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Appendix
The Chow ring of $\mathscr{M}_2$

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There is now a well developed theory of the Chow rings of moduli spaces of curves [M, F], [V]. Due to the singularities of these spaces, these rings are defined as **Q**-algebras.

The development of an equivariant interesection theory by Totaro, Edidin and Graham [E-G] allows one to define *integral* versions of these rings. More precisely, Edidin and Graham show that the equivariant Chow ring of a smooth algebraic scheme acted on by an algebraic group is a naturally defined integral Chow ring of the associated quotient stack [E-G], Proposition 19). When the group acts with finite stabilizers (as is the case for moduli of curves) this ring is naturally isomorphic to the previously defined rings after tensoring with **Q**. (Moreover, their definition also extends to situations where the "classical" theory collapses and the automorphism groups have infinite order.) In [E-G], Proposition 21, the integral Chow rings of the stacks $\mathscr{M}_{1,1}$ and $\overline{\mathscr{M}}_{1,1}$ of smooth (respectively stable) pointed curves of genus 1 are computed. In this note we give one further example by computing the Chow ring of the stack of smooth curves of genus 2.

Let $\mathscr{M}_2$ be the stack of smooth curves of genus 2 over a fixed field κ. There is a natural vector bundle $\mathscr{E}$ of rank 2 on M, called the *Hodge bundle*: if $\pi\colon C \to S$ is a flat family of curves of genus g corresponding to a morphism $S \to \mathscr{M}_2$, and ω_π is the relative dualizing sheaf, then the pullback of E to S is $\pi_*\omega_\pi$. The Chern classes $\lambda_i = c_i(\mathscr{E})$ are among the tautological classes introduced by Mumford.

We'll use the following notation. If R is a commutative ring, $x_1, \dots, x_n$ are elements of $R, f_1, \dots, f_r$ are integral polynomial in n variables $X_1, \dots, X_n$, we write

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$$R = \mathbf{Z}[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n))$$

to indicate that R is generated as a ring by the elements $x_1, \ldots, x_n$, and the polynomial $f_1, \ldots, f_r$ generate the ideal of relations of $x_1, \ldots, x_n$ in $\mathbf{Z}[X_1, \ldots, X_n]$.

The purpose of this appendix is to prove the following.

Theorem. *Assume that κ has characteristic difference from* 2 *and* 3. *Then*

$$A^*(\mathcal{M}_2) = \mathbf{Z}[\lambda_1, \lambda_2]/\left(10\lambda_1, 2\lambda_1^2 - 24\lambda_2\right) \ .$$

In characteristic 3 these two relations still hold, but they do not generate the ideal of relations.

The proof consists in expressing the stack $\mathcal{M}_2$ as a quotient of an open subscheme of a representation space of GL_2, thus showing that $A^*(\mathcal{M}_2)$ is generated by λ_1 and λ_2, then obtaining the relations coming from the complement of this open subscheme. For this last part, which is rather computational, I have used *Mathematica*, of Wolfram Research Inc.

Let Y be the stack whose objects are pairs (π, α), where $\pi\colon C \to S$ is a smooth proper morphism of schemes whose fibers are curves of genus 2, and α is an isomorphism of $\mathcal{O}_S$ sheaves $\alpha\colon \mathcal{O}_S^{\oplus 2} \simeq \pi_*\omega_\pi$, where ω_π is the relative dualizing sheaf of π, the arrows being the obvious ones. One can think of Y as the bundle of frames in the Hodge bundle of $\mathcal{M}_2$. It is easy to check that the objects of Y have no nontrivial automorphisms, so that Y is an algebraic space. There is natural left $\mathrm{GL}_{2,\kappa}$ action on Y: if (π, α) is an object of Y with basis S and $A \in \mathrm{GL}_2(S)$, we set $A \cdot (\pi, \alpha) = (\pi, \alpha \circ A^{-1})$. Clearly $\mathcal{M}_2$ is canonically isomorphic to the quotient $[Y/\mathrm{GL}_2]$, and the equivariant bundle on Y induced by the standard representation of GL_2 corresponds to the Hodge bundle on $\mathcal{M}_2$.

For the next result we only need to assume that the characteristic of κ is different from 2.

Consider the affine space $\mathbf{A}^7_\kappa$, considered as the space of all binary forms $\phi(x) = \phi(x_0, x_1)$ of degree 6. Denote by X the open subset consisting of nonzero forms with distinct roots.

Proposition 3.1. *The algebraic space Y is naturally isomorphic to X; the given action of* GL_2 *corresponds to the action of* GL_2 *on X defined by $A \cdot \phi(x) = \det(A)^2\phi(A^{-1}x)$. The canonical representation of* GL_2 *yields the Hodge bundle on $\mathcal{M}_2$.*

Proof. Let $(\pi\colon C \to S, \alpha)$ be an object of Y. The line bundle ω_π is generated by global sections on the fibers of π, so, together with the isomorphism α, yields an S-morphism $f\colon C \to \mathbf{P}^1_S$, which is a ramified covering of degree 2 on each fiber, together with an isomorphism $\omega_\pi \simeq f^*\mathcal{O}_{\mathbf{P}^1_S}(1)$. We use the well know description of covering of degree 2 of $\mathbf{P}^1$; the embedding $\mathcal{O}_S \hookrightarrow f_*\mathcal{O}_C$ has a splitting, given by the trace divided by 2, so we get an isomorphism of

$\mathcal{O}_{\mathbf{P}^1_S}$-modules $f_*\mathcal{O}_C \simeq \mathcal{O} \oplus \mathcal{L}$. The line bundle $\mathcal{L}$ on S is non-canonically isomorphic to $\mathcal{O}(-3)$ on the fibers of π. Multiplication in $f_*\mathcal{O}_{\mathscr{C}}$ yields a homomorphism $\mathcal{L}^{\otimes 2} \to \mathcal{O}$, which is injective on the fibers of π, and such that the quotient $\mathcal{O}/\mathcal{L}^{\otimes 2}$ is étale over S. The natural action of the cyclic group C_2 on C corresponds to the action of C_2 on $\mathcal{O} \oplus \mathcal{L}$ in which a generator of C_2 leaves $\mathcal{O}$ invariant, and changes sign on $\mathcal{L}$.

Conversely given a line bundle $\mathcal{L}$ which is isomorphic to $\mathcal{O}(-3)$ on each fiber of π and an injective homomorphism $\mathcal{L}^{\otimes 2} \to \mathcal{O}_S$ such that the quotient $\mathcal{O}/\mathcal{L}^{\otimes 2}$ is étale over S, we get an algebra structure on $\mathcal{O}_S \oplus \mathcal{L}$, and a smooth family of curves $C = \operatorname{Spec}(\mathcal{O}_S \oplus \mathcal{L}) \to \mathbf{P}^1_S \to S$ of genus 2. The line bundles ω_π and $f^*\mathcal{O}(1)$ are isomorphic when restricted to each of the fibers of the projection $f\colon \mathbf{P}^1_S \to S$. Giving an isomorphism $\omega_\pi \simeq f^*\mathcal{O}_{\mathbf{P}^1_S}(1)$ is equivalent to giving nowhere vanishing section of $\omega_\pi \otimes f_*f^*\mathcal{O}(-1)$, or a nowhere vanishing section of

$$f_*(\omega_\pi \otimes f^*\mathcal{O}(-1)) = f_*\omega_\pi(-1) \ .$$

But by Grothendieck duality, if we denote by ω the relative dualizing sheaf of $\mathbf{P}^1_S$ on S, there is a functorial isomorphism

$$f_*\omega_\pi \simeq \operatorname{Hom}(f_*\mathcal{O}_C, \omega) = \omega \oplus \left(\omega \otimes \mathcal{L}^{-1}\right) \ ;$$

so an isomorphism $\omega_\pi \simeq f^*\mathcal{O}_{\mathbf{P}^1_S}(1)$ corresponds functorially to a nowhere vanishing section of $\omega(-1) \oplus (\omega(-1) \otimes \mathcal{L}^{-1})$. Since $\omega(-1)$ does not have sections, this is the same as a nowhere vanishing section of $\omega(-1) \otimes \mathcal{L}^{-1}$, or an isomorphism of $\mathcal{L}$ with ω. Given this, a homomorphism $\mathcal{L}^2 \to \mathcal{O}$ corresponds to a homomorphism $\omega(-1)^{\otimes 2} \to \mathcal{O}$, or, equivalently, a section of $\mathcal{T}^{\otimes 2}(2)$, where $\mathcal{T}$ is the relative tangent bundle of $\mathbf{P}^1_S$ on S.

The conclusion of all this is that a morphism $S \to Y$ corresponds to a section of $\mathcal{T}^{\otimes 2}(2)$ whose subscheme of zeroes is étale on S. Fixing an isomorphism of $\mathcal{T}$ with $\mathcal{O}(2)$, we see that such sections correspond to morphisms $S \to X$; in this way we obtain an isomorphism of Y with X. It is not hard to see that the action of GL_2 on Y corresponds to the action on X induced by the natural action of GL_2 on $\mathcal{T}^{\otimes 2}(2)$. If we fix an isomorphism of $\mathcal{T}$ with $\mathcal{O}(2)$, this action corresponds to the action of GL_2 on $\mathcal{O}(2)$ defined by $A \cdot f(x) = \det(A) f(A^{-1}x)$, where $A \in \mathrm{GL}_2(S)$, $f \in X$ and $x \in \mathbf{A}^2_S$; therefore the action of GL_2 on Y corresponds to the given action on X.

The final statement is clear. □

Now we have to calculate the equivariant Chow ring $A^*_{\mathrm{GL}_2}(X)$ of X. Since X is an open subset in a representation space of GL_2, we see that $A^*_{\mathrm{GL}_2}(X)$ is generated by the Chern classes λ_1 and λ_2 of the Hodge bundle; to find the relations we have to analyze the discriminant hypersurface in $\mathbf{A}^7$, which is rather complicated. We write λ_1 and λ_2 for the first and second Chern classes of the standard representation of GL_2, and also for their pullbacks to the Chow ring of any smooth GL_2-scheme.

More generally, we will usually use the same symbol for a class in some Chow ring, and all of its pullbacks; this should not lead to confusion, and simplifies the notation considerably.

First of all, call F the dual of the basic representation of GL_2, namely, the space of linear forms on $\mathbf{A}^2$ with the action of GL_2 given by $A \cdot f(x) = f(A^{-1}x)$. Its Chern classes are $-\lambda_1$ and λ_2. Set $E = \mathrm{Sym}^6 F$. So E is $\mathbf{A}^7$, the space of sextic binary form, with the usual action defined by $A \cdot f(x) = f(A^{-1}x)$. Consider the space $\mathbf{P}^6 = \mathbf{P}(E)$ of lines in E, and the quotient Z of X by the diagonal subgroup $\mathbf{G}_m \subseteq \mathrm{GL}_2$. The GL_2-scheme Z is an invariant open subscheme of $\mathbf{P}^6$, and the projection $X \to Z \subseteq \mathbf{P}^6$ makes X into the total space of the principal $\mathbf{G}_m$ bundle on Z corresponding to the equivariant line bundle $\mathscr{D}^{\otimes 2} \otimes \mathscr{O}(-1)$, where $\mathscr{D}$ is the determinant of the standard representation of GL_2, and $\mathscr{O}(-1)$ is the tautological bundle on $\mathbf{P}^6$. If we denote by t the first Chern class of $\mathscr{O}(1)$ on $\mathbf{P}^6$ we see that the natural homomorphism

$$A^*_{\mathrm{GL}_2}(Z) \to A^*_{\mathrm{GL}_2}(X)$$

is surjective, and its kernel is generated by $2\lambda_1 - t$. This means that if $p_1, \dots p_r \in \mathbf{Z}[\lambda_1, \lambda_2, t]$ is a set of generators for the kernel of the surjective homomorphism

$$\mathbf{Z}[\lambda_1, \lambda_2, t] \to A^*_{\mathrm{GL}_2}(Z) \ ,$$

then $p_1(\lambda_1, \lambda_2, 2\lambda_1), \dots, p_r(\lambda_1, \lambda_2, 2\lambda_1)$ will be a set of generators for the kernel of the surjective homomorphism

$$\mathbf{Z}[\lambda_1, \lambda_2] \to A^*_{\mathrm{GL}_2}(X) \ .$$

The equivariant Chow ring of $\mathbf{P}^6$ is generated by λ_1 and λ_2, modulo a relation $p(\lambda_1, \lambda_2, t)$ in degree 7 which is determined by the Chern classes of $E = \mathrm{Sym}^6 F$. If ℓ_1 and ℓ_2 are the Chern roots of F, so that $\ell_1 + \ell_2 = -\lambda_1$ and $\ell_1 \ell_2 = \lambda_2$, the Chern roots of E are $6\ell_1, 5\ell_1 + \ell_2, 4\ell_1 + 2\ell_2,\ 3\ell_1 + 3\ell_2,$ $2\ell_1 + 4\ell_2, \ell_1 + 5\ell_2$ and $6\ell_2$.

We have, after a straightforward calculation,

$$\begin{aligned} p(\lambda_1, \lambda_2, t) &= (t + 6\ell_1)(t + 6\ell_2)(t + 5\ell_1 + \ell_2)(t + 5\ell_1 + \ell_2) \\ &\qquad (t + 4\ell_1 + 2\ell_2)(t + 2\ell_1 + 4\ell_2)(t + 3\ell_1 + 3\ell_2) \\ &= \left(t^2 - 6\lambda_1 t + 36\lambda_2\right)\left(t^2 - 6\lambda_1 t + 5\lambda_1^2 + 16\lambda_2\right) \\ &\qquad \left(t^2 - 6\lambda_1 t + 8\lambda_1^2 + 4\lambda_2\right)(t - 3\lambda_1) \ . \end{aligned}$$

We set

$$\beta = -p(\lambda_1, \lambda_2, 2\lambda_1) = 16\lambda_1\lambda_2\left(3\lambda_1^2 - 16\lambda_2\right)\left(2\lambda_1^2 - 9\lambda_2\right) \ ;$$

the expression β is 0 in $A^7_{\mathrm{GL}_2}(X)$. We have thus found our first relation.

For $r = 1, 2, 3$, call Δ_r the closed subset of $\mathbf{P}^6$ corresponding to forms divisible by the square of a polynomial of degree r over some extension of the base field. So Δ_1 is the discriminant locus in $\mathbf{P}^6$. There is a natural morphism

$$\pi_r \colon \mathbf{P}^r \times \mathbf{P}^{6-2r} \longrightarrow \mathbf{P}^6$$

induced by the map $\mathrm{Sym}^r F \times \mathrm{Sym}^{6-2r} F \to E$ which sends (f, g) into $f^2 g$. The image of π_r is, by definition, Δ_r.

Lemma 3.2. *For some $r = 1$, 2 or 3, let V be an irreducible subvariety of Δ_r which is not contained in Δ_{r+1}. Then there exists an irreducible subvariety V' of $\mathbf{P}^r \times \mathbf{P}^{6-2r}$ which maps birationally onto V.*

Here we implicitly set $\Delta_4 = \emptyset$.

Proof. The statement is equivalent to the following: if K is an extension of κ, then every K-valued point of $\Delta_r \setminus \Delta_{r+1}$ is the image of a K-valued point of $\mathbf{P}^r \times \mathbf{P}^{6-2r}$. Let p be a point in $\Delta_r \setminus \Delta_{r+1}$; p is represented by some form $f \in K[x_0, x_1]$ of degree 6. Write $f = u^2 v$, where $v \in K[x_0, x_1]$ is a square-free form. Obviously the degree of u must be at most r, because otherwise p would be in Δ_{r+1}. Furthermore v will remain square-free in any extension of K, because its degree is less than the characteristic, so the degree of u must be exactly r. Hence if a and b are the K-valued points of $\mathbf{P}^r$ and $\mathbf{P}^{6-2r}$ corresponding to u and v, we have $\pi_r(a, b) = p$. □

Lemma 3.3. *The kernel of the surjective homomorphism*

$$j^* \colon A^*_{\mathrm{GL}_2}(\mathbf{P}^6) \to A^*_{\mathrm{GL}_2}(Z)$$

is the sum of the images of the homomorphisms

$$\pi_{r*} \colon A^*_{\mathrm{GL}_2}(\mathbf{P}^r \times \mathbf{P}^{6-2r}) \longrightarrow A^*_{\mathrm{GL}_2}(\mathbf{P}^6) \ .$$

Proof. Follows by standard arguments from Lemma 3.2. □

Call ξ_i the pullback to $\mathbf{P}^r \times \mathbf{P}^{6-2r}$ of the first Chern class of the sheaf $\mathcal{O}(1)$ in the i^{th} factor. The Chow ring of $\mathbf{P}^r \times \mathbf{P}^{6-2r}$ is generated by $\lambda_1, \lambda_2, \xi_1$ and ξ_2. We have that

$$\pi_r^*(t) = 2\xi_1 + \xi_2$$

and that ξ_1 is a zero of a monic polynomial of degree $r + 1$ with coefficients in $\mathbf{Z}[\lambda_1, \lambda_2]$; therefore $A^*_{\mathrm{GL}_2}(\mathbf{P}^r \times \mathbf{P}^{6-2r})$ is generated as a $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$-module by 1, $\xi_1, \ldots, \xi_1^r$. Together with the projection formula, this implies that the image of π_{r*} is generated as an ideal in $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ by $\pi_{r*}1,\ \pi_{r*}\xi_1, \ldots, \pi_{r*}\xi_1^r$. From Lemma 3.3 it follows then that the kernel of j^* is generated by $\pi_{r*}\xi_1^i$ for $1 \le r \le 3$ and $0 \le i \le r$. For each r and i there is a unique homogeneous polynomial $p_{ri} \in \mathbf{Z}[\lambda_1, \lambda_2, t]$, of degree $r + i$, whose image into $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ is $\pi_{r*}\xi_1^i$. Set

$$\alpha_{ri} = p_{ri}(\lambda_1, \lambda_2, 2\lambda_2) \in \mathbf{Z}[\lambda_1, \lambda_2] \ .$$

The discussion above leads us to the following conclusion.

Lemma 3.4. *The kernel of the surjective homomorphism*

$$\mathbf{Z}[\lambda_1, \lambda_2] \longrightarrow A^*_{\mathrm{GL}_2}(X)$$

is generated by α_{10}, α_{11}, α_{20}, α_{21}, α_{22}, α_{30}, α_{31}, α_{32}, α_{33} *and* β.

So we need to compute the α_{ri}. We'll see that they are all in the ideal generated by α_{10} and α_{11}, and so is β.

Here is the set-up of the calculation. Set $\mathbf{P}^1 = \mathbf{P}(F)$, and consider the morphism $\rho\colon (\mathbf{P}^1)^6 \to \mathbf{P}^6$ induced by the multilinear map $F^6 \to E$, $(f_1, \ldots, f_6) \mapsto f_1 \ldots f_6$. Analogously one defines the map

$$\rho_r\colon \left(\mathbf{P}^1\right)^r \times \left(\mathbf{P}^1\right)^{6-2r} \to \mathbf{P}^r \times \mathbf{P}^{6-2r}$$

by multiplying separately representatives for the first r and the last $6 - 2r$ coordinates.

Finally, call $\delta_r\colon (\mathbf{P}^1)^r \times (\mathbf{P}^1)^{6-2r} \to (\mathbf{P}^1)^6$ the map that sends, in set-theoretic notation, $(f_1, \ldots, f_6)$ into $(f_1, f_1, \ldots, f_r, f_r, f_{r+1}, f_{r+2}, \ldots, f_6)$. We obtain our basic commutative diagram of GL_2-schemes

$$\begin{array}{ccc} \left(\mathbf{P}^1\right)^r \times \left(\mathbf{P}^1\right)^{6-2r} & \xrightarrow{\delta_r} & \left(\mathbf{P}^1\right)^6 \\ \downarrow \rho_r & & \downarrow \rho \\ \mathbf{P}^r \times \mathbf{P}^{6-2r} & \xrightarrow{\pi_r} & \mathbf{P}^6 \end{array}$$

in which all the maps are equivariant. The maps ρ and ρ_r are flat and finite, of degrees $6!$ and $r!(6 - 2r)!$ respectively.

We identify each class in $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ with its pullback to $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$ via ρ. Call x_i the pullback to $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$ of the first Chern class of the sheaf $\mathcal{O}(1)$ on $\mathbf{P}^1$. We have an equality $t = x_1 + \cdots + x_6$ in $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$, and for each i there is a relation

$$x_i^2 = \lambda_1 x_i - \lambda_2 \ .$$

We introduce some basic elements of $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$. For each positive integer k we set

$$t_k = \sum_{i=1}^{6} x_i^k \quad ;$$

furthermore for $1 \le k \le 6$ we call s_k the k-the symmetric function of $x_1, \dots, x_6$ multiplied by $k!$, that is

$$s_k = k! \sum_{1 \le i_1 < \cdots < i_k \le 6} x_{i_1} \dots x_{i_k} \quad .$$

By convention we set $s_0 = 1$. The fundamental relation $x_i^2 = \lambda_1 x_i - \lambda_2$ allows for each integer k to write x_i^k as $a_k + b_k x_i$, where a_k and b_k are integral polynomials in λ_1 and λ_2. By summing over i we see that $t_k = 6a_k + b_k t$. So we see that the t_k are all in $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$. After some computations we get

$$\begin{aligned}
t_1 &= t \quad , \\
t_2 &= -6\lambda_2 + \lambda_1 t \quad , \\
t_3 &= -6\lambda_1\lambda_2 + \left(\lambda_1^2 - \lambda_2\right)t \quad , \\
t_4 &= -6\lambda_1^2\lambda_2 + 6\lambda_2^2 + \left(\lambda_1^3 - 2\lambda_1\lambda_2\right)t \quad , \\
t_5 &= -6\lambda_1^3\lambda_2 + 12\lambda_1\lambda_2^2 + \left(\lambda_1^4 - 3\lambda_1^2\lambda_2 + \lambda_2^2\right)t \quad , \\
t_6 &= -6\lambda_1^4\lambda_2 + 18\lambda_1^2\lambda_2^2 - 6\lambda_2^3 + \left(\lambda_1^5 - 4\lambda_1^3\lambda_2 + 3\lambda_1\lambda_2^2\right)t \quad .
\end{aligned}$$

To compute the s_k we use Newton's formulas, which can be written as

$$s_k = \sum_{i=0}^{k} (-1)^{k-1} \frac{(k-1)!}{(k-i)!} t_i s_{k-i} \quad .$$

These show by recursion that the s_k are also elements of $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$. We can use the expressions for the t_k above to calculate the s_k inductively. The results are as follows.

Lemma 3.5.

$$\begin{aligned}
s_1 &= t \quad , \\
s_2 &= 6\lambda_2 - \lambda_1 t + t^2 \quad , \\
s_3 &= -12\lambda_1\lambda_2 + \left(2\lambda_1^2 + 16\lambda_2\right)t - 3\lambda_1 t^2 + t^3 \quad , \\
s_4 &= 36\lambda_1^2\lambda_2 + 72\lambda_2^2 - \left(6\lambda_1^3 + 72\lambda_1\lambda_2\right)t + \left(11\lambda_1^2 + 28\lambda_2\right)t^2 - 6\lambda_1 t^3 + t^4 \quad , \\
s_5 &= -144\lambda_1^3\lambda_2 - 432\lambda_1\lambda_2^2 + \left(24\lambda_1^4 + 60\lambda_1^2\lambda_2 + 552\lambda_2^2\right)t - \left(2\lambda_1^3 + 316\lambda_1\lambda_2\right)t^2 \\
&\quad + \left(35\lambda_1^2 + 40\lambda_2\right)t^3 - 10\lambda_1 t^4 + t^5 \quad , \\
s_6 &= 720\lambda_1^4\lambda_2 + 2520\lambda_1^2\lambda_2^2 + 720\lambda_2^3 - \left(120\lambda_1^5 + 1944\lambda_1^3\lambda_2 + 2472\lambda_1\lambda_2^2\right)t \\
&\quad + \left(274\lambda_1^4 + 1270\lambda_1^2\lambda_2 + 832\lambda_2^2\right)t^2 - \left(177\lambda_1^3 + 576\lambda_1\lambda_2\right)t^3 \\
&\quad + \left(85\lambda_1^2 + 50\lambda_2\right)t^4 - 15\lambda_1 t^5 + t^6 \quad .
\end{aligned}$$

Before proceeding to further calculations, let us remark that $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ is torsion free as an abelian group, and therefore to prove an identity in this ring it is enough to prove it in $A^*_{\mathrm{GL}_2}(\mathbf{P}^6) \otimes \mathbf{Q}$.

Let us also observe that as a $\mathbf{Z}[\lambda_1, \lambda_2]$-module, $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^6)$ is free with a basis formed by the monomials $x_{i_1} \dots x_{i_k}$, where $i_1, \dots, i_k$ are integers with $1 \le i_1 < \dots < i_k \le 6$. By the projection formula to understand the push-forward

$$\rho_*\colon A^*_{\mathrm{GL}_2}\left((\mathbf{P}^1)^6\right) \to A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$$

we only need to know what the $\rho_*(x_{i_1} \dots x_{i_k})$ are.

The importance of the s_k is revealed by the following lemma.

Lemma 3.6. *In $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ we have*

$$\rho_*(x_{i_1} \dots x_{i_k}) = (6-k)! s_k \ .$$

Proof. Since $s_k \in A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$ and ρ is flat and finite of degree 6!, we have $\rho_* s_k = 6! s_k$. On the other hand, because of the obvious action of the symmetric group S_6 on $(\mathbf{P}^1)^6$, we see that $\rho_*(x_{i_1} \dots x_{i_k})$ only depends on k, and not on the i_k. From the definition of the s_k we obtain

$$\rho_*(x_{i_1} \dots x_{i_k}) = \frac{(6-k)!}{6!} \rho_* s_k = (6-k)! s_k$$

as desired. □

Lemma 3.7. *The class of the image of δ_r in $(\mathbf{P}^1)^6$ is*

$$(x_1 + x_2 - \lambda_1) \dots (x_{2r-1} + x_{2r} - \lambda_1) \ .$$

Proof. Follows immediately from the next lemma. □

Lemma 3.8. *Let F be a vector bundle of rank 2 on a smooth variety S, $P = \mathbf{P}(F)$ the projective bundle of lines in F, Δ the image of the diagonal embedding $\delta\colon P \hookrightarrow P \times_S P$. Let x_1 and x_2 in $A^*(P \times_S P)$ be the two pullbacks of the first Chern class of $\mathcal{O}_P(1)$, $c_1 \in A^*(P \times_S P)$ the pullback of the first Chern class of F. Then the class of Δ is $x_1 + x_2 + c_1$.*

Proof. Denote by p_1 and p_2 the two projections of $P \times_S P$ onto P. Then we have the Beilinson resolution of the structure sheaf of the diagonal

$$0 \longrightarrow p_1^* \Omega_{P/S}(1) \otimes p_2^* \mathcal{O}(-1) \longrightarrow \mathcal{O}_{P \times_S P} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \ .$$

From the Euler sequence we can compute the first Chern class of $p_1^*\Omega_{P/S}(1) \otimes p_2^*\mathcal{O}(-1)$, and the result follows. □

Let us indicate how to calculate the α_{ij}. First one computes the $\delta_{r*}\xi_r^i$; this can be done using Lemma 3.7. Observe that in $A^*_{\mathrm{GL}_2}((\mathbf{P}^1)^r \times (\mathbf{P}^1)^{6-2r})$ we have $x_1 = x_2, \ldots, x_{2r-1} = x_{2r}$, and also

$$\xi_r = x_1 + x_3 + \cdots + x_{2r-1} \quad .$$

The results look extremely unwieldy, but everything will be pushed forward to $A^*_{\mathrm{GL}_2}(\mathbf{P}^6)$, where all the products of k distinct x_i all have the same image, thus becoming much more manageable.

Next we compute the classes $\pi_{r*}\xi_r^i$. The calculation is based on the following obvious result.

Lemma 3.9.

$$\pi_{r*}\xi_r^i = \frac{1}{r!(6-2r)!}\pi_*\delta_{r*}\xi_r^i \quad .$$

Having obtained the expressions for the $\delta_{r*}\xi_r^i$ and knowing the formula for ρ_* (Lemmas 3.5 and 3.6), we can calculate the $p_{ri} = \pi_{r*}\xi_r^i$ using Lemma 3.9; then we only have left to substitute $2\lambda_1$ for t in the expressions above, and we have computed the relations α_{ri}. The results are as follows.

Lemma 3.10.

$$\begin{aligned}
\alpha_{10} &= -10\lambda_1 \quad , \\
\alpha_{11} &= 2\lambda_1^2 - 24\lambda_2 \quad , \\
\alpha_{20} &= -12\lambda_1^2 + 144\lambda_2 \quad , \\
\alpha_{21} &= -24\lambda_1^3 + 168\lambda_1\lambda_2 \quad , \\
\alpha_{22} &= -24\lambda_1^4 + 148\lambda_1^2\lambda_2 \quad , \\
\alpha_{30} &= 24\lambda_1^3 - 128\lambda_1\lambda_2 \quad , \\
\alpha_{31} &= 24\lambda_1^4 - 128\lambda_1^2\lambda_2 \quad , \\
\alpha_{32} &= 408\lambda_1^5 - 2048\lambda_1^3\lambda_2 + 1152\lambda_1\lambda_2^2 \quad , \\
\alpha_{33} &= 1560\lambda_1^6 - 7808\lambda_1^4\lambda_2 + 4608\lambda_1^2\lambda_2^2 \quad .
\end{aligned}$$

Then one checks that all the α_{ri}, as well as

$$\beta = 16\lambda_1\lambda_2(3\lambda_1^2 - 16\lambda_2)(2\lambda_1^2 - 9\lambda_2) \quad ,$$

are in the ideal generated in $\mathbf{Z}[\lambda_1, \lambda_2]$ by α_{10} and α_{11}. This concludes the proof of the theorem.

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