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Inventiones mathematicae

Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients

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1. Introduction and main results

The problem of the uniqueness in the Cauchy problem for linear differential operators has been widely investigated during the last years (see [Z] for references). It is now well understood in the analytic framework, with Holmgren's theorem, where uniqueness always holds (at least for non characteristic surfaces) and in the C^∞ case, with Hörmander's theorem ([H1], IV, chap. 28) where the uniqueness is governed by principal normality and pseudo-convexity. The purpose of this work is to fill the gap between these two theorems by considering operators with C^∞ and partly analytic coefficients. In particular one of our results will contain both the theorems mentioned above. Let us be more precise. Let n_a, n_b be two non negative integers with $n = n_a + n_b \geq 1$. We shall set $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and, for x or ξ in $\mathbb{R}^n$, $x = (x_a, x_b)$, $\xi = (\xi_a, \xi_b)$. Let $P = P(x, D) = P(x_a, x_b, D_{x_a}, D_{x_b})$ be a linear differential operator of arbitrary order m, with principal symbol p_m. We shall assume that

$$(1.1) \qquad \begin{cases} \text{the coefficients of } P \text{ are } C^\infty \text{ in } x \text{ and analytic in } x_a. \\ \text{in a neighborhood of } \ x^0 \in \mathbb{R}^n \ . \end{cases}$$

Let S be a C^2 hypersurface through x^0 locally given by

$$S = \{x \colon \varphi(x) = \varphi(x^0)\}, \quad \varphi'(x^0) = (\varphi'_a(x^0), \varphi'_b(x^0)) \neq 0 \ .$$

As usual, $\{,\}$ will denote the Poisson bracket.

Our results are as follows

Theorem A. *Let us assume*

$$\text{(H.1)} \qquad \textit{transversal ellipticity: } p_m(x_a^0, x_b^0, 0, \xi_b) \textit{ is elliptic}$$

$$\text{(H.2)} \qquad \begin{cases} \textit{pseudo-convexity: let } \zeta = (x_a^0, x_b^0, i\varphi'_a(x^0), \\ \xi_b + i\varphi'_b(x^0)), \xi_b \in \mathbb{R}^{n_b}, \\ \textit{then } p_m(\zeta) = \{p_m, \varphi\}(\zeta) = 0 \textit{ implies} \\ \frac{1}{i}\{\overline{p}_m(x, \xi - i\varphi'(x)), p_m(x, \xi + i\varphi'(x))\}\big|_{\substack{x=x^0 \\ \xi=(0,\xi_b)}} > 0 \ . \end{cases}$$

Let V be a neighborhood of x^0 and $u \in C^\infty(V)$ be such that

$$\begin{cases} Pu = 0 \textit{ in } V \\ \operatorname{supp}\ u \subset \{x \in V : \varphi(x) \le \varphi(x^0)\} \ . \end{cases}$$

Then there exists a neighborhood W of x^0 in which $u \equiv 0$.

Theorem B. *Let us assume*

$$\text{(H.1)}' \qquad \begin{cases} \textit{principal normality} \\ |\{\overline{p}_m, p_m\}(x_a, x_b, 0, \xi_b)| \le C|\xi_b|^{m-1}|p_m(x_a, x_b, 0, \xi_b)|, \\ \textit{for all } x = (x_a, x_b) \textit{ in a neighborhood of } x^0 \textit{ and all } \xi_b \textit{ in } \mathbb{R}^{n_b}, \end{cases}$$

$$\text{(H.2)}' \qquad \begin{cases} & \textit{pseudo-convexity} \\ \text{i)} & n_b = 0 \textit{ or } n_b \ge 1 \textit{ and, with } X = (x_a^0, x_b^0, 0, \xi_b), \xi_b \in \mathbb{R}^{n_b} \setminus \{0\}, \\ & p_m(X) = \{p_m, \varphi\}(X) = 0 \textit{ implies } \mathrm{Re}\{\overline{p}_m, \{p_m, \varphi\}\}(X) > 0 \ . \\ \text{ii)} & \textit{Let } \zeta = (x_a^0, x_b^0, i\varphi'_a(x^0), \xi_b + i\varphi'_b(x^0)), \ \xi_b \in \mathbb{R}^{n_b} \ , \\ & \textit{then } p_m(\zeta) = \{p_m, \varphi\}(\zeta) = 0 \textit{ implies} \\ & \frac{1}{i}\{\overline{p}_m(x, \xi - i\varphi'(x)), p_m(x, \xi + i\varphi'(x))\}\big|_{\substack{x=x^0 \\ \xi=(0,\xi_b)}} > 0 \ . \end{cases}$$

$$\text{(H.3)}' \qquad \textit{On } \xi_a = 0, \ p_m \textit{ does not depend on } x_a \ .$$

Then the same conclusion, as in Theorem A, holds.

Let us give some applications of these results. First of all as we said before, Theorem B contains both the Holmgren and the Hörmander theorem. For operators with analytic coefficients Holmgren's theorem asserts that uniqueness holds for any non characteristic initial hypersurface. We take, in theorem B, $n_b = 0$ and $n_a \ge 1$; then (H.1)$'$, (H.3)$'$ follow from the fact that, by homogeneity, we have $p_m = \{\bar{p}_m, p_m\} = 0$ on $\xi_a = 0$, (H.2)$'$ *i*) is trivially satisfied and (H.2)$'$ *ii*) is empty since $p_m(\zeta) \ne 0$ if the initial hypersurface is non characteristic. For operators with C^∞ coefficients we take $n_a = 0$, $n_b \ge 1$; then (H.3)$'$ is empty and (H.1)$'$, (H.2)$'$ are exactly the hypotheses made by Hörmander in his theorem, [H1] Th. 28.3.4.

Here is an application of Theorem A. Let us consider, in a neighborhood V of a point $m_0 = (t_0, x_0)$ in $\mathbb{R}_t \times \mathbb{R}^n_x$, a second order strictly hyperbolic symbol of the form

$$p_2 = \tau^2 - \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j$$

where $(a_{ij}(t,x))$ is a symmetric positive definite matrix with entries which are analytic in time and C^∞ in space. Then uniqueness holds for any non characteristic initial hypersurface. (For a space-like hypersurface this result has been known for a long time even for coefficients merely C^∞ in time). Indeed let us set, in theorem A, $n_a = 1$ ($\xi_a = \tau$), $n_b = n \geq 1$ ($\xi_b = \xi$). On $\tau = 0$, p_2 is elliptic in ξ so (H.1) holds. Now a straightforward computation shows that the imaginary part of $\{p_2, \varphi\}(\zeta)$ is equal to $p_2(m_0, d\varphi(m_0))$, which does not vanish, so (H.2) is empty.

Let us now describe the background of this problem. The initial motivation for this kind of results came from control theory. Indeed Lions [Li] introduced the HUM method which relies partly on uniqueness results. In the case of second order hyperbolic operators $P = \partial_t^2 - A(t,x,\partial_x)$, the initial hypersurface is time-like and the corresponding uniqueness result is false if the coefficients are merely C^∞, as shown by the counterexamples of Alinhac-Baouendi [AB] (see also [R] for a detailed discussion of these counterexamples). However, when the coefficients of A do not depend on t, Rauch-Taylor [RT] and Lerner [L2] making a global vanishing assumption in t, proved uniqueness. Nevertheless this was not enough for control theory and Robbiano [R] was able to improve their result, using only a local vanishing assumption. His result was extended by Hörmander [H3] and then by Tataru [T] who was the first to consider operators with partially analytic coefficients as considered here. In fact Tataru proved our theorem A, when the coefficients of p_m are entire analytic functions of order 2 in x_a, and our theorem B when p_m is real and its coefficients are independent of x_a.

Let us give a sketch of the proofs. As usual uniqueness will follow from Carleman estimates; they are L^2 estimates with an exponential weight $e^{-\lambda\psi}$. Very roughly speaking, the principal normality and the pseudo-convexity can be viewed as a subelliptic condition on the operator $P_\lambda = e^{\lambda\psi} P e^{-\lambda\psi}$ and the proof of the estimates follows from Gårding type inequalities. Our problem here is that all our conditions are made on the set $\{\xi_a = 0\}$; this forces us to microlocalize our symbol on this set. This is the core of the proof which is achieved by the use of Sjöstrand's theory of FBI transform and pseudodifferential operators in the complex domain [S1], [S2]. (Although not very far in spirit, our method differs from Tataru's which uses real pseudodifferential weights). So making a partial FBI transformation (i.e. in the analytic variables only) we transfer our problem to the complex domain with the great advantage that, using the analyticity assumptions and several changes of contours, we can localize the symbol of the transferred operator around $\zeta_a = 0$, modulo some controlled errors (Theorems 3.1 and 3.3). As soon as this is achieved, we go back to the real domain and get a p.d.o with principal symbol localized near $\xi_a = 0$. We then use the C^∞ machinery (the Hörmander-Weyl calculus, the Fefferman-Phong inequality, see [H1], III, chap. 18, etc ...) to prove a Carleman estimate using some techniques of Lerner [L1]. The end of the proof is split according to whether N_a (the x_a component of the normal to the surface) vanishes or not. The case $N_a = 0$ is straightforward, while the case $N_a \neq 0$ requires use of the maximum principle according to an idea of Kashiwara (see also [S1]).

Finally we would like to thank Professors G. Lebeau and J. Sjöstrand for useful discussions during the preparation of this paper.

After the completion of the work, Professor L. Hörmander informed us that, using an extension of Tataru's method, he has very recently obtained the same results as described here (see [H4]).

2. Rewiew on Sjöstrand's theory

In this section we collect some material essentially taken from [S2], (see also [H2]).

2.1. The partial FBI transformation

Let n_a, n_b be two non negative integers with $n = n_a + n_b \geq 1$ and let us set $x = (x_a, x_b)$ if x is in $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$.

We introduce the partial Fourier-Bros-Iagolnitzer (FBI) transformation. It is defined for u in $\mathscr{S}(\mathbb{R}^n)$ by

$$Tu(z_a, x_b, \lambda) = K(\lambda) \int e^{-\frac{\lambda}{2}(z_a - y_a)^2} u(y_a, x_b)\, dy_a \tag{2.1}$$

where $z_a \in \mathbb{C}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $\lambda \geq 1$, $K(\lambda) = 2^{-\frac{n_a}{2}}\left(\frac{\lambda}{\pi}\right)^{\frac{3n_a}{4}}$ and $z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$.

Here are some properties of T which will be used later on. Let us first introduce

$$\Phi(z_a) = \frac{1}{2}(\mathrm{Im} z_a)^2, \quad z_a \in \mathbb{C}^{n_a} \ . \tag{2.2}$$

i) The function Tu is C^∞ on $\mathbb{R}^{2n_a} \times \mathbb{R}^{n_b} \times [1, +\infty[$ and entire-holomorphic in $z_a \in \mathbb{C}^{n_a}$ for all (x_b, λ) in $\mathbb{R}^{n_b} \times [1, +\infty[$.

Moreover for all M, N in $\mathbb{N}$, any α in $\mathbb{N}^{n_b}$ there exists $C = C_{N,M,\alpha} > 0$ such that

$$|D^\alpha_{x_b} Tu(z_a, x_b, \lambda)| \leq C K(\lambda) \langle x_b \rangle^{-M} \langle z_a \rangle^{-N}\, e^{\lambda \Phi(z_a)} \tag{2.3}$$

for all (z_a, x_b, λ) in $\mathbb{C}^{n_a} \times \mathbb{R}^{n_b} \times [1, +\infty[$. Here $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

ii) Conversely let $U(z_a, x_b, \lambda)$ be a C^∞ function on $\mathbb{R}^{2n_a} \times \mathbb{R}^{n_b} \times [1, +\infty[$ which is entire holomorphic in $z_a \in \mathbb{C}^{n_a}$ for all (x_b, λ) in $\mathbb{R}^{n_b} \times [1, +\infty[$ and assume that U satisfies estimates like (2.3). Then there exists a unique u in $\mathscr{S}(\mathbb{R}^n)$ such that $Tu = U$ (see [H2], prop. 6.1).

iii) Let now $(v_\varepsilon)_{\varepsilon\in]0,1]}$ be in $\mathscr{S}(\mathbb{R}^n)$ and $v \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\lim_{\varepsilon\to 0} v_\varepsilon = v \text{ in } \mathscr{S}'(\mathbb{R}^n) \text{ implies } \lim_{\varepsilon\to 0} \mathrm{e}^{-\lambda\Phi(z_a)} Tv_\varepsilon = \mathrm{e}^{-\lambda\Phi(z_a)} Tv \text{ in } \mathscr{S}'(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b}) \ . \tag{2.4}$$

iv) If u is in $C_0^\infty(\mathbb{R}^n)$ we can improve (2.3). Indeed, in that case, for all M, N in $\mathbb{N}$, any α in $\mathbb{N}^{n_b}$ there exists $C = C_{M,N,\alpha} > 0$ independent of u such that

$$\left|D^\alpha_{x_b} Tu(z_a, x_b, \lambda)\right| \le CK(\lambda)\langle x_b\rangle^{-M}\langle z_a\rangle^{-N}\,\mathrm{e}^{\lambda\Phi(z_a)-\frac{\lambda}{2}[d(\mathrm{Re}\, z_a,\mathrm{supp}\, u)]^2}\cdot \sup_{x_b} \|D^\alpha_{x_b} u(\cdot, x_b)\|_{H^N(\mathbb{R}^{n_a})} \tag{2.5}$$

for all z_a in $\mathbb{C}^{n_a}$, x_b in $\mathbb{R}^{n_b}$, $\lambda \ge 1$; here d is the Euclidian distance.
v) For fixed x_b, T can be viewed as a Fourier integral operator with associated (complex linear) canonical transformation

$$\kappa_T : \mathbb{C}^{2n_a} \ni (y_a, -\phi'_{y_a}(z_a, y_a)) \mapsto (z_a, \phi'_{z_a}(z_a, y_a)) \in \mathbb{C}^{2n_a}$$

where $\phi(z_a, y_a) = \frac{i}{2}(z_a - y_a)^2$.

Let us set

$$\Lambda_\Phi = \left\{(z_a, \xi_a) \in \mathbb{C}^{2n_a}\colon \xi_a = \frac{2}{i}\frac{\partial\Phi}{\partial z_a}(z_a)\right\} = \{(z_a, \xi_a) \in \mathbb{C}^{2n_a} : \xi_a = -\mathrm{Im}\, z_a\} \tag{2.6}$$

since $\Phi(z_a) = \frac{1}{2}(\mathrm{Im}\, z_a)^2$. Then $\kappa_T\colon T^*\mathbb{R}^{n_a} \to \Lambda_\Phi$ is a diffeomorphism. It is easy to see that

$$\kappa_T(x_a, \xi_a) = (x_a - i\xi_a, \xi_a) \ . \tag{2.7}$$

vi) In the sequel we shall also work with the partial FBI transformation T_η associated with the phase $\phi(z_a, y_a) = \frac{i}{2}(1+\eta)(z_a - y_a)^2$ where η is a small non negative real number. In that case we have

$$\kappa_{T_\eta}(x_a, \xi_a) = \left(x_a - \frac{i}{1+\eta}\xi_a, \xi_a\right) \ . \tag{2.8}$$

Let us introduce some notations. For $k \in \mathbb{N}$ we set

$$L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^k(\mathbb{R}^{n_b})) = L^2\Big(\Big(\mathbb{C}^{n_a},\, \mathrm{e}^{-2\lambda(1+\eta)\Phi(x_a)}L(dx_a)\Big), \quad H^k(\mathbb{R}^{n_b})\Big) \tag{2.9}$$

which is the space of square integrable functions defined on $\mathbb{C}^{n_a}$ equipped with the measure $\mathrm{e}^{-2\lambda(1+\eta)\,\Phi(x_a)}L(dx_a)$ (where $L(dx_a)$ denotes the Lebesgue measure in $\mathbb{C}^{n_a}$) and valued in $H^k(\mathbb{R}^{n_b})$ (the usual Sobolev space).

If $k = 0$ we shall set for short

$$L^2_{(1+\eta)\Phi}\big(\mathbb{C}^{n_a}, H^0(\mathbb{R}^{n_b})\big) = L^2_{(1+\eta)\Phi} \quad . \tag{2.10}$$

We also set

$$\mathscr{L}^2_{(1+\eta)\Phi} = L^2_{(1+\eta)\Phi} \cap \mathscr{H}(\mathbb{C}^{n_a}) \tag{2.11}$$

where $\mathscr{H}$ denotes the space of holomorphic functions.

Then we have:

Proposition 2.1. [S2]. i) *T_η is an isometry from $L^2(\mathbb{R}^{n_a}, H^k(\mathbb{R}^{n_b}))$ to $L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^k(\mathbb{R}^{n_b}))$.*

ii) *$T^*_\eta T_\eta$ is the identity on $L^2(\mathbb{R}^n)$, where T^*_η is the adjoint of T_η.*

iii) *$T_\eta T^*_\eta$ is the projection from $L^2_{(1+\eta)\Phi}$ to $\mathscr{L}^2_{(1+\eta)\Phi}$. In particular $T_\eta T^*_\eta \tilde{v} = \tilde{v}$ if $\tilde{v} = Tv$ where v is in $\mathscr{S}(\mathbb{R}^n)$.*

2.2. Transfer to the complex domain

Let $p = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha$, $(x, \xi) \in \mathbb{R}^{2n}$, be a polynomial with coefficients in $C_0^\infty(\mathbb{R}^n)$. Assume moreover that

$$\begin{cases} \text{there exists } c_0 > 0 \text{ such that if we set } \omega_a = \{z_a \in \mathbb{C}^{n_a} : |z_a| < c_0\} \\ \text{and } \omega_b = \{x_b \in \mathbb{R}^{n_b} : |x_b| < c_0\} \text{ then for all } \alpha \text{ in } \mathbb{N}^n, |\alpha| \le m, \\ \text{we have } a_\alpha \in C^\infty(\omega_b, \mathscr{H}(\omega_a)) \text{ where } \mathscr{H} \text{ denotes the space} \\ \text{of holomorphic functions.} \end{cases} \tag{2.12}$$

Let $P = \mathrm{Op}^w_\lambda(p)$ be the semi-classical Weyl quantized operator with symbol p, which means that, for $u \in C_0^\infty(\mathbb{R}^n)$, we have in the oscillatory sense

$$Pu(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \lambda\xi\right) u(y)\, dy\, d\xi \quad . \tag{2.13}$$

Let ψ be a real quadratic polynomial on $\mathbb{R}^n$. For any $\lambda \ge 1$ we shall denote by P_λ the differential operator defined by

$$P_\lambda = e^{\lambda\psi}\, P\, e^{-\lambda\psi} \quad . \tag{2.14}$$

It follows from Segal formula (see [H1]) that

$$P_\lambda v(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \lambda\xi + i\lambda\psi'\left(\frac{x+y}{2}\right)\right) u(y)\, dy\, d\xi \quad . \tag{2.15}$$

The main result of this section, which will follow from proposition 1.4 in [S2], is the following:

Proposition 2.2. *For v in $C_0^\infty(\mathbb{R}^n)$ we have $T P_\lambda v = \tilde{P}_\lambda\, Tv$ where*

$$\tilde{P}_\lambda Tv(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\xi_a=-\operatorname{Im}\frac{x_a+y_a}{2}} \omega\right) dy_b\, d\xi_b$$

where

$$\omega = e^{i\lambda(x_a-y_a)\xi_a} p\Bigg(\frac{x_a+y_a}{2}+i\xi_a, \frac{x_b+y_b}{2}, \lambda\xi + i\lambda\psi'\Big(\frac{x_a+y_a}{2}+i\xi_a, \frac{x_b+y_b}{2}\Big)\Bigg) Tv(y_a,y_b,\lambda)\, dy_a \wedge d\xi_a \tag{2.16}$$

and the above integral has to be taken in the oscillatory sense i.e.

$$\tilde{P}_\lambda Tv(x,\lambda) = \lim_{\varepsilon\to 0}\left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\chi(\varepsilon\xi_b)\left(\iint_{\xi_a=-\operatorname{Im}\frac{x_a+y_a}{2}} \omega\right) dy_b\, d\xi_b$$

in $\mathcal{S}'(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})$, where $\chi\in\mathcal{S}(\mathbb{R}^{n_b})$ and $\chi(0)=1$.

Proof. From (2.15) we have $P_\lambda v = \lim P_{\lambda,\varepsilon} v$ in $\mathcal{S}'(\mathbb{R}^n)$, where

$$P_{\lambda,\varepsilon}v(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi}\chi_1(\varepsilon\xi_a)\chi_2(\varepsilon\xi_b) \cdot p\Big(\frac{x+y}{2}, \lambda\xi + i\lambda\psi'\Big(\frac{x+y}{2}\Big)\Big) u(y)\, dy\, d\xi \tag{2.17}$$

where $\chi_j\in\mathcal{S}$, $\chi_j(0)=1$. It follows from (2.4) that

$$e^{-\lambda\Phi(z_a)} T P_\lambda v = \lim_{\varepsilon\to 0} e^{-\lambda\Phi(z_a)} T P_{\lambda,\varepsilon} v \quad \text{in } \mathcal{S}'(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b}) . \tag{2.18}$$

Now

$$T P_{\lambda,\varepsilon}v(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n_b} \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\chi_2(\varepsilon\xi_b) T\Bigg(\left(\frac{\lambda}{2\pi}\right)^{n_a}\iint e^{i\lambda(x_a-y_a)\cdot\xi_a}\chi_1(\varepsilon\xi_a) \cdot p\Big(\frac{x+y}{2}, \lambda\xi + i\lambda\psi'\Big(\frac{x+y}{2}\Big)\Big) v(y)\, dy_a\, d\xi_a\Bigg) dy_b\, d\xi_b \;.$$

Since, for fixed (x_b,y_b,ξ_b), the symbol $(x_a,\xi_a)\mapsto \chi_1(\varepsilon\xi_a)p(x_a,x_b,\lambda\xi+i\lambda\psi')$ belongs to $\mathcal{S}(\mathbb{R}^{2n_a})$ we can apply Proposition 1.4 in [S2]. It follows that

$$T\left(\left(\frac{\lambda}{2\pi}\right)^{n_a}\iint \dots\, dy_a\, d\xi_a\right) = \left(\frac{\lambda}{2\pi}\right)^{n_a}\iint_{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}} \chi_1(\varepsilon\xi_a)\,\omega$$

where ω is defined in (2.16).

We shall show that, in $\mathscr{S}'(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})$, we have

$$\begin{aligned}(2.19)\qquad &\lim_{\varepsilon\to 0} \mathrm{e}^{-\lambda\Phi}\, T\, P_{\lambda,\varepsilon} v\\ &= \mathrm{e}^{-\lambda\Phi}\lim_{\varepsilon\to 0}\left(\frac{\lambda}{2\pi}\right)^{n}\iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\chi_2(\varepsilon\xi_b)\\ &\quad\cdot\left(\iint_{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}}\omega\right) dy_b\, d\xi_b\ .\end{aligned}$$

According to (2.18) this will prove Proposition 2.2. Let us set

$$R_\varepsilon = \mathrm{e}^{-\lambda\Phi(x_a)}\iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\chi_2(\varepsilon\xi_b)\left(\iint_{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}}(1-\chi_1(\varepsilon\xi_a))\,\omega\right) dy_b\, d\xi_b$$
(2.20)

$$(2.21)\qquad S_\varepsilon = \iint R_\varepsilon(x_a,x_b,\lambda)\,\varphi(x_a,x_b)\, L(dx_a)\, dx_b$$

where $\varphi\in\mathscr{S}$ and $L(dx_a)$ is the Lebesgue measure on $\mathbb{C}^{n_a}$.

For fixed $(x_a,x_b,y_a,y_b,\xi_a,\xi_b,\lambda)$ the integrand in the right hand side of (2.20) tends to zero when ε goes to zero. Now since p is a polynomial in ξ, R_ε is a finite sum of terms of type (2.20) where, in ω, p is replaced by $a\left(\frac{x_a+y_a}{2}+i\xi_a,\frac{x_b+y_b}{2}\right)\cdot(\lambda\xi_a+i\lambda\psi'_a)^\alpha(\lambda\xi_b)^\beta$, where $a\in C_0^\infty(\mathbb{R}^n)$ and $|\alpha|+|\beta|\le m$. Since $(\lambda\xi_b)^\beta\,\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}=(-D_{y_b})^\beta\,\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}$ and since by (2.3) $Tv(y_a,y_b,\lambda)$ is in $\mathscr{S}(\mathbb{R}^{n_b})$ we can integrate by parts in y_b. We then use the equality $\int \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,\chi_2(\varepsilon\xi_b)\, d\xi_b = \varepsilon^{-n_b}\hat{\chi}_2\left(\lambda\frac{x_b-y_b}{\varepsilon}\right)$ and we deduce that R_ε is a finite sum of terms of the following kind

$$\begin{aligned}&\int \varepsilon^{-n_b}\hat{\chi}_2\left(\lambda\frac{x_b-y_b}{\varepsilon}\right) D^\beta_{y_b}\left(\iint_{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}} \mathrm{e}^{-\lambda\Phi(x_a)}\,\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\right.\\ (2.22)\qquad &(1-\chi_1(\varepsilon\xi_a))\quad\cdot a\left(\frac{x_a+y_a}{2}+i\xi_a,\frac{x_b+y_b}{2}\right)\\ &\left.(\lambda\xi_a+i\lambda\psi'_a)^\alpha\, Tv(y_a,y_b,\lambda)\, dy_a\wedge d\xi_a\right) dy_b\, d\xi_b\ .\end{aligned}$$

Now on the surface $\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}$ we have $dy_a\wedge d\xi_a = c_{n_a}L(dy_a)$, where $c_{n_a}\in\mathbb{C}$ and $L(dy_a)$ is the Lebesgue measure on $\mathbb{C}^{n_a}$, and $i(x_a-y_a)\cdot\xi_a=\Phi(x_a)-\Phi(y_a)$. It follows that the integral with respect to (y_a,ξ_a) in (2.22) is equal to

$$c_{n_a}\int\Big(1-\chi_1\Big(-\frac{\varepsilon}{2}\,\mathrm{Im}\,(x_a+y_a)\Big)\Big)\,a\Big(\mathrm{Re}\;\frac{x_a+y_a}{2},\frac{x_b+y_b}{2}\Big)$$
$$\cdot\Big(-\lambda\,\mathrm{Im}\;\frac{x_a+y_a}{2}+i\lambda\psi'_a\Big)^{\alpha}\,Tv(y_a,y_b,\lambda)\,\mathrm{e}^{-\lambda\Phi(y_a)}L(dy_a)\;\;.$$

Now (2.3) shows that we can differentiate this integral with respect to y_b under the sign integral. It follows from (2.22) that R_ε is a finite sum of terms of the following kind

$$\begin{aligned}&\iint \varepsilon^{-n_b}\hat{\chi}_2\Big(\lambda\frac{x_b-y_b}{\varepsilon}\Big)\Big(1-\chi_1\Big(-\frac{\varepsilon}{2}\;\mathrm{Im}(x_a+y_a)\Big)\Big)\\ &D^{\beta_1}_{y_b}\,a\Big(\mathrm{Re}\frac{x_a+y_a}{2},\frac{x_b+y_b}{2}\Big)\cdot\Big(-\frac{\lambda}{2}\;\mathrm{Im}(x_a+y_a)+i\lambda\psi'_a\Big)^{\alpha}\\ &\cdot\mathrm{e}^{-\lambda\Phi(y_a)}\,D^{\beta_2}_{y_b}Tv(y_a,y_b,\lambda)L(dy_a)\,dy_b\;\;.\end{aligned}\tag{2.23}$$

Setting $x_b-y_b=\varepsilon z_b$, using (2.3) and Lebesgue's theorem in (2.23) we deduce that for fixed (x_a,x_b,λ) in $\mathbb{C}^{n_a}\times\mathbb{R}^{n_b}\times[1,+\infty[$, $R_\varepsilon(x_a,x_b,\lambda)$ tends to zero with ε. Moreover this also shows that there exists $p,q\in\mathbb{N}$ such that for any $N\in\mathbb{N}$

$$\begin{aligned}|R_\varepsilon(x_a,x_b,\lambda)|&\le C_N(\lambda)\langle x_a\rangle^p\iint|\hat{\chi}_2(\lambda z_b)|\langle y_a\rangle^q\langle y_a\rangle^{-N}L(dy_a)\,dz_b\\ &\le C'_N(\lambda)\langle x_a\rangle^p\;\;.\end{aligned}$$

This implies that S_ε, which is defined by (2.21), tends to zero. This proves (2.19) and Proposition 2.2.

3. The localization procedure

In this section d is a positive real number such that $13d<c_0$, where c_0 is defined in (2.12), and v is a C^∞ function such that supp $v\subset C\{x\in\mathbb{R}^n\colon |x|\le d\}$. Let $\tilde{P}_\lambda$ be defined in Proposition 2.2.

3.1. Case of Theorem A

Theorem 3.1. *There exists* $\chi\in C_0^\infty(\mathbb{C}^{2n_a})$, $\chi(x_a,\xi_a)=1$ *if* $|x_a|+|\xi_a|\le 12d$, $\chi(x_a,\xi_a)=0$ *if* $|x_a|+|\xi_a|\ge 13d$ *such that if we set, for* $\eta\in]0,1]$,

$$\begin{aligned}&\tilde{Q}_\lambda\,Tv(x,\lambda)\\ &\quad=\Big(\frac{\lambda}{2\pi}\Big)^n\iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\xi_a=-(1+\eta)\mathrm{Im}\,\frac{x_a+y_a}{2}}\chi\Big(\frac{x_a+y_a}{2},\xi_a\Big)\,\omega\Big)\,dy_b\,d\xi_b\end{aligned}\tag{3.1}$$

where ω is defined in (2.16), *then*

$$\tilde{P}_\lambda Tv = \tilde{Q}_\lambda\, Tv + \tilde{R}_\lambda\, Tv + \tilde{g}_\lambda \tag{3.2}$$

with, for any N in $\mathbb{N}$,

$$||\tilde{R}_\lambda\, Tv||_{L^2_{(1+\eta)\Phi}} \leq \frac{C_N}{\lambda^N}\, ||Tv||_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))}\ , \tag{3.3}$$

$$||\tilde{g}_\lambda||_{L^2_{(1+\eta)\Phi}} = \mathcal{O}\big(e^{-\frac{\lambda}{3}\eta d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\,,\ \ \lambda \to +\infty\ . \tag{3.4}$$

where n_0 depends only on n and on the order m of P.

Proof. This proof requires several steps. Let us recall for convenience that

$$\begin{aligned}\omega = {} & e^{i\lambda(x_a - y_a)\cdot\xi_a}\, p\Big(\frac{x_a+y_a}{2} + i\xi_a, \frac{x_b+y_b}{2}\,, \\ & \lambda\xi + i\lambda\psi'\Big(\frac{x_a+y_a}{2} + i\xi_a, \frac{x_b+y_b}{2}\Big)\Big)\cdot Tv(y_a, y_b, \lambda)\, dy_a \wedge d\xi_a\end{aligned} \tag{3.5}$$

where $\xi = (\xi_a, \xi_b)$.

Step 1. Let us fix $(x_b, y_b, \xi_b, \lambda)$. Then we have

$$\iint_{\xi_a = -\mathrm{Im}\,\frac{x_a+y_a}{2}} \omega = \iint_{\xi_a = -\mathrm{Im}\,\frac{x_a+y_a}{2} + i\,\mathrm{Re}(x_a - y_a)} \omega\ . \tag{3.6}$$

To prove (3.6) we shall apply Stokes formula to the closed manifold $t \in [0,1]$, $y_a \in \mathbb{C}^{n_a}$, $\xi_a = -\mathrm{Im}\ (x_a + y_a)/2 + it\,\mathrm{Re}(x_a - y_a)$. On this manifold $(x_a + y_a)/2 + i\xi_a = \mathrm{Re}(x_a + y_a)/2 - t\,\mathrm{Re}(x_a - y_a) \in \mathbb{R}^{n_a}$. Therefore ω is well defined. Then (3.6) will follow from

$$\int_0^1 \iint_{\xi_a = -\mathrm{Im}\,\frac{x_a+y_a}{2} + it\,\mathrm{Re}(x_a - y_a)} d\omega = 0\ . \tag{3.7}$$

Since $\partial_{(y_a,\xi_a)}\omega = 0$ we have $d\omega = \overline{\partial}_{(y_a,\xi_a)}\omega$. Now $e^{i\lambda(x_a-y_a)\cdot\xi_a}\cdot Tv(y_a, y_b, \lambda)$ and $\psi'\big((x_a + y_a)/2 + i\xi_a,\ (x_b + y_b)/2\big)$ are holomorphic in (y_a, ξ_a) and since $(x_a + y_a)/2 + i\xi_a$ is real on our manifold we have

$$\begin{aligned}p(\dots) = p\Big(&\frac{1}{2}\Big[\frac{x_a+y_a}{2} + i\xi_a + \frac{\overline{x}_a + \overline{y}_a}{2} - i\,\overline{\xi}_a\Big], \\ &\frac{x_b+y_b}{2}\ , \lambda\xi + i\lambda\psi'\Big(\frac{x_a+y_a}{2} + i\xi_a, \frac{x_b+y_b}{2}\Big)\Big)\ .\end{aligned}$$

It follows that

$$\overline{\partial}_{(y_a,\xi_a)}\omega = \mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\, Tv(y_a,y_b,\lambda)\left(\frac{1}{4}\left(\frac{\partial p}{\partial x_a}\cdot d\overline{y}_a\right)\wedge dy_a\wedge d\xi_a - \frac{i}{2}\left(\frac{\partial p}{\partial x_a}\cdot d\overline{\xi}_a\right)\wedge dy_a\wedge d\xi_a\right) .$$

Now $\xi_a+\overline{\xi}_a = 2\,\mathrm{Re}\,\xi_a = -\frac{1}{2i}\,(y_a-\overline{y}_a)-\frac{1}{2i}\,(x_a-\overline{x}_a)$; therefore we have $d\xi_a+d\overline{\xi}_a = -\frac{1}{2i}\,(dy_a-d\overline{y}_a)$ i.e. $d\overline{\xi}_a = -d\xi_a-\frac{1}{2i}\,(dy_a-d\overline{y}_a)$. It follows that $\frac{i}{2}\left(\frac{\partial p}{\partial x_a}\cdot d\overline{\xi}_a\right)\wedge dy_a\wedge d\xi_a = \frac{1}{4}\left(\frac{\partial p}{\partial x_a}\cdot d\overline{y}_a\right)\wedge dy_a\wedge d\xi_a$ so $\overline{\partial}_{(y_a,\xi_a)}\omega = d\omega = 0$. This implies (3.7).

Let us set

$$(3.8)\quad \tilde{g}_1(x_a,x_b,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\substack{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}+i\,\mathrm{Re}(x_a-y_a)\\ |\mathrm{Re}(x_a-y_a)|\geq d}}\omega\right) dy_b\, d\xi_b .$$

Our purpose is to show that

$$(3.9)\qquad \left|\left|\mathrm{e}^{-\lambda(1+\eta)\Phi}\,\tilde{g}_1\right|\right|_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{2}d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\right) .$$

We proceed as in Proposition 2.2; $\tilde{g}_1$ is a limit as ε goes to zero of a finite sum of terms of the following kind

$$\left(\frac{\lambda}{2\pi}\right)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,\chi(\varepsilon\xi_b)\left(\iint_{\substack{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}+i\,\mathrm{Re}(x_a-y_a)\\ |\mathrm{Re}(x_a-y_a)|\geq d}}\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\,a\left(\frac{x_a+y_a}{2}+i\xi_a,\frac{x_b+y_b}{2}\right)\cdot(\lambda\xi_a+i\lambda\psi'_a)^\alpha\,(\lambda\xi_b)^\beta\,Tv(y_a,y_b,\lambda)\,dy_a\wedge d\xi_a\right) dy_b\,d\xi_b ,$$

where $a\in C_0^\infty(\mathbb{R}^n)$ and $|\alpha|+|\beta|\leq m$.

We write $(\lambda\xi_b)^\beta\,\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b} = (-D_{y_b})^\beta\,\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}$, we integrate by parts in the y_b integral, we use the equality $\int \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,\chi(\varepsilon\xi_b)\,d\xi_b = \varepsilon^{-n_b}\,\hat{\chi}\left(\lambda\,\frac{x_b-y_b}{\varepsilon}\right)$, we set $x_b-y_b=\varepsilon z_b$, we let ε go to zero and we deduce that $\tilde{g}_1$ is a finite sum of terms of the following kind

$$\left(\frac{\lambda}{2\pi}\right)^{n_a}\iint_{\substack{\xi_a=-\mathrm{Im}\,\frac{x_a+y_a}{2}+i\,\mathrm{Re}(x_a-y_a)\\ |\mathrm{Re}(x_a-y_a)|\geq d}}\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\,D_{x_b}^{\beta_1}\,a\left(\frac{x_a+y_a}{2}+i\xi_a,x_b\right)\cdot(\lambda\xi_a+i\lambda\psi'_a)^\alpha\,D_{x_b}^{\beta_2}\,Tv(y_a,x_b,\lambda)\cdot dy_a\wedge d\xi_a .$$

Then we use the following facts: on our contour we have

i) $\mathrm{Re}(i(x_a - y_a)\cdot\xi_a) = \Phi(x_a) - \Phi(y_a) - |\mathrm{Re}(x_a - y_a)|^2$ and $|\mathrm{Re}(x_a - y_a)| \geq d$,

ii) $\left| D^{\beta_1}_{x_b}\, a\left(\frac{x_a+y_a}{2} + i\xi_a, x_b\right)(\lambda\xi_a + i\lambda\psi'_a)^\alpha \right| \leq C\,\lambda^m(\langle x_a\rangle + \langle y_a\rangle + \langle x_b\rangle)^m$,

iii) $\left|\mathrm{e}^{-\lambda\Phi(y_a)}\, D^{\beta_2}_{x_b}\, Tv(y_a, x_b, \lambda)\right| \leq C_{M,N}\, K(\lambda)\, \langle x_b\rangle^{-M} \langle y_a\rangle^{-N} ||v||_{H^{n_0}(\mathbb{R}^n)}$, for all M, N in $\mathbb{N}$,

iv) $\langle x_a\rangle \leq \langle \mathrm{Re}(x_a - y_a)\rangle + \langle y_a\rangle + \langle \mathrm{Im}\, x_a\rangle$,

v) $dy_a \wedge d\xi_a = C_n\, L(dy_a)$.

It follows that we can find a constant C depending only on m, n, d and η such that

$$\langle x_a\rangle^{n_a+1}\, \langle x_b\rangle^{\frac{1}{2}(n_b+1)}\, \mathrm{e}^{-\lambda(1+\eta)\Phi(x_a)}\, |\tilde{g}_1| \leq C\, \mathrm{e}^{-\frac{\lambda}{2}d^2} ||v||_{H^{n_0}(\mathbb{R}^n)} \ .$$

This implies (3.9). Now it follows from (3.6), (3.8) and (3.9) that

$$(3.10)\quad \tilde{P}_\lambda\, Tv(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint \mathrm{e}^{i\lambda(x_b - y_b)\cdot\xi_b} \left(\iint_{\substack{\xi_a = -\mathrm{Im}\,\frac{x_a+y_a}{2} + i\,\mathrm{Re}(x_a - y_a) \\ |\mathrm{Re}(x_a - y_a)| \leq d}} \omega \right) dy_b\, d\xi_b + \tilde{g}_1$$

where $\tilde{g}_1$ satisfies (3.9).

Step 2. We want to prove

$$(3.11)\qquad \tilde{P}_\lambda\, Tv(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint \mathrm{e}^{i\lambda(x_b - y_b)\cdot\xi_b} \left(\iint_\Sigma \omega \right) dy_b\, d\xi_b + \tilde{g}_2$$

where $\Sigma = \left\{ y_a \in \mathbb{C}^{n_a},\ |\mathrm{Re}(x_a - y_a)| \leq d,\ |\mathrm{Re}\, y_a| \leq 2d,\ \xi_a = -\mathrm{Im}\ \frac{x_a+y_a}{2} + i\,\mathrm{Re}(x_a - y_a) \right\}$

$$(3.12)\qquad \left|\left|\mathrm{e}^{-\lambda(1+\eta)\Phi}\, \tilde{g}_2\right|\right|_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{3}d^2} ||v||_{H^{n_0}(\mathbb{R}^n)}\right) \ .$$

This will be proved if we show that the part, in the right hand side of (3.10), where $|\mathrm{Re}\, y_a| \geq 2d$ satisfies (3.12). This part is as before a finite sum of terms of the following type

$$\tilde{g} = \left(\frac{\lambda}{2\pi}\right)^{n_a} \iint_\Sigma \mathrm{e}^{i\lambda(x_a - y_a)\cdot\xi_a}\, D^{\beta_1}_{x_b}\, a\left(\frac{x_a + y_a}{2} + i\xi_a, x_b\right) (\lambda\xi_a + i\lambda\psi'_a)^\alpha\, D^{\beta_2}_{x_b} Tv(y_a, x_b, \lambda)\, dy_a \wedge d\xi_a \ .$$

We then use (2.5). Since $\mathrm{supp}\, v \subset \{|x| \leq d\}$ and $|\mathrm{Re}\, y_a| \geq 2d$ it follows that dist $(\mathrm{Re}\, y_a, \mathrm{supp}\ v) \geq d$. We also use the remarks i) to v) above and we deduce easily that

$$\langle x_a\rangle^{n_a+1}\, \langle x_b\rangle^{\frac{1}{2}(n_b+1)}\, \mathrm{e}^{-\lambda(1+\eta)\Phi(x_a)}\, |\tilde{g}| \leq C\, \mathrm{e}^{-\frac{\lambda}{3}d^2} ||v||_{H^{n_0}(\mathbb{R}^n)}$$

from which (3.12) follows.

Step 3. Our purpose is now to localize in $\operatorname{Im} y_a$.

Let t_0 be in $]0,1[$ and let us consider the manifold with boundary $G=[t_0,1]\times\Sigma_t$ where on Σ_t we have $y_a\in\mathbb{C}^{n_a}$, $|\operatorname{Re}(x_a-y_a)|\le d$, $|\operatorname{Re} y_a|\le 2d$, $|\operatorname{Im}(x_a-y_a)|\le\frac{d}{t}$ and $\xi_a=-\operatorname{Im}\frac{x_a+y_a}{2}+i\operatorname{Re}(x_a-y_a)+t\operatorname{Im}(x_a-y_a)$. On G we have

$$\left|\frac{x_a+y_a}{2}+i\xi_a\right|=\left|\operatorname{Re}\frac{x_a+y_a}{2}-\operatorname{Re}(x_a-y_a)+it\operatorname{Im}(x_a-y_a)\right|\le\frac{9}{2}d\ .$$

Since $\frac{9}{2}d<c_0$ we are, by (2.12), on a domain where the coefficients of p are holomorphic. We can apply Stokes formula to the differential form defined in (3.5) and we have $d\omega=0$. The difference between $\iint_{\Sigma_{t_0}}\omega$ and $\iint_{\Sigma_1}\omega$ consists then in boundary terms and we show now that each of them gives an exponentially decreasing contribution in the expression of $\tilde{P}\,Tv$ in (3.11).

i) $$|\operatorname{Re}(x_a-y_a)|=d,\quad |\operatorname{Re} y_a|\le 2d,\quad |\operatorname{Im}(x_a-y_a)|\le\frac{d}{t}\ ,$$

We use the same argument as in the proof of (3.9) in step 1. Indeed we just have an extra term in $|e^{i\lambda(x_a-y_a)\cdot\xi_a}|$ namely $e^{-\lambda t\,|\operatorname{Im}(x_a-y_a)|^2}$ which is bounded by one. Therefore the corresponding term satisfies an estimate like (3.12).

ii) $$|\operatorname{Re}(x_a-y_a)|\le d,\quad |\operatorname{Re} y_a|=2d,\quad |\operatorname{Im}(x_a-y_a)|\le\frac{d}{t}\ ,$$

The corresponding term can be handled exactly as in step 2.

iii) $$|\operatorname{Re}(x_a-y_a)|\le d,\quad |\operatorname{Re} y_a|\le 2d,\quad |\operatorname{Im}(x_a-y_a)|=\frac{d}{t}\ ,$$

In that case we have

$$\operatorname{Re}(i\lambda(x_a-y_a)\,\xi_a)=\lambda(\Phi(x_a)-\Phi(y_a))-\lambda\,|\operatorname{Re}(x_a-y_a)|^2-\lambda\,\frac{d^2}{t}\ .$$

Now $\langle x_a\rangle\le\langle\operatorname{Im} x_a\rangle+\langle\operatorname{Re}(x_a-y_a)\rangle+\langle y_a\rangle\le\langle\operatorname{Im} x_a\rangle+\langle y_a\rangle+\langle d\rangle$ and $e^{-\lambda\frac{d^2}{t}}\le e^{-\lambda d^2}$ since $t<1$. Therefore the corresponding term is also exponentially decreasing.

Summing up we have proved that

$$\iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\Sigma_{t_0}}\omega\right)dy_b\,d\xi_b=\iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\Sigma_1}\omega\right)dy_b\,d\xi_b+\tilde{g}_3$$

(3.13)

(3.14) $$\left\|e^{-\lambda(1+\eta)\Phi}\,\tilde{g}_3\right\|_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})}=\mathcal{O}\left(e^{-\frac{\lambda}{2}d^2}\|v\|_{H^{n_0}(\mathbb{R}^n)}\right)$$

where the above $\mathcal{O}$ is independant of t_0.

We want to prove now that when t_0 goes to zero the left hand side of (3.13) converges to $\iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_\Sigma \omega\right) dy_b\, d\xi_b$ where Σ is defined in (3.11). As usual this term is a finite sum of terms of the following kind

$$\iint_{\Sigma_{t_0}} e^{i\lambda(x_a-y_a)\cdot\xi_a} D_{x_b}^{\beta_1} a\Big(\frac{x_a+y_a}{2}+i\xi_a, x_b\Big)(\lambda\xi_a+i\lambda\psi'_a)^\alpha \cdot D_{x_b}^{\beta_2} Tu(y_a,x_b,\lambda)\, dy_a\wedge d\xi_a \ .$$

This integral can be written as

$$\int_{\substack{|\mathrm{Re}(x_a-y_a)|\le d\\ |\mathrm{Re} y_a|\le 2d}} e^{\lambda(\Phi(x_a)-\Phi(y_a)-|\mathrm{Re}(x_a-y_a)|^2-t_0\,\mathrm{Im}(x_a-y_a)|^2)}\,\mathbf{1}_{\{|\mathrm{Im}(x_a-y_a)|\le\frac{d}{t_0}\}}$$
$$\cdot D_{x_b}^{\beta_1} a\Big(\frac{x_a+y_a}{2}+i\xi_a,x_b\Big)(\lambda\xi_a+i\lambda\psi'_a)^\alpha D_{x_b}^{\beta_2} Tu(y_a,x_b,\lambda)\, C_n(t_0)\, L(dy_a)$$

where $\xi_a=-\mathrm{Im}\ \frac{x_a+y_a}{2}+i\,\mathrm{Re}(x_a-y_a)+t_0\,\mathrm{Im}(x_a-y_a)$, $\mathbf{1}_\Omega$ denotes the characteristic function of Ω and $C_n(t_0)$ converges to $C_n(0)$ as to goes to zero. Using (2.3) we can apply Lebesgue's theorem to reach the conclusion.

According to (3.13), (3.14), (3.11) and (3.12) it follows that

$$\text{(3.15)}\qquad \tilde{P}_\lambda\, Tv=\Big(\frac{\lambda}{2\pi}\Big)^n\iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\Sigma_1}\omega\Big)\, dy_b\, d\xi_b+\tilde{g}_4$$

where $\Sigma_1=\{y_a\in\mathbb{C}^{b_a}, \underline{|\mathrm{Re}(x_a-y_a)|}\le d,\ |\mathrm{Re} y_a|\le 2d,\ |\mathrm{Im}(x_a-y_a)|\le d$ and $\xi_a=-\mathrm{Im}\ \frac{x_a+y_a}{2}+i\,\overline{(x_a-y_a)}\}$ and

$$\text{(3.16)}\qquad \big|\big|e^{-\lambda(1+\eta)\Phi}\,\tilde{g}_4\big|\big|_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})}=\mathcal{O}\big(e^{-\frac{\lambda}{3}d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\ .$$

This will allow us to localize in Im y_a. Indeed let us consider the part of Σ_1 where $|\mathrm{Im}\, y_a|\ge 2d$. We shall show that its contribution in $\tilde{P}_\lambda\, Tv$ satisfies (3.16). To see that it is enough to consider the following term

$$\tilde{g}_{\alpha\beta\gamma}=\iint_{\Sigma_2} e^{i\lambda(x_a-y_a)\cdot\xi_a} D_{x_b}^\alpha a\Big(\frac{x_a+y_a}{2}+i\xi_a,x_b\Big) \cdot(\lambda\xi_a+i\lambda\psi'_a)^\beta D_{x_b}^\gamma Tu(y_a,x_b,\lambda)\cdot dy_a\wedge d\xi_a$$

where $\Sigma_2=\Sigma_1\cap\{|\mathrm{Im}\, y_a|\ge 2d\}$.

Since in Σ_1 we have $|\mathrm{Im}(x_a-y_a)|\le d$ we get $|\mathrm{Im}\, x_a|\ge d$. On the other hand, $\langle x_a\rangle\le\langle x_a-y_a\rangle+\langle y_a\rangle\le C\langle d\rangle+\langle y_a\rangle$ and $\mathrm{Re}(i(x_a-y_a)\cdot\xi_a)=\Phi(x_a)-\Phi(y_a)-|x_a-y_a|^2\le\Phi(x_a)-\Phi(y_a)$. Therefore using (2.3) we get

$$e^{-\lambda(1+\eta)\Phi(x_a)}\,\langle x_a\rangle^{n_a+1}\,\langle x_b\rangle^{\frac12(n_b+1)}\,|\tilde{g}_{\alpha\beta\gamma}|\le C\,\lambda^m\, e^{-\lambda\eta d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\ .$$

It follows then from (3.15), (3.16) that

$$\tilde{P}_\lambda\, Tv = \Big(\frac{\lambda}{2\pi}\Big)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_\Sigma \omega\Big)\, dy_b\, d\xi_b + \tilde{g}_5 \tag{3.17}$$

where $\Sigma = \big\{y_a \in \mathbb{C}^n,\ |\mathrm{Re}\ \ (x_a - y_a)| \le d,\ |\mathrm{Im}\ (x_a-y_a)| \le d,\ |\mathrm{Re}\ \ y_a| \le 2d,$ $|\mathrm{Im}\ y_a| \le 2d,\ \xi_a = -\mathrm{Im}\ \ \frac{x_a+y_a}{2} + i\,\overline{(x_a-y_a)}\big\}$ and

$$||\mathrm{e}^{-\lambda(1+\eta)\Phi}\ \tilde{g}_5||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} = \mathcal{O}\big(\mathrm{e}^{-\frac{\lambda}{3}\eta d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\ . \tag{3.18}$$

Step 4. Our goal is to write $\tilde{P}_\lambda$ in term of the contour

$$\Sigma_\eta = \Big\{y_a \in \mathbb{C}^{n_a}\ ,|\mathrm{Re}(x_a-y_a)| \le d\,,|\mathrm{Im}\ (x_a-y_a)| \le d\,,|\mathrm{Re}\ y_a| \le 2d\,, \\ |\mathrm{Im} y_a| \le 2d, \xi_a = -(1+\eta)\mathrm{Im}\ \ \frac{x_a+y_a}{2} + i\,\overline{(x_a-y_a)}\Big\}\ . \tag{3.19}$$

For that purpose we introduce for t in $[0,1]$ the contour $\Sigma_{t\eta}$ which is defined by (3.19) with $t\eta$ instead of η. Along these contours we have

$$\Big|\frac{x_a+y_a}{2} + i\xi_a\Big| = \Big|\mathrm{Re}\ \frac{x_a+y_a}{2} - t\eta\,\mathrm{Im}\ \ \frac{x_a+y_a}{2} - \overline{(x_a-y_a)}\Big| \le 7d\ \ .$$

Since $7d < c_0$ we are still on a domain where the coefficients of p are holomorphic. When $t=0$ we find the contour Σ defined in (3.17) and for $t=1$ we find the contour Σ_η. We apply Stokes formula to the differential form ω and we note that $d\omega = 0$. Our goal will be reached if we prove that the other boundary terms give exponentially decreasing contributions. As usual we just have to look at one term of the form

$$\tilde{g}_{\alpha\beta\gamma} = \iint_\partial \mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\, D^\alpha_{x_b}\, a\Big(\frac{x_a+y_a}{2} + i\xi_a, x_b\Big) \\ (\lambda\xi_a + i\lambda\psi')^\beta\, D^\gamma_{x_b}\ Tu(y_a,x_b,\lambda)\cdot dy_a \wedge d\xi_a$$

where ∂ is a part of the boundary of $\Sigma_{t\eta}$.

i) $|\mathrm{Re}(x_a-y_a)| = d$ or $|\mathrm{Im}(x_a-y_a) = d$. In that case $d \le |x_a-y_a| \le 2d$ and $\langle x_a\rangle \le \langle x_a-y_a\rangle + \langle y_a\rangle \le M(d)$. Now

$$\begin{aligned} &-\lambda(1+\eta)\,\Phi(x_a) + \mathrm{Re}(i\lambda(x_a-y_a)\cdot\xi_a) \\ &= \lambda(t-1)\eta\Phi(x_a) - \lambda t\eta\Phi(y_a) - \lambda\Phi(y_a)\ - \lambda\,|x_a-y_a|^2 \le -\lambda\Phi(y_a) - \lambda\, d^2 \end{aligned}$$

since $t \in [0,1]$. It follows from (2.3) that the corresponding term in $\tilde{g}_{\alpha\beta\gamma}$ satisfies

$$||\mathrm{e}^{-\lambda(1+\eta)\Phi}\ \tilde{g}_{\alpha\beta\gamma}||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} = \mathcal{O}\big(\mathrm{e}^{-\frac{\lambda}{2}d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\ . \tag{3.20}$$

ii) $|\mathrm{Re}(x_a - y_a)| \le d$, $|\mathrm{Im}\ (x_a - y_a)| \le d$, $|\ \mathrm{Re}\ y_a| \le 2d$, $|\mathrm{Im}\ y_a| = 2d$.

In that case $|\mathrm{Im}\ x_a| \ge d$ and as above

$$\begin{aligned} f(t) &= -\lambda(1+\eta)\,\Phi(x_a) + \mathrm{Re}(i\lambda(x_a - y_a)\cdot\xi_a) \\ &\le \lambda(t-1)\eta\,\Phi(x_a) - \lambda t\eta\,\Phi(y_a) - \lambda\Phi(y_a) \\ &\le \lambda(t-1)\eta\,d^2 - \lambda t\eta\,d^2 - \lambda\Phi(y_a) = -\lambda\eta\,d^2 - \lambda\Phi(y_a)\ . \end{aligned}$$

It follows that the corresponding term satisfies (3.20) with $\mathcal{O}\big(\mathrm{e}^{-\frac{\lambda}{2}d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)$.

iii) $|\mathrm{Re}(x_a - y_a) \le d$, $|\mathrm{Im}(x_a - y_a)| \le d$, $|\mathrm{Im}\ y_a| \le 2d$, $|\mathrm{Re}\ y_a| = 2d$.

For this case we use (2.5) instead and $f(t) \le -\lambda\Phi(y_a)$. Summing up we have proved

$$(3.21)\qquad \tilde{P}_\lambda\,Tv(x,\lambda) = \Big(\frac{\lambda}{2\pi}\Big)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\Sigma_\eta}\omega\Big)\,dy_b\,d\xi_b + \tilde{g}_6$$

where Σ_η is defined in (3.19) and

$$(3.22)\qquad \big|\big|\mathrm{e}^{-\lambda(1+\eta)\Phi}\,\tilde{g}_6\big|\big|_{L^2} = \mathcal{O}\big(\mathrm{e}^{-\frac{\lambda}{3}\eta d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\ .$$

Now on Σ_η we have $\big|\frac{x_a+y_a}{2}\big| + |\xi_a| \le 12d$. Let $\chi(z_a,\xi_a)$ be a C^∞ function on $\mathbb{C}^{2n_a}$ such that

$$(3.23)\qquad \begin{cases} \chi(z_a,\xi_a) = 1 & \text{if } |z_a| + |\xi_a| \le 12d \\ \chi(z_a,\xi_a) = 0 & \text{if } |z_a| + |\xi_a| \ge 13d \end{cases}$$

and χ is almost analytic on $\Lambda_{(1+\eta)\Phi} = \big\{(z_a,\xi_a)\in\mathbb{C}^{2n_a} : \xi_a = -(1+\eta)\,\mathrm{Im}\ z_a\big\}$ which means that

$$(3.24)\qquad |\overline{\partial}\chi(z_a,\xi_a)| \le C_N\,|\xi_a + (1+\eta)\,\mathrm{Im}\ z_a|^N \quad \text{for every } N\in\mathbb{N}\ .$$

According to (3.23) and (3.21) we can write

$$(3.25)\qquad \tilde{P}_\lambda\,Tv(x,\lambda) = \Big(\frac{\lambda}{2\pi}\Big)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\Sigma_\eta}\chi\omega\Big)\,dy_b\,d\xi_b + \tilde{g}_6$$

where $\tilde{g}_6$ satisfies (3.22).

Let us note that, since $13d < c_0$, $p\big(\frac{x_a+y_a}{2} + i\xi_a, \frac{x_b+y_b}{2}, \lambda\xi + i\lambda\psi'(\ldots)\big)$ is holomorphic in (y_a,ξ_a) on the support of χ.

Step 6. We want to remove the constraints $|\mathrm{Re}\ y_a| \le 2d$, $|\mathrm{Im}\ y_a| \le 2d$, $|\mathrm{Re}(x_a - y_a)| \le d$, $|\mathrm{Im}\ (x_a - y_a)| \le d$ and write

$$(3.26)\qquad \tilde{P}_\lambda\,Tv(x,\lambda) = \Big(\frac{\lambda}{2\pi}\Big)^n \iint \mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\Sigma'_\eta}\chi\omega\Big)\,dy_b\,d\xi_b + \tilde{g}_7$$

where $\Sigma'_\eta = \left\{y_a \in \mathbb{C}^{n_a},\ \xi_a = -(1+\eta)\,\mathrm{Im}\ \frac{x_a+y_a}{2} + i\overline{(x_a - y_a)}\right\}$ and $\tilde{g}_7$ satisfies (3.22).

Indeed on Σ_η we have

$$\begin{aligned} A &= -\lambda(1+\eta)\,\Phi(x_a) + \mathrm{Re}(i\lambda(x_a - y_a)\cdot\xi_a) + \lambda\Phi(y_a) \\ &= -\lambda\eta\,\Phi(y_a) - \lambda\,|x_a - y_a|^2 \ . \end{aligned}$$

If $|\mathrm{Re}\ y_a| \geq 2d$ we use (2.5) and we observe that $A \leq 0$. If $|\mathrm{Im}\ y_a| \geq 2d$ we use (2.3) and $A \leq -4\lambda\eta\, d^2$. If $|\mathrm{Re}(x_a - y_a)| \geq d$ or $|\mathrm{Im}\ (x_a - y_a)| \geq d$ then $|x_a - y_a| \geq d$ and $A \leq -\lambda d^2$. Therefore the contribution in the right hand side of (3.25) of $\Sigma_\eta \setminus \Sigma'_\eta$ is exponentially decreasing. Thus $\tilde{g}_7$ satisfies (3.22).

Step 7. In this last step we want to write $\tilde{P}_\lambda$ in term of the contour $\Sigma''_\eta = \left\{y_a \in \mathbb{C}^{n_a},\ \xi_a = -(1+\eta)\,\mathrm{Im}\ \frac{x_a+y_a}{2}\right\}$. For this purpose we state a lemma which will be also used later on. Recall that we have set

$$\Lambda_{(1+\eta)\Phi} = \left\{(z_a, \xi_a) \in \mathbb{C}^{2n_a} : \xi_a = -(1+\eta)\,\mathrm{Im}\ z_a\right\} \ .$$

Let χ be a C^∞ function on $\mathbb{C}^{2n_a}$ which is almost analytic on $\Lambda_{(1+\eta)\Phi}$. Let $b = b(z_a, \xi_a, x_b, \lambda)$ be a C^∞ function on $\mathbb{C}^{n_a} \times \mathbb{C}^{n_a} \times \mathbb{R}^{n_b} \times [1, +\infty[$ which is holomorphic with respect to (z_a, ξ_a) on the support of χ and such that $|b(z_a, \xi_a, x_b, \lambda)| \leq C\,\lambda^{m_0}$, $m_0 \in \mathbb{N}$, on the support of χ. Let $w = w(z_a, x_b, \lambda)$ be C^∞ on $\mathbb{C}^{n_a} \times \mathbb{R}^{n_b}$, entire holomorphic with respect to z_a such that $e^{-\lambda(1+\eta)\Phi(z_a)}w$ is in $L^2(\mathbb{C}^{n_a} \times \mathbb{R}^{n_b})$.

Lemma 3.2. *Let us consider the differential form*

$$\tilde{\omega} = e^{i\lambda(x_a - y_a)\cdot\xi_a}\,\chi\Big(\frac{x_a + y_a}{2}\ , \xi_a\Big)\, b\Big(\frac{x_a + y_a}{2}\ , \xi_a, x_b, \lambda\Big)\, w(y_a, x_b)\, dy_a \wedge d\xi_a$$

and the contours

$$\begin{aligned} \Sigma'_\eta &= \left\{(y_a, \xi_a) \in \mathbb{C}^{n_a} \times \mathbb{C}^{n_a}\colon \xi_a = -(1+\eta)\,\mathrm{Im}\ \frac{x_a + y_a}{2} + i\,\overline{(x_a - y_a)}\right\} \\ \Sigma''_\eta &= \left\{(y_a, \xi_a) \in \mathbb{C}^{n_a} \times \mathbb{C}^{n_a}\colon \xi_a = -(1+\eta)\,\mathrm{Im}\ \frac{x_a + y_a}{2}\right\} \ . \end{aligned}$$

If we set

$$\tilde{h} = \iint_{\Sigma'_\eta} \tilde{\omega} - \iint_{\Sigma''_\eta} \tilde{\omega}$$

then for any integer N one can find a positive constant C_N such that

$$||e^{-\lambda(1+\eta)\Phi}\,\tilde{h}||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} \leq \frac{C_N}{\lambda^N}\, ||e^{-\lambda(1+\eta)\Phi} w||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})}\ , \quad \textit{for}\ \lambda \geq 1 \ .$$

Proof. We follow the proof of Proposition 1.2 in [S2].

Let us consider for t in $[0,1]$ the contours

$$\Gamma_t = \left\{ (y_a, \xi_a) \in \mathbb{C}^{n_a} \times \mathbb{C}^{n_a} : \xi_a = -(1+\eta)\,\mathrm{Im}\ \frac{x_a + y_a}{2} + it\overline{(x_a - y_a)} \right\}$$

and $G = [0,1] \times \Gamma_t$. We apply Stokes formula to $\tilde{\omega}$ and G. Since $\Gamma_0 = \Sigma''_\eta$ and $\Gamma_1 = \Sigma'_\eta$ we have $\tilde{h} = \int_0^1 \int\int_{\Gamma_t} d\tilde{\omega}$. Noting that $\mathrm{e}^{i\lambda(x_a - y_a)\cdot\xi_a} b(\ldots)\, w(y_a, x_b, \lambda)$ is holomorphic in (y_a, ξ_a) on the support of χ we get

$$d\tilde{\omega} = \mathrm{e}^{i\lambda(x_a - y_a)\cdot\xi_a}\, b(\ldots)\, w\, \overline{\partial}_{(y_a,\xi_a)} \left[\chi\Big(\frac{x_a + y_a}{2}\ , \xi_a \Big)\, dy_a \wedge d\xi_a \right]\ .$$

Now $\overline{\partial}_{(y_a,\xi_a)} \left[\chi\big(\frac{x_a+y_a}{2}\ , \xi_a \big)\, dy_a \wedge d\xi_a \right]$ is a linear combination of terms as $\frac{\partial\chi}{\partial\overline{y}_{a,j}}\, d\overline{y}_{a,j} \wedge dy_a \wedge d\xi_a$ and $\frac{\partial\chi}{\partial\overline{\xi}_{a,j}}\, d\overline{\xi}_{a,j} \wedge dy_a \wedge d\xi_a$. On the other hand on Γ_t we have $\xi_a = -\frac{1+\eta}{4i}\ (x_a + y_a - \overline{x}_a - \overline{y}_a) + it\,\overline{(x_a - y_a)}$, therefore $d\xi_a$ and $d\overline{\xi}_a$ can be written as $\mathcal{O}(1)\, dy_a + \mathcal{O}(1)\, d\overline{y}_a + \mathcal{O}(|x_a - y_a|)\, dt$. It follows that $d\overline{y}_{a,j} \wedge dy_a \wedge d\xi_a$ and $d\overline{\xi}_{a,j} \wedge dy_a \wedge d\xi_a$ can be expressed as $\mathcal{O}(|x_a - y_a|)\, L(dy_a)\, dt$. Since on Γ_t we have, for every integer N, $|\overline{\partial}_{y_a}\chi| + |\overline{\partial}_{\xi_a}\chi| \le C_N \left| \xi_a + (1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2} \right|^N = C_N (t|x_a - y_a|)^N$ we can write

$$|\tilde{h}| \le C'_N\, \lambda^{m_0} \int_0^1 \int \mathrm{e}^{\lambda(1+\eta)[\Phi(x_a) - \Phi(y_a)] - \lambda t |x_a - y_a|^2}\ t^N\, |x_a - y_a|^{N+1}\, |w(y_a, x_b)|\, L(dy_a)\, dt\ .$$

It follows that

$$\mathrm{e}^{-\lambda(1+\eta)\Phi(x_a)}\, ||\tilde{h}(x_a,\cdot)||_{L^2(\mathbb{R}^{n_b})} \le C'_N\, \lambda^{m_0} \int\int_0^1 \mathrm{e}^{-\lambda t |x_a - y_a|^2}\, t^N\, |x_a - y_a|^{N+1}\, dt\ \mathrm{e}^{-\lambda(1+\eta)\Phi(y_a)}\, ||w(y_a,\cdot)||_{L^2(\mathbb{R}^{n_b})}\, L(dy_a)\ .$$

Now the right hand side is an integral operator with kernel

$$K(x_a, y_a) = \int_0^1 \mathrm{e}^{-\lambda t |x_a - y_a|^2}\, t^N\, |x_a - y_a|^{N+1}\, dt\ .$$

Since

$$\begin{aligned} \int |K(x_a, y_a)|\, L(dx_a) &= \int |K(x_a, y_a)|\, L(dy_a) \\ &= \lambda^{m_0 - n_a - \frac{N+1}{2}} \int_0^1 t^{\frac{N-1}{2} - n_a}\, dt \int \mathrm{e}^{-|\zeta_a|^2} |\zeta_a|^{N+1}\, L(d\zeta_a) \end{aligned}$$

Schur lemma ensures that for every large enough integer N we have

$$||\mathrm{e}^{-\lambda(1+\eta)\Phi}\, \tilde{h}||_{L^2(\mathbb{C}^{n_a} \times \mathbb{R}^{n_b})} \le C''_N\, \lambda^{m_0 - n_a - \frac{N+1}{2}}\, ||\mathrm{e}^{-\lambda(1+\eta)\Phi}\, w||_{L^2(\mathbb{C}^{n_a} \times \mathbb{R}^{n_b})}\ ,$$

and the lemma is proved.

Now lemma 3.2 ends the proof of theorem 3.1 since, as before the integral in the right hand side of (3.26) can be written as a finite sum of terms of the kind

$$A_{\alpha\beta\gamma} = \left(\frac{\lambda}{2\pi}\right)^{n_a} \iint_{\Sigma'_\eta} e^{i\lambda(x_a-y_a)\cdot\xi_a} \chi\left(\frac{x_a+y_a}{2}, \xi_a\right) D^\alpha_{x_b} a\left(\frac{x_a+y_a}{2} + i\xi_a, x_b\right) \cdot (\lambda\xi_a + i\lambda\psi'_a)^\beta D^\gamma_{x_b} Tv(y_a, x_b, \lambda)\, dy_a \wedge d\xi_a$$

where $|\alpha| + |\beta| + |\gamma| \leq m$ and a has compact support in x_b. Thus we can apply Lemma 3.2 with $w = D^\gamma_{x_b} Tv$ and $b = D^\alpha_{x_b} a \cdot (\lambda\xi + i\lambda\psi'_a)^\beta$.

3.2. Case of Theorem B

Recall that we have assumed

$$\text{(3.27)} \qquad \text{on } \xi_a = 0, \quad p_m \text{ does not depend on } x_a \ .$$

In that case we have

$$\text{(3.28)} \qquad p_m(x, \xi + i\psi'(x)) = p'_m(x_b, \xi_b) + p'_{m-1}(x_a, x_b, \xi_a, \xi_b)$$

where p'_m is a polynomial of order m in ξ_b and $p'_{m-1}(x, \xi)$ is a polynomial of order m in ξ but of order $m-1$ in ξ_b. Writing $p = \sum_{j=0}^{m} p_{m-j}$ we have,

Theorem 3.3. *There exists* $\chi \in C_0^\infty(\mathbb{C}^{2n_a})$, $\chi = 1$ *if* $|x_a| + |\xi_a| \leq 12d$, $\chi = 0$ *if* $|x_a| + |\xi_a| \geq 13d$, *such that, if we set* $X_b = \frac{x_b+y_b}{2}$, $Z_a = \frac{x_a+y_a}{2} + i\xi_a$ *and*

$$\text{(3.29)} \quad \tilde{\omega} = e^{i\lambda(x_a-y_a)\cdot\xi_a}\left\{\lambda^m p'_m(X_b, \xi_b) + \chi\left(\frac{x_a+y_a}{2}, \xi_a\right)\left[\lambda^m p'_{m-1}(Z_a, X_b, \xi_a, \xi_b) + \sum_{j=1}^{m} p_{m-j}(Z_a, X_b, \lambda\xi + i\lambda\psi'(Z_a, X_b))\right]\right\} Tv(y_a, y_b, \lambda)\, dy_a \wedge d\xi_a \ ,$$

$$\text{(3.30)} \quad \tilde{Q}_\lambda Tv(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\xi_a = -(1+\eta)\,\mathrm{Im}\,\frac{x_a+y_a}{2}} \tilde{\omega}\right) dy_b\, d\xi_b$$

then we have, with $\tilde{P}_\lambda$ *introduced in Proposition* 2.2,

$$\text{(3.31)} \qquad \tilde{P}_\lambda Tv = \tilde{Q}_\lambda Tv + \tilde{R}_\lambda Tv + \tilde{g}_\lambda$$

with

$$\text{(3.32)} \qquad ||e^{-\lambda(1+\eta)\Phi} \tilde{R}_\lambda Tv||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})} \leq \frac{C_N}{\lambda^N} ||e^{-\lambda(1+\eta)\Phi} Tv||_{L^2(\mathbb{C}^{n_a}, H^{m-1}(\mathbb{R}^{n_b}))}$$

$$(3.33)\qquad ||e^{-\lambda(1+\eta)\Phi}\,\tilde g_\lambda||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})}=\mathcal{O}\big(e^{-\frac{\lambda}{2}\eta d^2}||v||_{H^{n_0}(\mathbb{R}^n)}\big)\ .$$

Proof. It follows from (3.28) that the operator P_λ defined in (2.15) can be written as $P_\lambda = P'_m(x_b, D_{x_b}) + P''_\lambda$ where P''_λ is of order $\le m-1$ in D_{x_b}. Then $\tilde P_\lambda\, Tv = P'_m(x_b, D_{x_b})\, Tv + \tilde P''_\lambda\, Tv$. Then theorem 3.3 follows from theorem 3.1 applied to $\tilde P''_\lambda$ and from the equality

$$\Big(\frac{\lambda}{2\pi}\Big)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\Big(\iint_{\xi_a=-(1+\eta)\ \mathrm{Im}\,\frac{x_a+y_a}{2}} e^{i\lambda(x_a-y_a)\cdot\xi_a}\,p'_m(x_b,\lambda\xi_b)\; Tv(y_a,y_b,\lambda)\,dy_a\wedge d\xi_a\Big)\,dy_b dxi_a = P'_m(x_b,D_{x_b})\,Tv$$

(see formula (1.8) in [S2]).

Remark 3.4. A slight modification of these proofs shows that the estimates (3.4) and (3.33) can be precised as follows

$$(3.34)\qquad ||e^{-\lambda(1+\eta)\Phi}\,\tilde g_\lambda||_{L^2(\mathbb{C}^{n_a}\times\mathbb{R}^{n_b})}\le C\,e^{-\frac{\lambda}{3}\eta d^2}\,||v||_{L^2(\mathbb{R}^{n_a},H^k(\mathbb{R}^{n_b}))}\,,$$

where $k=m$ or $m-1$.

4. Back to the real domain. The main estimates

4.1. Pull back to the reals

Let $\tilde Q_\lambda$ be the operator defined in (3.1) (and (3.30)). It is complex in the (x_a,ξ_a) variable; we are going to pull it back to the reals by the canonical transformation κ_{T_η}, described in (2.8), which is associated with the FBI transformation T_η defined by

$$T_\eta v(z_a,x_b,\lambda)=K(\lambda)\int e^{-\lambda(1+\eta)(z_a-y_a)^2}\,v(y_a,x_b)\,dy_a\,,\quad v\in\mathscr{S}(\mathbb{R}^n)\ .$$

Let v be in $\mathscr{S}(\mathbb{R}^n)$ and set $w=T^*_\eta\,Tv$. Then it follows from Sect. 2.1 ii) and Proposition 2.1 iii) that

$$(4.1)\qquad w=T^*_\eta\,Tv\in\mathscr{S}(\mathbb{R}^n)\ \text{and}\ T_\eta w=Tv\ .$$

We deduce from Proposition 2.2 (see also Proposition 1.4 in [S2]),

$$(4.2)\qquad \tilde Q_\lambda\,Tv=\tilde Q_\lambda\,T_\eta w=T_\eta\,Q_\lambda w$$

where Q_λ is an operator on $\mathbb{R}^{n_a}\times\mathbb{R}^{n_b}$, pseudo-differential in x_a, differential in x_b. Moreover denoting by σ^w the Weyl-symbol

$$\sigma^w(Q_\lambda)(x_a,\xi_a,x_b,\xi_b) = \sigma^w(\tilde{Q}_\lambda)(\kappa_{T_\eta}(x_a,\xi_a),x_b,\xi_b) \tag{4.3}$$

where

$$\begin{cases} \sigma^w(\tilde{Q}_\lambda)(z_a,\xi_a,x_b,\xi_b) = \chi(z_a,\xi_a)\, p(z_a+i\xi_a,x_b,\lambda\xi \\ \qquad\qquad +i\lambda\psi'(x_a+i\xi_a,x_b)) \quad \text{(thm A)} \\ \sigma^w(\tilde{Q}_\lambda) = \lambda^m\, p'_m(x_b,\xi_b) + \chi(z_a,\xi_a)\, p''(z_a,x_b,\xi_a,\xi_b,\lambda) \\ \text{where } p''(z_a,x_b,\xi_a,\xi_b,\lambda) = \lambda^m\, p'_{m-1}(z_a,x_b,\xi_a,\xi_b) \\ \qquad\qquad + \sum_{j=1}^{m} p_{m-j}(z_a,x_b,\lambda\xi+i\lambda\psi'(z_a,x_b)) \quad \text{(thm B)} \end{cases} \tag{4.4}$$

Summing up we have by (4.1) to (4.4),

$$\begin{cases} \tilde{Q}_\lambda\, Tv = T_\eta\, Q_\lambda w, \\ w = T_\eta^*\, Tv \in \mathscr{S}(\mathbb{R}^n), \quad T_\eta w = Tv, \\ \sigma^w(Q_\lambda) = \chi\Big(x_a - \frac{i}{1+\eta}\,\xi_a,\xi_a\Big)\, p\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b,\lambda\xi \\ \qquad\qquad +i\lambda\psi'\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b\Big)\Big), \quad \text{(thm A)} \\ \sigma^w(Q_\lambda) = p'_m(x_b,\lambda\xi_b) \\ \qquad\qquad +\chi\Big(x_a-\frac{i}{1+\eta}\,\xi_a,\xi_a\Big)\, p''\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b,\xi_a,\xi_b,\lambda\Big) \quad \text{(thm B)} \\ Q_\lambda w(x) = \Big(\frac{\lambda}{2\pi}\Big)^n \iint \mathrm{e}^{i\lambda(x-y)\cdot\xi}\, \sigma^w(Q_\lambda)\Big(\frac{x+y}{2}\,,\xi\Big)\, w(y)\, dy\, d\xi \quad . \end{cases} \tag{4.5}$$

Moreover we have

$$\begin{cases} \sigma^w(Q_\lambda)(x,\xi) = \sum_{j=0}^{m} \lambda^{m-j}\, q_{m-j}(x,\xi) \\ q_{m-j}(x,\xi) = \chi\Big(x_a-\frac{i}{1+\eta}\,\xi_a,\xi_a\Big)\, p_{m-j}\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b,\xi \\ \qquad\qquad +i\psi'\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b\Big)\Big) \quad \text{(thm A)} \\ q_m(x,\xi) = p'_m(x_b,\xi_b) + \chi\Big(x_a-\frac{i}{1+\eta}\,\xi_a,\xi_a\Big)\, p'_{m-1}\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b,\xi_a,\xi_b\Big) \\ \qquad\qquad \text{and } q_{m-j}(x,\xi) = \chi(\ldots)\, p_{m-j}(\ldots) \quad \text{(thm B)} \ . \end{cases} \tag{4.6}$$

4.2. The estimates in case of Theorem A

We are now prepared to prove Carleman estimates for Q_λ. First of all we are going to precise our choice of ψ. Of course we may assume from now on that $x^0 = 0$ and $\varphi(0) = 0$. Let us recall our hypotheses on p_m

$$\begin{cases} n_b = 0 \text{ or } n_b \neq 0 \text{ and there is a positive constant } C \text{ such that} \\ |p_m(0,0,0,\xi_b)| \geq C\,|\xi_b|^m\,, \xi_b \in \mathbb{R}^{n_b} \end{cases} \tag{4.7}$$

$$(4.8)\begin{cases} p_m(0,0,i\varphi'_a(0),\xi_b+i\varphi'_b(0))=\varphi'(0)\cdot\dfrac{\partial p_m}{\partial\xi}\,(0,0,i\varphi'_a(0),\xi_b+i\varphi'_b(0))=0 \\ \textit{implies}\ \frac{1}{i}\Big\{\overline{p}_m(x,\xi-i\varphi'(x)),p_m(x,\xi+i\varphi'(x))\Big\}\Big|_{\substack{x=0\\ \xi_a=0}}>0\ . \end{cases}$$

Lemma 4.1. *Let φ be a C^2 function in a neighborhood of zero in $\mathbb{R}^n$ satisfying* (4.7), (4.8). *Then we can find a polynomial ψ of degree two in x such that*

$$\psi(0)=0\,,\quad \psi'(0)=\varphi'(0)\ , \tag{4.9}$$

and, *setting $X=(0,0,i\psi'_a(0),\xi_b+i\psi'_b(0))$, $\xi_b\in\mathbb{R}^{n_b}$*

$$p_m(X)=0\,\textit{implies}\,\frac{1}{i}\Big\{\overline{p}_m(x,\xi-i\psi'(x)),p_m(x,\xi+i\psi'(x))\Big\}\Big|_{\substack{x=0\\ \xi_a=0}}>0\ . \tag{4.10}$$

Moreover

$$\begin{cases}\textit{there exists a neighborhood of zero in which } \psi(x)=0\\ \textit{and } x\neq 0 \textit{ imply } \varphi(x)>0\ .\end{cases} \tag{4.11}$$

By homogeneity, (4.10) *is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.*

Proof. We shall take ψ of the following form

$$\psi(x)=x\cdot\varphi'(0)+A(x\cdot\varphi'(0))^2+\frac{1}{2}\,\varphi''(0)(x,x)-\frac{1}{A}\,|x|^2\,,\quad A>0\ . \tag{4.12}$$

Then (4.9) is obvious. Let us show (4.11). If $\psi(x)=0$ then $x\cdot\varphi'(0)=\mathcal{O}(|x|^2)$ and $x\cdot\varphi'(0)+\frac{1}{2}\,\varphi''(0)(x,x)=\frac{1}{A}\,|x|^2-A(x\cdot\varphi'(0))^2$. Then by Taylor formula

$$\varphi(x)=\frac{1}{A}\,|x|^2-A(x\cdot\varphi'(0))^2+o(|x|^2)=\frac{1}{A}\,|x|^2+\mathcal{O}(|x|^4)+o(|x|^2)$$

thus $\varphi(x)>0$ if x is small and $x\neq 0$. Let us prove (4.10). We set for convenience $Z=(x,\xi+i\varphi'(x))$, $\overline{Z}=(x,\xi-i\varphi'(x))$ and $p_m=p$. Then

$$\begin{aligned}\frac{1}{i}\,\{\overline{p}(\overline{Z}),p(Z)\}&=\frac{1}{i}\Big(\frac{\partial\overline{p}}{\partial\xi}\,(\overline{Z})\frac{\partial p}{\partial x}\,(Z)-\frac{\partial\overline{p}}{\partial x}\,(\overline{Z})\cdot\frac{\partial p}{\partial\xi}\,(Z)\Big)\\ &\quad+2\frac{\partial\overline{p}}{\partial\xi}(\overline{Z})\cdot\varphi''_{xx}(x)\cdot\frac{\partial p}{\partial\xi}\,(Z)\ .\end{aligned} \tag{4.13}$$

Now if we set $\zeta = (x = 0, i\varphi'_a(0), \xi_b + i\varphi'_b(0)), \overline{\zeta} = (0, -i\varphi'_a(0), \xi_b - i\varphi'_b(0))$ condition (4.8) reads

$$
(4.14) \quad \begin{cases} p(\zeta) = \dfrac{\partial p}{\partial \xi}(\zeta) \cdot \varphi'(0) = 0 \quad \textit{implies} \\ C_\varphi(\xi_b) = \frac{1}{i}\left(\dfrac{\partial \overline{p}}{\partial \xi}(\overline{\zeta}) \cdot \dfrac{\partial p}{\partial x}(\zeta) - \dfrac{\partial \overline{p}}{\partial x}(\overline{\zeta}) \cdot \dfrac{\partial p}{\partial \xi}(\zeta)\right) \\ +2\dfrac{\partial \overline{p}}{\partial \xi}(\overline{\zeta}) \cdot \varphi''_{xx}(0)\dfrac{\partial p}{\partial \xi}(\zeta) > 0 \ . \end{cases}
$$

We are looking for A in order to have (see (4.10))

$$
(4.15) \quad \begin{cases} p(X) = 0 \quad \textit{implies} \\ C_\psi(\xi_b) = \frac{1}{i}\left(\dfrac{\partial \overline{p}}{\partial \xi}(\overline{X})\dfrac{\partial p}{\partial x}(X) - \dfrac{\partial \overline{p}}{\partial x}(\overline{X})\dfrac{\partial p}{\partial \xi}(X)\right) \\ +2\psi''_{xx}(0) \cdot \dfrac{\partial \overline{p}}{\partial \xi}(\overline{X}) \cdot \dfrac{\partial p}{\partial \xi}(X) > 0 \ . \end{cases}
$$

Now by (4.9) and (4.12) we have

$$
(4.16) \qquad X = \zeta,\ \overline{X} = \overline{\zeta} \quad \text{and} \quad \psi''_{xx}(0) = \varphi''_{xx}(0) + 2A\varphi'(0)^t\varphi'(0) - \frac{2}{A}\ \mathrm{Id}
$$

from which we deduce

$$
(4.17) \qquad C_\psi(\xi_b) = C_\varphi(\xi_b) + 4A\left|\varphi'(0) \cdot \frac{\partial p}{\partial \xi}(X)\right|^2 - \frac{4}{A}\left|\frac{\partial p}{\partial \xi}(X)\right|^2 \ .
$$

We argue now by contradiction. Assume that for each A one can find ξ_b such that $p(X) = 0$ and $C_\psi(\xi_b) \leq 0$. Therefore there exist sequences $(A_j) + \infty$ and (ξ^j_b) such that

$$
(4.18) \quad p(X_j) = 0 \text{ and } \ C_\psi(\xi^j_b) \leq 0 \quad \text{where } X_j = (0, i\psi'_a(0), \xi^j_b + i\psi'_b(0)) \ .
$$

It follows from (4.16) that $p(\zeta_j) = 0$. Since $p(\zeta_j) = p(x = 0,\ \xi_a = 0,\ \xi^j_b) + \mathcal{O}(|\xi^j_b|^{m-1})$ we get $|p(x = 0, \xi_a = 0,\ \xi^j_b)| \leq C\,|\xi^j_b|^{m-1}$. If there is a subsequence of (ξ^j_b) which tends to $+\infty$ we would have by (4.7), $C_1\,|\xi^j_b|^m \leq C\,|\xi^j_b|^{m-1}$. Therefore the sequence (ξ^j_b) is bounded and there is a subsequence, still denoted by (ξ^j_b) which converges to ξ_b. Thus $\zeta_j \to \zeta = (x = 0, i\psi'_a(0), \xi_b + i\psi'_b(0))$ and

$$
(4.19) \qquad p(\zeta) = 0 \ .
$$

It follows from (4.16), (4.17) and (4.18) that,

$$\left|\varphi'(0)\cdot\frac{\partial p}{\partial\xi}\,(\zeta_j)\right|^2 \le \frac{1}{A_j^2}\left|\frac{\partial p}{\partial\xi}\,(\zeta_j)\right|^2 - \frac{1}{4A_j}\,C_\varphi(\xi_b^j)\ .$$

The right hand side tends to zero, thus

$$\varphi'(0)\cdot\frac{\partial p}{\partial\xi}\,(\zeta) = 0\ . \tag{4.20}$$

Using once more (4.16), (4.17) and (4.18) we get

$$C_\varphi(\xi_b^j) \le \frac{4}{A_j}\left|\frac{\partial p}{\partial\xi}\,(\zeta_j)\right|^2 \longrightarrow 0$$

so

$$C_\varphi(\xi_b) \le 0\ . \tag{4.21}$$

But (4.19), (4.20) and (4.21) contradict (4.14).

Lemma 4.2. *Under conditions* (4.7), (4.8) *there exist positive constants* η_0, ε, C_1 *and* C_2 *such that for all* η *in* $]0,\eta_0]$ *and all* (x,ξ) *in* $\mathbb{R}^{2n}$ *such that* $|x|+|\xi_a|\le\varepsilon$ *we have*

$$|q_m(x,\xi)| \ge C_1\,\langle\xi_b\rangle^m \ \ if\ \ |\xi_b|\ge C_2\ , \tag{4.22}$$

$$q_m(0,0,0,\xi_b)=0\ \ implies\ \ \frac{1}{i}\,\{\overline{q}_m(x,\xi),q_m(x,\xi)\}\big|_{\substack{x=0\\ \xi_a=0}} > 0\ . \tag{4.23}$$

Proof. We first take ε so small that $|x_a|+|\xi_a|\le\varepsilon$ implies $\left|x_a-\frac{i}{1+\eta}\,\xi_a\right|$ $+|\xi_a|\le 12d$. It follows then from (3.23) and (4.6) that

$$\begin{aligned} q_m(x,\xi) &= p_m\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b,\xi+i\psi'\Big(x_a+\frac{i\eta}{1+\eta}\,\xi_a,x_b\Big)\Big)\\ &= p_m(0,0,0,\xi_b)+\mathcal{O}(|x|+|\xi_a|)\langle\xi_a\rangle^m+\mathcal{O}(1)\langle\xi_b\rangle^{m-1}\ . \end{aligned}$$

Therefore $|q_m(x,\xi)|\ge C\,|\xi_b|^m - C_3(\varepsilon\langle\xi_b\rangle^m+\langle\xi_b\rangle^{m-1})$, and we get (4.22) if we still reduce ε and take $|\xi_b|$ large enough.

Let us look to (4.13) and let us set for convenience $p_m=p$ and

$$\begin{aligned} \zeta &= \Big(x_a+i\,\frac{\eta}{1+\eta}\,\xi_a,x_b,\xi+i\psi'\Big(x_a+i\,\frac{\eta}{1+\eta}\,\xi_a,x_b\Big)\Big)\\ \overline{\zeta} &= \Big(x_a-i\,\frac{\eta}{1+\eta}\,\xi_a,x_b,\xi-i\psi'\Big(x_a-i\,\frac{\eta}{1+\eta}\,\xi_a,x_b\Big)\Big)\ . \end{aligned}$$

Then

$$
\begin{aligned}
\{\overline{q}_m, q_m\}(x,\xi) =& \left[-i \,\frac{\eta}{1+\eta} \frac{\partial \overline{p}}{\partial x_a}\,(\overline{\zeta}) + \frac{\partial \overline{p}}{\partial \xi_a}\,(\overline{\zeta}) + i\left(i\,\frac{\eta}{1+\eta} \right) \psi''_{xx_a} \cdot \frac{\partial \overline{p}}{\partial \xi}\,(\overline{\zeta}) \right] \\
& \cdot \left[\frac{\partial p}{\partial x_a}\,(\zeta) + i\psi''_{xx_a}\,\frac{\partial p}{\partial \xi}\,(\zeta) \right] + \frac{\partial \overline{p}}{\partial \xi_b}\,(\overline{\zeta}) \left[\frac{\partial p}{\partial x_b}\,(\zeta) + i\psi''_{xx_b}\,\frac{\partial p}{\partial \xi}\,(\zeta) \right] \\
& - \left[\frac{\partial \overline{p}}{\partial x_a}\,(\overline{\zeta}) - i\psi''_{xx_a}\,\frac{\partial \overline{p}}{\partial \xi}\,(\overline{\zeta}) \right] \left[i\,\frac{\eta}{1+\eta} \frac{\partial p}{\partial x_a}\,(\zeta) + \frac{\partial p}{\partial \xi_a}\,(\zeta) \right. \\
& \left. + \, i\left(i\,\frac{\eta}{1+\eta} \right) \psi''_{xx_a}\,\frac{\partial p}{\partial \xi}\,(\zeta) \right] \\
& - \left[\frac{\partial \overline{p}}{\partial x_b}\,(\overline{\zeta}) - i\psi''_{xx_b}\,\frac{\partial \overline{p}}{\partial \xi}\,(\overline{\zeta}) \right] \frac{\partial p}{\partial \xi_b}\,(\zeta) \ .
\end{aligned}
$$

Moreover setting

$$
Z = (x, \xi + i\psi'(x)), \quad \overline{Z} = (x, \xi - i\psi'(x))
$$

we have

$$
\begin{aligned}
\{\overline{p}(\overline{Z}), p(Z)\} =& \frac{\partial \overline{p}}{\partial \xi}\,(\overline{Z}) \left[\frac{\partial p}{\partial x}\,(Z) + i\psi''_{xx} \cdot \frac{\partial p}{\partial \xi}\,(Z) \right] \\
& - \left[\frac{\partial \overline{p}}{\partial x}\,(\overline{Z}) - i\psi''_{xx} \frac{\partial \overline{p}}{\partial \xi}\,(\overline{Z}) \right] \cdot \frac{\partial p}{\partial \xi}\,(Z) \ .
\end{aligned}
$$

It follows that, for bounded $|\xi_b|$,

$$
\{\overline{q}_m, q_m\}\big|_{\substack{x=0 \\ \xi_a=0}} = \{\overline{p}_m(\overline{Z}), p_m(Z)\}\big|_{\substack{x=0 \\ \xi_a=0}} + \mathcal{O}(\eta) \ . \tag{4.24}
$$

Let ξ_b be such that $q_m(0,0,0,\xi_b) = 0$. Then $p_m(0,0,i\psi'_a(0), \xi_b + i\psi'_b(0)) = 0$ and (4.22) implies that $|\xi_b| \le C_2$. It follows from (4.10), by compactness on ξ_b, that

$$
\frac{1}{i}\,\{\overline{p}_m(\overline{Z}), p_m(Z)\}\big|_{\substack{x=0 \\ \xi_a=0}} \ge C_4 \ . \tag{4.25}
$$

Now (4.24) and (4.25) imply (4.23) if η is small enough. From now on η is a **fixed** number in $]0, \eta_0]$.

Lemma 4.3. *If q_m satisfies* (4.22) *and* (4.23) *there exist positive constants A, δ, ε_0 such that for all $(x,\xi) \in \mathbb{R}^{2n}$ such that $|x| + |\xi_a| \le \varepsilon_0$ we have*

$$
A\,|q_m(x,\xi)|^2 + \frac{1}{i}\,\{\overline{q}_m, q_m\}(x,\xi) \ge \delta \langle \xi_b \rangle^{2m} \ . \tag{4.26}
$$

Proof. We argue by contradiction. Otherwise there exist sequences $\varepsilon_j \to 0$, $\delta_j \to 0$, $A_j \to +\infty$, (x^j, ξ^j) with $|x^j| + |\xi^j_a| \leq \varepsilon_j$ and

$$A_j\,|q_m(x^j,\xi^j)|^2 + \frac{1}{i}\,\{\overline{q}_m, q_m\}(x^j,\xi^j) \leq \delta_j \langle \xi^j_b\rangle^{2m} \quad . \tag{4.27}$$

Case 1. There exists a subsequence, still denoted by (ξ^j_b), such that $|\xi^j_b| \to +\infty$. Since we have

$$\big|\{\overline{q}_m, q_m\}(x,\xi)\big| \leq C_0 \langle \xi_b\rangle^{2m} \tag{4.28}$$

and, by (4.22), $|q_m(x,\xi)|^2 \geq C_1^2 \langle \xi_b\rangle^{2m}$ if $|\xi_b| \geq C_2$, we deduce from (4.27), $(A_j C_1^2 - C_0)\langle \xi^j_b\rangle^{2m} \leq \delta_j \langle \xi^j_b\rangle^{2m}$ which is impossible since $A_j \to +\infty$ and $\delta_j \to 0$.

Case 2. The sequence (ξ^j_b) is bounded and therefore there exists a subsequence (still denoted by (ξ^j_b)) which converges to ξ^0_b. We deduce from (4.27) and (4.28) that

$$|q_m(x^j,\xi^j)|^2 \leq \frac{1}{A_j}\left(C_0\langle \xi^j_b\rangle^{2m} + \delta_j \langle \xi^j_b\rangle^{2m}\right) \to 0$$

thus, since $(x^j) \to 0$, $(\xi^j_a) \to 0$,

$$q_m(0,0,0,\xi^0_b) = 0 \quad . \tag{4.29}$$

Moreover (4.27) implies $\frac{1}{i}\,\{\overline{q}_m, q_m\}\,(x^j,\xi^j) \leq \delta_j \langle \xi^j_b\rangle^{2m}$, thus

$$\frac{1}{i}\,\{\overline{q}_m, q_m\}(0,0,0,\xi^0_b) \leq 0 \quad . \tag{4.30}$$

Now (4.29), (4.30) contradict (4.23). This ends the proof of lemma 4.3.

From now on ε_0 is fixed according to lemma 4.3.

Let $\tilde{\theta}_0 \in C^\infty(\mathbb{C}^{2n_a})$ be such that $0 \leq \tilde{\theta} \leq 1$ and

$$\begin{cases} \tilde{\theta}_0(z_a,\xi_a) = 1 & \text{if} \quad |z_a| + |\xi_a| \leq \frac{\eta}{1+\eta}\frac{\varepsilon_0}{4} \\ \tilde{\theta}_0(z_a,\xi_a) = 0 & \text{if} \quad |z_a| + |\xi_a| \geq \frac{\eta}{1+\eta}\frac{\varepsilon_0}{2} \\ \tilde{\theta}_0 \quad \text{is almost analytic on } \Lambda_{(1+\eta)\Phi} \quad . \end{cases} \tag{4.31}$$

Let us set, with κ_{T_η} defined in (2.8),

$$\theta_0 = \tilde{\theta}_0|_{\Lambda_{(1+\eta)\Phi}} \circ \kappa_{T_\eta} \quad . \tag{4.32}$$

It is easy to see that $\theta_0 \in C^\infty(\mathbb{R}^{2n_a})$ and there exists $\varepsilon_1 \in \left]0, \frac{\varepsilon_0}{2}\right[$ such that

$$\theta_0(x_a, \xi_a) = \begin{cases} 1 & \text{if } |x_a| + |\xi_a| \le \varepsilon_1 \\ 0 & \text{if } |x_a| + |\xi_a| \ge \frac{\varepsilon_0}{2} \end{cases} . \tag{4.33}$$

Let $h \in C_0^\infty(\mathbb{R}^{n_b})$ be such that $0 \le h \le 1$ and

$$h = \begin{cases} 1 & \text{if } |x_b| \le \frac{\varepsilon_0}{4} \\ 0 & \text{if } |x_b| \ge \frac{\varepsilon_0}{2} \end{cases} \tag{4.34}$$

Finally let us set

$$\theta(x, \xi_a) = h(x_b) \cdot \theta_0(x, \xi_a) . \tag{4.35}$$

Then

$$\theta(x, \xi_a) = \begin{cases} 1 & \text{if } |x| + |\xi_a| \le \varepsilon_1 \\ 0 & \text{if } |x| + |\xi_a| \ge \varepsilon_0 \end{cases} \tag{4.36}$$

We shall consider the semi classical norm on Sobolev space $H^m(\mathbb{R}^{n_b})$ which is defined by

$$||u||^2_{H^m_{sc}(\mathbb{R}^{n_b})} = \int \left(1 + \left|\frac{\xi}{\lambda}\right|^2\right)^m |\hat{u}(\xi)|^2 \, d\xi . \tag{4.37}$$

Lemma 4.4. *Let $Q = \mathrm{Op}^w_\lambda(q_m)$. There exist positive constants $C_0,\ C_1,\ \lambda_0$ such that for every u in $\mathscr{S}(\mathbb{R}^n)$ and $\lambda \ge \lambda_0$ we have*

$$\frac{C_1}{\lambda}\Big(\mathrm{Op}^w_\lambda((1-\theta)\langle\xi_b\rangle^{2m})\, u, u\Big)_{L^2(\mathbb{R}^n)} + ||Qu||^2_{L^2(\mathbb{R}^n)} \ge \frac{C_0}{\lambda}\, ||u||^2_{L^2(\mathbb{R}^{n_a}, H^m_{sc}(\mathbb{R}^{n_b}))} .$$

Proof. We write $Q = Q_R + i\, Q_I$ where $Q_R = \mathrm{Op}^w_\lambda(\mathrm{Re}\ q_m)$, $Q_I = \mathrm{Op}^w_\lambda(\mathrm{Im}\ q_m)$. Then $Q_K^* = Q_K$, $K = R, I$ and writing $||\cdot||$ for the $L^2(\mathbb{R}^n)$ norm

$$||Qu||^2 = ||Q_R u||^2 + ||Q_I u||^2 + \frac{1}{2}([Q^*, Q]\, u, u) . \tag{4.38}$$

Now the semi classical principal symbols of $[Q^*, Q]$ and $Q_K^*\, Q_K$ are $\frac{1}{i}\{\overline{q}_m, q_m\}$ and q_K^2, where $q_R = \mathrm{Re}\ q_m$, $q_I = \mathrm{Im}\ q_m$. We claim that one can find a positive constant B such that

$$B(1-\theta)\,\langle\xi_b\rangle^{2m} + A\,|q_m(x,\xi)|^2 + \frac{1}{i}\,\{\overline{q}_m, q_m\}(x,\xi) \ge \delta\,\langle\xi_b\rangle^{2m} \tag{4.39}$$

for all (x, ξ) in $\mathbb{R}^{2n}$.

Indeed Lemma 4.3 implies (4.39) if $|x| + |\xi_a| \leq \varepsilon_0$, since $0 \leq \theta \leq 1$, and if $|x| + |\xi_a| \geq \varepsilon_0$ then, by (4.36), $\theta = 0$ and $|q_m|^2 + |\{\overline{q}_m, q_m\}| \leq C \langle \xi_b \rangle^{2m}$, thus (4.39) is true if B is large enough.

Then we can apply the Gårding inequality in the following context. Let $g = dx_a^2 + dx_b^2 + d\xi_a^2 + \frac{d\xi_b^2}{\langle \xi_b \rangle^2}$. This is a metric which is temperate and slowly varying in the sense of Hörmander [H1]. Let $a \in S(\langle \xi_b \rangle^{2k}, g)$, $k \in \mathbb{N}$, be a symbol such that Re $a \geq \delta \langle \xi_b \rangle^{2k}$, and $A = \mathrm{Op}_\lambda^w(a)$. Then there exists $\lambda_0 > 0$ such that for every u in $\mathscr{S}(\mathbb{R}^n)$ and every $\lambda \geq \lambda_0$

$$\mathrm{Re}(Au, u)_{L^2} \geq \frac{\delta}{2} \, ||u||^2_{L^2(\mathbb{R}^{n_a}, H^k_{sc}(\mathbb{R}^{n_b}))} \quad . \tag{4.40}$$

Thus we may apply (4.40) with, for a, the left hand side of (4.39). It follows that for $\lambda \geq \lambda_0$

$$B\Big(\mathrm{Op}_\lambda^w\left((1-\theta)\langle \xi_b \rangle^{2m}\right) u, u\Big) + A\,||Q_R u||^2 + A\,||Q_I u||^2 + \lambda([Q^*, Q]u, u) \geq \frac{\delta}{2} \, ||u||^2_{L^2(\mathbb{R}^{n_a}, H^m_{sc}(\mathbb{R}^{n_b}))} \quad .$$

Now, we deduce from (4.38) that

$$2\lambda\,||Qu||^2_{L^2} \geq A\,||Q_R u||^2 + A\,||Q_I u||^2 + \lambda([Q^*, Q]\,u, u) \ \text{ if } \ 2\lambda \geq A\,,$$

and Lemma 4.4 follows.

Corollary 4.5. *Let Q_λ be defined in* (4.5). *Then one can find positive constants $C_0,\ C_1,\ \lambda_0$ such that for u in $\mathscr{S}(\mathbb{R}^n)$ and $\lambda \geq \lambda_0$*

$$C_1\,\lambda^{2m-1}\Big(\mathrm{Op}_\lambda^w\left((1-\theta)\,\langle \xi_b \rangle^{2m}\right) u, u\Big) + ||Q_\lambda u||^2_{L^2(\mathbb{R}^n)} \geq \frac{C_0}{\lambda}\,||u||^2_{L^2(\mathbb{R}^{n_a}, H^m_\lambda(\mathbb{R}^{n_b}))}$$

where

$$||v||^2_{H^m_\lambda} = \int (\lambda^2 + |\xi_b|^2)^m |\hat{v}(\xi_b)|^2 \, d\xi_b \quad . \tag{4.41}$$

Proof. Use (4.6).

We are now ready to prove the following estimate.

Proposition 4.6. *Let $\tilde{Q}_\lambda$ be defined in Theorem* 3.1. *Then there exist positive constants $C_1,\ C_2,\ \lambda_0,\ \varepsilon_2, n_0$ such that for $v \in C_0^\infty(\mathbb{R}^n)$,* $\ \mathrm{supp} v \subset \{x\colon |x| \leq \varepsilon_2\}$ *and $\lambda \geq \lambda_0$,*

$$||Tv||^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m_\lambda(\mathbb{R}^{n_b}))} \leq C_1\lambda\,||\tilde{Q}_\lambda\,Tv||^2_{L^2_{(1+\eta)\Phi}} + C_2\,\mathrm{e}^{-\lambda\sigma}||v||^2_{H^{n_0}(\mathbb{R}^n)}$$

where $\sigma > 0$ *depends only on* η *and* ε_0 *defined in lemma* 4.3. *The norms here have been defined in* (2.9), (2.10).

Proof. We apply corollary 4.5 to $u = T_\eta^* \, Tv$ which is in $\mathscr{S}(\mathbb{R}^n)$ (see Sect. 2.1 ii)). It follows from proposition 2.1 and (4.5)

$$||u||_{L^2(\mathbb{R}^{n_a}, H^m_\lambda(\mathbb{R}^{n_b}))} = ||T_\eta u||_{L^2_{(1+\eta)\Phi}(H^m_\lambda)} = ||Tv||_{L^2_{(1+\eta)\Phi}(H^m_\lambda)} \tag{4.42}$$

$$||Q_\lambda u||_{L^2(\mathbb{R}^n)} = ||T_\eta \, Q_\lambda \, T_\eta^* \, Tv||_{L^2_{(1+\eta)\Phi}} = ||\tilde{Q}_\lambda \, Tv||_{L^2_{(1+\eta)\Phi}} \ . \tag{4.43}$$

Let us set $R = \mathrm{Op}^w_\lambda((1-\theta)\,\langle \xi_b \rangle^{2m})$. Then proposition 1.4 in [S2] (see also Proposition 2.2) and Proposition 2.1 show that

$$T_\eta \, R \, u = \tilde{R} \, T_\eta u = \tilde{R} \, T_\eta \, T_\eta^* \, Tv = \tilde{R} \, Tv$$

with

$$\tilde{R} \, Tv(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b - y_b)\cdot \xi_b} \, \langle \xi_b \rangle^{2m} \left(\iint_{\xi_a = -(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2}} \tilde{\omega} \right) dy_b \, d\xi_b \tag{4.44}$$

$$\tilde{\omega} = e^{i\lambda(x_a - y_a)\cdot \xi_a} \left(1 - \tilde{\theta}\left(\frac{x_a+y_a}{2}, \frac{x_b+y_b}{2}, \xi_a\right)\right) Tv(y_a, y_b) \, dy_a \wedge d\xi_a \tag{4.45}$$

where $\tilde{\theta} = \theta \circ \kappa_{T_\eta}^{-1} = h(x_b)\,\tilde{\theta}_0$ is defined in (4.31) to (4.36). Therefore, we deduce from Proposition 2.1,

$$(Ru, u)_{L^2} = (T_\eta \, Ru, T_\eta u)_{L^2_{(1+\eta)\Phi}} = (\tilde{R} \, Tv, Tv)_{L^2_{(1+\eta)\Phi}} \ .$$

It follows that Proposition 4.6 will be proved if we show that for any integer N one can find a positive constant C_N such that

$$|(\tilde{R} \, Tv, Tv)| \le \frac{C_N}{\lambda^N} \, ||Tv||^2_{L^2_{(1+\eta)\Phi}(H^m_\lambda)} + \mathcal{O}(e^{-\lambda\sigma} ||v||^2_{H^{n_0}(\mathbb{R}^n)}) \,, \quad \sigma > 0 \ . \tag{4.46}$$

Proof of (4.46). First of all we see from (4.35) that

$$1 - \theta(x, \xi_a) = 1 - \theta_0(x_a, \xi_a)\, h(x_b) = (1 - \theta_0(x_a, \xi_a))\, h(x_b) + 1 - h(x_b) \ .$$

Now it follows from (4.34) that $(1 - h(x_b))\langle \xi_b \rangle^{2m}$ is the symbol of a differential operator with coefficients vanishing for $|x_b| \le \frac{\varepsilon_0}{4}$. If we take $\varepsilon_2 \le \frac{\varepsilon_0}{4}$ and $\mathrm{supp}\, v \subset \{x : |x| \le \varepsilon_2\}$ then $\mathrm{supp}\, u = \mathrm{supp}\, T_\eta^* \, Tv \subset \left\{|x_b| \le \frac{\varepsilon_0}{4}\right\}$, therefore $\mathrm{Op}^w_\lambda((1 - h(x_b))\langle \xi_b \rangle^{2m})\, u = 0$, which implies that

$$Ru = \mathrm{Op}^w_\lambda\left((1-\theta)\,\langle \xi_b \rangle^{2m}\right) u = \mathrm{Op}^w_\lambda\left((1 - \theta_0(x_a, \xi_a))\, h(x_b)\, \langle \xi_b \rangle^{2m}\right) u \ .$$

We deduce that, in the expression of $\tilde{R}$ in (4.44), (4.45), we can put $(1-\tilde{\theta}_0(x_a,\xi_a))\,h(x_b)$ instead of $1-\tilde{\theta}(x,\xi_a)$. We write $\langle\xi_b\rangle^{2m}=\sum_{|\alpha|\leq 2m} C_\alpha\,\xi_b^\alpha$ and we show, by induction that for $|\alpha|\leq 2m$

$$\lambda^{|\alpha|}\,\xi_b^\alpha\,\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,h\Big(\frac{x_b+y_b}{2}\Big)=\sum_{\substack{|\alpha_1|\leq m\\ |\alpha_2|\leq m\\ \alpha_1+\alpha_2=\alpha}} D_{x_b}^{\alpha_1}\,D_{y_b}^{\alpha_2}\Big(\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,h_{\alpha,\alpha_1,\alpha_2}\Big(\frac{x_b+y_b}{2}\Big)\Big)$$

where the $h_{\alpha,\alpha_1,\alpha_2}$ are derivatives of h.

We deduce that $\tilde{R}\,Tv$ is the limit, as ε goes to zero, of a finite sum of terms of the form

$$\begin{aligned} I_\varepsilon=\lambda^{N_0}\iint D_{x_b}^{\alpha_1}\,D_{y_b}^{\alpha_2}\Big\{\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,g\Big(\frac{x_b+y_b}{2}\Big)\,\zeta(\varepsilon\xi_b)\Big\}\\ \cdot\Big(\iint_{\xi_a=-(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2}}\tilde{\omega}_1\Big)\,dy_b\,d\xi_b \end{aligned}$$

where $N_0\in\mathbb{N}$ is fixed, $\zeta\in C_0^\infty(\mathbb{R}^{n_b})$, $\zeta(0)=1$, $|\alpha_1|\leq m$, $|\alpha_2|\leq m$, $g\in C_0^\infty(\mathbb{R}^{n_b})$ and

$$\tilde{\omega}_1=\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\Big(1-\tilde{\theta}_0\Big(\frac{x_a+y_a}{2}\,,\xi_a\Big)\Big)\,Tv(y_a,y_b)\,dy_a\wedge d\xi_a\ \ .$$

After integrating by parts in the y_b integral (which is possible by (2.3)) we can write $I_\varepsilon=D_{x_b}^{\alpha_1}\,J_\varepsilon$ with

$$J_\varepsilon=\lambda^{N_0}\iint\mathrm{e}^{i\lambda(x_b-y_b)\cdot\xi_b}\,g\Big(\frac{x_b+y_b}{2}\Big)\,\zeta(\varepsilon\xi_b)\Big(\iint_{\xi_a=-(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2}}\tilde{\omega}_2\Big)\,dy_b\,d\xi_b$$

$$\tilde{\omega}_2=\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\Big(1-\tilde{\theta}_0\Big(\frac{x_a+y_a}{2}\,,\xi_a\Big)\Big)\,D_{y_b}^{\alpha_2}\,Tv(y_a,y_b)\,dy_a\wedge d\xi_a\ \ .$$

As before we compute the integral in ξ_b then, in the y_b integral, we set $x_b-y_b=\varepsilon\,t_b$, we take the limit, when ε goes to zero, in $\mathscr{S}'$ and we get

$$\begin{aligned} \lim_{\varepsilon\to 0} I_\varepsilon=\lambda^{N_1}\,D_{x_b}^{\alpha_1}\iint_{\xi_a=-(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2}}\mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a}\,g(x_b)\\ \cdot\Big(1-\tilde{\theta}_0\Big(\frac{x_a+y_a}{2}\,,\xi_a\Big)\Big)\,D_{x_b}^{\alpha_2}\,Tv(y_a,x_b)\,dy_a\wedge d\xi_a\ \ . \end{aligned}$$

Moreover $\tilde{R}\,Tv$ is a finite sum of such terms. It follows that $(\tilde{R}\,Tv,Tv)_{L^2_{(1+\eta)\Phi}}$ is a finite sum of terms like $\lambda^{N_1}(\tilde{S}\,Tv,D_{x_b}^{\alpha_1}\,Tv)_{L^2_{(1+\eta)\Phi}}$ where

$$\tilde{S}\,Tv(x,\lambda) = \iint_{\xi_a = -(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2}} \mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a} \cdot \left(1 - \tilde{\theta}_0\left(\frac{x_a+y_a}{2}, \xi_a\right)\right) g(x_b)\, D_{x_b}^{\alpha_2}\, Tv(y_a, x_b)\, dy_a \wedge d\xi_a \ .$$

Therefore (4.46) will follow from the estimate

$$||\tilde{S}\,Tv||_{L^2_{(1+\eta)\Phi}} \leq \frac{C_N}{\lambda^N}\, ||Tv||_{L^2_{(1+\eta)\Phi}(H^m)} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}||v||_{H^{n_0}(\mathbb{R}^n)}) \,, \quad \forall N \in \mathbb{N}\,, \quad \sigma > 0 \ . \tag{4.47}$$

Proof of (4.47).

Step 1. Let us set

$$\tilde{\omega}_3 = \mathrm{e}^{i\lambda(x_a-y_a)\cdot\xi_a} \left(1 - \tilde{\theta}_0\left(\frac{x_a+y_a}{2}, \xi_a\right)\right) g(x_b)\, D_{x_b}^{\alpha_2}\, Tv(y_a, x_b)\, dy_a \wedge d\xi_a \ .$$

Then

$$\left\{\begin{array}{l} \tilde{S}\,Tv(x,\lambda) = \iint_{\xi_a = -(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2} + i\overline{(x_a-y_a)}} \tilde{\omega}_3 + \tilde{L}\,Tv \\ ||\tilde{L}\,Tv||_{L^2_{(1+\eta)\Phi}} \leq \frac{C_N}{\lambda^N}\, ||Tv||_{L^2_{(1+\eta)\Phi}(H^m)} \qquad \forall N \in \mathbb{N} \ . \end{array}\right. \tag{4.48}$$

This follows from Lemma 3.2 and (4.31).

Step 2. Assume $\varepsilon_2 \leq \frac{\eta}{1+\eta}\,\frac{\varepsilon_0}{100}$ and $\mathrm{supp}\ v \subset \{|x| \leq \varepsilon_2\}$. Then

$$\tilde{S}\,Tv(x,\lambda) = \iint_{\xi_a = -(1+\eta)\ \mathrm{Im}\ \frac{x_a+y_a}{2} + i\ \overline{(x_a-y_a)}, |x_a-y_a| \leq \varepsilon_2, |y_a| \leq 2\varepsilon_2} \tilde{\omega}_3 + \tilde{L}\,Tv + \tilde{g}_1 \tag{4.49}$$

where $\tilde{L}\,Tv$ satisfies (4.48) and there exists $\sigma = \sigma(\varepsilon_2, \eta)$ such that

$$||\tilde{g}_1||_{L^2_{(1+\eta)\Phi}} \leq C\,\mathrm{e}^{-\lambda\sigma}||v||_{H^{n_0}(\mathbb{R}^n)} \ . \tag{4.50}$$

To prove this we look at the part, in the integral in the right hand side of (4.48), where $|x_a - y_a| \geq \varepsilon_2$ or $|y_a| \geq \varepsilon_2$. The estimate (4.50) follows then from the argument in step 6 in the proof of Theorem 3.1.

Step 3. If $|x_a - y_a| \leq \varepsilon_2$ and $|y_a| \leq 2\varepsilon_2$ then $\left|\frac{x_a+y_a}{2}\right| + |\xi_a| \leq 10\varepsilon_2 < \frac{\eta}{1+\eta}\,\frac{\varepsilon_0}{4}$. Therefore, by (4.31), $\tilde{\theta}_0\left(\frac{x_a+y_a}{2}, \xi_a\right) = 1$ so $\tilde{\omega}_3 \equiv 0$ and $\tilde{S}\,Tv = \tilde{L}\,Tv + \tilde{g}_1$. By (4.48) and (4.50) we get (4.47) and the proof of proposition 4.6.

Corollary 4.7. *Let $\tilde{P}_\lambda$ be the operator occuring in proposition* 2.2. *One can find positive constants C_1, C_2, λ_0, ε_2, σ, n_0 such that for $v \in C_0^\infty(\mathbb{R}^n)$,* $\mathrm{supp}\, v \subset \{x\colon |x| \leq \varepsilon_2\}$ *and $\lambda \geq \lambda_0$ we have*

$$||Tv||^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m_\lambda(\mathbb{R}^{n_b}))} \leq C_1\,\lambda\,||\tilde{P}\,Tv||^2_{L^2_{(1+\eta)\Phi}} + C_2\,\mathrm{e}^{-\lambda\sigma}||v||^2_{H^{n_0}(\mathbb{R}^n)} \ .$$

Proof. This follows from Proposition 4.6 and Theorem 3.1.

4.3. The estimates in case of Theorem B

Let $Q^0 = \mathrm{Op}^w_\lambda(q_m)$ where q_m is defined in (4.6). We have

$$(4.51)\qquad \begin{cases} ||Q^0u||^2_{L^2} = ||Q_R u||^2_{L^2} + ||Q_I u||^2_{L^2} + \frac{1}{2}\,([Q^{0*}, Q^0]\,u, u) \\ \text{where } Q^0 = Q_R + i\,Q_I\ ,\ Q_R^* = Q_R\ ,\ Q_I^* = Q_I\ \ . \end{cases}$$

Let us introduce the following Hörmander's metrics

$$(4.52)\qquad g_1 = dx^2 + \frac{d\xi^2}{\langle\xi_b\rangle^2}\ , \qquad g_2 = dx^2 + d\xi_a^2 + \frac{d\xi_b^2}{\langle\xi_b\rangle^2}\ \ .$$

Then it is easy to see from (4.6) and (3.29) that

$$(4.53)\qquad \begin{cases} q_m(x,\xi) = p'_m(x_b,\xi_b) + \tilde{\chi}(x_a,\xi_a)(r_{m-1}(x,\xi) + \eta\,s_{m-1}(x,\xi)) \\ r_{m-1} \in S(\langle\xi_b\rangle^{m-1}, g_1), s_{m-1} \in S(\langle\xi_b\rangle^{m-1}, g_2)\ ,\ \text{where} \\ \tilde{\chi}(x_a,\xi_a) = \chi\Big(x_a - \frac{i}{1+\eta}\ \xi_a, \xi_a\Big)\ \ . \end{cases}$$

We shall write $Q^0 = P'_m + R_{m-1} + \eta\,S_{m-1}$ where $\sigma^w(P'_m) = p'_m(x_b,\xi_b)$, $\sigma^w\,(R_{m-1}) = \tilde{\chi}\,r_{m-1}$, $\sigma^w(S_{m-1}) = \tilde{\chi}\,s_{m-1}$. Let us set

$$(4.54)\qquad L = P'_m + R_{m-1}\ \ .$$

Since R_{m-1} and S_{m-1} belong to $\mathrm{Op}^w_\lambda(S(\langle\xi_b\rangle^{m-1}, g_2))$ and since p'_m depends only on (x_b,ξ_b), it is easy to see that

$$(4.55)\qquad [Q^{0*}, Q^0] - [L^*, L] \in \frac{\eta}{\lambda}\ \mathrm{Op}^w_\lambda(S(\langle\xi_b\rangle^{2m-2}, g_2))\ \ .$$

We shall set

$$(4.56)\qquad \begin{cases} \sigma^w(L) = \ell = \ell_1 + \ell_2 \quad \text{where} \\ \ell_1 = p'_m(x_b,\xi_b) + (\tilde{\chi}\,r_{m-1})|_{\xi_a=0}\ , \ell_2 = \tilde{\chi}\,r_{m-1} - (\tilde{\chi}\,r_{m-1})|_{\xi_a=0}\ \ . \end{cases}$$

Then

$$(4.57)\qquad \ell_1 \in S(\langle\xi_b\rangle^m, g_1)\ , \qquad \ell_2 \in S(\langle\xi_b\rangle^{m-1}, g_2)\ \ .$$

We shall also write

(4.58) $$\begin{cases} \sigma^w([L^*,L]) = \frac{1}{\lambda}\,(c_1+c_2) \text{ where} \\ c_1 = \frac{1}{i}\,\{\overline{\ell},\ell\}|_{\xi_a=0}\ . \end{cases}$$

Then since the symbol of L is a polynomial in ξ_b and p'_m depends only on (x_b,ξ_b) we have

(4.59) $$c_1 \in S(\langle\xi_b\rangle^{2m-1}, g_1),\ \ c_2 \in S(\langle\xi_b\rangle^{2m-2}, g_2),\ \text{ uniformly in } \lambda\ .$$

Lemma 4.8. *There exists a positive constant A such that if we set $\psi(x) = \varphi'(0)\cdot x + \frac{1}{2}\,\varphi''(0)x\cdot x - \frac{1}{A}\,|x|^2 - A(\varphi'(0)\cdot x)^2$ then*

$$A|\ell_1(x,\xi_b)|^2 + c_1(x,\xi_b) \geq \frac{1}{A}\,\langle\xi_b\rangle^{2m-2}\,, \quad \text{for } |x| \leq \frac{1}{A^2} \quad \text{and } \xi_b \text{ in } \mathbb{R}^{n_b}\ .$$
(4.60)

Moreover, by homogeneity, (4.60), *with possibly other constants, is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.*

Proof. We first take A so large that $\tilde{\chi} = 1$ if $|x_a| + |\xi_a| \leq \frac{1}{A^2}$. Then from (3.29) and (4.56) we have $\ell_1(x,\xi_b) = p_m(x, i\psi'_a(x), \xi_b + i\psi'_b(x))$ and

$$c_1(x,\xi_b) = \frac{1}{i}\,\left\{\overline{p}_m(x,\xi - i\psi'(x)), p_m(x,\xi + i\psi'(x))\right\}\Big|_{\xi_a=0}\ .$$

Now

$$\begin{aligned} c_1 = 2\,\mathrm{Im}&\left\{\frac{\partial\overline{p}_m}{\partial x}\,(x,-i\psi'_a(x),\xi_b - i\psi'_b(x))\,\frac{\partial p_m}{\partial\xi}\,(x,i\psi'_a(x),\xi_b + i\psi'_b(x))\right\} \\ &- i\psi''(x)\,\frac{\partial\overline{p}_m}{\partial\xi}(x,-i\psi'_a(x),\xi_b - i\psi'_b(x))\,\cdot\,\frac{\partial p_m}{\partial\xi}\,\big(x,i\psi'_a(x),\xi_b + i\psi'_b(x)\big)\ . \end{aligned}$$

If we multiply the inequality (4.60) by λ^{2m-2} and we divide both members by $\langle\lambda,\lambda\xi_b\rangle^{2m-2} = (\lambda^2 + \lambda^2\,|\xi_b|^2)^{m-1}$, we see, setting $\Xi_b = \frac{\lambda\xi_b}{\langle\lambda,\lambda\xi_b\rangle}$, $\Gamma = \frac{\lambda}{\langle\lambda,\lambda\xi_b\rangle}$, that (4.60) is equivalent to

$$\frac{A}{\Gamma^2}\,|p_m(Z)|^2 + \frac{2}{\Gamma}\,\mathrm{Im}\,\left(\frac{\partial\overline{p}_m}{\partial x}\,(\overline{Z})\frac{\partial p_m}{\partial\xi}\,(Z)\right) - \psi''(x)\cdot\frac{\partial\overline{p}_m}{\partial\xi}\,(\overline{Z})\cdot\frac{\partial p_m}{\partial\xi}\,(Z) \geq \frac{1}{A}$$
(4.61)

if $|x| \leq \frac{1}{A^2}$, where $Z = (x, i\Gamma\psi'_a(x),\ \Xi_b + i\Gamma\psi'_b(x))$, $\overline{Z} = (x,-i\Gamma\psi'_a(x),$ $\Xi_b - i\Gamma\psi'_b(x))$. We prove (4.61) by contradiction. If it is false one can find sequences $A_k \longrightarrow +\infty$, $|x_k| \leq \frac{1}{A_k^2}$, Ξ_b^k, Γ_k such that

$$\frac{A_k}{\Gamma_k}|p_m(Z_k)|^2+\frac{2}{\Gamma_k}\,\mathrm{Im}\,\left(\frac{\partial\overline{p}_m}{\partial x}\,(\overline{Z}_k)\,\frac{\partial p_m}{\partial\xi}\,(\overline{Z}_k)\right)$$
$$-\,\psi''(x_k)\cdot\frac{\partial p_m}{\partial\xi}(\overline{Z}_k)\cdot\frac{\partial p_m}{\partial\xi}\,(Z_k)\leq\frac{1}{A_k}\quad. \tag{4.62}$$

Since $|\Xi_b^k|\leq 1$, $\Gamma_k\leq 1$, taking subsequences, we may assume that

$$\Xi_b^k\longrightarrow\Xi_b^0\text{ and }\Gamma_k\longrightarrow\Gamma^0\quad. \tag{4.63}$$

On the other hand $\psi'(x_k)=\varphi'(0)+\varphi''(0)\,x_k-\frac{2}{A_k}\,x_k-2\,A_k(\varphi'(0)\cdot x_k)\,\varphi'(0)$ and $|x_k|\leq\frac{1}{A_k^2}$; therefore $\psi'(x_k)\longrightarrow\varphi'(0)$. It follows that

$$Z_k\to(0,i\Gamma^0\,N_a\,,\,\Xi_b^0+i\Gamma^0\,N_b)\,,\qquad\overline{Z}_k\to(0,-i\Gamma^0\,N_a,\,\Xi_b^0-i\Gamma^0\,N_b) \tag{4.64}$$

where $\varphi'(0)=(N_a,N_b)$.

Since $\psi''(x_k)=\varphi''(0)-\frac{2}{A_k}-2A_k\,\varphi'(0)^t\varphi'(0)$ the third term in the left hand side of (4.62) can be written

$$-\varphi''(0)\frac{\partial\overline{p}_m}{\partial\xi}\,(\overline{Z}_k)\frac{\partial p_m}{\partial\xi}\,(Z_k)+\frac{2}{A_k}\left|\frac{\partial p_m}{\partial\xi}\,(Z_k)\right|^2+2A_k\left|\varphi'(0)\,\frac{\partial p_m}{\partial\xi}\,(Z_k)\right|^2\quad. \tag{4.65}$$

Case 1. $\Gamma^0\neq 0$.
If we divide both members of (4.62) by A_k and if we use (4.64) and (4.65) we get with $Z^0=(x=0,i\Gamma^0\,N_a,\Xi_b^0+i\Gamma^0\,N_b)$

$$p_m(Z^0)=\varphi'(0)\cdot\frac{\partial p_m}{\partial\xi}\,(Z^0)=0\quad. \tag{4.66}$$

Coming back to (4.62), (4.65) we get

$$\frac{2}{\Gamma^0}\,\mathrm{Im}\left\{\frac{\partial\overline{p}_m}{\partial x}\,(\overline{Z}^0)\,\cdot\frac{\partial p_m}{\partial\xi}\,(Z^0)\right\}-\varphi''(0)\frac{\partial\overline{p}_m}{\partial\xi}\,(\overline{Z}^0)\cdot\frac{\partial p_m}{\partial\xi}\,(Z^0)\leq 0$$

which contradicts the hypothesis $(H.2)'$ ii) in theorem B.

Case 2. $\Gamma^0=0$ so $Z_k\longrightarrow Z^0=(x=0,\xi_a=0,\Xi_b^0)$, $\Xi_b^0\neq 0$.
In this case we write

$$\mathrm{Im}\left\{\frac{\partial\overline{p}_m}{\partial x}\,(\overline{Z}_k)\frac{\partial p_m}{\partial\xi}\,(Z_k)\right\}=\ \mathrm{Im}\left\{\frac{\partial\overline{p}_m}{\partial x}\,(x_k,0,\Xi_b^k)\frac{\partial p_m}{\partial\xi}\,(x,0,\Xi_b^k)\right\}$$
$$+\,\Gamma_k\mathrm{Im}\left\{-i\psi'(x_k)\frac{\partial^2\bar{p}_m}{\partial x\partial\xi}\,(x_k,0,\Xi_b^k)\,\cdot\,\frac{\partial p_m}{\partial\xi}\,(x_k,0,\Xi_b^k)\right.$$
$$\left.+\frac{\partial\overline{p}_m}{\partial x}\,(x_k,0,\Xi_b^k)\cdot\,\frac{\partial^2 p_m}{\partial\xi^2}\,(x_k,0,\Xi_b^k)\cdot i\psi'(x_k)\right\}+\mathcal{O}(\Gamma_k^2)\quad. \tag{4.67}$$

We use then the assumption $(H.1)'$ in theorem B. We get

$$\left|\operatorname{Im}\ \left\{\frac{\partial \overline{p}_m}{\partial x}\ (x_k, 0, \Xi_b^k)\frac{\partial p_m}{\partial \xi}\ (x_k, 0, \Xi_b^k)\right\}\right| \leq C\,|p_m(x_k, 0, \Xi_b^k)|$$
$$\leq C\,|p_m(Z_k)| + C\,\Gamma_k\left|\frac{\partial p_m}{\partial \xi}\ (Z_k)\cdot\ \psi'(x_k)\right| + \mathcal{O}(\Gamma_k^2)\ .$$

Therefore

$$\text{(4.68)}\qquad \left|\operatorname{Im}\ \left[\frac{\partial \overline{p}_m}{\partial x}\ (x_k, 0, \Xi_b^k)\ \cdot\ \frac{\partial p_m}{\partial \xi}\ (x_k, 0, \Xi_b^k)\right]\right| \leq \frac{\sqrt{A_k}}{\Gamma_k}\ |p_m(Z_k)|^2$$
$$+ C^2\,\frac{\Gamma_k}{\sqrt{A_k}} + C\Gamma_k\left|\frac{\partial p_m}{\partial \xi}\ (Z_k)\cdot\psi'(x_k)\right| + \mathcal{O}(\Gamma_k^2)\ .$$

It follows from (4.62), (4.65), (4.67) and (4.68) that

$$\text{(4.69)}\qquad \frac{A_k}{\Gamma_k^2}\,|p_m(Z_k)|^2 - \frac{\sqrt{A_k}}{\Gamma_k^2}\ |p_m(Z_k)|^2 - \frac{C^2}{\sqrt{A_k}} - C\left|\frac{\partial p_m}{\partial \xi}\ (Z_k)\cdot\psi'(x_k)\right|$$
$$- C'\,\Gamma_k + \operatorname{Im}\left\{-i\psi'(x_k)\ \frac{\partial^2\,\overline{p}_m}{\partial x\partial \xi}\ (x_k, 0, \Xi_b^k)\cdot\ \frac{\partial p_m}{\partial \xi}\ (x_k, 0, \Xi_b^k)\right.$$
$$\left.+\frac{\partial^2 p_m}{\partial \xi^2}\ (x_k, 0, \Xi_b^k)\cdot\ \frac{\partial \overline{p}_m}{\partial x}\ (x_k, 0, \Xi_b^k)\,i\psi'(x_k)\right\}$$
$$-\ \varphi''(0)\ \frac{\partial \overline{p}_m}{\partial \xi}\ (\overline{Z}_k)\cdot\ \frac{\partial p_m}{\partial \xi}\ (Z_k) + \frac{2}{A_k}\left|\frac{\partial p_m}{\partial \xi}\ (Z_k)\right|^2$$
$$+ 2A_k\left|\varphi'(0)\cdot\ \frac{\partial p_m}{\partial \xi}\ (Z_k)\right|^2 \leq \frac{1}{A_k}\ .$$

Dividing both members by $\frac{A_k}{\Gamma_k^2}$ we get, since $\Gamma_k \to 0,\ A_k \longrightarrow +\infty$,

$$\text{(4.70)}\qquad p_m(0, 0, \Xi_b^0) = 0\ .$$

Now, since $\left(\frac{A_k}{\Gamma_k^2} - \frac{\sqrt{A_k}}{\Gamma_k^2}\right)|p_m(Z_k)|^2 \geq 0$, dividing (4.69) by A_k we get

$$\text{(4.71)}\qquad \varphi'(0)\cdot\ \frac{\partial p_m}{\partial \xi}\ (0, 0, \Xi_b^0) = 0\ .$$

Removing all positive terms in (4.69) and letting k go to $+\infty$ we get

$$\left[\operatorname{Im}\ \left\{-i\ \frac{\partial^2\,\overline{p}_m}{\partial x\partial \xi}\cdot\frac{\partial p_m}{\partial \xi}\ \cdot\ N + i\ \frac{\partial^2 p_m}{\partial \xi^2}\cdot\frac{\partial \overline{p}_m}{\partial x}\ N\right\}\right.$$
$$\left.-\varphi''(0)\,\frac{\partial \overline{p}_m}{\partial \xi}\ \cdot\ \frac{\partial p_m}{\partial \xi}\right](0, 0, \Xi_b^0) \leq 0$$

which is contradiction with $(H.2)'$ i).

Lemma 4.9. *Let ℓ_2 and c_2 be defined in* (4.56) *and* (4.58). *Then there exists* $\sigma > 0$ *such that for any* $\varepsilon > 0$ *one can find a positive constant* C_ε *such that*

$$||\operatorname{Op}_\lambda^w(\ell_2)u||_{L^2(\mathbb{R}^n)} \le \varepsilon\,||u||_{L^2(H_{sc}^{m-1})} + \frac{C_\varepsilon}{\sqrt{\lambda}}\,||u||_{L^2(H_{sc}^{m-1})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}||v||_{H^{n_0}(\mathbb{R}^n)}) \ .$$

$$|(\operatorname{Op}_\lambda^w(c_2)u,u)| \le \varepsilon\,||u||^2_{L^2(H_{sc}^{m-1})} + \frac{C_\varepsilon}{\sqrt{\lambda}}\,||u||^2_{L^2(H_{sc}^{m-1})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}||v||_{H^{n_0}(\mathbb{R}^n)}))$$

for any $u = T_\eta^*\,Tv$, $v \in C_0^\infty(\mathbb{R}^n)$, *where* H_{sc}^m *has been defined in* (4.37).

Proof. Given $\varepsilon > 0$ let $\chi(x,\xi_a)$ in C^∞ with $0 \le \chi \le 1$ and $\operatorname{supp}\chi \subset \{|x| + |\xi_a| \le \varepsilon\}$. We claim that one can find $C_\varepsilon > 0$ such that

$$(4.72) \qquad ||\operatorname{Op}_\lambda^w(\xi_a\chi)u||_{L^2} \le \varepsilon\,||u||_{L^2(H_{sc}^{m-1})} + \frac{C_\varepsilon}{\sqrt{\lambda}}\,||u||_{L^2(H_{sc}^{m-1})} \ .$$

This follows from the sharp Gårding inequality in the class $S(1,g_2)$ ($h = 1$ for g_2). Indeed we have $\varepsilon^2\,\langle\xi_b\rangle^{2m-2} - \xi_a^2\,\chi^2\,\langle\xi_b\rangle^{2m-2} \ge 0$. Now (4.56) and (4.57) show that $\ell_2 \in S(\langle\xi_b\rangle^{m-1}, g_2)$ and $\ell_2|_{\xi_a=0} = 0$. Therefore taking $\chi = \theta(x_a,\xi_a)\cdot g(x_b)$, such that $\chi = 1$ if $|x| + |\xi_a| \le \frac{\varepsilon}{2}$ we write

$$||\operatorname{Op}_\lambda^w(\ell_2)u||_{L^2} \le ||\operatorname{Op}_\lambda^w(\ell_2\chi)u||_{L^2} + ||\operatorname{Op}_\lambda^w((1-\chi)\,\ell_2)u||_{L^2} = (1) + (2) \ .$$

We deduce from (4.72) that

$$||\operatorname{Op}_\lambda^w(\ell_2\chi)u||_{L^2} \le \varepsilon\,||u||_{L^2(H_{sc}^{m-1})} + \frac{C_\varepsilon}{\sqrt{\lambda}}\,||u||_{L^2(H_{sc}^{m-1})}\,,$$

and it follows from (4.47) that

$$||\operatorname{Op}_\lambda^w((1-\chi)\,\ell_2)u||_{L^2} \le \frac{C_N}{\lambda^N}\,||u||_{L^2(H_{sc}^{m-1})} + \mathcal{O}\Big(\mathrm{e}^{-\lambda\sigma}||v||_{H^{n_0}(\mathbb{R}^n)}\Big) \ .$$

This gives the first part of the lemma. For the second part we observe that c_2 is a sum of terms of the form $\xi_a\,c_2'(x,\xi_a)\,\xi_b^\alpha$ with $|\alpha| \le 2m-2$. Therefore $(\operatorname{Op}_\lambda^w(c_2)u,u)$ can be written as a sum of terms of the form $(\operatorname{Op}_\lambda^w(\xi_a\,c_2''(x,\xi_a)\,\xi_b^\beta)u, D_{x_b}^\gamma u)$, where $|\gamma| \le m-1$, $|\beta| \le m-1$, so

$$|(\operatorname{Op}_\lambda^w(c_2)\,u,u)| \le \varepsilon\,||u||^2_{L^2(H_{sc}^{m-1})} + \frac{C_\varepsilon}{\sqrt{\lambda}}\,||u||^2_{L^2(H_{sc}^{m-1})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}||v||^2_{H^{n_0}(\mathbb{R}^n)})) \ .$$

We are now ready to prove the Carleman estimate for Q^0.

Proposition 4.10. *Let* $Q^0 = \operatorname{Op}_\lambda^w(q_m)$ *be defined in* (4.6). *Then one can find positive constants* C_0, C_1, λ_0, σ *such that, for any* $u = T_\eta^*\,Tv$, $v \in C^\infty$, $\operatorname{supp} v \subset \left\{|x| \le \frac{1}{4A^2}\right\}$ *and* $\lambda \ge \lambda_0$, *we have*

$$\frac{C_0}{\lambda}\,||u||^2_{L^2(\mathbb{R}^{n_a},H^{m-1}_{sc}(\mathbb{R}^{n_b}))} \le C_1\,||Q^0u||^2_{L^2(\mathbb{R}^n)} + \mathcal{O}\Big(\mathrm{e}^{-\lambda\sigma}||v||^2_{H^{n_0}(\mathbb{R}^n)}\Big)\ .$$

Proof. First claim: let ℓ_1 and c_1 be defined in (4.56), (4.58). Then

$$\begin{aligned}&(A+2)\Big(||\operatorname{Op}^w_\lambda(\operatorname{Re}\ell_1)u||^2_{L^2}+||\operatorname{Op}^w_\lambda(\operatorname{Im}\ell_1)u||^2_{L^2}\Big)\\ &\quad+(\operatorname{Op}^w_\lambda(c_1)u,u)\ge\delta_0||u||^2_{L^2(H^{m-1}_{sc})}\end{aligned} \tag{4.73}$$

for large λ. (Here A has been fixed by lemma 4.8.)

Indeed let us set $a=A\,|\ell_1|^2+c_1$ (see lemma 4.8) and $a_0=a|_{x_a=0}$. Let $h_0\in C_0^\infty(\mathbb{R}^{n_a})$ be such that $h_0=\begin{cases}1 & \text{if } |x_a|\le\frac{1}{4A^2}\\ 0 & \text{if } |x_a|\ge\frac{1}{2A^2}\end{cases}$ and $0\le h_0\le 1$. Then we have

$$a+(1-h_0)(a_0-a)=h_0\,a+(1-h_0)\,a_0\ge\frac{1}{A}\,\langle\xi_b\rangle^{2m-2}\quad\text{if } |x_b|\le\frac{1}{2A^2}\ . \tag{4.74}$$

Indeed if $|x_a|\le\frac{1}{2A^2}$ then by lemma 4.8, a and a_0 satisfy (4.60) thus (4.74) is true. If $|x_a|\ge\frac{1}{2A^2}$ then $h_0=0$ and a_0 satisfies (4.60) and (4.74) is also true.

Now denoting by r_k a symbol in the class $S(\langle\xi_b\rangle^k,g_2)$ we have by (4.56) and (4.58)

$$a=|p'_m(x_b,\xi_b)|^2+2\operatorname{Im}\Big(\frac{\partial p'_m}{\partial x_b}\cdot\frac{\partial p'_m}{\partial\xi_b}\Big)(x_b,\xi_b)+\operatorname{Re}(\ell_1\cdot r_{m-1})+r_{m-2}\ .$$

Thus $a-a_0=\operatorname{Re}(\ell_1\cdot r_{m-1})+r_{m-2}$ so

$$|a-a_0|\le 2\,|\ell_1|^2+C\,\langle\xi_b\rangle^{2m-2}\ . \tag{4.75}$$

It follows from (4.60) and (4.75) that

$$(A+2)|\ell_1|^2+c_1+C\,(1-h_0)\,\langle\xi_b\rangle^{2m-2}\ge\frac{1}{A}\,\langle\xi_b\rangle^{2m-2}\quad\text{if } |x_b|\le\frac{1}{2A^2}\ . \tag{4.76}$$

Let $h_1(x_b)$ in $C^\infty(\mathbb{R}^{n_b})$ be such that $0\le h_1\le 1$ and $h_1=0$ if $|x_b|\ge\frac{1}{2A^2}$, $h=1$ if $|x_b|\le\frac{1}{4A^2}$. Thus we have, from (4.76)

$$\Big((A+2)\,|\ell_1|^2+c_1+C\,(1-h_0)\,\langle\xi_b\rangle^{2m-2}-\frac{1}{A}\,\langle\xi_b\rangle^{2m-2}\Big)\,h_1^2(x_b)\ge 0$$

for any (x,ξ_b) in $\mathbb{R}^n\times\mathbb{R}^{n_b}$, and this symbol belongs to $S(\langle\xi_b\rangle^{2m},g_1)$. Therefore we can apply the Fefferman-Phong inequality (see [H1]) and get

$$(4.77)\quad \Big(\mathrm{Op}_\lambda^w\left((A+2)|\ell_1|^2\, h_1^2\right)u,u\Big) + \left(\mathrm{Op}_\lambda^w(c_1\, h_1^2)u,u\right) \geq \frac{1}{A}\,\Big(\mathrm{Op}_\lambda^w\Big(h_1^2\,\langle \xi_b\rangle^{2m-2}\Big)u,u\Big) - C\left(\mathrm{Op}_\lambda^w(h_1^2(1-h_0))u,u)\right) - \frac{C}{\lambda^2}\,||u||^2_{L^2(H^{m-1}_{sc})}\ .$$

We can use the symbolic calculus in $S(\cdot,g_1)$. We get

$$I = \Big(\mathrm{Op}_\lambda^w\left((A+2)|\ell_1|^2\, h_1^2\right)u,u\Big) = (A+2)\Big(\left(\mathrm{Op}_\lambda^w(\ell_1^R\, h_1)^*\ \mathrm{Op}_\lambda^w(\ell_1^R\, h_1)\right. + \mathrm{Op}_\lambda^w(\ell_1^I h_1)^*\,\mathrm{OP}_\lambda^w(\ell_1^I h_1))u,u\Big) + \frac{1}{\lambda^2}\,\mathcal{O}\Big(||u||^2_{L^2(H^{m-1}_{sc})}\Big)\ .$$

Here $\ell_1^R = \mathrm{Re}\,\ell_1$ and $\ell_1^I =\ \mathrm{Im}\,\ell_1$. Thus

$$(4.78)\quad I = (A+2)\Big(||\,\mathrm{Op}_\lambda^w(\ell_1^R)u||^2_{L^2} + ||\,\mathrm{Op}_\lambda^w(\ell_1^I)u||^2_{L^2}\Big) + \mathcal{O}\left(\frac{1}{\lambda^2}\,||u||^2_{L^2(H^{m-1}_{sc})}\right)$$

because

$$\mathrm{Op}_\lambda^w(\ell_1^K)\cdot h_1 = \mathrm{Op}(\ell_1^K\, h_1) + \frac{1}{\lambda}\,\mathrm{Op}_\lambda^w(S(\langle\xi_b\rangle^{m-1},g_1))$$

for $K = R$ or I and $h_1 u = u$ since $\mathrm{supp}\,u \subset \left\{|x_b| \leq \frac{1}{4A^2}\right\}$. By the same way

$$\mathrm{Op}_\lambda^w(c_1\, h_1^2) = \mathrm{Op}(c_1)\, h_1^2 + \frac{1}{\lambda}\,\mathrm{Op}_\lambda^w\left(S(\langle\xi_b\rangle^{2m-2},g_1)\right)$$

thus

$$(4.79)\quad \left(\mathrm{Op}_\lambda^w(c_1\, h_1^2)u,u\right) = \left(\mathrm{Op}_\lambda^w(c_1)u,u\right) + \frac{1}{\lambda}\,\mathcal{O}\left(||u||^2_{L^2(H^{m-1}_{sc})}\right)\ .$$

We have also

$$(4.80)\quad \Big(\mathrm{Op}_\lambda^w(\langle\xi_b\rangle^{2m-2}\, h_1^2)u,u\Big) = ||u||^2_{L^2(H^{m-1}_{sc})} - \mathcal{O}\left(\frac{1}{\lambda}\,||u||^2_{L^2(H^{m-1}_{sc})}\right)$$

$$(4.81)\quad \Big(\mathrm{Op}_\lambda^w(h_1^2(1-h_0)\langle\xi_b\rangle^{2m-2})u,u\Big) = ||(1-h_0)u||^2_{L^2(H^{m-1}_{sc})} + \mathcal{O}\left(\frac{1}{\lambda}\,||u||^2_{L^2(H^{m-1}_{sc})}\right)\ .$$

$$(4.82)\quad ||(1-h_0)u||^2_{L^2(H^{m-1}_{sc})} \leq \frac{C_N}{\lambda^N}\,||u||^2_{L^2(H^{m-1}_{sc})}\ .$$

Thus (4.73) follows from (4.77) to (4.82).

Now from (4.53), (4.54), (4.56) we get

$$\|\operatorname{Op}^w_\lambda(\ell_1^R)u\|_{L^2} \le \|Q_R u\|_{L^2} + \|\operatorname{Op}^w_\lambda(\ell_2^R)u\|_{L^2} + \eta\,\|\operatorname{Op}^w_\lambda(\tilde\chi\, s^R_{m-1})u\|_{L^2}\ .$$

Therefore, applying Lemma 4.9, we deduce

$$\begin{cases} \|\operatorname{Op}^w_\lambda(\ell_1^R)u\|_{L^2} \le \|Q_R u\|_{L^2} + \left(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C'\eta\right)\|u\|_{L^2(H^{m-1}_{sc})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}\|v\|_{H^{n_0}(\mathbb{R}^n)}) \\ \|\operatorname{Op}^w_\lambda(\ell_1^I)u\|_{L^2} \le \|Q_I u\|_{L^2} + \left(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C'\eta\right)\|u\|_{L^2(H^{m-1}_{sc})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}\|v\|_{H^{n_0}(\mathbb{R}^n)}) \end{cases}$$
(4.83)

Using (4.55), (4.58) and lemma 4.9 we get

$$\begin{aligned} &\left|\left((\operatorname{Op}^w_\lambda(c_1) - \lambda[Q^{0*}, Q^0])u, u\right)\right| \\ &\quad \le \left(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + \eta C'\right)\|u\|^2_{L^2(H^{m-1}_{sc})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}\|v\|^2_{H^{n_0}(\mathbb{R}^n)})\ . \end{aligned} \tag{4.84}$$

It follows from (4.73), (4.83) and (4.84) that

$$\begin{aligned} \frac{\delta_0}{2}\|u\|^2_{L^2(H^{m-1}_{sc})} &\le C(A)\left(\|Q_R u\|^2_{L^2} + \|Q_I u\|^2_{L^2} + \frac{\lambda}{2}\left([Q^{0*}, Q^0]u, u\right)\right) + \\ &+ \left(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C'\eta\right)\|u\|^2_{L^2(H^{m-1}_{sc})} + \mathcal{O}(\mathrm{e}^{-\lambda\sigma}\|v\|^2_{H^{n_0}(\mathbb{R}^n)})\ . \end{aligned}$$

Taking ε and η small, then λ large we get, by (4.51), proposition 4.10.

Corollary 4.11. *Let $\tilde P_\lambda$ the operator occuring in Proposition* 2.2. *One can find positive constants* C_1, C_2, λ_0, ε_2, σ *such that for* $v \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} v \subset \{|x| \le \varepsilon_2\}$ *and* $\lambda \ge \lambda_0$ *we have*

$$\lambda\,\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^{m-1}_\lambda(\mathbb{R}^{n_b}))} \le C_1\,\|\tilde P\,Tv\|^2_{L^2_{(1+\eta)\Phi}} + C_2\,\mathrm{e}^{-\lambda\sigma}\|v\|^2_{H^{n_0}(\mathbb{R}^n)}\ . \tag{4.85}$$

Proof. By theorem 3.3, (4.85) will follow from the same estimate for $\tilde Q_\lambda$. Now $\|\tilde Q\,Tv\|_{L^2_{(1+\eta)\Phi}} = \|Q_\lambda u\|_{L^2}$ and by (4.6) we have $\sigma^w(Q_\lambda) = \lambda^{2m}\big(\sigma^w(Q^0) + \sum_{j=1}^m \lambda^{-j}\, q_{m-j}\big)$ where $q_{m-j} \in S(\langle\xi_b\rangle^{m-1}, g_2)$. Thus (4.84) follows from proposition 4.10 if λ is large enough.

5. End of the proof of the Theorems A and B

Without loss of generality we may assume that $x^0 = 0$, $\varphi(x^0) = 0$.

Let P be the differential operator under consideration in the theorems A and B and u be a C^∞ solution near the origin of the equation $Pu = 0$, with $\operatorname{supp} u \subset \{x\colon \varphi(x) \le 0\}$. Let ψ be the quadratic polynomial introduced in

(4.12) or in lemma 4.8 and $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = \begin{cases} 1 & \text{if } t \geq -\frac{\varepsilon}{2} \\ 0 & \text{if } t \leq -\varepsilon \end{cases}$ with $0 \leq \chi \leq 1$. We set

$$u_1 = \chi(\psi(x)) \cdot u \ . \tag{5.1}$$

It is classical that if ε is small enough we have $\operatorname{supp} u_1 \subset \{x \in \mathbb{R}^n\colon |x|^2 \leq C\,\varepsilon\}$ with a fixed constant C and we reduce ε in order that $\operatorname{supp} u_1 \subset \{x\colon |x| \leq \varepsilon_2\}$ where ε_2 has been fixed by corollary 4.7 (or 4.11). Now, since $Pu = 0$ we see that

$$P\,u_1 = f\ , f \in C^\infty\ , \ \operatorname{supp} f \subset \left\{x\colon -\varepsilon \leq \psi(x) \leq -\frac{\varepsilon}{2}\right\} \ . \tag{5.2}$$

We introduce a positive parameter ρ such that $\rho||\psi''|| \leq \frac{1}{2}$ and $\rho \sup_{|x|\leq 1} \frac{|\psi(x)|}{|x|} \leq \frac{1}{4}$. It follows that on the support of u_1 we have

$$\rho|\psi(x)| = \rho\,\frac{|\psi(x)|}{|x|}\,|x| \leq \frac{1}{4}\,\sqrt{C\varepsilon} \ .$$

Then we set

$$u_1 = \mathrm{e}^{-\lambda\rho\psi} v \ . \tag{5.3}$$

Then

$$P\,u_1 = \mathrm{e}^{-\lambda\rho\psi}\,P_\lambda v \tag{5.4}$$

where P_λ is defined by (2.14) with $\rho\psi$ instead of ψ. It follows that (5.2) can be written as

$$P_\lambda v = \mathrm{e}^{\lambda\rho\psi} f \ . \tag{5.5}$$

We apply proposition 2.2 and get

$$\tilde{P}_\lambda\,Tv = T\,\mathrm{e}^{\lambda\rho\psi} f \ . \tag{5.6}$$

Then corollary 4.7 (and 4.11) ensures that one can find $\sigma = \sigma(\rho) > 0$ such that

$$||Tv||^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m_\lambda(\mathbb{R}^{n_b}))} \leq C_1\lambda\,||T\,\mathrm{e}^{\lambda\rho\psi} f||^2_{L^2_{(1+\eta)\Phi}} + \mathcal{O}\left(\mathrm{e}^{-\lambda\sigma}||v||^2_{H^{n_0}(\mathbb{R}^n)}\right) \ . \tag{5.7}$$

We reduce ε in order that $\frac{1}{4}\,\sqrt{C\varepsilon} \leq \frac{1}{3}\,\sigma(\rho)$. We claim that

$$||T\,\mathrm{e}^{\lambda\rho\psi} f||_{L^2_{(1+\eta)\Phi}} = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{3}\,\varepsilon\rho}\right) \ . \tag{5.8}$$

Indeed we know from (5.2) that $\psi \leq -\frac{\varepsilon}{2}$ on the support of f. On the other hand $z_a^\alpha \, T \, \mathrm{e}^{\lambda\rho\psi} f$ is a finite sum of terms of the following kind

$$I = K(\lambda) \int \mathrm{e}^{-\frac{\lambda}{2}(z_a - y_a)^2} (z_a - y_a)^\beta \, y_a^\gamma \, \mathrm{e}^{\lambda\rho\psi(y_a, x_b)} f(y_a, x_b) \, dy_a \ .$$

Since $(z_a - y_a)_j \, \mathrm{e}^{-\frac{\lambda}{2}(z_a - y_a)^2} = \frac{1}{\lambda} \frac{\partial}{\partial y_{a_j}} \mathrm{e}^{-\frac{\lambda}{2}(z_a - y_a)^2}$ we can make integrations by parts and conclude that I is a finite sum of terms of the form

$$J = P(\lambda) \int \mathrm{e}^{-\frac{\lambda}{2}(z_a - y_a)^2 + \lambda\rho\psi(y_a, x_b)} g(y_a, x_b) \, y_a^{\gamma_1} \, D_{y_a}^{\gamma_2} f(y_a, x_b) \, dy_a$$

where P is a polynomial in λ and g a C^∞ function.

It is then easy to see that for large λ

$$\langle z_a \rangle^{n_a + 1} \, \mathrm{e}^{-\lambda(1+\eta) \, \Phi(z_a)} \, ||T \, \mathrm{e}^{\lambda\rho\psi} f(z_a, \cdot, \lambda)||_{L^2(\mathbb{R}^{n_b})} \leq C \, \mathrm{e}^{-\frac{\lambda}{3} \varepsilon\rho}$$

where C is independant of λ. Thus (5.8) follows.

We deduce from (5.7), (5.8) that

$$(5.9) \qquad ||T(\mathrm{e}^{\lambda\rho\psi} u_1)||^2_{L^2_{(1+\eta)\Phi}} = \mathcal{O}\left(\mathrm{e}^{-\lambda\delta}\right), \quad \delta = \min\left(\frac{\varepsilon\rho}{2}, \frac{1}{2} \, \sigma(\rho), \frac{1}{100}\right) \ .$$

Now since ψ is quadratic we have

$$\psi(y_a, x_b) = \psi(x_a, x_b) + \psi'_a(x_a, x_b) \cdot (y_a - x_a) + \frac{1}{2} \, A(y_a - x_a) \cdot (y_a - x_a) \, ,$$

where A is the symmetric matrix ψ''_{aa}. We have also, with $B = \psi''_{ab}$,

$$\psi'(x_a, x_b) = \psi'_a(0,0) + A \, x_a + B \, x_b = N_a + A \, x_a + B \, x_b$$

where

$$(5.10) \qquad N = (N_a, N_b) \text{ is the normal to } S \text{ at the origin} \ .$$

Thus

$$(5.11) \quad \psi(y_a, x_b) = \psi(x_a, x_b) + (N_a + A \, x_a + B \, x_b)(y_a - x_a) + \frac{1}{2} \, A(y_a - x_a)^2 \ .$$

We choosed ρ so small that

$$||\rho A|| \leq \frac{1}{2} \, , \quad ||\rho B|| \leq \frac{1}{2} \ .$$

It follows that

(5.12) $A_\rho = \mathrm{Id} - \rho A$ is symmetric and positive definite .

Let us set $X = y_a - x_a$ and $(1) = -\frac{\lambda}{2}\,(x_a - y_a)^2 + \lambda\rho\psi(y_a, x_b)$. We deduce from (5.11) that

$$(1) = -\frac{\lambda}{2}\left[X \cdot X - 2\rho V \cdot X - \rho A X \cdot X\right] + \lambda\rho\psi(x_a, x_b)$$

where

(5.13) $$V = N_a + A\,x_a + B\,x_b \ .$$

Then

$$\begin{aligned}(1) &= \lambda\rho\psi(x_a, x_b) - \frac{\lambda}{2}\left[A_\rho X \cdot X - 2\rho V \cdot X\right] \\ &= \lambda\rho\psi(x_a, x_b) - \frac{\lambda}{2}\left[\|A_\rho^{\frac{1}{2}} X\|^2 - 2\rho\, A_\rho^{-\frac{1}{2}} V \cdot A_\rho^{\frac{1}{2}} X\right] \\ &= \lambda\rho\psi(x_a, x_b) - \frac{\lambda}{2}\left[\|A_\rho^{\frac{1}{2}} X - \rho\, A_\rho^{-\frac{1}{2}} V\|^2 - \rho^2\,\|A_\rho^{-\frac{1}{2}} V\|^2\right] \ .\end{aligned}$$

Therefore

(5.14) $$T\,(\mathrm{e}^{\lambda\rho\psi}\,u_1)(x_a, x_b, \lambda) = K(\lambda)\,\mathrm{e}^{\lambda\rho\psi(x_a,x_b) + \frac{\lambda}{2}\rho^2\,\|A_\rho^{-\frac{1}{2}} V\|^2}\,S_\lambda\,u_1(x_a, x_b, \lambda)$$

(5.15) $$S_\lambda\,u_1(x_a, x_b, \lambda) = \int \mathrm{e}^{-\frac{\lambda}{2}\|A_\rho^{\frac{1}{2}}(y_a - x_a) - \rho\,A_\rho^{-\frac{1}{2}} V\|^2}\,u_1(y_a, x_b)\,dy_a \ .$$

We split the proof into two cases.

Case 1. $N_a = 0$.
Let $\tilde{\Omega} = \{(x_a, x_b) \in \mathbb{C}^{n_a} \times \mathbb{R}^{n_b}\colon |x_a| \le \delta,\ |x_b| \le \delta\}$. Then (5.9) implies that

$$\iint_{\tilde{\Omega}} \mathrm{e}^{-2\lambda(1+\eta)\,\Phi(x_a)}\,\left|T(\mathrm{e}^{\lambda\rho\psi} u_1)(x_a, x_b, \lambda)\right|^2 L(dx_a)\,dx_b = \mathcal{O}\left(\mathrm{e}^{-\lambda\delta}\right) \ .$$

Since in $\tilde{\Omega}$ we have $-\lambda(1+\eta)\,\Phi(x_a) \ge -2\lambda\delta^2 \ge -\frac{1}{2}\,\lambda\delta$ we get

(5.16) $$\iint_{\tilde{\Omega}} \left|T(\mathrm{e}^{\lambda\rho\psi} u_1)(x_a, x_b)\right|^2 L(dx_a)\,dx_b = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{2}\,\delta}\right) \ .$$

Let us set

$$\Omega = \left\{(x_a, x_b) \in \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}, |x_a| \le \frac{\delta}{2}\,, |x_b| \le \frac{\delta}{2}\right\} \ .$$

The function $x_a \mapsto T(\mathrm{e}^{\lambda\rho\psi}u_1)$ is holomorphic in $\mathbb{C}^{n_a}$. Therefore one can find a positive constant C independant of λ, x_b, ε, ρ such that

$$\iint\limits_{\Omega} \left|T(\mathrm{e}^{\lambda\rho\psi}u_1)(x_a, x_b, \lambda)\right|^2 dx_a\, dx_b \leq C \iint\limits_{\tilde{\Omega}} \left|T(\mathrm{e}^{\lambda\rho\psi}u_1)(x_a, x_b, \lambda)\right|^2 L(dx_a)\, dx_b \ .$$

According to (5.16) we get

$$\iint\limits_{\Omega} \left|T(\mathrm{e}^{\lambda\rho\psi}u_1)(x_a, x_b, \lambda)\right|^2 dx_a\, dx_b = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{2}\delta}\right) \ . \tag{5.17}$$

Using (5.14), (5.15), (5.17) and the fact that in Ω we have

$$\lambda\rho\psi(x_a, x_b) + \frac{\lambda}{2}\,\rho^2\,||A_\rho^{-\frac{1}{2}}V||^2 \geq -\lambda\rho\delta \sup_{|x|\leq 1} \frac{|\psi(x)|}{|x|} \geq -\frac{\lambda}{4}\,\delta$$

we deduce that for λ large enough

$$\iint\limits_{\Omega} |S_\lambda\, u_1(x_a, x_b)|^2\, dx_a\, dx_b = \mathcal{O}\left(\mathrm{e}^{-\frac{\lambda}{4}\delta}\right) \ . \tag{5.18}$$

Let us fix $(x_a, x_b) \in \Omega$ and set in (5.15)

$$y_a - x_a - \rho\, A_\rho^{-1}(A\, x_a + B\, x_b) = \frac{1}{\sqrt{\lambda}}\, t_a \ ,$$

we get

$$S_\lambda\, u_1(x_a, x_b, \lambda) = \frac{1}{\lambda^{\frac{n_a}{2}}} \int \mathrm{e}^{-\frac{1}{2}||A_\rho^{\frac{1}{2}}\, t_a||^2}\, u_1\left(x_a + \rho\, A_\rho^{-1}(A\, x_a + B\, x_b) + \frac{1}{\sqrt{\lambda}}\, t_a, x_b\right) dt_a$$

and Lebesgue's theorem shows that

$$\lim_{\lambda\to+\infty} \lambda^{\frac{n_a}{2}}\, S_\lambda\, u_1(x_a, x_b, \lambda) = \mathrm{Cte}\ \ u_1(A_\rho^{-1}(x_a + \rho B\, x_b), x_b) \ . \tag{5.19}$$

It follows then, from (5.18), (5.19) and Fatou's Lemma that $u_1(A_\rho^{-1}(x_a + \rho B\, x_b), x_b) = 0$. This implies that $u_1 = 0$ for $|x_a| \leq \frac{\delta}{4}$, $|x_b| \leq \frac{\delta}{4}$. Since $u_1 = u$ if δ is small enough we have proved theorem A.

Case 2. $N_a \neq 0$.
Assume $N_{a,1} = \frac{\partial\psi}{\partial x_{a,1}}(0,0) \neq 0$. In a neighborhood of the origin we can make the change of variables

$$\begin{cases} x'_{a,1} = \psi(x_a, x_b) \\ x'_{a,j} = x_{a,j} \, , & j \geq 2 \\ x'_b = x_b \quad . \end{cases}$$

The symbol of the operator P is transformed into a symbol whose coefficients are analytic in x'_a and C^∞ in x'_b in a neighborhood of the origin. Moreover all the hypotheses in the theorem are invariant. Therefore we still have the estimate (5.9) namely

$$\iint \mathrm{e}^{-2\lambda(1+\eta)\Phi(x_a)} \left|T(\mathrm{e}^{\lambda\rho x_{a,1}} u_1)(x_a, x_b, \lambda)\right|^2 L(dx_a)\, dx_b = \mathcal{O}\big(\mathrm{e}^{-\lambda\delta}\big) \tag{5.20}$$

where T is the FBI transform (2.1) where, for simplicity we have removed the factor $K(\lambda)$ i.e. with $v_a = (1, 0, \dots, 0)$

$$T\big(\mathrm{e}^{\lambda\rho\, x_a\cdot v_a} u_1\big)(x_a, x_b, \lambda) = \int \mathrm{e}^{-\frac{\lambda}{2}(x_a - y_a)^2 + \lambda\rho\, v_a\cdot y_a}\, u_1(y_a, x_b)\, dy_a \quad .$$

We see easily that

$$T\big(\mathrm{e}^{\lambda\rho\, x_a\cdot v_a} u_1\big)(x_a, x_b, \lambda) = \mathrm{e}^{\lambda\rho\, x_a\cdot v_a + \frac{\lambda}{2}\rho^2}\, T\, u_1(x_a + \rho v_a, x_b) \quad . \tag{5.21}$$

Inserting (5.21) in (5.20) and setting $x_a + \rho v_a = x'_a$ we get

$$\iint \mathrm{e}^{-2\lambda(1+\eta)\Phi(x'_a) - \lambda\rho^2 + 2\lambda\rho(\mathrm{Re}\, x'_a)\cdot v_a} \left|T\, u_1(x'_a, x_b, \lambda)\right|^2 L(dx'_a)\, dx_b = \mathcal{O}\big(\mathrm{e}^{-\lambda\delta}\big) \quad . \tag{5.22}$$

Let us consider

$$\tilde{\Omega} = \{(x_a, x_b) \in \mathbb{C}^{n_a} \times \mathbb{R}^{n_b} \colon |\mathrm{Re}\, x_a| < 2\rho,\ |\mathrm{Im} x_a| < 2\delta,\ |x_b| < \delta\} \quad .$$

For $(x_a, x_b) \in \tilde{\Omega}$ one has $2(1+\eta)\,\Phi(x_a) \leq 16\delta^2 \leq \frac{1}{2}\,\delta$ so (5.22) implies

$$\iint_{\tilde{\Omega}} \mathrm{e}^{-\lambda\rho^2 + 2\lambda\rho(\mathrm{Re}\, x_a)\cdot v_a} \left|T\, u_1(x_a, x_b, \lambda)\right|^2 L(dx_a)\, dx_b = \mathcal{O}\big(\mathrm{e}^{-\frac{\lambda}{2}\delta}\big) \quad . \tag{5.23}$$

Now since the function $x_a \mapsto \mathrm{e}^{\lambda\rho\, x_a\cdot v_a} T\, u_1(x_a, x_b, \lambda)$ is holomorphic in $\mathbb{C}^{n_a}$, it follows from Cauchy formula that we can find a positive constant C, independant of λ and x_b such that for $|\mathrm{Re}\, x_a| \leq \rho$ and $|\mathrm{Im}\, x_a| \leq \delta$ we have

$$\left|\mathrm{e}^{\lambda\rho\, x_a\cdot v_a}\, T\, u_1(x_a, x_b, \lambda)\right|^2 \leq C \int_{\substack{|\mathrm{Re}\, x_a| \leq 2\rho \\ |\mathrm{Im}\, x_a| \leq 2\delta}} \left|\mathrm{e}^{\lambda\rho\, x_a\cdot v_a}\, T\, u_1(x_a, x_b, \lambda)\right|^2 L(dx_a) \quad .$$

So we deduce from (5.23) that if $|\mathrm{Re}\, x_a| \leq \rho$, $|\mathrm{Im}\, x_a| \leq \delta$

$$\int_{|x_b| \leq \delta} \left|T\, u_1(x_a, x_b, \lambda)\right|^2 dx_b \leq C\, \mathrm{e}^{\lambda\rho^2 - 2\lambda\rho(\mathrm{Re}\, x_a)\cdot v_a - \frac{\lambda}{2}\delta} \quad . \tag{5.24}$$

On the other hand from its definition we have

$$(5.25) \qquad |T\,u_1(x_a,x_b,\lambda)| \le e^{\lambda\Phi(x_a)} \int e^{-\frac{\lambda}{2}(\mathrm{Re}\,x_a\cdot v_a - y_a\cdot v_a)^2} |u_1(y_a,x_b)|\,dy_a \ .$$

If $\mathrm{Re}\,x_a \cdot v_a < 0$ we bound the exponential, inside the integral, by one. If $\mathrm{Re}\,x_a \cdot v_a \ge 0$, since on the support of u_1 we have $y_a \cdot v_a \le 0$, we have $\mathrm{Re}\,x_a \cdot v_a - y_a \cdot v_a \ge \mathrm{Re}\,x_a \cdot v_a \ge 0$, therefore

$$(5.26) \qquad |T\,u_1(x_a,x_b,\lambda)| \le \begin{cases} C\ e^{\lambda\Phi(x_a)} & \text{if } \mathrm{Re}\,x_a\cdot v_a < 0 \\ C\ e^{\lambda\Phi(x_a)-\frac{\lambda}{2}(\mathrm{Re}\,x_a\cdot v_a)^2} & \text{if } \mathrm{Re}\,x_a\cdot v_a \ge 0. \end{cases}$$

For fixed λ let us introduce the subharmonic function

$$(5.27) \qquad w(x_a) = \int_{|x_b|\le\delta} \left| e^{\frac{\lambda}{2}(x_a\cdot v_a)^2}\, T\,u_1(x_a,x_b,\lambda)\right|^2 dx_b \ .$$

It follows from (5.24) and (5.26)

$$(5.28) \qquad w(x_a) \le C\, e^{\lambda\left[(\mathrm{Re}\,x_a\cdot v_a)^2 - (\mathrm{Im}\,x_a\cdot v_a)^2 - 2\rho\ \mathrm{Re}\,x_a\cdot v_a + \rho^2 - \frac{1}{2}\delta\right]}$$

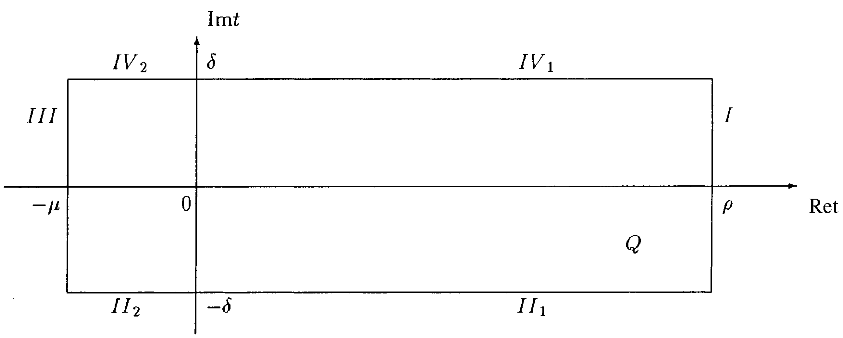
$$(5.29) \qquad w(x_a) \le \begin{cases} C\, e^{\lambda[(\mathrm{Re}\,x_a\cdot v_a)^2 + (\mathrm{Im}\,x'_a)^2]} & \text{if } \mathrm{Re}\,x_a\cdot v_a < 0 \\ C\, e^{\lambda(\mathrm{Im}\,x'_a)^2} & \text{if } \mathrm{Re}\,x_a\cdot v_a \ge 0 \end{cases}$$

where we have set $x_a = (x_a \cdot v_a, x'_a)$.

Let us fix x'_a and λ, let us set $t = x_a \cdot v_a \in \mathbb{C}$ and consider the subharmonic function

$$(5.30) \qquad \tilde{w}(t) = \frac{\mathrm{Ln}\,\frac{1}{C}\,w(t,x'_a)}{\lambda}, \quad |\mathrm{Re}\,t| \le \rho, \ \ \mathrm{Im}\,t \le \delta \ .$$

We introduce the rectangle Q drawn here the sides of which are denoted by I, II, III, IV as indicated below. Here μ is a fixed positive number such that $\mu \le \frac{1}{10}\,e^{-\frac{\pi\rho}{\delta}}$.



- On *II, III, IV* we use (5.29). We get

$$\tilde{w}(t) \leq (\text{Im } x'_a)^2 \text{ on } II_1, IV_1$$
$$\tilde{w}(t) \leq (\text{Im } x'_a)^2 + \mu^2 \text{ on } II_2, III \text{ and } IV_2 \ .$$

- On I we use (5.28). Here $\text{Re } x_a \cdot v_a = \rho$, thus

$$\tilde{w}(t) \leq \rho^2 - (\text{Im } x_a \cdot v_a)^2 - 2\rho^2 + \rho^2 - \frac{1}{2}\,\delta \leq -\frac{1}{2}\,\delta \ .$$

Summing up, we have

$$\tilde{w}(t) - (\text{Im } x'_a)^2 - \mu^2 \leq \begin{cases} 0 & \text{on } II, III, IV \\ -\frac{1}{2}\,\delta & \text{on } I \ . \end{cases} \tag{5.31}$$

Let us consider the harmonic function

$$g(t) = \frac{\cos\left(\frac{\pi}{2\delta}\text{ Im } t\right)\ \sinh\left(\frac{\pi}{2\delta}\,(\text{Re } t + \mu)\right)}{\sinh\left(\frac{\pi}{2\delta}\,(\rho + \mu)\right)} \ . \tag{5.32}$$

Then $g(t) = 0$ when $\text{Im } t = \mp\delta$ and when $\text{Re } t = -\mu$ thus $g(t) = 0$ on II, III, IV. On I we have $\text{Re } t = \rho$ so $g(t) = \cos\frac{\pi}{2\delta}\text{ Im } t \leq 1$ and $-\frac{1}{2}\,\delta \leq -\frac{1}{2}\,\delta\, g(t)$.

It follows from (5.31) that on the boundary of Q we have

$$\tilde{w}(t) - (\text{Im } x'_a)^2 - \mu^2 \leq -\frac{1}{2}\,\delta\, g(t) \ . \tag{5.33}$$

By the maximum principle we deduce from (5.33) that

$$\tilde{w}(t) - (\text{Im } x'_a)^2 - \mu^2 \leq -\frac{1}{2}\,\delta\, g(t)\,, \quad t \in Q \ .$$

Now it is easy to see that there exists a positive constant $M \geq 1$ independant of ρ such that

$$\sup_{t \in Q} ||g'(t)|| \leq \frac{M}{\delta} \ . \tag{5.35}$$

Since $g(0) = \frac{\sinh\left(\frac{\pi\mu}{2\delta}\right)}{\sinh\left(\frac{\pi}{2\delta}\,(\rho+\mu)\right)} \geq \frac{\pi\mu}{\delta}\,\mathrm{e}^{-\frac{\pi\rho}{\delta}}$, we deduce from (5.35) that

$$g(t) \geq \frac{1}{2}\,\frac{\pi\mu}{\delta}\,\mathrm{e}^{-\frac{\pi\rho}{\delta}} \text{ if } |t| \leq \frac{\pi\mu M}{2}\,\mathrm{e}^{-\frac{\pi\rho}{\delta}} \ . \tag{5.36}$$

It follows from (5.34) that

$$\tilde{w}(t) - (\mathrm{Im}\, x'_a)^2 - \mu^2 \le -\frac{1}{4}\, \pi\mu\, \mathrm{e}^{-\frac{\pi\rho}{\delta}} \text{ if } |t| \le \frac{\pi\mu M}{2}\, \mathrm{e}^{-\frac{\pi\rho}{\delta}} \ . \tag{5.37}$$

Since $\mu^2 \le \frac{1}{8}\, \pi\mu\, \mathrm{e}^{-\frac{\pi\rho}{\delta}}$, if $|\mathrm{Im}\, x'_a|^2 \le \frac{1}{16}\, \pi\mu\, \mathrm{e}^{-\frac{\pi\rho}{\delta}}$ we get

$$\tilde{w}(t) \le -\frac{1}{16}\, \pi\mu\, \mathrm{e}^{-\frac{\pi\rho}{\delta}} = -\mu_0 \ . \tag{5.38}$$

Using (5.30) and (5.27) we get if $|x_a|$ is small enough

$$\int_{|x_b| \le \delta} |T\, u_1(x_a, x_b, \lambda)|^2\, dx_b \le \mathrm{e}^{-\frac{\lambda}{2}\mu_0} \ . \tag{5.39}$$

Then we let λ go to $+\infty$, using, as in the proof of case 1, Fatou's lemma. We get $u_1 = 0$ in a neighborhood of zero. The proof of theorems A and B is complete.

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