "uniformization theorem" discussed in Sect. III. Conversely, that theorem is a reformulation of the argument in Lemma 2.9 below. However, it seemed sensible to give a non-stacky proof for a statement in classical topology.

We shall freely switch between $C^0(\Sigma; G)$ and $C^\infty(\Sigma; G)$: as Σ is smoothly deformable to a bouquet of N loops, the two spaces, and all C^k spaces with their compact-open C^k topologies, are homotopy-equivalent to $G \times \Omega G^{\times N}$.

Cauchy formulas show that all the above topologies agree on the subgroup $\mathrm{Hol}(\Sigma; G)$ of holomorphic maps, and *a fortiori* on each $G[\Sigma]_d$, where they restrict to the usual topology.

In the direct limit topology, the union G^Σ of the $G[\Sigma]_d$ is strongly equivalent to a *CW* complex, in a manner compatible with the d-filtration. This follows from standard results on cellular approximation, in light of the following remark: the successive inclusions are cofibrations, so G^Σ is strongly equivalent to the homotopy direct limit (mapping telescope) of the $G[\Sigma]_d$. Observe, in this connection, that the $H^k(G[\Sigma]_d; \mathbb{C})$ are finite-dimensional, so their $\mathbb{R}^1 \lim_{\leftarrow}$ vanishes; and we record the following, for later use.

Lemma (2.1) $H_s^*(G[\Sigma]; \mathbb{C}) \cong \lim_{\leftarrow d} H_s^*(G[\Sigma]_d; \mathbb{C})$. □

Remark (*2.2*) One might ask whether, given k, the pairs $(G[\Sigma], G[\Sigma]_d)$ are k-acyclic for large d (so that their cohomology stabilizes); but I don't know a way to verify this.

Now, if Σ is a singular curve and $\tilde{\Sigma}$ denotes its normalization, it turns out that the inclusions $G[\Sigma] \subset G[\tilde{\Sigma}]$ and $C^\infty(\Sigma; G) \subset C^\infty(\tilde{\Sigma}; G)$ give rise to principal bundles over equivalent quotient spaces (Cor. 3.10, combined with Lemma 2.6 below; note that singularities finer than crossings are homotopically irrelevant). Comparing the two bundles shows that it suffices to prove Theorem 1 for smooth curves. The proof has two steps, and uses the group $\mathrm{Hol}(\Sigma; G)$.

Proposition (2.3) *The inclusion* $\mathrm{Hol}(\Sigma; G) \subset C^0(\Sigma; G)$ *is a homotopy equivalence.*

Proposition (2.4) *The inclusion* $G[\Sigma] \subset \mathrm{Hol}(\Sigma; G)$ *is a homotopy equivalence.*

Remarks (*2.5*) (i) Weak equivalence in Proposition 2.3 is a result of H. Grauert's, and holds for G-valued maps on any Stein space ([Gra], Satz 3 and Satz 6 imply isomorphism of the homotopy groups). Strong equivalence is not clear in such generality; for instance, if the source does not have finite topology, the C^0 mapping space is not locally contractible, and thus not of *CW* type, in the compact-open topology. However, the case of curves admits some amusing proofs.

(ii) The result also holds for a compact Riemann surface with boundary, and C^∞ boundary conditions on the maps. (Sobolev boundary conditions[6] may be imposed, when Banach manifolds are needed, such as for the implicit function theorem.) My choice of an open surface was quite accidental, and, in fact, adds a difficulty, since it is not obvious that $\mathrm{Hol}(\Sigma; G)$ is a manifold! Interestingly enough, a proof of this fact will emerge from the first proof of (2.3) this is why I retained the original setup.

[6] Sup norms are not so good in this context; cf. [PS], Remark (i) following Proposition 8.3.3

We shall need to know whether a topological space A is a principal bundle over its quotient A/K by a continuous group action, so let us recall the following fact.

Lemma (2.6) *A is a locally trivial K-bundle over A/K iff*: (1) *the "action" $(k, a) \mapsto (k \cdot a, a)$ maps $K \times A$ homeomorphically onto its image in $A \times A$, with the induced topology*; *and* (2) *the projection has local sections.* □

In the examples, we shall only check the slice condition (2); the first condition will be clear. It holds, for instance, if A is a group and K a closed subgroup. Also, note that, once we have a candidate for the quotient, to which A maps K-invariantly and continuously, producing enough slices confirms the candidate to be right.

First proof of (2.3) It suffices to consider the inclusion $\mathrm{Hol}_\bullet \subset \Omega G^{\times N}$, of the group of based holomorphic maps into the space of based maps, from a bouquet of circles on Σ, to G. Now, $\mathrm{Hol}_\bullet$ acts freely, by gauge transformations, on the contractible space $\mathscr{A}$ of $\mathfrak{g}$-valued holomorphic connection forms. The quotient space can be identified with $G^{\times N}$, by sending a holomorphic connection to its holonomy around the loops of the bouquet. As the holonomy map χ is regular and surjective[7], with finite-dimensional base, it must admit local holomorphic sections.

Tracking the holonomy all the way along the loops lifts χ to a map from $\mathscr{A}$ to $\mathscr{P}G^{\times N}$, the total space of the path fibration $\mathscr{P}G^{\times N} \twoheadrightarrow G^{\times N}$ of $G^{\times N}$. This is a map of (strong) fibrations, inducing homotopy equivalences on the total spaces (both are contractible) and on the bases (the identity); it must, then, induce a strong equivalence between the fibers $\mathrm{Hol}_\bullet$ and $\Omega G^{\times N}$. □

Remark (*2.7*) The argument also shows that the fibers of χ are closed complex submanifolds of $\mathscr{A}$, with holomorphic tubular neighborhoods. Indeed, choosing a holomorphic section over some open U, the map $\mathrm{Hol}_\bullet \times$ section $\to \chi^{-1}(U)$ is not just bicontinuous, but biholomorphic, in all the Banach manifold structures on $\mathrm{Hol}_\bullet$ and $\mathscr{A}$, defined by a sequence of shrinking circles about the punctures. Projecting along the slice gives a biholomorphic map, in all these Banach metrics simultaneously, from some fixed open set in a gauge orbit to a fixed open set in its tangent space. It is also easy to see, from here, that $\mathrm{Hol}_\bullet$ and $\mathrm{Hol}(\Sigma; G)$ are Lie groups.

Second proof of (*2.3*) The quotient $C^\infty(\Sigma; G)/\mathrm{Hol}(\Sigma; G)$ can be identified with the contractible space of smooth, $\mathfrak{g}$-valued (0,1)-connection forms over Σ, by sending $\sigma \in C^\infty(\Sigma; G)$ to the form $\bar{\partial}\sigma \cdot \sigma^{-1}$. A local section of this map is obtained by sending a connection form α to the unique (multi-valued) solution to $\bar{\partial}\sigma \cdot \sigma^{-1} = \alpha, \partial\sigma \cdot \sigma^{-1} = 0$, corrected by a right factor which is holomorphic, multi-valued and has opposite periods. Locally, the correction

[7] In other words, flat G-bundles over Σ are holomorphically trivial. This is a special case of Grauert's results

factor can be chosen to vary holomorphically with α, as in the previous proof. $\square$

Definitions (2.8) *In the proof of Proposition* 2.4, *the following objects will be used.*

- Δ, *a union of disjoint open disks* Δ_i *centered at the points at infinity on* Σ^c.
- $\Delta^\times$, *the union of the punctured disks.*
- $\mathrm{Hol}(\Delta; G)$, $\mathrm{Hol}(\Delta^\times; G)$, *the holomorphic mapping groups, with the compact-open topology.*
- $G[\Delta^\times] \subset \mathrm{Hol}(\Delta^\times; G)$, *the subgroup of meromorphic maps, topologized as the limit of the subspaces* $G[\Delta^\times]_d$ *of maps with poles of order* $\leq d$.

Proof of (*2.4*) $\mathrm{Hol}(\Sigma; G)$ is a closed subgroup of $\mathrm{Hol}(\Delta^\times, G)$, and meets $G[\Delta^\times]$ in the closed subgroup G^Σ of the latter. Regarding elements of $\mathrm{Hol}(\Delta^\times, G)$ (resp. $G[\Delta^\times]$) as transition functions for holomorphic (therefore, algebraic) bundles over Σ^c, Harder's theorem [H] on the algebraic triviality of G-bundles over an affine curve implies that the two quotient spaces coincide, set-theoretically, with the set $\mathfrak{S}$ of isomorphism classes of "holomorphic G-bundles on Σ^c, with holomorphic sections over the disks Δ_i". I claim that the inclusion of $G[\Delta^\times]$ in $\mathrm{Hol}(\Delta^\times, G)$ again defines a map of principal bundles over the same base, inducing a strong equivalence on the total spaces; it will follow that the fibers are equivalent as well.

The homotopy equivalence of the inclusion $G[\Delta^\times] \subset \mathrm{Hol}(\Delta^\times; G)$ can be seen by considering the compatible, contractible open covers, provided by the Birkhoff factorization theorems. Local triviality of the projections to $\mathfrak{S}$ will be needed again, in slightly stronger form, so we shall state is separately. $\square$

Lemma (2.9) $\mathfrak{S}$ *is a complex manifold, over which* $G[\Delta^\times]$ *and* $\mathrm{Hol}(\Delta^\times, G)$ *are holomorphic principal bundles, with structure groups* G^Σ *and* $\mathrm{Hol}(\Sigma; G)$, *respectively.*

Proof. Briefly put, locally, $\mathrm{Hol}(\Delta^\times; G)/\mathrm{Hol}(\Sigma; G)$ differs finite-dimensionally from $\mathrm{Hol}(\Delta; G)$; this will allow the construction of a holomorphic slice through 1 for the action of $\mathrm{Hol}(\Sigma; G)$ on $\mathrm{Hol}(\Delta^\times; G)$, which will in fact sit in some $G[\Delta^\times]_d$. Such a slice inherits the same topologies from $G[\Delta^\times]$ and from $\mathrm{Hol}(\Delta^\times, G)$, and Lemma 2.6, with the comment following it, ensures that, topologically, we are done. The topological trivializations produced by the slice are holomorphic, because the slice itself is so, and the closed subgroups $G[\Sigma] \subset G[\Delta^\times]$ and $\mathrm{Hol}(\Sigma; G) \subset \mathrm{Hol}(\Delta^\times; G)$ carry the induced holomorphic structure.

Consider a smooth, quasi-projective *fine moduli space* M for "holomorphic G-bundles over Σ^c, with gauge-fixing at the centers of the Δ_i, that are not too far from the trivial bundle" (see Construction 3.19, Sect. III). The G-bundle over $\mathrm{Hol}(\Delta^\times, G) \times \Sigma^c$, defined by viewing elements of

$\mathrm{Hol}(\Delta^\times; G)$ as transition functions along Σ^c, determines a classifying map from a neighborhood of 1, in $\mathrm{Hol}(\Delta^\times; G)$, to M. The map is regular on tangent spaces, and left $\times$ right-invariant under $\mathrm{Hol}_b(\Delta; G) \times \mathrm{Hol}(\Sigma; G)$ ($\mathrm{Hol}_b(\Delta; G)$ is the group of maps based at the centers of the disks), so it can be defined on an invariant open set W. We can even arrange for the double quotient of W to be exactly M, because $\mathrm{Hol}(\Delta^\times; G)$ parametrizes all the bundles; but, most importantly, the fibers of the map consist of single orbits, and the group action is set-theoretically free in W.

Arguing as in Lemma 3.6 and Corollary 3.8, we can find a smooth, locally closed subvariety S of some $G[\Delta^\times]_d$, whose tangent space at 1 is complementary to $\mathfrak{g}[\Sigma] \oplus \mathrm{Hol}_b(\Delta; \mathfrak{g})$ in $\mathfrak{g}[\Delta^\times]$. The inverse function theorem applied to S and M, together with Lemma 2.6, show that, after possible shrinking, S is local slice for the $\mathrm{Hol}_b(\Delta; G) \times \mathrm{Hol}(\Sigma; G)$-action on W. It follows, then, that $S \times \mathrm{Hol}_b(\Delta; G)$ is a local cross-section for the action of $\mathrm{Hol}(\Sigma; G)$ on $\mathrm{Hol}(\Delta^\times; G)$: the multiplication map $\mathrm{Hol}(\Sigma; G) \times S \times \mathrm{Hol}_b(\Delta; G) \to \mathrm{Hol}(\Delta^\times; G)$ is biholomorphic. □

Remark (*2.10*) Reaching a bit ahead (see Sect. III), the quotient stack $\mathfrak{S}/\mathrm{Hol}(\Delta; G)$ is the stack $\mathfrak{M}_{\mathrm{hol}}$ of holomorphic G-bundles on Σ^c. However, in the previous proof, one may choose an *algebraic* (étale) slice S, and use it to define an algebraic structure on $\mathfrak{M}_{\mathrm{hol}}$. This algebraic structure stems from the fact that $\mathfrak{M}_{\mathrm{hol}}$ is the analytic stack underlying $\mathfrak{M}$.

Corollary (2.11) *The homotopy quotient $X/G[\Sigma]$ equivalent to $C^0(\Sigma^c; BG)$.*

Remark (*2.12*) The Corollary is equivalent to Theorem 1, since we know that the flag variety X is homotopy equivalent to its smooth version (a product of ΩGs).

First proof. Replacing Σ, for convenience, by a surface with n boundary circles, this is the homotopy double quotient $G^{\times n}\backslash C^0(\partial\Sigma; G)/C^0(\Sigma; G)$: on the left are the constant maps on $\partial\Sigma$, while the right group acts by restriction to $\partial\Sigma$. For simply connected G, this right action is transitive, so the quotient space is the classifying space of the stabilizer, $BC^0(\Sigma^c; G)$, which is also $C^0(\Sigma^c; BG)$. If G is connected, but not simply connected, the homotopy quotient $C^0(\partial\Sigma; G)\,/C^0(\Sigma; G)$ is the homotopy fiber of the restriction map $BC^0(\Sigma; G) \to BC^0(\partial\Sigma; G)$. This fiber is the space of maps from Σ^c to BG, based at the n points at infinity. However, it is better described as the total space of the principal $G^{\times n}$-bundle over $C^0(\Sigma^c; BG)$, obtained by pulling back the universal bundle EG under the evaluations at infinity. The left action of the constant maps on $\partial\Sigma$ is the structural $G^{\times n}$-action. Dividing it out, $C^0(\Sigma^c; BG)$ emerges as the correct description of $X/G[\Sigma]$. □

Second proof. Because the stack $\mathfrak{M}_{\mathrm{hol}}$ of holomorphic G-bundles over Σ^c is also the underlying analytic stack of $\mathfrak{M}$, its homotopy type is the homotopy quotient $X/G[\Sigma]$. But $\mathfrak{M}_{\mathrm{hol}}$ has an alternative presentation, by the Atiyah-Bott construction [AB], as the quotient stack of smooth (0,1)-connections over Σ^c by the smooth complex gauge group. As in [AB], this leads to the homotopy type $C^\infty(\Sigma^c; BG)$; the different connected components correspond to the topological types of holomorphic bundles. $\square$

Remark (*2.13*) The advantage of the second argument is that it applies to all reductive groups, even those for which the uniformization theorem fails (such as GL_N). It requires us to know that equivalent presentations of stacks give rise to equivalent homotopy types; but one could also compare homotopy types directly, by an explicit bi-simplicial argument (in which the second proof of Prop. 2.4 plays a rôle). However, the construction is somewhat similar to one we shall employ in Sect. VI, and the details are not as interesting as the general theorem.

III. Algebraic and analytic uniformization of the moduli stack

The "uniformization theorem" states that the quotient stack X/G^Σ is equivalent to $\mathfrak{M}$. The idea goes back to A. Weil's adèlic presentation of moduli spaces of vector bundles, and, language aside, a holomorphic version (due to Atiyah) can be found in [PS] (Proposition 8.10.2)[8]. However, a complete proof of the algebraic version was only given recently, in [BL1] for SL_N, and in [LS] for general G. We shall give a brief account of algebraic and analytic forms of the theorem; the proof is rewritten in simplicial language in Appendix B.

1. Generalities on mapping spaces

Algebraic properties of G^Σ are discussed in detail in [LS] (SL_N was worked out in [BL1]); but, to make this paper self-contained, I have extracted the statements we need, especially since Lemma 3.6 and some of its corollaries, though implied, are not overtly stated in *loc. cit.*

Let $\mathfrak{F}$ be the category of schemes of finite type over $\mathbb{C}$, with the étale topology. The formal loop group, and its standard parabolic subgroup, are the sheaves of $\mathfrak{F}$ sending an affine scheme Spec(R) to Hom(Spec($R((z))$); G), and to Hom(Spec($R[[z]]$); G), respectively. The latter is representable by a group scheme (of infinite type), while the loop group itself is a direct limit of such schemes.

Remark (*3.1*) The larger category $\mathfrak{Sch}$, for all schemes over $\mathbb{C}$, will be used on the rare occasions when we consider the group cohomology or sheaf cohomology of $G[[z]]$. $\mathfrak{F}$ has the benefit that the "underlying analytic space" and "homotopy type" carry their obvious meanings. However, schemes of infinite type, such as $G[[z]]$, are better left in their natural habitat. For stacks that are covered by finite type schemes, the restriction to $\mathfrak{F}$ does not alter

[8] Only genus zero is treated, but the method would work in general

their coherent sheaf cohomology, so the switch to $\mathfrak{Sch}$, when necessary, carries no hidden cost.

We are mostly concerned with the quotient "flag variety" $Q := G((z))/G[[z]]$. This was extensively studied in [K], [M], [M2], and we recall the following properties.

Proposition. (i) *Q is a direct limit of closed projective subvarieties.*

(ii) *The projection $G((z)) \to Q$ is a locally trivial fiber bundle in the Zariski topology.*

Part (i) is obtained by a Kodaira embedding [K]. Part (ii) follows from the "Birkhoff factorization theorem": $G[z^{-1}] \times_G G[[z]]$ is an open subsheaf of $G((z))$. This is one set of an explicit open cover, whose complete description amounts to the uniformization theorem in genus zero.

Given schemes Y, Z, the *mapping space* Y^z is the sheaf over $\mathfrak{F}$ sending a scheme U to $\mathrm{Hom}(U \times Z; Y)$. Accordingly, G^Σ is now defined as the group-valued sheaf taking U to

$$\mathrm{Hom}\big(U; G^\Sigma\big) := \mathrm{Hom}(U \times \Sigma; G) \ . \tag{3.2}$$

Remark (*3.3*) The original ind-group definition gives rise to the direct limit sheaf[9]

$$U \mapsto \lim\nolimits_{\to d} \mathrm{Hom}(U; G[\Sigma]_d) \ , \tag{3.4}$$

(when U is affine), using the reduced structure on the $G[\Sigma]_d$. The agreement of (3.2) and (3.4), which we shall see in Remark 3.7, amounts to *reducedness*[10] of the mapping sheaf (3.2), and was verified in [LS] for all semi-simple G; it fails for $G = \mathbb{C}^\times$. This distinction between semi-simple and reductive groups was, it seems, first emphasized in [BL1] (though it was also touched upon in [F]), and appears to capture the topological fact that, for semi-simple G, but not for a torus, G^Σ is dense in the holomorphic mapping group.

Proposition (3.5) *G^Σ is formally smooth, and its tangent bundle is isomorphic to $(TG)^\Sigma$.*

Proof. Formal smoothness means that the restriction $\mathrm{Hom}(Y'; G[\Sigma]) \to \mathrm{Hom}(Y; G[\Sigma])$ is subjective, for any nilpotent extension $Y \subset Y'$ of affine schemes. This is obvious from (3.2), by smoothness of G. The isomorphism $T(G^\Sigma) \cong (TG)^\Sigma$ follows from the definition $\mathrm{Hom}\big(U; T(G^\Sigma)\big) := \mathrm{Hom}\big(U[\varepsilon]/\varepsilon^2; G^\Sigma\big)$. □

[9] Over a category containing schemes of infinite type, this functor must be sheafified

[10] The sheaf direct limit of an ind-scheme is reduced iff it agrees with the limit of the reduced schemes

2. Finer properties of the algebraic mapping groups

Formal smoothness, by itself, is not too meaningful; it holds for the mapping space to any smooth target, even though, when, say, $G = \mathbb{C}^\times$ and Σ is the affine line, the space underlying $\mathbb{C}^\times[\Sigma]$ is just $\mathbb{C}^\times$. The following is more significant.

Lemma (3.6) *There is a morphism from TG^Σ* to *G^Σ with differential (I, I) on the zero-section.*

Proof. Pick a basis $\{v_k\}$ of $\mathfrak{g}$ consisting of nilpotent elements. For any function f on some $U \times \Sigma$, the exponentials $\exp(v_k \otimes f) : U \times \Sigma \to G$ are algebraic. A morphism as desired is obtained by sending the pair $(g, f) \in G^\Sigma \times \mathfrak{g}^\Sigma$, with $f = \sum v_k \otimes f_k$, to $g. \prod_k \exp(v_k \otimes f_k)$ (in order). □

Remarks (*3.7*) (i) Clearly, this morphism lands in the reduced group (3.2), and gives isomorphisms between the formal neighborhoods of 0 in $\mathfrak{g}^\Sigma$, and of 1, in either of the group sheaves (3.2) and (3.4). Now, it is easy to show that the mapping sheaf (3.2) is the direct limit of schemes with the same support as the $G[\Sigma]_d$. (It is a closed subsheaf of $M_N[\Sigma]$, for which the two definitions coincide). In this situation, isomorphism is detected at the level of the completed local rings, and we have just checked it.

(ii) It is tempting to believe that G^Σ is a complex manifold modeled on $\mathfrak{g}^\Sigma$, but this does not seem to be known.

Corollary (3.8) *Any finite-dimensional subspace S of $T_pG^\Sigma \cong \mathfrak{g}^\Sigma$ is tangent to some smooth subvariety of G^Σ through p. The same holds for the flag variety Q.*

Proof. This is the image of S under the morphism in (3.6). The result for Q follows from Birkhoff factorization. □

Corollary (3.9) *The étale slice theorem applies to G^Σ and to Q.* □

This means that a morphism from Q to a complex variety has a local étale cross-section through every point where it is formally smooth. It follows from the usual slice theorem, by restriction to a subvariety tangent to a large enough subspace of TG^Σ.

One consequence concerns subgroups of G^Σ defined by evaluation conditions – maps landing into specified subgroups of G at marked points on Σ. (One could also impose higher-order conditions.) Such subgroups define "obvious" smooth quotient varieties, to wit, the quotients of the corresponding adèle groups. The following lemma ensures that it is these obvious quotients that enter the Leray spectral sequences that will be used in Sect. VII.

Corollary (3.10) *The "obvious" quotient G^Σ/H agrees with the sheaf-theoretic quotient, in the étale (or any finer) topology.*

Proof. The map from the sheaf-theoretic to the adèlic quotient is a monomorphism, so it suffices to produce enough local sections. Clearly, the map from G^{Σ} to the adèlic quotient is formally smooth; by the approximation theorem in [H], it is surjective on points. Now apply (3.9). □

3. A theorem of Drinfeld and Simpson

The key step in the proof of uniformization is a result of Drinfeld and Simpson ([DS], Theorem 3; Proposition 3.14 below). As noted by the referee, Lemma (3.6) lead to a simpler proof of this (in characteristic 0), which I now describe. We need two constructions.

Construction (*3.11*) We define the "universal" G-bundle over $X \times \Sigma^c$ by a holomorphic shortcut (see [BL2] for an algebraic argument). By Birkhoff factorization, $X \cong G[\Delta^{\times}]/\mathrm{Hol}(\Delta; G)$, locally trivially. Viewing loop group elements as transition functions yields a holomorphic G-bundle over $X \times \Sigma^c$, equipped with a canonical section over $X \times \Sigma$, with finite-order poles at the punctures. Because the base is projective, everything is, in fact, algebraic, including the action of G^{Σ} (on the section, and on X).

Construction (*3.12*) For any G-bundle P over Σ^c, there exist smooth, quasi-projective, fine moduli spaces M for "G-bundles over Σ^c, near P, with suitable gauge-fixing". In any algebraic family of "G-bundles with suitable gauge fixing", there is an open part classified by a unique map to M; this open part contains all bundles that are freely isomorphic to P (ignoring the gauge condition). The meaning of "suitable" depends on P, and can be "at sufficiently many points", or "to sufficiently high order at a given point"; we label the type of the gauge-fixing by α. The tangent space to M at P is $H^1(\Sigma^c; \mathscr{A}d_P(\alpha))$, where $\mathscr{A}d_P(\alpha)$ is the Ad-sheaf of P, respecting the gauge restriction α; and the gauge-fixing is suitable for P if $H^0(\Sigma^c; \mathscr{A}d_P(\alpha)) = 0$.

For GL_N, M can be obtained by refining a construction of Gieseker's; vector bundles near P will be generated by their global sections, if poles to prescribed orders are allowed at certain points of Σ^c. In general, embed G in some GL_N. Because GL_N/G is affine, adequate gauge-fixing eliminates multiple G-reductions of GL_N-bundles near the isomorphism class P, in any family, and ensures that a local universal family of "G-bundles with gauge-fixing" appears as a smooth, closed subvariety of a local universal family of "GL_N-bundles with gauge-fixing".

Remark (*3.13*) The spaces $M = M_{\alpha}$ carry an action of the "residual local gauge group" G_{α}, whose orbits are the sets of α-gauge-fixing choices on principal bundles in a free isomorphism class. The quotient stack M_{α}/G_{α} is an open substack of $\mathfrak{M}$, and the latter is covered by the M_{α} (countably many suffice).

Proposition (3.14) ([DS], Prop. 3) *For semi-simple G, a principal bundle over $U \times \Sigma^c$ becomes trivial after pull-back to $\tilde{U} \times \Sigma$, for a suitable étale covering $\tilde{U}$* of U.

Remark (*3.15*) (i) Smoothness of G only ensures, a priori, triviality over étale covers of $U \times \Sigma^c$. (ii) The result generalizes a theorem of Harder's [H] on the triviality of G-bundles over an affine curve; it also extends [BL1], which verified the result for SL_n (in the Zariski topology).

Proof. X has a Zariski cover mapping to the M_α, and one checks on tangent spaces that these classifying morphisms are formally smooth. By Corollary (3.10), each M_α has an étale lifting $\tilde{M}_\alpha$ to X. Now, any family of G-bundles will be étale-covered by families mapping to the M_α (take local sections of the principal bundles of gauge-fixing choices). After further refinement, we can select a classifying morphism to X, and pull back, with its help, the trivialization of the universal bundle along Σ. □

Remark (*3.16*) In stack language: $\mathfrak{M}$ is locally of finite type and smooth, and the morphism $\Phi : X \to \mathfrak{M}$, classifying the universal bundle over $X \times \Sigma^c$, is formally smooth. Corollary (3.10) and Harder's theorem imply, then, that Φ has étale slices everywhere.

4. Algebraic uniformization

Let us briefly sketch the argument in [LS]. The authors start with the [DS] result, that the G^Σ-equivariant classifying morphism Φ of (3.16) admits a covering family of étale sections. To show that Φ is a principal G^Σ-bundle, they must further check that the "action" map from $G^\Sigma \times X$ to the (stack) fibered product $X \times_{\mathfrak{M}} X$ is an isomorphism. Now, X represents the functor classifying pairs, consisting of a bundle over Σ^c and a section over Σ: the germs of the section at infinity give the map, from the parameter space, to X. Loosely put, the morphism to $\mathfrak{M}$ forgets the section, and it follows that $X \times_{\mathfrak{M}} X$ classifies triples, consisting of a bundle over Σ^c and *two* sections over Σ. The ratio of the sections gives the G^Σ-component of the inverse map, and this implies the desired isomorphism.

From here, general theory implies that the morphism $X \to \mathfrak{M}$ "satisfies (étale) cohomological descent": that is, cohomology of a sheaf over the big étale site of $\mathfrak{M}$ is isomorphic to the cohomology of the restricted sheaf over $X_\bullet$, which is the simplicial space of fibered powers of X over $\mathfrak{M}$. (Topologists would say that cohomology over $\mathfrak{M}$ is G^Σ-equivariant cohomology over X.) It also follows that any simplicial vector bundle over $X_\bullet$ is pulled back from $\mathfrak{M}$; this is the descent theorem for first cohomology with $\mathrm{GL}_N(\mathcal{O})$ coefficients.

5. Analytic versions

There are several versions of the uniformization theorem over the category $\mathfrak{A}$ of complex analytic spaces. Recall the definitions (2.8), and note that the infinite-dimensional Lie groups $\mathrm{Hol}(\_;G)$ satisfy the "exponential law": e.g., $\mathrm{Hol}(U;\mathrm{Hol}(\Delta^\times G)) = \mathrm{Hol}(U\times\Delta^\times;G)$, for any analytic space U. (This is best seen in the gauge picture, in the first proof of Prop. 2.3). They thus represent the "holomorphic mapping space" functors over $\mathfrak{A}$, defined as in Sect. III.1. In addition, let X_{hol} be the "holomorphic flag variety" $\mathrm{Hol}(\Delta^\times;G)/\mathrm{Hol}(\Delta;G)$. It follows, from Birkhoff factorization again, that X_{hol} does not depend on the choice of the disks; $\mathrm{Hol}(\Delta^\times;G)$ is covered by open charts isomorphic to $\mathrm{Hol}(\Delta;G)\times_G\mathrm{Hol}(\mathbb{C};G)$. Because of that local factorization, the manifold X_{hol} also represents the sheaf-theoretical quotient of the two group sheaves.

Consider first the stack $\mathfrak{M}_{\mathrm{hol}}$ of holomorphic G-bundles over Σ^c. By Grauert's theorem, holomorphic G-bundles over a Stein space are classified by their topological type, for any complex Lie group G. Now, $U\times\Sigma$ is Stein as soon as U is so; thus, if G is connected, principal bundles over $U\times\Sigma$ will be trivial as soon as U is contractible. This holomorphic version of the Drinfeld-Simpson theorem leads to the following proposition (valid, this time, for any connected complex Lie group).

Proposition 3.17. $\mathfrak{M}_{\mathrm{hol}}\cong X_{\mathrm{hol}}/\mathrm{Hol}(\Sigma;G)$. □

On the other hand, considering the analytic objects underlying X and G^Σ, define $\mathfrak{M}_{\mathrm{an}}$ as the quotient stack X/G^Σ over $\mathfrak{A}$. This is the "analytic stack underlying $\mathfrak{M}$". (General theory ensures that its analytic equivalence class only depends on the algebraic equivalence class of $\mathfrak{M}$, and not on the choice of presentation.) There is a natural morphism $\mathfrak{M}_{\mathrm{an}}\to\mathfrak{M}_{\mathrm{hol}}$, induced by the inclusion $X\subset X_{\mathrm{hol}}$.

Proposition (3.18) *The natural morphism $\mathfrak{M}_{\mathrm{an}}\to\mathfrak{M}_{\mathrm{hol}}$ is an equivalence of stacks. In other words, the analytic stack underlying $\mathfrak{M}$ "is" the stack of holomorphic G-bundles over Σ^c.*

First proof. Note that $G[\Delta^\times]/\mathrm{Hol}(\Delta;G)\cong X$, by Birkhoff factorization. Further, by Lemma 2.9, the quotients $G[\Delta^\times]/G[\Sigma]\subset\mathrm{Hol}(\Delta^\times;G)/\mathrm{Hol}(\Sigma;G)$ are isomorphic complex manifolds. But they also represent the stack quotients, by local triviality of the bundles; therefore, we have

$$\mathfrak{M}_{\mathrm{an}} = \mathrm{Hol}(\Delta;G)\backslash G[\Delta^\times]/G[\Sigma]\cong\mathrm{Hol}(\Delta;G)\backslash\mathrm{Hol}(\Delta^\times;G)/\mathrm{Hol}(\Sigma;G)=\mathfrak{M}_{\mathrm{hol}}\ .$$

I am indebted to C. Simpson for outlining a second proof of the proposition, which applies to any reductive group. It is based on the following observation.

Construction (*3.19*) There is a holomorphic construction corresponding to (3.12), and GAGA implies that the moduli spaces obtained that way are the analytic varieties underlying M_α (and the "residual gauge groups" are the analytic G_α). To obtain, locally, a holomorphic classifying map to a Grassmannian, à la Gieseker, we must check that any holomorphic family of vector bundles over Σ^c can *locally* be twisted by a line bundle, in such a way that all bundles are generated by their global sections on Σ^c, and that these spaces of sections form a vector bundle over the parameter space. We can guarantee this, simply by making the curvature along Σ^c, on one bundle, sufficiently positive, so that standard vanishing theorems for H^1 apply; the positivity condition will then hold for nearby bundles as well. That the global sections form a holomorphic bundle is, then, routine Fredholm theory. Note that the argument applies even to an infinite-dimensional parameter space, as we needed in Lemma 2.9.

Second proof of (*3.18*) Construction (3.19) produces, as noted, the underlying analytic varieties of the M_α, and the corresponding residual gauge groups G_α. Thus, $\mathfrak{M}_{\mathrm{hol}}$ is the limit of the analytic stacks underlying M_α/G_α; but so is $\mathfrak{M}_{\mathrm{an}}$, the underlying analytic stack of $\mathfrak{M}$. □

Remark (*3.20*) Similarly, one establishes the double coset realization of $\mathfrak{M}_{\mathrm{hol}}$ as the smooth maps from a loop on Σ^c, to G, modulo the left and right action of holomorphic loops over the cut surface (maps with smooth boundary-values are handled as in the proof of [PS], Prop. 8.10.2). The cutting loop need not disconnect the surface (see, for example, the end of Sect. 1.5). Also, the "Atiyah-Bott stack", of (0,1)-connections modulo complex gauge transformations, agrees with $\mathfrak{M}_{\mathrm{hol}}$, essentially because the gauge group action has finite-dimensional slices.

IV. The Hodge structure of the moduli stack

This section was included after a suggestion of C. Simpson. It has some relevance to the other results (see the discussion on the degeneration of Σ, in the final section), but the remainder of the paper does not depend on it.

There is a well-defined theory of mixed Hodges structures ("m.H.s.") for stacks. This can be deduced, with mid effort, from the simplicial theory in [D] (an account is planned for [ST]). However, I shall also indicate, in Remark (4.7), a proof of the results – or, at least, of one interpretation of the statements – that does not require this fact.

Note, first, the following consequence of Corollary (2.11); it partially vindicates the statement of Theorem 1′, even in ignorance of the homotopy theory of stacks.

Proposition (4.1) *There is a natural isomorphism* $H^*(\mathfrak{M}_{\mathrm{an}};\mathbb{Z}) \cong H^*_s(C^\infty(\Sigma^c;BG);\mathbb{Z})$.

The first term stands for cohomology of the constant sheaf over the analytic site of $\mathfrak{M}_{\mathrm{an}}$.

Proof. By the uniformization theorem, we have $H^*(\mathfrak{M}_{\mathrm{an}};\mathbb{Z}) \cong \mathbb{H}^*_{\mathrm{an}}(X_\bullet;\mathbb{Z})$. But the latter is also the G^Σ-equivariant cohomology of X. □

For the rest of this section, H^* will denote cohomology with constant coefficients $\mathbb{C}$. Assuming that Σ is smooth, of genus, g, we have a ring isomorphism (already over $\mathbb{Q}$, cf. [AB])

$$H^*(\mathfrak{M}) \cong H^*(BG) \otimes H^*(G)^{\otimes 2g} \otimes H^*(\Omega G) \tag{4.2}$$

(however, the integer lattice can be more difficult to determine, when there is torsion). One obtains the algebra generators of (4.2) by pulling back those of $H^*(BG)$ to $\Sigma^c \times C^\infty(\Sigma^c; BG)$, via the evaluation map, and then integrating them against a basis of homology cycles in Σ^c. In particular, the space of primitive elements of degree $(2p-1)$ in $H^*(G)^{\otimes 2g}$ is naturally isomorphic to $H_1(\Sigma^c)$, twisted by the line of algebra generators of degree $2p$ of $H^*(BG)$. This also holds when Σ^c is singular, but the exponent $2g$ must be replaced by the rank of $H_1(\Sigma^c)$.

Now, as was mentioned in Remark (1.4), the evaluation map on homotopy types arises from the evaluation morphism on stacks $\Sigma^c \times \mathfrak{M} \to BG$, classifying the universal G-bundle. The pull-back map on cohomology is a morphism of mixed Hodge structures (see Remark 4.7). Consider the Hodge structures on $H^*(\mathfrak{M})$ and $H^*(\Sigma^c \times \mathfrak{M})$, and the dual m.H.s. on $H_*(\Sigma^c)$.

Proposition (4.3) *The slant product $H_m(\Sigma^c) \otimes H^n(\Sigma^c \times \mathfrak{M}) \to H^{n-m}(\mathfrak{M})$ is a morphism of mixed Hodge structures.*

Proof. Immediate from of the definition of the dual m.H.s on $H_*(\Sigma^c)$, and from the Künneth decomposition of the Hodge structure on $H^*(\Sigma^c \times \mathfrak{M})$ ([D], Cor. 8.2.11). □

Thus, (4.2) refines to a multiplicative splitting of $\mathbb{Q}$-Hodge structures. When the curve is smooth, the Hodge structure on each $H_m(\Sigma^c)$ is pure of weight $(-m)$. There is a natural pure Tate structure on $H^*(\Omega G)$ (when G is simple, this is a polynomial algebra over $\mathbb{Q}$ with generators in distinct even degrees). Since BG has a pure, Tate structure [D], we obtain the following.

Proposition (4.4) *When Σ is smooth, the Hodge structure of $\mathfrak{M}$ is pure. Specifically, in (4.2), BG and ΩG have the pure, Tate structure; whereas, in the factor $H^*(G)^{\otimes 2g}$, the space of primitive generators of degree $(2p-1)$ splits into $(p-1,p)$ and $(p,p-1)$ parts, according to the Hodge decomposition of $H_1(\Sigma^c)$.*

If Σ is singular, $H^1(\Sigma^c)$ is filtered by the weight-zero subspace of (0,0) classes, while the weight one quotient splits into (1,0) and (0,1) parts ([D], Sect. 10). The dual Hodge structure filters $H_1(\Sigma^c)$ by the subspace of $(-1,0)$ and $(0,-1)$ cycles (this is the image in homology of the normalization of Σ^c), the quotient having type (0,0). This induces, under the slant product

$H_1(\Sigma^c) \otimes H^{2p}(BG) \to H^{2p-1}(\mathfrak{M})$, a weight filtration $W_{2p-1} \subset W_{2p}$ on the space of primitive elements of degree (2p-1). Its excess-weight quotient Gr_{2p} has dimension equal to the number of self-crossings of Σ (multiply counted where appropriate). Let $\tilde{g}$ the genus of the normalization of Σ^c. □

Proposition (4.5) *When Σ is singular, the Hodge structure on $\mathfrak{M}$ is super-diagonal. Specifically, BG and ΩG in* (4.2) *still have the Tate structure, but the* "Gr" *of the middle factor splits into a pure tensor factor $H^*(G^{2\tilde{g}})$, with the Hodge structure in* (4.4), *and one factor of $H^*(G)$, with the usual structure, for each self-crossing of Σ* (*counted appropriately*). □

Remark (*4.6*) Recall from [D] that, in the usual structure on $H^*(G)$, the primitive generator of dimension $(2p-1)$ has type (p,p). To rephrase (4.5) more invariantly, consider the "pull-back" morphism, from $\mathfrak{M}$ to the stack $\tilde{\mathfrak{M}}$ of G-bundles over the normalization of Σ^c. Then, the pure subring of $H^*(\mathfrak{M})$ is the image of $H^*(\tilde{\mathfrak{M}})$, while the residual tensor factors $H^*(G)$ are the fiberwise cohomologies of this morphism.

Remark (*4.7*) Compatibility of the evaluation $\Sigma^c \times \mathfrak{M} \to BG$ with the Hodge structure follows directly from the results in [D], because this morphism of stacks can be realized as a morphism of simplicial ind-varieties. To exhibit such a morphism, one must take coverings of Σ^c, and we leave that for the end of Appendix B; but the propositions can also be proved without that. One simply notes that, when Σ has a single point at infinity, the Leray spectral sequence in singular cohomology for the projection $X_\bullet \to BG^\Sigma$ collapses at E_2; the first two factors in (4.2) come from the cohomology of the base. It suffices,* to determine the Hodge structures of X and of the simplicial ind-variety BG^Σ. The latter is determined, as above, from the evaluation $\Sigma \times BG^\Sigma \to BG$, which is now a map of simplicial ind-varieties. Further, the Hodge structure of X agrees with that of $G[z,z^{-1}]/G$, which fibers over it with contractible fibers. That the latter has a pure Tate structure is seen from the evaluation morphism $\mathbb{C}^\times \times G[z,z^{-1}] \to G$, noting that the 1-cocycle in $\mathbb{C}^\times$ has type (1,1). The disadvantage of this method is its reliance on the algebraic uniformization theorem; the earlier argument applies to any linear algebraic group.

One should mention, in summary, that there are three relevant algebraic objects whose homotopy type is ΩG, with different Hodge structures. The first is the standard flag variety Q, which has a pure Tate structure. The second is $G[A]/G$, where A is the affine line with a single self-intersection: its type is super-diagonal, a primitive generator of dimension $2p-2$ having type (p,p). Finally, the third type comes in pairs, as in $\Omega G^{\times 2g} \sim G[\Sigma]/G$,

*As $H^*(\Omega G)$ turns out to be pure, and $H^*(\mathfrak{M})$ super-diagonal at worst, the former can be lifted into the latter, so (4.2) does refine to a multiplicative splitting of $\mathbb{R}$-Hodge structures (but getting the $\mathbb{Q}$-splitting seems to require extra arguments) where Σ is a smooth curve with a single point at infinity. The primitive generators of dimension $2p-2$ split into type $(p, p-1)$ and $(p-1, p)$. (Additional punctures on Σ would contribute factors of ΩG with the Tate structure). Instead of G-quotients, we can also use the subgroups of maps based at some point; and these spaces, and their B's, are responsible for the Hodge structure of $\mathfrak{M}$.

V. The Picard group of $\mathfrak{M}$

This question has gathered some interest in recent literature. For the classical groups and for G_2, the expected result (Prop. 5.1 below) has been determined by case-by-case computations in [LS] (see also [BLS], and also [KNR], for complete accounts); but – one thing I had missed – the cases E and F were thought problematic. The simple holomorphic argument[11] for splitting the $G[\Sigma]$-extensions, given in the first proof below, was apparently unfamiliar in algebraic circles. Of course, knowing the low-degree homology of $G[\Sigma]$ allows for a completely algebraic rewriting. The several proofs below amount to overkill, but they help emphasize that the notion of a "line bundle over a stack" is not unusually subtle.

1. The simply connected case

One way to identify $\mathrm{Pic}(\mathfrak{M})$ is by "descent from X" (the method used in the cited papers). Assume that G is simple, the semi-simple analogue being obvious.

Proposition (5.1) *$H^1(\mathfrak{M}; \mathcal{O}^\times) \cong \mathbb{Z}$, and the generator lifts to the basic line bundle $\mathcal{L}$ on X.*

Remark (*5.2*) The first Chern class identifies $\mathrm{Pic}(\mathfrak{M})$ with $H^2_s(\mathfrak{M}_{\mathrm{an}}; \mathbb{Z})$. Indeed, for a singly punctured Σ, pull-back along the classifying morphism identifies this second group with $H^2_s(\Omega G; \mathbb{Z})$, whose generator is $c_1(\mathcal{L})$.

Remark (*5.3*) It is shown in [LS] that the line bundle $\mathcal{K}$ over $\mathfrak{M}$, the determinant of cohomology along Σ^c of the Ad-bundle $\mathcal{G} \times_G \mathfrak{g}$ of the universal bundle, has index 2c in $\mathrm{Pic}(\mathfrak{M})$, at least when G is not of type E or F. As expected, this holds in all (1-connected) cases, as follows from the computation of $c_1(\mathcal{K})$ (on ΩG) as $(-2c)$. (One way to obtain the latter uses Quillen's computation of the determinant bundle curvature over the "Atiyah-Bott stack" of GL_N-bundles, plus the effect, on second cohomology of $\mathfrak{M}$, of the Ad-homomorphism $G \to \mathrm{GL}(\mathfrak{g})$).

[11] Modulo 1-connectedness of $\mathrm{Hol}(\Sigma; G)$. The argument, I believe, is due to G. Segal

Proof. In words, we are asking which line bundles on X carry a lifting of the G^{Σ}-action. For X_{hol} and $\mathrm{Hol}(\Sigma; G)$, these are precisely the powers of $\mathscr{L}$, as they are the only ones for which the central extension of $\hat{L}G$ splits over $\mathrm{Hol}(\Sigma; G)$. Indeed, the connectedness and simple connectedness of $\mathrm{Hol}(\Sigma; G)$ reduces this to a Lie algebra question, which is settled by the well-known inspection of 2-cocycles, and the absence of non-trivial characters. The holomorphic splitting determines, by restriction, a holomorphic lifting of the G^{Σ}-action, the only question being whether these lifted actions are algebraic.

Now, this is true for certain powers of the basic line bundle (those pulled back, via a representation $G \to \mathrm{SL}_N$, from the "determinant" line bundle over $\Omega\mathrm{SL}_N$); but then, it must hold for $\mathscr{L}$ itself. To restate this in more familiar form, notice that $\mathscr{L}$ descends to a holomorphic line bundle over $\mathfrak{M}_{\mathrm{hol}}$, some power of which is algebraic. Since $H^0(\mathfrak{M}; \mathcal{O}^{\times}) = \mathbb{C}^{\times}$ (algebraically and holomorphically), the long exact cohomology sequence

$$0 \to H^1_{\acute{e}t}(\mathfrak{M}; \mathbb{Z}/(n)) \to H^1(\mathfrak{M}; \mathcal{O}^{\times}) \xrightarrow{(\cdot)^{\otimes n}} H^1(\mathfrak{M}; \mathcal{O}^{\times}) \xrightarrow{c_1} H^2_{\acute{e}t}(\mathfrak{M}; \mathbb{Z}/(n)) \ , \tag{5.4}$$

and the natural isomorphism $H^*_{\acute{e}t}(\mathfrak{M}; \mathbb{Z}/(n)) \cong H^*(\mathfrak{M}_{\mathrm{an}}; \mathbb{Z}/(n))$ (cf. Prop. 4.1), remind us that the obstruction to existence and uniqueness of algebraic roots of line bundles are purely topological, and agree with the holomorphic obstructions: they are the first Chern class (mod. n), and the flat line bundles. In this case, $H^1_{\acute{e}t}(\mathfrak{M}; \mathbb{Z}/n) = 0$, so $\mathscr{L}$ itself is algebraic, and uniquely so.

□

Algebraic reformulation: Since $H^0(X; \mathcal{O}^{\times}) \cong \mathbb{C}^{\times}$, the descent spectral sequence for cohomology of $\mathfrak{M}$ with $\mathcal{O}^{\times}$ coefficients leads, in low degrees, to the exact sequence

$$0 \to H^1(BG^{\Sigma}; \mathcal{O}^{\times}) \to H^1(\mathfrak{M}; \mathcal{O}^{\times}) \to H^0\big(BG^{\Sigma}; H^1(X; \mathcal{O}^{\times})\big) \to H^2(BG^{\Sigma}; \mathcal{O}^{\times}) \ . \tag{5.5}$$

Recalling the interpretations of H^1 and H^2 of BG^{Σ} as one-dimensional representations and central $\mathbb{C}^{\times}$-extensions, respectively, (5.5) is a typographically-enhanced way of stating the obvious: line bundles on $\mathfrak{M}$ are the same, by pull-back, as line bundles on X with a G^{Σ} action (the last arrow captures the central extension arising from an attempted lifting of the action); and the action on the lifted bundle is unique, up to a complex character. But, as noted before, all such characters are trivial. Also, G^{Σ} is connected, so taking invariants in the discrete set $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus n}$ removes nothing. As before, one argues that the kernel of the rightmost map is non-zero, and is contained in the diagonal copy of $\mathbb{Z}$ in $\mathrm{Pic}(X)$. We could be missing a torsion obstruction in $H^2(BG^{\Sigma}; \mathcal{O}^{\times})$; but such a class would come from $H^2_{\acute{e}t}(BG^{\Sigma}; \mathbb{Z}/n)$, and Theorem 1 shows this last group to be zero. □

2. The general semi-simple case

The addition of this section was inspired by [BLS], where the Picard group of $\mathfrak{M}$ was determined, case-by-case, for most simple groups (for positive genus curves). I shall give a general cohomological calculation; but first, I must explain how the statements of the previous sections are to be modified when G is not 1-connected. Let $\Pi := \pi_1(G)$, call $\hat{\Pi}$ the dual group, and $\tilde{G}$ the simply connected cover of G. Then:

• The standard flag variety $G((z))/G[[z]]$ is disconnected, with components labeled by Π. (All components are isomorphic to the flag variety of $L\tilde{G}$, but the $L\tilde{G}$-action on a component indexed by a central element $\exp(2\pi \mathrm{i} \cdot \zeta)$ is conjugated by the fractional loop $z \mapsto z^{\zeta}$.)

• The algebraic and holomorphic uniformization theorems hold without changes.

• The homotopy equivalences $G^{\Sigma} \sim C^{\infty}(\Sigma; G)$ and $\mathfrak{M} \sim C^{\infty}(\Sigma; BG)$ continue to hold.

• The Hodge theory of Sect. IV applies, but the components of $\mathfrak{M}$ are labeled by Π, and the complex cohomology acquires the tensor factor $H^0(\mathfrak{M}; \mathbb{C})$, the algebra $\mathbb{C}[\Pi]$ of functions on Π.

As expected, $\mathrm{Pic}(\mathfrak{M})$ is purely topological.

Proposition (5.6) $H^1(\mathfrak{M}; \mathcal{O}^{\times}) \cong H^1(\mathfrak{M}_{\mathrm{hol}}; \mathcal{O}^{\times}_{\mathrm{hol}})$.

Propostion (5.7) $H^1(\mathfrak{M}_{\mathrm{hol}}; \mathcal{O}^{\times}_{\mathrm{hol}}) \cong H^2(\mathfrak{M}_{\mathrm{hol}}; \mathbb{Z})$, *by the first Chern class.*

The last cohomology, which is also that of $C^{\infty}(\Sigma^c; BG)$, will be computed from the tower of fibrations over BG, with successive fibres $G^{\times 2g}$ and ΩG, associated to a bouquet of $2g$ loops on Σ^c (see [AB]). For simply connected G, we recover Proposition 5.1 from the absence of torsion in low degrees. In general, there is a natural pull-back $H^2(\mathfrak{M}_{\mathrm{hol}}; \mathbb{Z}) \to H^2(\Omega G; \mathbb{Z})$, and the latter is $H^2(\Omega\tilde{G}) \otimes \mathbb{Z}[\Pi]$. Recall now ([PS]), Sect. 4.6) that, to every integral class in $H^2(\Omega\tilde{G})$, one can assign a symmetric bilinear pairing $Z(\tilde{G}) \times Z(\tilde{G}) \to S^1$, by

$$\exp(2\pi \mathrm{i} \cdot \xi) \times \exp(2\pi \mathrm{i} \cdot \eta) \mapsto \exp(2\pi \mathrm{i} \cdot \langle \xi, \eta \rangle) \ , \tag{5.8}$$

where $\langle\ \rangle$ is the invariant inner product on $\mathfrak{g}$ associated to the class. (The integral generator of $H^2(\Omega\tilde{G})$ is sent to the *basic* inner product, in which the short co-roots have square-length 2.) Restricting (5.8) to $\Pi \subseteq Z(\tilde{G})$ yields a linear map $\alpha : H^2(\Omega G) \to \mathrm{Sym}^2(\hat{\Pi})^{\oplus|\Pi|}$. (This is the same map for every component of ΩG). By contracting the value of α with the indexing element, in Π, of components of ΩG, we also obtain a linear map $\beta : H^2(\Omega G) \to \hat{\Pi}^{\oplus|\Pi|}$.

Proposition (5.9) (i) *In genus zero,* $H^2(\mathfrak{M}_{\mathrm{hol}}; \mathbb{Z})$ *is the kernel of* β.

(ii) *In genus* $g > 0$, *the torsion subgroup of* $H^2(\mathfrak{M}_{\mathrm{hol}}; \mathbb{Z})$ *is a sum of copies of* $H^1(\Sigma^c; \hat{\Pi})$, *indexed by the components of* $\mathfrak{M}$. *The torsion-free quotient is the kernel of* α.

Remark (*5.10*) Recall that the centers of the simple Lie groups are: $\mathbb{Z}/n$ for SL_n, $\mathbb{Z}/2$ for $\mathrm{Sp}(n)$ and Spin $(2n+1)$, $\mathbb{Z}/2 \times \mathbb{Z}/2$ for Spin $(4n)$, $\mathbb{Z}/4$ for $\mathrm{Spin}(4n+2)$, $\mathbb{Z}/3$ for E_6, $\mathbb{Z}/2$ for E_7, and trivial for E_8, F_4 and G_2. The pairing (5.8) is non-degenerate for the simply laced groups and for the odd-rank symplectic groups, and null for all others. This determines Pic ($\mathfrak{M}$) for all semi-simple groups. For the classical ones, we recover the results of [BLS].

Recall also ([PS]), Proposition 4.6.3), that the kernel of α corresponds also to "levels" in $H^2(\Omega\tilde{G})$ at which the loop group LG has central extensions that are trivial over the subgroup of constant loops. Interestingly, in genus zero, different components of $\mathfrak{M}$ have basic line bundles of different levels; in higher genus, only those levels satisfying the most restrictive condition survive. We shall prove (5.9) in the next subsection; now we return to the previous propositions.

Proof of (*5.6*) We know that $H^1(BG^\Sigma; \mathcal{O}^\times) \cong H^1(BG^\Sigma; \mathcal{O}^\times_{\mathrm{hol}})$, because all one-dimensional representations of G^Σ come from the group of components; and $H^1(X; \mathcal{O}^\times) \cong H^1(X; \mathcal{O}^\times_{\mathrm{hol}})$, just as in the simply connected case. Comparing the algebraic and analytic versions of the sequence (5.5), one-half of the five lemma shows that the natural map $H^1(\mathfrak{M}; \mathcal{O}^\times) \to H^1(\mathfrak{M}_{\mathrm{hol}}; \mathcal{O}^\times_{\mathrm{hol}})$ is one-to-one. But, by the argument used in the first proof of (5.2), the quotient is torsion group. This is, then, detected by comparing the exact sequence (5.4) with its holomorphic counterpart; but cohomology with $\mathbb{Z}/n$ coefficients being the same in both, the algebraic and analytic Picard groups must agree. $\square$

Remark (*5.11*) If Σ has a single point at infinity, $H^1(BG^\Sigma, \mathcal{O}^\times)$ is dual to the group $H_1(\Sigma; \Pi)$ of components of G^Σ, and this is another way to explain the torsion summand in Pic($\mathfrak{M}$).

Proof of (*5.7*) This follows from the exponential sequence and the vanishing of $H^q(\mathfrak{M}_{\mathrm{hol}}; \mathcal{O}_{\mathrm{hol}})$ for $q = 1, 2$. This, of course, follows from Theorems 2 and 3 for $\mathcal{O}_{\mathrm{hol}}$ coefficients; but it can also be seen by descent from X, since G^Σ has no homomorphisms into, or extension by, $\mathbb{C}$. $\square$

Remark (*5.12*) The holomorphic argument can be avoided, if one argues, as in the second proof of (5.1), that the problem lies only in identifying the torsion subgroup and the integral generator of Pic($\mathfrak{M}$). Étale cohomology with torsion coefficients can then be used to show that Pic($\mathfrak{M}$) is $H^2(C^\infty(\Sigma^c; BG); \mathbb{Z})$.

3. Cohomological calculations

In genus zero, the homotopy type of $\mathfrak{M}$ is an ΩG-fibration over BG, with respect to the conjugation action of G. Thus, $H^2(\mathfrak{M}_{\mathrm{hol}}; \mathbb{Z}) \cong H^2_G(\Omega G; \mathbb{Z})$. From the Leray spectral sequence, the later is the kernel of the transgression

$$H^2(\Omega G; \mathbb{Z}) \xrightarrow{\delta_3} H^3(BG; \mathbb{Z}[\Pi]) \cong \hat{\Pi}^{\oplus|\Pi|} \ . \qquad (5.13)$$

(Recall that $H^3(BG;\mathbb{Z}) \cong H^2(B\Pi;\mathbb{Z})$, naturally isomorphic to the dual group $\hat{\Pi}$ of Π.)

Claim (5.14) *The transgression in* (5.12) *is the map* β *of* (5.8).

Proof. This is the computation of [PS], Sect. 4.6 (though it served a different purpose there). To translate, note that each component of the map (5.13), corresponding to some $p \in \Pi$, detects the obstruction for a line bundle over the component p of LG/G to carry a lifting of the left G-action. Now, all line bundles carry a unique $\tilde{G}$-action ($\tilde{G}$ has no non-trivial characters or central extensions), and this comes from a G-action precisely when $\pi_1(G) \subset Z(\tilde{G})$ acts trivially on the fibers of the bundle. But the computation in *loc. cit.* identifies this action with that character of Π which is associated to $p \in Z(\tilde{G})$ by means of the pairing (5.8). $\square$

In positive genus, $\mathfrak{M}$ is an ΩG-fibration over the homotopy quotient $G^{\times 2g}/G$ (again with respect to the conjugation action); specifically, it is the pull-back of the path fibration $\mathscr{P}G$ of G, via the "product of commutators" map $G^{\times 2g} \to G$. This last map lifts to $\tilde{G}$, over which the path fibration of G splits into $|\,\Pi\,|$ components, with globally constant cohomology sheaves; and everything is equivariant for the Ad action of G. (As before, the components of the fibre are not isomorphic as G-spaces, but his will now turn out to be unimportant.) We can then extract, from the Leray spectral sequence of the ΩG-fibration, the exact sequence

$$0 \to H^2_G(G^{\times 2g};\mathbb{Z}[\Pi]) \to H^2(\mathfrak{M};\mathbb{Z}) \to H^2(\Omega G;\mathbb{Z}) \xrightarrow{\delta_3} H^3_G(G^{\times 2g};\mathbb{Z}[\Pi]) \tag{5.15}$$

The first term is the advertised torsion subgroup $H^1(\Sigma^c;\hat{\Pi}) \otimes \mathbb{Z}[\Pi]$, and the image of $H^2(\mathfrak{M};\mathbb{Z})$ is free; so we must only identify δ_3. Now, the map $G^{\times 2g} \to G$ factors into g commutator maps, followed by the multiplication $G^{\times g} \to G$. The latter is split (equivariantly for the Ad action), and it follows that the kernel of δ_3 is correctly identified by the transgression to a single pair of G's, corresponding to a genus one curve. The commutator $G^{\times 2} \to G$; is rationally trivial, and factors through the smash product $G \wedge G$; so δ_3 is detected in the torsion subgroup $\hat{\Pi} \oplus \hat{\Pi} \otimes \hat{\Pi}$ of $H^3_G(G \wedge G;\mathbb{Z})$. (The first summand is $H^3(BG)$, the second is the torsion subgroup $\mathrm{Ext}^1_{\mathbb{Z}}(\Pi \otimes \Pi, \mathbb{Z})$ of $H^3(G \wedge G;\mathbb{Z})$. The splitting is natural, coming from the base-point section of the $G \wedge G$-bundle over BG.) So, projecting δ_3 onto the second summands gives a map $H^2(\Omega G;\mathbb{Z}) \to (\hat{\Pi} \otimes \hat{\Pi}) \otimes \mathbb{Z}[\Pi]$.

Claim (5.16) *The projected transgression* $H^2(\Omega G;\mathbb{Z}) \to (\hat{\Pi} \otimes \hat{\Pi}) \otimes \mathbb{Z}[\Pi]$ *is the map* α *of* (5.8).

Remark (*5.17*) Projection to the first summands $\hat{\Pi}^{\oplus|\Pi|}$ is, of course, the map β, as in (5.14); but it leads to no extra restrictions, because its kernel contains that of α.

Proof. The projected δ_3 arises by forgetting the Ad action on $G \wedge G$. Now, the generators of each fiber component of the pulled-back path fibration $\mathscr{P}G \times_G \tilde{G}$ transgress to the generators of $H^3(\tilde{G}; \mathbb{Z})$; so the question becomes to identify the generators of the latter, after pull-back to $H^3(G \wedge G)$ by the (lifted) commutator map. This map is rationally trivial, so the pull-back lives in the torsion subgroup $\mathrm{Ext}^1_{\mathbb{Z}}(\Pi \otimes \Pi, \mathbb{Z})$, as it should. The restriction $H^3(G \wedge G) \to H^3(G \wedge T)$ is injective, so it suffices to identify the pull-back to the latter. Now, the commutator $G \wedge T \to \tilde{G}$ factors through $F \wedge T$, where F is the flag variety of G. Since F has no torsion, we lose no information by lifting the class to $F \wedge \tilde{T}$ ($\tilde{T}$ is the maximal torus in $\tilde{G}$). Note that we have a natural isomorphism of $H^3(F \wedge \tilde{T})$ with $H^1(\tilde{T}) \otimes H^1(\tilde{T})$, by the transgression in the Leray spectral sequence of $\tilde{G}$ over F. From the definition in (5.9) of the pairing of Π, the proof of (5.16) will be completed, if we verify the following lemma. □

Lemma 5.18. *When G is simple, the generator of $H^3(\tilde{G})$ corresponds, under the above procedure, to the pairing on $H_1(\tilde{T})$ coming from the basic inner product in* $\mathfrak{t}$.

Proof. We show that the pairing is symmetric; since it is clearly Weyl-invariant, we only have to find the normalization factor, which is accomplished by restriction to the principal SL_2. For the latter, the commutator map $\mathbb{P}^1 \wedge S^1 \to \mathrm{SU}_2$ is generically 2-to-1, so the principal co-root acquires square-length 2, as predicted.

Now, in getting to $H^1(\tilde{T}) \otimes H^1(\tilde{T})$, we lift the cocycle from $F \wedge \tilde{T}$ to $\tilde{G} \wedge \tilde{T}$, write it as a co-boundary, and restrict to $\tilde{T} \wedge \tilde{T}$. We could also get there through the commutator map $\tilde{T} \wedge F \to \tilde{G}$, and symmetry is established by showing that the two classes thus constructed agree. But $H^3(\tilde{G})$ pulls back trivially to $\tilde{G} \times \tilde{G}$; so we can use the "same" trivialization of the cocycle both over $\tilde{G} \wedge \tilde{T}$ and over $\tilde{T} \wedge \tilde{G}$, and so the two pairings on $H_1(\tilde{T})$ must agree. □

VI. The van Est spectral sequence

The next two sections contain the proofs of Theorems 2 and 2′. Here, we show the following.

Proposition (6.1) *There is a spectral sequence with $E_2^{p,q} = H^p_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V} \otimes H^q_s(G^\Sigma))$, converging to $H^*(\mathfrak{g}[\Sigma]; \mathbf{H} \otimes \mathbf{V})$. It is compatible with products.*

Lie algebra cohomology with $\mathbb{C}$ coefficients is a ring, acting on the cohomology with arbitrary coefficients; and there are similar product structures on the E_2 term. The statement is that the differentials are derivations for these products, and that the structure on E_∞ is the "Gr" of the abutment. Compatibility with products will follow from the naturality of the constructions.

Remarks (*6.2*) (i) $H_s^q(G^\Sigma)$ carries the translations action of G^Σ. However, when G is simply connected, G^Σ is connected, so its action on cohomology is trivial; the factor $H_s^q(G^\Sigma)$ can then be pulled out of the group cohomology coefficients.

(ii) What appears naturally in the E_2 term is the *stratifying* (or *formal Alexander*) *cohomology* of G^Σ. Similarly, the abutment is naturally the cohomology of the *formal group* of G^Σ. In this generality, (6.1) holds for any group sheaf over $\mathfrak{F}$, and the true content of the Proposition lies in the identification of stratifying cohomology with $H_s^q(G^\Sigma)$; the identity between cohomology of the Lie algebra and of the formal group is obvious in characteristic zero (Lemma 6.10).

A direct proof is given in the final subsection, and the reader may skip ahead to it. However, it seemed sensible to start by reviewing the simple situation of a complex linear algebraic group. I then recall the *Shapiro spectral sequence*, analogous to (6.1), but where the Lie algebra is replaced by a subgroup. Afterwards, the morally correct proof of (6.1) is explained, in which the *crystalline cohomology* of G^Σ appears; some ingredients of this argument will resurface in the official proof anyway. Finally, the official proof of (6.1) uses only rudiments of the cohomology theory of simplicial objects. (See [D] for some background).

1. The case of a linear algebraic group

The Abelian categories $\mathrm{Rep}(K)$, or locally finite representations of a linear algebraic group K, and that of all its Lie algebra representations, $\mathrm{Rep}(\mathfrak{k})$, are related by a pair of adjoint functors (Res, Ind). The first functor is restriction to $\mathfrak{k}$, while $\mathrm{Ind}(\mathsf{V})$ is the largest locally finite, K-integrable subrepresentation of $\mathsf{V} \in \mathrm{Rep}(\mathfrak{k})$. This is the space $\mathsf{V}[K]^{\mathfrak{k}}$ of V-valued polynomials on K that are invariant under the action of $\mathfrak{k}$, simultaneously on V and, by right translation, on K. (K then acts of $\mathsf{V}[K]^{\mathfrak{k}}$ by left translations.)

The group cohomology functors H_K^* are the right derived functors of $(\cdot)^K$. Because each $\mathsf{V}[K]$ is injective in $\mathrm{Rep}(K)$, group cohomology is resolved by the complex of *algebraic* Eilenberg-MacLane cochains. The composition $(\cdot)^K$ Ind of Ind with the left exact functor of K-invariants is the left exact functor $(\cdot)^{\mathfrak{k}}$ of Lie algebra invariants, and there is a Grothendieck spectral sequence for the composition of functors

$$E_2^{p,q} = H_K^p(\mathbb{R}^q\ \mathrm{Ind}(\mathsf{V})) \Rightarrow H^*(\mathfrak{k};\mathsf{V}) \ . \tag{6.3}$$

It follows from the definition that $\mathbb{R}^q\mathrm{Ind}(\mathsf{V}) \cong H^*(\mathfrak{k};\mathsf{V}[K])$. For the trivial representation, this is the ordinary cohomology $H^q(K)$ of the underlying analytic space, as the Koszul resolution $\mathbb{C}[K]\otimes\Lambda^*(\mathfrak{k}^t)$ of $\mathfrak{k}$-cohomology is isomorphic to the algebraic de Rham complex of K. In general, if V is a restricted K-representation, the diagonal $\mathfrak{k}$-action on $\mathsf{V}[K]$ is isomorphic to the action on K alone: the untwisting isomorphism sends $\upsilon \in \mathsf{V}[K]$ to the function $(k \mapsto k \cdot v(k))$. Therefore $\mathbb{R}^q\mathrm{Ind}(\mathsf{V}) \cong H^q(\mathfrak{t}; \mathbb{C}[K]) \otimes \mathsf{V} \cong H^q(K) \otimes \mathsf{V}$, but with K now acting diagonally, and (6.3) becomes the *van Est spectral sequence*

$$E_2^{p,q} = H_K^p(H^q(K) \otimes V) \Rightarrow H^*(\mathfrak{t}; V) \quad . \tag{6.4}$$

When K is connected, it acts trivially on $H^*(K)$, and E_2 factors as $H_K^p(\mathsf{V}) \otimes H^q(K)$.

Remark (*6.5*) The spectral sequence (6.4) applies in great generality (real or complex Lie groups, and even to topological groups); but the choice of Abelian categories of representations may not be obvious. For linear algebraic groups, (6.4) always collapses at E_2, a property not shared by the analytic ones[12]. Collapse in the algebraic case can be shown from a unipotent $\times$ reductive semi-direct decomposition. (It can also be deduced from Hodge-theoretic properties of BK, specifically, from the vanishing of the map $H^p(BK; \mathbb{C}) \to H^p(BK; \mathcal{O})$).

2. *The Shapiro spectral sequence*

The reader may be familiar with a similar construction, in which the Lie algebra $\mathfrak{k}$ is replaced by a subgroup $L \subset K$. In that case, (6.3) is replaced by the following spectral sequence (text-book material when K, L are finite groups, in which case $q = 0$):

$$E^2_{p,q} = H_K^p(\mathbb{R}^q\mathrm{Ind}_L^K(\mathsf{V})) \Rightarrow H_L^*(\mathsf{V}) \quad . \tag{6.6}$$

From the descent spectral sequence $K \to K/L$, it follows that $\mathbb{R}^q\mathrm{Ind}_L^K(\mathsf{V}) \cong H^q(K/L; \mathscr{V})$, where $\mathscr{V}$ is the sheaf of (algebraic) sections of the vector bundle associated to V over K/L. In this picture, (6.6) is the "descent spectral sequence from K/L to BL".

A general proof of (6.6), covering the needed cases, mirrors the official proof of (6.1) below, and we shall not rewrite it. We only note that the relevant quotient K/L, in $\mathbb{R}^q\mathrm{Ind}_L^K$, is the sheaf-theoretic quotient in the étale topology; this is the obvious quotient when K and L are honest algebraic groups. The Shapiro spectral sequence will be used in Sect. VII.

3. *Relation to "crystalline cohomology"*

Before giving the official proof of (6.1), let me describe the "correct" argument that applies in much greater generality. It relies on the homotopy theory of simplicial sheaves over $\mathfrak{F}$, as in [Bro], [J] (or on a theory of 1-stacks

[12] The Heisenberg group with center $\mathbb{C}^\times$ gives a counterexample (using constant coefficients)

over big étale sites, thus a bit more general than [LM]); in the absence of a complete exposition, it would be inadequate by itself. The argument applies to any formally smooth group sheaf over $\mathfrak{F}$, in the étale topology[13] .

Unofficial proof of (*6.1*) Consider the formal group $\hat{K}$ of K. The short exact sequence of group sheaves $1 \to \hat{K} \to K \to K/\hat{K} \to 1$ leads to a fiber bundle $K/\hat{K} \subset B\hat{K} \twoheadrightarrow BK$. (This may seem like wishful thinking, but a model for these objects will be given below.) The 0-stack $\mathfrak{K} := K/\hat{K}$ is the **crystalline stack** of K. (If K is not formally smooth, this is, instead, the *stratifying stack* of App. A). For any scheme Y, we have Hom $(Y;\mathfrak{K}) =$ Hom$(Y_{\mathrm{red}};K)$, and the over category $\mathfrak{F}/\mathfrak{K}$ is the (big) crystalline site of K [Gro2]. This vocabulary comes with a theorem.

Proposition (6.7) ("*Crystalline cohomology*") $H^*(\mathfrak{K};\mathcal{O}) \cong H^*_s(K;\mathbb{C})$.

We postpone of the proof, and note, by Lemma A.18, that Proposition 6.7 will remain true for a direct product of copies of $\mathcal{O}$, if $\mathbb{C}$ is replaced by the corresponding vector space.

Remark (*6.8*) The functor Hom$(.;\mathfrak{K})$ is called by Simpson [Sim] the **de Rham functor** of K. Proposition 6.7 generalizes to a simplicial scheme (locally of finite type), instead of K, and can be proved using the methods of *loc. cit.* (an account is planned for [ST]). Any ind-variety, such as G^Σ, is weakly equivalent (in the context of App. B) to such a simplicial scheme, and this can replace, in the proof below, the use of Alexander cohomology.

To continue, consider the $\mathcal{O}$-module $\mathscr{F} := (\text{point} \times \mathbf{F})/K$ over BK, coming from a K-representation $\mathbf{F}$. When $K = G^\Sigma$ and $\mathbf{F} = \mathbf{H} \otimes \mathbf{V}$, one should take care that $\mathscr{F}$ is not a vector bundle; rather, it will be the push-down of $\mathscr{L}^{\otimes h} \otimes \mathscr{V}$ from the stack X/G^Σ to BG^Σ. The restriction of $\mathscr{F}$ to $B\hat{K}$ is trivial along the fiber $K/\hat{K}$ (for G^Σ, this means it is a *product* of copies of $\mathcal{O}$); Prop. 6.7 implies that the Leray spectral sequence takes the form

$$E_2^{p,q} = H^p(BK; H^q_s(K;\mathscr{F})) \Rightarrow H^{p+q}(B\hat{K};\mathscr{F}) \quad . \tag{6.9}$$

When $H^q_s(K;\mathbb{C})$ is finite-dimensional, the E_2 term is $H^p_K(\mathbf{F} \otimes H^q_s(K;\mathbb{C}))$, and the following proposition shows that (6.9) is the desired spectral sequence (6.1).

Proposition (6.10) $H^*(B\hat{K};\mathscr{F}) \cong H^*(\mathfrak{k};\mathbf{F})$, *canonically*.

Assuming the two propositions, the proof of (6.1) is completed by describing a world containing the mythical creatures $BK, B\hat{K}$ and $\mathfrak{K}$, with their advertised properties. The 2-category of 1-stacks over $\mathfrak{F}$ would do; but we need the simplicial objects for cohomology anyway.

[13] Or over $\mathfrak{A}$ in the analytic topology, for the complex analytic statement

For BK, we take the simplicial bar realization $EK_\bullet/K$. Next, let $\mathfrak{K}_\bullet$ be the Alexander stack of K of (A.4); this is the formal completion of $EK_\bullet$ about the diagonal copy of K. Finally, while the bar construction will not do for $B\hat{K}$ (the obvious map to $BK_\bullet$ is not a fiber bundle), we can take instead $B'\hat{K}_\bullet := (EK_\bullet \times \mathfrak{K}_\bullet)/K$. The projection to $B\hat{K}_\bullet$ (bar construction) is a weak equivalence (a "trivial local fibration", as defined in [Bro]). On the other hand, $B'\hat{K}_\bullet$ does map to $BK_\bullet$ through the first factor, with fiber $\mathfrak{K}_\bullet$.($B'\hat{K}_\bullet$ is a "twisted Cartesian product, with simplicially discrete structure group K and fiber $\mathfrak{K}_\bullet$"; see [May], Ch. 4, for the definition.) □

4. *Two useful lemmas proved*

Proof of (*6.7*). When K is a group variety, the proposition follows from [Gro2]. In general, $H^*(\mathfrak{K};\mathcal{O})$ is isomorphic to the Alexander cohomology AH^* (see App.A) of the ind-variety G^Σ: the map $K \to \mathfrak{K}$ is a covering (already in the Zariski topology), and the Alexander stack $\mathfrak{K}_\bullet$ of K consists of the fibered powers of K over $\mathfrak{K}$. By Lemma (A.20), $AH^*(G^\Sigma)$ is the inverse limit of the $AH^*(G[\Sigma]_d)$; by the Alexander cohomology theorem (A.5), these are isomorphic to the singular cohomologies, whose inverse limit is $H_s^*(G[\Sigma];\mathbb{C})$, by Prop. 2.1. □

Proof of (*6.10*). Here, we use the bar construction $B\hat{K}_\bullet$. Let $\pi : U(\mathfrak{k}) \to \mathbb{C}$ be the projection whose kernel is the maximal ideal $\mathfrak{k} \cdot U(\mathfrak{k})$ of the universal enveloping algebra. The free resolution of the trivial left $U(\mathfrak{k})$-module $\mathbb{C}$

$$(6.12) \qquad 0 \leftarrow \mathbb{C} \leftarrow U(\mathfrak{k}) \leftarrow U(\mathfrak{k}) \otimes U(\mathfrak{k}) \leftarrow \cdots \leftarrow U(\mathfrak{k})^{\otimes p} \leftarrow \cdots \quad ,$$

($\mathfrak{k}$ acts on $U(\mathfrak{k})^{\otimes p}$ diagonally by left multiplication), with differential

$$(6.13) \quad \partial(u_0 \otimes \cdots \otimes u_p) = \sum_i (-1)^i \pi(u_i) \cdot \otimes \cdots \otimes u_{i-1} \otimes u_{u+1} \otimes \cdots \otimes u_p \quad ,$$

leads to a $U(\mathfrak{k})$-injective resolution of $\mathbf{F}$ (where $\mathfrak{k}$ acts both on U and on $\mathbf{F}$):

$$(6.14) \qquad \begin{aligned} 0 \to \mathbf{F} \to \mathrm{Hom}(U(\mathfrak{k});\mathbf{F}) \to \mathrm{Hom}(U(\mathfrak{k}) \otimes U(\mathfrak{k});\mathbf{F}) \to \cdots \to \\ \mathrm{Hom}(U(\mathfrak{k})^{\otimes p};\mathbf{F}) \to \cdots \end{aligned}$$

Taking invariants for the $\mathfrak{k}$-action in (6.14) produces a resolution of $H^*(\mathfrak{k};\mathbf{F})$. Note that $\mathrm{Hom}(U(\mathfrak{k})^{\otimes p};\mathbf{F})$ is $\mathfrak{k}$-isomorphic to $\Gamma(\hat{K}^{\times p};\mathscr{F})$, the pairing $\Gamma(\hat{K}^{\times p};\mathscr{F}) \otimes U(\mathfrak{k})^{\otimes p} \to \mathbf{F}$ being right differentiation, followed by evaluation at the identity. Under this isomorphism, (6.14) is the chain complex associated to the co-simplicial complex of global sections of $\mathscr{F}$ over $B\hat{K}_\bullet$. □

5. Official proof of Proposition 6.1

The statement is purely cohomological, and the homotopy theory that was needed in the previous approach can be concealed by the use of bi-simplicial spaces. Let $K = G[\Sigma]$, recall that its action of X has a canonical lifting to $\mathscr{F} := \mathscr{L}^{\otimes h} \otimes \mathscr{V}$, and call $\mathscr{F}_\bullet$ the associated bundle on the simplicial homotopy quotient $X_\bullet$. We already noted the Leray spectral sequence (1.9), with

$$E_2^{p,q} = H^p(BK_\bullet; H^q(X; \mathscr{F}_\bullet)) \Rightarrow H^*(X_\bullet; \mathscr{F}_\bullet) \ . \tag{6.15}$$

The E_2 term is, by our definition, $H_K^p(H^q(X; \mathscr{F}))$. (When K is a group acting on a variety, one can show that $H^q(X; \mathscr{F})$ is locally K-finite, in agreement with the theory of Subsect. 1; cf. Prop. 8.10). Let $\mathbf{F} = \Gamma(X; \mathscr{F})$, and recall that $H^q(X; \mathscr{F})$ vanishes for $q > 0$, by the theorem of Kumar and Mathieu; we get, from the collapse of (6.15),

$$H^*(X_\bullet; \mathscr{F}_\bullet) \cong H_K^*(\mathbf{F}) \ . \tag{6.16}$$

For the analogous simplicial homotopy quotient $\hat{X}_\bullet := (X \times \mathfrak{K}_\bullet)/K$, the collapse of the spectral sequence for $\hat{X}_\bullet \to B\hat{K}_\bullet$ gives

$$H^p(\hat{X}_\bullet; \hat{\mathscr{F}}_\bullet) \cong H^p(B\hat{K}_\bullet; \Gamma(X; \mathscr{F})) \cong H^p(\mathfrak{k}; \mathbf{F}) \ . \tag{6.17}$$

Let now $\mathfrak{X}_{\bullet\bullet}$ be the *bi-simplicial* space $(X \times EK_\bullet \times \mathfrak{K}_\bullet)/K$. On one hand, $\mathfrak{X}_{\bullet\bullet}$ is fibered (in simplicial spaces) over $\hat{X}_\bullet$, with the cohomologically trivial fiber $EK_\bullet$. If $\mathscr{F}_{\bullet\bullet}$ denotes the pull-back of $\hat{\mathscr{F}}_\bullet$ to $\mathfrak{X}_{\bullet\bullet}$, this implies that $H^*(\mathfrak{X}_{\bullet\bullet}; \mathscr{F}_{\bullet\bullet}) \cong H^*(\mathfrak{k}; \mathbf{F})$. On the other hand, $\mathfrak{X}_{\bullet\bullet}$ is fibered in simplicial spaces over $X_\bullet$, with fiber $\mathfrak{K}_\bullet$. The corresponding spectral sequence is

$$E_2^{p,q} = H^p(X_K; H^q(\mathfrak{K}_\bullet; \mathscr{F}_{\bullet\bullet})) \Rightarrow H^*(\mathfrak{X}_{\bullet\bullet}; \mathscr{F}_{\bullet\bullet}) \cong H^*(\mathfrak{k}; \mathbf{F}) \ . \tag{6.18}$$

Because $\hat{\mathscr{F}}_\bullet$ is obtained by restriction of $\mathscr{F}_\bullet, \mathscr{F}_{\bullet\bullet}$ is isomorphic to the pull-back of $\mathscr{F}_\bullet$ from this second projection, so it restricts to a trivial bundle over $\mathfrak{K}_\bullet$. Thus, $H^q(\mathfrak{K}_\bullet; \mathscr{F}_{\bullet\bullet}) \cong H^q(\mathfrak{K}_\bullet; \mathscr{O}) \otimes \mathscr{F}_\bullet$. The isomorphism $H^q(\mathfrak{K}_\bullet; \mathscr{O}) \cong H_s^q(G^\Sigma)$ in Proposition 6.7, combined with (6.14) and (6.16) gives

$$H^p(BK_\bullet; H_s^q(G[\Sigma]) \otimes \mathbf{F}) \Rightarrow H^*(\mathfrak{g}[\Sigma] \ ; \mathbf{F}),$$

which is the desired spectral sequence, at least for $\mathbf{F} = \mathbf{H}_{0,h} \otimes \mathbf{V}$. To get other $\mathbf{H}$'s, one twists $\mathscr{F}$ with appropriate vector bundles on X. $\square$

VII. Collapse of the spectral sequence and its consequences

1. The edge-homomorphism

The collapse at E_2 of the spectral sequence will follow by considering products. The E_2 term is a module over $H^*_{G[\Sigma]}(\mathbb{C}) \otimes H^*_s(G^\Sigma)$, and the higher differentials are determined by their values on $E_2^{0,q} \cong H^q(G^\Sigma)$ in this second sequence. To allay any concerns, note that our knowledge of Lie algebra cohomology (Sect. I) already implies that all the $H^p_{G[\Sigma]}$ are finite-dimensional. The following lemma requires no smoothness assumption on Σ.

Lemma (7.1) *The vertical edge homomorphism* $H^*(\mathfrak{g}[\Sigma];\mathbb{C}) \to H^*_s(G^\Sigma)$ *is surjective.*

Proof. The homomorphism can be obtained by regarding a Koszul representative of a class in $H^*(\mathfrak{g}[\Sigma];\mathbb{C})$ as a left-invariant differential form on G^Σ. (The unceremonious switch, from the Alexander cocycles of App. A, to de Rham forms, is justified by the left G^Σ-equivariance of the double complex of (A.6) which realizes the quasi-isomorphism between the Alexander and de Rham complexes). Choose a loop on Σ; restriction to this loop gives the following commutative diagram (where LG stands for the smooth loop group).

$$\begin{array}{ccc} H^*(L_{\mathfrak{g}};\mathbb{C}) & \to & H^*_s(LG) \\ \downarrow & & \downarrow \\ H^*(\mathfrak{g}[\Sigma];\mathbb{C}) & \to & H^*_s(G^\Sigma) \end{array} \tag{7.2}$$

The horizontal maps are the edge homomorphisms for the van Est spectral sequence for LG (see [PS], Sect. 14.6); the vertical maps are induced by restriction. Now, the top arrow is an isomorphism (*loc. cit.*), and $H^*_s(G^\Sigma)$ is generated, as a ring, by the images of the right arrow. Surjectivity of the bottom arrow follows. □

2. Products in Lie algebra and group cohomology

When Σ is smooth, Lemma 7.1 and our formula for Lie algebra cohomology imply that the (shifted) edge homomorphism $H^{*+\ell}(\mathfrak{g}[\Sigma];\mathbf{H}\otimes\mathbf{V}) \to H^\ell_{G[\Sigma]}(\mathbf{H}\otimes\mathbf{V}) \otimes H^*_s(G^\Sigma)$ is bijective, whence the vanishing of group cohomology in degrees other than ℓ. Further, the van Est filtration on $H^*(\mathfrak{g}[\Sigma];\_\,)$ being trivial, the product structure is determined by that on E_2. We conclude the result stated in [T2], Proposition 2.8.

Proposition (7.3) $H^*(\mathfrak{g}[\Sigma];\mathbb{C}) \to H^*(G^\Sigma)$ *is a ring isomorphism, and* $H^*(\mathfrak{g}[\Sigma];\mathbf{H}\otimes\mathbf{V})$ *is freely generated by* $H^\ell(\mathfrak{g}[\Sigma];\mathbf{H}\otimes\mathbf{V})$ *under the cup-product action of* $H^*(\mathfrak{g}[\Sigma];\mathbb{C})$. □

When Σ degenerates to a nodal curve $\Sigma_0, H^*(G^\Sigma)$ loses one factor of $H^*(\Omega G)$ for each node. By [T2], the dimension of $H^*(\mathfrak{g}[\Sigma];.)$ remains unchanged, so the collapse of (7.1) at E_2 implies an (abstract) vector space isomorphism $H^{*+\ell}_{G[\Sigma_0]}(\mathbb{C}) \cong H^{\ell}_{G[\Sigma_0]} \otimes H^*(\Omega G^{\times N})$; but the ring structure is more delicate. The natural homomorphism $H^*_{G[\Sigma_0]}(\mathbb{C}) \to H^*(\mathfrak{g}[\Sigma_0];\mathbb{C})$ makes the second ring into a free module over the first, and our knowledge, to this point, is expressed by the existence of a canonical ring homomorphism

$$H^*(\mathfrak{g}[\Sigma_0];\mathbb{C}) \otimes_{H^*_{G[\Sigma_0]}} \mathbb{C} \cong H^*_s(G[\Sigma_0]) \tag{7.4}$$

Luckily, the right-hand side is a free graded algebra, so there exists a (non-canonical) lifting $H^*_s(G[\Sigma_0]) \to H^*(\mathfrak{g}[\Sigma_0];\mathbb{C})$. To describe the ring structure on $H^*_{G[\times_0]}(\mathbb{C})$, denote by "$\times$" the union of two formal disks crossing at the origin, and by $G[\times]$ the corresponding group of G-valued maps. An inclusion $G[\Sigma_0] \subset \Pi_{\text{nodes}} G[\times]$ is obtained by embedding a copy of "$\times$" at each node.

Lemma (7.5) (a) $H^*_{G[\times]}(\mathbb{C})$ *is a nilpotent ring.*

(b) *The inclusion induces an isomorphism in group cohomology with* $\mathbb{C}$ *coefficients.*

Remarks (7.6) (i) If we filter $\mathfrak{g}[z, z^{-1}]$ by the absolute value of the z-degree, the ring (a) is isomorphic to $\mathrm{Gr}(H^*(\Omega G))$, for the induced filtration on $H^*(\Omega G) \cong H^*(L\mathfrak{g}, \mathfrak{g}; \mathbb{C})$; but I do not know a geometric meaning for it, or a good way to compute it. (For SL_2, all products are zero; but in SL_3, there is a non-zero product $H^2 \wedge H^2 \to H^4$.)

(ii) The cohomology isomorphism illustrates a certain independence of geographically distinct regions of the surface (the same independence appears in the effect, on cohomology, of twisting by **V**). This fits in with G. Segal's cutting and sewing philosophy [S] in Conformal Field Theory; at level zero, the cup-product seems to be a cohomological version of the "fusion product".

Observe that the lemma and the discussion imply Theorem 2′ (and Proposition 2.8′ of [T2]).

Propostion (7.3′) $H^*(\mathfrak{g}[\Sigma];\mathbb{C}) \cong H^*_{G[\Sigma_o]}(\mathbb{C}) \otimes H^*(G^{\Sigma_0})$ (*non-canonically*), *and* $H^*_{G[\Sigma_0]}(\mathbb{C})$ *is a nilpotent algebra.* □

3. Proof of Lemma 7.5

Recall the Shapiro spectral sequence (7.6) (the Leray spectral sequence for the fiber bundle $K/L \subset BL \twoheadrightarrow BK$). When the quotient K/L is affine, so is the morphism $BL \twoheadrightarrow BK$, and the sequence collapses at E_2, absent any higher $\mathbb{R}^q\mathrm{Ind}$. Such is the case for $K = G[[z]] \times G[[z]]$ and $L = G[\times]$, with $K/L \cong G$, giving

$$H^*_{G[\times]}(\mathbb{C}) \cong H^*_K(\mathrm{Ind}_L^K(\mathbb{C})) = H^*_{G[[z]]\times G[[z]]}(\oplus_{\mathsf{U}}\mathsf{U}\otimes\mathsf{U}^t) = \bigoplus\nolimits_{\mathsf{U}} H^*_{G[[z]]}(\mathsf{U})\otimes H^*_{G[[z]]}(\mathsf{U}^t)\ . \tag{7.7}$$

The sum goes over the irreps of G, and K acts, by the evaluations at 0, on the two factors. The ring structure on the right-hand term comes from the Peter-Weyl isomorphism of $\bigoplus_{\mathsf{U}}\mathsf{U}\otimes\mathsf{U}^t$ with the algebra of functions of G. If we let $G_\bullet[[z]]$ be the based group of formal disks passing through 1 in G, there is the isomorphism

$$H^q_{G[[z]]}(\mathsf{U}) \cong (H^q_{G_\bullet[[z]]}(\mathbb{C})\otimes\mathsf{U})^G \tag{7.8}$$

either from (6.6), or from the Hochschild-Serre spectral sequence for the pair $(G_\bullet[[z]], G[[z]])$. Using this and van Est for $G_\bullet[[z]]$ gives the isomorphisms

$$H^q_{G[[z]]}(\mathsf{U}) \cong H^q(\mathfrak{g}[z],\mathfrak{g};\mathsf{U}) = \begin{cases} 0, & \text{if } \mathsf{U} \text{ is singular or } q\neq \ell(U); \\ \mathbb{C}, & \text{if } \mathsf{U} \text{ is regular and } q=\ell(\mathsf{U})\ . \end{cases} \tag{7.9}$$

"Regular" and "singular" refer to level zero. We can be casual about polynomials versus power series, because of the $\mathbb{C}^\times$ rotation action, with finite-dimensional eigenspaces.

Remark (*7.10*) The first isomorphism in (7.9) also follows from a relative van Est spectral sequence for the pair $(G[[z]], G)$, with topologically trivial quotient $G_\bullet[[z]]$.

Proof of 7.5(a) To see nilpotency of $H^*_{G[\times]}(\mathbb{C})$, it suffices to check the following.

Claim (7.11) *The cup-product map* $H^{\ell(\mathsf{V})}(\mathfrak{g}[z],\mathfrak{g};\mathsf{V})\otimes H^{\ell(\mathsf{W})}(\mathfrak{g}[z],\mathfrak{g};\mathsf{W}) \to H^{\ell(\mathsf{V})+\ell(\mathsf{W})}(\mathfrak{g}[z],\mathfrak{g};\mathsf{V}\otimes\mathsf{W})$ *vanishes, for all but finitely many pairs* (V,W), *when G is simple.*

Proof of (*7.11*) There is another quantity additive under products beside the degree, namely the *energy* (best defined on the Koszul complex). Now, while the length is roughly linear in the highest weight, the energy is quadratic: it equals the eigenvalue of the Casimir, divided by c. This leads to a mismatch when one of the degrees is large, forcing the product to vanish. (Note that, for simple G, the dot product of two dominant weights is bounded below by a constant multiple of the product of their norms). □

Proof of 7.5(*b*) Call $\tilde{\Sigma}_0$ the normalization of the nodal curve Σ_0, and note that, for the pair $K = G[\tilde{\Sigma}_0]$, $L = G[\Sigma_0], \mathrm{Ind}_L^K(\mathbb{C})$ is a tensor product of Peter-Weyl sums, with one tensor factor for each node. (Again there is no higher $\mathbb{R}^q$ Ind, because the quotient of the groups is a product of G's.) More precisely, the new induction-restriction isomorphism

$$(7.12) \qquad H_L^*(\mathbb{C}) \cong H_K^*(\mathrm{Ind}_L^K(\mathbb{C}))$$

is compatible with (7.7), via embedding the "$\times$" at the nodes of Σ_0. The proof is completed by verifying the following, final claim. Let Δ be a union of formal disks Δ_i at chosen points $x_i \in \tilde{\Sigma}_0, \mathbf{U} := \bigotimes_i \mathsf{U}_i(x_i)$ an evaluation representation of $G[\tilde{\Sigma}_0]$ (and $G[\Delta]$).

Claim (7.13) *The inclusion* $G[\tilde{\Sigma}_0] \subset G[\Delta]$ *induces an isomorphism* $H^*_{G[\Delta]}(\mathbf{U}) \tilde{\to} H^*_{G[\Sigma]}(\mathbf{U})$.

Proof. This is obvious when $\mathbf{U}$ is the trivial representation: both sides are $\mathbb{C}$, and confined to degree 0. Also, both cohomologies vanish when $\mathbf{U}$ is singular (at level zero). To extend it to all regular representations, recall from [T1], Sect. 3.5, that $H^*(\mathfrak{g}[z], \mathfrak{g}; \mathsf{U})$ can be determined inductively, over a minimal sequence of simple affine Weyl reflections relating the highest weight of U to 0. The inductive step is the "Bott reflection" from the original proof of BWB; it applies equally to our group cohomology setting, one factor of $\mathbf{U}$ at a time, and it is compatible with the inclusion $G[\Sigma] \subset G[\Delta]$. The joy of retracing all the steps is left to the reader. □

4. Application: computation of some Lie algebra cohomologies

Theorem 2 allows one to easily calculate of some of the Lie algebra cohomology groups that required spectral sequence calculations in [T2]. Let $\mathfrak{l} \subset \mathfrak{g}[\Sigma]$ be a subalgebra defined by evaluation conditions of order zero (vanishing or equality of the functions at selected points of Σ). Calling L the corresponding subgroup of G^Σ, the van Est spectral sequence implies that

$$(7.14) \qquad H^*(\mathfrak{l}; \mathbf{H} \otimes \mathbf{V}) \cong H_L^*(\mathbf{H} \otimes \mathbf{V}) \otimes H_s^*(L) \ ,$$

its collapse at E_2 following just as the first subsection. The fibration $L \hookrightarrow G^\Sigma \twoheadrightarrow B$, homotopy equivalent to the fibration of the topological mapping spaces (cf. Sect. III.2), determines the homotopy type of L as a product of ΩG's (and possibly a copy of G). The quotient B is a product of copies of G, and we get $H_L^*(\mathbf{H} \otimes \mathbf{V}) \cong H^*_{G[\Sigma]}(\mathbf{H} \otimes \mathbf{V} \otimes \ \mathrm{Ind}_L^{G[\Sigma]}(\mathbb{C}))$, from Shapiro's spectral sequence. This can be determined using Theorem 2, because $\mathrm{Ind}_L^{G[\Sigma]}(\mathbb{C})$ is a product of evaluation representations of G^Σ. Variations are possible, for subalgebras defined by parabolic evaluation conditions; finding the homotopy type of L becomes a brief and amusing exercise.

Remark (*7.15*) When L is defined by higher-order evaluation conditions, the induced representation is not of the type studied here, and we can, at best, set up a spectral sequence for group cohomology. There is a suggestion, due to B. Feigin, who has checked special cases, that Lie algebra cohomology is rigid under the coalescence of marked points into higher-order vanishing conditions; but the general case does not seem to be known.

Taking this approach to the calculations of [T2] might seem circular, since the Lie algebra cohomology (1.1) was used in the proof of Theorem 2. However, we shall now see how (1.1) reduces to the genus 0 result of [T1], bypassing the spectral sequence computations in [T2]. For simplicity, consider the case of a single puncture, and remove a factor $H^*(\mathfrak{g};\mathbb{C})$ from the cohomologies by switching to $H^*(\mathfrak{g}^\Sigma,\mathfrak{g};\mathbf{H}\otimes\mathbf{V})$, cohomology relative to the constants $\mathfrak{g}\subset\mathfrak{g}^\Sigma$. As in [T2], we deform Σ to an affine line Σ_0 with g simple nodes, and show that the "special" $(\mathfrak{g}[\Sigma_0],\mathfrak{g})$- cohomology vanishes in all odd (or in all even) degrees. This implies the rigidity of cohomology under specialization, and thus also computes $H^*\left(\mathfrak{g}^\Sigma,\mathfrak{g};\mathbf{H}\otimes\mathbf{V}\right)$. One can use group cohomology, as above, but let us proceed more geometrically, using the stack $\mathfrak{M}$.

The fibration of moduli stacks corresponding to the compactifications of Σ_0 and $\tilde{\Sigma}_0$

$$G^{\times g}\hookrightarrow\mathfrak{M}_0\twoheadrightarrow\tilde{\mathfrak{M}}_0 \tag{7.16}$$

is not a principal bundle; rather, each fiber factor G, corresponding to a node x_k with inverse images x_k' and x_k'' in $\tilde{\Sigma}_0$, is the diagonal quotient of $\mathscr{G}(x_k')\times\mathscr{G}(x_k'')$ by G. (7.16) is, however, an affine fiber bundle, and the Leray spectral sequence collapses at E_2, giving

$$H^*\left(\mathfrak{M}_0;\mathscr{L}^{\otimes h}\otimes\mathscr{V}\right)\cong H^*\left(\tilde{\mathfrak{M}}_0;\mathscr{L}^{\otimes h}\otimes\mathscr{V}\otimes\mathscr{U}\right)\ , \tag{7.17}$$

where $\mathscr{U}$ is the sheaf of fiberwise functions. Now, Lie algebra cohomology (relative to $\mathfrak{g}$) incorporates sheaf cohomology over the $\mathfrak{M}$'s and singular cohomology of $G[\Sigma_0]/G$. The fiber sequence $G[\Sigma_0]/G\to G[\tilde{\Sigma}_0]/G\twoheadrightarrow G^g$ is homotopy equivalent to the path fibration of $G^{\times g}$. All higher differentials in the Hochschild-Serre sequences of [T2] come from the Leray spectral sequence of this fibration. More precisely, $G[\tilde{\Sigma}_0]/G$ is contractible, so van Est and Theorem 3 give an isomorphism

$$H^*\left(\tilde{\mathfrak{M}}_0;\mathscr{L}^{\otimes h}\otimes\mathscr{V}\right)\cong H^*\left(\mathfrak{g}[\tilde{\Sigma}_0],\mathfrak{g};\mathbf{H}\otimes\mathbf{V}\right)\ , \tag{7.18}$$

which, by the main result of [T1], lives only in degree $\ell(\mathbf{V})$. Given (7.17), we see that

$$H^*\left(\mathfrak{M}_0;\mathscr{L}^{\otimes h}\otimes\mathscr{V}\right)\cong H^\ell\left(\tilde{\mathfrak{M}}_0;\mathscr{L}^{\otimes h}\otimes\mathscr{V}\right)H^{*-\ell}(\Omega G^{\times g})\ , \tag{7.19}$$

just as in [T2], Proposition 3.8, using (7.18) and Bott's Morse theory description of $H_*(\Omega G)$ ($\mathscr{U}$ splits into g tensor factors, each of which decomposes as a sum indexed by the Weyl alcoves in the positive chamber). But the spectral sequence

$$(7.20) \qquad H^p\left(\mathfrak{M}_0; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\right) \otimes H^q(G[\Sigma_0]/G) \Rightarrow H^*(\mathfrak{g}[\Sigma_0], \mathfrak{g}; \mathbf{H} \otimes \mathbf{V})$$

collapses at E_2, because, combining (7.18) and (7.19), we know that the left-hand side is

$$(7.21) \qquad H^\ell\left(\tilde{\mathfrak{M}}_0; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\right) \otimes H^{p-\ell}(\Omega G^{\times g}) \otimes H^q(\Omega G^{\times g}) \ ,$$

confined to dimensions having the parity of ℓ. Thus, (7.21), is also the abutment of (7.20).

VIII. Proof of Theorem 4

Recall that $\hat{L}G$ is the product of the $\hat{L}_iG$, and $\hat{L}^+G$ its subgroup of formal-holomorphic loops. The generalized flag variety $X_\Sigma := \hat{L}G/G^\Sigma$ is an étale-trivial, principal $\hat{L}^+G$-bundle over $\mathfrak{M}$, because the latter is equivalent to the double coset stack $\hat{L}^+G \setminus \hat{L}G/G^\Sigma$, by the uniformization theorem. Further, the pull-backs to X_Σ of the bundles $\mathscr{L}^{\otimes h}$ and $\mathscr{V}$ over $\mathfrak{M}$ agree with their homonyms in (0.5), since $\mathscr{L}$ descends from the basic line bundle over $\hat{L}G$, and $\mathscr{V}$ is, in both cases, associated to the evaluation representation $\mathbf{V}$ of G^Σ.

Choose G-irreps $\mathsf{U}_1, \ldots, \mathsf{U}_n$ (of length zero) and an HWR $\check{\mathbf{H}}$. Using the notations of Sect. I, note the following, from the well-known Lie algebra result and the van Est spectral sequence:

$$(8.1) \qquad H^*_{\hat{L}^+G}\left(\mathbf{U}^t \ \otimes \ \check{\mathbf{H}}\right) = \begin{cases} \mathbb{C}, & \text{if } \check{\mathbf{H}} = \check{\mathbf{H}}_{\mathrm{U}} \quad \text{and} \quad * = 0; \\ 0, & \text{otherwise} \ . \end{cases}$$

From here, we consider the descent spectral sequence for $X_\Sigma \to \mathfrak{M}$, with

$$(8.2) \quad E_2^{p,q} = H^p_{\hat{L}^+G}\left(\mathbf{U}^t \otimes H^q\left(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\right)\right) \Rightarrow H^*\left(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{V} \otimes \mathscr{U}^t\right) \ ,$$

where $\mathscr{U}$ is the evaluation bundle over $\mathfrak{M}$ obtained by attaching the U_k at the points at infinity. It will follow that the spectral sequence collapses at $E_2 = E_2^{0,q}$, which picks out the multiplicity space of $\check{\mathbf{H}}_{\mathrm{U}}$ in $H^q\left(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\right)$, once we prove the following claim. $\square$

Proposition (8.3) *All spaces $H^q\left(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\right)$ are sums of HWRs of $\hat{L}G$.*

Proof. We shall show, in Lemmata 8.6–8.8, that each H^q is a locally-finite representation for $\hat{L}^+G$; and the statement will then follow from the next observation.

Lemma (8.4) *The following two conditions on a $\mathfrak{g}((z))$-representation H are equivalent*:

(a) H *is locally $\mathfrak{g}[[z]]$-finite and has a countable vector space basis.*

(b) H *admits an increasing filtration, whose graded components are G-integrable generalized Verma modules, or quotients thereof.*
If, moreover, H *arises from a $G((z))$-representation, then it is actually a sum of HWRs.*

Remarks (8.5) (i) The conditions do not quite imply that H belongs to the Beilinson-Bernstein category $\mathcal{O}$; stronger finiteness assumptions are needed.

(ii) We say that H "arises from a $G((z))$-representation" if the sheaf it represents (over $\mathfrak{Sch}$) carries an action of the group sheaf $G((z))$, whose infinitesimal action is that of $\mathfrak{g}((z))$. However, we can restate this in human language: (a) implies that the action of $\mathfrak{g}[[z]]$ can be exponentiated to $G[[z]]$. As this subgroup and its conjugates generate $G((z))$, we can simply require that the exponentiated action and its conjugates combine to an action of $G((z))$.

Proof. That (b) $\Rightarrow$ (a) is obvious. To see (a) $\Rightarrow$ (b), choose a highest-weight subspace, from the generated $\mathfrak{g}((z))$-representation, note that the quotient will also have property (a), and keep going. (The "countable basis" assumption avoids the need for transfinite induction; otherwise, one should consider filtrations be indexed by arbitrary well-ordered sets.) For the last statement, recall that HWRs are the only quotients of Verma modules that carry the group action, and that all extensions of group-HWRs are split. $\square$

Lemma (8.6) *$H^q(X_\Sigma; \mathscr{L}^{\otimes h} \otimes \mathscr{V})$ is naturally isomorphic to $H^q(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{F} \otimes \mathscr{V})$, where $\mathscr{F}$ is the vector bundle over $\mathfrak{M}$ associated to the left regular representation* $\mathbf{F}$ *of $\hat{L}^+G$. The right $\hat{L}^+G$-action on* $\mathbf{F}$ *corresponds to the natural action of $\hat{L}^+G \subset \hat{L}G$ on cohomology.*

Proof. Clear from the fact that X_Σ is a principal bundle over $\mathfrak{M}$, with structure group the affine group scheme $\hat{L}^+G$. $\square$

Lemma (8.7) (a) *The cohomology of an increasingly filtered vector bundle over a finitely presented* 1*-stack is filtered, in every degree.*

(b) *An algebraic group action on the setting* (a) *leads to a locally-finite action on cohomology.*

Proof. (a) Such a stack is equivalent to a simplicial scheme, affine and of finite type in each dimension. The space of sections of a filtered bundle over such a scheme is filtered; so is, then, each component of the total cohomology of the bundle. To see (b), call K the group and $\mathfrak{S}$ the stack; arguing as in (a) shows that all higher direct image sheaves of the vector bundle, along the projection from the simplicial homotopy quotient $\mathfrak{S}/K$ to BK, are vector bundles over BK (possibly infinite-dimensional); they must then come from locally-finite K-representations. $\square$

Lemma (8.8) *$H^q\big(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{F} \otimes \mathscr{V}\big)$ is isomorphic to the cohomology of the same bundle over a finitely presented, open substack of $\mathfrak{M}$.*

Proof. It is known that $\mathfrak{M}$ has a smooth stratification by locally closed substacks, of finite type and increasing codimension. Removing the strata of codimension higher than $q + 1$ does not affect coherent sheaf cohomology in dimensions up to q. $\square$

End of Proof of Proposition 8.3. The bundle $\mathscr{F}$ is filtered by subbundles $\mathscr{F}_N$, on which the $\hat{L}^+G$-action factors through the algebraic group of G-valued germs to order N. (They correspond to the $\mathbf{F}$-subspaces of functions on $\hat{L}^+G$ that are pulled back from these groups of N-germs). By Lemmata 8.7 and 8.8, $H^q\big(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{F} \otimes \mathscr{V}\big)$ is filtered by locally-finite $\hat{L}^+G$-subrepresentations, and is thus locally $\hat{L}^+G$-finite. $\square$

Remark (*8.9*) The "finite type" restriction in Lemma 8.7 explains, in a way, the failure of the Peter-Weyl theorem for $\hat{L}G$. It is true that the space of sections of $\mathscr{L}^{\otimes h}$ over $\hat{L}G$ is the space of sections of the bundle $\mathscr{F} \otimes \mathscr{L}^{\otimes h}$ over the original flag variety X; however, this last space will be an *inverse limit* of locally finite $\hat{L}^+G$-representations, and this need not be locally $\hat{L}^+G$-finite.

Remark (*8.10*) The theorem applies to a larger class of vector bundles over X_Σ, such as the exterior powers of the cotangent bundle. It fails for the *tangent* bundle, whose space of regular sections, when Σ is the affine line, contains the Lie algebra $\mathfrak{g}((z))$.

Remark (*8.11*) By the same argument, the cohomologies of certain $\hat{L}^+G$-integrable $\mathscr{D}$-modules over X_Σ will satisfy condition (b) in Proposition 8.3. Such $\mathscr{D}$-modules are pulled back from $\mathfrak{M}$; and a sufficient condition for the proof to work is that, in any given degree, local cohomology supported on the far-away Atiyah-Bott strata should vanish. For adequate generality, we should consider $\mathscr{D}$-modules on stacks of bundles with additional structure, allowing gauge-fixing to finite order at prescribed points of Σ^c; for instance, parabolic bundles. They pull back to $\hat{L}^+G$-integrable $\mathscr{D}$-modules over "flag varieties" which are fiber bundles over X_Σ.

IX. Closing remarks

The use of Lie algebra cohomology seems unavoidable in the present determination of the sheaf cohomology $H^*\big(\mathfrak{M}; \mathscr{L}^{\otimes h} \otimes \mathscr{V}\big)$, because the latter is not rigid under the nodal degeneration of Σ, and there is no obvious way to control its jump. That is, the degeneration $\mathfrak{M} \to \mathfrak{M}_0$ is not "flat". This can already be suspected from the fibration (7.16): $\mathfrak{M}_0$ is "less compact" than $\mathfrak{M}$, because of the affine fibers G. This observation requires a large grain of salt, or else the fibration sequence $G \to * \twoheadrightarrow BG$ might suggest that the point is not compact. However, inspection of singular cohomology, for which the spectral sequence for (7.16) collapses rationally, and comparison with Proposition (4.5), shows that the fiber $G^{\times g}$ is the one responsible for the superdiagonal part of the Hodge structure of $\mathfrak{M}_0$. This suggests that the non-compactness of the fiber is "genuine", and raises the question of finding a "flat" degeneration of $\mathfrak{M}$, which "compactifies" $\mathfrak{M}_0$, and whose cohomology is computable.

As a final observation, the present proof the Borel-Weil-Bott theorem for $\mathfrak{M}$ is ultimately based on the genus zero Lie algebra theorem of [T1]. That really was Theorem 3 for the genus zero stack, but the paper was written within a different framework, and one suspects there should be a shorter proof of the result[14] (using the language of stacks). One approach is suggested by the result of Kumar and Narasimhan [KN], who prove the vanishing of higher cohomology of $\mathscr{L}^{\otimes h}$ on the GIT moduli *space* of semi-stable G-bundles over a smooth curve (in genus ≥ 2), using a theorem of Grauert and Riemenschneider (a version of Kodaira vanishing). Now, the agreement of the cohomology of positive line bundles over $\mathfrak{M}$, possibly enhanced with parabolic structures, with that over the semi-stable moduli spaces, is predicted by the *quantization conjecture*. In finite dimensions, there are some well-understood theorems on the subject (cf. [GS]). While the stack $\mathfrak{M}$ is not of finite type, the *Shatz stratification* [Sha] decomposes it into smooth, finite type substacks (corresponding to the Atiyah-Bott strata on the space of connections), closely related to semi-stable moduli spaces of bundles for subgroups of G; and these could be studied using traditional GIT methods.

Recall now that the result in [T1] was also based on a variant of the Kodaira vanishing theorem. It may be interesting to find a corresponding theorem for stacks, powerful enough to apply directly to $\mathfrak{M}$.

Appendix A: Alexander cohomology

When applying the van Est spectral sequence for G^{Σ}, we needed to know that the Lie algebra cohomology of its space of algebraic functions computed the cohomology of the underlying topological space. Absent an exhaustion of G^{Σ} by smooth subvarieties, a naïve de Rham argument does not apply[15], for the intrinsic de Rham complex of a singular variety may not compute the correct cohomology. (One does get the right answer by using smooth formal embeddings [Har], but finding an inductive system of such where the sheaf Mittag-Leffler conditions applied seemed non-trivial.) The solution was to use a different resolution for cohomology, known in the topological world as *Alexander-Spanier* cohomology, and to algebraic

[14] With the techniques of [T2], it can be reduced to the case of three marked points on the sphere, which, for SL_2, can be done by hand; but the higher-rank case is not clear

[15] Using the methods of [Sim], this can now be fixed (see [ST]). However, I have kept the original argument, as the proof of the Alexander cohomology theorem (A.5) may have some intrinsic interest

geometers as *stratifying cohomology*. The fact that the latter computes "classical" cohomology with complex coefficients was predicted in [Gro2], Sect. 6. Some experts regard this as known, but no published proof of the conjecture seems to exist, which is why I included the present section.

Recall first the definition of the **Alexander complex** for a paracompact Hausdorff space X, with coefficients in an Abelian group A ([Spa], Ch. 6). Its pth term is the sheaf over X of germs of *all* A-valued functions along the diagonal $X \to X^{\times(p+1)}$. (This is the sheafification of the presheaf $U \mapsto$ functions on U^{p+1}.) The differential is given by the formula

$$\partial_p \varphi(x_0, \ldots, x_p) := \sum_i (-1)^i \varphi(x_0, \ldots, \hat{x}_i, \ldots, x_p) \tag{A.1}$$

for x_i near some given $x \in X$ and an A-valued function φ on a neighborhood of $(x, \ldots, x)$ in $X^{\times p}$ (the hat denotes a missing argument). The complex gives a flabby resolution of the constant sheaf A, so the cohomology of its global sections, called **Alexander-Spanier cohomology**, agrees with the sheaf cohomology of A over X. When A is contractible, one obtains the same cohomology by considering the Alexander complex of *continuous* A-valued functions, since the latter gives a soft resolution of the constant sheaf. Finally, when X is an analytic space, one can resolve the constant sheaf $\mathbb{C}$ by using germs holomorphic functions, and the total cohomology ("hypercohomology") of this complex is the cohomology of X.

The algebraic version of Alexander cohomology runs as follows. Given a scheme X, of finite type over $\mathbb{C}$, let $\mathscr{C}^p$ be the formal completion of the structure sheaf of $X^{\times(p+1)}$ about the diagonal $X \to X^{\times(p+1)}$. All the $\mathscr{C}^p$ are sheaves over X, in the Zariski topology. By virtue of the following lemma, they form a complex – the **formal Alexander complex of** X – under a differential $\partial_p : \mathscr{C}^{p-1} \to \mathscr{C}^p$ given by the same formula (A.1).

Lemma (A.2) *The map ∂ of* (A.1) *extends by continuity to the formal completions $\mathscr{C}^*$, and $\partial^2 = 0$.*

Proof. An ideal of definition $\mathscr{D}_{p+1}$ for the diagonal $X \to X^{\times(p+1)}$ is the sum of the pull-backs, via all the projections of $X^{\times(p+1)}$ to pairs of factors, of $\mathscr{D}_2$, the ideal sheaf of the diagonal in $X \times X$. But then, clearly, $\partial_p\left(\mathscr{D}_p^n\right) \subseteq \left(\mathscr{D}_{p+1}\right)^n$ for all n. The vanishing of ∂^2 follows by continuity. □

Definition (A.3) The **Alexander cohomology** $AH^*(X)$ *is the hypercohomology* $\mathbb{H}^*(X; \mathscr{C}^*)$.

Remark (*A.4*) For a separated scheme X, let $\mathfrak{X}_p$ be the formal completion of $X^{\times(p+1)}$ along the diagonal X. The $\mathfrak{X}_\bullet$ form a simplicial formal scheme, and $AH^*(X)$ is the cohomology of its structure sheaf. The stack represented by $\mathfrak{X}_\bullet$ is the **Alexander stack** of X; it is equivalent (in the sense of App. B) to the quotient sheaf of X by the "equivalence relation" determined by the formal neighborhood of the diagonal in $X \times X$. The étale site of this quotient sheaf is the **stratifying site** of X, as defined in [Gro2], Sect. 5; it agrees with the crystalline site when X is formally smooth, but not otherwise. Nonetheless, we shall see that their cohomologies with $\mathcal{O}$ coefficients agree; thus, $AH^*(X)$ equals $H_s^*(X;\mathbb{C})$, by the crystalline cohomology theorem.

The construction of the Alexander complex also works for a Noetherian formal scheme $\mathfrak{Y}$. When $\mathfrak{Y}$ is the completion of a Noetherian scheme Y along a closed subscheme X (which we can always arrange locally), $\mathscr{C}^p(\mathfrak{Y})$ is the formal completion of the structure sheaf of $Y^{\times(p+1)}$ along a diagonally placed copy of X, and $\mathscr{C}^*(\mathfrak{Y})$ is the Alexander complex of the formal neighborhood of X in Y.

Proposition (A.5) *The Alexander cohomology of a scheme (or formal scheme) X, of finite type over $\mathbb{C}$, is naturally isomorphic to $H_s^*(X;\mathbb{C})$. When the X is affine, this is the cohomology of the Alexander complex of global sections $\Gamma(X;\mathscr{C}^*)$.*

Proof. When X is affine, $H^q(X;\mathscr{C}^p) = 0$ for $q > 0$ by (A.18), so the second statement follows from the first, and from the collapse at E_2 of spectral sequence $E_1^{p,q} = H^q(X;\mathscr{C}^p) \Rightarrow H^*(X^{\mathrm{an}};\mathbb{C})$.

Smooth X. We shall construct an *Alexander-de Rham* biocomplex $\mathscr{C}^{p,q}$ $(p,q \geq 0)$ of sheaves on X, which establishes a quasi-isomorphism between the Alexander and de Rham sheaf complexes; and it was shown in [Gro1] that de Rham hypercohomology was canonically isomorphic to $H_s^*(X;\mathbb{C})$, for smooth X. The double complex will have the following properties:

- the cohomology sheaves of the horizontal differential ∂ are confined to the vertical edge, where they give the algebraic de Rham complex $\Omega^q(X)$;
- the cohomology sheaves of the vertical differential d are confined to the horizontal edge, where they give the Alexander complex $\mathscr{C}^p$.

Let $\mathscr{C}^{p,q} := \hat{\mathcal{O}}_{X^{p+2}} \otimes \Lambda^q(T^*X)$, completed with respect to the diagonal $X \to X^{\times(p+2)}$, and let d be the de Rham operator with respect to the last variable. ∂ is defined by the formula (A.1), but ignoring the last variable:

$$\begin{aligned}
&\partial_p : \mathscr{C}^{p-1,q} \to \mathscr{C}^{p,q},\\
&\partial_p\varphi(x_0,\ldots,x_p,x_{p+1}) := \sum_{0\leq i\leq p} (-1)^i \varphi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}).
\end{aligned} \tag{A.6}$$

For $p = 0$, (A.6) also defines a ∂-augmentation $\mathscr{C}^{-1,q}$, isomorphic to $\Omega^q(X)$. A d-augmentation $\varepsilon : \mathscr{C}^{p,-1} \to \mathscr{C}^{p,*}$, isomorphic to $\mathscr{C}^p$, is given by the subsheaf of sections in $\mathscr{C}^{p,0}$ that are independent of the last argument.

A ∂-homotopy $H : \mathscr{C}^{p,q} \to \mathscr{C}^{p-1,q}$, proving acyclicity of the augmented ∂-complex, is given by $H\varphi(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p,x_p)$; we have $\partial \circ H - H \circ \partial = (-1)^p$. Id. To define a d-homotopy, locally on X, note the copy of $\mathfrak{X}_{p+1}$ inside $\mathfrak{X}_{p+2}$, embedded by repeating the last coordinate, and the projection $\pi : \mathfrak{X}_{p+2} \to \mathfrak{X}_{p+1}$ on the first $(p+1)$ coordinates. Given a deformation retraction $\rho : \mathbb{C} \times \mathfrak{X}_{p+2} \to \mathfrak{X}_{p+2}$ of $\mathfrak{X}_{p+2}$ onto $\mathfrak{X}_{p+1}$ (a morphism ρ such that $\rho|_0 = \pi$ and $\rho|_1 = \mathrm{Id}$), let $G : \mathscr{C}^{p,q} \to \mathscr{C}^{p,q-1}$ send φ to $\rho_*\rho^*(\varphi)$, where ρ_* is integration along any path in $\mathbb{C}$ from 0 to 1. (This is really an algebraic operation.) The augmentation ε is seen to be a cohomology isomorphism from the familiar homotopy formula $d \circ G - G \circ d = Id - \varepsilon$. When X is formally smooth, such deformations ρ always exist locally, over sets $U \subset X$ small enough to admit étale maps to affine space (one pulls back the obvious scaling homotopy).

General X. It suffices, by Čech resolution arguments, to prove (A.5) locally on X. Thus, we may embed X in some smooth variety Y, and denote by $\mathfrak{Y}$ the formal neighborhood of X. By [Har], the hypercohomology of the de Rham complex of $\mathfrak{Y}$ (the formal completion along X of the de Rham complex on Y) computes $H^*_s(X;\mathbb{C})$. But the completion along X of the double complex of Y establishes, as above, a quasi-isomorphism between de Rham and Alexander complexes of $\mathfrak{Y}$. The proof of (A.5) is completed by verifying the following "tautness" property:

Lemma (A.7) *$AH^*(\mathfrak{Y}) \cong AH^*(X)$, under the embedding $i : X \to \mathfrak{Y}$.*

Remark (A.8) This is an analogue of the following topological fact: if Y is paracompact and X is closed in Y, the restriction $\lim_{\to U} H^*(U) \to H^*(X)$ is an isomorphism in Alexander cohomology (U ranges over the neighborhoods of X in Y).

Proof. We first check (A.7) in the presence of a retraction $r : \mathfrak{Y} \to X$. Then, on the Alexander complex of X, $i^* \circ r^* =$ Id, whereas a homotopy between $r^* \circ i^*$ and Id is defined by a formula, which can be written just as in the topological case (cf. [Spa], Ch. 6, Sect. 6),

$$H\varphi(y_0, \ldots, y_{p-1}) = \sum_{o \le j \le p-1} (-1)^j \cdot \varphi\big(y_0, \ldots, y_j, i \circ r(y_j), \ldots, i \circ r(y_{p-1})\big) \quad , \tag{A.9}$$

if we pretend, for notational simplicity, that $\mathfrak{Y}$ is a neighborhood of X in Y and that r is a map. In the honest formula, to obtain the jth-summand, $r^* \circ i^*$ is applied to the last $(p-j)$ factors of $\mathfrak{Y}^{\times(p+1)}$, and the result is restricted to the copy of $\mathfrak{Y}^{\times p} \subset \mathfrak{Y}^{\times(p+1)}$ defined by equality of the jth and $(j+1)$st coordinates. Completeness of $\mathfrak{Y}$ along X ensures that the formula extends, by continuity, to the diagonal completions, because H preserves the diagonal, modulo an ideal of definition for $\mathfrak{Y}$.

Absent a retraction, let us place i in a flat, one-parameter family $i(t) : X \to \mathfrak{Y}(t)$, constant except at $t = 0$, where it specializes to the inclusion of X into its normal cone $\mathfrak{C}_X$ within $\mathfrak{Y}$. The latter retracts to X, so $i(0)$

induces an isomorphism in Alexander cohomology. If the condition "$i(t)^* : AH^*(\mathfrak{Y}(t)) \to AH^*(X)$ is an isomorphism" turns out to be open in t, then i^* itself must be an isomorphism. Openness would follow, if we showed that the fiberwise Alexander cohomologies of $\mathfrak{Y}(t)$ over Spec($\mathbb{C}[t]$) formed coherent sheaves. This can indeed be done, but there is an easier way to handle the algebra.

To show that $i^* : \mathscr{C}^*(\mathfrak{Y}) \to \mathscr{C}^*(X)$ is a quasi-isomorphism, call its kernel $\mathfrak{I}_q \subset \mathscr{C}^q(\mathfrak{Y})$, so that $\mathscr{C}^q(\mathfrak{Y})/\mathfrak{I}_q \cong \mathscr{C}^q(X)$. We shall see below that the complex $\mathscr{C}^*(\mathfrak{Y})$ is filtered by the powers of the ideals $\mathfrak{I}_q$, with associated graded complex the Alexander complex of $\mathfrak{C}_X$. Granting this, note that $i(0)^* : \mathscr{C}^*(\mathfrak{C}_X) \to \mathscr{C}^*(X)$ is a quasi-isomorphism, because $\mathfrak{C}_X$ retracts to X. Therefore, in the complex of sections of $\mathscr{C}^*(\mathfrak{Y})$ over an affine open set, the E_1 term of the filtration spectral sequence is confined to the vertical edge. So, the spectral sequence collapses there, and the original i^*, which induces the edge-homomorphism, is an isomorphism. Convergence questions do not arise if we assume that $\mathfrak{I}$ is nilpotent, as the filtration of $\mathscr{C}^*(\mathfrak{X})$ has finite length in every dimension. In the general case, isomorphism of cohomology groups follows by taking limits over the nilpotent neighborhoods of X (cf. Cor. A 20)[16]. □

It remains to verify the claim about the filtration, namely that the normal cone to $\mathfrak{X}_q$ in $\mathfrak{Y}_q$ is the qth diagonal completion of $\mathfrak{C}_X$. Because products of formal schemes are completed *ex oficio*, the algebraic notation is greatly simplified if we assume that $\mathfrak{Y}$ is a nilpotent extension of X; the general case follows, as before, by taking limits.

Locally then, $\mathfrak{Y} =$ Spec(A), $X =$ Spec(A/I), with A Noetherian and $I \subset A$ nilpotent. Let $\mathrm{Gr}^I_\bullet(A) := \bigoplus_p I^p/I^{p+1}$, and write $A^{\hat{\otimes} q}, \mathrm{Gr}^I_\bullet(A)^{\hat{\otimes} q}$ for the diagonally completed tensor powers. The diagonal ideal in $A^{\otimes q}$ is denoted D_q, while $I_q := \Sigma_{k+l=q-1} A^{\otimes k} \otimes I \otimes A^{\otimes l}$ corresponds to the sheaf $\mathfrak{I}_q$ used in the proof.

Proposition (A.10) $\mathrm{Gr}^I_\bullet(A)^{\hat{\otimes} q}$ *and* $\mathrm{Gr}^{\hat{I}_q}_\bullet(A^{\hat{\otimes} q})$ *are naturally isomorphic.*

Remark (*A.11*) This holds over any ground ring R, instead of $\mathbb{C}$, as long as A and $\mathrm{Gr}^I_\bullet(A)$ are flat over it. If I is nilpotent, flatness of $\mathrm{Gr}^I_\bullet(A)$ implies flatness of A, I and all of its powers.

Proof. We shall describe two natural ring isomorphisms

$$\text{(A.12)} \qquad \mathrm{Gr}^{\hat{I}_q}_\bullet\left(A^{\hat{\otimes} q}\right) \to \mathrm{Gr}^{I_q}_\bullet(A^{\otimes q})\hat{} \leftarrow \mathrm{Gr}^I_\bullet(A)^{\hat{\otimes} q}$$

where the middle ring is the completion of $\mathrm{Gr}^{I_q}_\bullet(A^{\otimes q})$ at $\mathrm{Gr}^{I_q}_\bullet(D_q)$.

[16] One can also check the Mittag-Leffler conditions directly in the spectral sequence

First Map: Both rings arise by completing the same, so we must only check the topologies. By the Artin-Rees lemma, the left-hand $A^{\hat{\otimes} q}$-topology on I_q^p/I_q^{p+1} is defined by the powers of D_q; as I_q acts trivially, these are also the powers of $\mathrm{Gr}_0^{I_q}(D_q)$. But, by the Artin-Rees lemma for the graded ring, the latter powers also define the right-hand topology, since $\mathrm{Gr}_+^{I_q}$ acts trivially.

Second Map: A natural isomorphism $\mathrm{Gr}_\bullet^I(A)^{\otimes q} \to \mathrm{Gr}_\bullet^{I_q}(A^{\otimes q})$ is induced by the obvious map

$$\text{(A.13)} \qquad \bigoplus_{p_1+\ldots p_q=p} I^{p_1} \otimes I^{p_2} \otimes \ldots \otimes I^{p_q} \to I_q^p \subset A^{\otimes q} \ .$$

(The flatness assumptions of Remark A.11 are needed here, if do not work over a field.) Because I is nilpotent, the diagonal ideal in $\mathrm{Gr}_\bullet^I(A)^{\otimes q}$, which is the kernel of the multiplication $(A/I)^{\otimes q} \to A/I$, differs only nilpotently from $\mathrm{Gr}^{I_q}(D_q)$, so the completions are isomorphic. □

Remark (*A.14*) The first isomorphism is an instance of the general fact that "normal cone formation commutes with completion" in a Noetherian, Cartesian diagram of closed embeddings. The proof is the same as above.

Finally, we check the following commutation property of sheaf cohomology with inverse limits, used several times in the paper (cf. [Har], Theorem 4.5). Recall that a *generating subcategory* of a Grothendieck site $\mathfrak{C}$ is a full subcategory containing a covering family for each object in $\mathfrak{C}$. For the category of open sets in a topological space, this is the same as a basis of neighborhoods.

Proposition (A.15) *Let $\mathscr{F}_n$ be an inverse system of Abelian sheaves over $\mathfrak{C}$. Assume there exists a generating subcategory in which every object U satisfies the following*:

(a) $H^q(U;\mathscr{F}_n) = 0$ *when* $q > 0$, *for all n*;

(b) *The system $\mathscr{F}_n(U)$ satisfies the Mittag-Leffler conditions.*

Then, the natural transformation $\lim_{\leftarrow n} \mathscr{F}_n \to \mathbb{R}\lim_{\leftarrow n} \mathscr{F}_n$ *is an isomorphism. For a complex $\mathscr{F}_n^\bullet$, bounded below*, *of sheaves*, *satisfying* (a), (b) *dimensionwise*, *we have a short exact sequence*

$$0 \to \mathbb{R}^1 \lim_{\leftarrow} \mathbb{H}^{q-1}(\mathfrak{C};\mathscr{F}_n) \to \mathbb{H}^q(\mathscr{C};\lim_{\leftarrow n}\mathscr{F}_n^\bullet) \to \lim_{\leftarrow n}\mathbb{H}^q(\mathfrak{C};\mathfrak{F}_n^\bullet) \to 0 \ .$$

Proof. The derived functor of $\lim_{\leftarrow}$ is represented by a complex of sheaves of the form

$$\text{(A.16)} \qquad 0 \to \prod_m \mathscr{F}_m \to \prod_{m<n} \mathscr{F}_m \to \prod_{m<n<p} \mathscr{F}_m \to \ldots$$

Because of (a), the cohomology sheaves of this complex, evaluated on U, agree with cohomology groups of the corresponding complex of sections over U. However, by the Mittag-Leffler theorem, the latter complex is acyclic in positive degrees. Thus, $\mathbb{R}^q \lim_{\leftarrow} \mathscr{F}_n$ vanishes on every U in the generating subcategory, so it vanishes altogether. Further, if Γ is the functor of sections over $\mathfrak{C}$, we have $\mathbb{R}^\bullet\Gamma \cong \mathbb{H}^\bullet$, and

$$\mathbb{R}\Gamma \circ \lim_{\leftarrow} \cong \mathbb{R}\Gamma \circ \mathbb{R}\lim_{\leftarrow} \cong \mathbb{R}(\Gamma \circ \lim_{\leftarrow}) = \mathbb{R}(\lim_{\leftarrow} \circ \Gamma) \cong \mathbb{R}\lim_{\leftarrow} \circ \mathbb{R}\Gamma \ . \tag{A.17}$$

The desired exact sequence then follows from the spectral sequence $E_2^{p,q} = \mathbb{R}^p \lim_{\leftarrow} \circ \mathbb{R}^q\Gamma$ of the composition of functors, as $\mathbb{R}^p \lim_{\leftarrow} = 0$ when $p > 1$. □

Consider an ind-scheme Y, union of closed subschemes Y_n, and the surjective system of push-forwards to Y of the restrictions of Y_n of a complex of vector bundles on Y. The following are consequences of (A.15)

(A.18) *If the Y_n are affine, hypercohomology of a complex of vector bundles is computed by the complex of global sections.* □

(A.19) *If the Y_n are projective, we have $H^*(Y;\mathscr{V}) \cong \lim_{\leftarrow} H^\bullet(Y_n;\mathscr{V}\,|_{Y_a})$.* □

In particular, the cohomology of vector bundles over the flag variety X, as computed in [K], agrees with the sheaf cohomology over its big site of finite-type schemes.

(A.20) *We have* $0 \to \mathbb{R}^1 \lim_{\leftarrow} AH^{q-1}(Y_n) \to AH^q(Y) \to \lim_{\leftarrow} AH^q(Y_n) \leftarrow 0.$ □

This proves the Alexander cohomology theorem for ind-varieties, because the singular cohomology of Y, which is $\operatorname{holim}_{\rightarrow} Y_n$, will be determined by the same short exact sequence.

Appendix B. stacks and simplicial objects

My preference for the simplicial theory of stacks comes from its expression in the familiar language of homotopy theory: the general results, which I shall use here without proof, look obvious enough. The proofs needed here are contained in the literature, or can be easily derived from it ([Bro], [BG], [J]; more recently [Sim]); but at present there seems to be no single, truly complete account of the material. This is partly because, whereas the general homotopy theory is well-understood, the finer algebro-geometric specifics are still under construction [Sim2].

1. Generalities

Stacks. I shall take the point of view that a stack is a simplicial scheme. Actually, stacks are objects in a homotopy category, obtained from the category of simplicial schemes by inverting a class of simplicial morphisms, the **weak equivalences** defined by Illusie in [I]. A simplicial morphism $f : Y_\bullet \to Z_\bullet$ is a weak equivalence if, for any pointed complex scheme[17] (U, u), the induced map on "stalks at u" $f_u : \lim_{\to V} \mathrm{Hom}(V, Y_\bullet) \to \lim_{\to V} \mathrm{Hom}(V, Z_\bullet)$, where V runs over the pointed étale neighborhoods of u in U, is a weak equivalence of simplicial sets.

For example, the simplicial $EG_\bullet$ is weakly equivalent to a point, for any group sheaf G. Another instance is that a scheme is weakly equivalent to the simplicial scheme formed by the fibered powers of some covering of it. For Zariski-open coverings, this equivalence is implicit in the Čech computation of sheaf cohomology, or, more generally, in the Mayer-Vietoris spectral sequence. There is a parallel theory for analytic spaces (of course, we can then use classical neighborhoods in the definition of weak equivalences).

Other presentations and "1-stacks". There are alternative ways to present a stack, much like a homotopy type in topology admits multiple descriptions. For instance, one could use cubical schemes. More interesting are small diagrams of schemes. (A simplicial scheme is obtained by taking the *nerve* of the diagram). One could also work with *category-schemes* – categories whose set of objects and of morphisms are schemes, so that the defining maps (source, target, composition, identity) are morphisms. Its nerve would also be a simplicial scheme. Thus, to a groupoid scheme, one associates a simplicial scheme by the bar construction, and obtains a socalled "1-stack". Restricted to such objects, the homotopy theory is roughly equivalent[18] to the standard "equivalence" theory of 1-stacks (as in [LM]), much as the notions of equivalence of categories and homotopy equivalence agree for groupoids.

Sheaves. One really works, instead of schemes, with sheaves over $\mathfrak{F}$, in the étale topology (or over $\mathfrak{A}$, in the analytic topology); the advantage is that the category $\mathbb{S}\mathrm{Sh}$ of simplicial sheaves (over any Grothendieck site) has a *closed simplicial model structure*, as defined by Quillen [Q]. (A proof is given in [J].) Any sheaf over $\mathfrak{F}$ can be covered by a disjoint union of schemes, and one can show from here that any stack is equivalent to a simplicial scheme (similarly for analytic spaces). However, there is no model structure on the subcategory of simplicial schemes.

Pull-backs. There is a "pull-back" functor from sheaves over $\mathfrak{F}$ to sheaves over $\mathfrak{A}$, extending the *underlying analytic space* functor. It preserves the weak equivalences of simplicial objects, and thus extends the underlying analytic space functor to stacks. (This holds, more generally, for the pull-back along any morphism of topoi.) Should we choose to work with a

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[17] U must belong to the appropriate test category; so it is of finite type, for stacks over $\mathfrak{F}$, and arbitrary, over $\mathfrak{Sch}$. Also, over $\mathfrak{F}$, we need only consider simplicial schemes that are, component-wise, locally of finite type

[18] To get a good theory, one must use algebraic spaces, instead of schemes

larger class of schemes, the absence of an obvious underlying analytic space can be circumvented, by pulling back the sheaves from the larger category to $\mathfrak{F}$.

Homotopy type. For a stack represented by a simplicial analytic space, its (topological) *homotopy type* is the geometric realization of the underlying simplicial topological space. Equivalences of simplicial schemes induce cohomology isomorphisms for a large class of sheaves, including all locally constant coefficient systems on the geometric realization; and Whitehead's theorem shows, then, that the resulting homotopy type does not depend on the chosen simplicial representative. The homotopy type of a stack over $\mathfrak{F}$ is that of its underlying analytic stack.

Internal Hom. There is an internal "mapping stack" functor $\mathscr{H}om_\bullet : \mathbb{S}\mathrm{Sh}^{\mathrm{op}} \times \mathbb{S}\mathrm{Sh} \to \mathbb{S}\mathrm{Sh}$. For any object U in the site, $\mathscr{H}om_k(\mathscr{Y}_\bullet, \mathscr{Z}_\bullet)(U)$ is the set of simplicial morphisms of sheaves from $U \times \mathscr{Y}_\bullet \times \Delta[k]_\bullet$ to $\mathscr{Z}_\bullet$, where $\Delta[k]_\bullet$ stands for the constant simplicial sheaf, with fiber the standard k-simplex. As one knows from ordinary simplicial homotopy theory, this is usually not the correct mapping stack[19] from $\mathscr{Y}_\bullet$ to $\mathscr{Z}_\bullet$; but $\mathscr{H}om_\bullet$ has a *total right derived functor* $\mathscr{E}xt$, in the sense of Quillen, defined on the homotopy categories. It is obtained by replacing the left argument by a *confibrant object* over it, and the right one by a *fibrant object* under it; the simplicial model axioms (esp. "SM7") ensure that the resulting object is well-defined, up to canonical equivalence. $\mathscr{E}xt$ has the usual adjointness property with respect to products.

The stack of G-bundles over Σ^c. $\mathfrak{M}$ is defined up to canonical equivalence, as follows:

Defintion (B.1) $\mathfrak{M} := \mathscr{E}xt(\Sigma^c; BG)$.

It follows that $[Y; \mathfrak{M}] \cong [Y \times \Sigma^c; BG] = \{$isomorphism classes of G-bundles over $Y \times \Sigma^c\}$, if we use brackets to denote morphisms in the homotopy category of stacks.

The universal bundle $\mathscr{G}$ over $\mathfrak{M} \times \Sigma^c$ is the principal G-bundle corresponding to the "evaluation arrow" $E : \Sigma^c \times \mathscr{E}xt(\Sigma^c; BG) \to BG$. This arrow corresponds to the identity, under the adjointness isomorphism

$$\text{(B.2)} \qquad [\Sigma^c \times \mathscr{E}xt(\Sigma^c; BG); BG] \cong [\mathscr{E}xt(\Sigma^c; BG); \mathscr{E}xt(\Sigma^c; BG)] \ .$$

Restricting the evaluation morphism to the marked points to Σ gives the classifying morphisms for the "evaluation" G-bundles $\mathscr{G}(z_k)$ of the Introduction.

[19] Unless the target is a *fibrant object*; in the homotopy theory of simplicial sets, this is a so-called *Kan complex* [May]

2. The uniformization theorem

The proof that follows is a sample application of homotopical techniques, and allows me to illustrate a point that is not always so evident. Problems about stacks often split into a "local" and a "local-to-global" part. The "local" part is pure algebraic geometry (or whatever category we work in), and "stacks" need not appear in the statements. The "local-to-global" part is pure homotopy theory. In this case, the local part was the Drinfeld-Simpson theorem, and I shall now describe the homotopy theory.

Choose a new point of Σ, and call the punctured curve $\Sigma^\times$; let $\Sigma^{c\times}$ be the punctured Σ^c.

Proposition (B.3) *If G is semi-simple, $G[\Sigma^\times]/G[\Sigma^{c\times}] \cong X$.*

Proof. Clearly, the map from the quotient to the adèlic quotient (at the old punctures) X is a monomorphism, so it suffices to check it is a covering. Étale-locally on X, the universal G-bundle over $X \times \Sigma^c$ can be trivialized over $X \times \Sigma^\times$, by the theorem of Drinfeld-Simpson. Dividing by the canonical section over $X \times \Sigma$ gives the desired étale lifting of X into $G[\Sigma^\times]$. □

Corollary(B.4) *The arrow $B(G[\Sigma^{c\times}]) \to B(G[\Sigma^\times])$ can be realized as the fiber bundle, with fiber X, associated to the obvious action of $G[\Sigma^\times]$ on X.* □

This is a refromulation of (B.3). "Fiber bundle" in the simplicial context means, in general, twisted Cartesian product, as in [May], but this case is especially obvious, since the fiber X is simplicially discrete. Clearly, the bar realization of $B(G[\Sigma^{c\times}])$ won't do, so having a notion of weak equivalence is essential for this proposition to hold.

Proposition (B.5) ("*Mayer-Vietoris*") *The natural arrow to the homotopy fibered product*

$$\mathscr{E}xt(\Sigma^c; BG) \to \mathscr{E}xt(\Sigma; BG) \times_{\mathscr{E}xt(\Sigma^\times;BG)} \mathscr{E}xt(\Sigma^{c\times}; BG)$$

is a weak equivalence. □

Remark (*B.6*) The evaluation bundle $\mathscr{G}(z_k)$ correspond to the evaluation bundles on either factor, when z_k differs from the punctures. With the next proposition, this implies that the bundle $\mathscr{V}$ over $\mathfrak{M}$ is associated to the evaluation representation $\mathsf{V}_1 \otimes \ldots \otimes \mathsf{V}_m$ of G^Σ in (B.9).

Proposition (B.7)[20] *The natural arrow $B(G[\Sigma^\times]) \to \mathscr{E}xt(\Sigma^\times; BG)$ is an equivalence, and the same holds for the other punctured curves.*

20 See the Correction at the end of the paper

This "natural arrow" is the structural transformation $\mathscr{H}om_\bullet(\Sigma^\times; BG) \to \mathscr{E}xt(\Sigma^\times; BG)$ from a functor to its right derived functor (cf. [Q], Ch. I Sect. 4). Explicitly, it corresponds to the evaluation $\Sigma^\times \times B(G[\Sigma^\times]) \to BG$ on standard simplicial presentations.

Proof. Let us construct the inverse in the homotopy category. Over $\Sigma^\times \times \mathscr{E}xt(\Sigma^\times; BG)$, there is a universal principal G-bundle, coming from the evaluation arrow to BG. It follows from the result of Drinfeld and Simpson that its pull-back, under some étale covering of $\mathscr{E}xt(\Sigma^\times; BG)$ can be trivialized. The sections along $\Sigma^\times$ on the local trivializations give an étale $G[\Sigma^\times]$-valued 1-cocycle over $\mathscr{E}xt(\Sigma^\times; BG)$, which correspond to an arrow from the latter to $B(G[\Sigma^\times])$; this is the desired homotopy inverse. □

Remark (*B.8*) It is known in ordinary homotopy theory that the potential difference between $B\mathrm{Map}(M; G)$ and $\mathrm{Map}(M; BG)$ is detected by π_0: that is, the natural map from the former to the latter is an equivalence iff the latter is connected. This remains true in the context of sheafified homotopy theory. If one defines the "homotopy sheaf" $\pi_0(\mathrm{Map}(M; BG))$ to be the *sheafification* of the presheaf $U \mapsto [U \times M; BG] \cong H^1_{et}(U \times M; G)$, triviality of π_0 is necessary and sufficient for (B.7) to hold. In the present case, this is exactly the Drinfeld-Simpson result.

Corollary (B.9) (*"Uniformization theorem"*) $\mathfrak{M}$ *is equivalent to the quotient stack* X/G^Σ.

Proof. The latter is, by definition, the X-bundle over $B(G[\Sigma])$ associated to the universal bundle by the obvious G^Σ-action. The Corollary follows from (B.4), (B.5) and (B.7), keeping in mind that the homotopy fibered product is represented by the strict fibered product, when one of the arrows has been realized as a (local) fibration (cf. the axioms of [Bro], Sect. 1). □

The proof of the first holomorphic version of uniformaization (Proposition 3.17) is identical, working over the category of analytic spaces.

Finally, let us realize the arrow $\Sigma^c \times \mathfrak{M} \to BG$, promised in Sect. IV, as a simplicial morphism. We replace Σ^c by the stack $S_\bullet$ of fibered powers of $\Sigma \amalg \Sigma^{c\times}$ over Σ^c (this is the homotopy quotient of the diagram $\Sigma^\times \rightrightarrows \Sigma \amalg \Sigma^{c\times}$), and we represent $\mathfrak{M}$ by the homotopy quotient of $G[\Sigma^\times]$ by $G[\Sigma^{c\times}] \times G[\Sigma]$ (left $\times$ right action). The later is $\mathrm{Hom}_\bullet(S_\bullet; BG)$, on the nose; and the desired arrow is realized by the "evaluation morphism" $S_\bullet \times \mathrm{Hom}_\bullet(S_\bullet; BG) \to BG$.

As all objects come from groupoids, the morphism is determined by the map on 1-simplices, from $(\Sigma \amalg \Sigma^\times \amalg \Sigma^\times \amalg \Sigma^{c\times}) \times G[\Sigma] \times G[\Sigma^\times] \times G[\Sigma^{c\times}]$ to G. On Σ and $\Sigma^{c\times}$, this is the obvious evaluation map. On each copy of $\Sigma^\times$, it is the ratio of two evaluation morphisms, one of which comes from $G[\Sigma^\times]$.

Correction. Theorem 3 of [DS] is misquoted in the proof of (B.7) and in (B.8). To conclude triviality, locally on the base, the original family of bundles should extend (locally) over the family of *complete* curves. This assumption is used in [DS], and also in my proof (cf. Proposition 3.14). Holomorphic local extensions exist, by a theorem of Grauert's, but the algebraic question is more delicate.

The discussion (B.7)–(B.8) of *algebraic* uniformization is incorrect, relying on a stronger form of the [DS] result, whose status is unclear. (B.5) is valid, but the $\mathscr{E}xt$ on the right side are unknown. To fix the computation of the homotopy fiber product, observe the following:

(B.10) *The images of* $\pi_0\mathscr{E}xt(\Sigma; BG)$ *and of* $\pi_0\mathscr{E}xt(\Sigma^{c\times}; BG)$ *in* $\pi_0\mathscr{E}xt(\Sigma^\times; BG)$ *meet only at the base-point of* $\pi_0\mathscr{E}xt(\Sigma^\times; BG)$

In words: a family of bundles over $\Sigma^\times$ which extends (locally) to a family of bundles over Σ and to one over $\Sigma^{c\times}$ extends, locally, to a family of bundles over Σ^c. The π_0 are taken sheaf-theoretically; they are the sheaves of isomorphism classes for families of bundles over the various affine curves. (B.7) claims (groundlessly) that these π_0 are trivial; but (B.10) means that we can limit ourselves to those families of bundles which admit local extensions to the complete curve. With that, the proof of (B.9) becomes correct.

The argument is not truly different from the one in [LS]; but it provides the needed link between the "1-stack" and the simplicial theories.

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