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On the motive of a reductive group

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Contents

In this paper, we attach a motive M of Artin-Tate type to a connected, reductive group G over a field k. The construction of M follows Steinberg, who used the twisted dual motive $M^\vee(1)$ to give a formula for the order of $G(k)$, when k is finite.

When k is a local field of characteristic zero, we show the L-function $L(M)$ is finite if and only if Serre's Euler-Poincaré measure μ_G on $G(k)$ is non-zero. In this case, we obtain a local functional equation, relating $L(M) \cdot \mu_G$ to $L(M^\vee(1)) \cdot |\omega_G|$, in the one-dimensional real vector space of invariant measures on $G(k)$. Here $|\omega_G|$ is a Haar measure on $G(k)$, defined using a differential form ω_G of top degree on G (specified in Sects. 4 and 7) and the normalized valuation of k^*. When k is non-Archimedean and G is quasi-split over k, we define a smooth model $\underline{G}^0$ for G over the ring A of integers of k using the theory of Bruhat and Tits, and ω_G is a volume form on $\underline{G}^0$ over A whose restriction to the special fibre is non-zero. In this case, the functional equation is:

$$L(M) \cdot \mu_G \cdot \#H^1(k, G) = L(M^\vee(1)) \cdot |\omega_G|.$$

The general case is given in Theorem 8.1; it also involves the sign $e(G)$ Kottwitz attaches to G, which is 1 in quasi-split case.

When k is a number field, we use the L-function of M to evaluate certain adèlic integrals which arise in the trace formula. For example, assume that the connected center of G is anisotropic, and that S is a finite set of places of k, which contains all infinite places and all finite places where G/k_v is not quasi-split. Let $\mathbb{A}$ be the ring of adèles of k, and define a measure $\mu_S = \otimes \mu_v$ on the locally compact group $G(\mathbb{A})$ by taking $\mu_v = L_v(M^\vee(1)) \cdot |\omega_{G_v}|$ on $G(k_v)$ for all v not in S, and $\mu_v =$ Euler-Poincaré measure μ_{G_v} on $G(k_v)$ for all v in S.

Let $L_S(M)$ be the value at $s = 0$ of the meromorphic continuation of the Euler product $\prod_{v \notin S} L_v(M, s)$, which converges in some right half plane. Results of Siegel show that $L_S(M)$ is a rational number, which is non-zero if and only if the measure μ_S is non-zero on $G(\mathbb{A})$. In this case, we show that

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_S = L_S(M) \cdot \tau(G) \Big/ \prod_{v \in S} c(G_v)$$

where $\tau(G)$ is the Tamagawa number of G, and $c(G_v) = \#H^1(k_v, G)$ for finite v. In general, let $T \to G$ be a maximal torus which is anisotropic over k_v. Then

$$c(G_v) = \frac{\#H^1(k_v, T)}{\#(\ker\colon H^1(k_v, T) \to H^1(k_v, G))}.$$

The general case also involves the local signs $e(G_v)$ for $v \in S$; it is given in Theorem 9.9. A related result on the global ε-factor $\varepsilon(M)$, occurring in the functional equation of the L-function of M, is given in Theorem 11.5.

The local results in this paper are reformulations of work of Kottwitz, Serre, Steinberg and Tate. The global results were suggested by the work of Harder and G. Prasad.

Notation

Throughout the paper, k is a field, k^s is a separable closure of k, and $\Gamma = \mathrm{Gal}(k^s/k)$. From Sect. 4 until the end, we assume that k has characteristic zero, so k^s is an algebraic closure of k.

We let G denote a connected, reductive group over k, C denote the connected component of the center of G (which is a torus), and G^{der} denote the derived subgroup of G. The group G^{der} is connected and semi-simple, and we let G^{sc} denote its simply-connected covering group.

If Y is a finite set, $\#Y$ denotes its cardinality.

1 The definition of M

We first define the motive M for quasi-split groups G over k. For general groups G, we will define M as the motive of the quasi-split inner form of G.

Let S be a maximal split torus in G, and let T be the centralizer of S in G. Since we are assuming that G is quasi-split, T is a maximal torus in G. Let $W = N_{G(k^s)}(T(k^s))/T(k^s)$ be the Weyl group of T in G, over the separable closure k^s of k. Then $\Gamma = \mathrm{Gal}(k^s/k)$ acts on W, and ${}_kW = W^\Gamma$ is the relative Weyl group of S in G [3, p. 13].

The $\mathbb{Q}$-vector space

$$E = X^\bullet(T) \otimes \mathbb{Q} = \mathrm{Hom}_{k^s}(T, \mathbb{G}_m) \otimes \mathbb{Q} \tag{1.1}$$

admits an action of $W \rtimes \Gamma$. Chevalley [9] proved that the algebra of W-invariants in the symmetric algebra on E is isomorphic to a symmetric algebra on a graded $\mathbb{Q}$-vector space V. If $R = \mathrm{Sym}^\bullet(E)^W$, and R_+ is the ideal of elements of degree ≥ 1 in R, then we define the graded vector space $V = \oplus_{d \geq 1} V_d$ by:

$$V = R_+/R_+^2. \tag{1.2}$$

One can show, using results of Steinberg [25, p.17], that V is isomorphic (as a representation of Γ over $\mathbb{Q}$) to E. The advantage of passing to the space V is that it is graded, and that each summand V_d (the primitive invariants of degree d) is a representation of Γ. Some useful formulae are [18. pg. 289]

$$\dim G = \sum_{d \geq 1} (2d - 1) \dim V_d \tag{1.3}$$

$$\# W = \prod_{d \geq 1} d^{\dim V_d} \tag{1.4}$$

Let $\mathbb{Q}(1) = H_1(\mathbb{G}_m)$ be the Tate motive, of rank 1 and weight -2 over k [11, p. 325]. If N is an Artin motive over k, given by rational representation of Γ, then $N(n) = N \otimes \mathbb{Q}(1)^{\otimes n}$ is an Artin-Tate motive, of weight $-2n$.

We define the motive M of G by:

$$M = \bigoplus_{d \geq 1} V_d(1 - d). \tag{1.5}$$

The rank of M is the rank of G over k^s. Since each representation V_d is self-dual, the twisted dual of M is the motive:

$$M^\vee(1) = \bigoplus_{d \geq 1} V_d(d). \tag{1.6}$$

The weights of M are all ≥ 0, and the weights of $M^\vee(1)$ are all ≤ -2.

2 Examples and properties of M

If $G = T$ is a torus, then $W = 1$ and $V = V_1 = E$. In this case, $M = X^\bullet(T) \otimes \mathbb{Q}$ is an Artin motive, which determines T up to isogeny over k [19, pp. 124–125].

If G is split over k, then $T = S$ and each V_d is the trivial representation of Γ. In this case, M is a Tate motive. For example, if $G = GL_n$ then

$$M = \mathbb{Q} + \mathbb{Q}(-1) + \mathbb{Q}(-2) + \ldots + \mathbb{Q}(1-n).$$

Here M does not even determine G up to isogeny: the split groups Sp_{2n} and SO_{2n+1} both give the motive

$$M = \mathbb{Q}(-1) + \mathbb{Q}(-3) + \ldots + \mathbb{Q}(1-2n),$$

but they are not isogenous if $n \geq 3$ and $\mathrm{char}(k) \neq 2$.

Lemma 2.1 1) *If G is isogenous to G' over k, then $M = M'$.*

2) *If $G = G_1 \times G_2$, then $M = M_1 + M_2$.*

3) *Let K be a finite separable extension of k, and G_K be a connected, reductive group over K with motive $M_K = \bigoplus_{d \geq 1} V_d(1-d)$. Then $G = \mathrm{Res}_{K/k}(G_K)$ has motive $M = \mathrm{Ind}_{K/k} M_K = \bigoplus_{d \geq 1} \mathrm{Ind}_{K/k} V_d(1-d)$.*

To prove this, one passes to the quasi-split inner forms, and compares the representations of $W \rtimes \Gamma$ on E.

Corollary 2.2 *The canonical isogeny $C \times G^{\mathrm{der}} \to G$ gives a decomposition of M by weight:*

$$M_C = V_1 \qquad \textit{of weight} = 0$$

$$M_{G^{\mathrm{der}}} = \bigoplus_{d \geq 2} V_d(1-d) \qquad \textit{of weights} \geq 2$$

When G is quasi-split over k, the Tate motive

$$M^\Gamma = \bigoplus_{d \geq 1} V_d^\Gamma(1-d) \tag{2.3}$$

can be computed from the relative root system of S. The restriction $X^\bullet(T) \to X^\bullet(S)$ identifies $X^\bullet(S) \otimes \mathbb{Q}$ with the Γ-coinvariants of E, as a representation of ${}_kW$, and the primitive generators of $\mathrm{Sym}^\bullet(E_\Gamma)^{{}_kW}$ can be identified with $V_\Gamma = V^\Gamma$ as a graded $\mathbb{Q}$-vector space.

For example, assume $G = U_{2n+1}$ is a quasi-split unitary group, associated to a Hermitian space of dimension $2n+1$ over the separable quadratic extension K of k. Let $\mathbb{Q}[\varepsilon]$ be the rank 1 Artin motive of $C = U_1$, so ε is the non-trivial quadratic character of $\mathrm{Gal}(K/k)$. Then

$$M = \mathbb{Q}[\varepsilon] + \mathbb{Q}(-1) + \mathbb{Q}[\varepsilon](-2) + \mathbb{Q}(-3) + \ldots + \mathbb{Q}[\varepsilon](-2n),$$

$$M^{\Gamma} = \mathbb{Q}(-1) + \mathbb{Q}(-3) + \ldots + \mathbb{Q}(1-2n).$$

The relative root system has type BC_n. Note that M^{Γ} is the motive associated to either of the maximal split subgroups SO_{2n+1} or Sp_{2n} of G (cf. [5, pp. 121–122]).

3 Finite fields

In this section, we assume that k is finite, of cardinality q. Then Γ is topologically generated by the geometric Frobenius element F, which has eigenvalue q^{-1} on the Tate motive $\mathbb{Q}(1)$.

The twisted dual $M^{\vee}(1)$ of M was introduced by Steinberg [25, p. 79], who obtained the formula:

$$\text{(3.1)} \qquad \begin{aligned} \#G(k)/q^{\dim G} &= \det(1 - F|M^{\vee}(1)) \\ &= \prod_{d\geq 1} \det(1 - F|V_d(d)) \\ &= \prod_{d\geq 1} \det(1 - Fq^{-d}|V_d) \end{aligned}$$

A nice formula involving M was shown to me by W-T. Gan. Assume that G is split, simply-connected, and simply-laced, and let R be the reflection representation of $G(k)$. Then R is defined over $\mathbb{Q}$, and Kilmoyer has observed that R is the unique irreducible complex representation of $G(k)$ with

$$\dim R^{P(k)} = \ell - \ell_P$$

for all parabolic subgroups $P \subset G$, where ℓ is the rank of G and ℓ_P is the semi-simple rank of a Levi factor of P.

Let γ be a semi-simple element in $G(k)$, and let G_γ be the centralizer of γ in G. Then G_γ is connected and reductive; let $M_\gamma = \bigoplus_{d\geq 1} V_{d,\gamma}(1-d)$ be the motive of G_γ over k. Then we have the formula

$$\text{(3.2)} \qquad \mathrm{Tr}(\gamma|R) = \mathrm{Tr}(F|M_\gamma) = \sum_{d\geq 1} \mathrm{Tr}(F \cdot q^{d-1}|V_{d,\gamma}).$$

When γ is regular, so $G_\gamma = T$ is a maximal torus in G, this is a restatement of a result of Lusztig [17, p. 334]. The general case was deduced from results of Lusztig by Gan. For example, when $\gamma = 1$, $G_\gamma = G$ and

$$\text{(3.3)} \qquad \dim(R) = \mathrm{Tr}(F|M) = \sum_{d\geq 1} \dim V_d \cdot q^{d-1} = \sum_{i=1}^{\ell} q^{m_i},$$

where $m_1, m_2, \ldots, m_\ell$ are the exponents of G.

4 *p*-adic fields

In the next three sections, we assume that k is a local, non-Archimedean field. Let A be the ring of integers of k, π a uniformizing element, and q be the cardinality of the finite residue field $A/\pi A$. To avoid questions of inseparable isogenies, we will assume that k has characteristic zero (although the results should be true in general). Let G be a connected, reductive group over k, with motive M.

Let k_1 be the maximal unramified extension of k contained in k^s, and let $I \lhd \Gamma$ be the inertia subgroup which fixes k_1. Our aim in this section is to give a geometric description of the Artin-Tate motive

$$M^I = \bigoplus_{d \geq 1} V_d^I(1-d) \tag{4.1}$$

over $A/\pi A$, using the theory of Bruhat and Tits [6]. As a by-product, we will define a canonical Haar measure $|\omega_G|$ on the group $G(k)$.

Assume that G is quasi-split over k, and let $X = X(G^{sc})$ be the (semi-simple) Bruhat-Tits building of G over k. This is a polysimplicial complex, which admits an action of $G(k)$. The center acts trivially, and the action extends to the larger group $G^{ad}(k)$ [28, p. 46].

Let $\mathscr{A} \subset X$ be the apartment associated to the maximal split torus S, and let T be the centralizer of S in G. Let x be a special vertex in $\mathscr{A}$, and let α be a root of S which lies in the reduced system [3, p. 12]

$$\Phi_s = \{\alpha \in \Phi : \alpha/2 \notin \Phi\} \tag{4.2}$$

Let

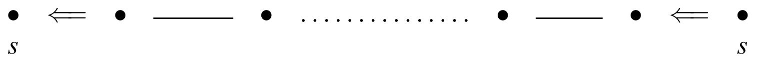
$$U_{\alpha,x} \subset U_\alpha(k)$$

be the compact open subgroup of the unipotent group U_α which is defined by the special vertex x [7, 292ff], and let $\underline{T}^0$ be the smooth group scheme over A which is the connected component of the Néron-Raynaud model of the torus T [6, ch.10].

Bruhat and Tits [7, 316ff], [16, Ch. 2] construct a smooth group scheme $\underline{G}_x^0$ over A, with general fibre G and connected special fibre, such that $\underline{G}_x^0(A)$ is the open compact subgroup of $G(k)$ generated by $\underline{T}^0(A)$ and the root subgroups $U_{\alpha,x}, \alpha \in \Phi_s$. If G is simply-connected, $\underline{G}_x^0(A)$ is the subgroup of $G(k)$ fixing the point x in X [7, p. 329]. In general, $\underline{G}_x^0$ is given by the root datum $(\underline{T}^0, \underline{U}_{\alpha,x})$ over A.

Let A_1 be the ring of integers in k_1. Then $\underline{G}_x^0(A_1)$ is the subgroup of $G(k_1)$ generated by $\underline{T}^0(A_1)$ and the root groups $\underline{U}_{\alpha,x}(A_1)$ in $U_\alpha(k_1)$. Since A_1 is strictly Henselian, this description of points determines the group scheme. The isomorphism class of $\underline{G}_x^0$ over A depends only on the $G^{ad}(k)$-orbit of the special vertex x in B. When the relative root system is reduced ($\Phi = \Phi_s$), there is a single orbit of $G^{ad}(k)$ on the special vertices [28, p. 47]. In this case, $\underline{G}^0 = \underline{G}_x^0$ is well-defined up to isomorphism.

When Φ is not reduced, we must fix an orbit. Our convention is as follows: for each component of the local Dynkin diagram of the type



we choose the special vertex at the right end of the diagram. For example, assume that G^{sc} has a factor $SU_{2n+1}(K/k)$. If the quadratic extension is unramified over k, we are choosing the hyperspecial vertex. If K is ramified over k, we are choosing the vertex whose reduction (mod π) is the group SO_{2n+1}, *not* the group Sp_{2n} (cf. [20, p. 95]). With this convention, $\underline{G}^0 = \underline{G}^0_x$ is well-defined over A in all cases. Let $\overline{G}$ denote the special fibre of $\underline{G}^0$, $R_u(\overline{G})$ the unipotent radical of $\overline{G}$, and $\overline{G}^{\mathrm{red}}$ the reductive quotient $\overline{G}/R_u(\overline{G})$. Both $R_u(\overline{G})$ and $\overline{G}^{\mathrm{red}}$ are smooth, connected group schemes over $A/\pi A$.

The reduction $\overline{S}$ of the smooth subgroup scheme $\underline{S}^0$ is a maximal split torus in $\overline{G}$ [28, p.52]. The reduction $\overline{T}$ of the smooth subgroup scheme $\overline{T}^0$ centralizes $\overline{S}$; its image $\overline{T}^{\mathrm{red}}$ in the quotient $\overline{G}^{\mathrm{red}}$ is a maximal torus in a Borel subgroup $\overline{B} \subset \overline{G}^{\mathrm{red}}$.

Proposition 4.5 *The motive of $\overline{T}^{\mathrm{red}}$ over $A/\pi A$ is E^I, and the (absolute) Weyl group of $\overline{T}^{\mathrm{red}}$ in $\overline{G}^{\mathrm{red}}$ is $W^I = {}_{k_1}W$. The motive of $\overline{G}^{\mathrm{red}}$ over $A/\pi A$ is M^I.*

This follows from the general theory of Bruhat and Tits (cf. [28, p. 52–53]). We emphasize that the group scheme $\underline{G}^0$ over A and its reduction $\overline{G}^{\mathrm{red}}$ over $A/\pi A$ are only considered when G is quasi-split over k

Corollary 4.6 *The following conditions are all equivalent.*

1) $M = M^I$.

2) *The quasi-split group G is split over k_1.*

3) *The group $\underline{G}^0(A)$ is a hyperspecial maximal compact subgroup of the locally compact group $G(k)$.*

4) *The group C is split over k_1, and the building X contains hyperspecial points.*

We continue to assume that G is quasi-split over k. The group scheme $\underline{G}^0$ over A with general fibre G gives rise to a canonical Haar measure on the locally compact group $G(k)$. Indeed, $\mathrm{Lie}(\underline{G}^0)$ is an A-lattice inside the k-vector space $\mathrm{Lie}(G)$, so $\overset{\mathrm{top}}{\wedge}\mathrm{Lie}(\underline{G}^0)$ is a free A-module of rank 1 inside the k-vector space $\overset{\mathrm{top}}{\wedge}\mathrm{Lie}(G)$ of dimension 1. Let ω_G be a differential of top degree on G over k which generates the A-submodule $\mathrm{Hom}(\overset{\mathrm{top}}{\wedge}\mathrm{Lie}(\underline{G}^0), \mathrm{A})$. Then ω_G is determined up to multiplication by a unit of A.

We say such differentials have good reduction (mod π), as $\overline{\omega}_G$ is non-zero on $\overline{G}$. Let $|\omega_G|$ be Haar measure on $G(k)$ which corresponds to a differential ω_G with good reduction (mod π), and to the canonical absolute value $\|_v : k^* \to \mathbb{R}^*_+$ with $|\pi|_v = q^{-1}$ [14, pp. 258–259].

Proposition 4.7 *We have*

$$\int_{\underline{G}^0(A)} |\omega_G| = \det(1 - F|M^\vee(1)^I).$$

Moreover, if $f : G' \to G$ is a central isogeny, then

$$f^*(|\omega_G|) = |\#\ \ker f(k^s)|_v \cdot |\omega'_G|$$

as Haar measures on $G'(k)$.

Proof. Let $\underline{G}^1(A)$ be the kernel of the surjective homomorphism $\underline{G}^0(A) \to \overline{G}(A/\pi A)$. If $\hat{G}$ is the formal group of dimension $n = \dim(G)$ over A associated to $\underline{G}^0$, then we have an analytic isomorphism

$$\hat{G}(\pi A) \xrightarrow{\sim} \underline{G}^1(A)$$

such that the pull-back of $|\omega_G|$ is the Haar measure $dx_1 \cdots dx_n$ on $(\pi A)^n$. Hence

$$\begin{aligned} \int_{\underline{G}^0(A)} |\omega_G| &= \#\overline{G}(A/\pi A)/q^n \\ &= \#\overline{G}^{\mathrm{red}}(A/\pi A)/q^{\dim \overline{G}^{\mathrm{red}}} \\ &= \det(1 - F|M^\vee(1)^I) \end{aligned}$$

by Proposition 4.5 and Steinberg's formula (3.1).

The second formula follows from the fact that $f^*(\omega_G) = (\#\ker f(k^s)) \cdot \omega_{G'}$ for a differential $\omega_{G'}$ with good reduction on G'. Indeed, the group schemes $\underline{G}_x^0$ and $\underline{G}_x'^0$ are associated to the same special point x in X. The isogeny f identifies $U'_{\alpha,x}$ with $U_{\alpha,x}$, so the cokernel of $f_* : \mathrm{Lie}(\underline{G}^{0'}) \hookrightarrow \mathrm{Lie}(\underline{G}^0)$ is the cokernel of $f_* : \mathrm{Lie}(\underline{T}^{0'}) \hookrightarrow \mathrm{Lie}(\underline{T}^0)$. The latter map has determinant equal to $\#(\ker f(k^s))$.

We have defined $\underline{G}^0$ and $|\omega_G|$ only for groups G which are quasi-split over k. The definition of the canonical Haar measure $|\omega_G|$ can be extended to general G as follows. Let H be the quasi-split inner form of G, and fix an inner twisting $\Psi : G \to H$ over k^s. Assume ω_H has good reduction (mod π), and define ω_G on G over k (cf. [15, pp. 68–69]) by pull-back

$$\omega_G = \Psi^*(\omega_H). \tag{4.8}$$

Again, this is unique up to multiplication by a unit in A, so $|\omega_G|$ is a well-defined Haar measure on $G(k)$.

The definition of $|\omega_G|$ on $G(k)$ by transfer from H makes the explicit calculation of local integrals difficult. However, on any connected reductive G we can define a measure $|v_G|$, where v_G is a volume form for the integral structure given by an Iwahori subgroup B of $G(k)$. Then $v_G = \Psi^*(v_H)$ [13, p. 632], and a short computation on the quasi-split inner form H shows that:

$$\omega_H = \pi^N \cdot v_H, \tag{4.9}$$

with

$$N = \sum_{d\geq 1} (d-1) \dim V_d^I. \tag{4.10}$$

Hence $|\omega_G| = q^{-N} \cdot |v_G|$, so

$$\int_B |\omega_G| = q^{-N} \det(1 - Fw_G | E(1)^I) \tag{4.11}$$

where w_G is an element of $W^I = {}_{k_1}W$ associated to Ψ [13, p.633].

5 The local L-function

We continue to assume that k is a local, non-Archimedean field of characteristic zero. If M is the motive of G over k, we have the L-functions

$$\begin{cases} L(M) & = \det(1 - F|M^I)^{-1} \\ L(M^\vee(1)) & = \det(1 - F|M^\vee(1)^I)^{-1} \end{cases} \tag{5.1}$$

The latter is a positive rational number, whose inverse appears in Proposition 4.7. The L-function of M can be infinite, positive, or negative.

Let μ_G be Serre's Euler-Poincaré measure on $G(k)$ [21, pp. 139–141]. This is an invariant measure, possibly negative or zero, with the property that, for all discrete, torsion-free, co-compact subgroups Δ of $G(k)$:

$$\int_{\Delta\backslash G(k)} \mu_G = \sum_{i=0}^{\dim G} (-1)^i \dim H^i(\Delta, \mathbb{Q}). \tag{5.2}$$

Proposition 5.3 *The following conditions are all equivalent.*

1) *The measure μ_G is non-zero on $G(k)$.*
2) *The L-function $L(M)$ is finite.*
3) *The connected center C of G is anisotropic.*
4) *There is a maximal anisotropic torus $T \to G$.*

Proof. The equivalence of 1), 3), and 4) is due to Serre [21, p. 151]. Since

$$\begin{aligned} L(M)^{-1} &= \det(1 - F|M^I) \\ &= \prod_{d\geq 1} \det(1 - F \cdot q^{d-1} | V_d^I) \end{aligned}$$

we see that $L(M)^{-1}$ is non-zero provided that $V_1^{\Gamma} = 0$. By Corollary 2.2, this is equivalent to condition 3) on the connected center.

Let $e(G) = \pm 1$ be the sign that Kottwitz attaches to the group G. If H is the quasi-split inner form of G over k, then [12, pp. 289–290]

$$e(G) = (-1)^{\mathrm{rank}(G/k) - \mathrm{rank}(H/k)} \tag{5.4}$$

We also recall that the pointed set $H^1(k, G)$ is finite, and has the structure of an abelian group. The following is our main result in the local non-Archimedean case.

Theorem 5.5 *Assume that the connected center C of G is anisotropic. Then*

$$L(M) \cdot \mu_G \cdot e(G) \cdot \# H^1(k, G) = L(M^{\vee}(1)) \cdot |\omega_G|$$

in the space of invariant measures on $G(k)$.

6 The functional equation

We now give the proof of Theorem 5.5, which we view as a functional equation for the L-function of M in the one dimensional real vector space of invariant measures on $G(k)$. We first prove the result for quasi-split groups G, when $e(G) = 1$, then deduce the general result.

When G is quasi-split and simply-connected, we have $H^1(k, G) = 1$ [14, p. 255]. Since

$$\int_{\underline{G}^0(A)} L(M^{\vee}(1)) \cdot |\omega_G| = 1$$

by Proposition 4.7, the identity in Theorem 5.5 is equivalent to the formula

$$\int_{\underline{G}^0(A)} \mu_G = \det(1 - F | M^I) = L(M)^{-1}. \tag{6.1}$$

Since G is simply-connected, $\underline{G}^0(A)$ is the maximal compact subgroup of $G(k)$ fixing the special vertex x in X.

Let $B \subset \underline{G}^0(A)$ be an Iwahori subgroup. Then B is the inverse image of a Borel subgroup $\overline{B} \subset \overline{G}^{\mathrm{red}}$ under the reduction map $\underline{G}^0(A) \to \overline{G}(A/\pi A)$ [28 pp.54–55,]. Hence the index of B in $\underline{G}^0(A)$ is equal to the index of $\overline{B}(A/\pi A)$ in $\overline{G}^{\mathrm{red}}(A/\pi A)$. Let $\sigma = F^{-1}$ be an arithmetic Frobenius in the Galois group Γ/I. Using Proposition 4.5, we find that:

$$(\underline{G}^0(A) : B) = \frac{\det(1 - \sigma | M^{\vee}(1)^I)}{\det(1 - \sigma | E(1)^I)}. \tag{6.2}$$

On the other hand, Serre [21, p. 148] evaluated the integral

$$\int_B \mu_G = 1/W(\underline{q}) \tag{6.3}$$

in terms of the Poincaré series $W(t)$ of the affine Weyl group. This series was, in turn, evaluated by Steinberg [25, p. 28]. Using the translation provided by Kottwitz [13, p. 634], we find that

$$W(\underline{q}) = \frac{\det(1-\sigma|M^\vee(1)^I)}{\det(1-\sigma|E(1)^I)} \cdot L(M). \tag{6.4}$$

Combining (6.2) – (6.4), we obtain a proof of (6.1). We note that (6.1) was proved by Serre in the case when G is split over k [21, p.151]; there we have:

$$\det(1-F|M^I) = \prod_{d\geq 1}(1-q^{d-1})^{\dim V_d} = \prod_{i=1}^{\ell}(1-q^{m_i}).$$

Next consider the case when $G = T$ is a torus. Since we are assuming the conditions of Proposition 5.3 hold, T is anisotropic over k and μ_T is the Haar measure on $T(k)$ with

$$\int_{T(k)} \mu_T = 1. \tag{6.5}$$

Since

$$\int_{\underline{T}^0(A)} L(M^\vee(1)) \cdot |\omega_T| = 1 \tag{6.6}$$

we see that Theorem 5.5 is equivalent to the formula

$$L(M) \cdot \#H^1(k,T) = (T(k) : \underline{T}^0(A)). \tag{6.7}$$

Since T is anisotropic, Tate local duality gives an isomorphism of finite abelian groups:

$$\begin{aligned} H^1(k,T) &\simeq X_\bullet(T)_\Gamma \\ &= \operatorname{coker}(1-F|X_\bullet(T)_I), \end{aligned} \tag{6.8}$$

where $X_\bullet(T) = \mathrm{Hom}_{k^s}(\mathbb{G}_m, T)$.

The abelian group $P = X_\bullet(T)_I$ is finitely generated, so we have an exact sequence of Γ/I-modules

$$0 \to P_{\mathrm{tor}} \to P \to P/P_{\mathrm{tor}} \to 0$$

with P_{tor} finite, and P/P_{tor} a free $\mathbb{Z}$-module of finite rank. The kernel of $(1-F)$ on P/P_{tor} is zero, as it is a free sub-module of rank equal to the dimension of E^Γ (which is zero, as T is anisotropic). Applying the snake lemma to the endomorphism $(1-F)$ of the above diagram, we find that:

$$\# \operatorname{coker}(1-F|P) = \# \operatorname{coker}(1-F|P_{\mathrm{tor}}) \cdot \#\operatorname{coker}(1-F|P/P_{\mathrm{tor}})$$
$$\#\ker(1-F|P) = \#\ker(1-F|P_{\mathrm{tor}}).$$

Since P_{tor} is finite, we also have

$$\# \ker(1-F|P_{\mathrm{tor}}) = \# \operatorname{coker}(1-F|P_{\mathrm{tor}}).$$

Hence we have:

$$\#H^1(k,T) = \# \ker(1-F|P) \cdot \# \operatorname{coker}(1-F|P/P_{\mathrm{tor}}) \tag{6.9}$$

The group $T(k)/\underline{T}^0(A)$ is isomorphic to

$$\underline{T}(A)/\underline{T}^0(A) = \overline{T}/\overline{T}^0(A/\pi A),$$

where $\underline{T}$ is the Néron-Raynaud model for T over A (which is a smooth, commutative group scheme, locally of finite type over A) [6, Ch. 10]. As an étale group scheme over $A/\pi A$

$$\overline{T}/\overline{T}^0 \simeq X_\bullet(T)_I = P.$$

Hence

$$(T(k) : \underline{T}^0(A)) = \# \ker(1-F|P) \tag{6.10}$$

Since P/P_{tor} is dual to $X^\bullet(T)^I$, we have

$$\begin{aligned} \# \operatorname{coker}\left(1-F\middle|P/P_{\mathrm{tor}}\right) &= \det\left(1-F\middle|P/P_{\mathrm{tor}}\right) \\ &= \det\left(1-F\middle|X^\bullet(T)^I\right) \\ &= L(M)^{-1}. \end{aligned}$$

Combining (6.9) – (6.11) gives a proof of (6.7).

To complete the proof of the functional equation in the quasi-split case, we show it is compatible with products and central isogenies. Since G is isogenous to the product $G^{sc} \times C$, where G^{sc} is simply-connected and C is a torus, and we have checked the functional equation in those two cases, we will be done. The compatibility with isogenies hinges on Tate's formula for the Euler characteristic of the kernel [22, p. 109].

Proposition 6.12 1) *If Theorem 5.5 is true for the groups G_1 and G_2, it is true for the product $G = G_1 \times G_2$.*

2) *If Theorem 5.5 is true for the group G_K over the finite separable extension K/k, it is true for the group $G = \mathrm{Res}_{K/k}(G_K)$.*

3) *Let $1 \to F \to G' \xrightarrow[f]{} G \to 1$ be the exact sequence of separable isogeny, so $F = \ker f$ is a finite commutative étale group scheme over k. If Theorem 5.5 is true for either G or G', it is true for the other group. Moreover, if Theorem 5.5 is true for both G and G', then we have Tate's formula:*

$$\chi(F) = \#H^0(k,F) \cdot \#H^2(k,F)/\#H^1(k,F) = |\#F(k^s)|_v.$$

Proof. 1) In this case, $\mu_G = \mu_{G_1} \otimes \mu_{G_2}$ [21, p. 143], and $|\omega_G| = |\omega_{G_1}| \otimes |\omega_{G_2}|$. The latter follows from the fact that $\underline{G}^0 = \underline{G}_1^0 \times \underline{G}_2^0$ as smooth group schemes over A. Since

$$L(M) = L(M_1)L(M_2), L(M^\vee(1)) = L(M_1^\vee(1))L(M_2^\vee(1)), e(G) = e(G_1)e(G_2),$$

and $H^1(k,G) = H^1(k,G_1) \times H^1(k,G_2)$, the result follows.

2) In this case, $L(M) = L(M_K)$ and $L(M^\vee(1)) = L(M^\vee(1))$ by Artin's formalism for induction. Similarly, $e(G_K) = e(G)$ [12, p. 295], and $H^1(k,G) = H^1(K,G_K)$. Since $G(k) = G_K(K)$, we clearly have μ_G identified with the Haar measure μ_{G_K}. But $\underline{G}^0 = \mathrm{Res}_{A_K/A}\underline{G}_K^0$, as this gives a smooth connected group scheme over A with the correct points in A_1 [7, p. 218ff]. Hence $|\omega_G|$ is identified with $|\omega_{G_K}|$.

3) Serre shows that $f^*(\mu_G)$ is the measure [21, p. 152]

$$\frac{\#H^0(k,F)}{\#\big(G(k)/fG'(k)\big)} \cdot \mu_{G'} \qquad \text{on } G'(k)$$

On the other hand, since C is anisotropic, the cohomology sequence (of finite abelian groups):

$$1 \to G(k)/fG'(k) \to H^1(k,F) \to H^1(k,G') \to H^1(k,G) \to H^2(k,F) \to 1$$

is exact. Hence

$$\#H^1(k,G) \cdot f^*(\mu_G) = \chi(F) \cdot \#H^1(k,G') \cdot \mu_{G'}$$

as measures on $G'(k)$.

By Proposition 4.7, we have

$$f^*(|\omega_G|) = |\#F(k^s)|_v \cdot |\omega_{G'}|$$

Since $e(G) = e(G')$ and the L-function is unchanged under isogeny, we deduce that the combination of Theorem 5.5 for G (resp. G') and Tate's formula $\chi(F) = |\#F(k^s)|_v$ implies Theorem 5.5 for G' (resp. G). Moreover if one has Theorem 5.5 for both G and G', this implies Tate's formula for $\chi(F)$. (Thus, our proof of Theorem 5.5 for tori gives a proof of the formula for $\chi(F)$, similar to Tate's original argument).

We have now completed the proof of Theorem 5.5 for quasi-split groups G. To finish the proof in the general case, we let H be the quasi-split inner form of G and appeal to Kottwitz's results on the compatibility of the measures $e(G) \cdot \mu_G$ and $\mu_H = e(H) \cdot \mu_H$ [13, p. 631]. Let $\Psi : G \to H$ be an inner twisting over k^s. Then by definition we have

$$\Psi^*(|\omega_H|) = |\omega_G|,$$

and by Kottwitz's results

$$\Psi^*(\mu_H) = \Psi^*(e(H) \cdot \mu_H) = e(G) \cdot \mu_G.$$

Since $H^1(k, G)$ and $H^1(k, H)$ are both dual to $Z(\hat{G})^\Gamma = Z(\hat{H})^\Gamma$, they are isomorphic. Since the motive M of G is, by definition, the motive of H, we are done.

7 The real case

We now consider the local functional equation when k is Archimedean, although only the case when $k = \mathbb{R}$ is non-trivial. Assume G is connected and reductive over k with motive M. Again, we have the local L-function value in $\mathbb{R}^* \cup \{\infty\}$. When $k \simeq \mathbb{C}$ we have

$$(7.1) \qquad \begin{cases} L(M) &= \prod_{d \geq 1} \Gamma_{\mathbb{C}}(1-d)^{\dim V_d} \\ L(M^\vee(1)) &= \prod_{d \geq 1} \Gamma_{\mathbb{C}}(d)^{\dim V_d} \end{cases}$$

with $\Gamma_{\mathbb{C}}(s) = 2 \cdot 2\pi^{-s} \cdot \Gamma(s)$. When $k = \mathbb{R}$, let $V_d^\pm$ be the subspace of V_d on which complex conjugation acts by ± 1. Then

$$(7.2) \qquad \begin{cases} L(M) &= \prod_{d \geq 1} \Gamma_{\mathbb{R}}(1-d)^{\dim V_d^+} \Gamma_{\mathbb{R}}(2-d)^{\dim V_d^-} \\ L(M^\vee(1)) &= \prod_{d \geq 1} \Gamma_{\mathbb{R}}(d)^{\dim V_d^+} \Gamma_{\mathbb{R}}(d+1)^{\dim V_d^-} \end{cases}$$

with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

Lemma 7.3 The following conditions are all equivalent

1) *The Euler-Poincaré measure μ_G is non-zero on $G(k)$.*
2) *The L-function $L(M)$ is finite.*
3) *The group G has a compact inner form over k.*
4) *There is a maximal anisotropic torus $T \to G$.*

Proof. This is clear when $k = \mathbb{C}$, when the conditions hold only when $G = 1$. When $k = \mathbb{R}$, the equivalence of 1), 3), and 4) was shown by Serre [21, p. 136]. It is well-known that condition 3) is equivalent to the fact that $V_d = V_d^{(-1)^d}$ for all $d \geq 1$, which is precisely what is needed to show that $L(M)$ is finite.

We henceforth assume that $k = \mathbb{R}$, and that G has a compact inner form G^c. Let $\Psi : G \to G^c$ be an inner twisting over $\mathbb{C}$. We will define a canonical Haar measure $|\omega_G|$ on $G(\mathbb{R})$ by first defining a measure $|\omega_{G^c}|$ on the compact group $G^c(\mathbb{R})$, following Bourbaki, and then transferring that measure to $G : |\omega_G| = \Psi^*(|\omega_{G^c}|)$.

For simplicity in notation, first assume $G = G^c$ is compact. Let $\mathfrak{g} = \mathrm{Lie}(G/\mathbb{R})$; by Bourbaki [1, p. 122] it suffices to define a Haar measure $|\omega_{\mathfrak{g}}|$ on $\mathfrak{g}$. Let $T \to G$ be a maximal torus, and $\mathfrak{t} \to \mathfrak{g}$ the corresponding Cartan sub-algebra. Let $\Gamma(T)$ be the kernel of the exponential map $\mathfrak{t} \to T$; then $\Gamma(T) \simeq 2\pi i X_\bullet(T)$ in $\mathfrak{t} \otimes \mathbb{C}$. Let Φ be the roots of $\mathfrak{t} \otimes \mathbb{C}$ on $\mathfrak{g} \otimes \mathbb{C}$, and let $\{X_\alpha\}_{\alpha \in \Phi}$ be a Chevalley system in $\mathfrak{g} \otimes \mathbb{C}$ which satisfies: $\overline{X}_\alpha = X_{-\alpha}$. Then [1, p. 17]:

$$
\begin{aligned}
u_\alpha &= X_\alpha + X_{-\alpha} \\
v_\alpha &= i(X_\alpha - X_{-\alpha})
\end{aligned}
$$

are elements of $\mathfrak{g}$, with $u_\alpha \wedge v_\alpha = -2iX_\alpha \wedge X_{-\alpha}$ in $\bigwedge^2 \mathfrak{g}$. Bourbaki [1, p. 112] shows that

$$
\mathfrak{g}_{\mathbb{Z}} = \left\{ \frac{1}{2\pi} \Gamma(T), \; u_\alpha, v_\alpha \right\}_{\alpha \in \Phi}
$$

gives a Lie algebra over $\mathbb{Z}$ with $\mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{R} = \mathfrak{g}$. There is therefore a unique Haar measure $|\omega_{\mathfrak{g}}|$ on $\mathfrak{g}$ such that

$$
\int_{\mathfrak{g}/\mathfrak{g}_{\mathbb{Z}}} |\omega_{\mathfrak{g}}| = 2^{\#(\Phi^+)}
$$

The associated Haar measure $|\omega_G|$ on $G(\mathbb{R})$ satisfies [1, p. 122]:

$$
\int_{G(\mathbb{R})} |\omega_G| = \prod_{d \geq 1} \frac{(2\pi)^{d \cdot \dim V_d}}{(d-1)!}. \tag{7.4}
$$

Having defined $|\omega_G|$ on the compact group $G = G^c$, we can transfer it to any inner form of G^c by an inner twisting. We henceforth assume G is an arbitrary inner twist of G^c.

Let $e(G) = \pm 1$ be the sign that Kottwitz attaches to G. If H is the quasi-split inner form of G, then [12, p. 289]

$$
e(G) = (-1)^{\frac{1}{2}\dim(X_G) - \frac{1}{2}\dim(X_H)} \tag{7.5}
$$

where X_G and X_H are the symmetric spaces of G and H respectively.

Proposition 7.6 *Assume that G has an anisotropic maximal torus $T \to G$, and let W^c be the compact Weyl group of T in G. Then*

$$L(M) \cdot \mu_G \cdot e(G) \cdot \frac{2^{\dim T}}{\#(W/W^c)} = L(M^\vee(1)) \cdot |\omega_G|$$

in the space of Haar measures on $G(k)$.

Proof. Again, Kottwitz [13, p. 631] has shown that the measures $e(G) \cdot \mu_G/\#(W/W^c)$ and $e(G^c)\mu_{G^c}$ are compatible. Hence, it suffices to prove Proposition 7.6. when $G = G^c$ is compact. In this case, μ_G has volume 1 on $G(\mathbb{R})$, and the volume of $|\omega_G|$ is given by formula (7.4).

For $d \geq 2$ even, we find that

$$\frac{L(\mathbb{Q}(1-d))}{L(\mathbb{Q}(d))} = \frac{1}{2}\frac{(2\pi i)^d}{(d-1)!}$$

For $d \geq 1$ odd, and ε the sign character of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ we find that

$$\frac{L(\mathbb{Q}[\varepsilon](1-d))}{L(\mathbb{Q}[\varepsilon](d))} = \frac{1}{2i}\frac{(2\pi i)^d}{(d-1)!}$$

Since $V_d = V_d^{(-1)^d}$ by hypothesis, we have

$$\frac{L(M)}{L(M^\vee(1))} \cdot 2^{\dim T} = (-1)^m \cdot \prod_{d \geq 1} \frac{(2\pi)^{d \cdot \dim V_d}}{(d-1)!} \tag{7.7}$$

where $m = \frac{1}{4}(\dim G + \dim M^+ - \dim M^-)$. Checking case by case, we find that $m = \frac{1}{2}\dim(X_H)$, where H is the quasi-split inner form of G. Hence $(-1)^m = e(G)$. Since $W = W^c$ in this case, a combination of (7.4) and (7.7) proves Proposition 7.6.

8 The general local result

We end by restating Theorem 5.5 and Proposition 7.6 in a common manner.

Theorem 8.1 *Let k be a local field of characteristic zero, and let G be a connected reductive group over k. Assume that G has a maximal anisotropic torus T, and let $D(T, G)$ be the finite set* $\ker\,(H^1(k, T) \to H^1(k, G))$. *Then*

$$L(M) \cdot \mu_G \cdot e(G) \cdot \frac{\#H^1(k, T)}{\#D(T, G)} = L(M^\vee(1)) \cdot |\omega_G|$$

in the real vector space of invariant measures on $G(k)$, where μ_G is Euler-Poincaré measure, and $|\omega_G|$ is the Haar measure defined in sects. 4 *and* 7.

We will henceforth denote the cohomological invariant

$$c(G) = \frac{\#H^1(k,T)}{\#D(T,G)}. \tag{8.1}$$

It depends only on G. In the non-Archimedean case

$$c(G) = \#H^1(k,G) \tag{8.2}$$

as the map $H^1(k,T) \to H^1(k,G)$ is a surjective group homomorphism. Since C is anisotropic, we also have the formula

$$c(G) = \#Z(\hat{G})^{\Gamma}, \tag{8.3}$$

where $\hat{G}$ is the complex dual group. In the real case [23, p. 13–14]:

$$c(G) = \frac{2^{\dim T}}{\#(W/W^c)}. \tag{8.4}$$

9 Adèlic integrals

We now assume that k is a number field, and let $\mathbb{A}$ be the ring of adèles of k. If G is connected and reductive over k, then $G(k)$ is a discrete subgroup of the locally compact group $G(\mathbb{A})$ [26 p. 116].

Let S be a finite set of places of k, which includes the infinite places. For v not in S, let μ_v be the Haar measure $L_v(M^\vee(1)) \cdot |\omega_{G_v}|$ on $G(k_v)$. For v in S, let μ_v be Euler-Poincaré measure on $G(k_v)$.

For almost all places v, the group G is unramified over k_v (i.e. G is quasi-split, and split by an unramified extension) [28, pp. 55–56]. For almost all places v where G is unramified, the measure μ_v defined above has volume 1 on any hyperspecial maximal compact subgroup of $G(k_v)$. We may therefore define the product measure

$$\mu_S = \bigotimes_v \mu_v \qquad \text{on} \quad G(\mathbb{A}). \tag{9.1}$$

Our aim is to calculate the integral of μ_S over $G(k) \setminus G(\mathbb{A})$, using the global L-function of M.

The L-function $L_S(M)$ is defined as the value at $s = 0$ of the function

$$L_S(M,s) = \prod_{v \notin S} \det(1 - F_v \cdot q_v^{-s} | M^{I_v})^{-1}. \tag{9.2}$$

This Euler product converges for $\mathrm{Re}(s) \gg 0$, and has a meromorphic continuation to the entire complex plane [27, p. 16]. We make a similar definition for the L-function $L_S(M^\vee(1))$. If S consists only of infinite places, we will write $L(M)$ and $L(M^\vee(1))$ for $L_S(M)$ and $L_S(M^\vee(1))$ respectively. Then we have:

$$(9.3)\qquad \begin{cases} L(M) &= \prod_{d\geq 1} L(V_d, 1-d) \\ L(M^\vee(1)) &= \prod_{d\geq 1} L(V_d, d) \end{cases}$$

where $L(V_d, s)$ is the Artin L-series of the rational representation V_d of $\mathrm{Gal}(k^s/k)$.

Proposition 9.4 *The following conditions are equivalent.*

1) *The connected center C of G is anisotropic.*

2) *The quotient space $G(k) \setminus G(\mathbb{A})$ has finite volume, for any Haar measure μ on $G(\mathbb{A})$.*

3) *The L-function $L(M^\vee(1))$ is finite.*

Proof. The equivalence of 1) and 2) is due to Borel and Harish-Chandra [4]. We will show that 1) is equivalent to 3). The product $\prod_{d\geq 2} L(V_d, d)$ is given by an absolutely convergent Euler product, so is finite and non-zero. Hence $L(M^\vee(1))$ is finite if and only if $L(V_1, 1)$ is finite. But by Corollary 2.2, $L(V_1, s)$ is the Artin L-function of the representation $V_1 = X^\bullet(C) \otimes \mathbb{Q}$ of $\mathrm{Gal}(k^s/k)$. This is finite at $s = 1$ if and only if $V_1^\Gamma = 0$ [27, p. 16], [30, p. 124], which is equivalent to the condition that C is anisotropic.

Proposition 9.5 *The value $L_S(M)$ is finite and rational, for any S.*

If $L(M^\vee(1))$ is finite, the following conditions are all equivalent.

1) *The measure μ_S is non-zero on $G(\mathbb{A})$.*

2) *The Euler-Poincaré measure $\prod_{v\in S} \mu_v$ is non-zero on $\prod_{v\in S} G(k_v)$.*

3) *The value $L_S(M)$ is non-zero.*

Proof. The first statement follows from (9.3), and Siegel's results on the values of Artin L-series at negative integers [24].

The conditions 1) and 2) are equivalent without any hypothesis on $L(M^\vee(1))$ as for $v \notin S$ μ_v is a positive measure on $G(k_v)$. To see the equivalence of 2) and 3), define the complete L-functions

$$(9.6)\qquad \begin{cases} \Lambda(M) &= \prod_{v\in S} L_v(M) \cdot L_S(M) \\ \Lambda(M^\vee(1)) &= \prod_{v\in S} L_v(M^\vee(1)) \cdot L_S(M^\vee(1)) \end{cases}$$

Then Artin's functional equation is given by [pg.18,26], [pg.328,10].

$$(9.7)\qquad \Lambda(M) = \varepsilon(M) \cdot \Lambda(M^\vee(1)),$$

with $\varepsilon(M)$ a positive real number, whose square is an integer divisible only by the primes ramified in M or k. More precisely, if $f(\mathrm{Ind}\ V_d)$ is the Artin conductor of the representation $\mathrm{Ind}_{k/\mathbb{Q}}(V_d)$, $\mathfrak{f}(V_d)$ is the Artin conductor of V_d, and d_k is discriminant of k over $\mathbb{Q}$, then

$$(9.8) \qquad \varepsilon(M) = |d_k|^{\dim G/2} \cdot \prod_{d \geq 1} \mathbb{N}(\mathfrak{f}(V_d))^{d-\frac{1}{2}} = \prod_{d \geq 1} f(\mathrm{\ Ind\ } V_d)^{d-\frac{1}{2}}.$$

In any case, from (9.7) and the hypothesis that $L(M^\vee(1))$ is finite, we deduce that $\Lambda(M)$ is finite and non-zero. Hence $L_S(M)$ is non-zero precisely when $\prod_{v \in S} L_v(M)$ is finite. By Proposition 5.3, this is equivalent to the condition that $\prod_{v \in S} \mu_v$ is non-zero. Our main global result is the following integral formula, which holds when the connected center C of G is anisotropic over k.

Theorem 9.9 *Assume that $L(M^\vee(1))$ is finite and that $L_S(M)$ is non-zero. For v in S, let $e(G_v)$ and $c(G_v)$ be the signs and cohomological invariants attached to G/k_v in sect.* 8, *and let $\tau(G)$ denote the Tamagawa number of G.*
Then

$$\int_{G(k)\backslash G(\mathbb{A})} \mu_S = L_S(M) \cdot \tau(G) \Big/ \prod_{v \in S} e(G_v) c(G_v).$$

Note 9.10 We may replace the product $\prod_{v \in S} e(G_v)$ of signs in the integral formula by the product $\prod_{v \notin S} e(G_v)$, as

$$(9.10) \qquad \prod_v e(G_v) = 1 \qquad [12, \text{p. } 297].$$

If G/k_v is quasi-split for all $v \notin S$, then $e(G_v) = 1$ for all $v \notin S$, and hence $\prod_{v \in S} e(G_v) = 1$. This gives the formula stated in the introduction.

10 The integral formula

We now present a proof of Theorem 9.9. The method is similar to the local case: by a series of reductions we are left with the cases where either G is quasi-split, simply-connected, and absolutely quasi-simple over k or G is a one dimensional anisotropic torus over k. We check these cases by explicit calculation.

Throughout this section, the hypothesis that $L(M^\vee(1))$ is finite is assumed. Thus, the connected center C of G is anisotropic. We will not always assume that $L_S(M)$ is non-zero; if it is zero then $\mu_S = 0$ by Proposition 9.5. In all cases we will refer to the integral formula of Theorem 9.9 as $i(G, S)$. If S consists only of infinite places, we refer to the formula as $i(G)$, and write the measure μ_S simply as μ_G.

Proposition 10.1 1) *Assume that $i(G,S)$ is true, and that $S' \supset S$. Then $i(G,S')$ is true.*

2) *Assume that H is an inner form of G over k, and that $i(G,S)$ is true. Then $i(H,S)$ is true.*

Proof. 1) Write $S' = S \cup T$. Then

$$L_{S'}(M) = L_S(M) \cdot \prod_{v \in T} \det(1 - F_v | M^{I_v}).$$

On the other hand, by Theorem 8.1,

$$\mu_{S'} = \mu_S \cdot \prod_{v \in T} \det(1 - F_v | M^{I_v}) / e(G_v) c(G_v).$$

Hence $i(G,S)$ implies $i(G,S')$. The converse is also true, provided that $\prod_{v \in T} \det(1 - F_v | M^{I_v}) \neq 0$.

2) Let ω be a non-zero differential form of top degree on G over k. We define the Tamagawa measure $|\omega_G|$ on $G(\mathbb{A})$ (assuming that C is anisotropic) by the formula:

$$|\omega_G| = \bigotimes_v L_v(M^\vee(1)) |\omega|_v / \Lambda(M^\vee(1)) \cdot |d_k|^{\dim(G)/2} \tag{10.2}$$

where d_k is the discriminant of k over $\mathbb{Q}$. Then $|\omega_G|$ is independent of the choice of $\omega \neq 0$, and we have the formula [19, p. 124].

$$\int_{G(k) \backslash G(\mathbb{A})} |\omega_G| = \tau(G). \tag{10.3}$$

Hence $i(G,S)$ is equivalent to the formula:

$$\prod_{v \in S} e(G_v) c(G_v) \cdot \mu_S = L_S(M) \cdot |\omega_G| \tag{10.4}$$

in the space of measures on $G(\mathbb{A})$. This, in turn, is equivalent to the following identity of real numbers:

$$\begin{aligned} \prod_v |\omega_{G_v}| / |\omega|_v &= \frac{\Lambda(M)}{\Lambda(M^\vee(1)) \cdot |d_k|^{(\dim G)/2}} \\ &= \varepsilon(M) / |d_k|^{(\dim G)/2} \\ &= \prod_{d \geq 1} \mathbb{N}(\mathfrak{f}(V_d))^{d - \frac{1}{2}}. \end{aligned} \tag{10.5}$$

In the product on the left hand side of (10.5), almost all of the terms are equal to 1.

Let $\Psi : G \to H$ be an inner twisting over k^s. If $\mu_H = \otimes\mu_v$ is a measure on $H(\mathbb{A})$, we define $\Psi^*(\mu_H) = \otimes\Psi^*(\mu_v)$ on $G(\mathbb{A})$. Here we transfer $\mu_v = c|\omega|_v$ on $H(k_v)$ to a measure on $G(k_v)$ by the usual formula: $\Psi^*(\mu_v) = c \cdot |\Psi^*\omega|_v$. Clearly, the transfer of Tamagawa measure $|\omega_H|$ on $H(\mathbb{A})$ is Tamagawa measure on $G(\mathbb{A})$: $\Psi^*(|\omega_H|) = |\omega_G|$ (cf. [15, p. 69–71]).

But by Theorem 8.1, we have

$$\Psi^*\left(\prod_{v\in S} e(H_v)c(H_v)\cdot\mu_{S,H}\right) = \prod_{v\in S} e(G_v)c(G_v)\cdot\mu_{S,G}.$$

If $i(G,S)$ is true, we have (10.4) on $G(\mathbb{A})$. This implies the corresponding equality of measures on $H(\mathbb{A})$. Therefore $i(H,S)$ is true.

Note. In the proof of part 2) of Proposition 10.1, we do not use the deeper fact, due to Kottwitz [13] that $\tau(G) = \tau(H)$. Nor is the theorem that $\tau(G) = 1$ for simply-connected groups used anywhere in the proof of Theorem 9.9. However, it is critical in the applications of the integral formula to the trace formula.

By Proposition 10.1, we may reduce to the case when G is quasi-split over k, and S contains only the infinite places. Then $e(G_v) = 1$ for all places v, and we must show that (10.4) holds:

$$\prod_{v|\infty} c(G_v)\cdot\mu_G = L(M)\cdot|\omega_G|.$$

If $G \neq 1$, and k has a complex place, both sides of this identity are zero. Hence, we may assume that k is totally real, and that the real Lie group $G(k\otimes\mathbb{R}) = \prod_{v|\infty} G(k_v)$ has a compact inner form. Then both sides of (10.4) are non-zero measures on $G(\mathbb{A})$.

Proposition 10.6 1) *Assume that* $G = G_1 \times G_2$. *If* $i(G_1)$ *and* $i(G_2)$ *are true, so is* $i(G)$. *If* $i(G)$ *and* $i(G_1)$ *are true, so is* $i(G_2)$. *If* $i(G^n)$ *is true for any* $n \geq 1$, *then* $i(G)$ *is true.*

2) *Assume that* $G = \mathrm{Res}_{K/k}(G_K)$. *Then* $i(G)$ *is true if and only if* $i(G_K)$ *is true.*

3) *Let* $1 \to F \to G' \xrightarrow{f} G \to 1$ *be the exact sequence of a central isogeny from* G' *to* G. *Then* $i(G)$ *is true if and only if* $i(G')$ *is true.*

Proof. 1) Since $M = M_1 + M_2$, the terms in (10.5) for G are just the products of the terms for G_1 and G_2. Since both sides of (10.5) are positive real numbers, $i(G^n)$ for any $n \geq 1$ implies $i(G)$.

2) If we identify $G(\mathbb{A}) = G_K(\mathbb{A}_K)$, then μ_G is identified with μ_{G_K} and $|\omega_G|$ is identified with $|\omega_{G_K}|$. The other terms are equal, as $M = \mathrm{Ind}\ M_K$.

3) Let $\omega' = f^*(\omega)$, so $|\omega_{G'}| = f^*(|\omega_G|)$. To show that $i(G)$ is invariant under isogeny, we use the fact that $L(M)$ is an isogeny invariant, and show that:

$$f^*\left(\prod_{v|\infty} c(G_v) \cdot \mu_G\right) = \prod_{v|\infty} c(G'_v) \cdot \mu_{G'}. \tag{10.7}$$

This follows from our local results and product formula. By Proposition 4.7, we have

$$f^*(\mu_v) = |\#F(k^s)|_v \cdot \mu'_v$$

for all finite v, and similarly one shows that

$$f^*(c(G_v)\mu_v) = |\#F(k^s)|_v \cdot c(G'_v)\mu'_v$$

for all real v. Then (10.7) follows from the product formula:

$$\prod_v |\#F(k^s)|_v = 1.$$

By Proposition 10.6, we are now reduced to proving $i(G)$ in the following two cases:

a) G is an absolutely quasi-simple, simply-connected group, which is quasi-split over k.
b) G is a one dimensional torus, which is anisotropic over $k \otimes \mathbb{R}$, and is split by totally imaginary quadratic extension K/k.

To see this, we use the isogeny $G^{sc} \times C \to G$ to reduce to simply-connected groups and tori. Since $G^{sc} = \prod_i \mathrm{Res}_{k_i/k}(G_{k_i})$, with G_{k_i} simply-connected and absolutely quasi-simple, this reduces simply-connected groups to case a). Similarly, since C is anisotropic over $k \otimes \mathbb{R}$, we have an isogeny from the product $\prod_i \mathrm{Res}_{k_i/k}(T_i) \times C^n$ to $\prod_j \mathrm{Res}_{k_j/k}(T_j)$, where all T_i have dimension 1 and are split by totally imaginary quadratic extensions K_i/k_i (cf. [19, p. 125], [10, p. 548]). Thus the case of a torus reduced to b).

In case a) $i(G)$ is equivalent to a result of G. Prasad. Fix a non-zero differential ω on G over k. For finite v, Prasad defines the real number γ_v by

$$\gamma_v = |\omega_{G_v}| / |\omega|_v.$$

This ratio is equal to 1 for almost all v. For real v, he defines γ_v by

$$\gamma_v = c(G_v) \cdot \mu_v / c(G^c) e(G^c) |\omega|_v$$

where μ_v is Euler-Poincaré measure on $G(k_v)$, and G^c is the compact inner form of G over k_v. Thus $\gamma_v \cdot |\omega|_v$ is the unique Haar measure on $G(k_v)$ whose transfer to $G^c(k_v)$ has volume $= 1$. Prasad's main result is a formula for the product [20, p. 96]:

$$\prod_v \gamma_v = \mathbb{N}_{k/\mathbb{Q}}(d_{L/k})^{\frac{1}{2}s(G)} \cdot \left(\prod_{i=1}^{r} \frac{(m_i)!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]}.$$

Here L is a minimal field extension (of degree ≤ 3 over k) whose Galois closure splits G, $d_{L/k}$ is the discriminant of L over k, $s(G)$ is an integer associated to G [20, pp. 93–94], and $m_1, \ldots, m_r$ are the exponents of G [20 p. 96].

Since for all real v,

$$\mu_v \cdot c(G_v) = \frac{L_v(M^\vee(1))}{L_v(M)} \cdot |\omega_{G_v}|$$

by Proposition 7.6, and

$$\prod_{i=1}^{r} \frac{(m_i)!}{(2\pi)^{m_i+1}} = \frac{L_v(M^\vee(1))}{L_v(M) \cdot e(G^c) \cdot c(G^c)}$$

by (7.7) (the degrees d_i of the invariants are given by $d_i = m_i + 1$), Prasad's result is equivalent to the statement:

$$\prod_v |\omega_{G_v}|/|\omega|_v = \mathbb{N}_{k/\mathbb{Q}}(d_{L/k})^{\frac{1}{2}s(G)}.$$

By (10.5), we are therefore reduced to proving that:

$$\prod_{d \geq 1} \mathfrak{f}(V_d)^{2d-1} = d_{L/k}^{s(G)}. \tag{10.8}$$

This follows a calculation of the ramification of M. If G is quasi-split but not split, the Galois group of the splitting field acts non-trivially on V_d, when d is odd. It acts trivially on V_d when d is even, except when G is of type D_{2r} and $d = 2r$, when it acts faithfully. This gives the formula (10.8).

In case b), we have $L(M) = L(\chi, 0)$, where χ is the non-trivial quadratic character of $\mathrm{Gal}(K/k)$. This can be evaluated as a ratio of zeta functions, whose leading terms at $s = 0$ are given by the class-number formula [27, pg. 48]:

$$\begin{aligned} L(M) &= -\frac{h_K R_K}{w_K} \Big/ -\frac{hR}{w} \\ &= \frac{h_K}{h} \cdot \frac{w}{w_K} \cdot \frac{2^{[k:\mathbb{Q}]-1}}{(U_K : \mu_K \cdot U)} \end{aligned}$$

Here h_K and h are the class numbers of K and k, U_K and U are their unit groups, $w = 2$ is the order of $\mu_k = \langle \pm 1 \rangle$, and w_K is the order of μ_K, the group of roots of unity in K. Since $c_v(G) = 2$ for all real v, to prove $i(G)$ we must show that

(10.9) $$\mu_G = \frac{h_K}{h \cdot w_K \cdot (U_K : \mu_K \cdot U)} \cdot |\omega_G|$$

as Haar measures on $G(\mathbb{A})$.

Let $M = \prod_v M_v$ be the maximal compact subgroup of $G(\mathbb{A})$. Since $M_v = G(k_v)$ for all real v, $\int_{M_v} \mu_v = 1$. Similarly, if v is unramified in K/k, then $M_v = \underline{G}^0(A_v)$ and $\int_{M_v} \mu_v = 1$. However, if v is ramified in K/k, then $\underline{G}^0(A_v)$ has index 2 in $M_v = \underline{G}(A_v) = G(k_v) = K_v^*/k_v^*$, and consists of the elements $A_{K,v}^*/A_v^*$ with even valuation. Hence $\int_{M_v} \mu_v = 2$, and $\int_M \mu_G = 2^t$, with t the number of finite places v in k which are ramified in K.

Let $h(G)$ be the cardinality of the finite quotient group $G(\mathbb{A})/G(k) \cdot M$. Since $G(k) \cap M = G(k)_{\mathrm{tor}}$ has order w_K, we find that

$$\int_{G(k)/G(\mathbb{A})} \mu_G = \frac{2^t \cdot h(G)}{w_K}.$$

On the other hand, since $\tau(G) = 2$ by [19, p. 128], we have

$$\int_{G(k)/G(\mathbb{A})} |\omega_G| = 2,$$

and (10.9) is equivalent to the formula:

(10.10) $$h(G) = \frac{h_K}{2^{t-1} \cdot h \cdot (U_K : \mu_K U)}$$

for the class-number of G. This is standard genera theory, and we sketch the proof.

Since $G(k) = (K^*)_{\mathbb{N}=1}$ and $G(\mathbb{A}) = (\mathbb{A}_K^*)_{\mathbb{N}=1}$, we consider the snake lemma for the diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K^* \cdot \left(\prod_{v|\infty} K_v^* \times \hat{A}_K^*\right) & \longrightarrow & \mathbb{A}_K^* & \longrightarrow & C_K & \longrightarrow & 1 \\
 & & \Big\downarrow \mathbb{N} & & \Big\downarrow \mathbb{N} & & \Big\downarrow \mathbb{N} & & \\
1 & \longrightarrow & k^* \cdot \left(\prod_{v|\infty} k_v^* \times \hat{A}^*\right) & \longrightarrow & \mathbb{A}^* & \longrightarrow & C & \longrightarrow & 1
\end{array}$$

Here A is the integers of k, A_K is the integers of K, C is the class group of k, and C_K is the class group of K.

This gives a long exact sequence of abelian groups:

$$1 \to G(\mathbb{A})/G(k) \cdot M \longrightarrow \ker\,(\mathbb{N} : C_K \to C) \searrow$$

$$1 \leftarrow \mathrm{coker}(\mathbb{N}\colon C_K \to C) \leftarrow \mathbb{A}^*\big/\mathbb{N}\mathbb{A}_K^* \underset{f}{\leftarrow} k^*\left(\prod_{v|\infty} k_v^* \times \hat{A}^*\right)\Big/\mathbb{N}K^*\left(\prod_{v|\infty} \mathbb{N}K_v^* \times \mathbb{N}\hat{A}_K^*\right)$$

These groups are all finite, with the exception of the source and target of f. Replacing those with $(\ker f)$ and $(\operatorname{coker} f)$ respectively, we get:

$$\begin{aligned} h(T) &= \frac{\#\ker(\mathbb{N} : C_K \to C) \cdot \#\operatorname{coker} f}{\#\operatorname{coker}(\mathbb{N} : C_K \to C) \cdot \#\ker f} \\ &= \frac{h_K}{h} \cdot \frac{\#\operatorname{coker} f}{\#\ker f}. \end{aligned} \tag{10.11}$$

By local class field theory, the quotient

$$\prod_{v|\infty} k_v^* \times \hat{A}^* \Big/ \prod_{v|\infty} \mathbb{N}K_v^* \times \mathbb{N}\hat{A}_K^*$$

is an elementary abelian 2-group of order $2^{t+[k:\mathbb{Q}]}$. By global class field theory, the map

$$k^*/\mathbb{N}K^* \to \mathbb{A}^*/\mathbb{N}A_K^*$$

is an injection (as K/k is cyclic), with cokernel of order $2 = [K : k]$. On the other hand:

$$k^* \cap \prod_{v|\infty} k_v^* \cdot \hat{A}^* = U$$

$$\mathbb{N}K^* \cap \prod_{v|\infty} \mathbb{N}K_v^* \cdot \mathbb{N}\hat{A}_K^* = \mathbb{N}U_K$$

$$(U : \mathbb{N}U_K) = \frac{(U : U^2)}{(\mathbb{N}U_K : U^2)} = \frac{2^{[k:\mathbb{Q}]}}{(U_K : \mu_K \cdot U)}$$

Putting these together gives:

$$\frac{\#\operatorname{coker} f}{\#\ker f} = \frac{2}{2^t \cdot (U_K : \mu_K U)},$$

which, combined with (10.11), gives the desired formula (10.10) for the class-number of G.

This concludes the proof of Theorem 9.9.

11 A general global result

The identity (10.5), in its final form:

$$\prod_v |\omega_{G_v}|/|\omega|_v = \varepsilon(M)/|d_k|^{\dim G/2}$$

makes no reference to $L(M^\vee(1))$, or to $L_S(M)$, and is true in greater generality than Theorem 9.9. However, to make sense of the left hand side, we must define the canonical Haar measures $|\omega_{G_v}|$ on $G(k_v)$ in all cases.

When k_v is non-Archimedean, $|\omega_{G_v}|$ is defined in Sect.4. We recall that when G is quasi-split over k_v, we have

$$\int_{\underline{G}^0(A)} |\omega_{G_v}| \cdot L_v(M^\vee(1)) = 1 \tag{11.1}$$

by Proposition 4.7. In general, if H is the quasi-split inner form of G over k_v and $\Psi : G \to H$ is an inner twisting over k_v^s, we define $|\omega_{G_v}| = \Psi^*|\omega_{H_v}|$ by transfer.

In the Archimedean case, we have only defined $|\omega_{G_v}|$ when $G(k_v)$ has a compact inner form. When $G(k_v)$ is compact, we have

$$\int_{G(k_v)} |\omega_{G_v}| = \prod_{d\geq 1} \frac{(2\pi)^{d\cdot \dim V_d}}{(d-1)!} \tag{11.2}$$

by formula (7.4). We first do the general case G when $k_v = \mathbb{R}$. Let G^c be the compact form of G over $\mathbb{R}$, and choose an isomorphism $\Psi : G \to G^c$ over $\mathbb{C}$. Then $\overline{\Psi} = \alpha \circ \Psi$ with α an automorphism of G^c over $\mathbb{C}$. We have the formula

$$\det(\alpha|\mathrm{Lie}(G^c)) = (-1)^{\dim(G/K)} \tag{11.3}$$

where K is the maximal compact subgroup of G. In particular, the differential

$$\omega_{G_v} = \Psi^*(\omega_{G_v^c}) \cdot i^{\dim(G/K)} \tag{11.4}$$

is defined over $\mathbb{R}$, and $|\omega_{G_v}|$ is the canonical Haar measure on $G(\mathbb{R})$. When $k_v = \mathbb{C}$, we have $G(k_v) = G'(\mathbb{R})$ with $G' = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, G$, and we define $|\omega_{G_v}|$ to be equal to the canonical Haar measure on $G'(\mathbb{R})$.

The general global theorem, which is an extension of Prasad's work [20 pp. 95–96], is the following.

Theorem 11.5 *Let k be a number field, and let G be a connected, reductive group over k with motive M. Let ω be a non-zero differential of top degree on G over k and let $|\omega_{G_v}|$ be the canonical Haar measure on $G(k_v)$, for each place v of k. Then $|\omega_{G_v}| = |\omega|_v$ for almost all v, and*

$$\prod_v |\omega_{G_v}|/|\omega|_v = \varepsilon(M)/|d_k|^{\frac{\dim G}{2}}.$$

This formula should also hold when $k = F(X)$ is a function field of a curve X over a finite field F, replacing $|d_k|$ with q^{2g-2}, where g is the genus of X, and q is the cardinality of F.

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