Invent. math. 128, 501–588 (1997)

Inventiones mathematicae

Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras

V. Tarasov★, A. Varchenko★★

★ Laboratoire de Physique Théorique ENSLAPP★★★, École Normale Supérieure de Lyon, 46, Allée d'Italie, F-69364 Lyon Cedex 07, France

★ Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA

Oblatum 20-V-1996 & 7-VIII-1996

1. Introduction

In this paper we consider the rational quantized Knizhnik–Zamolodchikov equation (qKZ equation) associated with the Lie algebra $\mathfrak{sl}_2$ and solve it. The rational qKZ equation associated with $\mathfrak{sl}_2$ is a system of difference equations for a function $\Psi(z_1,\dots,z_n)$ with values in a tensor product $V_1 \otimes \cdots \otimes V_n$ of $\mathfrak{sl}_2$-modules. The system of equations has the form

$$\Psi(z_1,\dots,z_m + p,\dots,z_n) = R_{m,m-1}(z_m - z_{m-1} + p)\cdots R_{m,1}(z_m - z_1 + p)\kappa^{-H_m} \times R_{m,n}(z_m - z_n)\cdots R_{m,m+1}(z_m - z_{m+1})\Psi(z_1,\dots,z_n)\,,$$

$m = 1,\dots,n$, where p and κ are parameters of the qKZ equation, H is a generator of the Cartan subalgebra of $\mathfrak{sl}_2, H_m$ is H acting in the m-th factor, $R_{l,m}(x)$ is the rational R-matrix $R_{V_l V_m}(x) \in \mathrm{End}(V_l \otimes V_m)$ acting in the l-th and m-th factors of the tensor product of $\mathfrak{sl}_2$-modules. In this paper we consider only the negative steps p. The case of other values of the step can be treated by analytic continuation.

The qKZ equation is an important system of difference equations. The qKZ equations had been introduced in [FR] as equations for matrix elements of vertex operators of the quantum affine algebra. An important special case of the qKZ equation had been introduced earlier in [S] as equations for form factors in integrable quantum field theory; relevant solutions for these equations had been given therein. Later, the qKZ equations were derived as equations for correlation functions in lattice integrable models, cf. [JM] and references therein.

★ On leave of absence from St. Petersburg Branch of Steklov Mathematical Institute, supported in part by MAE-MICECO-CNRS Fellowship. e-mail: vtarasov@enslapp.ens-lyon.fr

★★ Supported in part by NSF grant DMS-9501290, e-mail: av@math.unc.edu

★★★ URA 14-36 du CNRS associée à l'E.N.S. de Lyon, au LAPP d'Annecy et à l'Universitè de Savoie

In the quasiclassical limit the qKZ equation turns into the differential Knizhnik–Zamolodchikov equation for conformal blocks of the Wess–Zumino–Witten model of conformal field theory on the sphere.

Asymptotic solutions to the qKZ equation as p tends to zero are closely related to diagonalization of the transfer-matrix of the corresponding lattice integrable model by the algebraic Bethe ansatz method [TV2].

We describe the space of solutions to the qKZ equation in terms of representation theory. Namely, we consider the quantum group $U_q(\mathfrak{sl}_2)$ with $q = e^{\pi i/p}$ and the $U_q(\mathfrak{sl}_2)$-modules $V_1^q, \ldots, V_n^q$ where V_m^q is the deformation of the $\mathfrak{sl}_2$-module V_m. For every permutation $\tau \in \mathbb{S}^n$ we consider the tensor product $V_{\tau 1}^q \otimes \cdots \otimes V_{\tau n}^q$ and establish a natural isomorphism of the space $\mathbb{S}$ of solutions to the qKZ equation with values in $V_1 \otimes \cdots \otimes V_n$ and the space $V_{\tau 1}^q \otimes \cdots \otimes V_{\tau n}^q \otimes \mathbb{F}$, where $\mathbb{F}$ is the space of functions in $z_1, \ldots, z_n$ which are p-periodic with respect to each of the variables,

$$C_\tau : V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q \otimes \mathbb{F} \to \mathbb{S} .$$

Notice that if $\Psi(z)$ is a solution to the qKZ equation and $F(z)$ is a p-periodic function, then also $F(z)\Psi(z)$ is a solution to the qKZ equation.

We call the isomorphisms C_τ the tensor coordinates on the space of solutions. The compositions of the isomorphisms are linear maps

$$C_{\tau,\tau'}(z_1, \ldots, z_n) : V_{\tau'_1}^q \otimes \cdots \otimes V_{\tau'_n}^q \to V_{\tau_1}^q \otimes \cdots \otimes V_{\tau n}^q$$

depending on $z_1, \ldots, z_n$ and p-periodic with respect to all variables. We call these compositions the transition functions. It turns out that the transition functions are defined in terms of the trigonometric R-matrices $R^q_{V_l V_m}(\zeta) \in \mathrm{End}(V_l^q \otimes V_m^q)$ acting in tensor products of $U_q(\mathfrak{sl}_2)$-modules. Namely, for any permutation τ and for any transposition $(m, m+1)$ the transition function

$$C_{\tau,\tau \,\cdot\, (m,m+1)}(z_1, \ldots, z_n) : V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_{m+1}}^q \otimes V_{\tau_m}^q \otimes \cdots \otimes V_{\tau_n}^q \to V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q$$

equals the operator $P_{V^q_{\tau_{m+1}} V^q_{\tau_m}} R^q_{V^q_{\tau_{m+1}} V^q_{\tau_m}}(\exp(2\pi i(z_{\tau_{m+1}} - z_{\tau_m})/p))$ acting in the m-th and $(m+1)$-th factors, here $P_{V_l V_m}$ is the transposition of the tensor factors; cf. Theorem 4.22.

We consider asymptotic zones $\mathrm{Re}\, z_{\tau_1} \ll \cdots \ll \mathrm{Re}\, z_{\tau_n}$ labelled by permutations $\tau \in \mathbb{S}^n$. For every asymptotic zone we define a basis of asymptotic solutions to the qKZ equation. We show that for every permutation τ the basis of the corresponding asymptotic solutions is the image of the standard monomial basis in $V_1^q \otimes \cdots \otimes V_n^q$ under the map

$$V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q \to V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q \otimes 1 \hookrightarrow V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q \otimes \mathbb{F} \xrightarrow{C_\tau} \mathbb{S} ,$$

cf. Theorem 6.4. The last two statements express the transition functions between the asymptotic solutions via the trigonometric R-matrices.

The rational R-matrix $R_{V_l V_m}(x) \in \mathrm{End}(V_l \otimes V_m)$ is defined in terms of the action of the Yangian $Y(\mathfrak{gl}_2)$ in the tensor product of $\mathfrak{sl}_2$-modules. The

Yangian $Y(\mathfrak{gl}_2)$ is a Hopf algebra which contains the universal enveloping algebra $U(\mathfrak{sl}_2)$ as a Hopf subalgebra and has a family of homomorphisms $Y(\mathfrak{gl}_2) \to U(\mathfrak{sl}_2)$ depending on a parameter. Therefore, each $\mathfrak{sl}_2$-module V_m carries a $Y(\mathfrak{gl}_2)$-module structure $V_m(x)$ depending on a parameter. For irreducible $\mathfrak{sl}_2$-modules V_l, V_m the Yangian modules $V_l(x) \otimes V_m(y)$ and $V_m(y) \otimes V_l(x)$ are irreducible and isomorphic for generic x, y. The map

$$P_{V_l V_m} R_{V_l V_m}(x-y) : V_l(x) \otimes V_m(y) \to V_m(y) \otimes V_l(x)$$

is the unique suitably normalized intertwiner [T], [D1].

Similarly, the trigonometric R-matrix $R^q_{V_l V_m}(\zeta) \in \operatorname{End}(V_l^q \otimes V_m^q)$ is defined in terms of the action of the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ in the tensor product of $U_q(\mathfrak{sl}_2)$-modules. The quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ contains $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra and has a family of homomorphisms $U_q'(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$ depending on a parameter. Therefore, each $U_q(\mathfrak{sl}_2)$-module V_m^q has a $U_q'(\widetilde{\mathfrak{gl}_2})$-module structure $V_m^q(\zeta)$ depending on a parameter. For irreducible $U_q(\mathfrak{sl}_2)$-modules V_l, V_m the $U_q'(\widetilde{\mathfrak{gl}_2})$-modules $V_l^q(\xi) \otimes V_m^q(\zeta)$ and $V_m^q(\zeta) \otimes V_l^q(\xi)$ are irreducible and isomorphic for generic ξ, ζ. The map

$$P_{V_l V_m} R^q_{V_l V_m}(\xi/\zeta) : V_l^q(\xi) \otimes V_m^q(\zeta) \to V_m^q(\zeta) \otimes V_l^q(\xi)$$

is the unique suitably normalized intertwiner [T], [CP].

Our result on the transition functions between asymptotic solutions together with the indicated construction of R-matrices shows that the *qKZ* equation establishes a connection between representation theories of the Yangian $Y(\mathfrak{gl}_2)$ and the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$. Our result is analogous to the Kohno–Drinfeld theorem on the monodromy group of the differential Knizhnik–Zamolodchikov equation [K], [D2].

The differential Knizhnik–Zamolodchikov equation (*KZ* equation) with values in a tensor product of $\mathfrak{sl}_2$-modules $V = V_1 \otimes \cdots \otimes V_n$ is a system of differential equations for a V-valued function $\Psi(z_1, \dots, z_n)$ and has the form

$$d\Psi = \frac{1}{p} \sum_{l \neq m} \frac{\Omega_{lm}}{z_l - z_m} \Psi \, d(z_l - z_m)$$

where p is a parameter of the equation, $\Omega_{lm} \in \operatorname{End}(V_l \otimes V_m)$ is the Casimir operator. The *KZ* equation defines an integrable connection over the complement in $\mathbb{C}^n$ to the union of the diagonal hyperplanes. The fundamental group of the complement is the pure braid group $\mathbb{P}_n$. The monodromy group of the equation is the representation $\mathbb{P}_n \to \operatorname{End}(V)$ defined by analytic continuation of solutions over loops. The Kohno–Drinfeld theorem says that this representation is isomorphic to the R-matrix representation of $\mathbb{P}_n$ in the tensor product of $U_q(\mathfrak{sl}_2)$-modules $V^q = V_1^q \otimes \cdots \otimes V_n^q, q = e^{\pi i/p}$, where the R-matrix representation is defined as follows. Let $R^q_{V_l V_m} \in \operatorname{End}(V_l^q \otimes V_m^q)$ be the action of the universal R-matrix of the quantum group $U_q(\mathfrak{sl}_2)$ in the tensor product of

$U_q(\mathfrak{sl}_2)$-modules. Then the R-matrix representation of $\mathbf{P}_n$ in V^q is defined by elementary transformations

$$V^q_{\tau_1} \otimes \cdots \otimes V^q_{\tau_n} \xrightarrow{P_{V_{\tau_m} V_{\tau_{m+1}}} R^q_{V_{\tau_m} V_{\tau_{m+1}}}} V^q_{\tau_1} \otimes \cdots \otimes V^q_{\tau_{m+1}} \otimes V^q_{\tau_m} \otimes \cdots \otimes V^q_{\tau_n} .$$

The Kohno–Drinfeld theorem establishes a connection between representation theories of a Lie algebra and the corresponding quantum group, see [D2]. Using the ideas of the Kohno-Drinfeld result it was proved in [KL] that the category of representations of a quantum group is equivalent to a suitably defined fusion category of representations of the corresponding affine Lie algebra. Similarly to the Kazhdan–Lusztig theorem one could expect that our result for the difference qKZ equation could be a base for a Kazhdan–Lusztig type result connecting certain categories of representations of Yangians and quantum affine algebras, cf. [KS].

In this paper we consider the rational qKZ equation. There are other types of the qKZ equation: the trigonometric qKZ equation [FR] and the elliptic $qKZB$ equation [F]. Here KZB stands for Knizhnik–Zamolodchikov-Bernard, and the difference $qKZB$ equation is a discretization of the differential KZB equation for conformal blocks on the torus.

The trigonometric qKZ equation with values in a tensor product of $U_q(\mathfrak{sl}_2)$-modules $V^q = V^q_1 \otimes \cdots \otimes V^q_n$ is a system of difference equations for a V^q-valued function $\Psi(z_1, \dots, z_n)$ and has the form

$$\begin{aligned}\Psi(z_1, \dots, pz_m, \dots, z_n) = {} & R^q_{m,m-1}(pz_m/z_{m-1}) \cdots R^q_{m,1}(pz_m/z_1) \kappa^{-H_m} \\ & \times R^q_{m,n}(z_m/z_n) \cdots R^q_{m,m+1}(z_m/z_{m+1}) \Psi(z_1, \dots, z_n) ,\end{aligned}$$

$m = 1, \dots, n$, where p and κ are parameters of the qKZ equation, q^H is a generator of the Cartan subalgebra of $U_q(\mathfrak{sl}_2), H_m$ is H acting in the m-th factor, $R_{l,m}(x)$ is the trigonometric R-matrix $R^q_{V^q_l V^q_m}(x) \in \mathrm{End}(V^q_l \otimes V^q_m)$ acting in the l-th and m-th factors of the tensor product of $U_q(\mathfrak{sl}_2)$-modules. In the next paper [TV3] we will describe for the trigonometric qKZ equation the analogues of the above results for the rational qKZ equation. Namely, we will describe the space of solutions to the trigonometric qKZ equation in terms of modules of the elliptic quantum group associated to the Lie algebra $\mathfrak{sl}_2$ [F], [FV] and will get the transition functions between asymptotic solutions in the same way as we did for the rational case. This result for the trigonometric qKZ equation gives a connection between representation theories of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ and the elliptic quantum group associated to $\mathfrak{sl}_2$.

In the paper [FTV] we will describe solutions to the elliptic difference $qKZB$ equation. The construction of solutions for the elliptic $qKZB$ equation is similar to the construction of solutions to the rational qKZ equation described in this paper and to the solutions of the trigonometric qKZ in [M], [R], [V3], [TV1]. Nevertheless, we do not know yet how to define asymptotic solutions for the elliptic $qKZB$ equation and what could be an elliptic analogue of our result on transition functions.

There are three different proofs for the Kohno–Drinfeld theorem. Roughly speaking, they are analytic [K], algebraic [D2], and geometric [SV2], [V2]. In the initial proof [K], Kohno expands a monodromy operator as a series of iterated integrals and studies such expansions. Drinfeld in [D2] formalizes algebraic properties of transition functions between asymptotic solutions and proves that the monodromy group could be nothing else but the R-matrix representation.

The leading idea of the geometric proof [SV2], [V2], [V4] was the principle that the monodromy of a differential equation could be computed only if the differential equation is the equation of the Gauss–Manin connection. The Gauss–Manin connection is a connection associated to a locally trivial bundle of algebraic manifolds with a local system on the space of the bundle. One considers the associated holomorphic vector bundle which fiber is the homology group of the fiber of the initial locally trivial bundle. Then the vector bundle has a canonical connection called the Gauss–Manin connection. Having a trivialization of the vector bundle one realises the connection as a system of differential equations. Its solutions are parametrized by elements of the homology group of the fiber. The monodromy group of that differential equation is the monodromy group of cycles of the fiber of the initial locally trivial bundle under continuous deformations over loops in the base. The description of the monodromy group of cycles is a geometric problem which is easier than studying analytic continuation of solutions of an abstract differential equation. In order to apply this idea to the proof of the Kohno–Drinfeld theorem the differential KZ equation was solved explicitly in terms of multidimensional hypergeometric integrals and solutions were represented as integrals of closed differential forms over cycles depending on parameters, then the space of cycles was identified with a tensor product of $U_q(\mathfrak{sl}_2)$-modules and the monodromy of cycles was computed in term of R-matrices.

In this paper, in order to establish a connection between representation theories of Yangians and quantum loop algebras we quantize the geometric picture for the KZ equation. First we solve the qKZ equation in terms of suitable multidimensional hypergeometric integrals of Mellin–Barnes type. We define a discrete analogue of a locally trivial bundle and a local system on the space of bundle. We define a discrete analogue of the Gauss–Manin connection for the discrete locally tirvial bundle with a discrete local system and consider the corresponding difference equation. We identify that difference equation with the difference qKZ equation. To realize this idea we introduce a suitable discrete de Rham complex and its cohomology group in the spirit of [A], then we define the homology group as the dual space to the cohomology group and construct a family of discrete cycles, elements of the discrete homology group, using ideas of [S]. We construct the space of discrete cycles as a certain space of functions. Having a representative of a discrete cohomology class (a function) and a discrete cycle (a function again) we define the pairing (the hypergeometric pairing) between the cohomology class and the cycle as an integral of their product with a certain fixed "hypergeometric phase function" over a certain fixed contour of the middle dimension. We show that there is enough discrete cycles and they form the space dual to the quotient space of the space of our discrete closed forms modulo discrete coboundaries. To prove this we compute the determinant of the period matrix and surprisingly get an explicit formula (5.14) for the determinant analogous to the determinant formulae for the "continuous" hypergeometric functions [V1], cf. the Loeser determinant formula for the Frobenius transformation [L]. The form of our discrete cycles suggests a natural identification of the space of our discrete cycles with a tensor product of $U_q(\mathfrak{sl}_2)$-modules and this identification allows us to prove the result on transition functions between asymptotic solutions.

As we know the *qKZ* equation turns into the differential *KZ* equation under the quasiclassical limit. We show that our discretization of geometry under the quasiclassical limit turns into the geometry of the differential *KZ* equation: representatives of our discrete cohomology classes turn into closed differential forms, our discrete cycles turn into "honest" topological cycles.

Note in conclusion, that our solutions to the *qKZ* equation in the special case considered in [S] are close to the solutions constructed therein, but different. It is also worth mentioning that our description of transition functions indicates quantum loop algebra symmetries in the model of quantum field theory considered in [S].

The paper is organized as follows. Sections 2–7 contain constructions and statements. In Sect. 8 we consider the special case of one-dimensional hypergeometric functions of the Mellin–Barnes type to illustrate ideas and proofs. Section 9 contains proofs in the multidimensional case.

Parts of this work had been written when the authors visited the University of Tokyo, the Kyoto University, the University Paris VI, École Normale Supérieure de Lyon, the MSRI at Berkeley. The authors thank those institutions for hospitality. The authors thank G. Felder and P. Etingof for valuable discussions.

2. Discrete flat connections and local systems

Discrete flat connections

Consider a complex vector space $\mathbb{C}^n$ called the *base space*. Fix a nonzero complex number p called the *step*. The lattice $\mathbb{Z}^n$ acts on the base space by translations $z \mapsto z + pl$ where $l \in \mathbb{Z}^n \subset \mathbb{C}^n$. Let $\mathbb{B}$ be an invariant subset of the base space. Say that there is a *bundle with a discrete connection* over $\mathbb{B}$ if for any $z \in \mathbb{B}$ there are a vector space $V(z)$ and linear isomorphisms

$$A_m(z_1,\dots,z_n) : V(z_1,\dots,z_m + p,\dots,z_n) \to V(z_1,\dots,z_n), \quad m = 1,\dots,n\,.$$

The connection is called *flat* (or *integrable*) if the isomorphisms $A_1,\dots,A_n$ commute:

$$(2.1)\qquad \begin{aligned} &A_l(z_1,\dots,z_n)A_m(z_1,\dots,z_l + p,\dots,z_n) \\ &\quad = A_m(z_1,\dots,z_n)A_l(z_1,\dots,z_m + p,\dots,z_n)\,. \end{aligned}$$

Say that a *discrete subbundle* in $\mathbb{B}$ is given if a subspace in every fiber is distinguished and the family of subspaces is invariant with respect to the connection.

A section $s : z \mapsto s(z)$ is called *periodic* (or *horizontal*) if its values are invariant with respect to the connection:

$$(2.2)\qquad A_m(z_1,\dots,z_n)s(z_1,\dots,z_m+p,\dots,z_n) = s(z_1,\dots,z_n), \quad m = 1,\dots,n\,.$$

A function $f(z_1,\dots,z_n)$ on the base space is called a *quasiconstant* if

$$f(z_1,\dots,z_m+p,\dots,z_n) = f(z_1,\dots,z_n), \quad m = 1,\dots,n\,.$$

Periodic sections form a module over the ring of quasiconstants.

The *dual bundle* with the *dual connection* has fiber $V^*(z)$ and isomorphisms

$$A^*_m(z_1,\dots,z_n) : V^*(z_1,\dots,z_n) \to V^*(z_1,\dots,z_m+p,\dots,z_n)\,.$$

Let $s_1,\dots,s_N$ be a basis of sections of the initial bundle. Then the isomorphisms A_m of the connection are given by matrices $A^{(m)}$:

$$A_m(z_1,\dots,z_n)s_k(z_1,\dots,z_m+p,\dots,z_n) = \sum_{l=1}^{N} A^{(m)}_{kl}(z_1,\dots,z_n)s_l(z_1,\dots,z_n)\,.$$

For any section $\psi : z \mapsto \psi(z)$ of the dual bundle, denote by $\Psi : z \mapsto \Psi(z)$ its coordinate vector, $\Psi_k(z) = \langle \psi(z), s_k(z)\rangle$.

The section ψ is periodic if and only if its coordinate vector satisfies the system of difference equations

$$\Psi(z_1,\dots,z_m+p,\dots,z_n) = A^{(m)}(z_1,\dots,z_n)\Psi(z_1,\dots,z_n), \quad m = 1,\dots,n\,.$$

Moreover, all solutions to the system have this form. This system of difference equations is called the *periodic section equation.*

Say that functions $\varphi_1,\dots,\varphi_n$ in variables $z_1,\dots,z_n$ form a *system of connection coefficients* if

$$\varphi_l(z_1,\dots,z_m+p,\dots,z_n)\varphi_m(z_1,\dots,z_n) = \varphi_m(z_1,\dots,z_l+p,\dots,z_n)\varphi_l(z_1,\dots,z_n)\,.$$

for all l,m. These functions define a connection on the trivial complex one-dimensional vector bundle.

There is a simple construction of connection coefficients. Fix arbitrary functions ϕ_{lm}, $l < m$, in one variable and nonzero complex numbers κ_m. Set

$$\varphi_m(z_1,\dots,z_n) = \kappa_m \left[\prod_{1\leqq l<m} \phi_{lm}(z_l - z_m - p)\right]^{-1} \prod_{m<l\leqq n} \phi_{ml}(z_m - z_l)\,.$$

The system of connection coefficients of this form is called *decomposable*, the functions ϕ_{lm} are called *primitive factors* and κ_m are called *scaling parameters.*

A function $\Phi(z_1,\dots,z_n)$ is called a *phase function* of a system of connection coefficients if

$$\Phi(z_1,\dots,z_m+p,\dots,z_n)=\varphi_m(z_1,\dots,z_n)\Phi(z_1,\dots,z_n),\quad m=1,\dots,n\,.$$

Similarly, a function $\Phi(x)$ is called a *phase function* of a function $\phi(x)$ in one variable if $\Phi(x+p)=\phi(x)\Phi(x)$. Note that the phase functions are not unique.

If the connection coefficients are decomposable, if Φ_{lm} are phase functions of primitive factors, and if K_m are phase functions of scaling parameters, then

$$\Phi(z_1,\dots,z_n)=\prod_{m=1}^{n}K_m(z_m)\prod_{l<m}\Phi_{lm}(z_l-z_m)$$

is a phase function of the system of connection coefficients.

For any function $f(z_1,\dots,z_n)$ define new functions $Q_1f,\dots,Q_nf$ and $D_1f,\dots,D_nf$ by

$$(Q_mf)(z_1,\dots,z_n)=\varphi_m(z_1,\dots,z_n)f(z_1,\dots,z_m+p,\dots,z_n)\,,$$

and

$$D_mf=Q_mf-f\,.$$

The functions $D_1f,\dots,D_nf$ are the *discrete partial derivatives* of the function f. We have $D_lD_mf=D_mD_lf$.

Let F be a vector space of functions in $z_1,\dots,z_n$ such that the operators $Q_1,\dots,Q_n$ induce linear isomorphisms of F:

$$Q_m:\mathrm{F}\to\mathrm{F}\,.$$

Say that the space F and the connection coefficients $\varphi_1,\dots,\varphi_n$ form a one-dimensional *discrete local system* on $\mathbb{C}^n$. F is called the *functional space* of the local system.

Define the *de Rham complex* $(\Omega^\bullet(\mathrm{F}),D)$ of the local system in a standard way. Namely, set

$$\Omega^a=\left\{\omega=\sum_{k_1,\dots,k_a}f_{k_1,\dots,k_a}Dz_{k_1}\wedge\dots\wedge Dz_{k_a}\right\}$$

where $Dz_1,\dots,Dz_n$ are formal symbols and the coefficients $f_{k_1,\dots,k_a}$ belong to F. Define the differential of a function by $Df=\sum_{m=1}^{n}D_mfDz_m$, and the differential of a form by

$$D\omega=\sum_{k_1,\dots,k_a}Df_{k_1,\dots,k_a}\wedge Dz_{k_1}\wedge\dots\wedge Dz_{k_a}\,.$$

The cohomology groups $H^1,\dots,H^n$ of this complex are called the *cohomology groups of* $\mathbb{C}^n$ *with coefficients in the discrete local system*. In particular, the top cohomology group is $H^n=\mathrm{F}/D\mathrm{F}$ where $D\mathrm{F}=\sum_{m=1}^{n}D_m\mathrm{F}$. The dual spaces $H_a=(H^a)^*$ are called the *homology groups*.

There is a geometric construction of bundles with discrete flat connections. This is a discrete version of the Gauss–Manin connection construction.

Let $\pi : \mathbb{C}^{\ell+n} \to \mathbb{C}^n$ be an affine projection onto the base with fiber $\mathbb{C}^\ell$. $\mathbb{C}^{\ell+n}$ will be called the *total space*. Let $z_1,\dots,z_n$ be coordinates on the base, $t_1,\dots,t_\ell$ coordinates on the fiber, so that $t_1,\dots,t_\ell$, $z_1,\dots,z_n$ are coordinates on the total space. When it is convenient, we will denote the coordinates $z_1,\dots,z_n$ by $t_{\ell+1},\dots,t_{\ell+n}$.

Let F, $\varphi_1,\dots,\varphi_{\ell+n}$ be a local system on $\mathbb{C}^{\ell+n}$. For a point $z \in \mathbb{C}^n$ define a local system F(z), $\varphi_a(\cdot;z)$, $a = 1,\dots,\ell$, on the fiber over z. Set

$$\mathrm{F}(z) = \{f\,|_{\pi^{-1}(z)} \,|\, f \in \mathrm{F}\} \quad \text{and} \quad \varphi_a(\cdot\,;z) = \varphi_a|_{\pi^{-1}(z)}\,.$$

The de Rham complex, cohomology and homology groups of the fiber are denoted by $(\Omega^\bullet(z),D(z))$, $H^a(z)$ and $H_a(z)$, respectively.

There is a natural homomorphism of the de Rham complexes

$$(\Omega^\bullet(\mathbb{C}^{\ell+n},\mathrm{F}),D) \to (\Omega^\bullet(z),D(z)), \quad \omega \mapsto \omega|_{\pi^{-1}(z)}\,,$$

where the restriction of a form is defined in a standard way: all symbols $Dz_1,\dots,Dz_n$ are replaced by zero and all coefficients of the remaining monomials $Dt_{k_1} \wedge \cdots \wedge Dt_{k_a}$ are restricted to the fiber.

For a fixed a the vector spaces $H^a(z)$ form a bundle with a discrete flat connection. The linear maps

$$A_m(z_1,\dots,z_n) : H^a(z_1,\dots,z_m + p,\dots,z_n) \to H^a(z_1,\dots,z_n)$$

are defined as follows. Define $Q_m : \Omega^a(\mathbb{C}^{\ell+n},\mathrm{F}) \to \Omega^a(\mathbb{C}^{\ell+n},\mathrm{F})$ by

$$\omega \mapsto \sum_{k_1,\dots,k_a} Q_m f_{k_1,\dots,k_a} Dz_{k_1} \wedge \cdots \wedge Dz_{k_a}\,.$$

Then Q_m induces a homomorphism of the de Rham complexes

$$\begin{aligned}&(\Omega^\bullet(z_1,\dots,z_m + p,\dots,z_n),D(z_1,\dots,z_m + p,\dots,z_n))\\ &\quad\to (\Omega^\bullet(z_1,\dots,z_n),D(z_1,\dots,z_n))\,.\end{aligned}$$

We set $A_m(z_1,\dots,z_n)$ to be equal to the induced map of the cohomology spaces. This connection is called the *discrete Gauss–Manin connection.*

The Gauss–Manin connection on the cohomological bundle induces the dual flat connection on the homological bundle:

$$A_m^*(z_1,\dots,z_n) : H_a(z_1,\dots,z_n) \to H_a(z_1,\dots,z_m + p,\dots,z_n)\,.$$

In this paper we study the Gauss–Manin connection for a class of discrete local systems.

Connection coefficients of local systems

There are three important classes of local systems: rational, trigonometric and elliptic.

Consider a local system with decomposable connection coefficients and primitive factors of the form

$$\phi_{ab}(x) = \frac{\tau(x + \alpha_{ab})}{\tau(x + \beta_{ab})}$$

where $\tau(x)$ is a function in one variable and α_{ab}, β_{ab} are suitable complex numbers. A local system is called *rational, trigonometric* or *elliptic* if

$$\tau(x) = x, \qquad \tau(x) = \sin(\gamma x), \qquad \tau(x) = \theta(\gamma x),$$

respectively. Here $\theta(x)$ is a theta-function and γ is a nonzero complex number. Note that $\tau(x) = \gamma x$ for all $\gamma \neq 0$ gives the same primitive factors.

Say that a decomposable system of connection coefficients on the total space is of the $\mathfrak{sl}_2$*-type* if the constants α_{ab}, β_{ab}, and the scaling parameters $\kappa_1, \dots, \kappa_{\ell+n}$ have the following form:

$$\begin{aligned}
&\alpha_{ab} = -\beta_{ab} = -h && \text{for } a < b \leqq \ell, \\
&\alpha_{ab} = -\beta_{ab} = \Lambda_{b-\ell} && \text{for } a \leqq \ell < b, \\
&\alpha_{ab} = -\beta_{ab} = 0 && \text{for } \ell < a < b, \\
&\kappa_a = \kappa && \text{for } a \leqq \ell, \\
&\kappa_a = 1 && \text{for } \ell < a.
\end{aligned} \tag{2.3}$$

Such a system of connection coefficients depends on $n + 2$ complex numbers $\Lambda_1, \dots, \Lambda_n$, κ, h.

In this paper we study rational systems of the $\mathfrak{sl}_2$-type, for the trigonometric case see [TV3] and for the elliptic case see [FTV].

The primitive factors of a rational $\mathfrak{sl}_2$-type local system have the form

$$\begin{aligned}
&\phi_{ab}(x) = \frac{x - h}{x + h} && \text{for } a < b \leqq \ell, \\
&\phi_{ab}(x) = \frac{x + \Lambda_{b-\ell}}{x - \Lambda_{b-\ell}} && \text{for } a \leqq \ell < b, \\
&\phi_{ab}(x) = 1 && \text{for } \ell < a < b.
\end{aligned}$$

Rescaling $\Lambda_1, \dots, \Lambda_n$ and x we can set $h = 1$, so we assume that the primitive factors of a rational $\mathfrak{sl}_2$-type local system have the form

$$\begin{aligned}
&\phi_{ab}(x) = \frac{x - 1}{x + 1} && \text{for } a < b \leqq \ell, \\
&\phi_{ab}(x) = \frac{x + \Lambda_{b-\ell}}{x - \Lambda_{b-\ell}} && \text{for } a \leqq \ell < b, \\
&\phi_{ab}(x) = 1 && \text{for } \ell < a < b.
\end{aligned}$$

The connection coefficients of a rational $\mathfrak{sl}_2$-type local system have the form

$$\varphi_a(t,z) = \kappa \prod_{m=1}^{n} \frac{t_a - z_m + \Lambda_m}{t_a - z_m - \Lambda_m} \prod_{a<b\leqq\ell} \frac{t_a - t_b - 1}{t_a - t_b + 1}$$

$$\times \prod_{1\leqq b<a} \frac{t_a - t_b - 1 + p}{t_a - t_b + 1 + p}, \quad a = 1,\dots,\ell,$$

$$\varphi_{\ell+m}(t,z) = \prod_{a=1}^{\ell} \frac{t_a - z_m - \Lambda_m - p}{t_a - z_m + \Lambda_m - p}, \quad m = 1,\dots,n.$$

A phase function of a primitive factor $(x+\alpha)/(x-\alpha)$ has the form

$$\Phi(x;\alpha) = \frac{\Gamma((x+\alpha)/p)}{\Gamma((x-\alpha)/p)} \tag{2.4}$$

and, therefore, a phase function of the system of connection coefficients is given by

$$\Phi(t_1,\dots,t_\ell,z_1,\dots,z_n) = \exp\left(\mu \sum_{a=1}^{\ell} t_a/p\right) \prod_{m=1}^{n} \prod_{a=1}^{\ell} \Phi(t_a - z_m; \Lambda_m) \times \prod_{1\leqq a<b\leqq\ell} \Phi(t_a - t_b; -1) \tag{2.5}$$

where parameters κ and μ are connected by the equation $\kappa = e^{\mu}$.

The Stirling formula gives the following asymptotics for the phase function (2.4) of the primitive factor as $x \to \infty$:

$$\Phi(x;\alpha) = (x/p)^{2\alpha}(1 + o(1)), \quad |\arg(x/p)| < \pi. \tag{2.6}$$

This formula defines asymptotics at infinity of the phase function of the system of connection coefficients.

The phase function (2.4) of the primitive factor has a symmetry property

$$\Phi(-x;\alpha) = \Phi(x;\alpha)\frac{(x+\alpha)\sin(\pi(x+\alpha)/p)}{(x-\alpha)\sin(\pi(x-\alpha)/p)}$$

which leads to a symmetry property

$$\Phi(t_1,\dots,t_{a+1},t_a,\dots,t_\ell,z_1,\dots,z_n) = \Phi(t_1,\dots,t_\ell,z_1,\dots,z_n)\frac{(t_a - t_{a+1} - 1)\sin(\pi(t_a - t_{a+1} - 1)/p)}{(t_a - t_{a+1} + 1)\sin(\pi(t_a - t_{a+1} + 1)/p)} \tag{2.7}$$

of the phase function of the system of connection coefficients. This property later motivates definitions (2.9) and (2.25) of certain actions of the symmetric group.

The functional space of a rational $\mathfrak{sl}_2$-type local system

Define the functional space $\widehat{\mathscr{F}}$ of a rational $\mathfrak{sl}_2$-type local system as the space of rational functions on the total space with at most simple poles at the following hyperplanes

$$\begin{aligned}(2.8)\qquad & t_a = z_m - \Lambda_m + (s+1)p, \qquad && t_a = z_m + \Lambda_m - sp\,,\\ & t_a = t_b - 1 - (s+1)p, && t_a = t_b + 1 + sp\,,\end{aligned}$$

$1 \leqq b < a \leqq \ell$, $m = 1,\dots,n$, $s \in \mathbb{Z}_{\geqq 0}$. It is easy to check that the functional space is invariant with respect to all operators $Q_m^{\pm 1}$.

Define an action of the symmetric group $\mathbb{S}^\ell$ on the functional space:

$$(2.9)\qquad \sigma : \widehat{\mathscr{F}} \to \widehat{\mathscr{F}}, \qquad f \mapsto [f]_\sigma, \quad \sigma \in \mathbb{S}^\ell\,,$$

by the following action of simple transpositions:

$$[f]_{(a,a+1)}(t_1,\dots,t_\ell,z_1,\dots,z_n) = f(t_1,\dots,t_{a+1},t_a,\dots,t_\ell,z_1,\dots,z_n)\frac{t_a - t_{a+1} - 1}{t_a - t_{a+1} + 1}\,,$$

$a = 1,\dots,\ell-1$. The operators $Q_1,\dots,Q_{\ell+n}$ and $D_1,\dots,D_{\ell+n}$ commute with the action of the symmetric group.

We extend the $\mathbb{S}^\ell$-action to the de Rham complex assuming that it respects the exterior product and

$$\sigma : Dt_a \mapsto Dt_{\sigma_a}, \qquad \sigma : Dz_m \mapsto Dz_m, \quad \sigma \in \mathbb{S}^\ell\,.$$

The same formulae define an action of the symmetric group on the de Rham complex of a fiber. The homomorphism of the restriction of the de Rham complex of the total space to the de Rham complex of a fiber commutes with the action of the symmetric group. The action of the symmetric group induces an action of the symmetric group on the homology and cohomology groups. The Gauss–Manin connection commutes with this action.

If a symmetric group acts on a vector space V, we will denote by V_Σ the subspace of invariant vectors and by V^A or by V_A the subspace of skew-invariant vectors.

In this paper we are interested in the skew-invariant part $H_A^\ell(z)$ of the top cohomology group of a fiber. This subspace is generated by forms $f Dt_1 \wedge \dots \wedge Dt_\ell$ where f runs through the space $\widehat{\mathscr{F}}_\Sigma(z)$ of invariant functions.

Introduce an important *rational hypergeometric space* $\mathscr{F} \subset \widehat{\mathscr{F}}_\Sigma$ as the subspace of functions of the form

$$P(t_1,\dots,t_\ell,z_1,\dots,z_n)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{t_a - z_m - \Lambda_m}\prod_{1\leqq a<b\leqq\ell}\frac{t_a - t_b}{t_a - t_b + 1}$$

where P is a polynomial with complex coefficients which is symmetric in variables $t_1,\dots,t_\ell$, and has degree less than n in each of the variables $t_1,\dots,t_\ell$. The restriction of the hypergeometric space to a fiber defines the *rational* *hypergeometric space* $\mathscr{F}(z) \subset \widehat{\mathscr{F}}_{\Sigma}(z)$ of the fiber which is a complex finite-dimensional vector space. A form $f\,Dt_1 \wedge \cdots \wedge Dt_\ell$ with the coefficient in the hypergeometric space is called a *hypergeometric form.*

The subspace $\mathscr{H}(z) \subset H^\ell_A(z)$ of the top cohomology group of a fiber generated by the hypergeometric forms is called the *hypergeometric space* or the *hypergeometric cohomology group.*

The union of the hyperplanes

$$(2.10) \qquad z_l + \Lambda_l - z_m + \Lambda_m = r + ps, \quad r = 0,\dots,\ell-1, \; s \in \mathbb{Z}\,,$$

$l, m = 1,\dots,n$, $l \neq m$, in the base space $\mathbb{C}^n$ is called the *discriminant*. The complement to the discriminant will be denoted by $\mathbb{B}$.

(2.11) Theorem. [V3], [TV1] *The family of subspaces* $\{\mathscr{H}(z)\}_{z\in\mathbb{B}}$ *is invariant with respect to the Gauss–Manin connection and, therefore, defines a discrete subbundle.*

This subbundle will be called the *hypergeometric subbundle.*

Later on we often make the following assumptions. We assume that the step p is real negative and such that

$$(2.12) \qquad \{1,\dots,\ell\} \not\subset p\mathbb{Z}\,,$$

the weights $\Lambda_1,\dots,\Lambda_n$ are such that

$$(2.13) \qquad 2\Lambda_m - s \notin p\mathbb{Z}, \quad m = 1,\dots,n, \quad s = 1-\ell,\dots,\ell-1\,,$$

and the coordinates $z_1,\dots,z_n$ obey the condition

$$(2.14) \qquad z_l \pm \Lambda_l - z_m \pm \Lambda_m - s \notin p\mathbb{Z}, \quad l, m = 1,\dots,n, \quad l \neq m\,,$$

for any $s = 1-\ell,\dots,\ell-1$ and for an arbitrary combination of signs.

(2.15) Theorem. *Let* $\kappa \neq 1$. *Let* $p < 0$. *Let* (2.12)–(2.14) *hold. Then*

$$\dim \mathscr{H}(z) = \dim \mathscr{F}(z) = \binom{n+\ell-1}{n-1}\,.$$

This means that

$$(2.16) \qquad \mathscr{H}(z) \simeq \mathscr{F}(z)\,.$$

(2.17) Theorem. *Let* $\kappa = 1$. *Let* $p < 0$. *Let* (2.12)–(2.14) *hold. If* $2\sum_{m=1}^n \Lambda_m - s \notin p\mathbb{Z}_{<0}$ *for all* $s = \ell-1,\dots,2\ell-2$, *then* $\dim \mathscr{H}(z) = \binom{n+\ell-2}{n-2}$.

Theorems 2.15 and 2.17 are proved in Sect. 9.

Theorem 2.15 means that if the scaling parameter κ is not equal to 1, then every nonzero hypergeometric form defines a nonzero cohomology class. On the contrary, if $\kappa = 1$, then by Theorem 2.17 there are exact hypergeometric forms. We describe them in Lemma 2.21.

Bases in the rational hypergeometric space of a fiber

The finite-dimensional rational hypergeometric space $\mathscr{F}(z)$ of a fiber has $n!$ remarkable bases. These bases will allow us to identify geometry of an $\mathfrak{sl}_2$-type local system with representation theory. The bases are labelled by elements of the symmetric group $\mathbb{S}^n$. First we define the basis corresponding to the unit element of the symmetric group.

Let

(2.18) $$\mathscr{Z}_\ell^n = \left\{ \mathfrak{l} \in \mathbb{Z}_{\geqq 0}^n \,\middle|\, \sum_{m=1}^{n} \mathfrak{l}_m = \ell \right\} .$$

Set $\mathfrak{l}^m = \sum_{k=1}^m \mathfrak{l}_k$. In particular, $\mathfrak{l}^0 = 0$, $\mathfrak{l}^n = \ell$. For any $\mathfrak{l} \in \mathscr{Z}_\ell^n$ define a rational function $w_\mathfrak{l} \in \mathscr{F}$ as follows:

(2.19) $$w_\mathfrak{l}(t_1,\dots,t_\ell,z_1,\dots,z_n) = \sum_{\sigma\in\mathbb{S}^\ell} \left[\prod_{m=1}^{n} \frac{1}{\mathfrak{l}_m!} \prod_{a\in\Gamma_m} \left(\frac{1}{t_a - z_m - \Lambda_m} \prod_{1\leqq l<m} \frac{t_a - z_l + \Lambda_l}{t_a - z_l - \Lambda_l} \right) \right]_\sigma$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1,\dots,n$. The functions $w_\mathfrak{l}$ are called the *rational weight functions.*

Example. For $\ell = 1$ the functions have the form

$$w_{\mathfrak{e}(m)}(t,z_1,\dots,z_n) = \frac{1}{t - z_m - \Lambda_m} \prod_{1\leqq l<m} \frac{t - z_l + \Lambda_l}{t - z_l - \Lambda_l}$$

where $\mathfrak{e}(m) = (0,\dots,1_{m\text{-}th},\dots,0)$, $m = 1,\dots,n$.

Example. For $n = 1$ the function has the form

$$w_{(\ell)}(t_1,\dots,t_\ell,z_1) = \prod_{a=1}^{\ell} \frac{1}{t_a - z_1 - \Lambda_1} \prod_{1\leqq a<b\leqq \ell} \frac{t_a - t_b}{t_a - t_b + 1} .$$

Example. For $\ell = 2$ and $n = 2$ the functions have the form

$$w_{(2,0)}(t_1,t_2,z_1,z_2) = \frac{1}{(t_1 - z_1 - \Lambda_1)(t_2 - z_1 - \Lambda_1)} \frac{t_1 - t_2}{t_1 - t_2 + 1} ,$$

$$\begin{aligned} w_{(1,1)}(t_1,t_2,z_1,z_2) &= \frac{1}{(t_1 - z_1 - \Lambda_1)(t_2 - z_2 - \Lambda_2)} \frac{t_2 - z_1 + \Lambda_1}{t_2 - z_1 - \Lambda_1} \\ &\quad + \frac{1}{(t_2 - z_1 - \Lambda_1)(t_1 - z_2 - \Lambda_2)} \frac{t_1 - z_1 + \Lambda_1}{t_1 - z_1 - \Lambda_1} \frac{t_1 - t_2 - 1}{t_1 - t_2 + 1} , \end{aligned}$$

$$\begin{aligned} w_{(0,2)}(t_1,t_2,z_1,z_2) &= \frac{1}{(t_1 - z_2 - \Lambda_2)(t_2 - z_2 - \Lambda_2)} \frac{(t_1 - z_1 + \Lambda_1)(t_2 - z_1 + \Lambda_1)}{(t_1 - z_1 - \Lambda_1)(t_2 - z_1 - \Lambda_1)} \\ &\quad \times \frac{t_1 - t_2}{t_1 - t_2 + 1} . \end{aligned}$$

(2.20) Lemma. *The functions $w_{\mathfrak{l}}$, $\mathfrak{l} \in \mathscr{Z}_\ell^n$, restricted to the fiber over z form a basis in the rational hypergeometric space $\mathscr{F}(z)$ of the fiber provided that for any $s = 0, \dots, \ell - 1$,*

$$z_l - \Lambda_l - z_m - \Lambda_m + s \neq 0, \quad 1 \leqq l < m \leqq n .$$

Lemma 2.20 is proved in Sect. 9.

(2.21) Lemma. *Let $\kappa = 1$. Then for any $\mathfrak{l} \in \mathscr{Z}_{\ell-1}^n$ the following relation holds:*

$$\sum_{m=1}^{n} (\mathfrak{l}_m + 1)(2\Lambda_m - \mathfrak{l}_m) w_{\mathfrak{l}+\mathfrak{e}(m)} = \sum_{a=1}^{\ell} D_a[w_{\mathfrak{l}}(t_2, \dots, t_\ell)]_{(1,a)} ,$$

where $(1, a) \in \mathbb{S}^\ell$ are transpositions. Moreover, if $\mathscr{R}(z)$ is the subspace in $\mathscr{F}(z)$ generated by the elements in the left hand side of the relations, then

$$\dim \mathscr{F}(z)/\mathscr{R}(z) = \binom{n + \ell - 2}{n - 2}$$

provided that $z_l - \Lambda_l - z_m - \Lambda_m + s \neq 0$, $1 \leqq l \leqq m \leqq n$, for any $s = 0, \dots, \ell - 1$.

The subspace $\mathscr{R}(z) \subset \mathscr{F}(z)$ is called the *coboundary subspace.*

The relations (2.21) induce relations

$$\sum_{m=1}^{n} \lfloor (\mathfrak{l}_m + 1)(2\Lambda_m - \mathfrak{l}_m) w_{\mathfrak{l}+\mathfrak{e}(m)} Dt_1 \wedge \cdots \wedge Dt_\ell \rfloor = 0 , \quad \mathfrak{l} \in \mathscr{Z}_{\ell-1}^n ,$$

in the cohomology group $H^\ell(z)$, where $\lfloor \alpha \rfloor$ denotes the cohomological class of a form α. For $\kappa = 1$ under assumptions of Theorem 2.17 we have

$$\mathscr{H}(z) \simeq \mathscr{F}(z)/\mathscr{R}(z) . \tag{2.22}$$

For any permutation $\tau \in \mathbb{S}^n$ define a basis $\{w_{\mathfrak{l}}^\tau\}_{\mathfrak{l} \in \mathscr{Z}_\ell^n}$ in the rational hypergeometric space of a fiber by similar formulae. Namely,

(2.23)

$$w_{\mathfrak{l}}^\tau(t_1, \dots, t_\ell, z_1, \dots, z_n; \Lambda_1, \dots, \Lambda_n) = w_{{}^\tau\mathfrak{l}}(t_1, \dots, t_\ell, z_{\tau_1}, \dots, z_{\tau_n}; \Lambda_{\tau_1}, \dots, \Lambda_{\tau_n})$$

where ${}^\tau\mathfrak{l} = (\mathfrak{l}_{\tau_1}, \dots, \mathfrak{l}_{\tau_n})$.

Example. For $\ell = 1$ and permutation $\tau = (n, n-1, \dots, 1)$ the functions have the form

$$w_{\mathfrak{e}(m)}^\tau(t, z_1, \dots, z_n) = \frac{1}{t - z_m - \Lambda_m} \prod_{m < l \leqq n} \frac{t - z_l + \Lambda_l}{t - z_l - \Lambda_l} .$$

The trigonometric hypergeometric space

In our study of the Gauss–Manin connection an important role is played by the following *trigonometric hypergeometric space.* The trigonometric hypergeometric space is a trigonometric counterpart of the rational hypergeometric space introduced above.

The trigonometric hypergeometric space $\mathcal{F}_q$ is the space of functions in variables $t_1,\dots,t_\ell,z_1,\dots,z_n$ which have the form

(2.24)
$$P(\xi_1,\dots,\xi_\ell,\zeta_1,\dots,\zeta_n) \times \prod_{m=1}^{n}\prod_{a=1}^{\ell} \frac{\exp(\pi i(z_m-t_a)/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)} \prod_{1\leqq a<b\leqq \ell} \frac{\sin(\pi(t_a-t_b)/p)}{\sin(\pi(t_a-t_b+1)/p)}$$

where

$$\xi_a = \exp(2\pi i t_a/p), \qquad \zeta_m = \exp(2\pi i z_m/p)\,,$$

and P is a polynomial with complex coefficients which is symmetric in variables $\xi_1,\dots,\xi_\ell$ and has degree less than n in each of the variables $\xi_1,\dots,\xi_\ell$.

Introduce the *singular trigonometric hypergeometric space* $\mathcal{F}_q^{\mathrm{sing}}\subset\mathcal{F}_q$ as the space of functions of the form (2.24) such that the polynomial P is divisible by the product $\xi_1,\dots,\xi_\ell$.

The restriction of the trigonometric hypergeometric spaces to a fiber defines the *trigonometric hypergeometric spaces* $\mathcal{F}_q^{\mathrm{sing}}(z)\subset\mathcal{F}_q(z)$ of the fiber. The trigonometric hypergeometric space $\mathcal{F}_q(z)$ is a complex finite-dimensional vector space of the same dimension as the rational hypergeometric space of the fiber.

The trigonometric hypergeometric spaces of fibers over z and z' are naturally identified if the points z and z' lie in the same orbit of the $\mathbb{Z}^n$-action on the base space, since all elements of the trigonometric hypergeometric space are p-periodic functions.

Introduce a new action of the symmetric group $\mathbb{S}^\ell$ on functions,

(2.25)
$$f\mapsto [\![f]\!]_\sigma, \quad \sigma\in\mathbb{S}^\ell\,,$$

by the following action of simple transpositions:

$$[\![f]\!]_{(a,a+1)}(t_1,\dots,t_\ell,z_1,\dots,z_n) = f(t_1,\dots,t_{a+1},t_a,\dots,t_\ell,z_1,\dots,z_n)\frac{\sin(\pi(t_a-t_{a+1}-1)/p)}{\sin(\pi(t_a-t_{a+1}+1)/p)}\,,$$

$a=1,\dots,\ell-1$. The trigonometric hypergeometric space is invariant with respect to this action. The action commutes with the restriction of functions to a fiber.

The trigonometric hypergeometric space of a fiber has $n!$ remarkable bases. The bases are labelled by elements of the symmetric group $\mathbb{S}^n$. First we define the basis corresponding to the unit element of the symmetric group. For any $\mathfrak{l} \in \mathbb{Z}_\ell^n$ define a function $W_\mathfrak{l} \in \mathscr{F}_q$ as follows:

(2.26) $$W_\mathfrak{l}(t_1,\dots,t_\ell,z_1,\dots,z_n) = \prod_{m=1}^{n}\prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \sum_{\sigma\in\mathbb{S}^\ell} \Bigg[\prod_{m=1}^{n}\prod_{a\in\Gamma_m}\Bigg(\frac{\exp(\pi i(z_m-t_a)/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)} \times \prod_{1\leqq l<m}\frac{\sin(\pi(t_a-z_l+\Lambda_l)/p)}{\sin(\pi(t_a-z_l-\Lambda_l)/p)}\Bigg)\Bigg]_\sigma$$

where $\Gamma_m = \{1+\mathfrak{l}^{m-1},\dots,\mathfrak{l}^m\}$, $m=1,\dots,n$. Also for any $\mathfrak{l}\in\mathscr{Z}_\ell^{n-1}$ define a function $ciW_\mathfrak{l}\in\mathscr{F}_q^{\rm sing}$ as follows:

(2.27)
$$\mathring{W}_\mathfrak{l}(t_1,\dots,t_\ell,z_1,\dots,z_n) = \prod_{m=1}^{n-1}\prod_{s=1}^{\mathfrak{l}_m}\frac{\sin(\pi/p)}{\sin(\pi s/p)}\sin(\pi(z_m-\Lambda_m-z_{m+1}-\Lambda_{m+1}+s-1)/p) \times \sum_{\sigma\in\mathbb{S}^\ell}\Bigg[\prod_{m=1}^{n-1}\prod_{a\in\Gamma_m}\Bigg(\frac{1}{\sin(\pi(t_a-z_m-\Lambda_m)/p)\sin(\pi(t_a-z_{m+1}-\Lambda_{m+1})/p)} \times \prod_{1\leqq l<m}\frac{\sin(\pi(t_a-z_l+\Lambda_l)/p)}{\sin(\pi(t_a-z_l-\Lambda_l)/p)}\Bigg)\Bigg]_\sigma .$$

The functions $W_\mathfrak{l}$ and $\mathring{W}_\mathfrak{l}$ are called the *trigonometric weight functions.*

(2.28) Lemma. *The functions $W_\mathfrak{l}$, $\mathfrak{l}\in\mathbb{Z}_\ell^n$, restricted to the fiber over z form a basis in the trigonometric hypergeometric space $\mathscr{F}_q(z)$ of the fiber, provided that for any $s=0,\dots,\ell-1$,*

$$z_l-\Lambda_l-z_m-\Lambda_m+s\notin p\mathbb{Z},\quad 1\leqq l<m\leqq n .$$

(2.29) Lemma. *The functions $\mathring{W}_\mathfrak{m}$, $\mathfrak{m}\in\mathscr{Z}_\ell^{n-1}$, restricted to the fiber over z form a basis in the singular trigonometric hypergeometric space $\mathscr{F}_q^{\rm sing}(z)$ of the fiber, provided that for any $s=0,\dots,\ell-1$,*

$$z_l-\Lambda_l-z_m-\Lambda_m+s\notin p\mathbb{Z},\quad 1\leqq l<m\leqq n .$$

Lemmas 2.28, 2.29 are proved in Sect. 9.

Example. For $\ell=1$ the functions $W_\mathfrak{l}$ have the form

$$W_{\mathrm{e}(m)}(t,z_1,\dots,z_n) = \frac{\exp(\pi i(z_m-t)/p)}{\sin(\pi(t-z_m-\Lambda_m)/p)}\prod_{1\leqq l<m}\frac{\sin(\pi(t-z_l+\Lambda_l)/p)}{\sin(\pi(t-z_l-\Lambda_l)/p)} .$$

The singular trigonometric hypergeometric space $\mathscr{F}_q^{\rm sing}(z) \subset \mathscr{F}_q(z)$ has dimension $(n-1)$ and is generated by the functions

$$\mathring{W}_{e(m)} = W_{e(m)} \exp(-\pi i \Lambda_m / p) - W_{e(m+1)} \exp(\pi i \Lambda_{m+1}/p), \quad m = 1, \dots, n-1 .$$

Example. For $n = 1$ the function $W_{(\ell)}$ has the form

$$W_{(\ell)}(t_1, \dots, t_\ell, z_1) = \prod_{a=1}^{\ell} \frac{\exp(\pi i(z_1 - t_a)/p)}{\sin(\pi(t_a - z_1 - \Lambda_1)/p)} \prod_{1 \leqq a < b \leqq \ell} \frac{\sin(\pi(t_a - t_b)/p)}{\sin(\pi(t_a - t_b + 1)/p)} .$$

Example. For $\ell = 2$ and $n = 2$ the functions $W_{\mathfrak{l}}$ have the form

$$W_{(2,0)}(t_1, t_2, z_1, z_2)$$
$$= \frac{\exp(\pi i(2z_1 - t_1 - t_2)/p)}{\sin(\pi(t_1 - z_1 - \Lambda_1)/p)\sin(\pi(t_2 - z_1 - \Lambda_1)/p)} \frac{\sin(\pi(t_1 - t_2)/p)}{\sin(\pi(t_1 - t_2 + 1)/p)},$$

$$W_{(1,1)}(t_1, t_2, z_1, z_2)$$
$$= \frac{\exp(\pi i(z_1 + z_2 - t_1 - t_2)/p)}{\sin(\pi(t_1 - z_1 - \Lambda_1)/p)\sin(\pi(t_2 - z_2 - \Lambda_2)/p)} \frac{\sin(\pi(t_2 - z_1 + \Lambda_1)/p)}{\sin(\pi(t_2 - z_1 - \Lambda_1)/p)}$$
$$+ \frac{\exp(\pi i(z_1 + z_2 - t_1 - t_2)/p)}{\sin(\pi(t_2 - z_1 - \Lambda_1)/p)\sin(\pi(t_1 - z_2 - \Lambda_2)/p)}$$
$$\times \frac{\sin(\pi(t_1 - z_1 + \Lambda_1)/p)}{\sin(\pi(t_1 - z_1 - \Lambda_1)/p)} \frac{\sin(\pi(t_1 - t_2 - 1)/p)}{\sin(\pi(t_1 - t_2 + 1)/p)},$$

$$W_{(0,2)}(t_1, t_2, z_1, z_2) = \frac{\exp(\pi i(2z_2 - t_1 - t_2)/p)}{\sin(\pi(t_1 - z_2 - \Lambda_2)/p)\sin(\pi(t_2 - z_2 - \Lambda_2)/p)}$$
$$\times \frac{\sin(\pi(t_1 - z_1 + \Lambda_1)/p)\sin(\pi(t_2 - z_1 + \Lambda_1)/p)}{\sin(\pi(t_1 - z_1 + \Lambda_1)/p)\sin(\pi(t_2 - z_1 - \Lambda_1)/p)}$$
$$\times \frac{\sin(\pi(t_1 - t_2)/p)}{\sin(\pi(t_1 - t_2 + 1)/p)},$$

The singular trigonometric hypergeometric space $\mathscr{F}_q^{\rm sing}(z) \subset \mathscr{F}_q(z)$ is one-dimensional and is generated by the function

$$\mathring{W}_{(2)} = W_{(2,0)} \exp(\pi i(1 - 2\Lambda_1)/p) - W_{(1,1)} \exp(\pi i(\Lambda_2 - \Lambda_1)/p)$$
$$+ W_{(0,2)} \exp(\pi i(2\Lambda_2 - 1)/p) .$$

For any permutation $\tau \in \mathbb{S}^n$ define a basis $\{W_{\mathfrak{l}}^{\tau}\}_{\mathfrak{l}\in\mathscr{Z}_\ell^n}$ in the trigonometric hypergeometric space of a fiber by similar formulae. Namely,

(2.30)
$$W_{\mathfrak{l}}^{\tau}(t_1,\dots,t_\ell,z_1,\dots,z_n;\Lambda_1,\dots,\Lambda_n) = W_{\tau_{\mathfrak{l}}}(t_1,\dots,t_\ell,z_{\tau_1},\dots,z_{\tau_n};\Lambda_{\tau_1},\dots,\Lambda_{\tau_n})$$

where ${}^{\tau}\mathfrak{l} = (\mathfrak{l}_{\tau_1},\dots,\mathfrak{l}_{\tau_n})$.

Example. For $\ell = 1$ and permutation $\tau = (n, n-1, \dots, 1)$ the functions have the form

$$W_{\mathfrak{e}(m)}^{\tau}(t,z_1,\dots,z_n) = \frac{\exp(\pi i(z_m - t)/p)}{\sin(\pi(t - z_m - \Lambda_m)/p)} \prod_{m<l\leqq n} \frac{\sin(\pi(t - z_l + \Lambda_l)/p)}{\sin(\pi(t - z_l - \Lambda_l)/p)} .$$

3. R-matrices and the qKZ connection

Highest weight $\mathfrak{sl}_2$*-modules*

Let E, F, H be generators of the Lie algebra $\mathfrak{sl}_2$, $[H,E]=E$, $[H,F]=-F$, $[E,F] = 2H$.

For an $\mathfrak{sl}_2$-module V let $V = \bigoplus_\lambda V_\lambda$ be its weight decomposition. Let $V^* = \bigoplus_\lambda V_\lambda^*$ be its restricted dual. Define a structure of an $\mathfrak{sl}_2$-module on V^* by

$$\langle E\varphi, x\rangle = \langle \varphi, Fx\rangle, \qquad \langle F\varphi, x\rangle = \langle \varphi, Ex\rangle, \qquad \langle H\varphi, x\rangle = \langle \varphi, Hx\rangle .$$

This $\mathfrak{sl}_2$-module structure on V^* will be called the *dual* module structure.

Let $V_1,\dots,V_n$ be $\mathfrak{sl}_2$-modules with highest weights $\Lambda_1,\dots,\Lambda_n$, respectively. We have the weight decompositions

$$V_1 \otimes \dots \otimes V_n = \bigoplus_{\ell=0}^{\infty} (V_1 \otimes \dots \otimes V_n)_\ell$$

and

$$(V_1 \otimes \dots \otimes V_n)^* = \bigoplus_{\ell=0}^{\infty} (V_1 \otimes \dots \otimes V_n)_\ell^*$$

where $()_\ell$ denotes the eigenspace of H with eigenvalue $\sum_{m=1}^n \Lambda_m - \ell$.

Let $F(V_1 \otimes \dots \otimes V_n)_{\ell-1}^* \subset (V_1 \otimes \dots \otimes V_n)_\ell^*$ be the image of the operator F. Let $(V_1 \otimes \dots \otimes V_n)_\ell^{\text{sing}} \subset V_1 \otimes \dots \otimes V_n$ be the kernel of the operator E. There is a natural pairing

(3.1) $$(V_1 \otimes \dots \otimes V_n)_\ell^{\text{sing}} \otimes (V_1 \otimes \dots \otimes V_n)_\ell^* / F(V_1 \otimes \dots \otimes V_n)_{\ell-1}^* \to \mathbb{C}$$

Let $V_1,\dots,V_n$ be Verma modules, then this pairing is nondegenerate provided

$$\prod_{m=1}^{n} \prod_{s=0}^{\ell-1} (2\Lambda_m - s) \neq 0 .$$

The rational R-matrix

Let V_1, V_2 be Verma modules for $\mathfrak{sl}_2$ with highest weights Λ_1, Λ_2 and generating vectors v_1, v_2, respectively. Consider an $\mathrm{End}(V_1 \otimes V_2)$-valued meromorphic function $R_{V_1V_2}(x)$ with the following properties:

$$(3.2) \qquad [R_{V_1V_2}(x), F \otimes \mathrm{id} + \mathrm{id} \otimes F] = 0\,,$$

$$R_{V_1V_2}(x)(H \otimes F - F \otimes H + xF \otimes \mathrm{id}) = (F \otimes H - H \otimes F + xF \otimes \mathrm{id})R_{V_1V_2}(x)\,,$$

in $\mathrm{End}(V_1 \otimes V_2)$ and

$$(3.3) \qquad R_{V_1V_2}(x)v_1 \otimes v_2 = v_1 \otimes v_2\,.$$

Such a function $R_{V_1V_2}(x)$ exists and is uniquely determined. $R_{V_1V_2}(x)$ is called the $\mathfrak{sl}_2$ *rational R-matrix* for the tensor product $V_1 \otimes V_2$.

It turns out that $R_{V_1V_2}(x)$ commutes with the standard diagonal action of $\mathfrak{sl}_2$ in $V_1 \otimes V_2$:

$$(3.4) \qquad [R_{V_1V_2}(x), X \otimes \mathrm{id} + \mathrm{id} \otimes X] = 0, \quad X \in \mathfrak{sl}_2\,.$$

In particular, $R_{V_1V_2}(x)$ respects the weight decomposition of $V_1 \otimes V_2$. $R_{V_1V_2}(x)$ also satisfies the following relation

$$R_{V_1V_2}(x)(E \otimes H - H \otimes E + xE \otimes \mathrm{id}) = (H \otimes E - E \otimes H + xE \otimes \mathrm{id})R_{V_1V_2}(x)\,.$$

The rational R-matrix $R_{V_1V_2}(x)$ satisfies the symmetry relation

$$P_{V_1V_2}R_{V_1V_2}(x) = R_{V_2V_1}(x)P_{V_1V_2}$$

where $P_{V_1V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is the permutation map: $P_{V_1V_2}(v \otimes v') = v' \otimes v$, and the inversion relation

$$R_{V_1V_2}(x) = R^{-1}_{V_1V_2}(-x)\,.$$

The following asymptotics holds as $x \to \infty$:

$$\begin{aligned} &R_{V_1V_2}(x) \\ &\quad = \mathrm{id} \otimes \mathrm{id} + x^{-1}(2\Lambda_1\Lambda_2\, \mathrm{id} \otimes \mathrm{id} - 2H \otimes H - E \otimes F - F \otimes E) + O(x^{-2})\,. \end{aligned}$$

Let $V_1 \otimes V_2 = \bigoplus_{l=0}^{\infty} V^{(l)}$ be the decomposition of the $\mathfrak{sl}_2$-module $V_1 \otimes V_2$ into the direct sum of irreducibles, where the irreducible module $V^{(l)}$ is generated by a singular vector of weight $\Lambda_1 + \Lambda_2 - l$. Let $\Pi^{(l)}$ be the projector onto $V^{(l)}$ along the other summands. Then we have

$$(3.5) \qquad R_{V_1V_2}(x) = \sum_{l=0}^{\infty} \Pi^{(l)} \cdot \prod_{s=0}^{l-1} \frac{x + \Lambda_1 + \Lambda_2 - s}{x - \Lambda_1 - \Lambda_2 + s}\,.$$

Let V_1, V_2, V_3 be Verma modules. The corresponding R-matrices satisfy the Yang–Baxter equation:

$$(3.6) \quad R_{V_1V_2}(x-y)R_{V_1V_3}(x)R_{V_2V_3}(y) = R_{V_2V_3}(y)R_{V_1V_3}(x)R_{V_1V_2}(x-y)\,.$$

All of the properties of $R_{V_1V_2}(x)$ given above are well known (cf. [KRS], [FTT], [T]).

The Yangian $Y(\mathfrak{gl}_2)$

The rational R-matrix is connected with an action of the Yangian $Y(\mathfrak{gl}_2)$ in a tensor product of $\mathfrak{sl}_2$-modules. The Yangian $Y(\mathfrak{gl}_2)$ is a remarkable Hopf algebra which contains $U(\mathfrak{sl}_2)$ as a Hopf subalgebra. We recall the necessary facts about $Y(\mathfrak{gl}_2)$ in this section.

The *Yangian* $Y(\mathfrak{gl}_2)$ is a unital associative algebra with an infinite set of generators $T_{ij}^{(s)}$, $i,j=1,2$, $s=1,2,\dots$, subject to the relations

$$[T_{ij}^{(r)},T_{kl}^{(s+1)}]-[T_{ij}^{(r+1)},T_{kl}^{(s)}]=T_{kj}^{(r)}T_{il}^{(s)}-T_{kj}^{(s)}T_{il}^{(r)}\,, \tag{3.7}$$

$i,j,k,l=1,2$, $r,s=1,2\dots$. Here $T_{ij}^{(0)}=\delta_{ij}$ and δ_{ij} is the Kronecker symbol.

The Yangian $Y(\mathfrak{gl}_2)$ is a Hopf algebra with a coproduct $\Delta: Y(\mathfrak{gl}_2)\to Y(\mathfrak{gl}_2)\otimes Y(\mathfrak{gl}_2)$:

$$\Delta: T_{ij}^{(s)}\mapsto \sum_{k=1}^{2}\sum_{r=0}^{s} T_{ik}^{(r)}\otimes T_{kj}^{(s-r)}\,.$$

There is an important one-parametric family of automorphisms $\rho_x: Y(\mathfrak{gl}_2)\to Y(\mathfrak{gl}_2)$:

$$\rho_x: T_{ij}^{(s)}\mapsto \sum_{r=1}^{s}\binom{s-1}{r-1}x^{s-r}T_{ij}^{(r)}\,.$$

The Yangian $Y(\mathfrak{gl}_2)$ contains $U(\mathfrak{sl}_2)$ as a Hopf subalgebra; the embedding is given by

$$E\mapsto T_{21}^{(1)},\qquad F\mapsto T_{12}^{(1)},\qquad H\mapsto (T_{11}^{(1)}-T_{22}^{(1)})/2\,.$$

There is also an *evaluation homomorphism* $\varepsilon: Y(\mathfrak{gl}_2)\to U(\mathfrak{sl}_2)$:

$$\begin{aligned}\varepsilon: T_{11}^{(s)}&\mapsto H\delta_{1s}, & \varepsilon: T_{12}^{(s)}&\mapsto F\delta_{1s}\,,\\ \varepsilon: T_{21}^{(s)}&\mapsto E\delta_{1s}, & \varepsilon: T_{22}^{(s)}&\mapsto -H\delta_{1s}\,,\end{aligned}$$

$s=1,2,\dots$. Both the automorphisms ρ_x and ε restricted to the subalgebra $U(\mathfrak{sl}_2)$ are the identity maps.

Introduce the generating series $T_{ij}(u)=\delta_{ij}+\sum_{s=1}^{\infty}T_{ij}^{(s)}u^{-s}$. In terms of these series the coproduct, the automorphisms ρ_x and the evaluation homomorphism look like

$$\begin{gathered}\Delta: T_{ij}(u)\mapsto \sum_k T_{ik}(u)\otimes T_{kj}(u)\,,\\ \rho_x: T(u)\mapsto T(u-x)\,,\end{gathered}$$

$$\begin{aligned}\varepsilon: T_{11}(u)&\mapsto Hu^{-1}, & \varepsilon: T_{12}(u)&\mapsto Fu^{-1}\,,\\ \varepsilon: T_{21}(u)&\mapsto Eu^{-1}, & \varepsilon: T_{22}(u)&\mapsto -Hu^{-1}\,.\end{aligned}$$

Let e_{ij}, $i,j=1,2$, be the 2×2 matrix with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$R(x)=\sum_{i,j=1}^{2}(xe_{ii}\otimes e_{jj}+e_{ij}\otimes e_{ji})\,.$$

Then relations (3.7) in the Yangian $Y(\mathfrak{gl}_2)$ have the form

$$R(x-y)T_{(1)}(x)T_{(2)}(y) = T_{(2)}(y)T_{(1)}(x)R(x-y)\,,$$

where $T_{(1)}(u) = \sum_{ij} e_{ij} \otimes 1 \otimes T_{ij}(u)$ and $T_{(2)}(u) = \sum_{ij} 1 \otimes e_{ij} \otimes T_{ij}(u)$.

For any $\mathfrak{sl}_2$-module V denote by $V(x)$ the $Y(\mathfrak{gl}_2)$-module which is obtained from the module V via the homomorphism $\varepsilon \circ \rho_x$. The module $V(x)$ is called the *evaluation module.*

Let V_1, V_2 be Verma modules for $\mathfrak{sl}_2$ with generating vectors v_1, v_2, respectively. For generic complex numbers x, y the $Y(\mathfrak{gl}_2)$-modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$ are isomorphic and the rational R-matrix $P_{V_1V_2}R_{V_1V_2}(x-y)$ intertwines them [T], [D1]. The vectors $v_1 \otimes v_2$ and $v_2 \otimes v_1$ are respective generating vectors of the $Y(\mathfrak{gl}_2)$-modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$. The rational R-matrix $R_{V_1V_2}(x-y)$ can be defined as the unique element of $\mathrm{End}(V_1 \otimes V_2)$ with property (3.3) and such that

$$P_{V_1V_2}R_{V_1V_2}(x-y) : V_1(x) \otimes V_2(y) \to V_2(y) \otimes V_1(x) \tag{3.8}$$

is an isomorphism of the $Y(\mathfrak{gl}_2)$-modules.

For a $Y(\mathfrak{gl}_2)$-module V let $V = \bigoplus_\lambda V_\lambda$ be its weight decomposition as an $\mathfrak{sl}_2$-module. Let $V^* = \bigoplus_\lambda V_\lambda^*$ be its restricted dual. Define a structure of a $Y(\mathfrak{gl}_2)$-module on V^* by

$$\langle T_{11}(u)\varphi, x\rangle = \langle \varphi, T_{11}(u)x\rangle, \qquad \langle T_{12}(u)\varphi, x\rangle = \langle \varphi, T_{21}(u)x\rangle\,,$$

$$\langle T_{21}(u)\varphi, x\rangle = \langle \varphi, T_{12}(u)x\rangle, \qquad \langle T_{22}(u)\varphi, x\rangle = \langle \varphi, T_{22}(u)x\rangle\,.$$

This $Y(\mathfrak{gl}_2)$-module structure on V^* will be called the *dual* module structure.

The rational qKZ connection associated with $\mathfrak{sl}_2$

Let $V_1, \dots, V_n$ be $\mathfrak{sl}_2$-modules. The *qKZ* connection is a discrete connection on the trivial bundle over $\mathbb{C}^n$ with fiber $V_1 \otimes \dots \otimes V_n$. We define it below.

Let $V_1, \dots, V_n$ be Verma modules with highest weights $\Lambda_1, \dots, \Lambda_n$, respectively. Let $R_{V_iV_j}(x)$ be the rational R-matrices. Let $R_{ij}(x) \in \mathrm{End}(V_1 \otimes \dots \otimes V_n)$ be defined in a standard way:

$$R_{ij}(x) = \sum \mathrm{id} \otimes \dots \otimes \underset{i\text{-th}}{r(x)} \otimes \dots \otimes \underset{j\text{-th}}{r'(x)} \otimes \dots \otimes \mathrm{id} \tag{3.9}$$

provided that $R_{V_iV_j}(x) = \sum r(x) \otimes r'(x) \in \mathrm{End}(V_i \otimes V_j)$. For any $X \in \mathfrak{sl}_2$ set

$$X_m = \mathrm{id} \otimes \dots \otimes \underset{m\text{-th}}{X} \otimes \dots \otimes \mathrm{id}\,.$$

Let p, κ be complex numbers. For any $m = 1, \dots, n$ set

(3.10)

$$\begin{aligned} K_m(z_1, \dots, z_n) = {} & R_{m,m-1}(z_m - z_{m-1} + p) \cdots R_{m,1}(z_m - z_1 + p)\kappa^{\Lambda_m - H_m} \\ & \times R_{m,n}(z_m - z_n) \cdots R_{m,m+1}(z_m - z_{m+1})\,, \end{aligned}$$

(3.11) Theorem. [FR] *The linear maps $K_m(z)$ obey the flatness conditions*

$$K_l(z_1,\dots,z_m+p,\dots,z_n)K_m(z_1,\dots,z_n)$$
$$= K_m(z_1,\dots,z_l+p,\dots,z_n)K_l(z_1,\dots,z_n), \quad l,m=1,\dots,n\,.$$

The maps $K_1(z),\dots,K_n(z)$ define a flat connection on a trivial bundle over $\mathbb{C}^n$ with fiber $V_1\otimes\cdots\otimes V_n$. This connection is called the qKZ connection.

By (3.4) the operators $K_m(z)$ commute with the diagonal action of H in $V_1\otimes\cdots\otimes V_n$:

$$[K_m(z_1,\dots,z_n),H]=0, \quad m=1,\dots,n\,,$$

and, therefore, preserve the weight decomposition of $V_1\otimes\cdots\otimes V_n$. Hence the qKZ connection induces the dual flat connection on the trivial bundle over $\mathbb{C}^n$ with fiber $(V_1\otimes\cdots\otimes V_n)^*$. This connection will be called the dual qKZ connection.

Let $\mathbb{B}\subset\mathbb{C}^n$ be the complement to the discriminant (2.10).

(3.12) Lemma. *For any $z\in\mathbb{B}$ the linear maps $K_1^*(z),\dots,K_n^*(z)$ define isomorphisms of $(V_1\otimes\cdots\otimes V_n)^*_\ell$.*

This statement follows from (3.5) and (3.10).

If $\kappa=1$, then the dual qKZ connection commutes with the diagonal action of $\mathfrak{sl}_2$ in $(V_1\otimes\cdots\otimes V_n)^*$:

$$[K_m^*(z_1,\dots,z_n),X]=0, \quad X\in\mathfrak{sl}_2,\ m=1,\dots,n\,,$$

and, therefore, admits a trivial discrete subbundle with fiber $F(V_1\otimes\cdots\otimes V_n)^*_{\ell-1}$, moreover, it induces a flat connection on the trivial bundle with fiber $(V_1\otimes\cdots\otimes V_n)^*_\ell/F(V_1\otimes\cdots\otimes V_n)^*_{\ell-1}$.

Let $V_1,\dots,V_n$ be $\mathfrak{sl}_2$-modules. The *qKZ equation* for a $V_1\otimes\cdots\otimes V_n$-valued function $\Psi(z_1,\dots,z_n)$ is the following system of equations

$$\Psi(z_1,\dots,z_m+p,\dots,z_n)=K_m(z_1,\dots,z_n)\Psi(z_1,\dots,z_n), \quad m=1,\dots,n\,.$$

The qKZ equation is a remarkable difference equation, see [S], [FR], [JM], [Lu].

The trigonometric R-matrix

Let q be a nonzero complex number which is not a root of unity. Let $E_q,F_q,q^{\pm H}$ be generators of $U_q(\mathfrak{sl}_2)$:

$$q^Hq^{-H}=q^{-H}q^H=1\,,$$

$$q^HE_q=qE_qq^H, \qquad q^HF_q=q^{-1}F_qq^H\,,$$

$$[E_q,F_q]=\frac{q^{2H}-q^{-2H}}{q-q^{-1}}\,.$$

A comultiplication $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is given by

$$\Delta(q^H) = q^H \otimes q^H, \quad \Delta(q^{-H}) = q^{-H} \otimes q^{-H},$$
$$\Delta(E_q) = E_q \otimes q^H + q^{-H} \otimes E_q, \quad \Delta(F_q) = F_q \otimes q^H + q^{-H} \otimes F_q .$$

The comultiplication defines a module structure on the tensor product of $U_q(\mathfrak{sl}_2)$-modules.

Let V_1, V_2 be Verma modules for $U_q(\mathfrak{sl}_2)$ with highest weights $q^{\Lambda_1}, q^{\Lambda_2}$ and generating vectors v_1, v_2, respectively. Consider an $\mathrm{End}(V_1 \otimes V_2)$-valued meromorphic function $R^q_{V_1V_2}(\zeta)$ with the following properties:

$$(3.13) \quad \begin{aligned} R^q_{V_1V_2}(\zeta)(F_q \otimes q^H + q^{-H} \otimes F_q) &= (F_q \otimes q^{-H} + q^H \otimes F_q)R^q_{V_1V_2}(\zeta) \\ R^q_{V_1V_2}(\zeta)(F_q \otimes q^{-H} + \zeta q^H \otimes F_q) &= (F_q \otimes q^H + \zeta q^{-H} \otimes F_q)R^q_{V_1V_2}(\zeta) \end{aligned}$$

in $\mathrm{End}(V_1 \otimes V_2)$ and

$$(3.14) \qquad R^q_{V_1V_2}(\zeta) v_1 \otimes v_2 = v_1 \otimes v_2 .$$

Such a function $R^q_{V_1V_2}(\zeta)$ exists and is uniquely determined. $R^q_{V_1V_2}(\zeta)$ is called the $\mathfrak{sl}_2$ *trigonometric R-matrix* for the tensor product $V_1 \otimes V_2$.

The trigonometric R-matrix $R^q_{V_1V_2}(\zeta)$ also satisfies the following relations

(3.15)
$$R^q_{V_1V_2}(\zeta)(E_q \otimes q^H + q^{-H} \otimes E_q) = (E_q \otimes q^{-H} + q^H \otimes E_q)R^q_{V_1V_2}(\zeta)$$
$$R^q_{V_1V_2}(\zeta)(\zeta E_q \otimes q^{-H} + q^H \otimes E_q) = (\zeta E_q \otimes q^H + q^{-H} \otimes E_q)R^q_{V_1V_2}(\zeta),$$
$$R^q_{V_1V_2}(\zeta) q^H \otimes q^H = q^H \otimes q^H R^q_{V_1V_2}(\zeta) .$$

In particular, $R^q_{V_1V_2}(\zeta)$ respects the weight decomposition of $V_1 \otimes V_2$.

$R^q_{V_1V_2}(\zeta)$ satisfies the inversion relation

$$P_{V_1V_2} R^q_{V_1V_2}(\zeta) = (R^q_{V_2V_1}(\zeta^{-1}))^{-1} P_{V_1V_2}$$

where $P_{V_1V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ is the permutation map.

Let $V_1 \otimes V_2 = \bigoplus_{l=0}^{\infty} V^{(l)}$ be the decomposition of the $U_q(\mathfrak{sl}_2)$-module $V_1 \otimes V_2$ into the direct sum of irreducibles, where the irreducible module $V^{(l)}$ is generated by a singular vector of weight $q^{\Lambda_1+\Lambda_2-l}$. Let $\Pi^{(l)}$ be the projector onto $V^{(l)}$ along the other summands. Then we have

$$(3.16) \qquad R^q_{V_1V_2}(\zeta) = R^q_{V_1V_2}(0) \sum_{l=0}^{\infty} \Pi^{(l)} \cdot \prod_{s=0}^{l-1} \frac{1 - \zeta q^{2s-2\Lambda_1-2\Lambda_2}}{1 - \zeta q^{2\Lambda_1-2\Lambda_2-2s}} .$$

where

$$R^q_{V_1V_2}(0) = q^{2\Lambda_1\Lambda_2 - 2H\otimes H} \sum_{k=0}^{\infty} (q^2-1)^{2k} \prod_{s=1}^{k} (1-q^{2s})^{-1} (q^H F_q \otimes q^{-H} E_q)^k .$$

Let V_1, V_2, V_3 be Verma modules. The corresponding R-matrices satisfy the Yang–Baxter equation:

$$(3.17) \qquad R^q_{V_1V_2}(\xi/\zeta)R^q_{V_1V_3}(\xi)R^q_{V_2V_3}(\zeta) = R^q_{V_2V_3}(\zeta)R^q_{V_1V_3}(\xi)R^q_{V_1V_2}(\xi/\zeta)\,.$$

All of the properties of $R^q_{V_1V_2}(\zeta)$ given above are well known (cf. [T], [D1], [J], [CP]).

Similar to the rational case one can define the qKZ connection associated with the trigonometric R-matrix (cf. [FR]). We study this trigonometric qKZ connection in [TV3].

The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$

The trigonometric R-matrix is connected with an action of the quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ in a tensor product of $U_q(\mathfrak{sl}_2)$-modules. The quantum loop algebra $U'_q(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra which contains $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra. We recall the necessary facts about $U'_q(\widetilde{\mathfrak{gl}_2})$ in this section.

Let q be a complex number, $q \neq \pm 1$. The *quantum loop algebra* $U_q(\widetilde{\mathfrak{gl}_2})$ is a unital associative algebra with generators $L^{(+0)}_{ij}$, $L^{(-0)}_{ji}$, $1 \leqq j < i \leqq 2$, and $L^{(s)}_{ij}$, $i,j = 1,2$, $s = \pm 1, \pm 2, \ldots$, subject to relations (3.18) [RS], [DF].

Let e_{ij} $i,j = 1,2$, be the 2×2 matrix with the only nonzero entry 1 at the intersection of the i-th row and j-th column. Set

$$\begin{aligned} R(\xi) = {} & (\xi q - q^{-1})(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + (\xi - 1)(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}) \\ & + \xi(q - q^{-1})e_{12} \otimes e_{21} + (q - q^{-1})e_{21} \otimes e_{12}\,. \end{aligned}$$

Introduce the generating series $L^{\pm}_{ij}(u) = L^{(\pm 0)}_{ij} + \sum_{s=1}^{\infty} L^{(\pm s)}_{ij} u^{\pm s}$. The relations in $U_q(\widetilde{\mathfrak{gl}_2})$ have the form

$$(3.18) \qquad L^{(+0)}_{ii}L^{(-0)}_{ii} = 1, \qquad L^{(-0)}_{ii}L^{(+0)}_{ii} = 1, \quad i = 1,2\,,$$

$$R(\xi/\zeta)L^{+}_{(1)}(\xi)L^{+}_{(2)}(\zeta) = L^{+}_{(2)}(\zeta)L^{+}_{(1)}(\xi)R(\xi/\zeta)\,,$$

$$R(\xi/\zeta)L^{+}_{(1)}(\xi)L^{-}_{(2)}(\zeta) = L^{-}_{(2)}(\zeta)L^{+}_{(1)}(\xi)R(\xi/\zeta)\,,$$

$$R(\xi/\zeta)L^{-}_{(1)}(\xi)L^{-}_{(2)}(\zeta) = L^{-}_{(2)}(\zeta)L^{-}_{(1)}(\xi)R(\xi/\zeta)\,,$$

where $L^{\nu}_{(1)}(\xi) = \sum_{ij} e_{ij} \otimes 1 \otimes L^{\nu}_{ij}(\xi)$ and $L^{\nu}_{(2)}(\xi) = \sum_{ij} 1 \otimes e_{ij} \otimes L^{\nu}_{ij}(\xi)$, $\nu = \pm$.

Elements $L^{(+0)}_{11}L^{(+0)}_{22}$, $L^{(+0)}_{22}L^{(+0)}_{11}$, $L^{(-0)}_{11}L^{(-0)}_{22}$, $L^{(-0)}_{22}L^{(-0)}_{11}$ are central in $U_q(\widetilde{\mathfrak{gl}_2})$. Impose the following relations:

$$L^{(+0)}_{11}L^{(+0)}_{22} = 1, \qquad L^{(+0)}_{22}L^{(+0)}_{11} = 1, \qquad L^{(-0)}_{11}L^{(-0)}_{22} = 1, \qquad L^{(-0)}_{22}L^{(-0)}_{11} = 1$$

in addition to relations (3.18). Denote the corresponding quotient algebra by $U'_q(\widetilde{\mathfrak{gl}_2})$.

The quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ is a Hopf algebra with a coproduct $\Delta^q : U_q'(\widetilde{\mathfrak{gl}_2}) \to U_q'(\widetilde{\mathfrak{gl}_2}) \otimes U_q'(\widetilde{\mathfrak{gl}_2})$:

$$\Delta^q : L_{ij}^{\nu}(\xi) \mapsto \sum_k L_{kj}^{\nu}(\xi) \otimes L_{ik}^{\nu}(\xi), \quad \nu = \pm .$$

Remark. Notice that we take the coproduct Δ^q for the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ which is in a sense opposite to the coproduct Δ taken for the Yangian $Y(\mathfrak{gl}_2)$ (cf. Theorems 4.25, 4.26).

There is an important one-parametric family of automorphisms $\rho_\zeta^q : U_q'(\widetilde{\mathfrak{gl}_2}) \to U_q'(\widetilde{\mathfrak{gl}_2})$:

$$\rho_\zeta^q : L_{ij}^{\nu}(\xi) \mapsto L_{ij}^{\nu}(\xi/\zeta), \quad \nu = \pm ,$$

that is

$$\rho_\zeta^q : L_{ij}^{(\pm 0)} \mapsto L_{ij}^{(\pm 0)} \quad \text{and} \quad \rho_\zeta^q : L_{ij}^{(s)} \mapsto \zeta^{-s} L_{ij}^{(s)}, \quad s \in \mathbb{Z}_{\neq 0} .$$

The quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ contains $U_q(\mathfrak{sl}_2)$ as a Hopf subalgebra; the embedding is given by

$$E_q \mapsto -L_{21}^{(+0)}/(q-q^{-1}), \qquad F_q \mapsto L_{12}^{(-0)}/(q-q^{-1}), \quad q^H \mapsto L_{11}^{(-0)} .$$

There is also an *evaluation homomorphism* $\varepsilon^q : U_q'(\widetilde{\mathfrak{gl}_2}) \to U_q(\mathfrak{sl}_2)$:

$$\begin{aligned}
&\varepsilon^q : L_{11}^{+}(\xi) \mapsto q^{-H} - q^H \xi, &&\varepsilon^q : L_{12}^{+}(\xi) \mapsto -F_q(q-q^{-1})\xi ;\\
&\varepsilon^q : L_{21}^{+}(\xi) \mapsto -E_q(q-q^{-1}), &&\varepsilon^q : L_{22}^{+}(\xi) \mapsto q^H - q^{-H}\xi ,\\
&\varepsilon^q : L_{11}^{-}(\xi) \mapsto q^H - q^{-H}\xi^{-1}, &&\varepsilon^q : L_{12}^{-}(\xi) \mapsto F_q(q-q^{-1}) ,\\
&\varepsilon^q : L_{21}^{-}(\xi) \mapsto E_q(q-q^{-1})\xi^{-1}, &&\varepsilon^q : L_{22}^{-}(\xi) \mapsto q^{-H} - q^H\xi^{-1} ,
\end{aligned}$$

that is

$$\varepsilon^q : L_{11}^{(+0)} \mapsto q^{-H}, \qquad \varepsilon^q : L_{11}^{(1)} \mapsto -q^H, \qquad \varepsilon^q : L_{12}^{(1)} \mapsto -F_q(q-q^{-1}) ,$$

$$\varepsilon^q : L_{21}^{(+0)} \mapsto -E_q(q-q^{-1}), \qquad \varepsilon^q : L_{22}^{(+0)} \mapsto q^H, \qquad \varepsilon^q : L_{22}^{(1)} \mapsto -q^{-H} ,$$

$$\varepsilon^q : L_{11}^{(-0)} \mapsto q^H, \qquad \varepsilon^q : L_{11}^{(-1)} \mapsto -q^{-H}, \qquad \varepsilon^q : L_{12}^{(-0)} \mapsto F_q(q-q^{-1}) ,$$

$$\varepsilon^q : L_{21}^{(-1)} \mapsto E_q(q-q^{-1}), \qquad \varepsilon^q : L_{22}^{(-0)} \mapsto q^{-H}, \qquad \varepsilon^q : L_{22}^{(+1)} \mapsto -q^H ,$$

and $\varepsilon^q : L_{ij}^{(s)} \mapsto 0$ for all other generators $L_{ij}^{(s)}$.

Both the automorphisms ρ_ξ^q and ε^q restricted to the subalgebra $U_q(\mathfrak{sl}_2)$ are the identity maps.

For any $U_q(\mathfrak{sl}_2)$-module V denote by $V(\xi)$ the $U_q'(\widetilde{\mathfrak{gl}_2})$-module which is obtained from the module V via the homomorphism $\varepsilon^q \circ \rho_\xi^q$. The module $V(\xi)$ is called the *evaluation module*.

Let V_1, V_2 be Verma modules for $U_q(\mathfrak{sl}_2)$ with generating vectors v_1, v_2, respectively. For generic complex numbers ξ, ζ the $U_q'(\widetilde{\mathfrak{gl}_2})$-modules $V_1(\xi) \otimes V_2(\zeta)$ and $V_2(\zeta) \otimes V_1(\xi)$ are isomorphic and the trigonometric R-matrix $P_{V_1 V_2} R^q_{V_1 V_2}(\xi/\zeta)$ intertwines them [T], [CP]. The vectors $v_1 \otimes v_2$ and $v_2 \otimes v_1$ are respective generating vectors of the $U_q'(\widetilde{\mathfrak{gl}_2})$-modules $V_1(\xi) \otimes V_2(\zeta)$ and $V_2(\zeta) \otimes V_1(\xi)$. The trigonometric R-matrix $R^q_{V_1 V_2}(\xi/\zeta)$ can be defined as the unique element of $\mathrm{End}(V_1 \otimes V_2)$ with property (3.14) and such that

$$P_{V_1 V_2} R^q_{V_1 V_2}(\xi/\zeta) : V_1(\xi) \otimes V_2(\zeta) \to V_2(\zeta) \otimes V_1(\xi) \tag{3.19}$$

is an isomorphism of the $U_q'(\widetilde{\mathfrak{gl}_2})$-modules.

4. Tensor coordinates and module structures on the hypergeometric spaces

In this section we identify the Gauss–Manin connection and the qKZ connection. In addition we also describe a structure of a $Y(\mathfrak{gl}_2)$-module on the rational hypergeometric space and a structure of a $U_q'(\widetilde{\mathfrak{gl}_2})$-module on the trigonometric hypergeometric space, respectively.

The rational hypergeometric module

The $\mathscr{F}[l]$ be the rational hypergeometric space defined for the projection $\mathbb{C}^{l+n} \to \mathbb{C}^n$. In particular, $\mathscr{F}[0] = \mathbb{C}$ and, in our previous notations, we have $\mathscr{F}[\ell] = \mathscr{F}$. Consider the direct sum

$$\mathfrak{F} = \bigoplus_{l \geqq 0} \mathscr{F}[l]$$

which will be called the *rational hypergeometric Fock space*.

Let $T_{ij}(u)$, $i,j = 1,2$, be the generating series for the Yangian $Y(\mathfrak{gl}_2)$ introduced in Sect. 3. Set

$$\widetilde{T}_{ij}(u) = T_{ij}(u) \prod_{m=1}^{n} \frac{u}{u - z_m - \Lambda_m}, \quad i,j = 1,2\,,$$

where the rational function in the right hand side is understood as its Laurent series expansion at $u = \infty$. It is clear that the coefficients of the series $\widetilde{T}_{ij}(u)$ generate $Y(\mathfrak{gl}_2)$. Introduce an action of the coefficients of the series $\widetilde{T}_{ij}(u)$ in the space $\mathfrak{F}$. Namely, for any $f \in \mathscr{F}[l]$ set:

$$\begin{aligned}(\widetilde{T}_{11}(u)f)(t_1,\dots,t_l) = {} & f(t_1,\dots,t_l) \prod_{m=1}^{n} \frac{u - z_m + \Lambda_m}{u - z_m - \Lambda_m} \\ & \times \prod_{a=1}^{l} \frac{u - t_a - 1}{u - t_a} + \prod_{a=1}^{l} \frac{u - t_a - 1}{u - t_a} \sum_{a=1}^{l} \\ & \times \left[\frac{f(t_1,\dots,t_{l-1},u)}{u - t_l - 1} \prod_{m=1}^{n} \frac{t_l - z_m + \Lambda_m}{t_l - z_m - \Lambda_m} \right]_{(a,l)},\end{aligned} \tag{4.1}$$

$$(\widetilde{T}_{22}(u)f)(t_1,\dots,t_l) = f(t_1,\dots,t_l)\prod_{a=1}^{l}\frac{u-t_a+1}{u-t_a}$$

$$-\prod_{a=1}^{l}\frac{u-t_a+1}{u-t_a}\sum_{a=1}^{l}\left[\frac{f(u,t_2,\dots,t_l)}{u-t_1+1}\right]_{(1,a)},$$

$$(\widetilde{T}_{12}(u)f)(t_1,\dots,t_{l+1})$$

$$=\sum_{a=1}^{l+1}\left[\frac{f(t_2,\dots,t_{l+1})}{u-t_1}\left(\prod_{m=1}^{n}\frac{t_1-z_m+\Lambda_m}{t_1-z_m-\Lambda_m}\prod_{b=2}^{l+1}\frac{u-t_b+1}{u-t_b}\,\frac{t_1-t_b-1}{t_1-t_b+1}\right.\right.$$

$$\left.\left.-\prod_{m=1}^{n}\frac{u-z_m+\Lambda_m}{u-z_m-\Lambda_m}\prod_{b=2}^{l+1}\frac{u-t_b-1}{u-t_b}\right)\right]_{(1,a)} - \prod_{a=1}^{l+1}\frac{u-t_a+1}{u-t_a}$$

$$\times\sum_{\substack{a,b=1\\ a\neq b}}^{l+1}\left[\frac{f(u,t_2,\dots,t_l)}{(u-t_1+1)(u-t_{l+1}+1)}\prod_{m=1}^{n}\frac{t_{l+1}-z_m+\Lambda_m}{t_{l+1}-z_m-\Lambda_m}\right]_{\sigma^{ab}},$$

$$(\widetilde{T}_{21}(u)f)(t_1,\dots,t_{l-1}) = f(t_1,\dots,t_{l-1},u)\prod_{a=1}^{l-1}\frac{u-t_a-1}{u-t_a},\qquad l>0,$$

and $\widetilde{T}_{21}(u)f=0$ for $f\in\mathscr{F}[0]$. Here $(1,a),(a,l)$ are transpositions and $\sigma^{ab}\in\mathbb{S}^{l+1}$ is the following permutation

$$\sigma^{ab}: i\mapsto i\quad\text{for } i=2,\dots,l,\qquad \sigma^{ab}:1\mapsto a,\qquad \sigma^{ab}: l+1\mapsto b\,.$$

The right hand sides of formulae (4.1) are rational functions in u, and the precise meaning of each of the formulae is that the left hand side equals the Laurent series expansion of the respective right hand side at $u=\infty$.

(4.2) Lemma. *Formulae* (4.1) *define a* $Y(\mathfrak{gl}_2)$*-module structure in the rational hypergeometric Fock space* $\mathfrak{F}$.

The proof is given by direct verification.

Let $\mathfrak{F}(z)=\bigoplus_{l\geqq 0}\mathscr{F}[l](z)$ be the rational hypergeometric Fock space of a fiber. The $Y(\mathfrak{gl}_2)$-module structure in $\mathfrak{F}$ clearly induces a $Y(\mathfrak{gl}_2)$-module structure in $\mathfrak{F}(z)$. This module will be called the *rational hypergeometric module.*

For the action of the generators of the subalgebra $U(\mathfrak{sl}_2)$ (4.1) simplify and for $f\in\mathscr{F}[l]$ look as follows:

(4.3) $$(Hf)(t_1,\dots,t_l)=\left(\sum_{m=1}^{n}\Lambda_m-l\right)f(t_1,\dots,t_l)\,,$$

$$(Ff)(t_1,\dots,t_{l+1})$$

$$=\sum_{a=1}^{l+1}\left[f(t_2,\dots,t_{l+1})\left(\prod_{m=1}^{n}\frac{t_1-z_m+\Lambda_m}{t_1-z_m-\Lambda_m}\prod_{b=2}^{l+1}\frac{t_1-t_b-1}{t_1-t_b+1}-1\right)\right]_{(1,a)},$$

$$(Ef)(t_1,\dots,t_{l-1})=(t_l f(t_1,\dots,t_l))|_{t_l=\infty},\qquad l>0\,,$$

and $Ef=0$ for $f\in\mathscr{F}[0]$. Here $(1,a)\in\mathbb{S}^{l+1}$ are transpositions.

Remark. It is worth to mention that for any function $w_{\mathfrak{l}}$ we have

$$Fw_{\mathfrak{l}} = \sum_{m=1}^{n} (\mathfrak{l}_m + 1)(2\Lambda_m - \mathfrak{l}_m) w_{\mathfrak{l}+e(m)} \,.$$

cf. (2.21). Hence $R[l](z) \subset F(\mathscr{F}[l-1](z))$, where $R[l](z)$ is the coboundary subspace.

Remark. Let $\kappa = 1$. Then for any function $f \in \mathscr{F}[\ell - 1]$ we have

$$(Ff)(t_1, \ldots, t_\ell) = \sum_{a=1}^{\ell} D_a [f(t_2, \ldots, t_\ell)]_{(1,a)} \,. \tag{4.4}$$

Tensor coordinates on the rational hypergeometric spaces of fibers

Let $V_1, \ldots, V_n$ be $\mathfrak{sl}_2$ Verma modules with highest weights $\Lambda_1, \ldots, \Lambda_n$ and generating vectors $v_1, \ldots v_n$, respectively. Consider the weight subspace $(V_1 \otimes \cdots \otimes V_n)_\ell$ with a basis given by monomials $F^{\mathfrak{l}_1} v_1 \otimes \cdots \otimes F^{\mathfrak{l}_n} v_n$. The dual space $(V_1 \otimes \cdots \otimes V_n)^*_\ell$ has the dual basis denoted by $(F^{\mathfrak{l}_1} v_1 \otimes \cdots \otimes F^{\mathfrak{l}_n} v_n)^*$.

For any $z \in \mathbb{C}^n$ and for any $\tau \in \mathbb{S}^n$ denote by $B_\tau(z)$ the following homomorphism:

$$B_\tau(z) : (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \to \mathscr{F}(z) \,,$$

$$B_\tau(z) : (F^{\mathfrak{l}_{\tau_1}} v_{\tau_1} \otimes \cdots \otimes F^{\mathfrak{l}_{\tau_n}} v_{\tau_n})^* \to w^\tau_{\mathfrak{l}}(t,z), \quad \mathfrak{l} \in \mathscr{Z}^n_\ell \,,$$

where $\mathscr{F}(z)$ is the rational hypergeometric space of a fiber (cf. (2.19), (2.23)). The homomorphisms $B_\tau(z)$ are called the *tensor coordinates* on the rational hypergeometric space of a fiber. The composition maps

$$B_{\tau,\tau'}(z) : (V_{\tau'_1} \otimes \cdots \otimes V_{\tau'_n})^*_\ell \to (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \,,$$

$$B_{\tau,\tau'}(z) = B^{-1}_\tau(z) \circ B_{\tau'}(z) \,,$$

are called the *transition functions*, cf. [V3].

(4.5) Lemma. *Let $z_l + \Lambda_l - z_m + \Lambda_m \notin \{0, \ldots, \ell - 1\}$ for any $l \neq m$, $l,m = 1, \ldots, n$. Then for any permutation τ the linear map $B_\tau(z) : (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \to \mathscr{F}(z)$ is nondegenerate.*

The statement follows from Lemma 2.20.

Consider the evaluation module $V_{\tau_1}(z_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}(z_{\tau_n})$ over $Y(\mathfrak{gl}_2)$ coinciding with $V_{\tau_1} \otimes \cdots \otimes V_{\tau_n}$ as an $\mathfrak{sl}_2$-module.

(4.6) Lemma. *For any $\varphi \in (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell$ we have*

$$\langle \varphi, T_{12}(t_1) \cdots T_{12}(t_\ell) v_1 \otimes \cdots \otimes v_n \rangle$$
$$= (B_\tau(z)\varphi)(t_1, \ldots, t_\ell) \prod_{a=1}^{\ell} \prod_{m=1}^{n} (t_a - z_m - \Lambda_m)/t_a \prod_{1 \leq a < b \leq \ell} \frac{t_a - t_b + 1}{t_a - t_b} \,.$$

It is easy to see that the right hand side above is a polynomial in $t_1^{-1}, \dots, t_\ell^{-1}$, so the formula makes sense without additional prescriptions.

(4.7) Theorem. *For any permutation $\tau \in \mathbb{S}^n$ the map*

$$B_\tau(z) : (V_{\tau_1}(z_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}(z_{\tau_n}))^* \to \mathfrak{F}(z)$$

is an intertwiner of $Y(\mathrm{gl}_2)$-modules.

(4.8) Corollary. *Let $z_l + \Lambda_l - z_m + \Lambda_m \notin \mathbb{Z}$ for any $l \neq m$, $l, m = 1, \dots, n$. Then for any permutation $\tau \in \mathbb{S}^n$ the map $B_\tau(z) : (V_{\tau_1}(z_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}(z_{\tau_n}))^* \to \mathfrak{F}(z)$ is an isomorphism of $Y(\mathrm{gl}_2)$-modules.*

The statement follows from Theorem 4.7 and Lemma 4.5.

(4.9) Theorem. [V3] *For any $\tau \in \mathbb{S}^n$ and any transposition $(m, m+1)$, $m = 1, \dots, n-1$, the transition function*

$$B_{\tau, \tau \cdot (m,m+1)}(z) : ((V_{\tau_1} \otimes \cdots \otimes V_{\tau_{m+1}} \otimes V_{\tau_m} \otimes \cdots \otimes V_{\tau_n})^*)_\ell \to (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell$$

equals the operator $(P_{V_{\tau_m} V_{\tau_{m+1}}} R_{V_{\tau_m} V_{\tau_{m+1}}}(z_{\tau_m} - z_{\tau_{m+1}}))^*$ acting in the m-th and $(m+1)$-th factors.

The theorem follows from Lemma 4.6 and (3.8).

Each $B_\tau(z)$ induces a linear map $(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \to \mathscr{H}(z)$ which also will be denoted by $B_\tau(z)$.

(4.10) Theorem. *Let $\kappa \neq 1$. Let $p < 0$. Let (2.12)–(2.14) hold. Then for any $\tau \in \mathbb{S}^n$ the map $B_\tau(z) : (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \to \mathscr{H}(z)$ is an isomorphism.*

This statement follows from Theorem 2.15 and Lemma 4.5.

It is easy to see that for any $\tau \in \mathbb{S}^n$ the image of $F(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_{\ell-1}$ under the map $B_\tau(z)$ coincides with the coboundary subspace $\mathscr{R}(z) \subset \mathscr{F}(z)$.

(4.11) Theorem. *Let $\kappa = 1$. Let $p < 0$. Let (2.12)–(2.14) hold. If $2\sum_{m=1}^n \Lambda_m - s \notin p\mathbb{Z}_{<0}$ for all $s = \ell - 1, \dots, 2\ell - 2$, then for any $\tau \in \mathbb{S}^n$ the map $B_\tau(z)$ induces an isomorphism*

$$(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell / F(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_{\ell-1} \to \mathscr{H}(z)\,.$$

The statement follows from Theorem 2.17 and Lemmas 2.20, 2.21.

Taking into account (3.1) we get an isomorphism

$$((V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell^{\mathrm{sing}})^* \to \mathscr{H}(z) .$$

(4.12) Theorem. [V3], [TV1] *For any $m = 1,\dots,n$, the following diagram is commutative*:

$$\begin{array}{ccc}
(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell^* & \xrightarrow{K_m^*(z_1,\dots,z_n)} & (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell^* \\
{\scriptstyle B_\tau(z_1,\dots,z_m+p,\dots,z_n)}\downarrow & & \downarrow{\scriptstyle B_\tau(z_1,\dots,z_n)} \\
\mathscr{H}(z_1,\dots,z_m+p,\dots,z_n) & \xrightarrow[A_m(z_1,\dots,z_n)]{} & \mathscr{H}(z_1,\dots,z_n)
\end{array}$$

Here $A_m(z)$ are the operators of the Gauss–Manin connection, $K_m^*(z)$ are the operators dual to $K_m(z)$, and $K_m(z)$ are the operators of the qKZ connection in $(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell$ defined by (3.10).

(4.13) Corollary. *The construction above identifies the qKZ connection and the Gauss–Manin connection restricted to the hypergeometric subbundle.*

The trigonometric hypergeometric module

Let $\mathscr{F}_q[l]$ be the trigonometric hypergeometric space defined for the projection $\mathbb{C}^{l+n} \to \mathbb{C}^n$. In particular, $\mathscr{F}_q[0] = \mathbb{C}$ and, in our previous notations, we have $\mathscr{F}_q[\ell] = \mathscr{F}_q$. Consider the direct sum

$$\mathfrak{F}_q = \bigoplus_{l \geqq 0} \mathscr{F}_q[l]$$

which will be called the *trigonometric hypergeometric Fock space.*

Let $q = \exp(\pi i/p)$. Let $L_{jk}^{\pm}(u)$, $j,k = 1,2$, be the generating series for the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ introduced in Sect. 3. Set

$$\widetilde{L}_{jk}^{\pm}(\xi) = L_{jk}^{\pm}(\xi) \prod_{m=1}^{n} \frac{\pm i \exp(\pm\pi i(z_m - u)/p)}{2\sin(\pi(u - z_m - \Lambda_m)/p)}, \quad j,k = 1,2 ,$$

where $\xi = \exp(2\pi i u/p)$. The products in the right hand side are rational functions in ξ. The precise meaning is that $\widetilde{L}_{jk}^{-}(\xi)$ equals the Laurent series expansion of the corresponding right hand side at $\xi = \infty$, and $\widetilde{L}_{jk}^{+}(\xi)$ equals the Taylor series expansion of the corresponding right hand side at $\xi = 0$. It is clear that the coefficients of the series $\widetilde{L}_{jk}^{\pm}(\xi)$ generate $U_q'(\widetilde{\mathfrak{gl}_2})$. Introduce an action of the coefficients of the series $\widetilde{L}_{jk}^{\pm}(\xi)$ in the space $\mathfrak{F}_q$. Namely, for any

$f \in \mathscr{F}_q[l]$ set:

$$
\begin{aligned}
(4.14)\quad (\widetilde{L}_{11}^{\pm}(\xi)f)(t_1,\dots,t_l) &= f(t_1,\dots,t_l)\prod_{m=1}^{n}\frac{\sin(\pi(u-z_m+\Lambda_m)/p)}{\sin(\pi(u-z_m-\Lambda_m)/p)} \\
&\quad\times\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a-1)/p)}{\sin(\pi(u-t_a)/p)} \\
&\quad+\sin(\pi/p)\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a-1)/p)}{\sin(\pi(u-t_a)/p)} \\
&\quad\times\sum_{a=1}^{l}\Bigg\llbracket f(t_1,\dots,t_{l-1},u)\frac{\exp(\pi i(u-t_l)/p)}{\sin(\pi(u-t_l-1)/p)} \\
&\quad\times\prod_{m=1}^{n}\frac{\sin(\pi(t_l-z_m+\Lambda_m)/p)}{\sin(\pi(t_l-z_m+\Lambda_m)/p)}\Bigg\rrbracket_{(a,l)}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{L}_{22}^{\pm}(\xi)f)(t_1,\dots,t_l) &= f(t_1,\dots,t_l)\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)} \\
&\quad-\sin(\pi/p)\prod_{a=1}^{l}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)} \\
&\quad\times\sum_{a=1}^{l}\Bigg\llbracket f(u,t_2,\dots,t_l)\frac{\exp(\pi i(u-t_l)/p)}{\sin(\pi(u-t_1+1)/p)}\Bigg\rrbracket_{(1,a)}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{L}_{12}^{\pm}(\xi)f)(t_1,\dots,t_{l+1}) &= \sin(\pi/p)\sum_{a=1}^{l+1}\Bigg\llbracket f(t_2,\dots,t_{l+1})\frac{\exp(\pi i(u-t_1)/p)}{\sin(\pi(u-t_1)/p)} \\
&\quad\times\Bigg(\prod_{m=1}^{n}\frac{\sin(\pi(t_1-z_m+\Lambda_m)/p)}{\sin(\pi(t_1-z_m-\Lambda_m)/p)} \\
&\quad\times\prod_{b=2}^{l+1}\frac{\sin(\pi(u-t_b+1)/p)}{\sin(\pi(u-t_b)/p)}\frac{\sin(\pi(t_1-t_b-1)/p)}{\sin(\pi(t_1-t_b+1)/p)} \\
&\quad-\prod_{m=1}^{n}\frac{\sin(\pi(u-z_m+\Lambda_m)/p)}{\sin(\pi(u-z_m-\Lambda_m)/p)} \\
&\quad\times\prod_{b=2}^{l+1}\frac{\sin(\pi(u-t_b-1)/p)}{\sin(\pi(u-t_b)/p)}\Bigg)\Bigg\rrbracket_{(1,a)} \\
&\quad-\sin^2(\pi/p)\prod_{a=1}^{l+1}\frac{\sin(\pi(u-t_a+1)/p)}{\sin(\pi(u-t_a)/p)} \\
&\quad\times\sum_{\substack{a,b=1\\ a\neq b}}^{l+1}\Bigg\llbracket f(u,t_2,\dots,t_l)
\end{aligned}
$$

$$\times \frac{\exp(\pi i(2u - t_1 - t_{l+1})/p)}{\sin(\pi(u - t_1 + 1)/p)\sin(\pi(u - t_{l+1} + 1)/p)}$$

$$\times \prod_{m=1}^{n} \frac{\sin(\pi(t_{l+1} - z_m + \Lambda_m)/p)}{\sin(\pi(t_{l+1} - z_m - \Lambda_m)/p)}\Bigg]\Bigg]_{\sigma^{ab}},$$

$$(\widetilde{L}_{21}^{\pm}(\xi)f)(t_1,\dots,t_{l-1}) = f(t_1,\dots,t_{l-1},u)\prod_{a=1}^{l-1} \frac{\sin(\pi(u - t_a - 1)/p)}{\sin(\pi(u - t_a)/p)}, \quad l > 0,$$

and $\widetilde{L}_{21}^{\pm}(u)f = 0$ for $f \in \mathscr{F}_q[0]$. Here $\xi = \exp(2\pi iu/p)$, $(1,a),(a,l)$ are transpositions and $\sigma^{ab} \in \mathbb{S}^{l+1}$ is the following permutation

$$\sigma^{ab} : i \mapsto i \quad \text{for } i = 2,\dots,l, \qquad \sigma^{ab} : 1 \mapsto a, \qquad \sigma^{ab} : l+1 \mapsto b.$$

The right hand sides of (4.14) are rational functions in ξ, and the precise meaning of each of the formulae is that $\widetilde{L}_{jk}^{-}(\xi)$ equals the Laurent series expansion of the corresponding right hand side at $\xi = \infty$, and $\widetilde{L}_{jk}^{+}(\xi)$ equals the Taylor series expansion of the corresponding right hand side at $\xi = 0$.

(4.15) Lemma. *Formulae* (4.14) *define an* $U_q'(\widetilde{\mathfrak{gl}_2})$*-module structure in the trigonometric hypergeometric Fock space* $\mathfrak{F}_q$.

The proof is given by direct verification.

Let $\mathfrak{F}_q(z) = \bigoplus_{l\geqq 0} \mathscr{F}_q[l](z)$ be the trigonometric hypergeometric Fock space of a fiber. The $U_q'(\widetilde{\mathfrak{gl}_2})$-module structure in $\mathfrak{F}_q$ clearly induces an $U_q'(\widetilde{\mathfrak{gl}_2})$-module structure in $\mathfrak{F}_q(z)$. This module will be called the *trigonometric hypergeometric module.*

For the action of the generators of the subalgebra $U_q(\mathfrak{sl}_2)$ (4.14) simplify and for $f \in \mathscr{F}_q[l]$ look as follows:

(4.16)

$$(q^{\pm H}f)(t_1,\dots,t_l) = q^{\pm(\sum_{m=1}^{n}\Lambda_m - l)} f(t_1,\dots,t_l),$$

$$(F_q f)(t_1,\dots,t_{l+1}) = \exp\left(-\pi i\left(l + \sum_{m=1}^{n}\Lambda_m\right)\Big/p\right)\sum_{a=1}^{l+1}\Bigg[f(t_2,\dots,t_{l+1})$$

$$\times\left(\exp(2\pi il/p)\prod_{m=1}^{n}\frac{\sin(\pi(t_1 - z_m + \Lambda_m)/p)}{\sin(\pi(t_1 - z_m - \Lambda_m)/p)}\right.$$

$$\times\prod_{b=2}^{l+1}\frac{\sin(\pi(t_1 - t_b - 1)/p)}{\sin(\pi(t_1 - t_b + 1)/p)}$$

$$\left.\left. - \exp\left(2\pi i\sum_{m=1}^{n}\Lambda_m\Big/p\right)\right)\right]_{(1,a)},$$

$$(E_q f)(t_1,\dots,t_{l-1}) = -(2i\sin(\pi/p))^{-1}\exp\left(\pi i\left(l-1+\sum_{m=1}^{n}\Lambda_m\right)\Big/p\right)$$

$$\times f(t_1,\dots,t_l)|_{\exp(2\pi i t_\ell/p)=0}, \quad l>0\,,$$

and $E_q f = 0$ for $f\in\mathscr{F}_q[0]$. Here $(1,a)\in\mathbb{S}^{l+1}$ are transpositions.

Tensor coordinates on the trigonometric hypergeometric spaces of fibers

Let $q=\exp(\pi i/p)$. Let $V_1^q,\dots,V_n^q$ be $U_q(\mathfrak{sl}_2)$ Verma modules with highest weights $q^{\Lambda_1},\dots,q^{\Lambda_n}$ and generating vectors $v_1^q,\dots,v_n^q$, respectively. Consider a weight subspace $(V_1^q\otimes\dots\otimes V_n^q)_\ell$ with a basis given by monomials $F_q^{\mathfrak{l}_1}v_1^q\otimes\dots\otimes F_q^{\mathfrak{l}_n}v_n^q$. For any $z\in\mathbb{B}$ and for any $\tau\in\mathbb{S}^n$ denote by $C_\tau(z)$ the following homomorphism:

$$C_\tau(z):(V_{\tau_1}^q\otimes\dots\otimes V_{\tau_n}^q)_\ell\to\mathscr{F}_q(z)\,,$$

$$C_\tau(z):F_q^{\mathfrak{l}_{\tau_1}}v_{\tau_1}^q\otimes\dots\otimes F_q^{\mathfrak{l}_{\tau_n}}v_{\tau_n}^q\to c_{\mathfrak{l}}W_{\mathfrak{l}}^\tau(t,z),\quad \mathfrak{l}\in\mathscr{Z}_\ell^n\,,$$

where

$$c_{\mathfrak{l}}=\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}\frac{\sin(\pi(s+1)/p)\sin(\pi(2\Lambda_m-s)/p)}{\sin(\pi/p)}\,,$$

where $\mathscr{F}_q(z)$ is the trigonometric hypergeometric space of the fiber (cf. (2.19), (2.23)). The homomorphisms $C_\tau(z)$ are called the *tensor coordinates* on the trigonometric hypergeometric space of a fiber. The composition maps

$$C_{\tau,\tau'}(z):(V_{\tau'_1}^q\otimes\dots\otimes V_{\tau'_n}^q)_\ell\to(V_{\tau_1}^q\otimes\dots\otimes V_{\tau_n}^q)_\ell,\qquad C_{\tau,\tau'}(z)=C_\tau^{-1}(z)\circ C_{\tau'}(z)\,,$$

are called the *transition functions*, cf. [V3].

(4.17) Lemma. *Let $z_l+\Lambda_l-z_m+\Lambda_m-s\notin p\mathbb{Z}$ for any $s=0,\dots,\ell-1$, and for any $l,m=1,\dots,n$. Then for any permutation τ the linear map $C_\tau(z)$: $(V_{\tau_1}^q\otimes\dots\otimes V_{\tau_n}^q)_\ell\to\mathscr{F}_q(z)$ is nondegenerate.*

The statement follows from Lemma 2.28.

Let $\zeta_m=\exp(2\pi i z_m/p)$. Consider the evaluation module $V_{\tau_1}^q(\zeta_{\tau_1})\otimes\dots\otimes V_{\tau_n}^q(\zeta_{\tau_n})$ over $U_q'(\widetilde{\mathfrak{gl}_2})$ coinciding with $V_1^q\otimes\dots\otimes V_n^q$ as an $U_q(\mathfrak{sl}_2)$-module.

(4.18) Lemma. *For any $v\in(V_{\tau_1}^q\otimes\dots\otimes V_{\tau_n}^q)_\ell$ we have*

$$L_{21}^{\pm}(\xi_1)\cdots L_{21}^{\pm}(\xi_\ell)v=(C_\tau(z)v)(t_1,\dots,t_\ell)$$

$$\times\prod_{a=1}^{\ell}\prod_{m=1}^{n}\frac{2\sin(\pi(u-z_m-\Lambda_m)/p)}{\pm i\exp(\pm\pi i(z_m-u)/p)}$$

$$\times\prod_{1\le a<b\le\ell}\frac{\sin(\pi(t_a-t_b+1)/p)}{\sin(\pi(t_a-t_b)/p)}v_{\tau_1}^q\otimes\dots\otimes v_{\tau_n}^q\,,$$

where $\xi_a=\exp(2\pi i t_a/p)$, $a=1,\dots,\ell$.

It is easy to see that the right hand side above is a polynomial in $\xi_1, \dots, \xi_\ell$ for the case of the upper signs, and is a polynomial in $\xi_1^{-1}, \dots, \xi_\ell^{-1}$ for the case of the lower signs, so the formula makes sense without additional prescriptions.

(4.19) Theorem. *For any permutation* $\tau \in \mathbb{S}^n$ *the map*

$$C_\tau(z) : V_{\tau_1}^q(\zeta_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}^q(\zeta_{\tau_n}) \to \mathfrak{F}_q(z)$$

is an intertwiner of $U_q'(\widetilde{\mathfrak{gl}_2})$*-modules.*

(4.20) Corollary. *Let* $z_l + \Lambda_l - z_m + \Lambda_m - s \notin p\mathbb{Z}$ *for any* $s \in \mathbb{Z}_{\geq 0}$, *and for any* $l, m = 1, \dots, n$. *Then for any permutation* $\tau \in \mathbb{S}^n$ *the map* $C_\tau(z) : V_{\tau_1}^q(\zeta_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}^q(\zeta_{\tau_n}) \to \mathfrak{F}_q(z)$ *is an isomorphism of* $U_q'(\widetilde{\mathfrak{gl}_2})$*-modules.*

The statement follows from Theorem 4.19 and Lemma 4.17.

(4.21) Corollary. *For any* $\tau \in \mathbb{S}^n$ *the homomorphism* $C_\tau(z)$ *maps* $(V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q)_\ell^{\mathrm{sing}}$ *into the singular trigonometric hypergeometric space* $\mathscr{F}_q^{\mathrm{sing}}(z)$ *of a fiber. The map*

$$C_\tau(z) : (V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q)_\ell^{\mathrm{sing}} \to \mathscr{F}_q^{\mathrm{sing}}(z)$$

is an isomorphism provided that $z_l + \Lambda_l - z_m + \Lambda_m - s \notin p\mathbb{Z}$ *for any* $s = 0, \dots, \ell - 1$, *and for any* $l, m = 1, \dots, n$.

The statement follows the last formula in (4.16).

(4.22) Theorem. [V3] *For any* $\tau \in \mathbb{S}^n$ *and any transposition* $(m, m+1)$, $m = 1, \dots, n-1$, *the transition function*

$$C_{\tau, \tau \cdot (m,m+1)}(z) : V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_{m+1}}^q \otimes V_{\tau_m}^q \otimes \cdots \otimes V_{\tau_n}^q \to V_{\tau_1}^q \otimes \cdots \otimes V_{\tau_n}^q$$

equals the operator $P_{V_{\tau_{m+1}}^q V_{\tau_m}^q} R_{V_{\tau_{m+1}}^q V_{\tau_m}^q}^q(\exp(2\pi i(z_{\tau_{m+1}} - z_{\tau_m})/p))$ *acting in the* m*-th and* $(m+1)$*-th factors.*

The theorem follows from Lemma 4.18 and (3.19).

Tensor products of the hypergeometric modules

Let $\mathscr{F}[z_1, \dots, z_m; \Lambda_1, \dots, \Lambda_m; l]$ and $\mathscr{F}_q[z_1, \dots, z_m; \Lambda_1, \dots, \Lambda_m; l]$ be respectively the rational and the trigonometric hypergeometric spaces defined for the projection $\mathbb{C}^{l+m} \to \mathbb{C}^m$. In particular, in our previous notations we have

$$\mathscr{F} = \mathscr{F}[z_1, \dots, z_n; \Lambda_1, \dots, \Lambda_n; \ell] \quad \text{and} \quad \mathscr{F}_q = \mathscr{F}_q[z_1, \dots, z_n; \Lambda_1, \dots, \Lambda_n; \ell] .$$

There are maps

$$\chi : \mathscr{F}[z_1, \dots, z_k; \Lambda_1, \dots, \Lambda_k; j] \otimes \mathscr{F}[z_{k+1}, \dots, z_{k+m}; \Lambda_{k+1}, \dots, \Lambda_{k+m}; l]$$
$$\to \mathscr{F}[z_1, \dots, z_{k+m}; \Lambda_1, \dots, \Lambda_{k+m}; j + l] ,$$

and

$$\chi_q : \mathscr{F}_q[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k;j] \otimes \mathscr{F}_q[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots\Lambda_{k+m};l]$$
$$\to \mathscr{F}_q[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m};j+l]\,,$$

which are respectively defined by $\chi : f\otimes g \mapsto f\star g$ and $\chi_q : f\otimes g\mapsto f*g$, where

$$(f\star g)(t_1,\dots,t_{j+l})$$
$$= \frac{1}{j!l!}\sum_{\sigma\in\mathbb{S}^{j+l}}\left[f(t_1,\dots,t_j)g(t_{j+1},\dots,t_{j+l})\prod_{i=1}^{k}\prod_{a=1}^{l}\frac{t_{a+j}-z_i+\Lambda_i}{t_{a+j}-z_i-\Lambda_i}\right]_\sigma .$$

and

$$(f*g)(t_1,\dots,t_{j+l}) = \frac{1}{j!l!}\sum_{\sigma\in\mathbb{S}^{j+l}}\left[\!\!\left[f(t_1,\dots,t_j)g(t_{j+1},\dots,t_{j+l})\right.\right.$$
$$\left.\left.\times\prod_{i=1}^{k}\prod_{a=1}^{l}\frac{\sin(\pi(t_{a+j}-z_i+\Lambda_i)/p)}{\sin(\pi(t_{a+j}-z_i-\Lambda_i)/p)}\right]\!\!\right]_\sigma .$$

We have the next lemmas.

(4.23) Lemma. *Assume that* $(z_i-\Lambda_i-z_{k+j}-\Lambda_{k+j}+s)\neq 0$ *for any* $i=1,\dots,k,\ j=1,\dots,m,\ s=0,\dots,l-1$. *Then the map*

$$\chi : \bigoplus_{i+j=l}\mathscr{F}[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k;i]((z_1,\dots,z_k))$$
$$\otimes\mathscr{F}[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m};j]((z_{k+1},\dots,z_{k+m}))$$
$$\to\mathscr{F}[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m};l]((z_1,\dots,z_{k+m}))$$

defined by linearity is an isomorphism of the rational hypergeometric spaces of fibers.

(4.24) Lemma. *Assume that* $(z_i-\Lambda_i-z_{k+j}-\Lambda_{k+j}+s)\notin p\mathbb{Z}$ *for any* $i=1,\dots,k, j=1,\dots,m,\ s=0,\dots,l-1$. *Then the map*

$$\chi_q : \bigoplus_{i+j=l}\mathscr{F}_q[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k;i]((z_1,\dots,z_k))$$
$$\otimes\mathscr{F}_q[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m};j]((z_{k+1},\dots,z_{k+m}))$$
$$\to\mathscr{F}_q[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m};l]((z_1,\dots,z_{k+m}))$$

defined by linearity is an isomorphism of the trigonometric hypergeometric spaces of fibers.

Let

$$\mathfrak{F}[z_1,\dots,z_m;\Lambda_1,\dots,\Lambda_m]=\bigoplus_{l=0}^{\infty}\mathscr{F}[z_1,\dots,z_m;\Lambda_1,\dots,\Lambda_m;l]$$

and

$$\mathfrak{F}_q[z_1,\dots,z_m;\Lambda_1,\dots,\Lambda_m]=\bigoplus_{l=0}^{\infty}\mathscr{F}_q[z_1,\dots,z_m;\Lambda_1,\dots,\Lambda_m;l]$$

be the rational and the trigonometric hypergeometric Fock spaces, respectively. Extend the maps χ,χ_q to the respective maps

$$\begin{aligned}\chi:\ &\mathfrak{F}[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k]((z_1,\dots,z_k))\\ &\otimes\mathfrak{F}[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m}]((z_{k+1},\dots,z_{k+m}))\\ &\quad\to\mathfrak{F}[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m}]((z_1,\dots,z_{k+m}))\,,\\ \chi_q:\ &\mathfrak{F}_q[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k]((z_1,\dots,z_k))\\ &\otimes\mathfrak{F}_q[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m}]((z_{k+1},\dots,z_{k+m}))\\ &\quad\to\mathfrak{F}_q[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m}]((z_1,\dots,z_{k+m}))\,.\end{aligned}$$

(4.25) Theorem. *The map*

$$\begin{aligned}\chi\circ P:\ &\mathfrak{F}[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m}]((z_{k+1},\dots,z_{k+m}))\\ &\otimes\mathfrak{F}[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k]((z_1,\dots z_k))\\ &\quad\to\mathfrak{F}[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m}]((z_1,\dots,z_{k+m}))\end{aligned}$$

is an intertwiner of $Y(\mathfrak{gl}_2)$-modules. Here P is the permutation map. The map $\chi\circ P$ is an isomorphism provided that $(z_i-\Lambda_i-z_{k+j}-\Lambda_{k+j})\notin\mathbb{Z}_{\leqq 0}$ for any $i=1,\dots,k$, $j=1,\dots,m$.

(4.26) Theorem. *The map*

$$\begin{aligned}\chi_q:\ &\mathfrak{F}_q[z_1,\dots,z_k;\Lambda_1,\dots,\Lambda_k]((z_1,\dots,z_k))\\ &\otimes\mathfrak{F}_q[z_{k+1},\dots,z_{k+m};\Lambda_{k+1},\dots,\Lambda_{k+m}]((z_{k+1},\dots,z_{k+m}))\\ &\quad\to\mathfrak{F}_q[z_1,\dots,z_{k+m};\Lambda_1,\dots,\Lambda_{k+m}]((z_1,\dots,z_{k+m}))\,.\end{aligned}$$

is an intertwiner of $U_q'(\widetilde{\mathfrak{gl}_2})$-modules. The map χ_q is an isomorphism provided that $(z_i-\Lambda_i-z_{k+j}-\Lambda_{k+j}+s)\notin p\mathbb{Z}$ for any $i=1,\dots,k$, $j=1,\dots,m$, $s\in\mathbb{Z}_{\geqq 0}$.

It is clear that for any functions f,g,h we have $(f\bigstar g)\bigstar h=f\bigstar(g\bigstar h)$ and for any functions f,g,h we have $(f*g)*h=f*(g*h)$. Lemmas 4.23, 4.24 and Theorems 4.25, 4.26 can be extended naturally to an arbitrary number of factors.

5. The hypergeometric pairing

In this section we define the main object of this paper, the hypergeometric pairing. We define a pairing between the rational and the trigonometric hypergeometric spaces of a fiber. For any functions $w \in \mathscr{F}(z)$ and $W \in \mathscr{F}_q(z)$ we define the *hypergeometric integral* by

$$I(W, w) = \int_{\widetilde{\mathbb{I}}^\ell} \Phi(t) w(t) W(t)\, d^\ell t \tag{5.1}$$

where Φ is the phase function (2.5) and $\widetilde{\mathbb{I}}^\ell$ is a suitable deformation of the imaginary subspace

$$\mathbb{I}^\ell = \{t \in \mathbb{C}^\ell \,|\, \operatorname{Re} t_1 = 0, \ldots, \operatorname{Re} t_\ell = 0\}\,.$$

We always assume that the step p is real and negative. The case of arbitrary step can be treated by analytic continuation.

The phase function Φ has a factor $\exp(\mu \sum_{a=1}^{\ell} t_a/p)$ where the parameter μ is connected with the parameter κ in the definition of the connection coefficients by $\kappa = e^\mu$. We choose the parameter μ so that it satisfies

$$0 \leqq \operatorname{Im} \mu < 2\pi\,. \tag{5.2}$$

We define the hypergeometric integral as follows. First we assume that the real parts of the weights $\Lambda_1, \ldots, \Lambda_n$ are large negative and set

$$I(W, w) = \int_{\mathbb{I}^\ell} \Phi(t) w(t) W(t)\, d^\ell t\,. \tag{5.3}$$

(5.4) Lemma. *Let* $0 < \operatorname{Im} \mu < 2\pi$. *Let the real parts of the weights* $\Lambda_1, \ldots, \Lambda_n$ *be large negative. Then the hypergeometric integral* $I(W, w)$ *is well defined for any functions* $w \in \mathscr{F}(z)$ *and* $W \in \mathscr{F}_q(z)$.

Proof. It follows from (2.5), (2.6) and (2.24) that the integrand of the hypergeometric integral decays exponentially as t goes to infinity. □

Let $\mathscr{F}_q^{\mathrm{sing}}(z) \subset \mathscr{F}_q(z)$ be the singular trigonometric hypergeometric space.

(5.5) Lemma. *Let* $\operatorname{Im} \mu = 0$. *Let the real parts of the weights* $\Lambda_1, \ldots, \Lambda_n$ *be large negative. Then the hypergeometric integral* $I(W, w)$ *is well defined for any functions* $w \in \mathscr{F}(z)$ *and* $W \in \mathscr{F}_q^{\mathrm{sing}}(z)$.

The proof is similar to the previous lemma.

The hypergeometric integral for generic $\Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ and arbitrary negative p is defined by analytic continuation with respect to $\Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ and p. This analytic continuation makes sense since the integrand is analytic in $\Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ and p, cf. (2.5), (2.19), (2.26). More precisely, first we define the hypergeometric integral for basic functions w_{l}, W_{m} and then extend the definition by linearity to arbitrary functions $w \in \mathscr{F}(z)$, $W \in \mathscr{F}_q(z)$. The result of analytic continuation can be represented as an integral of the integrand over a suitably deformed imaginary subspace. Namely, the poles of the integrand of the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ are located at the hyperplanes

$$t_a = z_m \pm (\Lambda_m + sp), \qquad t_a = t_b \pm (1 - sp), \tag{5.6}$$

$1 \leqq b < a \leqq \ell$, $m = 1, \ldots, n$, $s \in \mathbb{Z}_{\geqq 0}$. We deform $\Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ and p in such a way that the topology of the complement in $\mathbb{C}^\ell$ to the union of hyperplanes (5.6) does not change. We deform accordingly the imaginary subspace $\mathbb{I}^\ell$ so that it does not intersect the hyperplanes (5.6) at every moment of the deformation. The deformed imaginary subspace is denoted by $\widetilde{\mathbb{I}}^\ell$ and called the *deformed imaginary subspace*. Then the analytic continuation of the integral (5.3) is given by (5.1).

(5.7) Theorem. *Let* $0 < \operatorname{Im} \mu < 2\pi$. *Then for any* $\mathfrak{l}, \mathfrak{m} \in \mathscr{Z}_\ell^n$ *the hypergeometric integral* $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ *can be analytically continued as a holomorphic univalued function of complex variables* $p, \Lambda_1, \ldots, \Lambda_n, z_1, \ldots, z_n$ *to the region*:

$$p < 0, \quad \{1, \ldots, \ell\} \not\subset p\mathbb{Z},$$

$$2\Lambda_m - s \notin p\mathbb{Z}, \quad m = 1, \ldots, n, \; s = 1 - \ell, \ldots, \ell - 1,$$

$$z_l \pm \Lambda_l - z_m \pm \Lambda_m - s \notin p\mathbb{Z}, \quad l, m = 1, \ldots, n, \; l \neq m,$$

for an arbitrary combination of signs (*cf.* (2.14)).

(5.8) Theorem. *Let* $\operatorname{Im} \mu = 0$. *Then for any* $\mathfrak{l} \in \mathscr{Z}_\ell^{n-1}$, $\mathfrak{m} \in \mathscr{Z}_\ell^n$ *the hypergeometric integral* $I(\mathring{W}_{\mathfrak{l}}, w_{\mathfrak{m}})$ *can be analytically continued as a holomorphic univalued function of complex variables* $p, \Lambda_1, \ldots, \Lambda_n$, $z_1, \ldots, z_n$ *to the region*:

$$p < 0, \quad \{1, \ldots, \ell\} \not\subset p\mathbb{Z},$$

$$2\Lambda_m - s \notin p\mathbb{Z}, \quad m = 1, \ldots, n, \; s = 1 - \ell, \ldots, \ell - 1,$$

$$z_l \pm \Lambda_l - z_m \pm \Lambda_m - s \notin p\mathbb{Z}, \quad l, m = 1, \ldots, n, \; l \neq m,$$

for an arbitrary combination of signs (*cf.* (2.14)).

The theorems are proved in Sect. 9.

Let $\mathscr{R}(z) \subset \mathscr{F}(z)$ be the coboundary subspace.

(5.9) Lemma. *Let $\mu = 0$. Let $p < 0$. Let (2.12)–(2.14) hold. Then the hypergeometric integral $I(W, w)$ equals zero for any $w \in \mathscr{R}(z)$ and $W \in \mathscr{F}_q^{\text{sing}}(z)$.*

The lemma is proved in Sect. 9.

The hypergeometric integral defines a *hypergeometric pairing*

$$I : \mathscr{F}_q(z) \otimes \mathscr{F}(z) \to \mathbb{C} \tag{5.10}$$

for $0 < \operatorname{Im} \mu < 2\pi$, and

$$I^\circ : \mathscr{F}_q^{\text{sing}}(z) \otimes \mathscr{F}(z)/\mathscr{R}(z) \to \mathbb{C} \tag{5.11}$$

for $\mu = 0$. According to (2.16) and (2.22) this can be written as

$$I : \mathscr{F}_q(z) \otimes \mathscr{H}(z) \to \mathbb{C} \tag{5.12}$$

and

$$I^\circ : \mathscr{F}_q^{\text{sing}}(z) \otimes \mathscr{H}(z) \to \mathbb{C}\,, \tag{5.13}$$

respectively.

(5.14) Theorem. *Let $0 < \operatorname{Im} \mu < 2\pi$. Let $p < 0$. Let (2.12)–(2.14) hold. Then the hypergeometric pairing $I : \mathscr{F}_q(z) \otimes \mathscr{F}(z) \to \mathbb{C}$ is nondegenerate. Moreover*

$$\begin{aligned}
\det[I(W_{\mathrm{l}}, w_{\mathrm{m}})]_{\mathrm{l},\mathrm{m} \in \mathscr{Z}_\ell^n} &= (2i)^{\ell\binom{n+\ell-1}{n-1}} \ell!^{\binom{n+\ell-1}{n-1}} \\
&\quad \times (e^\mu - 1)^{-2\sum_{m=1}^n \Lambda_m/p \cdot \binom{n+\ell-1}{n} + 2n/p \cdot \binom{n+\ell-1}{n+1}} \\
&\quad \times \exp\left(\mu \sum_{m=1}^n z_m/p \cdot \binom{n+\ell-1}{n}\right) \\
&\quad \times \exp\left((\mu + \pi i)\left(\sum_{m=1}^n \Lambda_m/p \cdot \binom{n+\ell-1}{n}\right.\right. \\
&\quad \left.\left. - n/p \cdot \binom{n+\ell-1}{n+1}\right)\right) \\
&\quad \times \prod_{s=0}^{\ell-1} \left[\Gamma(-(s+1)/p)^n \Gamma(-1/p)^{-n} \prod_{m=1}^n \Gamma((2\Lambda_m - s)/p)\right. \\
&\quad \left. \times \prod_{1 \leqq l < m \leqq n} \frac{\Gamma((z_l + \Lambda_l - z_m + \Lambda_m - s)/p)}{\Gamma((z_l - \Lambda_l - z_m - \Lambda_m + s)/p)}\right]^{\binom{n+\ell-s-2}{n-1}}.
\end{aligned}$$

Here $0 \leqq \arg(e^\mu - 1) < 2\pi$.

(5.15) Theorem. *Let $\mu = 0$. Let $p < 0$. Let (2.12)–(2.14) hold. If $2\sum_{m=1}^n \Lambda_m - s \notin p\mathbb{Z}_{<0}$ for all $s = \ell - 1, \ldots, 2\ell - 2$, then the hypergeometric pairing*

$I^\circ : \mathscr{F}_q^{\mathrm{sing}}(z) \otimes \mathscr{F}(z)/\mathscr{R}(z) \to \mathbb{C}$ *is nondegenerate. Moreover*

$$\det[I(\mathring{W}_{\mathfrak{l}}, w_{\mathfrak{m}})]_{\mathfrak{l},\mathfrak{m}\in\mathscr{Z}_\ell^{n-1}} = (2i)^{\ell\binom{n+\ell-2}{n-2}} \ell!^{\binom{n+\ell-2}{n-2}}$$
$$\times \prod_{s=0}^{\ell-1} \Big[\Gamma(-(s+1)/p)^{n-1}\Gamma(-1/p)^{1-n}$$
$$\times \Gamma\left(1 + 2\sum_{m=1}^{n} \Lambda_m/p + (s+2-2\ell)/p\right)^{-1} \Gamma(1 + (2\Lambda_n - s)/p)$$
$$\times \prod_{m=1}^{n-1} \Gamma((2\Lambda_m - s)/p) \prod_{1\leqq l<m\leqq n} \frac{\Gamma((z_l + \Lambda_l - z_m + \Lambda_m - s)/p)}{\Gamma((z_l - \Lambda_l - z_m - \Lambda_m + s)/p)}\Big]^{\binom{n+\ell-s-3}{n-2}} .$$

Here we identify $\mathfrak{m} \in \mathscr{Z}_\ell^{n-1}$ *with* $(\mathfrak{m}, 0) \in \mathscr{Z}_\ell^n$.

Theorems 5.14 and 5.15 are proved in Sect. 9.

Example. Theorem 5.14 for $n = 1$, $\ell = 1$ and Theorem 5.15 for $n = 2$, $\ell = 1$ give

$$(5.16) \qquad \int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(a-s)u^{2s}\, ds = 2\pi i \Gamma(2a)(u + u^{-1})^{-2a} ,$$
$$\int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\, ds$$
$$= 2\pi i \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} ,$$

which are formulae for the Barnes integrals [WW].

For arbitrary ℓ, Theorem 5.14 for $n = 1$ and Theorem 5.15 for $n = 2$ give the following Mellin–Barnes integrals, which are generalizations of the famous Selberg integral:

(5.17)
$$\int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{k=1}^{\ell} \left(u^{2s_k}\Gamma(a+s_k)\Gamma(a-s_k) \prod_{j=1}^{k-1} \frac{\Gamma(s_j - s_k + x)\Gamma(s_k - s_j + x)}{\Gamma(s_j - s_k)\Gamma(s_k - s_j)} \right) d^\ell s$$
$$= (2\pi i)^\ell (u + u^{-1})^{-\ell(2a+(\ell-1)x)} \prod_{k=1}^{\ell} \frac{\Gamma(1+kx)}{\Gamma(1+x)} \Gamma(2a + (k-1)x) ,$$
$$\int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{k=1}^{\ell} \Big(\Gamma(a+s_k)\Gamma(b+s_k)\Gamma(c-s_k)\Gamma(d-s_k)$$
$$\times \prod_{j=1}^{k-1} \frac{\Gamma(s_j - s_k + x)\Gamma(s_k - s_j + x)}{\Gamma(s_j - s_k)\Gamma(s_k - s_j)} \Big) d^\ell s$$

$$= (2\pi i)^\ell \prod_{k=1}^{\ell} \left(\frac{\Gamma(1+kx)}{\Gamma(1+x)} \right.$$

$$\times \frac{\Gamma(a+c+(k-1)x)\Gamma(a+d+(k-1)x)}{\Gamma(a+b+c+d+(2\ell-k-1)x)}$$

$$\left. \times \frac{\Gamma(b+c+(k-1)x)\Gamma(b+d+(k-1)x)}{\Gamma(a+b+c+d+(2\ell-k-1)x)} \right),$$

where $\mathrm{Re}\, a, b, c, d, u, x > 0$.

Remark. After this paper was written we found out that the second formula in (5.17) had appeared in [G]. In Sect. 9 we give a proof of the first formula in (5.17) and use the formula to prove Theorems 5.14, 5.15.

Remark. We also obtain determinant formulae similar to (5.14) and (5.15) for the hypergeometric pairing in the trigonometric case [TV3]. Under the same specialization as above, those formulae give multidimensional generalizations of the Askey–Roy formula [GR, (4.11.2)], and, on the other hand, can be viewed as a generalization of the famous q-Selberg integral, cf. [Ka, AK].

Remark. It is plausible that the assumptions on p, $\Lambda_1, \dots, \Lambda_n$, $z_1, \dots, z_n$ of Theorems 5.14 and 5.15 as well as of Theorems 2.15, 2.17, 4.10, 4.11, 5.9, 6.4, 6.6, 6.7 could be replaced by the following weaker assumptions: the step p is such that $\{2, \dots, \ell\} \not\subset p\mathbb{Z}_{>0}$, the weights $\Lambda_1, \dots, \Lambda_n$ are such that

$$2\Lambda_m - s \notin p\mathbb{Z}, \quad m = 1, \dots, n, \; s = 0, \dots, \ell - 1 \, .$$

and the coordinates $z_1, \dots, z_n$ obey the condition

$$z_l + \Lambda_l - z_m + \Lambda_m - s \notin p\mathbb{Z}, \quad l, m = 1, \dots, n, \; l \neq m \, ,$$

for any $s = 0, \dots, \ell - 1$, so that $z \in \mathbb{B}$.

Let W be any element of the trigonometric hypergeometric space $\mathscr{F}_q$. The restriction of the function W to a fiber defines an element $W|_z \in \mathscr{F}_q(z)$ of the trigonometric hypergeometric space of the fiber. The hypergeometric pairing allows us to consider the element $W|_z \in \mathscr{F}_q(z)$ as an element $s_W(z)$ of the space $\mathscr{H}^*(z)$ dual to the hypergeometric cohomology group $\mathscr{H}(z)$. This construction defines a section of the bundle over $\mathbb{C}^n$ with fiber $\mathscr{H}^*(z)$.

There is a simple but important statement.

(5.18) Theorem. *Let either* $0 < \mathrm{Im}\, \mu < 2\pi$ *and* $W \in \mathscr{F}_q$ *or* $\mu = 0$ *and* $W \in \mathscr{F}_q^{\mathrm{sing}}$. *Let* $p < 0$. *Let* $\Lambda_1, \dots, \Lambda_n$ *obey* (2.13). *Then the section* s_W *is a periodic section with respect to the Gauss–Manin connection.*

The theorem is proved in Sect. 9.

The section s_W and the tensor coordinates B_τ induce a section

$$\Psi_W : z \mapsto B_\tau^* \cdot W|_z \in (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell \tag{5.19}$$

of the trivial bundle with fiber $(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell$. Theorem (5.18) and Theorem (4.12) imply

(5.20) Corollary. *The section Ψ_W is a solution to the qKZ equation.*

The tensor coordinates $B_\tau(z)$, $C_{\tau'}(z)$ induce a *hypergeometric pairing*

$$I_{\tau,\tau'}(z) : (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})_\ell \otimes (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell \to \mathbb{C} \tag{5.21}$$

if $0 < \operatorname{Im} \mu < 2\pi$ and

(5.22)

$$I^\circ_{\tau,\tau'}(z) : (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})^{\rm sing}_\ell \otimes (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_\ell / F(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^*_{\ell-1} \to \mathbb{C}$$

if $\mu = 0$, which also can be considered as maps

$$\check{I}_{\tau,\tau'}(z) : (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})_\ell \to (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell \tag{5.23}$$

and

$$\check{I}^\circ_{\tau,\tau'}(z) : (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})^{\rm sing}_\ell \to (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^{\rm sing}_\ell . \tag{5.24}$$

If $v \in (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})_\ell$, then the hypergeometric pairing defines a section

$$\Psi_v : z \mapsto \check{I}_{\tau,\tau'}(z) \cdot v \in (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell$$

and, if $v \in (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})^{\rm sing}_\ell$, then the hypergeometric pairing defines a section

$$\Psi_v : z \mapsto \check{I}^\circ_{\tau,\tau'}(z) \cdot v \in (V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^{\rm sing}_\ell$$

(5.25) Corollary. *Let $0 < \operatorname{Im} \mu < 2\pi$ and, therefore, $\kappa \neq 1$. Then for any $v \in (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})_\ell$ the section Ψ_v is a solution to the qKZ equation with values in $(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})_\ell$. Under conditions of Theorem* 5.14 *all solutions are constructed in this way.*

Therefore, for $\kappa \neq 1$ we constructed the hypergeometric maps

$$\check{I}_{\tau,\tau'}(z) : V^q_{\tau'_1}(\zeta_{\tau'_1}) \otimes \cdots \otimes V^q_{\tau'_n}(\zeta_{\tau'_n}) \to V_{\tau_1}(z_{\tau_1}) \otimes \cdots \otimes V_{\tau_n}(z_{\tau_n})$$

from quantum loop algebra modules to Yangian modules. Here $\zeta_m = \exp(2\pi i z_m / p)$, $m = 1, \ldots, n$. The maps have the following properties:

$$\check{I}_{\tau \cdot (m,m+1), \tau'}(z) = P_{V_{\tau_m} V_{\tau_{m+1}}} R_{V_{\tau_m} V_{\tau_{m+1}}}(z_{\tau_m} - z_{\tau_{m+1}})) \check{I}_{\tau,\tau'}(z) ,$$

$$\check{I}_{\tau,\tau' \cdot (m,m+1)}(z) = \check{I}_{\tau,\tau'}(z) P_{V^q_{\tau'_{m+1}} V^q_{\tau'_m}} R^q_{V^q_{\tau'_{m+1}} V^q_{\tau'_m}}(\exp(2\pi i (z_{\tau_{m+1}} - z_{\tau_m})/p)) ,$$

and as functions of z they satisfy the *qKZ* equations:

$$\check{I}_{\tau,\tau'}(z_1, \ldots, z_{\tau_m} + p, \ldots, z_n) = K_m(z_{\tau_1}, \ldots, z_{\tau_n}) \check{I}_{\tau,\tau'}(z_1, \ldots, z_n) .$$

(5.26) Corollary. *Let $\mu = 0$ and, therefore, $\kappa = 1$. Then for any $v \in (V^q_{\tau'_1} \otimes \cdots \otimes V^q_{\tau'_n})^{\mathrm{sing}}_{\ell}$ the section Ψ_v is a solution to the qKZ equation with values in $(V_{\tau_1} \otimes \cdots \otimes V_{\tau_n})^{\mathrm{sing}}_{\ell}$. Under conditions of Theorem 5.15 all solutions are constructed in this way.*

Remark. Let $V_1 \otimes \cdots \otimes V_n$ be a tensor product of $\mathfrak{sl}_2$ Verma modules, $\widetilde{V}_1 \otimes \cdots \otimes \widetilde{V}_n$ the tensor product of the corresponding irreducible $\mathfrak{sl}_2$-modules, and $S: V_1 \otimes \cdots \otimes V_n \to \widetilde{V}_1 \otimes \cdots \otimes \widetilde{V}_n$ the natural projection. If $\Psi(z)$ is a solution to the *qKZ* equation with values in $V_1 \otimes \cdots \otimes V_n$ then $S\Psi(z)$ is a solution to the *qKZ* equation with values in $\widetilde{V}_1 \otimes \cdots \otimes \widetilde{V}_n$.

This observation shows that the previous constructions give all solution to the *qKZ* equation with values in $(\widetilde{V}_1 \otimes \cdots \otimes \widetilde{V}_n)_\ell$ if $\ell \leqq \dim \widetilde{V}_m$ for all $m = 1,\dots,n$. Moreover, the space of solutions to the *qKZ* equation with values in $(\widetilde{V}_1 \otimes \cdots \otimes \widetilde{V}_n)_\ell$ in this case is identified with the space $(\widetilde{V}^q_1 \otimes \cdots \otimes \widetilde{V}^q_n)_\ell \otimes \mathbb{F}$ where $\widetilde{V}^q_1 \otimes \cdots \otimes \widetilde{V}^q_n$ is the tensor product of the corresponding irreducible $U_q(\mathfrak{sl}_2)$-modules, and $\mathbb{F}$ is the space of functions in $z_1,\dots,z_n$ which are p-periodic with respect to each of the variables.

In a separate paper we shall explain how the construction of this paper gives all solutions to the *qKZ* equation with values in a tensor product of irreducible $\mathfrak{sl}_2$-modules.

6. Asymptotic solutions to the *qKZ* equation

One of the most important characteristics of a differential equation is the monodromy group of its solutions. For the differential *KZ* equation with values in a tensor product of representations of a simple Lie algebra its monodromy group is described in terms of the corresponding quantum group. This fact establishes a remarkable connection between representation theories of simple Lie algebras and their quantum groups, see [K, D2, KL, SV, V2, V4].

The substitution of the monodromy group for difference equations is the set of transition functions between asymptotic solutions. For a difference equation one defines suitable asymptotic zones in the domain of the definition of the equation and then an asymptotic solution for every zone. Thus, for every pair of asymptotic zones one gets a transition function between the corresponding asymptotic solutions.

In this section we describe asymptotic zones, asymptotic solutions, and their transition functions for the *qKZ* equation with values in a tensor product of $\mathfrak{sl}_2$-modules when the parameter κ is different from 1. A remarkable fact is that the transition functions are described in terms of the trigonometric R-matrices acting in the tensor product of the corresponding $U_q(\mathfrak{sl}_2)$-modules. This fact establishes a correspondence between representation theories of Yangians and quantum loop algebras, since the *qKZ* equation is defined in terms of the rational R-matrix action in the tensor product $\mathfrak{sl}_2$-modules (and, therefore, in terms of the Yangian action), and the trigonometric R-matrix action in the tensor product of $U_q(\mathfrak{sl}_2)$-modules is defined in terms of the action of the quantum loop algebra.

Let V be a vector space of dimension N for some N. Consider an integrable system of difference equations for a V-valued function $\Psi(z_1,\dots,z_n)$:

$$(6.1)\qquad \Psi(z_1,\dots,z_m+p,\dots,z_n)=A_m(z_1,\dots,z_n)\Psi(z_1,\dots,z_n),\qquad m=1,\dots,n\,.$$

Let $\mathbb{A}$ be a domain in $\mathbb{C}^n$. Say that a basis $\Psi_1,\dots,\Psi_N$ of solutions to system (6.1) form an *asymptotic solution* in the domain if

$$(6.2)\qquad \Psi_j(z)=\exp\left(\sum_{m=1}^{n}a_{mj}z_m/p\right)\prod_{1\leqq m<l\leqq n}(z_l-z_m)^{b_{jlm}}(v_j+o(1))\,,$$

where a_{mj} and b_{jlm} are suitable numbers, $v_1,\dots,v_N$ are vectors which form a basis in V, and $o(1)$ tends to 0 as z tends to infinity in $\mathbb{A}$. We will call the domain an *asymptotic zone*.

Consider the qKZ equation with parameter $\kappa\neq 1$ and values in $(V_1\otimes\cdots\otimes V_n)_\ell$. We describe its asymptotic solutions in suitable asymptotic zones.

For every permutation $\tau\in\mathbb{S}^n$ we consider an asymptotic zone in $\mathbb{C}^n$ given by

$$(6.3)\qquad \mathbb{A}_\tau=\{z\in\mathbb{C}^n\mid \operatorname{Re}z_{\tau_1}\ll\cdots\ll\operatorname{Re}z_{\tau_n}\}\,.$$

Say that $z\to\infty$ in $\mathbb{A}_\tau$ if $\operatorname{Re}(z_{\tau_m}-z_{\tau_{m+1}})\to-\infty$ for all $m=1,\dots,n-1$.

Recall that for every permutation $\tau\in\mathbf{S}^n$ we constructed a basis $W_{\mathfrak{l}}^\tau$, $\mathfrak{l}\in\mathscr{Z}_\ell^n$, in the trigonometric hypergeometric space. This basis defines a basis $\Psi_{W_{\mathfrak{l}}^\tau}$, $\mathfrak{l}\in\mathscr{Z}_\ell^n$, of solutions to the qKZ equation, cf. (5.19).

(6.4) Theorem. *Let $p<0$. Assume that the weights $\Lambda_1,\dots,\Lambda_n$ obey condition* (2.13). *Let $0<\operatorname{Im}\mu<2\pi$ and, therefore, $\kappa\neq 1$. Then for any permutation $\tau\in\mathbb{S}^n$ the basis $\Psi_{W_{\mathfrak{l}}^\tau}$, $\mathfrak{l}\in\mathscr{Z}_\ell^n$, is an asymptotic solution in the asymptotic zone $\mathbb{A}_\tau$. Namely,*

$$\Psi_{W_{\mathfrak{l}}^\tau}(z)=\Theta_{\mathfrak{l}}\exp\left(\mu\sum_{m=1}^{n}\mathfrak{l}_mz_m\Big/p\right)\prod_{1\leqq l<m\leqq n}((z_{\tau_l}-z_{\tau_m})/p)^{2(\mathfrak{l}_{\tau_l}\Lambda_{\tau_m}+\mathfrak{l}_{\tau_m}\Lambda_{\tau_l}-\mathfrak{l}_{\tau_l}\mathfrak{l}_{\tau_m})/p}$$
$$\times(F^{\mathfrak{l}_1}v_1\otimes\cdots\otimes F^{\mathfrak{l}_n}v_n+o(1))$$

as $z\to\infty$ in $\mathbb{A}_\tau$ so that at any moment assumption (2.14) *holds. Here the branches of the multivalued functions are fixed by the agreement that $|\arg((z_{\tau_l}-z_{\tau_m})/p)|<\pi$ for $l<m$ and $\Theta_{\mathfrak{l}}$ is a constant independent of the permutation τ and given by*

$$\Theta_{\mathfrak{l}}=(2i)^\ell\ell!\Gamma(-1/p)^{-\ell}\prod_{m=1}^{n}\Bigg[(e^\mu-1)^{(\mathfrak{l}_m(\mathfrak{l}_m-1)-2\mathfrak{l}_m\Lambda_m)/p}$$
$$\times\exp((\mu+\pi i)(\mathfrak{l}_m\Lambda_m-\mathfrak{l}_m(\mathfrak{l}_m-1)/2)/p)$$
$$\times\prod_{s=0}^{\mathfrak{l}_m-1}\Gamma((2\Lambda_m-s)/p)\Gamma(-(s+1)/p)\Bigg]\,,$$

where $0\leqq\arg(e^\mu-1)<2\pi$.

The theorem is proved in Sect. 9.

Remark. The qKZ operators $K_m(z)$ have the following asymptotics in the asymptotic zone $\mathbb{A}_\tau$,

$$K_m(z) = \kappa^{\Lambda_m - H_m}(1 + o(1)), \quad m = 1, \dots, n .$$

The vectors $F^{\mathfrak{l}_1} v_1 \otimes \dots \otimes F^{\mathfrak{l}_n} v_n$ form an eigenbasis of the operator $\kappa^{\Lambda_m - H_m}$ with eigenvalues $\kappa^{\mathfrak{l}_m}$.

Remark. The qKZ equation and the basis of solutions $\Psi_{W_\mathfrak{l}^\tau}$, $\mathfrak{l} \in \mathscr{Z}_\ell^n$, depend meromorphically on parameters μ, $\Lambda_1, \dots, \Lambda_n$. The asymptotics of the basis $\Psi_{W_\mathfrak{l}^\tau}$, $\mathfrak{l} \in \mathscr{Z}_\ell^n$, determine the basis uniquely. Namely, if a basis of solutions meromorphically depends on the parameters μ, $\Lambda_1, \dots, \Lambda_n$ and has asymptotics in $\mathbb{A}_\tau$ described in Theorem 6.4, then such a basis coincides with the basis $\Psi_{W_\mathfrak{l}^\tau}$. In fact, elements of any such a basis are linear combinations of the functions $\Psi_{W_\mathfrak{l}^\tau}$ with coefficients meromorphically depending on μ, $\Lambda_1, \dots, \Lambda_n$ and p-periodic in $z_1, \dots, z_n$. To preserve the asymptotics one can add to an element $\Psi_{W_\mathfrak{l}^\tau}$ any other functions $\Psi_{W_{\mathfrak{l}'}^\tau}$ having smaller asymptotics. If $\mu < 0$, then one can add only the functions $\Psi_{W_{\mathfrak{l}'}^\tau}$ with $\mathfrak{l}'$ lexicographically greater than $\mathfrak{l}$, and if $\mu > 0$, then one can add only the functions $\Psi_{W_{\mathfrak{l}'}^\tau}$ with $\mathfrak{l}'$ lexicographically smaller than $\mathfrak{l}$. Since the coefficients of added terms are meromorphic they have to be zero.

Example. Theorem (6.4) allows us to write a trigonometric R-matrix as an infinite product of rational R-matrices. Namely, consider the qKZ equation with values in the tensor product of two $\mathfrak{sl}_2$ Verma modules $V_1 \otimes V_2$. Then there are two asymptotic zones $\operatorname{Re} z_1 \ll \operatorname{Re} z_2$ and $\operatorname{Re} z_1 \not\ll \operatorname{Re} z_2$. Our result on the transition function from the first asymptotic zone to the second is the following statement.

For any $\mathfrak{sl}_2$ Verma module V let V^q be $U_q(\mathfrak{sl}_2)$ Verma module corresponding to V. Let Λ be the highest weight of module V and let v, v^q be the respective generating vectors of modules V, V^q. Define a map $G : V \to V^q$:

$$G : F^l v \mapsto F_q^l v^q \prod_{s=0}^{l-1} \Gamma(1 + (s - 2\Lambda)/p) \Gamma(1 + (s+1)/p) .$$

Let p, μ be complex numbers such that $p < 0$ and $0 < \operatorname{Im} \mu < 2\pi$. Let $q = e^{\pi i/p}$. Set

$$R_{V_1 V_2}(x; \mu, p) = \exp(\mu x (\mathrm{id} \otimes H)/p) R_{V_1 V_2}(x) \exp(-\mu x (\mathrm{id} \otimes H)/p) .$$

and $J(s, \mu) = (G \otimes G)(-is(e^{\mu/2} - e^{-\mu/2}))^{2H \otimes H/p}$, where $|\arg(-i(e^{\mu/2} - e^{-\mu/2}))| < \pi/2$. Then

(6.5)
$$\lim_{s \to \infty} \left(J(s, \mu) \left(\prod_{r=-s}^{s} R_{V_1 V_2}(x + rp; \mu, p) \right) J(s, \mu)^{-1} \right) = R^q_{V_1^q V_2^q}(\exp(-2\pi i x/p)) .$$

Here the factors of the product are ordered in such a way that r grows from right to left.

Notice that the minus sign in the argument of the R-matrix in the right hand side of (6.5) above reflects the fact that we use the coproducts Δ and Δ^q for the Yangian $Y(\mathfrak{gl}_2)$ and the quantum loop algebra $U_q'(\widetilde{\mathfrak{gl}_2})$ which are in a sense opposite to each other.

The restriction of (6.5) to the weight subspace $(V_1 \otimes V_2)_1$ of weight $\Lambda_1 + \Lambda_2 - 1$ can be transformed to the infinite product formula for 2×2 matrices (cf. [RF]), which looks as follows.

Let a, b, c, d, ϑ be complex numbers, $\operatorname{Re} \vartheta > 0$. Set $\lambda = \sqrt{a^2 - bc}$,

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A(u) = \frac{1}{d+u} \begin{pmatrix} a+u & b \\ c & a-u \end{pmatrix}.$$

and $A(u;\vartheta) = \vartheta^{uh} A(u) \vartheta^{-uh}$. Assume that $-bc \neq s(s+2a)$ for any $s \in \mathbb{Z}$. Then

$$\lim_{s \to \infty} \left(s^{-ah} h^s \left(\prod_{r=-s}^{s} A(u+r;\vartheta) \right) h^s s^{ah} \right) = A^q(u)$$

where the factors of the product are ordered in such a way that r grows from right to left and

$$A^q(u) = \frac{1}{\sin(\pi(d+u))} \times \begin{pmatrix} \sin(\pi(a+u)) & \dfrac{\pi b(\vartheta + \vartheta^{-1})^{2a}}{\Gamma(1+a+\lambda)\Gamma(1+a-\lambda)} \\ \dfrac{\pi c(\vartheta + \vartheta^{-1})^{-2a}}{\Gamma(1+\lambda-a)\Gamma(1-\lambda-a)} & \sin(\pi(a-u)) \end{pmatrix}.$$

Theorem (6.4) admits the following generalization. Fix a nonnegative integer k not greater than n. Let $n_0, \dots, n_k$ be nonnegative integers such that

$$0 = n_0 < n_1 < \cdots < n_k = n .$$

Set

$$\mathscr{F}^i[l] = \mathscr{F}[z_{n_{i-1}+1}, \dots, z_{n_i}; \Lambda_{n_{i-1}+1}, \dots, \Lambda_{n_i}; l]$$

and

$$\mathscr{F}_q^i[l] = \mathscr{F}_q[z_{n_{i-1}+1}, \dots, z_{n_i}; \Lambda_{n_{i-1}+1}, \dots, \Lambda_{n_i}; l]$$

so that

$$\mathscr{F} = \bigoplus_{\substack{\ell_1 + \cdots + \ell_k = \ell \\ \ell_1 \geqq 0, \dots, \ell_k \geqq 0}} \mathscr{F}^1[\ell_1] \otimes \cdots \otimes \mathscr{F}^k[\ell_k]$$

and

$$\mathscr{F}_q = \bigoplus_{\substack{\ell_1 + \cdots + \ell_k = \ell \\ \ell_1 \geqq 0, \dots, \ell_k \geqq 0}} \mathscr{F}_q^1[\ell_1] \otimes \cdots \otimes \mathscr{F}_q^k[\ell_k] .$$

with respect to the tensor products introduced in Sect. 4. We consider an asymptotic zone in $\mathbb{C}^n$ given by

$$\mathbb{A}^n = \left\{ z \in \mathbb{C}^n \,\middle|\, \begin{array}{l} \operatorname{Re} z_{m_1} \ll \cdots \ll \operatorname{Re} z_{m_k}, \text{ for all } m_1, \ldots, m_k \text{ such that} \\ n_{i-1} < m_i \leqq n_i,\ i = 1, \ldots, k \end{array} \right\}.$$

We say that $z \to \infty$ in $\mathbb{A}^n$ if $\operatorname{Re}(z_l - z_m) \to -\infty$ for all l, m such that $n_{i-1} < l \leqq n_i < m \leqq n_{i+1}$, $i = 1, \ldots, k-1$, and $z_l - z_m$ remains bounded for all l, m such that $n_{i-1} < l,\, m \leqq n_i$, $i = 1, \ldots, k$.

For any $W \in \mathscr{F}_q^i[l]$ let $\Psi_W(z_{n_{i+1}+1}, \ldots, z_{n_i})$ be the solution to the *qKZ* equation with values in $(V_{n_{i-1}+1} \otimes \cdots \otimes V_{n_i})_l$ corresponding to W (cf. (5.19)).

(6.6) Theorem. *Let $p < 0$. Assume that the weights $\Lambda_1, \ldots, \Lambda_n$ obey condition* (2.13). *Let $0 < \operatorname{Im} \mu < 2\pi$ and, therefore, $\kappa \neq 1$. Let $\ell_1, \ldots, \ell_k$ be nonnegative integers such that $\ell_1 + \cdots + \ell_k = \ell$. Let $W_i \in \mathscr{F}_q^i[\ell_i]$, $i = 1, \ldots, k$. Let $W = W_1 * \cdots * W_k$. Then the solution $\Psi_W(z_1, \ldots, z_n)$ to the qKZ equation with values in $(V_1 \otimes \cdots \otimes V_n)_\ell$ has the following asymptotics as $z \to \infty$ in $\mathbb{A}^n$ such that at any moment assumption* (2.14) *holds*:

$$\Psi_W(z_1, \ldots, z_n) = \frac{\ell!}{\ell_1! \ldots \ell_n!} \prod_{1 \leqq i < j \leqq k} ((z_{n_i} - z_{n_j})/p)^{2(\ell_i \sum_{m \in \Gamma_j} \Lambda_m + \ell_j \sum_{m \in \Gamma_i} \Lambda_m - \ell_i \ell_j)/p}$$
$$\times (\Psi_{W_1}(z_1, \ldots, z_{n_1}) \otimes \cdots \otimes \Psi_{W_k}(z_{n_{k-1}}, \ldots, z_n) + o(1)).$$

Here $\Gamma_i = \{n_{i-1} + 1, \ldots, n_i\}$ and $|\arg((z_l - z_m)/p)| < \pi$ for $l < m$.

Theorem 6.4 for $\tau = \mathrm{id}$ follows from Theorem 6.6 for $k = n$ so that $n_j = j$, $j = 0, \ldots, n$, and the first formula in (5.17). Theorem 6.4 for a general permutation τ reduces to the same theorem for $\tau = \mathrm{id}$.

Theorem 6.6 follows from the next statement on asymptotics of the hypergeometric pairing.

(6.7) Theorem. *Let $p < 0$. Assume that the weights $\Lambda_1, \ldots, \Lambda_n$ obey condition* (2.13). *Let $0 < \operatorname{Im} \mu < 2\pi$ and, therefore, $\kappa \neq 1$. Let $\ell_1, \ldots, \ell_k$ and $\ell'_1, \ldots, \ell'_k$ be nonnegative integers such that $\ell_1 + \cdots + \ell_k = \ell$ and $\ell'_1 + \cdots + \ell'_k = \ell$. Let $w_i \in \mathscr{F}_q^i[\ell_i]$ and $W_i \in \mathscr{F}_q^i[\ell'_i]$, $i = 1, \ldots, k$. Let $w = w_1 \star \cdots \star w_k$ and $W = W_1 * \cdots * W_k$. Then the hypergeometric integral $I(W, w)$ has the following asymptotics as $z \to \infty$ in $\mathbb{A}^n$ so that at any moment assumption* (2.14) *holds*:

$$I(W, w) = \frac{\ell!}{\ell_1! \ldots \ell_n!} \prod_{1 \leqq i < j \leqq k} ((z_{n_i} - z_{n_j})/p)^{2(\ell_i \sum_{m \in \Gamma_j} \Lambda_m + \ell_j \sum_{m \in \Gamma_i} \Lambda_m - \ell_i \ell_j)/p}$$
$$\times \left(\prod_{i=1}^k \delta_{\ell_i \ell'_i} I(W_i, w_i) + o(1) \right).$$

Here $\Gamma_i = \{n_{i-1} + 1, \ldots, n_i\}$, $|\arg((z_i - z_m)/p)| < \pi$ for $i < m$ and δ_{lm} is the Kronecker symbol.

Remark. In a separate paper we will describe asymptotic zones and asymptotic solutions for the qKZ equation, if the parameter κ of the equation equals 1. In this case the asymptotic zones are essentially the same as the asymptotic zones for the KZ differential equation and the asymptotic solutions are similar, cf. [V4]. If $\kappa = 1$, then the asymptotic zones of the qKZ equation are labelled by permutations in $\mathbb{S}^n$ and suitable planar trees T. For every permutation τ and a tree T we define an asymptotic zone and a basis $\mathfrak{B}_{T,\tau}$ in the space of singular vectors $(V_1 \otimes \dots \otimes V_n)_\ell^{\mathrm{sing}}$, a basis of "iterated singular vectors", see [V4]. For every permutation τ and a tree T we also define a basis $\mathring{W}_{T,\tau}$ in the singular trigonometric hypergeometric space. This basis defines a basis of solutions to the qKZ equation with values in $(V_1 \otimes \dots \otimes V_n)_\ell^{\mathrm{sing}}$. This basis gives an asymptotic solution to the qKZ equation in the asymptotic zone corresponding the permutation and the tree. Moreover, the leading terms of asymptotics in this case are proportional to elements of the basis $\mathfrak{B}_{T,\tau}$ and the coefficients of proportionality are products of powers of linear functions like in (6.2) with no exponential factors unlike in the case of $\kappa \neq 1$.

If $\kappa = 1$ then the qKZ operators $K_m(z)$ have the following asymptotics

$$K_m(z) = 1 + o(1)_m, \quad m = 1, \dots, n,$$

as all differences $z_i - z_j$ tend to infinity. In every asymptotic zone the leading terms of $o(1)_m$ form a system of commuting operators, see (2.2.3) in [V4]. The vectors of the basis $\mathfrak{B}_{T,\tau}$ form an eigenbasis of those commuting operators.

As an illustrating example consider the equation $f(z+p) = (1 + a/z)f(z)$. The equation has a solution $\Gamma((z+a)/p)/\Gamma(z/p)$ with asymptotics $(z/p)^{\alpha/p}$ as z tends to infinity.

7. Quasiclassical asymptotics

Consider a system of difference equations

$$\Psi(z_1, \dots, z_m + p, \dots, z_n) = A^{(m)}(z_1, \dots, z_n; h)\Psi(z_1, \dots, z_n), \quad m = 1, \dots, n,$$

depending on a parameter h and assume that

$$A^{(m)}(z_1/h, \dots, z_n/h; h) = 1 + hB^{(m)}(z_1, \dots, z_n) + o(h) \tag{7.1}$$

as $h \to 0$. Introduce new coordinates $y_m = hz_m$, $m = 1, \dots, n$, and a new function

$$\widetilde{\Psi}(y_1, \dots, y_n) = \Psi(y_1/h, \dots, y_n/h).$$

Then the system of difference equations takes the form

$$\widetilde{\Psi}(y_1, \dots, y_m + hp, \dots, y_n) = (1 + hB^{(m)}(y_1, \dots, y_n) + o(h))\widetilde{\Psi}(y_1, \dots, y_n),$$

$m = 1, \dots, n$, and turns into a system of differential equations

$$p\frac{\partial}{\partial y_m}\widetilde{\Psi}(y_1, \dots, y_n) = B^{(m)}(y_1, \dots, y_n)\widetilde{\Psi}(y_1, \dots, y_n), \quad m = 1, \dots, n,$$

as h tends to zero. We call this system of differential equations the *quasiclassical asymptotics* of the initial system of difference equations.

Consider the *qKZ* equation with values in $(V_1 \otimes \cdots \otimes V_n)_\ell$ and parameter $\kappa = e^{h\eta}$ where η is a given number and h is an additional parameter. Then the *qKZ* equation has property (7.1) and its quasiclassical asymptotics is the *KZ* differential equation

$$p\frac{\partial}{\partial y_m}\widetilde{\Psi}(y_1,\dots,y_n) = \eta H_m \widetilde{\Psi}(y_1,\dots,y_n) + \sum_{\substack{l=1\\ l\neq m}}^{n} \frac{\Omega_{lm}}{y_m - y_l}\widetilde{\Psi}(y_1,\dots,y_n),$$

$m = 1,\dots,n$, where $\Omega_{lm} = 2\Lambda_l\Lambda_m - 2H_lH_m - E_lF_m - F_lE_m$.

In the previous sections we constructed solutions to the *qKZ* equation. The solutions were labelled by elements of a suitable subspace of a tensor product of $U_q(\mathfrak{sl}_2)$-modules. We show that these solutions have quasiclassical asymptotics and turn into the hypergeometric solutions to the *KZ* differential equation which are described in [SV1]. To show this fact we study quasiclassical asymptotics of the hypergeometric pairing.

Let h be a real positive number. Assume that $\operatorname{Im}\eta \geqq 0$. We connect the parameter μ in the phase function (2.5) with the parameter η by an equation $\mu = h\eta$.

The case $\operatorname{Im}\eta < 0$ can be treated similarly. The parameters μ and η have to be connected by an equation $\mu = 2\pi i - h\eta$, if $\operatorname{Im}\eta < 0$.

The asymptotics (2.6) of the phase function of a primitive factor gives the following asymptotics for the phase function (2.5) as $h \to +0$:

$$\Phi(u/h, y/h) = h^{\ell(\ell-1-2\sum_{m=1}^{n}\Lambda_m)/p}\,\widetilde{\Phi}(u,y)(1+o(1)), \tag{7.2}$$

where

$$\widetilde{\Phi}(u_1,\dots,u_\ell,y_1,\dots,y_n) = \exp\left(\eta\sum_{a=1}^{\ell}u_a/p\right)\prod_{m=1}^{n}\prod_{a=1}^{\ell}((u_a-y_m)/p)^{2\Lambda_m/p} \times \prod_{1\leqq a<b\leqq\ell}((u_a-u_b)/p)^{-2/p}. \tag{7.3}$$

Here we fix a branch of the function $(x/p)^\alpha$ by $|\arg(x/p)| < \pi$.

Consider a domain $\mathbb{Y}$ given by

$$\mathbb{Y} = \{y \in \mathbb{C}^n \mid \operatorname{Im} y_1 < \cdots < \operatorname{Im} y_n\}. \tag{7.4}$$

For every $y \in \mathbb{Y}$ and each $m = 1,\dots,n$ we consider an imaginary interval

$$U_m = \{x \in \mathbb{C} \mid \operatorname{Re} u = 0,\ \operatorname{Im} y_{m-1} \leqq \operatorname{Im} x \leqq \operatorname{Im} y_m\}, \quad y_0 = -i\infty,$$

and a chain

$$\overline{U_m} = \sum_{l=1}^{m}\exp\left(4\pi i\sum_{1\leqq k<l}\Lambda_k/p\right)U_l.$$

For any $\mathfrak{l} \in \mathscr{Z}_\ell^n$ we define a chain $\overline{\mathbb{U}}_\mathfrak{l}$ in the imaginary subspace in $\mathbb{C}^\ell$ by

$$\overline{\mathbb{U}}_\mathfrak{l} = \underbrace{\overline{U}_1 \times \cdots \times \overline{U}_1}_{\mathfrak{l}_1} \times \cdots \times \underbrace{\overline{U}_n \times \cdots \times \overline{U}_n}_{\mathfrak{l}_n} .$$

For any $\mathfrak{l} \in \mathscr{Z}_\ell^n$ we also define a rational function $\tilde{w}_\mathfrak{l}(u, y)$ by

$$\tilde{w}_\mathfrak{l}(u_1, \ldots, u_\ell, y_1, \ldots, y_n) = \sum_{\sigma \in \mathbf{S}^\ell} \prod_{m=1}^{n} (\mathfrak{l}_m!)^{-1} \prod_{a \in \Gamma_m} (u_{\sigma_a} - y_m)^{-1} \tag{7.5}$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m\}$, $m = 1, \ldots, n$.

(7.6) Theorem. *Let* $p < 0$. *Let* $\operatorname{Re} \Lambda_m < 0$ *and let* $\operatorname{Re} y_m = 0$ *for all* $m = 1, \ldots, n$. *Let* $\mu = h\eta$, $\operatorname{Im} \eta > 0$. *Then for any* $\mathfrak{l}, \mathfrak{m} \in \mathscr{Z}_\ell^n$ *the hypergeometric integral* $I(W_\mathfrak{l}, w_\mathfrak{m})$ *has the following asymptotics as* $h \to +0$ *and* $y \in \mathbb{Y}$:

$$\begin{aligned} I(W_\mathfrak{l}, w_m) = (-2i)^\ell \ell! h^{\ell(\ell-1-2\sum_{m=1}^n \Lambda_m)/p} \prod_{m=1}^{n} \prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \\ \times \exp\left(\pi i \sum_{m=1}^{n} \Lambda_m (\mathfrak{l}^{m-1} + \mathfrak{l}^m - 2\ell)/p\right) \\ \times \int_{\overline{\mathbb{U}}_\mathfrak{l}} \tilde{\Phi}(u, y) \tilde{w}_\mathfrak{m}(u, y)\, d^\ell u\, (1 + o(1)) . \end{aligned}$$

Remark. Recall that the hypergeometric integral $I(W_\mathfrak{l}, w_\mathfrak{m})$ is defined by (5.3), the functions $W_\mathfrak{l}$ and w_m are given by (2.26) and (2.19), respectively, and we replace in these formulae $z_1, \ldots, z_n$ by $y_1/h, \ldots, y_n/h$.

For any $\mathfrak{l} \in \mathscr{Z}_\ell^{n-1}$ consider a domain $\mathbb{U}_\mathfrak{l}$ in the imaginary subspace in $\mathbb{C}^\ell$ defined by

(7.7)

$$\mathbb{U}_\mathfrak{l} = \left\{ u \in \mathbb{C}^\ell \;\middle|\; \begin{array}{l} \operatorname{Re} u_a = 0,\ a = 1, \ldots, \ell,\ \operatorname{Im} y_m \leqq \operatorname{Im} u_{1+\mathfrak{l}^{m-1}} \\ \leqq \cdots \leqq \operatorname{Im} u_{\mathfrak{l}^m} \leqq \operatorname{Im} y_{m+1},\ m = 1, \ldots, n-1 \end{array} \right\} .$$

(7.8) Theorem. *Let* $p < 0$. *Let* $\operatorname{Re} \Lambda_m < 0$ *and let* $\operatorname{Re} y_m = 0$ *for all* $m = 1, \ldots, n$. *Let* $\mu = h\eta$, $\operatorname{Im} \eta = 0$. *Then for any* $\mathfrak{l} \in \mathscr{Z}_\ell^{n-1}$ *and any* $\mathfrak{m} \in \mathscr{Z}_\ell^n$ *the hypergeometric integral* $I(\mathring{W}_\mathfrak{l}, w_m)$ *has the following asymptotics as* $h \to +0$ *and* $y \in \mathbb{Y}$:

$$\begin{aligned} I(\mathring{W}_\mathfrak{l}, w_m) = (2i)^\ell \ell! h^{\ell(\ell-1-2\sum_{m=1}^n \Lambda_m)/p} \exp\left(2\pi i \sum_{m=1}^{n} \Lambda_m (\ell - \mathfrak{l}^{m-1})/p\right) \\ \times \int_{\mathbb{U}_\mathfrak{l}} \tilde{\Phi}(u, y) \tilde{w}_\mathfrak{m}(u, y)\, d^\ell u\, (1 + o(1)) . \end{aligned}$$

Remark. Recall that the hypergeometric integral $I(\mathring{W}_\mathfrak{l}, w_\mathfrak{m})$ is defined by (5.3), the functions $\mathring{W}_\mathfrak{l}$ and $w_\mathfrak{m}$ are given by (2.27) and (2.19), respectively, and we replace in those formulae $z_1, \ldots, z_n$ by $y_1/h, \ldots, y_n/h$.

Theorems 7.6 and 7.8 essentially follow from (2.26), (2.27) and (7.2).

(7.9) Conjecture. *The claims of Theorems* 7.6 *and* 7.8 *remain valid for any* $\Lambda_1, \dots, \Lambda_n$ *which obey condition* (2.13) *if other assumptions of the theorems hold and the integrals in the right hand sides of* (7.6), (7.8) *are defined by analytic continuation.*

Remark. If $\eta = 0$, that is $\kappa = 1$, then the limiting phase function (7.3) has no exponential factor and is a product of powers of linear functions. In particular, if the numbers Λ_m/p and $2/p$ are all rational, then the limiting integral is an integral of an algebraic function. From this point of view our initial hypergeometric integrals are a deformation of periods of algebraic differential forms, and the subject of our study is a p-deformation of algebraic geometry.

8. The one-dimensional case

In this section we consider in detail the one-dimensional case $\ell = 1$. So we consider the affine projection $\pi : \mathbb{C}^{1+n} \to \mathbb{C}^n$ and a discrete rational $\mathfrak{sl}_2$-type local system on $\mathbb{C}^{1+n}$ and study its de Rham complex. Our main goal of doing this is methodological. Since this case is technically simpler than the general case, the ideas of the proofs become more clear and visual. The case $\ell = 1$ can be viewed as a p-deformation of the following example.

Let $z_1, \dots, z_n$ be pairwise distinct points in $\mathbb{C}$. Let $\widehat{\mathscr{F}}$ be the space of rational functions in t which are regular in $\mathbb{C} \backslash \{z_1, \dots, z_n\}$. Consider the holomorphic de Rham complex $\Omega^\bullet$ on $\mathbb{C} \backslash \{z_1, \dots, z_n\}$ with coefficients in $\widehat{\mathscr{F}}$ associated with the differential $\nabla = \mathrm{d} + \omega \wedge \cdot$, $\omega = \eta\, dt + \sum_{m=1}^n \lambda_m \omega_m$, where $\omega_m = dt/(t - z_m)$.

(8.1) Theorem. *Let* $\eta \neq 0$. *Then for generic* $\lambda_1, \dots, \lambda_n$ *the forms* $\omega_1, \dots, \omega_n$ *form a basis in* $H^1(\Omega^\bullet, \nabla)$.

For $\eta = 0$ the differential of 1 gives a relation in $H^1(\Omega^\bullet, \nabla)$

$$\sum_{m=1}^n \lambda_m \omega_m \sim 0 \,. \tag{8.2}$$

(8.3) Theorem. *Let* $\eta = 0$. *Then for generic* $\lambda_1, \dots, \lambda_n$ *the forms* $\omega_1, \dots, \omega_n$ *span* $H^1(\Omega^\bullet, \nabla)$. *Moreover, relation* (8.2) *is the only independent relation between them.*

Let $z_1, \dots, z_n \in i\mathbb{R}$, $\operatorname{Im} z_1 < \dots < \operatorname{Im} z_n$, $z_0 = -i\infty$, $z_{n+1} = +i\infty$. Consider the following intervals:

$$I_k = \{t \in \mathbb{C} \mid \operatorname{Re} t = 0,\ \operatorname{Im} z_k \leqq \operatorname{Im} t \leqq \operatorname{Im} z_{k+1}\}, \quad k = 0, \dots, n \,.$$

Set

$$I_k(\omega) = \int_{I_k} \exp(\eta t) \prod_{m=1}^n (t - z_m)^{\lambda_m} \omega$$

(the integral must be appropriately regularized). Here we assume that $0 \leqq \arg(t - z_m) < 2\pi$, thus fixing a branch of the integrand. The intervals I_k become linear functionals on the space of differential forms. For a function f we have

$$I_k(\nabla f) = 0, \quad k = 0, \dots, n .$$

This means that the linear functionals on differential forms defined by intervals I_k can be considered as elements of the space $H_1(\Omega^\bullet, \nabla)$ of linear functionals on $H^1(\Omega^\bullet, \nabla)$.

(8.4) Theorem. *Let $\lambda_1, \dots, \lambda_n$ be generic. Then*

a) *For any η, $\operatorname{Im} \eta > 0$, the intervals $I_1, \dots, I_n$ form a basis in $H_1(\Omega^\bullet, \nabla)$.*
b) *For any η, $\operatorname{Im} \eta < 0$, the intervals $I_0, \dots, I_{n-1}$ form a basis in $H_1(\Omega^\bullet, \nabla)$.*

(8.5) Theorem. *Let $\eta = 0$. Let $\lambda_1, \dots, \lambda_n$ be generic. Then the intervals $I_1, \dots, I_{n-1}$ form a basis in $H_1(\Omega^\bullet, \nabla)$.*

Remark. Theorems 8.4 and 8.5 follow from elementary topological considerations. Theorem 8.5 can be also deduced from the following formula [V1]:

$$\det \left[\int_{z_k}^{z_{k+1}} \frac{\lambda_l}{t - z_l} \prod_{m=1}^{n} (t - z_m)^{\lambda_m} \, dt \right]_{l,m=1}^{n-1} = \Gamma\left(1 + \sum_{m=1}^{n} \lambda_m\right)^{-1} \prod_{m=1}^{n} \Gamma(1 + \lambda_m) \prod_{l \neq m} (z_l - z_m)^{\lambda_m} . \tag{8.6}$$

One-dimensional discrete cohomologies

Consider the affine projection $\pi : \mathbb{C}^{1+n} \to \mathbb{C}^n$ and a discrete rational $\mathfrak{sl}_2$-type local system on $\mathbb{C}^{1+n}$. In this case the connection coefficients are equal to

$$\varphi_1(t, z) = \kappa \prod_{m=1}^{n} \frac{t - z_m + \Lambda_m}{t - z_m - \Lambda_m} ,$$
$$\varphi_{m+1}(t, z) = \frac{t - z_m - \Lambda_m - p}{t - z_m + \Lambda_m - p} ,$$

$m = 1, \dots, n$, and the phase function takes the form

$$\Phi(t) = \exp(\mu t/p) \prod_{m=1}^{n} \frac{\Gamma((t - z_m + \Lambda_m)/p)}{\Gamma((t - z_m - \Lambda_m)/p)} . \tag{8.7}$$

The functional space $\widehat{\mathscr{F}}$ is the space of rational functions in t and $z_1, \dots, z_n$ with at most simple poles at the following hyperplanes

$$t = z_m - \Lambda_m + (s+1)p, \qquad t = z_m + \Lambda_m - sp ,$$

$m = 1, \dots, n$, $s \in \mathbb{Z}_{\geqq 0}$. The rational hypergeometric space $\mathscr{F} \subset \widehat{\mathscr{F}}$ is the subspace consisting of functions of the form

$$P(t, z_1, \dots, z_n) \prod_{m=1}^{n} \frac{1}{t - z_m - \Lambda_m}$$

where P is a polynomial of degree less than n in the variable t. The discriminant $\mathbb{B} \subset \mathbb{C}^n$ is the union of the hyperplanes

$$z_l - z_m + \Lambda_l + \Lambda_m = ps, \quad s \in \mathbb{Z},$$

$l, m = 1, \dots, n$, $l \neq m$, in the base space $\mathbb{C}^n$.

To simplify notations in this section we write $w_m(t, z)$ instead of $w_{\mathfrak{e}(m)}(t, z)$. Recall that

$$w_m(t, z_1, \dots, z_n) = \frac{1}{t - z_m - \Lambda_m} \prod_{1 \leqq l < m} \frac{t - z_l + \Lambda_l}{t - z_l - \Lambda_l}, \quad m = 1, \dots, n. \tag{8.8}$$

(8.9) Lemma. (cf. (2.20)) *For any $z \in \mathbb{B}$ the functions $w_1, \dots, w_n$ restricted to the fiber over z form a basis in the rational hypergeometric space $\mathscr{F}(z)$ of the fiber.*

Proof. Consider functions

$$g_m(t, z) = t^{m-1} \prod_{m=1}^{n} \frac{1}{t - z_m - \Lambda_m}, \quad m = 1, \dots, n.$$

Their restrictions to the fiber over z form a basis of the space $\mathscr{F}(z)$. Define a matrix $M(z)$ by

$$w_l(t, z) = \sum_{m=1}^{n} M_{lm}(z) g_m(t, z), \quad l = 1, \dots, n.$$

The lemma follows from the formula

$$\det M = \prod_{1 \leqq l < m \leqq n} (z_l - \Lambda_l - z_m - \Lambda_m). \tag{8.10}$$

The last formula is similar to the Vandermonde determinant formula. □

The coboundary subspace $\mathscr{R}(z)$ is one-dimensional and is spanned by $\sum_{m=1}^{n} \Lambda_m w_m$. Relation (2.21) has the form

$$D(z) \cdot 1 = 2 \sum_{m=1}^{n} \Lambda_m w_m \, dt, \tag{8.11}$$

where $D(z)$ is the differential of the de Rham complex of the fiber over z. Consider the de Rham complex of a fiber,

$$0 \to \Omega^0(z) \to \Omega^1(z) \to 0.$$

Let $\mathscr{H}(z) \subset H^1(z)$ be the image of the rational hypergeometric space of a fiber.

(8.12) Theorem. *Let $\ell = 1$. Let $\kappa \neq 1$. Assume that $p < 0$ and $2\Lambda_m \notin p\mathbb{Z}$ for any $m = 1, \dots, n$. Let $z \in \mathbb{B}$. Then $\dim \mathscr{H}(z) = n$, that is $\mathscr{H}(z) \simeq \mathscr{F}(z)$.*

(8.13) Theorem. *Let $\ell = 1$. Let $\kappa = 1$. Assume that $p < 0$ and $2\Lambda_m \notin p\mathbb{Z}$ for any $m = 1, \dots, n$. Let $z \in \mathbb{B}$. If $2\sum_{m=1}^{n} \Lambda_m \notin p\mathbb{Z}_{<0}$, then $\dim \mathscr{H}(z) = n - 1$, that is $\mathscr{H}(z) \simeq \mathscr{F}(z)/\mathscr{R}(z)$.*

Theorems 8.12 and 8.13 can be proved by rather straightforward calculations. Nevertheless, we will give further another proof which can be naturally extended to the general case.

Remark. Assume that the weights $\Lambda_1, \dots, \Lambda_n$ are such that $2\Lambda_m \notin p\mathbb{Z}_{\geqq 0}$ for any $m = 1, \dots, n$. Let $z \in \mathbb{B}$. Then it is easy to check the following.

a) If $\kappa \neq 1$, then we have $\mathscr{H}(z) = H^1(z)$ and $\dim \mathscr{H}(z) = n$.

b) If $\kappa = 1$ and $2\sum_{m=1}^{n} \Lambda_m \notin p\mathbb{Z}_{<0}$, then also $\mathscr{H}(z) = H^1(z)$, but $\dim \mathscr{H}(z) = n - 1$.

Otherwise, we have $\dim H^1(z)/\mathscr{H}(z) = 1$ and $\dim \mathscr{H}(z)$ can be $n - 2$ or $n - 1$.

One-dimensional discrete homologies

The trigonometric hypergeometric space $\mathscr{F}_q$ is the space of functions of the form

$$P(\xi, \zeta_1, \dots, \zeta_n) \prod_{m=1}^{n} \frac{\exp(\pi i(z_m - t)/p)}{\sin(\pi(t - z_m - \Lambda_m)/p)}$$

where

$$\xi = \exp(2\pi i t/p), \qquad \zeta_m = \exp(2\pi i z_m/p),$$

and P is a polynomial of degree less than n in the variable ξ.

We write $W_m(t, z)$ instead of $W_{\mathfrak{e}(m)}(t, z)$ and $\mathring{W}_m(t, z)$ instead of $\mathring{W}_{\mathfrak{e}(m)}(t, z)$. Recall that

$$(8.14) \qquad W_m(t, z_1, \dots, z_n) = \frac{\exp(\pi i(z_m - t)/p)}{\sin(\pi(t - z_m - \Lambda_m)/p)} \prod_{1 \leqq l < m} \frac{\sin(\pi(t - z_m + \Lambda_m)/p)}{\sin(\pi(t - z_m - \Lambda_m)/p)},$$

$m = 1, \dots, n$, and

$$(8.15) \qquad \mathring{W}_m = W_m \exp(-\pi i \Lambda_m/p) - W_{m+1} \exp(\pi i \Lambda_{m+1}/p), \quad m = 1, \dots, n-1.$$

(8.16) Lemma. (cf. (2.28)) *For any $z \in \mathbb{B}$ the functions $W_1, \dots, W_n$ restricted to the fiber over z form a basis in the trigonometric hypergeometric space $\mathscr{F}_q(z)$ of the fiber.*

Proof. Consider functions

$$G_m(t,z) = \exp(2\pi i(m-1)t/p) \prod_{m=1}^{n} \frac{\exp(-\pi i t/p)}{\sin(\pi(t-z_m-\Lambda_m)/p)}, \quad m = 1,\dots,n.$$

The restrictions of these functions to the fiber over z form a basis of the space $\mathscr{F}_q(z)$. Define a matrix $M^q(z)$ by

$$W_l(t,z) = \sum_{m=1}^{n} M^q_{lm}(z) G_m(t,z), \quad l = 1,\dots,n.$$

The lemma follows from the formula

$$\det M^q = (2i)^{n(1-n)/2} \exp\left(\pi i \sum_{m=1}^{n} z_m/p\right) \prod_{1 \leqq l < m \leqq n} \sin(\pi(z_l - \Lambda_l - z_m - \Lambda_m)/p),$$

(cf. (8.10)). □

(8.17) Lemma. (cf. (2.29)) *For any* $z \in \mathbb{B}$ *the functions* $\mathring{W}_1,\dots,\mathring{W}_{n-1}$ *restricted to the fiber over* z *form a basis in the singular trigonometric hypergeometric space* $\mathscr{F}_q^{\mathrm{sing}}(z)$ *of the fiber.*

The proof is similar to the proof of Lemma 8.16.

Let $\mathbb{I}$ be the imaginary axis in the space $\mathbb{C}$ with coordinate t oriented from $-i\infty$ to $+i\infty$. Recall that the hypergeometric integral $I(W,w)$ for functions $w \in \mathscr{F}(z)$, $W \in \mathscr{F}_q(z)$ is defined as the analytic continuation of the integral

$$I(W,w) = \int_{\mathbb{I}} \Phi(t) w(t) W(t)\, dt \tag{8.18}$$

with respect to $\Lambda_1,\dots,\Lambda_n$ and $z_1,\dots,z_n$, starting from large real negative $\Lambda_1,\dots,\Lambda_n$ and imaginary $z_1,\dots,z_n$. The analytic continuation can be written as an integral over a deformed imaginary space

$$I(W,w) = \int_{\widetilde{\mathbb{I}}} \Phi(t) w(t) W(t)\, dt. \tag{8.19}$$

The deformation of the imaginary space is not unique. Below we describe an example of the deformed imaginary axis $\widetilde{\mathbb{I}}$ which is involved in the integral (8.19).

The deformed imaginary axis $\widetilde{\mathbb{I}}$ is a sum of three terms:

$$\widetilde{\mathbb{I}} = \widetilde{\mathrm{I}} + \mathrm{C}^+ + \mathrm{C}^-, \tag{8.20}$$

which are defined below. First we assume that all the points

$$z_m \pm (\Lambda_m + sp), \quad m = 1,\dots,n, \quad s \in \mathbb{Z}_{\geqq 0}, \tag{8.21}$$

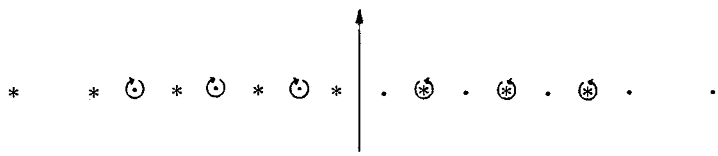
are not imaginary. In this case we set $\widetilde{\mathrm{I}} = \mathbb{I}$. To define the terms $\mathrm{C}^{\pm}$ consider the following sets:

$$Z^{+} = \{z_m + \Lambda_m + ps \mid \mathrm{Re}(z_m + \Lambda_m + ps) > 0,\ m = 1,\dots,n,\ s \in \mathbb{Z}_{\geqq 0}\},$$
$$Z^{-} = \{z_m - \Lambda_m - ps \mid \mathrm{Re}(z_m - \Lambda_m - ps) < 0,\ m = 1,\dots,n,\ s \in \mathbb{Z}_{\geqq 0}\},$$
$$Z_{+} = \{z_m - \Lambda_m - ps \mid \mathrm{Re}(z_m - \Lambda_m - ps) > 0,\ m = 1,\dots,n,\ s \in \mathbb{Z}_{\geqq 0}\},$$
$$Z_{-} = \{z_m + \Lambda_m + ps \mid \mathrm{Re}(z_m + \Lambda_m + ps) < 0,\ m = 1,\dots,n,\ s \in \mathbb{Z}_{\geqq 0}\}.$$

We define C^{+} to be the sum of small circles with centers at the points of Z^{+} oriented anticlockwise. Similarly, C^{-} is the union of small circles with centers at the points of Z^{-} oriented clockwise. We assume that the circles are so small that there are no points of the sets Z_{+}, Z_{-} inside them and they do not intersect the imaginary axis.

If some of the points (8.21) are imaginary, then we take $\widetilde{\mathrm{I}}$ to be an appropriate deformation of the imaginary axis. Namely, if $\mathrm{Re}(z_m + \Lambda_m + ps) = 0$, then we replace the small interval $\mathrm{Re}\, t = 0$, $|\mathrm{Im}(t - z_m - \Lambda_m - ps)| \leqq \varepsilon$, by a small semicircle $|(t - z_m - \Lambda_m - ps| = \varepsilon$, $\mathrm{Re}(t - z_m - \Lambda_m - ps) \geqq 0$. Similarly, if $\mathrm{Re}(z_m - \Lambda_m - ps) = 0$, then we replace the small interval $\mathrm{Re}\, t = 0$, $\mathrm{Im}(t - z_m + \Lambda_m + ps)| \leqq \varepsilon$, by a small semicircle $|t - z_m + \Lambda_m + ps| = \varepsilon$, $\mathrm{Re}(t - z_m + \Lambda_m + ps) \leqq 0$. The terms $\mathrm{C}^{\pm}$ remain the same.

Example. Let $n = 1$. In this case the deformed imaginary axis $\widetilde{\mathbb{I}}$ looks like



where asterisks and dots stay for points $z_1 + \Lambda_1 + ps$ and $z_1 - \Lambda_1 - ps$, $s \in \mathbb{Z}_{\geqq 0}$, respectively.

(8.22) Lemma. *Let* $0 < \mathrm{Im}\,\mu < 2\pi$. *Then for any* $l, m = 1,\dots,n$ *the hypergeometric integral* $I(W_l, w_m)$ *can be analytically continued as a univalued holomorphic function of complex variables* $p, \Lambda_1,\dots,\Lambda_n, z_1,\dots,z_n$ *to the region*

$$p < 0, \qquad z \in \mathbb{B}, \qquad 2\Lambda_m \notin p\mathbb{Z}_{\leqq 0}, \quad m = 1,\dots,n.$$

Proof. The only thing to be shown is convergence of the integral in the right hand side of (5.1) for functions $W = W_l$, $w = w_m$. The convergence is clear since

$$\Phi(t) = t^{2\sum_{m=1}^{n}\Lambda_m/p} \exp(\mu t/p)(1 + o(1)), \quad t \to \pm i\infty,$$

and therefore, under the assumptions of the lemma the integrand decays exponentially as t goes to infinity. □

(8.23) Lemma. *Let* $\operatorname{Im}\mu = 0$. *Then for any* $l = 1, \dots, n-1$, $m = 1, \dots, n$ *the hypergeometric integral* $I(\mathring{W}_l, w_m)$ *can be analytically continued as a univalued holomorphic function of complex variables* $p, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n$ *to the region*

$$p < 0, \qquad z \in \mathbb{B}, \qquad 2\Lambda_m \notin p\mathbb{Z}_{\leqq 0}, \quad m = 1, \dots, n\,.$$

The proof is similar to the proof of the previous lemma.

In what follows we need to consider the hypergeometric integral $I(W, w)$ for functions w from the functional space $\widehat{\mathscr{F}}(z)$ of a fiber. The definition is similar to the definition of the hypergeometric integral for $w \in \mathscr{F}(z)$. Below we describe explicitly the analytic continuation of the hypergeometric integral $I(W, w)$ for any function $w \in \widehat{\mathscr{F}}(z)$ as an integral over a suitable deformation of the imaginary line.

For any integer s let $\widetilde{\mathbb{I}}[s]$ be the deformation of the imaginary axis which is defined similarly to $\widetilde{\mathbb{I}}$ but the parameters $\Lambda_1, \dots, \Lambda_n$ are replaced by $\Lambda_1 + ps, \dots, \Lambda_n + ps$, respectively. In particular, $\widetilde{\mathbb{I}}[0] = \widetilde{\mathbb{I}}$.

For a function $w \in \widehat{\mathscr{F}}(z)$ we have

$$I(W, w) = \int\limits_{\widetilde{\mathbb{I}}[s]} \Phi(t) w(t) W(t)\, dt \tag{8.24}$$

where the integer s is chosen so that the integrand has no poles at the points $z_m \pm (\Lambda_m + pr)$ for $r < s$, $r \in \mathbb{Z}$. Under this assumption the right hand side of (8.24) does not depend on s.

Let $D\widehat{\mathscr{F}}(z) = \{Dw \mid w \in \widehat{\mathscr{F}}(z)\}$.

(8.25) Lemma. *Let either* $0 < \operatorname{Im}\mu < 2\pi$ *and* $W \in \mathscr{F}_q(z)$ *or* $\mu = 0$ *and* $W \in \mathscr{F}_q^{\text{sing}}(z)$. *Assume that* $p < 0$ *and* $2\Lambda_m \notin p\mathbb{Z}$ *for any* $m = 1, \dots, n$. *Let* $z \in \mathbb{B}$. *Then*

a) *The hypergeometric integral* $I(W, w)$ *is well defined for any function* $w \in \widehat{\mathscr{F}}(z)$.

b) *The hypergeometric integral* $I(W, w)$ *equals zero for any function* $w \in D\widehat{\mathscr{F}}(z)$.

Proof. The proof of claim a) is similar to the proof of Lemma 8.22. Claim b) follows from the next observation. Let $\widetilde{\mathbb{I}}_p[s]$ be the contour obtained from $\widetilde{\mathbb{I}}[s]$ by the translation $t \mapsto t + p$. Then for a given function $w \in \widehat{\mathscr{F}}(z)$ and a large negative s the contour $\widetilde{\mathbb{I}}[s]$ and $\widetilde{\mathbb{I}}_p[s]$ are homologous in the complement of the set of poles of the function $\Phi(t) w(t) W(t)$. $\square$

(8.26) Lemma. *Let* $\mu = 0$. *Assume that* $p < 0$ *and* $2\Lambda_m \notin p\mathbb{Z}_{\leqq 0}$ *for any* $m = 1, \dots, n$. *Then the hypergeometric integral* $I(W, w)$ *equals zero for any* $w \in \mathscr{R}(z)$ *and* $W \in \mathscr{F}_q^{\text{sing}}(z)$.

Proof. The lemma follows from formula (8.11) and Lemma 8.25. $\square$

The hypergeometric integral defines linear functionals $I(W,\,\cdot\,)$ on the functional space of a fiber. Lemma 8.25 means that these linear functions can be considered as elements of the homology group $H_1(z)$, the dual space to the cohomology group of the de Rham complex of the discrete local system of the fiber.

Let W be any element of the trigonometric hypergeometric space $\mathscr{F}_q$. Let $W|_z \in \mathscr{F}_q(z)$ be its restriction to a fiber. Consider an element $s_W(z) = I(W|_z,\,\cdot\,)$ of the homology group $H_1(z)$.

(8.27) Theorem. (cf. (5.18)) *Let $\ell = 1$. Let either $0 < \operatorname{Im}\mu < 2\pi$ and $W \in \mathscr{F}_q$ or $\mu = 0$ and $W \in \mathscr{F}_q^{\text{sing}}$. Assume that $p < 0$ and $2\Lambda_m \notin p\mathbb{Z}$ for any $m = 1,\dots,n$. Then the section s_W is a periodic section with respect to the Gauss–Manin connection.*

Proof. Let the contour $\widetilde{\mathbb{I}}_m[s]$ be defined similar to $\widetilde{\mathbb{I}}[s]$ but the parameter z_m is replaced by $z_m - p$. The statement of the theorem means that for any function $w \in \mathscr{F}(z)$ and each $m = 1,\dots,n$ we have the equality

$$I(W,w) = \int\limits_{\widetilde{\mathbb{I}}_m[s]} \Phi(t)w(t)W(t)\,dt\,,$$

where s is a sufficiently large negative integer. The last equality holds since the integrand $\Phi(t)w(t)W(t)$ has no poles separating the contours $\widetilde{\mathbb{I}}_m[s]$ and $\widetilde{\mathbb{I}}[s]$. □

Consider a section Ψ_W of the trivial bundle over $\mathbb{C}^n$ with fiber $(V_1 \otimes \cdots \otimes V_n)_\ell$:

$$\Psi_W(z) = \sum_{m=1}^{n} I(W|_z, w_m|_z)v_1 \otimes \cdots \otimes Fv_m \otimes \cdots \otimes v_n\,.$$

(8.28) Corollary. (cf. (5.20)) *The section Ψ_W is a solution to the qKZ equation.*

Our further strategy is as follows. First we show that if $0 < \operatorname{Im}\mu < 2\pi$, then the basis of sections Ψ_{W_m}, $m = 1,\dots,n$, is an asymptotic solutions to the *qKZ* equation, (cf. Theorem 8.29). Using this fact we prove that the hypergeometric pairing $I : \mathscr{F}_q(z) \otimes \mathscr{F}(z) \to \mathbb{C}$ is nondegenerate if $0 < \operatorname{Im}\mu < 2\pi$ (cf. Theorem 8.33). Studying the asymptotic behaviour of the hypergeometric integral as μ tends to zero we will show that for $\mu = 0$ the hypergeometric pairing $I^\circ : \mathscr{F}_q^{\text{sing}}(z) \otimes \mathscr{F}(z)/\mathscr{R}(z) \to \mathbb{C}$ is nondegenerate (cf. Theorem 8.34). At the end of the section we will describe the quasiclassical asymptotics of the hypergeometric integral for $\ell = 1$ (cf. Theorems 8.39, 8.40).

For every permutation $\tau \in \mathbb{S}^n$, consider the asymptotic zone in $\mathbb{C}^n$ given by

$$\mathbb{A}_\tau = \{z \in \mathbb{C}^n \mid \operatorname{Re} z_{\tau_1} \ll \cdots \ll \operatorname{Re} z_{\tau_n}\}\,,$$

and say that $z \to \infty$ in $\mathbb{A}_\tau$ if $\operatorname{Re}(z_{\tau_m} - z_{\tau_{m+1}}) \to -\infty$ for all $m = 1,\dots,n-1$.

(8.29) Theorem. (cf. (6.4)) *Let* $\ell = 1$. *Let* $0 < \operatorname{Im}\mu < 2\pi$ *and, therefore,* $\kappa \neq 1$. *Assume that* $p < 0$ *and* $2\Lambda_m \notin p\mathbb{Z}$ *for any* $m = 1,\dots,n$. *Then for any permutation* $\tau \in \mathbb{S}^n$ *the basis* $\Psi_{W_m^\tau}$, $m = 1,\dots,n$, *is an asymptotic solution in the asymptotic zone* $\mathbb{A}_\tau$. *Namely,*

$$\Psi_{W_m^\tau}(z) = \Theta_m \exp(\mu z_m/p) \prod_{1 \leqq l < \tau_m^{-1}} ((z_{\tau_l} - z_m)/p)^{2\Lambda_{\tau_l}} \prod_{\tau_m^{-1} < l \leqq n} ((z_m - z_{\tau_l})/p)^{2\Lambda_{\tau_l}}$$
$$\times (v_1 \otimes \cdots \otimes Fv_m \otimes \cdots \otimes v_n + o(1)) .$$

as $z \to \infty$ *in* $\mathbb{A}_\tau$ *so that* $z \in \mathbb{B}$ *at any moment. Here* $|\arg((z_k - z_l)/p)| < \pi$ *and* Θ_m *is a constant independent of the permutation* τ *and given by*

$$\Theta_m = 2i(e^\mu - 1)^{-2\Lambda_m/p} \exp((\mu + \pi i)\Lambda_m/p)\Gamma(2\Lambda_m/p) ,$$

where $0 \leqq \arg(e^\mu - 1) < 2\pi$.

Proof. To simplify notations we will give a proof only for $\tau = \mathrm{id}$. A simple but important fact is that for any $W \in \mathcal{F}_q$

$$\Psi_W(z_1 + p,\dots,z_n + p) = \kappa\Psi_W(z_1,\dots,z_n) . \tag{8.30}$$

It allows us to fix freely the real part of one of the coordinates $z_1,\dots,z_n$.

Consider the hypergeometric integral $I(W_m, w_m)$. The corresponding integrand $\Phi(t)w_m(t)W_m(t)$ can be rewritten as follows:

(8.31)
$$\Phi(t)w_m(t)W_m(t) = (-\pi p)^{-1} \exp((\mu - \pi i)t/p + \pi i z_m/p)$$
$$\times \Gamma((t - z_m + \Lambda_m)/p)\Gamma((z_m + \Lambda_m - t)/p)$$
$$\times \prod_{1 \leqq l < m} \frac{\Gamma((z_l + \Lambda_l - t)/p)}{\Gamma((z_l - \Lambda_l - t)/p)} \prod_{m < l \leqq n} \frac{\Gamma((t - z_l + \Lambda_l)/p)}{\Gamma((t - z_l - \Lambda_l)/p)} .$$

This function has no poles at points $z_l - \Lambda_l - sp$, $s \in \mathbb{Z}$, for $l < m$ and has no poles at points $z_l + \Lambda_l + sp$, $s \in \mathbb{Z}$, for $l > m$. Moreover, due to (8.30), without loss of generality, we can assume that z tends to infinity in $\mathbb{A}_{\mathrm{id}}$ so that $\operatorname{Re} z_l \to -\infty$ for $l < m$, $\operatorname{Re} z_m$ remain finite, and $\operatorname{Re} z_l \to +\infty$ for $l > m$. Under this assumption the integrand has no poles at the points $z_l + \Lambda_l + sp$, $s \in \mathbb{Z}$, for $l < m$ in the halfplane $\operatorname{Re} t > 0$ and has no poles at the points $z_l - \Lambda_l - sp$, $s \in \mathbb{Z}$, for $l > m$ in the halfplane $\operatorname{Re} t > 0$. Therefore, we can "straighten" the contour and write

$$I(W_m, w_m) = \int_{\widetilde{\mathbb{I}}_m} \Phi(t)w_m(t)W_m(t)\, dt \tag{8.32}$$

where the contour $\widetilde{\mathbb{I}}_m$ is the contour defined above for analytic continuation of the integral

$$\int_{\mathbb{I}} \exp((\mu - \pi i)t/p)\Gamma((t - z_m + \Lambda_m)/p)\Gamma((z_m + \Lambda_m - t)/p)\, dt .$$

The remaining part of the calculation is a standard exercise. We replace the integrand in the integral (8.32) by its asymptotics as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$ and obtain

$$I(W_m, w_m) = (-\pi p)^{-1} \exp(\mu z_m/p) \prod_{1 \leqq l < m} ((z_l - z_m)/p)^{2\Lambda_l/p}$$
$$\times \prod_{m < l \leqq n} ((z_m - z_l)/p)^{2\Lambda_l/p} \int_{\tilde{\mathbb{I}}_m} \exp((\mu - \pi i)(t - z_m)/p)$$
$$\times \Gamma((t - z_m + \Lambda_m)/p)\Gamma((z_m - t + \Lambda_m/p)\, dt\, (1 + o(1))\,.$$

The last integral reduces to the Barnes integral (5.16) and is calculated explicitly. Finally, we have

$$I(W_m, w_m) = 2i(e^\mu - 1)^{-2\Lambda_m/p} \exp((\mu + \pi i)\Lambda_m/p)\Gamma(2\Lambda_m/p) \exp(\mu z_m/p)$$
$$\times \prod_{1 \leqq l < m} ((z_l - z_m)/p)^{2\Lambda_l/p} \prod_{m < l \leqq n} ((z_m - z_l)/p)^{2\Lambda_l/p}(1 + o(1))\,,$$

as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$. Here $0 \leqq \arg(e^\mu - 1) < 2\pi$.

The hypergeometric integral $I(W_m, w_l)$ for $l \neq m$ can be treated similarly to the hypergeometric integral $I(W_m, w_m)$ considered above. The final answer is

$$I(W_m, w_l) = I(W_m, w_m)\, o(1)\,,$$

which completes the proof of Theorem 8.29. □

(8.33) Theorem. (cf. (5.14)) *Let* $\ell = 1$. *Let* $0 < \operatorname{Im} \mu < 2\pi$. *Assume that* $p < 0$ *and* $2\Lambda_m \notin p\mathbb{Z}_{\leqq 0}$ *for any* $m = 1, \dots, n$. *Let* $z \in \mathbb{B}$. *Then the hypergeometric pairing* $I : \mathscr{F}_q(z) \otimes \mathscr{F}(z) \to \mathbb{C}$ *is nondegenerate. Moreover,*

$$\det[I(W_l, w_m)]_{l,m=1}^n = (2i)^n (e^\mu - 1)^{-2\sum_{m=1}^n \Lambda_m/p}$$
$$\times \exp\left((\mu + \pi i) \sum_{m=1}^n \Lambda_m/p + \mu \sum_{m=1}^n z_m/p\right)$$
$$\times \prod_{m=1}^n \Gamma(2\Lambda_m/p) \prod_{1 \leqq l < m \leqq n} \frac{\Gamma((z_l + \Lambda_l - z_m + \Lambda_m)/p)}{\Gamma((z_l - \Lambda_l - z_m - \Lambda_m)/p)}\,.$$

Here $0 \leqq \arg(e^\mu - 1) < 2\pi$.

Proof. Denote by $F(z)$ the determinant $\det[I(W_l, w_m)]_{l,m=1}^n$ and by $G(z)$ the right hand side of the formula above. Since for every $l = 1, \dots, n$ the section Ψ_{W_l} is a $(V_1 \otimes \dots \otimes V_n)_1$-valued solution to the qKZ equation, $F(z)$ solves the next system of difference equations

$$F(z_1, \dots, z_m + p, \dots, z_n) = \overset{(1)}{\det}\, K_m(z_1, \dots, z_n) F(z_1, \dots, z_n)\,.$$

Here $\overset{(1)}{\det} K_m(z)$ stays for the determinant of the operator $K_m(z)$ (3.10) acting in the weight subspace $(V_1 \otimes \cdots \otimes V_n)_1$. Using (3.5) we see that

$$\overset{(1)}{\det} K_m(z_1,\dots,z_n) = \kappa \prod_{1 \leqq l < m} \frac{z_m + \Lambda_m - z_l + \Lambda_l + p}{z_m - \Lambda_m - z_l - \Lambda_l + p} \times \prod_{m < l \leqq n} \frac{z_m + \Lambda_m - z_l + \Lambda_l}{z_m - \Lambda_m - z_l - \Lambda_l} .$$

Therefore, the ratio $F(z)/G(z)$ is a p-periodic function in each of the variables $z_1,\dots,z_n$:

$$\frac{F}{G}(z_1,\dots,z_m + p,\dots,z_n) = \frac{F}{G}(z_1,\dots,z_n) .$$

Theorem 8.29 implies that the ratio $F(z)/G(z)$ tends to 1 as z tends to infinity in the asymptotic zone $\mathbb{A}_{\mathrm{id}}$. Hence, this ratio equals 1 identically, which completes the proof. □

(8.34) Theorem. *Let $\ell = 1$. Let $\mu = 0$. Assume that $p < 0$ and $2\Lambda_m \notin p\mathbb{Z}$ for any $m = 1,\dots,n$. Let $z \in \mathbb{B}$. If $2\sum_{m=1}^n \Lambda_m \notin p\mathbb{Z}_{<0}$, then the hypergeometric pairing $I^\circ : \mathscr{F}_q^{\mathrm{sing}}(z) \otimes \mathscr{F}(z)/\mathscr{R}(z) \to \mathbb{C}$ is nondegenerate. Moreover,*

$$\det[I(\mathring{W}_l, w_m)]_{l,m=1}^{n-1} = (2i)^{n-1} \Gamma\left(1 + 2\sum_{m=1}^n \Lambda_m/p\right)^{-1} \Gamma(1 + 2\Lambda_n/p) \times \prod_{m=1}^{n-1} \Gamma(2\Lambda_m/p) \prod_{1 \leqq l < m \leqq n} \frac{\Gamma((z_l + \Lambda_l - z_m + \Lambda_m)/p)}{\Gamma((z_l - \Lambda_l - z_m - \Lambda_m)/p)} .$$

Proof. Since both sides of the formula above are analytic functions in $\Lambda_1,\dots,\Lambda_n$, it suffices to prove the formula under the assumption

$$0 < 2\sum_{m=1}^n \Lambda_m/p < 1 .$$

To prove the theorem we first assume that $\mu \neq 0$ and study asymptotics of the determinant $\det[I(W_l, w_m)]_{l,m=1}^n$ as μ tends to zero. We will show that

(8.35)
$$\det[I(W_l, w_m)]_{l,m=1}^n = (ip/\Lambda_n)\exp\left(\pi i \sum_{m=1}^n \Lambda_m/p\right) \mu^{-2\sum_{m=1}^n \Lambda_n/p} \times \Gamma\left(1 + 2\sum_{m=1}^n \Lambda_m/p\right)^{-1} \det[I(\mathring{W}_l, w_m)]_{l,m=1}^{n-1}(1 + o(1))$$

as $\mu \to 0$, $0 < \arg \mu < \pi$. Due to (8.33) the last formula will imply the required formula for $\det[I(\mathring{W}_l, w_m)]$.

First we change bases in the rational and trigonometric hypergeometric spaces of a fiber. We set

$$W'_m = \mathring{W}_m, \quad m = 1,\dots,n-1, \qquad W'_n = W_n ,$$

and

$$w'_m = w_m, \quad m = 1, \dots, n-1, \qquad w'_m = \sum_{m=1}^{n} \Lambda_m w_m .$$

We have

$$(8.36) \qquad \det[I(W'_l, w'_m)]_{l,m=1}^n = \Lambda_n \exp\left(-\pi i \sum_{m=1}^{n-1} \Lambda_m\right) \det[I(W_l, w_m)]_{l,m=1}^n .$$

As μ tends to zero, the entries $I(W'_l, w'_m)$, $l, m = 1, \dots, n-1$, have finite limits $I(\mathring{W}_l, w_m)$, respectively. Similarly, the entries $I(W'_l, w'_n)$, $l = 1, \dots, n-1$, tend to zero since $D(z) \cdot 1 = 2w'_n \, dt$ at $\mu = 0$ and, therefore, $I(\mathring{W}_l, w_n) = 0$ at $\mu = 0$. More precisely, we have $I(W'_n, w'_m) = O(\mu)$ as $\mu \to 0$. The behaviour of the entries $I(W'_n, w'_m)$, $m = 1, \dots, n$, is described in the next lemma.

(8.37) Lemma. *Let* $0 < 2\sum_{m=1}^n \Lambda_m/p < 1$. *Let* $\mu \to 0$, $0 < \arg \mu < \pi$. *Then*

$$I(W_n, w_m) = 2i \exp(\pi i \Lambda_n/p) \mu^{-2\sum_{m=1}^n \Lambda_m/p} \Gamma\left(2 \sum_{m=1}^n \Lambda_m/p\right)(1 + o(1)) .$$

Proof. As $t \to -i\infty$, the integrand of the hypergeometric integral $I(W_n, w_m)$ has the following asymptotics:

$$\Phi(t) w_m(t) W_n(t) = (-2i/p) \exp\left(\mu t/p - \pi i \Lambda_n/p - 2\pi i \sum_{m=1}^{n-1} \Lambda_m/p\right)$$
$$\times (t/p)^{-1+2\sum_{m=1}^n \Lambda_m/p}(1 + o(1)) .$$

Denote by $F(t)$ the left hand side of the equality above and by $G(t)$ the right hand side without the factor $1 + o(1)$.

Let s be a positive number such that $s > \max\{|z_1|, \dots, |z_n|\}$. Let $\widetilde{\mathbb{I}}_s$ be the part of the deformed imaginary axis $\widetilde{\mathbb{I}}$ in the halfplane $\operatorname{Im} t > -s$. We have

$$I(W_n, w_m) = \left(\int_{-i\infty}^{0} - \int_{-is}^{0}\right) G(t)\, dt + \int_{-i\infty}^{-is} (F(t) - G(t))\, dt + \int_{\widetilde{\mathbb{I}}_s} F(t)\, dt .$$

The first integral in the right hand side above can be calculated explicitly since

$$\int_{-i\infty}^{0} \exp(\mu t/p)(t/p)^{-1+2\sum_{m=1}^n \Lambda_m/p} \, dt/p$$
$$= -\exp\left(2\pi i \sum_{m=1}^n \Lambda_m\right) \mu^{-2\sum_{m=1}^n \Lambda_m/p} \Gamma\left(2 \sum_{m=1}^n \Lambda_m/p\right) ,$$

and the three other integrals have finite limits as $\mu \to 0$. The lemma is proved. □

(8.38) Corollary. *Let* $0 < 2\sum_{m=1}^n \Lambda_m/p < 1$. *Let* $\mu \to 0$, $0 < \arg \mu < \pi$. *Then*

$$I(W'_n, w'_n) = 2i \exp(\pi i \Lambda_n/p) \mu^{-2\sum_{m=1}^n \Lambda_m/p} \Gamma\left(1 + 2 \sum_{m=1}^n \Lambda_m/p\right)(1 + o(1)) .$$

Finally, we have

$$\det[I(W_l', w_m')]_{l,m=1}^{n} = \det[I(\mathring{W}_l, w_m)]_{l,m=1}^{n-1} I(W_n', w_m')(1 + o(1)) .$$

Using (8.36) and Corollary 8.38 we get (8.35). Theorem 8.34 is proved. □

Proof of Theorems 8.12 and 8.13. Theorems 8.12 and 8.13 follow from Theorems 8.33 and 8.34, respectively, and Lemma 8.25. □

Quasiclassical asymptotics

Recall that to study the quasiclassical asymptotics of the hypergeometric integral we introduced new parameters h and $\eta = \mu/h$, and new coordinates $u = ht$ and $y_m = hz_m$, $m = 1, \ldots, n$. The quasiclassical asymptotics of a hypergeometric integral is the asymptotics of the integral as $h \to 0$ while the coordinates $y_1, \ldots, y_n$ and the parameter η remain fixed.

For each $m = 1, \ldots, n$, we defined an imaginary interval

$$U_m = \{u \in \mathbb{C} \mid \operatorname{Re} u = 0,\ \operatorname{Im} y_{m-1} \leqq \operatorname{Im} u \leqq \operatorname{Im} y_m\}, \quad y_0 = -i\infty ,$$

a chain

$$\overline{U}_m = \sum_{l=1}^{m} \exp\left(4\pi i \sum_{1 \leqq k < l} \Lambda_k/p\right) U_l .$$

and a rational function

$$\tilde{w}_m(u, y_1, \ldots, y_n) = \frac{1}{u - y_m} .$$

Set

$$\widetilde{\Phi}(u, y_1, \ldots, y_n) = \exp(\eta u/p) \prod_{m=1}^{n} ((u - y_m)/p)^{2\Lambda_m/p}$$

where $|\arg((u - y_m)/p)| < \pi$.

(8.39) Theorem. (cf. (7.6)) *Let* $\ell = 1$. *Let* $p < 0$. *Let* $\operatorname{Re} \Lambda_m < 0$ *and let* $\operatorname{Re} y_m = 0$ *for all* $m = 1, \ldots, n$. *Let* $\mu = h\eta$, $\operatorname{Im} \eta > 0$. *Then for any* $l, m = 1, \ldots, n$ *the hypergeometric integral* $I(W_l, w_m)$ *has the following asymptotics as* $h \to +0$ *and* $y \in \mathbb{Y}$:

$$I(W_l, w_m) = -2ih^{-2\sum_{k=1}^{n} \Lambda_k/p} \exp\left(-\pi i \Lambda_l/p - 2\pi i \sum_{1 \leqq k < l} \Lambda_k/p\right)$$
$$\times \int_{\overline{U}_l} \widetilde{\Phi}(u, y) \tilde{w}_m(u, y)\, du\, (1 + o(1)) ,$$

Remark. Recall that hypergeometric integral $I(W_l, w_m)$ is defined by (5.3) where $\ell = 1$, the functions W_l and w_m are given by (8.14) and (8.8), respectively, and we replace in these formulae $z_1, \ldots, z_n$ by $y_1/h, \ldots, y_n/h$.

(8.40) Theorem. (cf. (7.8), (7.9)) *Let $\ell = 1$. Let $p < 0$. Let* $\operatorname{Re} \Lambda_m < 0$ *and let* $\operatorname{Re} y_m = 0$ *for all $m = 1, \ldots, n$. Let $\mu = h\eta$,* $\operatorname{Im} \eta = 0$. *Then for any $l = 1, \ldots, n-1$ and any $m = 1, \ldots, n$ the hypergeometric integral $I(\mathring{W}_l, w_m)$ has the following asymptotics as $h \to +0$ and $y \in \mathbb{Y}$:*

$$I(W_l, w_m) = 2ih^{-2\sum_{k=1}^{n} \Lambda_k/p} \exp\left(2\pi i \sum_{1 \leqq k \leqq l} \Lambda_k/p\right)$$
$$\times \int_{U_{l+1}} \tilde{\Phi}(u, y) \tilde{w}_m(u, y)\, du\, (1 + o(1)) .$$

Remark. Recall that the hypergeometric integral $I(\mathring{W}_l, w_m)$ is defined by (5.3), the functions $\mathring{W}_l$ and w_m are given by (8.15) and (8.8), respectively, and we replace in these formulae $z_1, \ldots, z_n$ by $y_1/h, \ldots, y_n/h$.

Remark. The claims of Theorems 8.39 and 8.40 remain valid for any $\Lambda_1, \ldots, \Lambda_n$ such that $\Lambda_m \notin p\mathbb{Z}_{\leqq 0}$ for all $m = 1, \ldots, n$, if the other assumptions of the theorems hold and the integrals in the right hand sides of (8.39), (8.40) are regularized in the standard way. We omit the proof since it is not essential for our purpose in this paper.

The idea of the proofs of Theorems 8.39, 8.40 is simple. After a suitable renormalization, the quasiclassical asymptotics of the function W_l is given by a linear combination of the characteristic functions of the intervals $U_1, \ldots, U_l$ with the coefficients defined by the chain $\overline{U}_l$. Similarly, after a suitable renormalization, the quasiclassical asymptotics of the function $\mathring{W}_l$ is given by the characteristic function of the interval U_{l+1}. Therefore, modulo renormalization factors the quasiclassical asymptotics of the hypergeometric integrals $I(W_l, w_m)$, $I(\mathring{W}_l, w_m)$ are given by integrals of products of powers of linear functions over the chain $\overline{U}_l$ or over the interval U_{l+1}, respectively.

Proof of Theorem 8.39. To simplify notations we will give a proof only for $l = m$. Consider the hypergeometric integral $I(W_m, w_m)$. It is given by

$$I(W_m, w_m) = \int_{\mathbb{I}} \Phi(t) w_m(t) W_m(t)\, dt . \tag{8.41}$$

Let h be a positive number. The factors of the integrand above have the following quasiclassical asymptotics as $h \to +0$ while the parameter $\eta = \mu/h$, the variable $u = ht$ and the coordinates $y_m = hz_m$, $m = 1, \ldots, n$, remain fixed:

$$\Phi(t, z_1, \ldots, z_n) = \exp(\mu t/p) \prod_{m=1}^{n} ((t - z_m)/p)^{2\Lambda_m/p} (1 + o(1)) ,$$

$$w_m(t, z_1, \ldots, z_n) = \frac{1}{t - z_m}(1 + o(1)) ,$$

$$W_m(t, z_1, \ldots, z_n)$$
$$= 2i \exp\left(2\pi i(z_m - t)/p + \pi i \Lambda_m/p + 2\pi i \sum_{1 \leqq k < m} \Lambda_k/p\right) (1 + o(1))$$

if $\operatorname{Im} z_m < \operatorname{Im} t$ and

$$W_m(t, z_1, \ldots, z_n)$$
$$= -2i \exp\left(-\pi i \Lambda_m/p + 2\pi i \sum_{1 \leqq k < l} \Lambda_k/p - 2\pi i \sum_{l \leqq k < m} \Lambda_k/p\right)(1 + o(1))$$

If $\operatorname{Im} z_l < \operatorname{Im} t < \operatorname{Im} z_{l+1}$, $l = 0, \ldots, m-1$. Here $z_0 = -i\infty$.

To compute the quasiclassical asymptotic of the hypergeometric integral $I(W_m, w_m)$ we replace the integrand in the right hand side of (8.41) by its quasiclassical asymptotics, and after simple transformations we obtain that

$$I(W_l, w_m) = -2ih^{-2\sum_{k=1}^{n} \Lambda_k/p} \exp\left(-\pi i \Lambda_l/p - 2\pi i \sum_{1 \leqq k < l} \Lambda_k/p\right)$$
$$\times \int_{\overline{U}_l} \widetilde{\Phi}(u, y) \tilde{w}_m(u, y)\, du\, (1 + o(1))\,.$$

This step can be justified in a standard way using the next lemma.

(8.42) Lemma. *Let* $\operatorname{Re}\alpha > 0$. *Then there is a constant* A *such that for any real* s *the following estimates hold*:

$$|(\alpha^2 + s^2)^{1-\alpha} \exp(-\pi|s|)\Gamma(\alpha + is)\Gamma(\alpha - is)| < A\,,$$

$$|(\alpha^2 + s^2)^{-\alpha}\Gamma(is + \alpha)/\Gamma(is - \alpha)| < A\,.$$

Proof. The required formula follow from the next specialization of the Stirling formula

$$|\log \Gamma(x) - (x - 1/2)\log x + x - \log\sqrt{2\pi}| < \frac{K}{\operatorname{Re} x}, \quad \operatorname{Re} x > 0\,,$$

where K is some constant [WW]. □

Theorem 8.39 is proved. □

The proof of Theorem 8.40 is similar to the proof of Theorem 8.39.

9. The multidimensional case

This section contains proofs of the statements formulated in Sects. 2–7. We start from the lemmas which describe bases in the rational and trigonometric hypergeometric spaces of a fiber.

Proof of Lemmas 2.20, 2.28, 2.29. First we have to show that functions $w_{\mathfrak{l}}$, $W_{\mathfrak{l}}$ and $\mathring{W}_{\mathfrak{l}}$ lie in the rational, in the trigonometric and in the singular trigonometric hypergeometric spaces, respectively. The arguments in all the cases are similar, so we will consider only the rational case.

It is clear from Definition (2.19) that the function $w_{\mathfrak{l}}(t,z)$ has the form

$$Q(t_1,\dots,t_\ell,z_1,\dots,z_n)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{t_a-z_m-\Lambda_m}\prod_{1\leqq a<b\leqq \ell}\frac{1}{t_a-t_b+1}$$

where Q is a polynomial which has degree less than $n+\ell-1$ in each of the variables $t_1,\dots,t_\ell$. Furthermore, by construction the function $w_{\mathfrak{l}}$ as a function of $t_1,\dots,t_\ell$ is invariant with respect to the action (2.9) of the symmetric group $\mathbf{S}^\ell$, which means that the polynomial Q is skewsymmetric with respect to the variables $t_1,\dots,t_\ell$. Hence, the polynomial Q is divisible by $\prod_{1\leqq a<b\leqq\ell}(t_a-t_b)$ and the ratio is a polynomial which is symmetric in variables $t_1,\dots,t_\ell$ and has degree less than n in each of the variables $t_1,\dots,t_\ell$; that is the function $w_{\mathfrak{l}}$ is in the rational hypergeometric space.

For any $\mathfrak{l}\in\mathscr{Z}_\ell^n$ (cf. (2.18)), let $P_{\mathfrak{l}}(u_1,\dots,u_\ell)$ be the following symmetric polynomial

$$P_{\mathfrak{l}}(u_1,\dots,u_\ell)=\frac{1}{\mathfrak{l}_1!\dots\mathfrak{l}_n!}\sum_{\sigma\in\mathbf{S}^\ell}\prod_{m=1}^{n}\prod_{a\in\Gamma_m}u_{\sigma_a}^{m-1}\,.$$

Here $\Gamma_m=\{1+\mathfrak{l}^{m-1},\dots,\mathfrak{l}^m\}$, $m=1,\dots,n$. Consider the following functions

$$g_{\mathfrak{l}}(t,z)=P_{\mathfrak{l}}(t_1,\dots,t_\ell)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{t_a-z_m-\Lambda_m}\prod_{1\leqq a<b\leqq\ell}\frac{t_a-t_b}{t_a-t_b+1}\,,$$

$$G_{\mathfrak{l}}(t,z)=P_{\mathfrak{l}}(\xi_1,\dots,\xi_\ell)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{\exp(-\pi i t_a/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)}$$

$$\times\prod_{1\leqq a<b\leqq\ell}\frac{\sin(\pi(t_a-t_b)/p)}{\sin(\pi(t_a-t_b+1)/p)}$$

$$\mathring{G}_{\mathfrak{l}}(t,z)=P_{\mathfrak{l}}(\xi_1,\dots,\xi_\ell)\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{\exp(\pi i t_a/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)}$$

$$\times\prod_{1\leqq a<b\leqq\ell}\frac{\sin(\pi(t_a-t_b)/p)}{\sin(\pi(t_a-t_b+1)/p)}$$

where $\xi_a=\exp(2\pi i t_a/p)$, $a=1,\dots,\ell$. Restrictions of the functions $g_{\mathfrak{l}}(t,z)$, $\mathfrak{l}\in\mathscr{Z}_\ell^n$, to the fiber over z form a basis of the rational hypergeometric space of the fiber. Restrictions of the functions $G_{\mathfrak{l}}(t,z)$, $\mathfrak{l}\in\mathscr{Z}_\ell^n$, (resp.$\mathring{G}_{\mathfrak{l}}(t,z)$, $\mathfrak{l}\in\mathscr{Z}_\ell^{n-1}$) to the fiber over z form a basis of the trigonometric (resp. the singular trigonometric) hypergeometric space of the fiber.

Define matrices $M(z)$, $M^q(z)$ and$\mathring{M}(z)$ by

$$w_{\mathfrak{l}}(t,z)=\sum_{\mathfrak{m}\in\mathscr{Z}_\ell^n}M_{\mathfrak{l}\mathfrak{m}}(z)g_{\mathfrak{m}}(t,z),\qquad \mathfrak{l}\in\mathscr{Z}_\ell^n\,,$$

$$W_{\mathfrak{l}}(t,z)=\sum_{\mathfrak{m}\in\mathscr{Z}_\ell^n}M^q_{\mathfrak{l}\mathfrak{m}}(z)G_{\mathfrak{m}}(t,z),\qquad \mathfrak{l}\in\mathscr{Z}_\ell^n\,,$$

$$\mathring{W}_{\mathfrak{l}}(t,z)=\sum_{\mathfrak{m}\in\mathscr{Z}_\ell^{n-1}}\mathring{M}_{\mathfrak{l}\mathfrak{m}}(z)\mathring{G}_{\mathfrak{m}}(t,z),\qquad \mathfrak{l}\in\mathscr{Z}_\ell^{n-1}\,,$$

(9.1) Lemma.

$$\det M = \prod_{s=0}^{\ell-1} \prod_{1 \leq \ell < m \leq n} (z_l - \Lambda_l - z_m - \Lambda_m + s)^{\binom{n+\ell-s-2}{n-1}},$$

$$\det M^q = (2i)^{n(1-n)/2 \cdot \binom{n+\ell-1}{n}} \exp\left(\pi i \sum_{m=1}^{n} z_m/p \cdot \binom{n+\ell-1}{n}\right)$$
$$\times \prod_{s=0}^{\ell-1} \prod_{1 \leq l < m \leq n} \sin(\pi(z_l - \Lambda_l - z_m - \Lambda_m + s)/p)^{\binom{n+\ell-s-2}{n-1}},$$

$$\det \mathring{M} = (2i)^{(1-n)(n-2)/2 \cdot \binom{n+\ell-2}{n-1}}$$
$$\times \prod_{s=0}^{\ell-1} \prod_{1 \leq l < m \leq n} \sin(\pi(z_l - \Lambda_l - z_m - \Lambda_m + s)/p)^{\binom{n+\ell-s-3}{n-2}}.$$

Proof. The first and the second formulae are equivalent to Lemmas 5.2 and 2.2 in [T], respectively. The third formula can be reduced to the second one by a suitable change of variables. □

Lemmas 2.20, 2.28, 2.29 clearly follow from Lemma 9.1. □

Proof of Lemma 2.21. The right hand side of formula (2.21) can be rewritten as

(9.2)
$$\prod_{m=1}^{n} \prod_{s=0}^{\mathfrak{l}_m - 1} (2\Lambda_m - s)/p \sum_{\sigma \in S^{\ell}} \left[\left(\prod_{m=1}^{n} \frac{t_1 - z_m + \Lambda_m}{t_1 - z_m - \Lambda_m} \prod_{a=2}^{\ell} \frac{t_1 - t_a - 1}{t_1 - t_a + 1} - 1 \right) \times \prod_{a \in \Gamma'_m} \left(\frac{1}{t_a - z_m - \Lambda_m} \prod_{1 \leq l < m} \frac{t_a - z_l + \Lambda_l}{t_a - z_l - \Lambda_l} \right) \right]_\sigma$$

where $\Gamma'_m = \{2 + \mathfrak{l}^{m-1}, \ldots, \mathfrak{l}^m, 1 + \mathfrak{l}^m\}$, $m = 1, \ldots, n$. Set

$$f_m(t,z) = \left(\frac{t_1 - z_m + \Lambda_m}{t_1 - z_m - \Lambda_m} \prod_{a \in \Gamma'_m} \frac{t_1 - t_a - 1}{t_1 - t_a + 1} - 1 \right) \times \prod_{1 \leq l < m} \left[\frac{t_1 - z_l + \Lambda_l}{t_1 - z_l - \Lambda_l} \prod_{a \in \Gamma'_l} \frac{t_1 - t_a - 1}{t_1 - t_a + 1} \right],$$

so that

$$\sum_{m=1}^{n} f_m(t,z) = \prod_{m=1}^{n} \frac{t_1 - z_m + \Lambda_m}{t_1 - z_m - \Lambda_m} \prod_{a=2}^{\ell} \frac{t_1 - t_a - 1}{t_1 - t_a + 1} - 1,$$

and expression (9.2) equals

$$\sum_{k=1}^{n}\left\{\prod_{m=1}^{n}\prod_{s=0}^{\mathfrak{l}_m-1}(2\Lambda_m-s)/p\right.$$
$$\left.\times\sum_{\sigma\in\mathbf{S}^{\mathfrak{l}}}\left[f_k(t,z)\prod_{m=1}^{n}\prod_{a\in\Gamma'_m}\left(\frac{1}{t_a-z_m-\Lambda_m}\prod_{1\leq l<m}\frac{t_a-z_l+\Lambda_l}{t_a-z_l-\Lambda_l}\right)\right]_\sigma\right\}.$$

Lemma 2.21 now follows from the formulae

$$(\mathfrak{l}_k+1)(2\Lambda_k-\mathfrak{l}_k)w_{\mathfrak{l}+\mathfrak{e}(k)}$$
$$=\sum_{\sigma\in\mathbf{S}^{\mathfrak{l}}}\left[f_k(t,z)\prod_{m=1}^{n}\frac{1}{\mathfrak{l}_m!}\prod_{a\in\Gamma'_m}\left(\frac{1}{t_a-z_m-\Lambda_m}\prod_{1\leq l<m}\frac{t_a-z_l+\Lambda_l}{t_a-z_l-\Lambda_l}\right)\right]_\sigma$$

(cf. (2.19)). □

Lemmas 4.6, 4.18 follow from formulae (A.3), (A.5) in [IK], respectively, and the definitions of the evaluation modules, by induction with respect to the number of factors of the tensor products.

Theorems 4.7, 4.19 follow from formulae (A.5)–(A.8) in [Ko] and Lemmas 4.26, 4.18, respectively.

Lemmas 4.23, 4.24 follow from formulae (2.19), (2.26) for the rational and trigonometric weight functions, respectively, and Lemma 9.1.

The claims of Theorems 4.25, 4.26 that the maps $\chi\circ P$ and χ_q are intertwiners can be verified directly from formulae (4.1) and (4.14), respectively, though the calculations are cumbersome. These claims also follow from Theorems 4.7, 4.19, respectively. The claims that the maps $\chi\circ P$ and χ_q are isomorphisms follow from Lemmas 4.23, 4.24, respectively.

The proof of Theorems 5.7 and 5.8 are based on the following simple lemma.

(9.3) Lemma. *Consider a configuration of hyperplanes in* $\mathbb{C}^\ell$

$$t_a=z_m\pm\Lambda_m+sp,\qquad t_a=t_b\pm 1+sp\,,$$

$1\leqq b<a\leqq\ell$, $m=1,\dots,n$, $s\in\mathbb{Z}$. *The dimensions of all edges of the configuration do not depend on* p, $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ *provided that assumptions* (2.12)–(2.14) *hold.*

Proof. The initial configuration of hyperplanes induces a configuration of hyperplanes in any edge of the initial configuration. The dimensions of all edges of the initial configuration remain the same if and only if all the induced configurations do not have nonstandard coinciding hyperplanes. This is obviously true if assumptions (2.12)–(2.14) hold. □

This lemma implies that the topology of the complement of configuration (9.3) of hyperplanes in $\mathbb{C}^\ell$ remains the same for all p, $\Lambda_1,\dots,\Lambda_n$ and $z_1,\dots,z_n$ satisfying conditions (2.12)–(2.14).

Proof of Theorem 5.7. The theorem is proved by induction with respect to the number of integration variables in the hypergeometric integral.

Recall that the hypergeometric integral $I(W_\mathrm{l}, w_m)$ is defined by

$$I(W_\mathrm{l}, w_\mathrm{m}) = \int_{\mathrm{I\!I}^\ell} \Phi(t) w_\mathrm{m}(t) W_\mathrm{l}(t)\, d^\ell t\,,$$

if $\Lambda_1,\dots,\Lambda_n$ are large negative. (cf. (5.3)). We can replace the imaginary subspace $\mathrm{I\!I}^\ell$ in the last formula by any subspace of the form

$$\mathrm{I\!I}_x = \{t \in \mathbb{C}^\ell \mid \operatorname{Re} t_a = x_a,\ a = 1,\dots,\ell\}\,, \tag{9.4}$$

where $x_1,\dots,x_\ell$ are small pairwise distinct real numbers without changing the integral.

For the analytic continuation we move $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ and p and preserve the integration contour $\mathrm{I\!I}_x$ as long as it does not touch the hyperplanes of configuration (5.6). If a hyperplane Π of configuration (5.6) goes through the integration contour $\mathrm{I\!I}_x$, the integration contour should be deformed to avoid the intersection. Deforming the integration contour we add a tube over the intersection of $\mathrm{I\!I}_x$ and the hyperplane Π. The result of the deformation is the sum $\mathrm{I\!I}_x + \mathrm{I\!I}'_x \times C_\Pi$, where $\mathrm{I\!I}'_x \subset \Pi$ is a suitable subspace of real dimension $(\ell - 1)$, and C_Π is a small circle around the hyperplane Π. For example, if Π is given by an equation $t_\ell - z_n - \Lambda_n - ps = 0$, then Π has coordinates $t_1,\dots,t_{\ell-1}$, in these coordinates

$$\mathrm{I\!I}'_x = \{t \in \Pi \mid \operatorname{Re} t_a = x_a,\ a = 1,\dots,\ell - 1\}\,.$$

and the circle C_Π is given by

$$C_\Pi = \{t_\ell \in \mathbb{C} \mid |t_\ell - z_n - \Lambda_n - ps| = \rho\}\,,$$

ρ is a small positive number. The analytic continuation of the initial hypergeometric integral $I(W_\mathrm{l}, w_\mathrm{m})$ equals the sum of two integrals

$$\int_{\mathrm{I\!I}_x} \Phi(t) w_\mathrm{m}(t) W_\mathrm{l}(t)\, d^\ell t + \int_{\mathrm{I\!I}'_x} \operatorname*{Res}_{t\in\Pi} (\Phi(t) w_\mathrm{m}(t) W_\mathrm{l}(t))\, d^{\ell-1} t\,,$$

and the second integral is of the same type as the first one but of a smaller dimension. Therefore, under the analytic continuation the passage of a hyperplane of the configuration through the integration contour results in appearance of a new hypergeometric integral with a smaller number of integrations. This reason shows that the hypergeometric integral can be analytically continued to the region described in the Theorem 5.7.

Now we show the univaluedness of the hypergeometric integral by induction on the number of integration variables. Denote the domain described in Theorem 5.7 by U. Consider its fundamental group $\pi_1(U, z^*)$. The generators of the fundamental group can be chosen of a special form. Namely, for any hyperplane Π lying at the boundary of the domain U choose a curve α_Π in U from the base point z^* to a generic point z_Π of the hyperplane Π and fix a loop γ_Π in U which goes from z^* to Π along α_Π, then turns around Π along a small circle β_Π and returns back to z^* along the same curve α_Π. The loops γ_Π generate the fundamental group.

Let us show that the hypergeometric integral $I(W_{\mathrm{I}}, w_{\mathrm{m}})$ has the trivial monodromy under the analytic continuation along the curve γ_Π. In fact, under the analytic continuation from the base point z^* to the hyperplane Π along the curve α_Π we create smaller dimensional integrals each time one of the hyperplanes of singularities hits the integration contour. Under the analytic continuation of the integral along the circle β_Π the hyperplanes of singularities do not touch the integration contour if the point z_Π is generic. Now under the analytic continuation along the curve α_Π from Π to z^* we create again smaller dimensional integrals each time one of the hyperplanes of singularities hits the integration contour. But the corresponding integrals created on the way to Π and on the way from Π come with the opposite signs. Moreover, they are equal according to the induction assumptions. Hence the monodromy of the integral along the loop γ_Π is trivial. Theorem 5.7 is proved. □

The proof of Theorem 5.8 is similar to the proof of Theorem 5.7.

As in the case $\ell = 1$ we extend the notion of the hypergeometric integral $I(W, w)$ and consider the hypergeometric integral for any function w in the functional space $\widehat{\mathscr{F}}(z)$ of a fiber. Namely, let $w(t,z) \in \widehat{\mathscr{F}}(z)$ be a function of the form

$$P(t_1,\dots,t_\ell,z_1,\dots,z_n,\Lambda_1,\dots,\Lambda_n)$$
$$\times \prod_{s=0}^{r}\left[\prod_{m=1}^{n}\prod_{a=1}^{\ell}\frac{1}{(t_a - z_m - \Lambda_m + sp)(t_a - z_m + \Lambda_m - (s+1)p)}\right.$$
$$\left.\times \prod_{1\le a<b\le \ell}\frac{1}{(t_a - t_b + 1 + sp)(t_a - t_b - 1 - (s+1)p)}\right]$$

where P is a polynomial. If the real parts of the weights $\Lambda_1,\dots,\Lambda_n$ are large negative and p is small, then we define the hypergeometric integral $I(W,w)$ by (5.3). The hypergeometric integral is well defined if either $0 < \operatorname{Im}\mu < 2\pi$ and $W \in \mathscr{F}_q(z)$ or $\operatorname{Im}\mu = 0$ and $W \in \mathscr{F}_q^{\mathrm{sing}}(z)$, since the integrand exponentially vanishes at infinity. For generic $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ and p we define respectively the hypergeometric integral $I(W_{\mathrm{I}}, w)$ or the hypergeometric integral $I(\mathring{W}_{\mathrm{I}}, w)$ by analytic continuation with respect to $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ and p. Similar to the proof of Theorem 5.7 one can show that these hypergeometric integrals can be analytically continued as holomorphic univalued functions of complex variables p, $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$ to the region described in Theorem 5.7. For arbitrary functions $w \in \widehat{\mathscr{F}}(z)$, $W \in \mathscr{F}_q(z)$ we define the hypergeometric integral by linearity.

Let $D\widehat{\mathscr{F}}(z) = \{Dw \mid w \in \widehat{\mathscr{F}}(z)\}$.

(9.5) Lemma. *Let $p<0$. Let* (2.12)–(2.14) *hold. Let either* $0<\operatorname{Im}\mu<2\pi$ *and* $W\in\mathscr{F}_q(z)$ *or* $\mu=0$ *and* $W\in\mathscr{F}_q^{\text{sing}}(z)$. *Then*

a) *The hypergeometric integral $I(W,w)$ is well defined for any function* $w\in\widehat{\mathscr{F}}(z)$.

b) *The hypergeometric integral $I(W,w)$ equals zero for any function* $w\in D\widehat{\mathscr{F}}(z)$.

Proof. Claim a) holds by the definition of the hypergeometric integral $I(W,w)$. Claim b) is clear, if the real parts of the weights $\Lambda_1,\dots,\Lambda_n$ are large negative and p is small. Then the analytic continuation of the integral gives Claim b) for generic p, $\Lambda_1,\dots,\Lambda_n$, $z_1,\dots,z_n$. □

Lemma 5.9 follows from Lemmas 2.21 and 9.5.

The hypergeometric integral defines linear functionals $I(W,\cdot)$ on the functional space of a fiber. Lemma 9.5 means that these linear functionals can be considered as elements of the top homology group $H_\ell(z)$, the dual space to the top cohomology group of the de Rham complex of the discrete local system of the fiber.

Proof of Theorem 5.18. The section s_W is defined by

$$s_W(z)=I(W|_z,\cdot)$$

where $W|_z$ denotes the restriction of the function $W(t,z)$ to the fiber over z. The theorem is a direct corollary of the periodicity of the function W with respect to each of the variables $z_1,\dots,z_n$:

$$W(t,z_1,\dots,z_m+p,\dots,z_n)=W(t,z_1,\dots,z_n),\quad m=1,\dots,n\,,$$

cf. the case $\ell=1$ in Sect. 8. □

Our further strategy is the same as in the case $\ell=1$. First we prove Theorem 6.7 which imply Theorem 6.4. Using Theorem 6.4 we prove that the hypergeometric pairing $I:\mathscr{F}_q(z)\otimes\mathscr{F}(z)\to\mathbb{C}$ is nondegenerate if $0<\operatorname{Im}\mu<2\pi$ (cf. Theorem 5.14). Studying the asymptotic behaviour of the hypergeometric integral as μ tends to zero we show that for $\mu=0$ the hypergeometric pairing $I^\circ:\mathscr{F}_q^{\text{sing}}(z)\otimes\mathscr{F}(z)/\mathscr{R}(z)\to\mathbb{C}$ is nondegenerate (cf. Theorem 5.15).

Theorems 2.15 and 2.17 will follow from Theorems 5.14 and 5.15, respectively, and Lemma 9.5.

Proof of Theorem 6.7. To simplify notations we will give a proof only for the case $k=n$, so that $n_m=m$, $m=1,\dots,n$. The general case is similar.

Let $w_{(l)}^{(m)}\in\mathscr{F}[z_m;\Lambda_m;l]$ and $W_{(l)}^{(m)}\in\mathscr{F}_q[z_m;\Lambda_m;l]$ be the following functions:

$$(9.6)\qquad w_{(l)}^{(m)}(t_1,\dots,t_l,z_m)=\frac{1}{l!}\sum_{\sigma\in S^l}\left[\prod_{a=1}^{l}\frac{1}{t_a-z_m-\Lambda_m}\right]_\sigma,$$

$$W_{(l)}^{(m)}(t_1,\dots,t_l,z_m)=\prod_{s=1}^{l}\frac{\sin(\pi/p)}{\sin(\pi s/p)}\sum_{\sigma\in S^l}\left[\!\left[\prod_{a=1}^{l}\frac{\exp(\pi i(z_m-t_a)/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)}\right]\!\right]_\sigma,$$

(cf. (2.19), (2.26)). We have the equalities

$$w_{\mathfrak{l}} = w^{(1)}_{(\mathfrak{l}_1)} \star \cdots \star w^{(n)}_{(\mathfrak{l}_n)} \quad \text{and} \quad W_{\mathfrak{l}} = W^{(1)}_{(\mathfrak{l}_1)} * \cdots * W^{(n)}_{(\mathfrak{l}_n)}$$

Therefore, we have to study the asymptotics of the hypergeometric integrals $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$.

Consider the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$. Due to property (2.7) all the terms in the Definition (2.26) of the function $W_{\mathfrak{l}}$ give the same contribution to the integral. So we can replace the integrand $\Phi(t) w_{\mathfrak{m}}(t) W_{\mathfrak{l}}(t)$ by the following integrand

$$\begin{aligned} F(t) = \pi^{-\ell} \ell! w_{\mathfrak{m}}(t) \exp\Bigl(\pi i \sum_{m=1}^{n} \mathfrak{l}_m z_m / p\Bigr) & \prod_{1 \leqq a < b \leqq \ell} \frac{\Gamma((t_a - t_b - 1)/p)}{((t_a - t_b + 1)/p)} \\ \times \prod_{m=1}^{n} \Biggl[\prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} & \prod_{a \in \Gamma_m} \exp((\mu - \pi i) t_a / p) \Gamma((t_a - z_m + \Lambda_m)/p) \\ \times \Gamma(1 - (t_a - z_m - \Lambda_m)/p) & \prod_{1 \leqq l < m} \frac{\Gamma(1 - (t_a - z_l - \Lambda_l)/p)}{\Gamma(1 - (t_a - z_l + \Lambda_l)/p)} \\ \times \prod_{m < l \leqq n} \frac{\Gamma((t_a - z_l + \Lambda_l)/p)}{\Gamma((t_a - z_l - \Lambda_l)/p)} \Biggr] & d^\ell t , \end{aligned}$$

where $\Gamma_m = \{1 + \mathfrak{l}^{m-1}, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$.

Assume that the real parts of the weights $\Lambda_1, \dots, \Lambda_n$ are negative. If all $z_1, \dots, z_n$ are imaginary, then we have

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \int_{\mathbb{I}^\ell} F(t)\, d^\ell t .$$

The analytic continuation of $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ to the region $\operatorname{Re} z_1 < \cdots < \operatorname{Re} z_n$ is given by

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \int_{\mathbb{I}_1^{\mathfrak{l}_1} \times \cdots \times \mathbb{I}_n^{\mathfrak{l}_n}} F(t)\, d^\ell t$$

where

$$\mathbb{I}_m^{\mathfrak{l}_m} = \{ (t_{1+\mathfrak{l}^{m-1}}, \dots, t_{\mathfrak{l}^m}) \in \mathbb{C}^{\mathfrak{l}_m} \mid \operatorname{Re} t_a = \operatorname{Re} z_m,\ \mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m \}$$

since the integrand has no poles at the hyperplanes $t_a = z_l - \Lambda_l - sp$, $s \in \mathbb{Z}$, for $\mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m$, $m > l$, has no poles at the hyperplanes $t_a = z_l + \Lambda_l + sp$, $s \in \mathbb{Z}$, for $\mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m$, $m < l$, and has no poles at the hyperplanes $t_a = t_b + 1 + sp$, $s \in \mathbb{Z}$, for $a > b$.

Let $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$ so that $\operatorname{Re}(z_m - z_{m+1}) \to -\infty$ for all $m = 1, \dots, n-1$. Consider the case $\mathfrak{l} = \mathfrak{m}$. Transform the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{l}})$ as above and replace the integrand by its asymptotics as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$. Since

$$w_{\mathfrak{l}}(t_1, \dots, t_\ell) = \prod_{m=1}^{n} w^{(m)}_{(\mathfrak{l}_m)}(t_{\mathfrak{l}^{m-1}+1}, \dots, t_{\mathfrak{l}_m}) + o(1)$$

as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$ and $t \in \mathbb{I}_1^{\mathfrak{l}_1} \times \cdots \times \mathbb{I}_n^{\mathfrak{l}_n}$, we obtain that

$$I(W_{\mathfrak{l}}, w_{\mathfrak{l}}) = \pi^{-\ell}\ell! \exp\left(\pi i \sum_{m=1}^{n} \mathfrak{l}_m z_m\right) \prod_{1 \leqq l < m \leqq n} ((z_l - z_m)/p)^{2(\mathfrak{l}_l \Lambda_m + \mathfrak{l}_m \Lambda_l - \mathfrak{l}_l \mathfrak{l}_m)/p}$$

$$\times \prod_{m=1}^{n} \left[\prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \int_{\mathbb{I}_m^{\mathfrak{l}_m}} w_{(\mathfrak{l}_m)}^{(m)}(t_{\mathfrak{l}^{m-1}+1}, \dots, t_{\mathfrak{l}^m}) \prod_{a \in \Gamma_m} \left(\exp((\mu - \pi i)t_a/p) \right.\right.$$

$$\times \Gamma((t_a - z_m + \Lambda_m)/p)\Gamma(1 - (t_a - z_m - \Lambda_m)/p)$$

$$\left.\left. \times \prod_{\substack{b<a \\ b \in \Gamma_m}} \frac{\Gamma((t_b - t_a - 1)/p)}{\Gamma((t_b - t_a + 1)/p)} \right) d^{\mathfrak{l}_m}t \right] (1 + o(1)),$$

as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$. Here $|\arg((z_l - z_m)/p)| < \pi$. Due to (2.7) the integrals over $\mathbb{I}_m^{\mathfrak{l}_m}$ are the hypergeometric integrals $I(W_{(\mathfrak{l}_m)}^{(m)}, w_{(\mathfrak{l}_m)}^{(m)})$ up to simple factors. Hence, we finally obtain that

$$I(W_{\mathfrak{l}}, w_{\mathfrak{l}}) = \frac{\ell!}{\mathfrak{l}_1! \cdots \mathfrak{l}_n!} \prod_{1 \leqq l < m \leqq n} ((z_l - z_m)/p)^{2(\mathfrak{l}_l \Lambda_m + \mathfrak{l}_m \Lambda_l - \mathfrak{l}_l \mathfrak{l}_m)/p}$$

$$\times \left(\prod_{m=1}^{n} I(W_{(\mathfrak{l}_m)}^{(m)}, w_{(\mathfrak{l}_m)}^{(m)}) + o(1) \right).$$

The hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ for $\mathfrak{l} \neq \mathfrak{m}$ can be treated similarly to the hypergeometric integral $I(W_{\mathfrak{l}}, w_{\mathfrak{l}})$ considered above. The final answer is

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = I(W_{\mathfrak{l}}, w_{\mathfrak{l}})o(1),$$

which completes the proof if the real parts of the weights $\Lambda_1, \dots, \Lambda_n$ are negative.

For general $\Lambda_1, \dots, \Lambda_n$ the proof is similar. The analytic continuation of $I(W_{\mathfrak{l}}, w_{\mathfrak{m}})$ to the region $\operatorname{Re} z_1 \ll \cdots \ll \operatorname{Re} z_n$ is given by

$$I(W_{\mathfrak{l}}, w_{\mathfrak{m}}) = \int_{\widetilde{\mathbb{I}}_1^{\mathfrak{l}_1} \times \cdots \times \widetilde{\mathbb{I}}_n^{\mathfrak{l}_n}} F(t)\, d^{\ell}t$$

where $\widetilde{\mathbb{I}}_m^{\mathfrak{l}_m}$ is the respective deformation of $\mathbb{I}_m^{\mathfrak{l}_m}$. On every contour $\widetilde{\mathbb{I}}_m^{\mathfrak{l}_m}$ the quantities $\operatorname{Re}(t_a - z_m)$ remain bounded as $z \to \infty$ in $\mathbb{A}_{\mathrm{id}}$ for all a such that $\mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m$, and the rest part of the proof remains the same as before.

Theorem 6.7 is proved. □

Further in the proofs we will make use of the following identities:

(9.7) $$\sum_{\sigma \in \mathbf{S}^l} \prod_{1 \leqq j < k \leqq l} \frac{y_{\sigma_k} - \beta y_{\sigma_j}}{y_{\sigma_k} - y_{\sigma_j}} = \prod_{s=1}^{l} \frac{1 - \beta^s}{1 - \beta},$$

(9.8)

$$\sum_{\sigma\in S^l}\left(\prod_{k=1}^{l}\frac{1}{y_{\sigma_k}-\beta y_{\sigma_{k-1}}}\prod_{1\leqq j<k\leqq l}\frac{y_{\sigma_k}-\beta y_{\sigma_j}}{y_{\sigma_k}-y_{\sigma_j}}\right)=\prod_{k=1}^{l}\frac{1}{y_k-\beta y_0}\,,\qquad \sigma_0=0\,.$$

Proof of formula (5.17). Consider the integral in the left hand side of (5.17) as a function in u and denote it by $F(u)$. We will show that $F(u)$ satisfies a differential equation

$$(u+u^{-1})\frac{d}{du}F(u)=\ell(2a+(\ell-1)x)(1-u^{-2})F(u)\,. \tag{9.9}$$

The right hand side of (5.17) solves the same differential equation. Therefore both sides are proportional. The proportionality coefficient equals 1 since, as it is shown below, both sides have the same asymptotics as $u\to+0$.

Denote by $f(u;s_1,\dots,s_\ell)$ the integrand of integral (5.17):

$$f(u;s_1,\dots,s_\ell)=\prod_{k=1}^{\ell}u^{2s_k}\Gamma(a+s_k)\Gamma(a-s_k)\prod_{\substack{j,k=1\\ j\neq k}}^{\ell}\frac{\Gamma(s_k-s_j+x)}{\Gamma(s_k-s_j)}$$

so that

$$F(u)=\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}f(u;s_1,\dots,s_\ell)\,d^\ell s\,. \tag{9.10}$$

Differentiating the integral with respect to u and using the identity

$$\ell(a+(\ell-1)x/2)+\sum_{k=1}^{\ell}s_k=\sum_{k=1}^{\ell}(a+s_k)\prod_{\substack{j=1\\ j\neq k}}^{\ell}\frac{s_k-s_j+x}{s_k-s_j}$$

we obtain that

$$\left(\ell(2a+(\ell-1)x)+u\frac{d}{du}\right)F(u)=2\sum_{k=1}^{\ell}F_k(u)$$

where

$$F_k(u)=\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}(a+s_k)\prod_{\substack{j=1\\ j\neq k}}^{\ell}\frac{s_k-s_j+x}{s_k-s_j}f(u;s_1,\dots,s_\ell)\,d^\ell s\,.$$

The space $\{(s_1,\dots,s_\ell)\in\mathbb{C}^\ell\mid \operatorname{Re}s_k=1,\ \operatorname{Re}s_j=0,\ j\neq k\}$ is homologous to the imaginary space in the complement of the poles of the integrand for $F_k(u)$. Therefore, changing the k-th integration variable $s_k\to s_k-1$ and using the functional equation for the gamma-function we obtain that

$$F_k(u)=u^{-2}\int_{-i\infty}^{i\infty}\cdots\int_{-i\infty}^{i\infty}(a-s_k)\prod_{\substack{j=1\\ j\neq k}}^{\ell}\frac{s_j-s_k+x}{s_j-s_k}f(u;s_1,\dots,s_\ell)\,d^\ell s\,.$$

Now the identity

$$\ell(a+(\ell-1)x/2)-\sum_{k=1}^{\ell} s_k=\sum_{k=1}^{\ell}(a-s_k)\prod_{\substack{j=1\\ j\neq k}}^{\ell}\frac{s_j-s_k+x}{s_j-s_k}$$

implies that

$$2\sum_{k=1}^{\ell}F_k(u)=u^{-2}\left(\ell(2a+(\ell-1)x)-u\frac{d}{du}\right)F(u),$$

which complete the proof of equality (9.9).

To compute asymptotics of the integral (5.17) as $u\to+0$ we first suitably transform the integrand. Taking identity (9.8) for $l=\ell$, $\beta=\exp(2\pi ix)$ and $y_k=\exp(2\pi is_k)$, $k=0,\dots,\ell$, we obtain that

$$\begin{aligned}1&=\exp(\pi i\ell(\ell-1)x/2)\prod_{k=1}^{\ell}\sin(\pi(s_k-a))\\&\times\sum_{\sigma\in S^{\ell}}\left(\prod_{k=1}^{\ell}\frac{\exp(\pi i(a-s_{\sigma_{k-1}}-x))}{\sin(\pi(s_{\sigma_k}-s_{\sigma_{k-1}}-x))}\prod_{1\leqq j<k\leqq\ell}\frac{\sin(\pi(s_{\sigma_k}-s_{\sigma_j}-x))}{\sin(\pi(s_{\sigma_k}-s_{\sigma_j}))}\right)\end{aligned}$$

where $\sigma_0=0$, $s_0=a-x$. Substitute the right hand side of the identity above into the integral (9.10). Since $f(u;s_1,\dots,s_\ell)$ is a symmetric function in the variables $s_1,\dots,s_\ell$ and the imaginary space is invariant under permutations of $s_1,\dots,s_\ell$, we can keep in the integral only one term of the sum, multiplying then the result by $\ell!$. Taking the term corresponding to the identity permutation we obtain that

$$F(u)=\int_{\mathbb{I}_0^{\ell}} g(u;s_1,\dots,s_\ell)\,d^{\ell}s \tag{9.11}$$

where

$$\begin{aligned}g(u;s_1,\dots,s_\ell)&=\pi^{\ell}\ell!\exp(\pi i\ell(\ell-1)x/2)\\&\times\prod_{k=1}^{\ell}\left(u^{2s_k}\frac{\Gamma(a+s_k)}{\Gamma(1+s_k-a)}\frac{\exp(\pi i(a-s_{k-1}-x))}{\sin(\pi(s_{k-1}-s_k+x))}\right.\\&\left.\times\prod_{j=1}^{k-1}\frac{(s_k-s_j)\Gamma(s_k-s_j+x)}{\Gamma(1+s_k-s_j-x)}\right).\end{aligned}$$

Here we use a notation

$$\mathbb{I}_y^{\ell}=\{(s_1,\dots,s_\ell)\in\mathbb{C}^{\ell}\,|\,\operatorname{Re}s_k=y,\ k=1,\dots,\ell\}.$$

To compute the proportionality coefficient which we are interested in, it suffices to study the asymptotics of $F(u)$ as $u\to+0$ only for small real x and real a because both sides of (5.17) are analytic functions in x and a. Moreover, we can assume that u is real. To find the asymptotics of $F(u)$ we deform continuously the integration contour in integral (9.11). Namely we replace there $\mathbb{I}_0^\ell$ by $\mathbb{I}_y^\ell$ and move y from zero to the positive direction.

Since $\operatorname{Re} a > 0$ and $\operatorname{Re} x > 0$, there are no obstacles for the deformation of the integration contour until y becomes equal to a. At this moment the integration contour touches the singularity hyperplane of the integrand given by $s_1 = a$. The next intersection of $\mathbb{I}_y^\ell$ with a singularity hyperplane of the integrand appears at $y = a + 1$ with the hyperplane $s_1 = a + 1$. Therefore, we have

$$F(u) = -2\pi i \int_{\mathbb{I}_a^{\ell-1}} \operatorname*{Res}_{s_1=a} g(u; s_1, \dots, s_\ell)\, d^{\ell-1}s + \int_{\mathbb{I}_{a+\delta}^\ell} g(u; s_1, \dots, s_\ell)\, d^\ell s\,,$$

where $0 < \delta < 1$ and $\mathbb{I}_y^{\ell-1} = \{(s_2, \dots, s_\ell) \in \mathbb{C}^{\ell-1} \mid \operatorname{Re} s_k = y,\ k = 2, \dots, \ell\}$. We can estimate the second term from above

$$\left| \int_{\mathbb{I}_{a+\delta}^\ell} g(u; s_1, \dots, s_\ell)\, d^\ell s \right| \leqq u^{a+\delta} \int_{\mathbb{I}_{a+\delta}^\ell} g(1; s_1, \dots, s_\ell)\, d^\ell s\,.$$

In the first term we continue the deformation of the integration contour; we replace there $\mathbb{I}_a^{\ell-1}$ by $\mathbb{I}_y^{\ell-1}$ and move y from a to the positive direction. The first obstacle to the deformation appears at $y = a + x$; at this moment $\mathbb{I}_y^{\ell-1}$ touches the hyperplane $s_2 = a + x$. Repeating the consideration ℓ times we finally obtain the following asymptotics for $F(u)$ as $u \to +0$:

$$F(u) = (-2\pi i)^\ell \operatorname*{Res}_{a \lhd x} g(u; s_1, \dots, s_\ell)(1 + o(1))\,,$$

where $\operatorname{Res}_{a \lhd x}$ means the residue at the point $s_k = a + (k-1)x$, $k = 1, \dots, \ell$. Calculating the residue explicitly we find that

$$F(u) = (2\pi i)^\ell u^{\ell(2a + (\ell-1)x)} \prod_{k=1}^{\ell} \frac{\Gamma(1 + kx)}{\Gamma(1 + x)} \Gamma(2a + (k-1)x)(1 + o(1))\,,$$

as $u \to +0$, which clearly coincide with the asymptotics of the right hand side of (5.17). This means that the proportionality coefficient between $F(u)$ and the right hand side of (5.17) equals 1. Formula (5.17) is proved. □

Proof of Theorem 5.14. The proof is similar to the proof of Theorem 8.33. Since both sides of formula (5.14) are analytic functions in $\Lambda_1, \dots, \Lambda_n$, it suffices to prove the formula only for real negative $\Lambda_1, \dots, \Lambda_n$.

Denote by $F(z)$ the determinant $\det[I(W_{\mathfrak{l}}, w_{\mathfrak{m}})]_{\mathfrak{l},\mathfrak{m} \in \mathcal{Z}_\ell^n}$ and by $G(z)$ the right hand side of (5.14). Since for every $\mathfrak{l} \in \mathcal{Z}_\ell^n$ the section $\Psi_{W_{\mathfrak{l}}}$ is a $(V_1 \otimes \dots \otimes V_n)_\ell$-valued solution to the qKZ equation, $F(z)$ solves the next system of difference equations

$$F(z_1, \dots, z_m + p, \dots, z_n) = \overset{(\ell)}{\det}\, K_m(z_1, \dots, z_n) F(z_1, \dots, z_n)\,.$$

Here $\overset{(\ell)}{\det} K_m(z)$ stays for the determinant of the operator $K_m(z)$ (3.10) acting in the weight subspace $(V_1 \otimes \cdots \otimes V_n)_\ell$. Using (3.5) we see that

$$\overset{(\ell)}{\det} K_m(z_1,\ldots,z_n) = \kappa^{\binom{n+\ell-1}{n}} \prod_{s=0}^{\ell-1} \left[\prod_{1 \leq l < m} \frac{z_m + \Lambda_m - z_l + \Lambda_l + s + p}{z_m - \Lambda_m - z_l - \Lambda_l + s + p} \right.$$
$$\left. \times \prod_{m < l \leq n} \frac{z_m + \Lambda_m - z_l + \Lambda_l - s}{z_m - \Lambda_m - z_l - \Lambda_l - s} \right]^{\binom{n+\ell-s-2}{n-1}} .$$

Therefore, the ratio $F(z)/G(z)$ is a p-periodic function in each of the variables $z_1,\ldots,z_n$:

$$\frac{F}{G}(z_1,\ldots,z_m + p,\ldots,z_n) = \frac{F}{G}(z_1,\ldots,z_n) .$$

Theorem 6.4 implies that the ratio $F(z)/G(z)$ tends to 1 as z tends to infinity in the asymptotic zone $\mathbb{A}_{\mathrm{id}}$. Hence, this ratio equals 1 identically, which completes the proof. □

Proof of Theorem 5.15. The idea of the proof is the same as in the case $\ell = 1$. Since both sides (5.15) are analytic functions in $\Lambda_1,\ldots,\Lambda_n$ and p, it suffices to prove the formula for large negative $\Lambda_1,\ldots,\Lambda_n$ and large negative p. More precisely, we assume that

$$0 < 2\sum_{m=1}^{n} \Lambda_m/p - \ell(\ell-1)/p < 1 . \tag{9.12}$$

We will construct certain bases in the rational and the trigonometric hypergeometric spaces of fibers such that in these bases the hypergeometric pairing has triangular asymptotics as $\mu \to 0$. Using this fact we will show that

$$\det[I(W_{\mathrm{l}}, w_{\mathrm{m}})]_{\mathrm{l},\mathrm{m}\in\mathscr{Z}_\ell^n} = \Xi \mu^{-2\sum_{m=1}^n \Lambda_m/p \cdot \binom{n+\ell-1}{n} + 2n/p \cdot \binom{n+\ell-1}{n+1}} \tag{9.13}$$
$$\times \prod_{k=1}^{\ell} \det[I(\mathring{W}_{\mathrm{i}}, w_{\mathrm{j}})]_{\mathrm{i},\mathrm{j}\in\mathscr{Z}_k^{n-1}} (1 + o(1)) ,$$

where $0 < \arg \mu < \pi$, $\mathrm{j} \in \mathscr{Z}_k^{n-1}$ is identified with $(\mathrm{j}, 0) \in \mathscr{Z}_k^n$, and Ξ is a constant given by

(9.14)
$$\Xi = (2i)^{\binom{n+\ell-1}{n}} \exp\left(\pi i \left(\sum_{m=1}^{n} \Lambda_m/p \cdot \binom{n+\ell-1}{n} - n/p \cdot \binom{n+\ell-1}{n+1} \right) \right)$$
$$\times \prod_{s=0}^{\ell-1} \left(\left[\frac{p(\ell-s)\Gamma(-(s+1)/p)}{(2\Lambda_n - s)\Gamma(-1/p)} \right]^{\binom{n+\ell-s-2}{n-1}} \right.$$
$$\left. \times \prod_{s<r\leq\ell} \Gamma\left(1 + 2\sum_{m=1}^{n} \Lambda_m/p(s+2-2r)/p \right)^{\binom{n+r-s-2}{n-2}} \right) .$$

Formulae (9.13) and (9.14) imply Theorem 5.15.

In the proof we use the $\mathfrak{sl}_2$-module structure in the rational hypergeometric Fock space which was defined in Sect. 4. We define functions $w_{\mathfrak{l}}, W_{\mathfrak{l}}$ for an arbitrary vector $\mathfrak{l} \in \mathbb{Z}^n_{\geqq 0}$ respectively by (2.19), (2.26), where we replace ℓ in the left hand sides by the sum $\mathfrak{l}_1 + \dots + \mathfrak{l}_n$. Similarly, we define functions $\mathring{W}_{\mathfrak{l}}$ for arbitrary vector $\mathfrak{l} \in \mathbb{Z}^{n-1}_{\geqq 0}$ by (2.27) replacing there ℓ in the left hand side by the sum $\mathfrak{l}_1 + \dots + \mathfrak{l}_{n-1}$.

For any vector $\mathfrak{l} = (\mathfrak{l}_1, \dots, \mathfrak{l}_n)$ set $\mathfrak{l}' = (\mathfrak{l}_1, \dots, \mathfrak{l}_{n-1}, 0)$ and $\mathfrak{l}^! = (\mathfrak{l}_1, \dots, \mathfrak{l}_{n-1})$.

The required bases in the rational and the trigonometric hypergeometric spaces of a fiber are given by functions $w'_{\mathfrak{l}}$, $\mathfrak{l} \in \mathscr{Z}^n_\ell$, and $W'_{\mathfrak{l}}$, $\mathfrak{l} \in \mathscr{Z}^n_\ell$, respectively. The functions $w'_{\mathfrak{l}}$ are defined by rule:

$$w'_{\mathfrak{l}} = F^{\mathfrak{l}_n} w_{\mathfrak{l}'} \,.$$

where F is the generator of $\mathfrak{sl}_2$ acting in the rational hypergeometric Fock space $\mathfrak{F}$ (cf. (4.3)). The functions $W'_{\mathfrak{l}}$ are given by

$$\begin{aligned}(9.15)\quad & W'_{\mathfrak{l}}(t_1, \dots, t_\ell) \\ &= \frac{1}{(\ell - \mathfrak{l}_n)!} \prod_{s=1}^{\mathfrak{l}_n} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \\ &\times \sum_{\sigma \in S^\ell} \Bigg[\!\!\Bigg[\mathring{W}_{\mathfrak{l}^!}(t_1, \dots, t_{\ell - \mathfrak{l}_n}) \prod_{\ell - \mathfrak{l}_n < a \leqq \ell} \left(\frac{\exp(\pi i (z_n - t_a)/p)}{\sin(\pi(t_a - z_n - \Lambda_n)/p)} \right. \\ &\qquad \left. \times \prod_{m=1}^{n-1} \frac{\sin(\pi(t_a - z_m + \Lambda_m)/p)}{\sin(\pi(t_a - z_m - \Lambda_m)/p)} \right) \Bigg]\!\!\Bigg]_\sigma \,,\end{aligned}$$

By Lemmas 2.20, 2.28, for any $z \in \mathbb{B}$ there are matrices $N(z), N^q(z)$ such that

$$w'_{\mathfrak{l}}(t,z) = \sum_{\mathfrak{m} \in \mathscr{Z}^n_\ell} N_{\mathfrak{l}\mathfrak{m}}(z) w_{\mathfrak{m}}(t,z), \quad \mathfrak{l} \in \mathscr{Z}^n_\ell \,,$$

$$W'_{\mathfrak{l}}(t,z) = \sum_{\mathfrak{m} \in \mathscr{Z}^n_\ell} N^q_{\mathfrak{l}\mathfrak{m}}(z) W_{\mathfrak{m}}(t,z), \quad \mathfrak{l} \in \mathscr{Z}^n_\ell \,.$$

(9.16) Lemma.

$$\det N(z) = \prod_{s=0}^{\ell-1} ((s+1)(2\Lambda_n - s))^{\binom{n+\ell-s-2}{n-1}} \,,$$

$$\det N^q(z) = \exp\left(-\pi i \sum_{m=1}^{n-1} \Lambda_m/p \cdot \binom{n+\ell-1}{n} + \pi i (n-1)/p \cdot \binom{n+\ell-1}{n+1} \right) \,.$$

Now we study asymptotics of the hypergeometric pairing as $\mu \to 0$. We will consider the total family of hypergeometric pairings $I : \mathscr{F}_q[l] \otimes \mathscr{F}[l] \to \mathbb{C}$, $l = 0, \dots, \ell$, not indicating explicitly dependence on l. Recall, that we assumed that $\Lambda_1, \dots, \Lambda_n$ are large negative.

The first observation is that for any $w \in \mathcal{F}[l]$, $W \in \mathcal{F}_q^{\text{sing}}[l]$ the hypergeometric integral $I(W, w)$ has a finite limit as $\mu \to 0$, cf. Lemma 5.5. Furthermore, for any $w \in \mathcal{F}[l-1]$ and $W \in \mathcal{F}_q^{\text{sing}}[l]$ we have

$$(9.17) \qquad I(W, Fw) = O(\mu)$$

as $\mu \to 0$, because $F(\mathcal{F}[l-1](z))$ is the coboundary subspace $\mathcal{R}[l](z) \subset \mathcal{F}[l](z)$, cf. (4.4) and Lemma 5.9. The asymptotic behaviour of the hypergeometric integral $I(W, w)$ for a general function W is described by the following lemma.

(9.18) Lemma. *Let $\Lambda_1, \dots, \Lambda_n$ be large negative. Assume that condition* (9.12) *hold. Then for any $\mathfrak{l} \in \mathcal{Z}_\ell^n$ and for any $w \in \mathcal{F}$ the hypergeometric integral $I(W_\mathfrak{l}', w)$ has the following asymptotics as $\mu \to 0$, $0 < \arg \mu < \pi$:*

$$I(W_\mathfrak{l}', w) = \Xi' \mu^{\mathfrak{l}_n(2\ell - \mathfrak{l}_n - 1 - 2\sum_{m=1}^n \Lambda_m)/p} I(\mathring{W}_{\mathfrak{l}'}, E^{\mathfrak{l}_n} w)(1 + o(1)) .$$

Here E is the generator of $\mathfrak{sl}_2$ acting in the rational hypergeometric Fock space $\mathfrak{F}$ (cf. (4.3))

$$\Xi' = \frac{(2i)^{\mathfrak{l}_n} \ell!}{(\ell - \mathfrak{l}_n)!} \exp(\pi i \mathfrak{l}_n(\Lambda_n - (\mathfrak{l}_n - 1)/2)/p) \times \prod_{s=0}^{\mathfrak{l}_n - 1} \left[\frac{\Gamma(-(s+1)/p)}{\Gamma(-1/p)} \Gamma\left(2 \sum_{m=1}^n \Lambda_m/p + (2\mathfrak{l}_n - 2\ell - s)/p\right)\right] .$$

To obtain the required formulae (9.13), (9.14), we also need the next lemma.

(9.19) Lemma. *Let $\mu = 0$. Then for any $k, l \in \mathbb{Z}_{\geqq 0}$ we have*

$$\det [I(\mathring{W}_\mathfrak{l}, E^k F^k w_\mathfrak{m})]_{\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_l^{n-1}} = \prod_{s=0}^{k-1} \left((s+1) \left(2 \sum_{m=1}^n \Lambda_m - 2l - s \right) \right)^{\binom{n+l-2}{n-2}} \det [I(\mathring{W}_\mathfrak{l}, w_\mathfrak{m})]_{\mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_l^{n-1}} .$$

Here we identify $\mathfrak{m} \in \mathcal{Z}_l^{n-1}$ and $(\mathfrak{m}, 0) \in \mathcal{Z}_l^n$.

Now we will complete the proof of Theorem 5.15 assuming that Lemmas 9.16, 9.18 and 9.19 hold, and then we will prove the lemmas.

Consider a matrix U with the entries

$$U_{\mathfrak{l}\mathfrak{m}} = I(W_\mathfrak{l}', w_\mathfrak{m}), \quad \mathfrak{l}, \mathfrak{m} \in \mathcal{Z}_\ell^n .$$

Lemmas 4.7, 9.18 and (9.17) imply that the matrix U has a block-triangular asymptotics as $\mu \to 0$, namely

$$U_{\mathfrak{l}\mathfrak{m}} = O(\mu^{\delta_{\mathfrak{l}_n}}), \quad \text{for } \mathfrak{l}_n \geqq \mathfrak{m}_n ,$$

and

$$U_{\mathfrak{l}\mathfrak{m}} = O(\mu^{1+\delta_{\mathfrak{l}_n}}), \quad \text{for } \mathfrak{l}_n < \mathfrak{m}_n ,$$

where $\delta_l = l(2\ell - l - 1 - 2\sum_{m=1}^{n} \Lambda_m)/p$. Therefore, we see that

$$\det U = O\left(\mu^{\sum_{l=0}^{\ell} d_l \delta_l}\right), \qquad d_l = \binom{n+\ell-l-2}{n-2},$$

as $\mu \longrightarrow 0$, where $d_0, \ldots, d_\ell$ are the dimensions of the diagonal blocks. Furthermore, the leading term of the asymptotics of $\det U$ is given by the product of the determinants of the diagonal blocks, which are described by Lemma 9.19. Finally, formula (9.13) follows from a simple relation

$$\det [I(W_{\mathfrak{l}}, w_{\mathfrak{m}}]_{\mathfrak{l},\mathfrak{m} \in \mathscr{Z}_\ell^n} = \frac{\det U}{\det N \ \det N^q}$$

and easy calculations. □

Proof of Lemma 9.16. The formula for $\det N$ is a corollary of Lemma 4.7.

To prove the formula for $\det N^q$, consider the points $y^{(\mathfrak{l})} \in \mathbb{C}^\ell$ defined below:

$$y_a^{(\mathfrak{l})} = z_m - \Lambda_m + \mathfrak{l}^m - a, \quad \mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m, \ a = 1, \ldots, \ell\,.$$

Recall that $\mathfrak{l}^m = \mathfrak{l}_1 + \cdots + \mathfrak{l}_m$, $m = 1, \ldots, n$. Let L and L' be the matrices with the entries

$$L_{\mathfrak{l}\mathfrak{m}} = W_{\mathfrak{l}}(y^{(\mathfrak{m})}), \qquad L'_{\mathfrak{l}\mathfrak{m}} = W'_{\mathfrak{l}}(y^{(\mathfrak{m})}), \quad \mathfrak{l}, \mathfrak{m} \in \mathscr{Z}_\ell^n\,,$$

respectively. The matrices L and L' are triangular with respect to the following lexicographical order in $\mathscr{Z}_\ell^n : \mathfrak{l} < \mathfrak{m}$ if $\mathfrak{l}_1 < \mathfrak{m}_1$ or $\mathfrak{l}_1 = \mathfrak{m}_1$, $\mathfrak{l}_2 < \mathfrak{m}_2$ etc. Namely, $L_{\mathfrak{l}\mathfrak{m}} = 0$ and $L'_{\mathfrak{l}\mathfrak{m}} = 0$ for $\mathfrak{l} < \mathfrak{m}$. Since $L'_{\mathfrak{l}\mathfrak{l}} = \exp(\pi i \mathfrak{l}_m((\mathfrak{l}_m - 1)/2 - \Lambda_m)/p)L_{\mathfrak{l}\mathfrak{l}}$ and $N = L'L^{-1}$, the formula for $\det N^q$ is proved.

Proof of Lemma 9.19. By Theorem 4.7 the rational hypergeometric module is isomorphic to $(V_1 \otimes \cdots \otimes V_n)^*$. We also have $I(W, Fw) = 0$ for any $w \in \mathscr{F}[l-1]$ and $W \in \mathscr{F}_q^{\text{sing}}[l]$. Therefore, the coefficient of proportionality equals the determinant of the operator $E^k F^k$ acting in the quotient space $((V_1 \otimes \cdots \otimes V_n)^*)_l / F((V_1 \otimes \cdots \otimes V_n)^*)_{l-1}$. This operator is isomorphic to the operator $E^k F^k$ acting in the space of singular vectors $(V_1 \otimes \cdots \otimes V_n)_l^{\text{sing}}$. The last operator is simply the multiplication by $\prod_{s=0}^{k-1}((s+1)(2\sum_{m=1}^{n} \Lambda_m - 2l - s))$ in the space of dimension $\binom{n+l-2}{n-2}$. The lemma is proved. □

Proof of Lemma 9.18. First we will give another expression for the function $W'_{\mathfrak{l}}$ which will be more convenient for our purpose.

Taking identities (9.7), (9.8) for $\beta = \exp(-2\pi i/p)$ and $y_k = \exp(-2\pi i t_k/p)$, $k = 0, \ldots, l$, we transform them respectively to the following form

$$\sum_{\sigma \in S^l} [\![1]\!]_\sigma = \prod_{s=1}^{l} \frac{\sin(\pi s/p)}{\sin(\pi/p)} \prod_{1 \leqq j < k \leqq l} \frac{\sin(\pi(t_j - t_k)/p)}{\sin(\pi(t_j - t_k + 1)/p)}\,,$$

$$\sum_{\sigma\in S^l}\left[\!\!\left[\prod_{k=1}^{l}\frac{\exp(\pi i(t_{k-1}-t_k+1)/p)}{\sin(\pi(t_{k-1}-t_k+1)/p)}\right]\!\!\right]_\sigma$$
$$=\exp(\pi il(l-1)/(2p))\prod_{k=1}^{l}\frac{\exp(\pi i(z_n+\Lambda_n-t_k)/p)}{\sin(\pi(z_n+\Lambda_n-t_k)/p)}$$
$$\times\prod_{1\leqq j<k\leqq l}\frac{\sin(\pi(t_j-t_k)/p)}{\sin(\pi(t_j-t_k+1)/p)}\,,$$

where $t_0=z_n+\Lambda_n-1$. Subsequently using the identities above we replace expression (9.15) for the function $W_{\mathfrak{l}}'$ by the following expression:

$$(9.20)\qquad W_{\mathfrak{l}}'(t_1,\dots,t_\ell)$$
$$=\frac{(-1)^{\mathfrak{l}_n}}{(\ell-\mathfrak{l}_n)!}\exp(\pi i\mathfrak{l}_n((1-\mathfrak{l}_n)/2-\Lambda_n)/p))$$
$$\times\sum_{\sigma\in S^\ell}\left[\!\!\left[\mathring{W}_{\mathfrak{l}^!}(t_1,\dots,t_{\ell-\mathfrak{l}_n})\prod_{\ell-\mathfrak{l}_n<a\leqq\ell}\left(\frac{\exp(\pi i(t'_{a-1}-t'_a+1)/p)}{\sin(\pi(t'_{a-1}-t'_a+1)/p)}\right.\right.\right.$$
$$\left.\left.\left.\times\prod_{m=1}^{n-1}\frac{\sin(\pi(t_a-z_m+\Lambda_m)/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)}\right)\right]\!\!\right]_\sigma\,,$$

where $t'_a=t_a$ for $\ell-\mathfrak{l}_n<a\leqq\ell$ and $t'_{\ell-\mathfrak{l}_n}=z_n+\Lambda_n-1$.

Consider the hypergeometric integral $I(W_{\mathfrak{l}}',w)$,

$$I(W_{\mathfrak{l}}',w)=\int_{\mathbb{I}^\ell}\Phi(t)w(t)W_{\mathfrak{l}}'(t)\,d^\ell t\,.$$

The imaginary space is invariant under permutations of the variables $t_1,\dots,t_\ell$. By property (2.7) of function $\Phi(t)$, we can keep in the integral only the term of the sum in the right hand side of (9.20) which corresponds to the identity permutation, multiplying then the result by $\ell!$. Hence

$$(9.21)\qquad I(W_{\mathfrak{l}}',w)=\int_{\mathbb{I}^\ell}F(t)\,d^\ell t\,,$$

$$F(t_1,\dots,t_\ell)=\frac{(-1)^{\mathfrak{l}_n}\ell!}{(\ell-\mathfrak{l}_n)!}\exp(\pi i\mathfrak{l}_n((1-\mathfrak{l}_n)/2-\Lambda_n)/p))$$
$$\times\Phi(t_1,\dots,t_\ell)w(t_1,\dots,t_\ell)\,\mathring{W}_{\mathfrak{l}^!}(t_1,\dots,t_{\ell-\mathfrak{l}_n})$$
$$\times\prod_{\ell-\mathfrak{l}_n<a\leqq\ell}\prod_{m=1}^{n-1}\frac{\sin(\pi(t_a-z_m+\Lambda_m)/p)}{\sin(\pi(t_a-z_m-\Lambda_m)/p)}$$
$$\times\prod_{\ell-\mathfrak{l}<a\leqq\ell}\frac{\exp(\pi i(t'_{a-1}-t'_a+1)/p)}{\sin(\pi(t'_{a-1}-t'_a+1)/p)}\,d^\ell t\,,$$

with the same convention about the variables $t'_{\ell-\mathfrak{l}_n},\dots,t'_\ell$ as in (9.20).

Consider the asymptotics of the integrand $F(t)$ for large t. Namely, assume that

(9.22) $\quad t_a = i(x_a - Au_a), \qquad \operatorname{Im} u_a = 0 \quad \operatorname{Im} x_a = 0, \ a = 1, \dots, \ell,$

and $A \to +\infty$. From the Stirling formula we find that $F(t)$ exponentially decays for any $u = (u_1, \dots, u_\ell)$ which does not belong to the cone

$$\{u \in \mathbb{R}^\ell \,|\, 0 = u_1 = \cdots = u_{\ell-\mathfrak{l}_n} \leqq u_{\ell-\mathfrak{l}_n+1} \leqq \cdots \leqq u_\ell\}\,.$$

The decay takes place for any μ including $\mu = 0$. On the contrary, if u belongs to the cone

$$(9.23) \qquad \{u \in \mathbb{R}^\ell \,|\, 0 = u_1 = \cdots = u_{\ell-\mathfrak{l}_n} < u_{\ell-\mathfrak{l}_n+1} < \cdots < u_\ell\}\,,$$

then the asymptotics of $F(t)$ essentially depends on if μ equals zero or not. If $\mu \neq 0$, then $F(t)$ exponentially decays due to the factor $\exp(\mu \sum_{a=1}^{\ell} t_a/p)$ and integral (9.21) converges. If $\mu = 0$, then $F(t)$ grows as a positive power of A and integral (9.21) diverges.

So the leading term of the asymptotics of the hypergeometric integral $I(W_{\mathfrak{l}}', w)$ as $\mu \to 0$ is given by the integral of the asymptotics of the integrand $F(t)$ for large t in the cone (9.23) (the justification of this fact is given below). Explicitly computing the asymptotics of $F(t)$ for large t, we obtain that

$$\begin{aligned}
&F(t_1, \dots, t_\ell) \\
&\quad = \frac{(-2i/p)^{\mathfrak{l}_n} \ell!}{(\ell - \mathfrak{l}_n)!} \exp\left(\pi i \mathfrak{l}_n \left((1 - \mathfrak{l}_n)/2 + 2(\ell - \mathfrak{l}_n) + \Lambda_n - 2\sum_{m=1}^{n} \Lambda\right)\Big/ p\right) \\
&\quad \times \Phi_{[\ell-\mathfrak{l}_n]}(t_1, \dots, t_{\ell-\mathfrak{l}_n})(E^{\mathfrak{l}_n} w)(t_1, \dots, t_{\ell-\mathfrak{l}_n}) \mathring{W}_{\mathfrak{l}'}(t_1, \dots, t_{\ell-\mathfrak{l}_n}) \\
&\quad \times \prod_{\ell-\mathfrak{l}_n < a \leqq \ell} \exp(\mu t_a/p)(t_a/p)^{2\sum_{m=1}^{n} \Lambda_m/p - 2(\ell-\mathfrak{l}_n)/p - 1} \\
&\quad \times \prod_{\ell-\mathfrak{l} < a, b \leqq \ell} ((t_a - t_b)/p)^{-2/p}(1 + o(1))\,,
\end{aligned}$$

as $A \to +\infty$ and $t_1, \dots, t_\ell$ are described by (9.22). Here the function $\Phi_{[l]}$ $(t_1, \dots, t_l)$ is defined by (2.5) where ℓ is replaced by l.

Denote by $G(t)$ the right hand side of the formula above without the factor $1 + o(1)$. Thus we have

$$(9.24) \qquad I(W'\mathfrak{l}, w) = \int\limits_{\mathbb{I}^{\ell-\mathfrak{l}_n} \times \mathbb{I}_{\mathrm{T}}^{\mathfrak{l}_n}} G(t)\, d^\ell t\, (1 + o(1))\,,$$

as $\mu \to 0$, where $\mathbb{I}_{\mathrm{T}}^{\mathfrak{l}_n} = \{(t_{\ell-\mathfrak{l}_n+1}, \dots, t_\ell \,|\, \operatorname{Re} t_a \leqq 0, \ \operatorname{Im} t_a = 0, \ \ell - \mathfrak{l}_n < a \leqq \ell\}$. The integral with respect to the variables $t_1, \dots, t_{\ell-\mathfrak{l}_n}$ clearly gives $I(\mathring{W}_{\mathfrak{l}'}, E^{\mathfrak{l}_n} w)$. The integral with respect to the variables $t_{\ell-\mathfrak{l}_n+1}, \dots, t_\ell$ can be calculated explicitly via the Selberg integral and we obtain formula (9.18).

The asymptotic (9.24) can be justified in a standard way. The main idea is the same as in the one-dimensional case, cf. Lemma 8.37. We explain the details for the example $n = 1$, $\ell = 2$. The general case is similar.

To make formulae shorter we cahnge notations and consider the following integral

$$J(\alpha) = \int_{\mathbb{R}^2} F(s_1, s_2) \exp(-\alpha(s_1 + 2s_2))\, ds_1\, ds_2 \,,$$

$$F(s_1,s_2) = \frac{\Gamma(a+is_1)}{\Gamma(1-a+is_1)} \frac{\Gamma(b+is_2)}{\Gamma(-b+is_2)} \frac{\Gamma(c+is_1+is_2)}{\Gamma(1-c+is_1+is_2)} \times \frac{\exp(\pi(s_1+s_2+ia+ib))}{4\sin\mathrm{h}(\pi(s_1+ia))\sin\mathrm{h}(\pi(s_2+ib))}$$

Our assumptions mean that parameters a, b, c are small positive numbers such that

$$0 < a + b + c < 1/2 \,.$$

For $s_1 = Au_1$, $s_2 = Au_2$ and $A \to +\infty$ the function $F(s_1, s_2)$ has the following asymptotics:

$$F(s_1,s_2) = s_1^{2a-1} s_2^{2b} (s_1 + s_2)^{2c-1} (1 + o(1))$$

if $u_1 > 0$ and $u_2 > 0$, and $F(s_1, s_2)$ decays exponentially if either $u_1 \leqq 0$ or $u_2 \leqq 0$. We have to show that

$$J(\alpha) = \int_{\mathbb{R}^2_{\geqq 0}} s_1^{2a-1} s_2^{2b} (s_1+s_2)^{2c-1} \exp(-\alpha(s_1+2s_2))\, ds_1\, ds_2\, (1+o(1)) \tag{9.25}$$

as $\alpha \to 0$, $\operatorname{Re}\alpha > 0$,

Fix a small positive number ε and decompose $\mathbb{R}^2$ into four parts:

$$\begin{aligned}
Q_1(\varepsilon) &= \{(u_1,u_2) \in \mathbb{R}^2 \mid u_1 \geqq 0,\ u_2 \geqq 0,\ u_1 \geqq \varepsilon u_2,\ u_2 \geqq \varepsilon u_1\}\,,\\
Q_2(\varepsilon) &= \{(u_1,u_2) \in \mathbb{R}^2 \mid u_1 \geqq |u_2|/\varepsilon\}\,,\\
Q_3(\varepsilon) &= \{(u_1,u_2) \in \mathbb{R}^2 \mid u_2 \geqq |u_1|/\varepsilon\}
\end{aligned}$$

and $Q_4(\varepsilon)$ is the closure of $\mathbb{R}^2 \backslash (Q_1(\varepsilon) \cup Q_2(\varepsilon) \cup Q_3(\varepsilon))$. The respective decomposition of the integral $J(\alpha)$ is

$$\begin{aligned}
J(\alpha) = &\int_{Q_1(\varepsilon)} s_1^{2a-1} s_2^{2b} (s_1+s_2)^{2c-1} \exp(-\alpha(s_1+2s_2))\, ds_1\, ds_2\\
&+ \int_{Q_1(\varepsilon)} (F(s_1,s_2) - s_1^{2a-1} s_2^{2b} (s_1+s_2)^{2c-1}) \exp(-\alpha(s_1+2s_2))\, ds_1\, ds_2\\
&+ \left(\int_{Q_2(\varepsilon)} + \int_{Q_3(\varepsilon)} + \int_{Q_4(\varepsilon)} \right) F(s_1,s_2) \exp(-\alpha(s_1+2s_2))\, ds_1\, ds_2 \,.
\end{aligned}$$

The first integral equals

$$\alpha^{-2(a+b+c)} \int_{Q_1(\varepsilon)} s_1^{2a-1} s_2^{2b} (s_1+s_2)^{2c-1} \exp(-s_1 - 2s_2)\, ds_1\, ds_2 \,,$$

the second and the fifth integrals have finite limits as $\alpha \to 0$, the third and the forth integrals can be respectively estimated from above by

$$A_1 \varepsilon^{2b+1} \alpha^{-2(a+b+c)} \int\limits_0^{+\infty} s_1^{2(a+b+c)-1} \exp(-s_1)\, ds$$

and

$$A_2 \varepsilon^{2a} \alpha^{-2(a+b+c)} \int\limits_0^{+\infty} s_2^{2(a+b+c)-1} \exp(-2s_2)\, ds$$

as $\alpha \to 0$, and the constants A_1, A_2 do not depend on ε. The estimates can be obtained by means of Lemma 8.42. Therefore,

$$\left| \lim_{\alpha \to 0} (\alpha^{2(a+b+c)} J(\alpha)) - \int\limits_{Q_1(\varepsilon)} s_1^{2a-1} s_2^{2b} (s_1 + s_2)^{2c-1} \exp(-s_1 - 2s_2))\, ds_1\, ds_2 \right|$$
$$< \varepsilon^{2b+1} + A_2 \varepsilon^{2a} .$$

Moving ε to zero we see that

$$J(\alpha) = \alpha^{-2(a+b+c)} \int\limits_{\mathbb{R}^2_{\geqq 0}} s_1^{2a-1} s_2^{2b} (s_1 + s_2)^{2c-1} \exp(-s_1 - 2s_2))\, ds_1\, ds_2\, (1 + o(1)) ,$$

which coincide with (9.25).

Lemma 9.18 is proved. □

Proof of Theorem 7.8. Let h be a positive number. Consider the hypergeometric integral

$$I(\mathring{W}_\mathfrak{l}, w_\mathfrak{m}) = \int\limits_{\mathbb{I}^\ell} \Phi(t,z) w_\mathfrak{m}(t,z) \mathring{W}_\mathfrak{l}(t,z)\, d^\ell t ,$$

where we substitute into formulae (2.5), (2.19), (2.27), defining the functions $\Phi, w_\mathfrak{m}, \mathring{W}_\mathfrak{l}$, a new parameter η and new coordinates $y_1, \ldots, y_n$:

$$\mu = h\eta, \qquad z_m = y_m/h, \quad m = 1, \ldots, n .$$

We study the quasiclassical asymptotics of the hypergeometric integral $I(\mathring{W}_\mathfrak{l}, w_\mathfrak{m})$ as $h \to +0$ while the parameter η and the coordinates $y_1, \ldots, y_n$ remain fixed.

For any $\mathfrak{l} \in \mathscr{Z}_\ell^{n-1}$ consider a region $\mathring{\mathbb{U}}_\mathfrak{l}$ and a domain $\widehat{\mathbb{U}}_\mathfrak{l}$ in the imaginary subspace $\mathbb{I}^\ell$ given by

$$\mathring{\mathbb{U}}_\mathfrak{l} = \left\{ u \in \mathbb{I}^\ell \,\middle|\, \begin{array}{l} \operatorname{Im} y_m < \operatorname{Im} u_{1+\mathfrak{l}^{m-1}} < \cdots < \operatorname{Im} u_{\mathfrak{l}^m} < \operatorname{Im} y_{m+1} , \\ m = 1, \ldots, n-1 \end{array} \right\} ,$$

$$\widehat{\mathbb{U}}_\mathfrak{l} = \left\{ u \in \mathbb{I}^\ell \,\middle|\, \begin{array}{l} \operatorname{Im} y_m \leqq \operatorname{Im} u_a \leqq \operatorname{Im} y_{m+1}, \ \mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m , \\ m = 1, \ldots, n-1 \end{array} \right\} .$$

Recall that $\mathfrak{l}^m = \mathfrak{l}_1 + \cdots + \mathfrak{l}_m$.

Let $\mathbf{S}^{\mathfrak{l}} \subset \mathbf{S}^{\ell}$ be the following subgroup isomorphic to $\mathbf{S}^{\mathfrak{l}_1} \times \cdots \times \mathbf{S}^{\mathfrak{l}_n}$:

$$\mathbf{S}^{\mathfrak{l}} = \{\sigma \in \mathbf{S}^{\ell} \mid \mathfrak{l}^{m-1} < \sigma_a \leqq \mathfrak{l}^m \quad \text{for } \mathfrak{l}^{m-1} < a \leqq \mathfrak{l}^m, \ m = 1, \dots, n\}.$$

Set $\mathbb{U}_{\mathfrak{l}} = \{u \in \mathbb{C}^{\ell} \mid (u_{\sigma_1}, \dots, u_{\sigma_\ell}) \in \mathring{\mathbb{U}}_{\mathfrak{l}} \text{ for some } \sigma \in \mathbf{S}^{\mathfrak{l}}\}$.

The imaginary subspace $\mathbb{I}^{\ell}$ is invariant under permutations of the variables $t_1, \dots, t_\ell$. Using the property (2.7) of function $\Phi(t)$ we see that

$$I(\mathring{W}_{\mathfrak{l}}, w_{\mathfrak{m}}) = \ell! \int_{\mathbb{I}^{\ell}} \Phi(t,z) w_{\mathfrak{m}}(t,z) \overline{W_{\mathfrak{l}}}(t,z)\, d^{\ell}t\,, \tag{9.26}$$

where

$$\begin{aligned}
&\overline{W_{\mathfrak{l}}}(t_1, \dots, t_\ell, z_1, \dots, z_n) \\
&= \prod_{m=1}^{n-1} \prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \sin(\pi(z_m - \Lambda_m - z_{m+1} - \Lambda_{m+1} + s - 1)/p) \\
&\quad \times \prod_{m=1}^{n-1} \prod_{a \in \Gamma_m} \left(\frac{1}{\sin(\pi(t_a - z_m - \Lambda_m)/p) \sin(\pi(t_a - z_{m+1} - \Lambda_{m+1})/p)} \right. \\
&\quad \left. \times \prod_{1 \leqq l < m} \frac{\sin(\pi(t_a - z_l + \Lambda_l)/p)}{\sin(\pi(t_a - z_l - \Lambda_l)/p)} \right)
\end{aligned}$$

and $\Gamma_m = \{\mathfrak{l}^{m-1} + 1, \dots, \mathfrak{l}^m\}$, $m = 1, \dots, n$.

The factors of the integrand above have the following quasiclassical asymptotics as $h \to +0$ while the parameter η, the coordinates $y_1, \dots, y_n$ and the variables $u_a = h t_a$, $a = 1, \dots, \ell$, remain fixed:

$$\Phi(u/h, y/h) = h^{\ell(\ell - 1 - 2\sum_{m=1}^{n} \Lambda_m)/p}\, \widetilde{\Phi}(u,y)(1 + o(1))\,,$$

$$w_{\mathfrak{m}}(u/h, y/h) = h^{\ell} \tilde{w}(u,h)(1 + o(1))\,,$$

$$\begin{aligned}
\overline{W_{\mathfrak{l}}}(u/h, y/h) = \prod_{m=1}^{n-1} &\left(\exp\left(\pi i \left(2 \sum_{m=1}^{n} \Lambda_m(\ell - \mathfrak{l}^{m-1}) - \mathfrak{l}_m(\mathfrak{l}_m - 1)/2 \right) \Big/ p \right) \right. \\
&\left. \times \prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \right) (1 + o(1))
\end{aligned}$$

for $u \in \mathring{\mathbb{U}}_{\mathfrak{l}}$ and $\overline{W_{\mathfrak{l}}}(u/h, y/h) = o(1)$ for $u \notin \mathring{\mathbb{U}}_{\mathfrak{l}}$. Here the functions $\widetilde{\Psi}(u,y)$ and $\tilde{w}_{\mathfrak{m}}(u,y)$ are given by (7.3) and (7.5), respectively.

The quasiclassical asymptotics of the integral (9.26) is given by the integral of the quasiclassical asymptotics of the integrand, that is

(9.27)

$$\begin{aligned}
I(\overline{W_{\mathfrak{l}}}, w_{\mathfrak{m}}) = \prod_{m=1}^{n-1} &\left(\exp\left(\pi i \left(2 \sum_{m=1}^{n} \Lambda_m(\ell - \mathfrak{l}^{m-1}) - \mathfrak{l}_m(\mathfrak{l}_m - 1)/2 \right) \Big/ p \right) \right. \\
&\left. \times \prod_{s=1}^{\mathfrak{l}_m} \frac{\sin(\pi/p)}{\sin(\pi s/p)} \right) \int_{\mathring{\mathbb{U}}_{\mathfrak{l}}} \widetilde{\Phi}(u,y) \tilde{w}(u,y)\, d^{\ell}u\, (1 + o(1))\,.
\end{aligned}$$

Taking into account that

$$\int_{\mathbb{U}_\mathfrak{l}} \widetilde{\Phi}(u,y)\tilde{w}(u,y)\,d^\ell u = \prod_{m=1}^{n-1}\left(\exp(\pi i \mathfrak{l}_m(1-\mathfrak{l}_m)/(2p))\prod_{s=1}^{\mathfrak{l}_m}\frac{\sin(\pi/p)}{\sin(\pi s/p)}\right) \times \int_{\overset{*}{\mathbb{U}}_\mathfrak{l}} \widetilde{\Phi}(u,y)\tilde{w}(u,y)\,d^\ell u$$

where $\mathbb{U}_\mathfrak{l}$ is given by (7.7), we obtain (7.8).

Formula (9.27) can be justified in a standard way, similarly to the proof of formula (9.25).

Theorem 7.8 is proved. □

References

[A] K. Aomoto: q-analogue of de Rham cohomology associated with Jackson Integrals, I, Proceedings of Japan Acad. **66** Ser. A (1990), 161–164; II, Proceedings of Japan Acad. **66** Ser. A, 240–244 (1990)

[AK] K. Aomoto, Y. Kato: Gauss decomposition of connection matrices for symmetric A-type Jackson integrals. Preprint 1–45 (1995)

[CP] V. Chari, A. Pressley: A guide to quantum groups. Cambrige University Press, Cambridge, 1994

[D1] V.G. Drinfeld: Quantum groups. In: Proceedings of ICM, Berkley, 1986 (A.M. Gleason, ed.), AMS, Providence, 798–820 (1987)

[D2] V.G. Drinfeld: Quasi-Hopf algebras. Leningrad Math. J. 1 1419–1457 (1990)

[DF] J. Ding, I.B. Frenkel: Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(n))$, Comm. Math. Phys. **156**, 277–300

[F] G. Felder: Elliptic quantum groups. Preprint hep-th/9412207 (1994); Proceedings of the ICMP, Paris, 1994 (to appear)
G. Felder: Conformal field theory and integrable systems associated to elliptic curves. Preprint hep-th/9407154 (1994); Proceedings of the ICM, Zurich, 1994 (to appear)

[FR] I.B. Frenkel, N.Yu. Reshetikhin: Quantum affine algebras and holonomic difference equations, Comm. Math. Phys. **146**, 1–60 (1992)

[FTT] L.D. Faddeev, L.A. Takhtajan, V.O. Tarasov: Local Hamiltonians for integrable models on a lattice, Theor. Math. Phys. **57**, 1059–1072 (1983)

[FTV] G. Felder, V. Tarasov, A. Varchenko: Bethe ansatz for interaction-round-a-face elliptic models and solutions to $qKZB$ equations. In preparation

[FV] G. Felder, A. Varchenko: On representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. Preprint 1–21 (1995)

[G] R.A. Gustafson: A generalization of Selbergs's beta integral, Bull. Amer. Math. Society **22**, 97–105 (1990)

[GR] G. Gasper, M. Rahman: Basic hypergeometric series, Encycl, Math. Appl., Cambrige University Press, Cambridge, 1990

[JM] M. Jimbo, T. Miwa: Algebraic analysis of solvable lattice models, CBMS Regional Conf. Series in Math. **85** (1995)

[IK] A. Izergin, V.E. Korepin: The quantum scattering method approach to correlation functions, Comm. Math. Phys. **94**, 67–92 (1984)

[K] T. Kohno: Monodromy representations of braid groups and Yang–Baxter equations, Ann. Inst. Fourier **37**, 139–160 (1987)
T. Kohno: Linear representations of braid groups and classical Yang–Baxter equations, Contemp. Math. **78**, 339–363 (1988)

[Ka] K. Kadell: A proof of Askey's conjectured q-analogue of Selberg's integral and a conjecture of Morris, SIAM J. Math. Anal. **19**, 969–986 (1988)

[KL] D. Kazhdan, G. Lusztig: Affine Lie algebras and quantum groups, Intern. Math. Research Notices **2**, 21–29 (1991)
D. Kazhdan, G. Lusztig: Tensor structures arising from affine Lie algebras, I. J. Amer. Math. Society **6** (1993), 905–947; II, J. Amer. Math. Society **6**, 949–1011 (1993)

[Ko] V.E. Korepin: Calculations of norms of Bethe wave functions, Comm. Math. Phys. **86**, 391–418 (1982)

[KRS] P.P. Kulish, N.Yu. Reshetikhin, E.K. Sklyanin: Yang–Baxter equation and representation theory I, Lett. Math. Phys. **5**, 393–403 (1981)

[KS] D. Kazhdan, Ya.S. Soibelman: Representations of the quantized function algebras. 2-categories and Zamolodchikov tetrahedra equation. Preprint, Harvard University 1–66 (1992)

[L] F. Loeser: Arrangements d'hyperplans et sommes de Gauss, Ann. Scient. École Normale Super. 4-e serie, t. 24, 379–400 (1991)

[Lu] S. Lukyanov: Free field representation for massive integrable models, Comm. Math. Phys. **167**, 183–226 (1995)

[M] A. Matsuo: Jackson integrals of Jordan–Pockhgammer type and quantum Knizhnik–Zamolodchikov equation, Comm. Math. Phys. **151**, 263–274 (1993)

[R] N.Yu. Reshetikhin: Jackson-type integrals. Bethe vectors, and solutions to a difference analogue of the Knizhnik–Zamolodchikov system, Lett. Math. Phys. **26**, 153–165 (1992)

[RF] N.Yu. Reshetikhin, L.D. Faddeev: Hamiltonian structures for integrable field theory models, Theor. Math. Phys. **56**, 847–862 (1983)

[RS] N.Yu. Reshetikhin, M.A. Semenov–Tyan–Shansky: Central extensions of the quantum current groups, Lett. Math. Phys. **19**, 133–142 (1990)

[S] F.A. Smirnov: Form factors in completely integrable models of quantum field theory, Advanced Series in Math. Phys., vol. 14, World Scientific, Singapore, 1992

[Se] A. Selberg: Bemerkninger om et multipelt integral, Norsk. Mat. Tidsskr. **26**, 71–78 (1944)

[SV1] V.V. Schechtman, A.N. Varchenko: Hypergeometric solutions of Knizhnik–Zamolodchikov equations, Lett. Math. Phys. **20**, 279–283 (1990)

[SV2] V.V. Schechtman, A.N. Varchenko: Arrangements of hyperplanes and Lie algebras homology. Invent. Math. **106**, 139–194 (1991)
V.V. Schechtman, A.N. Varchenko: Quantum groups and homology of local systems. Algebraic Geometry and Analytic Geometry, ICM-90, Satellite Confer. Proceedings, Tokyo, 1990, Springer-Verlag, 182–191, 1991

[T] V.O. Tarasov: Irreducible monodromy matrices for the R-matrix of the XXZ-model and lattice local quantum Hamiltonians, Theor. Math. Phys. **63**, 440–454 (1985)

[TV1] V.O. Tarasov, A.N. Varchenko: Jackson integral representations for solutions to the quantized Knizhnik–Zamolodchikov equation, St. Petersburg Math. J. **6**, no. 2, 275–313 (1995)

[TV2] V. Tarasov, A. Varchenko: Asymptotic solution to the quantized Knizhnik–Zamolodchikov equation and Bethe vectors. Preprint HU-TFT-94-21, UTMS 94–26 (1994); Advances in Soviet Math. (to appear)

[TV3] V. Tarasov, A. Varchenko: Geometry of q-hypergeometric functions as a bridge between quantum affine algebras and elliptic quantum affine algebras. In preparation

[V1] A.N. Varchenko: The Euler beta-function, the Vandermonde determinant, Legendre's equation, and critical values of linear functions on a configuration of hyperplanes, I. Math. of the USSR, Izvestia **35**, 543–571 (1990); II, Math. of the USSR Izvestia **36**, 155–168 (1991)
A.N. Varchenko: Determinant formula for Selberg type integrals, Funct. Anal. Appl. **4**, 65–66 (1991)

[V2] A. Varchenko: Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, Advanced Series in Math. Phys., vol. 21, World Scientific, Singapore, 1995

[V3] A. Varchenko: Quantized Knizhnik–Zamolodchikov equations, quantum Yang–Baxter equation, and difference equations for q-hypergeometric functions, Comm. Math. Phys. **162**, 499–528 (1994)

[V4] A. Varchenko: Asymptotic solutions to the Knizhnik–Zamolodchikov equation and crystal base, Comm. Math. Phys. **171**, 99–137 (1995)

[WW] E.T. Whittaker, G.N. Watson: A Course of Modern Analysis. Cambrige University Press, Cambribge, 1927