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Tight contact structures and Seiberg–Witten invariants

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1. Introduction and statement of results

Contact structures are the odd-dimensional analogue of symplectic structures. Although much is known, the present understanding of both kinds of structures is far from complete, even in low dimensions. Due to the work of Cliff Taubes is it now a fact that there is a close relationship between symplectic structures and Seiberg–Witten monopole equations on 4-manifolds [Ta1, Ta2, Ta3, Ta4]. The results contained in this paper may be thought of as evidence that the 3-dimensional reduction of the Seiberg–Witten equations is related with contact structures. Although we will not work directly with the 3-dimensional version of the equations, we will use the 4-dimensional Seiberg–Witten theory to prove new results about contact structures on 3-manifolds.

Let M be a smooth 3-dimensional manifold. A contact structure on M is a distribution ξ of tangent 2-planes locally defined by a 1-form α ($\xi = \{\alpha = 0\}$) such that $\alpha \wedge d\alpha$ is nowhere vanishing. Therefore $\alpha \wedge d\alpha$ defines an orientation on M and this orientation is independent of the choice of the sign of α. If M is already oriented and $\alpha \wedge d\alpha$ is a positive multiple of the volume form, then ξ is called *positive*, otherwise it is called *negative*. When M is oriented, α defines a coorientation, and hence an orientation of the plane field ξ. Changing from α to $-\alpha$ changes the orientation of ξ. On a generically embedded surface $S \subset M$, ξ induces a line field which integrates

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to a foliation S_ξ with isolated singularities (at points where S is tangent to ξ) which is called the *characteristic foliation* [Ae]. There is an essential dichotomy among contact structures: they are either *tight* or *overtwisted*. A contact structure is called overtwisted if there is an embedded disc $D \subset M$ such that its characteristic foliation contains a closed orbit with exactly one singular point inside it. Otherwise the contact structure is called tight ([Ae,El5]). An important result due to Eliashberg [El1] is that the classification up to isotopy of overtwisted contact structures on closed 3-manifolds coincides with their homotopy classification as 2-plane fields. On the other hand, the classification of tight contact structures is far from understood.

A source of examples of tight contact structures is provided by boundaries of pseudo-convex domains inside 2-dimensional complex manifolds. Let X be a smooth 4-manifold with an almost complex structure $J: TX \to TX$. Any smooth, oriented hypersurface $M \subset X$ has a canonically induced distribution of J-invariant tangent 2-planes $\xi = TM \cap J(TM)$. Notice that J induces a complex structure on ξ. Suppose that M is defined as the zero set of a smooth function $f: X \to \mathbb{R}$, with $df|_M \neq 0$. Then, ξ is the kernel of the 1-form $\alpha = J^* df$ and, up to changing f into $-f$, one may assume that α defines the coorientation of ξ inside M determined by the complex orientation of ξ and the orientation of M. M is called *J-convex* if the quadratic form $d\alpha(v, Jv)$ restricted to ξ is everywhere positive definite. Thus, ξ is a contact structure. When M is J-convex and J is integrable, M is also called *strictly pseudo-convex*, and the induced contact structure ξ is called *holomorphically fillable*. It is a theorem of Gromov and Eliashberg [El3,Gro] that fillable contact structures are tight. Holomorphically fillable contact structures are naturally oriented, since they are distributions of complex lines. This orientation agrees with the orientation determined by $\alpha = -J^* df$ and the orientation of M as the boundary of $\{f < 0\}$. Since we will mainly consider fillable structures, from now on we shall always implicitly assume that a contact structure is oriented.

Generalizing Bennequin's work, Eliashberg [El3] proved that if ξ is an oriented tight contact structure on a 3-manifold M, $e(\xi) \in H^2(M; \mathbb{Z})$ is its Euler class as a 2-plane bundle and $S \subset M$ is a closed oriented surface, then either S is a sphere and $\langle e(\xi), [S] \rangle = 0$ or

$$(1) \qquad |\langle e(\xi), [S] \rangle| \leq -\chi(S)$$

This inequality clearly implies that for any compact 3-manifold there are only finitely many classes that can be Euler classes of tight contact structures (cf. 4.3 in [El3], 2.2.2 in [El5]). It is an open question whether all the cohomology classes allowed by (1) are Euler classes of tight contact structures.

We will discuss here various notions of equivalence of contact structures. Two contact structures ξ_1 and ξ_2 are said to be *isomorphic*, or equivalently, two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are called *contactomorphic* if there is a diffeomorphism sending one contact structure to the other. Such a diffeomorphism is called a *contactomorphism*. There are two other notions of equivalence. They deal with the question of path-connectedness of the space of 2-plane fields and of the space of contact structures. We will say that the contact structures are *homotopic* if they belong to the same connected component of the space of 2-plane fields, i.e. if they are connected by a smooth family of 2-plane fields (which are not necessarily contact). Two contact structures belonging to the same component of the space of contact structures are called *isotopic*. Namely, by a classical result of Gray [Gra], two contact structures on a closed 3-manifold are connected by a smooth family of contact structures if and only if there is a diffeomorphism isotopic to the identity which sends one contact structure to the other.

Clearly, two contact structures which are isotopic are both homotopic and isomorphic. We will discuss here to what extent the converse fails and these notions are different. Since, as we said before, classifications up to homotopy and isotopy agree for overtwisted contact structures, the question is only interesting in the case of tight contact structures.

There is one example where the space of contact structures is understood, namely S^3. By the work of Eliashberg [El1], the isotopy classification of overtwisted structures coincides with their homotopy classification as 2-plane fields. The classification of tight structures is simple: any tight contact structure on S^3 is isotopic to the standard one (see Sect. 2).

Consistently with the examples available at the time, a few years ago Eliashberg [El4] formulated the conjecture that two tight contact structures with the same Euler class are isotopic. It is now known that two tight contact structures with the same Euler class do not have to be even homotopic. Gompf's work [Go2] implies the existence of non-homotopic tight contact structures on homology 3-spheres (having necessarily the same Euler class, namely 0). It is also known that two tight contact structures which are homotopic can be non-isomorphic. Moreover, homotopic tight contact structures could be isomorphic but non-isotopic. The first counterexamples were provided by Giroux [Gi1, Gi2, Gi3], who used rigidity results about Lagrangian submanifolds of cotangent bundles to construct examples of homotopic but not isomorphic (and therefore certainly non-isotopic) tight contact structures on T^3. Eliashberg and Polterovich, using results of Luttinger on Lagrangian tori in $\mathbb{C}^2$, constructed homotopic, isomorphic but non-isotopic contact structures on T^3 [EP]. In both of the Giroux and Eliashberg–Polterovich counterexamples the tools used to distinguish the structures relied on the fact that the underlying manifold was T^3.

The two kinds of counterexamples we discussed above are of a qualitatively different nature. While the obstruction to the existence of a path connecting two contact structures in the space of 2-plane fields can be described purely in topological terms (see e.g. [Go2]), a finer tool is needed to distinguish up to isotopy two tight contact structures which are homotopic as 2-plane fields.

The proof of Theorem 1.1 below shows that 4-dimensional Seiberg–Witten theory is subtle enough to distinguish many pairs of homotopic, non-isomorphic tight contact structures. The theorem gives new counterexamples (of the second kind) to the conjecture on infinitely many homology 3-spheres.

Theorem 1.1. *Given any positive integer n, there exist homology* 3*-spheres with at least n homotopic, but non-isomorphic tight contact structures.*

In order to explain the construction of the counterexamples, we will now make a little digression.

A *Stein manifold* is a complex manifold which can be embedded as a proper submanifold of $\mathbb{C}^n$. Any Stein manifold X admits a smooth, strictly plurisubharmonic function ϕ, namely a smooth function which is strictly subharmonic on any holomorphic curve in X (cf [EG]). A *Stein manifold with boundary* can be defined as a smooth manifold with boundary W having a Stein structure in its interior and admitting a smooth, strictly plurisubharmonic function which has ∂W as a level set (cf. [Go2]). Regular level sets of a strictly plurisubharmonic function are strictly pseudo-convex, thus so is ∂W. In complex dimension $n > 2$ Eliashberg [El2] has shown that any almost complex manifold having a proper Morse function with the indices of all of its critical points $\leq n$ carries a genuine complex structure which makes its interior a Stein manifold with boundary. Thus the boundary carries a tight contact structure. The main idea of the proof is that the standard Stein structure on the 4-ball can be extended over handles of index $\leq n$. In the case $n = 2$ the obstructions Eliashberg introduces in [El2] and which vanish in higher dimensions do not always vanish. A sufficient condition (Theorem 6.1 in [El4]) was brought to our attention by Emmanuel Giroux, as well as by recent work of Bob Gompf [Go2] on the existence of Stein structures on fake $\mathbb{R}^4$'s. The condition is the following: surgery on a framed Legendrian link in S^3 will define a Stein structure on the resulting 4-manifold if the framing of each component K is $fr(K) = tb(K) - 1$, where $tb(K)$ is the Thurston–Bennequin invariant of the Legendrian knot K. To prove theorem 1.1 we consider contact structures which are induced on the boundary of Stein manifolds defined by surgery on framed Legendrian links (see Sect. 2). Different Stein manifolds are shown to induce homotopic non-isomorphic contact structures on the same homology sphere.

It seems to be a natural question to ask whether for a smooth 4-manifold with boundary W there is a relationship among the various contact structures on ∂W which are induced by Stein structures with boundary on W. The proof of Theorem 1.1 can be adapted to prove the following Theorem 1.2. A different proof has been recently announced by Kronheimer and Mrowka [KM3]. Recall that an almost complex structure J on W has a canonically associated Spin^c-structure S_J. When $H^2(W;\mathbb{Z})$ has no 2-torsion S_J is determined up to isomorphism by the Chern class of its determinant line bundle $c_1(J) \in H^2(W;\mathbb{Z})$, but in general there are non-isomorphic Spin^c-structures corresponding to the same c_1.

Theorem 1.2. *Let X be a smooth* 4*-manifold with boundary. Suppose J_1, J_2 are two Stein structures with boundary on X with associated* Spin^c*-structures* Θ_1

and Θ_2. *If the induced contact structures* ξ_1 *and* ξ_2 *on* ∂X *are isotopic, then* Θ_1 *and* Θ_2 *are isomorphic* (*and in particular have the same* c_1).

The contents of the paper are as follows. In Sect. 2 we exploit the above mentioned result of Eliashberg to construct certain families of tight contact structures, showing that all the contact structures in the same family are homotopic as 2-plane fields. In Sect. 3 we prove the following general result, which seems to be interesting on its own: given an exhausting strictly plurisubharmonic function ϕ on a Stein manifold Ω, any sublevel set $\Omega_c = \{\phi < c\}$ can be holomorphically embedded as a domain inside a smooth projective variety with ample canonical bundle S having a Kähler form whose pull-back to Ω equals $\omega_\phi = dJ^*(d\phi)$, the symplectic form induced by ϕ. Moreover, when Ω has complex dimension two S may be chosen so that $b_2^+(S) > 1$. The contact structures constructed in Sect. 2 are those induced by the complex structure on the boundary of such domains Ω_c. In Sect. 4 we prove Theorems 1.1 and 1.2 by exploiting the following idea. Let $\Omega_i = (X, J_i)$, $i = 1, 2$, be two Stein structures on the same smooth 4-manifold with boundary X, and let ξ_i, $i = 1, 2$ be the induced contact structures on ∂X. If ξ_1 and ξ_2 are isotopic (or just isomorphic, in special cases) then one can cut Ω_1 out of a compact Kähler surface S containing it as a symplectic domain and glue back Ω_2 in such a way that the resulting closed 4-manifold X is diffeomorphic to S and admits a symplectic structure. Finally, results from Seiberg–Witten theory applied to X are used to relate the first Chern classes and the Spin^c-structures of Ω_1 and Ω_2, and prove the theorems. In Sect. 5 we deduce some final consequences of Corollary 3.3 and we end the paper with a question.

2. Construction of contact structures

In this section we will define certain families of contact structures, and we shall use Proposition 2.2 to prove that members of the same family are homotopic as 2-plane fields.

A knot K in a contact 3-manifold M is *Legendrian* if it is tangent to the plane field ξ at every point. For a Legendrian knot K which is homologous to zero in M and a relative homology class $\beta \in H_2(M, K)$ the *Thurston–Bennequin invariant* $tb(K, \beta)$ is defined as follows: if S is an oriented surface which represents β and K' is a "parallel copy" of the knot K (obtained by pushing off K along a vector field transversal to ξ) then $tb(K, \beta)$ is the homological intersection of K' and S. This invariant is independent of the orientation of the knot. It was originally defined by Bennequin [Be] by using the push-off of the curve in the direction of the vector field normal to the curve and contained in ξ, which gives the same result.

Another invariant associated to an oriented Legendrian curve K in a contact 3-manifold and a surface S bounded by K representing $\beta \in H_2(M, K)$ is the *rotation number* $\mathrm{r}(K, \beta)$. To define it choose a trivialization of the bundle $\xi|_S$. Let t be the vector field tangent to K determining the orientation. The degree of t with respect to the chosen trivialization depends only on K and β and is the rotation number. It follows from the definition that when the orientation of K is reversed, $r(K, \beta)$ changes sign.

Let us recall a few useful properties of the standard contact structures on S^3 and $\mathbb{R}^3$. Consider the sphere $S^3 \subset \mathbb{C}^2$. The complex lines tangent to S^3 define the *standard contact structure* ξ_0. When restricted to the complement of a point, ξ_0 is isomorphic to the standard structure on $\mathbb{R}^3$ defined by the 1-form $\alpha = xdy + dz$ (cf. [Be, Erl]). A *front* $C \subset \mathbb{R}^2 = \{(y, z)\}$ is a piecewise-smooth immersed curve with finitely many singularities which are either ordinary double points or horizontal cusps, such that the cusps are exactly the local extrema of $y|C$. Moreover, the lines tangent to C are nowhere vertical. The projection of a generic Legendrian knot $K \subset \mathbb{R}^3$ to the yz-plane is a front C_K, and the knot K can be reconstructed from C_K using the differential equation $dz/dy = -x$. In particular, the diagram obtained from a front C by letting the over-arc at any double point be the one with the most negative slope is a knot diagram for the Legendrian knot K_C constructed from C via the differential equation (see Fig. 1 for an example of such diagrams).

Given an oriented Legendrian knot $K \subset \mathbb{R}^3 \subset S^3$, it is not hard to check that its Thurston-Bennequin invariant $tb(K)$ and rotation number $\mathrm{r}(K)$ (notice that, since $H_2(S^3, K) = 0$, β can be omitted) can be calculated from such a generic projection in the following way. Let w denote the writhe of K, namely the algebraic number of crossings. Let c denote the number of cusps, a the number of ascending cusps and d the number of descending cusps. Then $tb(K) = w - \frac{1}{2}c$, while $\mathrm{r}(K) = \frac{1}{2}(d - a)$. In Fig. 1 the diagram of a Legendrian right-handed trefoil is shown, and tb and r are computed.

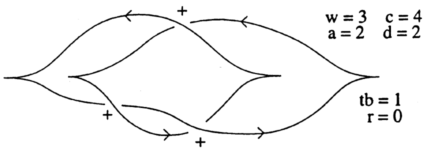

Fig. 1

The following result, due to Eliashberg, is implicitly contained in [El2] (and explained in detail in [Go2]).

Theorem 2.1. ([El2],[El4, Theorem 6.1]). *Let W be a smooth 4-manifold with boundary having a handlebody decomposition $B^4 \bigcup_i H_i^2$ with only 2-handles. Suppose that there exists a Legendrian link $\mathscr{L} = \bigcup_i K_i$ in the framed isotopy class of the union of the attaching circles of the 2-handles such that $fr(K_i) = tb(K_i) - 1$ for all i. Then W admits a structure of Stein manifold with boundary such that, if h_i denotes the 2-homology class supported by H_i^2, then, $\langle c_1(W), h_i \rangle = \mathrm{r}(K_i)$.*

Proof. (Sketch) One can easily check that the conditions $fr(K_i) = tb(K_i) - 1$ insure that the attaching maps of the 2-handles give special HAT's in the sense of [El2, Sect. 2]. Hence, the proof of 1.3.2 in [El2], which deals with complex dimension > 2, carries over to show that a standard plurisubharmonic function on the 4-ball can be extended to a smooth strictly plurisubharmonic function on W having ∂W as a level set and exactly one critical point of index 2 inside each 2-handle. Moreover, the extension can be constructed independently over each 2-handle. This proves the first part of the statement. The property of the resulting Chern class follows from the construction and the definition of r. □

We are now ready to define our families of contact structures. We will use the so-called nuclei N_n (studied e.g. in [Go1]), which are the 4-manifolds with boundary having the framed link description given in Fig. 2a. The boundary of N_n is isomorphic to the Brieskorn homology sphere $\Sigma(2,3,6n-1)$, endowed with orientation opposite to the one as a boundary of the Milnor fiber of the corresponding singularity.

These framed links can be realized as Legendrian in several ways (Fig. 2b). For $n \geq 2$, denote by W_n^k, $1 \leq k \leq n-1$, the Stein manifolds with boundary constructed by attaching handles along these Legendrian links and applying Theorem 2.1. Using the theorem, from Fig. 2b we read off the values $\langle c_1(W_n^k), T\rangle = 0$, $\langle c_1(W_n^k), S\rangle = 2k - n$. Hence $c_1(W_n^k) = (2k-n)$ PD(T). Thus, although all the W_n^k's are the same as smooth manifolds, they

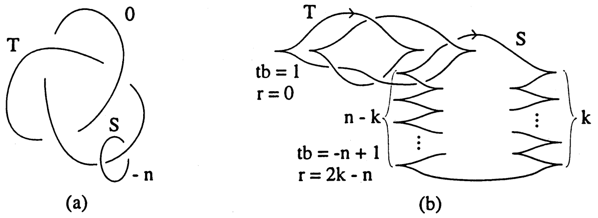

Fig. 2

are different complex manifolds. We denote by ξ_n^k the holomorphically fillable (hence tight) contact structures induced on ∂N_n by the complex structure on W_n^k.

We learned the statement of the following proposition from Bob Gompf. A forthcoming preprint of Gompf contains a more detailed analysis for a general 3-manifold [Go2].

Proposition 2.2. *Let X_i, $i = 1, 2$, be two almost complex 4-manifolds with boundary. Let M be an oriented integral homology 3-sphere, and suppose there exist orientation-preserving diffeomorphisms $\phi_i : M \to \partial X_i$. Let ξ_i, $i = 1, 2$ be the 2-plane fields induced on ∂X_i by the almost complex structures on X_i. Then, $\phi_1^*(\xi_1)$ and $\phi_2^*(\xi_2)$ are homotopic as 2-plane fields if and only if*

$$c_1(X_1)^2 - 2\chi(X_1) - 3\sigma(X_1) = c_1(X_2)^2 - 2\chi(X_2) - 3\sigma(X_2).$$

Proof. Notice that, if X is an almost complex 4-manifold with a homology sphere boundary, $c_1(X) \in H^2(X;\mathbb{Z}) \cong H^2(X, \partial X;\mathbb{Z})$ is well-defined, and

$c_1(X)^2 \in H^4(X, \partial X; \mathbb{Z})$ can be identified with the number $\langle c_1(X) \cup c_1(X), [X, \partial X] \rangle \in \mathbb{Z}$.

Denote by J_i, $i = 1, 2$, the almost complex structure on X_i. Fix J_i-invariant Riemannian metrics g_i on X_i, and let $\{v_i, J_i v_i, w_i\}$ be orthonormal global frames for $T\partial X_i$ such that $v_i \in \xi_i$. Let $\overline{X}_2$ denote X_2 with the opposite orientation and let $X = X_1 \cup \overline{X}_2$ be the smooth closed oriented manifold obtained by gluing via the diffeomorphism $\Phi = \phi_2 \circ \phi_1^{-1} : \partial X_1 \to \partial X_2$. J_2 gives $T\overline{X}_2$ a structure of a complex bundle (note that this is not an almost complex structure on $\overline{X}_2$, since it does not define the same orientation). Define a complex bundle $E = TX_1 \cup_\mu T\overline{X}_2 \to X$ by gluing the restrictions of TX_1 and $T\overline{X}_2$ via a complex bundle isomorphism $\mu : TX_1|\partial X_1 \to T\overline{X}_2|\partial X_2$ covering Φ and sending v_1 to v_2 and w_1 to w_2. If we look at the two clutching maps defining the bundles E and TX we see that

$$p_1(TX) - p_1(E) = \pm 2 \deg(F), \tag{2}$$

where p_1 denotes the first Pontrjagin number, i.e. the evaluation of the first Pontrjagin class on the fundamental class of X, and F: $\partial X_1 \to SO(3)$ is the map defined by associating to every point of ∂X_1 the matrix expressing the frame $\{d\Phi(v_1), d\Phi(J_1 v_1), d\Phi(w_1)\}$ in terms of the frame $\{v_2, J_2 v_2, w_2\}$ for $T\partial\overline{X}_2$. $\phi_1^*(\xi_1)$ and $\phi_2^*(\xi_2)$ are homotopic as 2-plane fields if and only if F is null-homotopic, and since ∂X_1 is an integral homology sphere the map F is null-homotopic if and only if F has degree zero, i.e. if and only if $p_1(TX) = p_1(E)$.

Since $\mu(J_1(w_1)) = J_2(w_2)$, the vector field $v = J_1(w_1)$ can be considered as a section of $E|\partial X_1$. On the other hand, v is also a section of $TX_1|\partial X_1$, and $\mu(v)$ a section of $TX_2|\partial X_2$. Now we observe that the obstruction to extend v as a nonzero section of TX_1 is equal to $\chi(X_1)$, while the obstruction to extend $\mu(v)$ as a nonzero section of TX_2 is equal to $-\chi(X_2)$ (because the orientation of $T\overline{X}_2$ is the opposite of the one compatible with J_2). Hence, the obstruction to extend v as a nonzero section of E, i.e. the Euler number of E, is equal to $e(E) = \chi(X_1) - \chi(X_2)$.

By the Mayer–Vietoris sequence there is a direct sum decomposition $H^2(X) = H^2(X_1) \oplus H^2(X_2)$, and under this decomposition $c_1(E) = c_1(X_1) - c_1(X_2)$. Hence,

$$p_1(E) = c_1(E)^2 - 2e(E) = c_1(X_1)^2 - c_1(X_2)^2 - 2\chi(X_1) + 2\chi(X_2). \tag{3}$$

On the other hand, by the Hirzebruch signature theorem,

$$p_1(TX) = 3\sigma(X) = 3\sigma(X_1) - 3\sigma(X_2). \tag{4}$$

Therefore, it follows from (2), (3) and (4) that ξ_1 and ξ_2 are homotopic if and only if the condition stated is satisfied. □

Finally, applying proposition 2.2, we have the following:

Corollary 2.3. *Given any k, k' with $1 \le k, k' \le n-1$, ξ_n^k and $\xi_n^{k'}$ are homotopic as fields of 2-planes tangent to ∂N_n.*

Proof. Apply the proposition to the manifolds $X_1 = W_n^k$ and $X_2 = W_n^{k'}$. Since the underlying smooth manifold is the same, we only need to check that $c_1(W_n^k)^2 = c_1(W_n^{k'})^2$. This follows because $c_1(W_n^k) = (2k-n)\mathrm{PD}(T)$ for all k and since $T \cdot T = 0$ both squares are 0. □

3. Symplectic compactifications of Stein surfaces

The fact that a Stein manifold is biholomorphic to a domain in a projective manifold has been known for some time. It was originally proved by Stout for Stein manifolds [S] and later extended to Stein spaces (see e.g. [DLS, L]). Using the approach of [DLS], Theorem 3.2 improves Stout's result by showing that the biholomorphism can be chosen to be a symplectomorphism, and the projective manifold to have ample canonical bundle, which is what we need for our applications.

We learned the following lemma from Peter Kronheimer.

Lemma 3.1. *Let X be a complex manifold and $\phi, \psi : X \to \mathbb{R}$ smooth strictly plurisubharmonic functions on X. Suppose that for some $c \in \mathbb{R}$ $\{\psi \le c\} \subset X$ is compact. Then, for every $\epsilon > 0$ there exist constants a, b and a strictly plurisubharmonic function τ on X which coincides with ϕ on $\{\psi \le c - \epsilon\}$, and with $a\psi + b$ on $\{\psi \ge c + \epsilon\}$.*

Proof. Since $\{\psi \le c\}$ is compact, we can find a C^∞ function β which is equal to 1 on $\{\psi \le c\}$ and to 0 on $\{\psi \ge c + \epsilon\}$. We look for a τ of the form

$$\tau(x) = \beta(x)\phi(x) + f(\psi(x)), \quad x \in X, \tag{5}$$

where f: $\mathbb{R} \to \mathbb{R}$ has positive first and second derivative. It suffices to show that, for a suitable choice of f, the hessian form $H(\tau) = \sum_{i,j} \partial^2\tau/\partial z_i \partial \bar{z}_j dz_i d\bar{z}_j$ is positive definite. Since ψ is strictly plurisubharmonic, $H(\psi) > 0$, and a straightforward calculation shows that $H(f \circ \psi) \ge f' H(\psi)$. We are free to choose f to be zero on $(-\infty, c - \epsilon]$ and f' big enough on $[c, c + \epsilon]$ so that $f' H(\psi)$ overcomes the negative terms coming from $H(\beta\phi)$. Moreover, we can make $f(t)$ linear for $t \ge c + \epsilon$. Clearly, τ satisfies the conditions of the statement. □

Recall that a symplectic form on a smooth 4-manifold X is a closed, non-degenerate 2-form ω. If X has also an almost complex structure J, ω is called *J-positive* if $\omega(v, Jv) > 0$ for every tangent vector v. An almost complex structure J is called *ω-compatible* if ω is J-positive and $\omega(Jv, Jv') = \omega(v, v')$. The space of almost complex structures compatible with a given symplectic form is well known to be contractible [Ae], so the canonical class K of the almost complex structure is uniquely determined by the symplectic structure, and sometimes called the canonical class of ω. Let X be a Stein manifold, and $\phi\colon X \to \mathbb{R}$ a smooth strictly plurisubharmonic function. If we denote by J^* the dual of J, the 2-form $\omega_\phi = dJ^*(d\phi)$ is non-degenerate and closed, hence it defines a symplectic structure. X is therefore Kähler with a Kähler metric defined by $g_\phi(v, v') = \omega_\phi(v, Jv')$. A vector field Θ is called *contracting* for a symplectic form ω if the Lie derivative $\mathscr{L}_\Theta \omega = v\omega$ with v a negative, locally constant function. If ω is understood, then we will simply say that Θ is contracting. For a domain W with smooth boundary M contained in (X, ω) we say that M is (locally) *ω-convex* if there is a contracting vector field defined on a neighborhood of M which is transverse to M and pointing into W. The gradient vector field of ϕ is contracting for the symplectic form ω_ϕ, and if we look at the domain $W = \{\phi \le c\}$ its boundary is J-convex and ω_ϕ-convex. It is a theorem of Eliashberg and Gromov (see 1.4.A in [EG]) that for any two plurisubharmonic functions ϕ and ψ on the complex manifold X, the two symplectic manifolds (X, ω_ϕ) and (X, ω_ψ) are symplectomorphic.

Theorem 3.2. *Let X be a Stein manifold and $\phi : X \to \mathbb{R}$ a smooth strictly plurisubharmonic function. Let $r \in \mathbb{R}$ be a regular value of ϕ and $X_r = \{\phi < r\} \subset X$. Then there exists a holomorphic embedding of X_r as a domain inside a smooth projective variety with ample canonical bundle S having a Kähler form whose pull-back to X equals $\omega_\phi = dJ^*(d\phi)$. Moreover, when X has complex dimension two S may be chosen so that $b_2^+(S) > 1$.*

Proof. When X has complex dimension two, in order to insure the condition $b_2^+ > 1$ we need to perform a preparatory "enlargement" of X_r in the following way. Choose a regular value t of ϕ, with $t > r$. If $X_t = \{\phi < t\} \subset X$, then ∂X_t is endowed with a holomorphically fillable contact structure ξ. On sufficiently small 3-balls in ∂X_t, ξ is isomorphic to the standard structure on $\mathbb{R}^3$. Consider two copies of the Legendrian link L defining W_3^1 inside two disjoint such 3-balls. By [El2], ϕ may be extended to a smooth strictly plurisubharmonic function on the 4-manifold W obtained by attaching 2-handles along the two copies of L with framings -1 with respect to the canonical framings induced by ξ (cf. the proof of Theorem 2.1). The resulting function, which we shall keep calling ϕ, has ∂W as a level set. Hence, the interior of W is Stein. Notice that, since $b_2^+(N_3) = 1$, $b_2^+(W) \ge 2$. When X is not complex two-dimensional, the construction just described is not needed, and one may just take W to be X_t, and start the proof from this point.

Let c be a regular value of ϕ slightly smaller than the value taken on ∂W, and let $\Omega_c = \{\phi < c\}$. Then, Ω_c is a relatively compact Stein domain of W. In view of Lemma 5.2 of [DLS], there exists an affine algebraic manifold A and a proper holomorphic embedding $\Omega_c \hookrightarrow A$ with trivial normal bundle. Moreover, by a quick inspection of the proof one sees that A has a smooth projective compactification $\tilde{A}$. Using the embedding inside A, ϕ can be pushed forward to a strictly plurisubharmonic function on (the image of) Ω_c, which we will keep calling ϕ. We shall denote by $\Omega_r = \{\phi < r\}$ the sublevel sets of ϕ. Let $\psi = \log(1 + \sum_i |z_i|^2)$ be the standard strictly plurisubharmonic function on the affine space containing A, so that the associated Kähler form $\omega_\psi = dJ^*(d\psi)$ is the $(1,1)$-form associated to the Fubini-Study metric. Of course, ϕ also defines a Kähler form ω_ϕ on Ω_c. We will denote the restriction of ψ to Ω_c again by ψ, to simplify the notation. Both ϕ and ψ are exhausting functions on Ω_c with relatively compact sublevel sets, hence for any $r < c$ there is $t \in \mathbb{R}$ such that $\Omega_r \subset \{\psi < t\}$, and $\epsilon > 0$, s with $r < s < c$ such that $\psi^{-1}([t-\epsilon, t+\epsilon]) \subset \Omega_s \setminus \Omega_r$. Since Ω_s is a relatively compact Runge domain of Ω_c (by, e.g. [GR, Theorem IX.C.8]), by Lemma 5.3 in the same paper [DLS], for every neighborhood V of Ω_s in A there exists a Runge domain Ω in A with $\Omega_s \subset \Omega \subset V$, a holomorphic retraction $\rho : \Omega \to \Omega_s$ and a closed algebraic submanifold $Y \subset A$ such that ρ maps $Y \cap \rho^{-1}(\Omega_s)$ biholomorphically onto Ω_s. Moreover, Y is the intersection of the zero sets of generic polynomial functions $\{P_j\}$ on A. The polynomials P_j can be chosen to be of arbitrarily large degree. Thus, Y is an affine Zariski open subset of the projective algebraic submanifold $\tilde{Y}$ obtained by intersecting the hypersurfaces defined by the homogenizations of the generic polynomials P_j with the projective algebraic manifold $\tilde{A}$. For any $r < s$, let $Y_r = Y \cap \rho^{-1}(\Omega_r) \subset Y$. The function $\phi' = \phi \circ \rho$ is strictly plurisubharmonic on Y_s and Y_r are its sublevel sets. Up to choosing a smaller neighborhood V, we will still have $Y \cap \psi^{-1}([t-\epsilon, t+\epsilon]) \subset Y_s \setminus Y_r$. Lemma 3.1 applied to the Stein manifold Y_s and the two strictly plurisubharmonic functions ϕ' and ψ yields a strictly plurisubharmonic function which coincides with ϕ' on $\{\psi \le t - \epsilon\}$ and with $a\psi + b$ on $\{\psi \ge t + \epsilon\}$. It follows that on the smooth projective surface $\tilde{Y}$ there is a Kähler form which is equal to $\omega_{\phi'} = dJ^*(d\phi')$ on an open subset containing Y_r, and to a times the standard Fubini-Study Kähler form on the complement of Y_s. Y_r has ω-convex boundary and is a biholomorphic image of Ω_r. Therefore, to finish the proof we need to argue that $\tilde{Y}$ can be chosen to have ample canonical bundle. This is because $\tilde{Y}$ can be chosen to be the intersection of the smooth projective variety $\tilde{A}$ with hypersurfaces of arbitrarily high degree. In fact, the adjunction formula applied to the inclusion $\tilde{Y} \subset \tilde{A}$ says that the canonical divisor $K_{\tilde{Y}}$ is equal to the restriction of $K_{\tilde{A}} + \left(\sum_i d_i\right)H$, where H is the hyperplane class and the d_i's are the degrees of the hypersurfaces. But if $\sum_i d_i$ is sufficiently large $K_{\tilde{A}} + \left(\sum_i d_i\right)H$ is ample on $\tilde{A}$, hence $K_{\tilde{Y}}$ is ample on $\tilde{Y}$. □

Corollary 3.3. *Let X be a complex 2-dimensional Stein manifold and $\phi : X \to \mathbb{R}$ a smooth strictly plurisubharmonic function. Let $r \in \mathbb{R}$ be a regular value of ϕ and $X_r = \{\phi < r\} \subset X$. Then there exists a holomorphic embedding of X_r as a domain inside a compact Kähler minimal surface S of general type, with $b_2^+ > 1$, such that the pull-back of the Kähler form of S to X equals $\omega_\phi = dJ^*(d\phi)$.*

Proof. All we need to do is to observe that a smooth complex surface with ample canonical bundle is minimal of general type. □

4. Proofs of Theorems 1.1 and 1.2

In this section we will use the symplectic embeddings of Corollary 3.3 to prove Theorems 1.2 and 1.1. The following lemma shows that we can cut out a domain with ω-convex boundary from a symplectic manifold and glue in another such domain in a symplectic way (thus constructing a new symplectic manifold), provided that the two corresponding contact structures are isomorphic.

Lemma 4.1. *Let X_0 and X_1 be complex 2-dimensional Stein manifolds with boundary, and suppose that ∂X_0 and ∂X_1 are diffeomorphic to the connected 3-manifold M. Let $\mu\colon X_1 \to \mathbb{R}$ be a strictly plurisubharmonic Morse function on X_1 having ∂X_1 as a level set, and endow X_1 with the symplectic form ω_1 associated to μ. Suppose that the contact structures ξ_0 and ξ_1 induced on M are isomorphic. Then, there exist:*

1. *collars $U_i \subset X_i$, $i = 0, 1$, around ∂X_i,*
2. *a J-compatible symplectic form ω_0 on the interior of X_0,*
3. *a symplectic embedding Φ of the interior of U_1 as a subcollar of the interior of U_0.*

Proof. Let us first recall the following facts. Let ϕ be a strictly plurisubharmonic function, and let $\omega_\phi = dJ^*(d\phi)$. Let θ_ϕ denote minus the gradient vector field of ϕ with respect to the metric $\omega_\phi(-, J-)$. Then, it is straightforward to check that the 1-form $\omega_\phi(\theta_\phi, -)$ is equal to $-J^*(d\phi)$. Also, the diffeomorphisms between regular level sets of ϕ defined by the flow of θ_ϕ preserve the induced contact structures [We]. Let θ_μ be minus the gradient vector field of $\mu : X_1 \to \mathbb{R}$. The flow generated by θ_μ defines a diffeomorphism ϕ_1 from a neighborhood U_1 of ∂X_1 onto $M \times (-\epsilon_1, 0]$, for some $\epsilon_1 > 0$. Moreover, ϕ_1 sends θ_μ to $-d/dt$. Denote by α_1 the restriction to $M \times \{0\}$ of the push-forward of $-J_1^*(d\mu)$ under ϕ_1. Then, $e^t\alpha_1$ can be thought of as a 1-form on $M \times (-\epsilon_1, 0]$. Since both $e^t\alpha_1$ and the push-forward of $-J_1^*(d\mu)$ satisfy the equation $\frac{d\beta}{dt} = -\beta$, and coincide along $M \times \{0\}$, they must be equal on $M \times (-\epsilon_1, 0)$. This shows that the 1-form α_1 defines ξ_1, and that $\phi_1^*(d(e^t\alpha_1))|U_1 = \omega_1|U_1$. Let $v\colon X_0 \to \mathbb{R}$ be a strictly plurisubharmonic Morse function having constant value c on ∂X_2. As before, the flow generated by θ_v defines a diffeomorphism from a neighborhood U_0 of ∂X_0 onto $M \times (-\epsilon_2, 0]$, for some $\epsilon_2 > 0$. By composing v with a suitable function $g : (-\infty, c) \to \mathbb{R}$ such that $g(r) = r$ for $r \le -\epsilon_2$ and g is increasing and convex for $r > -\epsilon_2$ we get an unbounded strictly plurisubharmonic function τ on the interior of X_0 with the same level sets as v. τ defines a symplectic structure on X_0 given by $dJ^*(d\tau)$. The flow of θ_τ defines a diffeomorphism ϕ_0 from the interior of U_0 with $M \times (-\epsilon_2, \infty)$ and $\phi_0^*(d(e^t(\alpha_0))) = dJ^*(d\tau)$, where α_0 is a 1-form on $M = M \times \{0\}$ defining ξ_0 up to isomorphisms. ξ_0 and ξ_1 are isomorphic if and only if there is a diffeomorphism $f\colon M \to M$ such that $f^*(\alpha_0) = \lambda\alpha_1$, where $\lambda\colon M \to \mathbb{R}$ is a non-vanishing function. We can find a constant $a \in \mathbb{R}$ such that

$G : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $G(p,t) = (f(p), t - \ln|\lambda(p)| + a)$ maps $M \times (-\epsilon_1, 0)$ diffeomorphically onto $M \times (c_1, c_2)$, where (c_1, c_2) is contained in $(-\epsilon_2, \infty)$. Moreover,

$$G^*(d(e^t\alpha_0)) = d(G^*(e^t\alpha_0)) = d\left(e^{t-\ln|\lambda|+a}\lambda\alpha_1\right) = \pm e^a d(e^t\alpha_1).$$

Finally, set $\omega_0 = \pm e^{-a} dJ^*(d\tau)$ and $\Phi = \phi_0^{-1} \circ G \circ \phi_1$. □

Let us now recall a few results from Seiberg–Witten theory [FS, KM2, Kr, Mo, Wi]. For a smooth 4-manifold with $b_2^+ \geq 2$ the Seiberg–Witten monopole equations [SW1, SW2] give rise to invariants of its differentiable structure. The *Seiberg–Witten invariant* is a map SW from the set of Spin^c-structures to the integers. It takes nonzero values at only finitely many Spin^c-structures. The first Chern classes of the determinant line bundles associated to the Spin^c-structures for which SW is nonzero are called *basic classes* by analogy to the similar notion in Donaldson theory introduced by Kronheimer and Mrowka in [KM1]. The set of basic classes is finite and it is a differentiable invariant of the 4-manifold. Moreover, when a class c is basic then so is $-c$. Witten [Wi] showed that this set contains the canonical class in the case of a Kähler surface. Taubes proved [Ta1] the same to be the case for symplectic 4-manifolds. For minimal Kähler surfaces of general type a more precise result holds. In this case there are only two Spin^c-structures with nontrivial Seiberg–Witten invariants, i.e. those canonically associated to the complex structure and its conjugate.

Let ξ_n^k denote the tight contact structure induced on the homology 3-sphere ∂N_n by the complex structure of W_n^k constructed in Sect. 2. Theorem 1.1 is an immediate consequence of Corollary 2.3 together with the following result, which shows that, for $n \geq 2$, there are at least $[\frac{n}{2}]$ non-isomorphic contact structures among the ξ_n^k's.

Theorem 4.2. *Let* $1 \leq k, k' \leq n-1$. *Suppose that* ξ_n^k *and* $\xi_n^{k'}$ *are isomorphic. Then, either* $k = k'$ *or* $k = n - k'$.

Proof. Let S be a compact Kähler minimal surface of general type (with $b_2^+(S) > 1$) containing W_n^k according to Corollary 3.3. Suppose that ξ_n^k and $\xi_n^{k'}$ are isomorphic. We can build a new symplectic manifold $S' = N_n \bigcup_\Phi (S \setminus N_n)$ by gluing via the symplectomorphism Φ of the collars constructed in Lemma 4.1. Choose an almost complex structure on S' compatible with the newly built symplectic structure. Now recall that there are only two isotopy classes of self-diffeomorphisms of ∂N_n and that they both extend to orientation-preserving self-diffeomorphisms of N_n which induce plus or minus the identity map on (co)homology [Go1]. Therefore, there is an orientation-preserving diffeomorphism $f: S \to S'$ which extends the identity on $S \setminus N_n$. Hence, f^* sends any class in $H^2(N_n) \subset H^2(S')$ to plus or minus the same class in $H^2(N_n) \subset H^2(S)$, and is the identity on $H^2(S \setminus N_n)$. Since the first Chern class of a symplectic manifold (with

$b_2^+ > 1$) is a basic class, $c_1(S')$ is a basic class, and therefore it is mapped by ϕ^* to the set of basic classes of S which, S being a minimal Kähler surface of general type, is $\{\pm c_1(S)\}$. But $c_1(S)|H^2(N_n) = (2k-n)\mathrm{PD}(T)$, $c_1(S')|H^2(N_n) = (2k'-n)\mathrm{PD}(T)$, and $f^* = \pm\mathrm{id}$ on $H^2(N_n)$. Hence, either $k = k'$ or $k = n - k'$. □

Remark 4.3. If instead of the trefoil knot in Fig. 2a we consider a (p,q)-torus knot K (with p, q relatively prime) the figure becomes a framed link presentation of a smooth 4-manifold W whose boundary is the Seifert fibered homology sphere $\Sigma(p,q,pqn-1)$. It is easy to see that K can be realized by Legendrian knots having Thurston–Bennequin invariant $tb = 1$ and any even rotation number r with $|\mathrm{r}| \le (p-1)(q-1)-2$. The Chern classes of the resulting Stein structures on W are of the form $c_1(W) = (s+rn)F + rS$, where F and S are the generators corresponding to K and to the unknot, respectively, while $|s| \le n-2$, $s \equiv n \bmod 2$, $|r| \le (p-1)(q-1)-2$, $r \equiv 0 \bmod 2$. Thus if we take p and q sufficiently large and calculate the values of $c_1(W)^2$ for different allowable s and r, Proposition 2.2 immediately implies that among the contact structures induced on the boundary of W there are homotopic as well as non-homotopic ones. Moreover, among the homotopic contact structures there are non-isotopic ones by an application of Theorem 1.2. This shows that the same 3-manifold $\Sigma(p,q,pqn-1)$ supports both kinds of counterexamples to the conjecture that the Euler class determines the tight contact structure up to isotopy.

Proof of Theorem 1.2. The proof is very similar to the proof of Theorem 4.2. Suppose that there exists a diffeomorphism $f\colon \partial X \to \partial X$ isotopic to the identity with $f_*(\xi) = \xi'$. Let S be a compact Kähler minimal surface of general type with $b_2^+(S) > 1$ containing (X, J_1) as a complex domain, according to Corollary 3.3, and let Θ_S denote the Spin^c-structure associated to the complex structure of S. We may apply Lemma 4.1 to build a symplectic structure ω' on the smooth manifold $S' = X \bigcup_\Phi (S \setminus X)$ obtained by gluing via the symplectomorphism of collars Φ constructed in the lemma. Choose an almost complex structure J' on S' compatible with this symplectic structure, and denote by $\Theta_{S'}$ the associated Spin^c-structure. Since f is isotopic to the identity on ∂X, there is an orientation-preserving diffeomorphism $\psi : S \to S'$ (which is equal to the identity outside of a small neighborhood of ∂X). Since S is a surface of general type, the pull-back of $\Theta_{S'}$ under ψ is isomorphic either to Θ_S or to the Spin^c-structure associated to the conjugate complex structure. But the symplectic structure ω' is equal to the symplectic structure ω on $S' \setminus X = S \setminus X$ by construction, and $\psi = \mathrm{id}$ on the same set, so $\psi^*(\Theta_{S'}) = \Theta_S$ on $X \setminus S$. The Spin^c-structure associated to the conjugate complex structure has basic class equal to $-K_S$. Since ψ induces the identity map on cohomology and the restriction of K_S to the complement of X is nontrivial (as one can check by going through the construction of S in Theorem 3.2) this implies $\psi^*(\Theta_{S'}) = \Theta_S$. Moreover, since the restrictions of Θ_S and $\Theta_{S'}$ to X are equal, respectively, to Θ_1 and Θ_2 we see that these have to be isomorphic. □

5. Final remarks

In this last section we collected a few intriguing consequences of Corollary 3.2 together with Seiberg–Witten theory, which seem to point to phenomena worth investigating. We hope to return to this in a future paper.

Let us recall that basic classes satisfy the following *adjunction inequality* [KM2, MST]. Let X be a smooth 4-manifold with $b_2^+(X) > 1$. If $c \in SWB(X)$ and $\Sigma \hookrightarrow X$ is a smoothly embedded surface of positive genus such that $\Sigma \cdot \Sigma \geq 0$, the genus of S is bounded from below, as

$$2g(\Sigma) - 2 \geq |c \cdot \Sigma| + \Sigma \cdot \Sigma \tag{6}$$

Remark 5.1. The adjunction inequality (6) has an interesting consequence. If a 2-plane field ξ on a 3-manifold M is induced by an almost-complex structure on a 4-manifold X in which M is a hypersurface, the Euler class $e(\xi)$ is the restriction of $c_1(X)$. If the almost-complex structure of X is compatible with a symplectic structure, $c_1(X)$ is basic, and since for any $S \subset M$ the self-intersection in X is $S \cdot S = 0$ we see that for such structures the Bennequin–Eliashberg inequality (1) follows trivially from the adjunction inequality (6). The relationship of such 2-plane fields to the tight contact structures on M is not clear.

Theorem 5.2. *Let W be a 2-dimensional Stein manifold with boundary. Then, $c_1(W)$ satisfies the adjunction inequality for any smooth surface $\Sigma \subset W$ having positive genus and satisfying $\Sigma \cdot \Sigma \geq 0$.*

Proof. By Corollary 3.3 the Stein manifold W embeds holomorphically inside a closed Kähler surface S with $b_2^+(S) > 1$. Thus, the image of Σ under this embedding satisfies (6) with $c = c_1(S)$. But since the embedding $W \subset S$ is holomorphic, $c_1(S)$ restricts to $H^2(W)$ as $c_1(W)$. □

We may now apply this theorem to compute, for $n \geq 2$, all the Chern classes of Stein structures on the manifold N_n defined by the framed link of Fig. 2a. Recall that the homology of N_n is generated by the two classes T and S, where T is represented by a smoothly embedded torus of square zero, S is represented by a smoothly embedded sphere of square $-n$ and $T \cdot S = 1$.

Corollary 5.3. *The set of first Chern classes of Stein structures with boundary on N_n is $\{a\mathrm{PD}(T) | a \equiv n \bmod 2, |a| \leq n-2\}$.*

Proof. The class $S + hT$ has non-negative square for $h \geq \frac{n}{2}$, and it is representable by a smoothly embedded surface of genus h. If $c \in H^2(N_n; \mathbb{Z})$ is the first Chern class of some Stein structure on N_n then it is a characteristic class, and by Theorem 5.2 it satisfies (6) with respect to both T and $S + hT$ for $h \geq \frac{n}{2}$. This implies $c = a\mathrm{PD}(T)$, with $a \equiv n \bmod 2$, and $|a| \leq n - 2$. On the other hand, we know that every such class is the first Chern class of a Stein structure, because $c_1(W_n^k) = (2k - n)\mathrm{PD}(T)$, $k = 1, \ldots n - 1$. □

Looking at the previous results, it seems natural to raise a question. Given a smooth 4-manifold with boundary W, denote by $\mathscr{B}(W)$ the set of first Chern classes of Stein structures with boundary on W.

Question 5.4. Let W be a smooth 4-manifold with boundary. Given some description of W, as for example a framed link presentation, does there exist an algorithmic procedure to compute $\mathscr{B}(W)$?

Finally, we ask two more informal questions. We have seen in Corollary 5.3 that the possible Legendrian realizations of the framed link representing the 4-manifold N_n give us all the possible Chern classes of Stein structures with boundary on N_n. Does this happen for every 4-manifold with boundary? Having picked another framed link presentation for N_n, would the same procedure give the same classes?

References

[Ae] B. Aebischer et al., Symplectic Geometry: and introduction based on the seminar in Bern, 1992, Progr. in Math. **124**, Birkhäuser Verlag, 1994

[Be] D. Bennequin, Entrelacements et equations de Pfaff, Astèrisque **107–108**, 83–161 (1983)

[DLS] J. Demailly, L. Lempert, B. Shiffman, Algebraic approximations of holomorphic maps from Stein domains to projective manifolds, Duke Math. Jour. **76**, No. 2, 333–363 (1994)

[El1] Y. Eliashberg, Classification of overtwisted contact structures on 3-manifolds, Invent. Math. **98**, 623–637 (1989)

[El2] Y. Eliashberg, Topological characterization of Stein manifolds of dimension > 2, Intern. Journal of Math. **1**, No. 1, 29–46 (1990)

[El3] Y. Eliashberg, Filling by holomorphic discs and its applications, London Math. Soc. Lecture Notes Series **151**, 45–67 (1991)

[El4] Y. Eliashberg, Legendrian and transversal knots in tight contact manifolds, in Topological Methods in Modern Mathematics, Publish or Perish, Stony Brook, NY, 171–193 (1991)

[El5] Y. Eliashberg, Contact 3-manifolds twenty years since J. Martinet's work, Ann. Inst. Fourier **42**, 165–192 (1992)

[EG] Y. Eliashberg, M. Gromov, Convex symplectic manifolds, Proc. of Symposia in Pure Math **52**, part 2, 135–162 (1991)

[EP] Y. Eliashberg, L. Polterovich, New applications of Luttinger's surgery, Comment. Math. Helvetici **65**, 512–522 (1994)

[Erl] T. Erlandsson, Geometry of contact tranformations in dimension 3, Doctoral Dissertation, Uppsala, 1981

[FS] R. Fintushel, R.J. Stern, Immersed spheres in 4-manifolds and the immersed Thom conjecture,Turkish J. Math.**19**, 145–157 (1995)

[Gi1] E. Giroux, Topologie de contact en dimension 3, Sèminaire Bourbaki **760**, 7–33 (1992–93)

[Gi2] E. Giroux, Une structure de contact meme tendue, est plus ou moins tordue, Ann. scient. Éc. Norm. Sup., 4[e] série **27**, 697–705 (1994)

[Gi3] E. Giroux, Structures de contact en dimension trois et bifurcation des feuilletages de surfaces, in preparation
[Go1] R.E. Gompf, Nuclei of Elliptic Surfaces, Topology **30**, 479–511 (1991)
[Go2] R.E. Gompf, Handlebody construction of Stein surfaces, preprint
[Goo] J.E. Goodman, Affine open subsets of algebraic varieties and ample divisors, Ann. of Math. **89**, 160–183 (1969)
[Gra] J.W. Gray, Some global properties of contact structures, Ann. of Math. **69**, 421–450 (1959)
[Gro] M. Gromov, Pseudo-holomorphic curves in symplectic manifolds, Inv. Math. **82**, 307–347 (1985)
[GR] R. Gunning, H. Rossi, Analytic functions of several complex variables, Prentice Hall 1965
[KM1] P.B. Kronheimer, T.S. Mrowka, Embedded surfaces and the structure of Donaldson's polynomial invariants, J. Differential Geom. **41**, 573–734 (1995)
[KM2] P.B. Kronheimer, T.S. Mrowka, The genus of embedded surfaces in the projective plane, Math. Res. Lett. **1**, 797–808 (1995)
[KM3] P.B. Kronheimer, T.S. Mrowka, Monopoles and contact structures, to appear in Inv. Math
[Kr] P.B. Kronheimer, Lecture Notes, Oxford University, 1995
[L] L. Lempert Algebraic approximations in analytic geometry Invent. Math. **121**, 335–354 (1995)
[Mo] J.W. Morgan, The Seiberg–Witten Equations and Applications to the Topology of Smooth Four-Manifolds, preprint
[MST] J.W. Morgan, Z. Szabo, C.H. Taubes, The generalized Thom Conjecture, preprint
[S] E.L. Stout, Algebraic domains in Stein manifolds, Comtemp. Math. **32**, 259–266 (1984)
[SW1] N. Seiberg, E. Witten, Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory, Nucl. Phys. **B426**, 19–52 (1994)
[SW2] N. Seiberg, E. Witten, Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD, Nucl. Phys. **B431**, 581–640 (1994)
[Ta1] C.H. Taubes, The Seiberg–Witten invariants and symplectic forms, Math. Res. Lett. **1**, 809–822 (1995)
[Ta2] C.H. Taubes, More constraints on symplectic manifolds from Seiberg–Witten equations, Math. Res. Lett. **2**, 9–14 (1995)
[Ta3] C.H. Taubes, The Seiberg–Witten and the Gromov Invariants, Math. Res. Lett. **2**, 221–238 (1995)
[Ta4] C.H. Taubes, SW $\Rightarrow$ Gr From the Seiberg–Witten equations to pseudo-holomorphic curves, J. Amer. Math. Soc. **9**, 845–918 (1996)
[We] A. Weinstein, Contact surgery and symplectic handlebodies, Hokkaido Math. Journal **20**, 241–251 (1991)
[Wi] E. Witten, Monopoles and four-manifolds, Math. Res. Lett. **1**, 769–796 (1995)