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A high fibered power of a family of varieties of general type dominates a variety of general type

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0 Introduction

We work over $\mathbb{C}$.

0.1 Statement

We prove the following theorem:

Theorem 0.1 (Fibered power theorem). *Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with B irreducible. Then there exists an integer $n > 0$, a positive dimensional variety of general type W_n, and a dominant rational map $X^n_B \dashrightarrow W_n$.*

Specifically, let $m_n : X^n_B \dashrightarrow W_n$ be the n-pointed birational-moduli map. Then for sufficiently large n, W_n is a variety of general type.

The latter statement will be explained in Sect. 3. This solves "Conjecture H" of [CHM], Sect. 6.1 as well as the question at the end of Remark 1.3 in [ℵ-V].

Following Viehweg's suggestions in [V3], the fibered power theorem is proved by way of the following theorem:

Theorem 0.2. *Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with B irreducible, and* $\mathrm{Var}(X/B) = \dim B$*. Then there exists an integer $n > 0$ such that the fibered power X^n_B is of general type.*

0.2 Main ingredients

The starting point is a theorem of Kollár, which roughly speaking says that given $f : X \to B$ a morphism between smooth irreducible projective varieties, whose

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generic fiber is a variety of general type, and $\mathrm{Var}(X/B) = \dim B$, then for large m the saturation of $f_*(\omega_f^m)$ has many sections. A very useful trick of Viehweg shows that this implies that for large m the sheaf ω_f^m itself has many sections, that is, ω_f is big.

Following [CHM], one would like to use these sections pulled back to the fibered powers $f_n : X_B^n \to B$ of X over B to overcome any possible negativity in ω_B. Unfortunately, the fibered powers may become increasingly singular, and it is not easy to tell who wins in the race between the positivity of ω_{f_n} and the so called adjoint conditions imposed by the singularities of X_B^n. The fact that ω_f may have positivity "by accident", as shown by the example in [CHM], Sect. 6.1 of plane quartics, shows that something more is needed - the fiber space $X \to B$ should be "straightened out" before we can use sections coming from Kollár's theorem.

Semistable reduction would be sufficient for this purpose, but unfortunately semistable reduction for families of fiber dimension > 2 over a base of dimension > 1 is yet unknown. It is not known whether unipotent monodromies would suffice. The case of curves was done in [CHM] using the moduli space of stable curves, and the case of surfaces was done in [Has] using the moduli space of stable surfaces.

Lacking such constructions in higher dimensions, we will use a variant of semistable reduction, introduced by de Jong [dJ]. This variant involves, after a suitable base change and birational modification, a Galois cover $Y \to X$, such that $Y \to B$ is a composition of families of curves with at most nodes. The fibers now are much better controlled, but we are left with descending differential forms from Y_B^n to X_B^n. This is done by studying the behavior of the ramification ideals in the fibered powers.

0.3 Arithmetic background and applications

Results of this type are motivated by Lang's conjecture. See, *e.g.*, [CHM], [Has], [ℵ], [ℵ-V], [Pac].

Let K be a number field (or any field finitely generated over $\mathbb{Q}$), and let X be a variety of general type over K. According to a well known conjecture of Lang (see [L], Conjecture 5.7), the set of K-rational points $X(K)$ is not Zariski dense in K. In [CHM], it is shown that this conjecture of Lang implies the existence of a uniform bound $B(K, g)$ on the number of K rational points on curves of genus g (a stronger implication arises if one assumes a stronger version of Lang's conjecture). This result was later refined in [ℵ], and the ultimate result of this type was recently obtained by Pacelli in [Pac], to wit:

Theorem 0.3 (Pacelli [Pac], Theorem 1.1). *Assume that Lang's conjecture is true. Let $g \geq 2$ and $d \geq 1$ be integers, and let K be a field as above. Then there exists an integer $P_K(d, g)$, such that for any extension L of K of degree d, and any curve C of genus g defined over L, one has*

$$\#C(L) \leq P_K(d, g).$$

The main geometric ingredient in the above mentioned results is the "Correlation Theorem" of [CHM], which is Theorem 0.1 for curves. In [CHM], Sect. 6, a version of Theorem 0.1 was conjectured ("Conjecture H"), with the suggestion that strong uniformity results would follow from such a theorem. This was further investigated in [ℵ-V], where it is shown that Theorem 0.1 gives an alternative proof for Pacelli's theorem, as well as other strong implication results for curves and higher dimensional varieties. The general result may be stated as follows:

$$\text{Lang's conjecture} \Longrightarrow \text{uniform Lang's conjecture.}$$

As an example of a result about curves, which does not follow from Pacelli's theorem, we have (see [ℵ-V], Corollary 3.4 and Theorem 1.7):

Corollary 0.4. *Assume that the weak Lang conjecture holds. Fix a number field K and an integer d. Then there is a uniform bound N for the number of points of degree d over K on any curve C of genus g and gonality $> 2d$ over K. In fact, N depends only on g, d and the degree $[K : \mathbb{Q}]$.*

It would be interesting to obtain analogous statements using the geometric case of Lang's conjecture (see some results for curves in [Pac], Corollary 5.4, and [ℵ-V] Sect. 3). Another possible direction for extending the results is the logarithmic case (see [ℵ1] for definitions and results for stably integral points on elliptic curves). One suspects that in the future a fibered power theorem will be available for log-varieties. At the moment, the main difficulties seem to lie in obtaining an appropriate generalization of the theorems of Kollár and Viehweg.

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1 Preliminaries

1.1 Definitions

A variety is called a *rational-Gorenstein variety* if it has only rational Gorenstein (and hence canonical) singularities. For a Gorenstein variety X to be rational-Gorenstein, it is necessary and sufficient that for any resolution of singularities $r : Y \to X$ one has $r_*\omega_Y = \omega_X$ (see [E], II).

We say that a flat morphism of irreducible varieties $Y \to B$ is *mild*, if for any dominant $B_1 \to B$ where B_1 is rational - Gorenstein, we have that $Y_1 = Y \times_B B_1$ is rational - Gorenstein. Note that, by induction, if $Y \to B$ is mild then the fibered powers $Y_B^n \to B$ are mild as well.

An *alteration* is a projective, surjective, generically finite morphism of irreducible varieties. This is slightly different from the definition in [dJ], where properness is assumed rather than projectivity. An alteration $B_1 \to B$ is *Galois* if there exists a finite group $G \subset \mathrm{Aut}_B B_1$ such that $B_1/G \to B$ is birational.

A *fiber space* is a projective morphism of irreducible normal varieties whose general fibers are irreducible and normal.

Given a fiber space $X \to B$ and an alteration $B_1 \to B$ we denote by $X \tilde{\times}_B B_1$ the *main component* of $X \times_B B_1$. Namely, if η_{B_1} is the generic point of B_1, then $X \tilde{\times}_B B_1 = \overline{X \times_B \eta_{B_1}}$.

A *flat family* is a flat fiber space.

A flat family $Y \to Y_1$ is called a *nodal family* if every fiber is a curve with at most ordinary nodes. A flat family $Y \to B$ is called a *pluri-nodal family* if it is a composition of nodal families $Y \to Y_1 \to \cdots \to B$. Note that while nodal families are generically smooth, this is not the case with pluri-nodal families.

Given a line bundle L and an ideal sheaf $\mathcal{I}$ on a variety X, we say that $L \otimes \mathcal{I}$ is *big* if for some $k > 0$ the rational map associated to $H^0(X, L^{\otimes k} \otimes \mathcal{I}^k)$ is birational to the image. It readily follows that, if $L \otimes \mathcal{I}$ is big, and J is an ideal sheaf, then for sufficiently large k we have that $L^{\otimes k} \otimes \mathcal{I}^k J$ is big. The definition differs somewhat from Kollár's definition in [Ko]. In Sect. 3 we will refer to sheaves which are "big" in Kollár's sense as *weakly big*: we say that a sheaf $\mathcal{F}$ is *weakly big*, if for any ample L there is a positive integer a such that $Sym^a(\mathcal{F}) \otimes L^{(-1)}$ is weakly positive (see [Ko], p.367, (vii)).

1.2 Group theory

For a finite group G let $e(G) = \mathrm{lcm}\{\mathrm{ord}(g) | g \in G\}$. We will use the following obvious lemma:

Lemma 1.1. *Let G be a finite group. Then for any n, we have $e(G^n) = e(G)$.*

2 Ramification

Let V be a quasi projective rational - Gorenstein variety, $G \subset \mathrm{Aut}(V)$ a finite group. Let $W = V/G$, and $q : V \to W$ the quotient map. Let $r : W_1 \to W$ be a resolution of singularities. Note that W is normal, therefore it is regular in codimension 1. We can pull back sections of pluricanonical sheaves on the nonsingular locus W_{ns} and extend them into the pluricanonical sheaf of V. Thus, for an integer $n > 0$ we have a morphism $\phi_n : q^* r_* \omega_{W_1}^n \to \omega_V^n$, which is an isomorphism away from the fixed points of G.

Define the *n-th ramification ideal* $\mathcal{F}_n = \mathcal{F}_n(G, V) = \mathrm{Ann}\,\mathrm{Coker}\,\phi_n$.

Lemma 2.1. *1. We have* $\mathcal{F}_n \otimes \omega_V^n \cong q^* r_* \omega_{W_1}^n/\mathrm{torsion}$.
2. For any integer $k > 0$ we have $\mathcal{F}_n^k \subset \mathcal{F}_{nk}$.
3. The ideals $\mathcal{F}_n$ are locally defined: if $V' \subset V$ is a G-invariant open subset, then $\mathcal{F}_n(G, V') = \mathcal{F}_n(G, V)|_{V'}$.
4. The ideals $\mathcal{F}_n$ are independent of the choice of resolution $r : W_1 \to W$.

5. The ideals $\mathscr{J}_n$ can be also obtained if we use a partial resolution $r : W_1 \to W$ where W_1 is rational - Gorenstein.

Proof. Since ω_V is by assumption invertible, we have (1). For the same reason (2) follows: if $\omega = \prod_{i=1}^k \omega_i$ where ω_i are local sections of ω_V^n and if $f = \prod_{i=1}^k f_i$ where $f_i \in \mathscr{J}_n$, then we can write $f_i \cdot \omega_i = \sum g_{i\,j} \cdot (q^* r_* \eta_{i\,j})$ and expanding we get that $f\omega$ is a local section of $\phi_{nk}(q^* r_* \omega_{W_1}^{nk})$. It would be nice to have an actual equality for high n. Part (3) follows by definition. Parts (4) and (5) follow by noticing that for a birational morphism $r' : W_2 \to W_1$ with W_2 nonsingular, we have $r'_* \omega_{W_2}^n = \omega_{W_1}^n$ in both cases. □

Traditionally, one studies ramification by reducing to the case where both V and W are regular. Most of the results below follow that line, with the exception of Proposition 2.7, where the author finds it liberating, if not essential, to avoid unnecessary blowups.

The ideals $\mathscr{J}_n$ give conditions for invariant differential forms to descend to regular forms on the quotient:

Proposition 2.2. *Given an integer $n > 0$ we have*

$$(q_*(\omega_V^n \otimes \mathscr{J}_n))^G = r_* \omega_{W_1}^n .$$

Proof. A local section of $(q_*(\omega_V^n \otimes \mathscr{J}_n))^G$ can be written as $\sum q_*(f_i) r_*(s_i)$, where f_i are G invariant, therefore $f_i = q^* g_i$. □

The above property of an ideal, giving *sufficient conditions for invariant n-differentials to descend*, can be bounded below in terms of multiplicities (here we first use the assumption on rational singularities):

Proposition 2.3 ([CHM] Sect. 4.2, Lemma 4.1). *Let $\Sigma_{G,V} = \Sigma \subset V$ be the locus of fixed points:*

$$\Sigma = \{x \in V \,|\, \exists g \in G, g(x) = x\},$$

viewed as a closed reduced subscheme, with ideal $\mathscr{I}_\Sigma$. Then $(q_*(\omega_V^n \otimes \mathscr{I}_\Sigma^{n\cdot(e(G)-1)}))^G \subset r_* \omega_{W_1}^n$.

Proof. Let V_1 be the normalization of W_1 in $\mathbb{C}(V)$. Let $W_1' \subset W_1$ be the open subset over which both V_1 and the branch locus B_{V_1/W_1} are nonsingular. The codimension of $W_1 \setminus W_1'$ is at least 2. Let V_1' be the inverse image of W_1'. We have a diagram

$$\begin{array}{ccc} V_1' & \xrightarrow{s} & V \\ \downarrow q_1 & & \downarrow q \\ W_1' & \xrightarrow{r'} & W \end{array}$$

Let ω be a G-invariant n-canonical form on V, vanishing to order $n \cdot (e(G) - 1)$ on Σ. To show that ω descends to W_1 it suffices to descend it to W_1', since the codimension of the complement is at least 2. Since V is rational - Gorenstein, $\omega' = s^*\omega$ is a regular n-canonical form on V_1', vanishing to order $n \cdot (e(G) - 1)$ on B_{V_1/W_1}. The subgroup fixing a general point of a component of B_{V_1/W_1}

is cyclic, and the action is given formally by $u_1 \mapsto \zeta_k u_1, u_i \mapsto u_i$ for some root of unity $\zeta_k, k \le e(G)$, where u_i are local parameters, u_1 a uniformizer for B_{V_1/W_1}. Formally at such a point, the quotient map is given by $w_1 = u_1^k, w_i = u_i, i > 1$. By assumption, ω' can be written in terms of local parameters as $\omega' = f(u)(u^{k-1}du_1 \wedge \cdots \wedge du_m)^n = f(u)q_1{}^*(dw_1 \wedge \cdots \wedge dw_m)^n$. The invariance implies that $f(u) = q_1{}^*g(w)$ and therefore $\omega' = q_1{}^*g(w)(dw_1 \wedge \cdots \wedge dw_m)^n$. □

Remark. It is not difficult to obtain the following refinement of this proposition (see analogous case in [Ko], Lemma 3.2): let $B = q(\Sigma)_{\text{red}}$, and let $\mathcal{I}_B$ be the defining ideal. Then $q^{-1}\mathcal{I}_B^{\lfloor n(1-\frac{1}{e(G)})\rfloor}$ gives sufficient conditions for invariant n-differentials to descend.

Recall that if a group G acts on a variety V, a line bundle L and an ideal $\mathcal{I}$ then the ring of invariant sections $\oplus_{k\ge 0}H^0(Y, L^{\otimes k} \otimes \mathcal{I}^k)^G$ has the same dimension as the ring of sections $\oplus_{k\ge 0}H^0(Y, L^{\otimes k} \otimes \mathcal{I}^k)$. This allows us to have:

Corollary 2.4 (See more general statement in [Pac], Lemma 4.2). *Let X be a variety of general type and let $G = \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(X))$ be its birational automorphism group. Then for some $n > 0$ the quotient variety X^n/G, where G acts diagonally, is of general type.*

Proof. Applying Hironaka's equivariant resolution of singularities, we may assume that X is regular and $G = \mathrm{Aut}X$. Let $p_i : X^n \to X$ be the projection onto the i-th factor. Choose n large enough so that $\omega_X^n \otimes \mathcal{I}_{\Sigma_{G,X}}^{(e(G)-1)}$ is big. Therefore $\omega_{X^n}^n \otimes (\sum p_i^{-1}\mathcal{I}_{\Sigma_{G,X}})^{(e(G)-1)n}$ is big. But

$$(\sum p_i^{-1}\mathcal{I}_{\Sigma_{G,X}})^{(e(G)-1)n} \subset \mathcal{I}_{\Sigma_{G,X^n}}^{(e(G)-1)n},$$

and 2.3 gives the result. □

Let $\Sigma \subset V$ be the locus of fixed points, and let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a closed decomposition. Then $\mathcal{J}_n$ is supported along Σ, and can be written as $\mathcal{J}_n = \mathcal{J}_{n,\Sigma_1} \cap \mathcal{J}_{n,\Sigma_2}$. Applying 2.3 we obtain:

Corollary 2.5. *The ideal $(\mathcal{I}_{\Sigma_2}^{e(G)})^n \cdot \mathcal{J}_{n,\Sigma_1}$ gives sufficient conditions for invariant n-differentials to descend.*

Our goal is to apply our propositions to powers of mild families. First, let $f : V \to B$ be mild. Assume that B is rational - Gorenstein. As before, let $G \subset \mathrm{Aut}_B(V)$, $W = V/G$, and $q : V \to W$ the quotient map.

Let $p_i : V_B^m \to V$ be the i-th projection. We naturally have $G^m \subset \mathrm{Aut}_B(V_B^m)$ acting on all components. We denote by $q_m : V_B^m \to W_B^m$ the associated map. Let $r : W_1 \to W$ be a resolution of singularities.

Define $\mathcal{J}_{m,n} = \prod p_i^{-1}\mathcal{J}_n$.

Lemma 2.6. *Assume that $W_1 \to B$ is mild. Then $\mathcal{J}_{m,n} \subset \mathcal{J}_n(G^m, V_B^m)$.*

Proof. Denote $r_m : W_m = (W_1)_B^m \to W_B^m$ and $p_{i,W} : W_m \to W_1$ the i-th projection. Since $V \to B$ and $W_1 \to B$ are mild, we have that

$$\omega^n_{V^m_B/B} = \otimes_i p_i^*(\omega^n_{V/B}) \quad \text{and} \quad \omega^n_{W_m/B} = \otimes_i p_{i,W}^*(\omega^n_{W_1/B}).$$

Suppose a local section w of $\omega^n_{V^m_B/B}$ is a monomial written as $w = \prod p_i^* w_i$, and suppose $f \in \mathscr{J}_{m,n}$ is a monomial written as $f = \prod p_i^* f_i$. Then $fw = \prod p_i^*(f_i w_i)$ is a local section in the image of $q_m^* r_{m*}\omega^n_{W_m/B}$. □

Proposition 2.7. *There exists a closed subset $F \subset B$ such that $(\mathscr{I}_F^{e(G)n} \cdot \mathscr{J}_{m,n})^G$ gives sufficient conditions for invariant n-differentials to descend.*

Proof. Let $F \subset B$ be the discriminant locus of $W_1 \to B$, and $U = B \setminus F$. Now apply 2.6 and 2.5. □

Remark. It follows from the remark after 2.3 that already

$$(\mathscr{I}_F^{\lfloor n(1-\frac{1}{e(G)})\rfloor}) \cdot \mathscr{J}_{m,n}$$

suffices.

We will need to perform base changes for fiber spaces. We need to find a condition on the base changed fiber space which guarantees that the original variety is of general type. This is provided by the following proposition (which is probably well known):

Proposition 2.8. *Given an alteration $\rho : B_1 \to B$ between smooth projective varieties, there exists an ideal sheaf $\mathscr{I} \subset \mathscr{O}_{B_1}$ with the following property: given a fiber space $f : Y \to B$, with $Y_1 \to Y \tilde{\times}_B B_1$ a resolution of singularities, $f_1 : Y_1 \to B_1$ the induced projection, such that $\omega_{f_1} \otimes f_1{}^{-1}\mathscr{I}$ is big, then Y is of general type.*

First a lemma:

Lemma 2.9. *1. Let $g : Y_1 \to Y$ be a generically finite morphism of smooth projective varieties. Let $B \subset Y$ be the branch locus. Then there exists an effective g-exceptional divisor E on Y_1 and an injection $\omega_{Y_1}(-g^*B) \to g^*\omega_Y \otimes \mathscr{O}_{Y_1}(E)$.*
2. If $\omega_{Y_1}(-g^*B)$ is big, then ω_Y is big as well.

Proof. The pull-back morphism $g^*\omega_Y \to \omega_{Y_1}$ gives $g^*\omega_Y = \omega_{Y_1}(-R-E)$ where E is an effective exceptional divisor and R is the ramification divisor. Clearly $R < g^*B$.

Assume that $\omega_{Y_1}(-g^*B)$ is big. Then $g^*\omega_Y \otimes \mathscr{O}_{Y_1}(E)$ is big. Let $Y_1 \xrightarrow{g_1} Y' \xrightarrow{s} Y$ be the Stein factorization. Since Y' is normal and E is g_1-exceptional we have that $s^*\omega_Y \otimes g_{1*}\mathscr{O}_{Y_1}(E) = s^*\omega_Y$ therefore $s^*\omega_Y$ is big. Since s is finite we have that ω_Y is big.

Proof of 2.8. Choose a nonzero ideal $\mathscr{I}_0 \subset \mathscr{O}_{B_1}$ with an injection $\mathscr{I}_0 \subset \omega_{B_1}$, and an ideal $\mathscr{I}_1 \subset \mathscr{O}_B$ such that $\omega_{B_1} \otimes \rho^{-1}\mathscr{I}_1 \subset \rho^*\omega_B$. Given a fiber space $f : Y \to B$, with $g : Y_1 \to Y$ as above, we have that the ideal $\mathscr{I}_1$ vanishes on the branch locus of g. Set $\mathscr{I} = \mathscr{I}_1\rho^{-1}\mathscr{I}_2$. Assume that $\omega_{Y_1/B_1} \otimes g^{-1}\mathscr{I}$ is big, then $\omega_{Y_1} \otimes (\rho \circ g)^{-1}\mathscr{I}_2$ is big, therefore $\omega_{Y_1}(-g^*B)$ is big, and by the lemma we have that ω_Y is big. □

3 Maximal variation and Kollár's theorems

3.1 Pointed birational moduli

The following is an immediate generalization of Kollár's generic moduli theorem ([Ko], 2.4):

Theorem 3.1 (Pointed birational moduli theorem). *Let $f : X \to B$ be a smooth family of varieties of general type. There exist open sets $U \subset B$ and $V \subset f^{-1}U$, and projective varieties Z and W_n, $n \geq 1$, with a diagram:*

$$\begin{array}{ccccccc} V_B^n & \to & V_B^{n-1} & \to & \cdots & \to & U \\ \downarrow m_n & & \downarrow m_{n-1} & & & & \downarrow m_0 \\ W_n & \to & W_{n-1} & \to & \cdots & \to & Z \end{array}$$

satisfying the following requirements:
1. The morphisms m_n are dominant.
2. If $P = (P_1, \ldots, P_n), P' = (P'_1, \ldots, P'_n) \in V_B^n, f_n(P) = b, f_n(P') = b' \in U$, then $m_n(P) = m_n(P')$ if and only if there exists a birational map $g : V_b \dashrightarrow V_{b'}$ which is defined and invertible at P_i, such that $g(P_i) = P'_i$.
3. For general $b \in U$, let G be the birational automorphism group of X_b, then the fiber of W_n over $m_0(b)$ is birational to X_b^n/G, where G acts diagonally.
4. There are canonical generically finite rational maps $W_{nk} \dashrightarrow (W_n)_Z^k$.

Sketch of proof. Parts (3) and (4) follow from (2). The proof of (1) and (2) is a simple modification of [Ko], 2.4, where we let PGL act on the universal family over the Hilbert scheme and its fibered powers. □

3.2 Reduction of Theorem 0.1 to Theorem 0.2

Recall by Corollary 2.4 that for sufficiently large n the general fiber of $W_n \to Z$ is of general type. Also, a simple lemma below shows that for large n the family $W_n \to Z$ is of maximal variation. Assuming that theorem 0.2 holds true, we have that for large k the variety $(W_n)_Z^k$ is of general type, therefore W_{nk} is of general type. For any $n' > nk$, applying the additivity theorem (Satz III of [V1]) to $W_{n'} \dashrightarrow W_{nk}$ we have that the variety $W_{n'}$ is of general type. Therefore $X_B^{n'}$ dominates a variety of general type.

Lemma 3.2. *Suppose $X \to B$ is a flat family of varieties of general type, where* $\dim B = \mathrm{Var}(X/B) = 1$, *and $G \subset \mathrm{Aut}_B X$. Then for sufficiently large n, the quotient by the diagonal action $W_n = X_B^n/G \to B$ has* $\mathrm{Var}(W_n/B) = 1$.

Proof. This is immediate from the theorems of Kobayashi-Ochiai (see [D-M]) and Maehara ([Mae], appendix; see also [Mor]). Using Proposition 2.4, choose n so that the general fiber of W_{n_0} over B is of general type. We show that $\mathrm{Var}(W_{n_0+1}/B) = 1$, and by induction this follows for any higher n. Assume the opposite. We have the projection map $W_{n_0+1} \to W_{n_0}$. The theorem of Maehara implies that $\mathrm{Var}(W_{n_0}/B) = 0$: a family of varieties of genral type dominated by a fixed variety is isotrivial. The theorem of Kobayashi-Ochiai (which is also included in Maehara's paper) implies that the map $W_{n_0+1} \to W_{n_0}$ is birationally isotrivial: a family of rational maps from a fixed variety to a fixed variety of general type is isotrivial. But the general fiber of $(W_{n_0+1})_b \to (W_{n_0})_b$ is isomorphic to X_b (one only needs to avoid the fixed point set of the group action) - implying that $Var(X/B) = 0$. □

3.3 *Kollár's bigness theorem*

Here we introduce the main source for global sections.

Theorem 3.3 (Kollár's bigness theorem, [Ko], I, p. 363). *Suppose that* $\pi: X \to B$ *is a fiber space of positive dimensional varieties of general type, and* $\mathrm{Var}(X/B) = \dim B$. *Assume both* X *and* B *are smooth. There is an integer* $n > 0$ *such that the sheaf* $\pi_*\omega_\pi^n$ *is* weakly big. □

Kollár's use of *weakly big* requires saturations, which means that the sections obtained may have poles over exceptional divisors of the map $X \to B$. From this one first deduces:

Corollary 3.4 ([V2], Corollary 7.2). *Suppose that* $\pi : X \to B$ *is as above. There is a divisor* D *on* X *such that* $\mathrm{codim}(\pi(\mathrm{supp}D)) > 1$, *and such that* $\omega_\pi(D)$ *is big.* □

We still have the annoying divisor D. Our method below will allow us to ignore it, but actually a trick of Viehweg ([V2], Lemma 7.3) makes it easier. Viehweg simply applies the theorem above to $X' \to B'$ where X' is a desingularization of a flattening of X, where any exceptional divisor for X'/B' is exceptional for X'/X. Since $\omega_{B'/B}$ is effective, one immediately obtains:

Theorem 3.5 (Kollár-Viehweg). *Suppose that* $\pi : X \to B$ *is as above. Then* ω_π *is big.* □

4 Pluri-nodal reduction

4.1 *Statement*

Let $X_0 \to B_0$ be a fiber space. We need to dominate it by a pluri-nodal family, so that it becomes a quotient by the action of a finite group.

To this end, we prove the following lemma, which is a variant of de Jong's results in [dJ], Sects. 6 and 7. The proof is based on that of de Jong.

Lemma 4.1 (Galois pluri-nodal reduction lemma). *There exists a diagram*

$$\begin{array}{ccc} Y & \rightarrow & X_0 \\ \downarrow & & \downarrow \\ B_1 & \rightarrow & B_0 \end{array},$$

and a finite group $G \subset \mathrm{Aut}_{X_0 \tilde{\times}_{B_0} B_1} Y$ *such that* $B_1 \rightarrow B_0$ *is an alteration,* $Y/G \rightarrow X_0 \tilde{\times}_{B_0} B_1$ *is birational and* $Y \rightarrow B_1$ *is a pluri-nodal family.*

Proof. We proceed by induction. The setup is as follows: suppose we have $X \rightarrow Z \rightarrow B$ a pair of fiber spaces, where $X \rightarrow Z$ is pluri-nodal, and a finite group $G_0 \subset \mathrm{Aut}_B(X \rightarrow Z)$. We also assume that we have a birational morphism $X/G_0 \rightarrow X_0 \tilde{\times}_{B_0} B$. We will produce a diagram

$$\begin{array}{ccccccc} X' & \rightarrow & Z'' & \rightarrow & Z' & \rightarrow & B' \\ \downarrow & & \downarrow & & & & \downarrow \\ X & \rightarrow & Z & & \longrightarrow & & B \end{array}$$

with the following properties:

1. the vertical arrows are alterations,
2. the horizontal arrows are fiber spaces,
3. the morphism $Z'' \rightarrow Z'$ is a nodal family,
4. $X' = X \times_Z Z''$, and therefore $X' \rightarrow Z'$ is pluri-nodal,
5. there is a finite group $G' = G_0 \times G'' \subset \mathrm{Aut}_{B'}(X' \rightarrow Z'' \rightarrow Z')$, and
6. the morphism $X'/G'' \rightarrow X \tilde{\times}_B B'$ is birational, and therefore $X'/G' \rightarrow X_0 \tilde{\times}_{B_0} B'$ is birational.

The basis of the induction is $X_0 \rightarrow X_0 \rightarrow B_0$ with G_0 trivial. The induction ends with $Z' \rightarrow B'$ being birational, in which case we set $Y := X'$, $B_1 := Z'$, $G := G'$ and the lemma will be proved.

Let $G_Z \subset \mathrm{Aut}_B Z$ be the image of G_0, and denote $W = Z/G_Z$.

Lemma 4.2. *There exists a dominant rational map* $Z/G_Z \dashrightarrow P \rightarrow B$, *where* $P \rightarrow B$ *is a projective bundle, such that* $\dim(P) = \dim(Z) - 1$, *and such that the generic fiber of* Z *over* P *is geometrically irreducible.*

Proof. This is obvious in case $\mathrm{rel.}\dim(Z/B) = 1$, so assume $\mathrm{rel.}\dim(Z/B) > 1$. Denote this relative dimension by r. Since we are looking for a rational map, we may replace B by its generic point η, and replace Z by Z_η. Let $W = Z/G$, choose an embedding $W \subset \mathbb{P}^N$, and let $f : Z \rightarrow \mathbb{P}^N$ be the induced morphism. According to [J], 6.3(4), for general hyperplane $H \subset \mathbb{P}^N$ we have $f^{-1}H$ geometrically irreducible. Continuing by induction, there is a linear series $(\mathbb{P}^{r-1})^*$ of dimension $r-1$ of hyperplanes in $\mathbb{P}^N$ such that the general fiber of $Z \dashrightarrow \mathbb{P}^{r-1}$ is a geometrically irreducible curve. □

The normalization of the closure of the graph of the rational map $Z \dashrightarrow P$ gives a G_Z-equivariant resolution of indeterminacies

$$\begin{array}{ccc} Z_1 & \rightarrow & P \\ \downarrow & & \\ Z & & \end{array}$$

Let $X_1 = X \times_Z Z_1$. Then $X_1 \to Z_1$ is pluri-nodal, and the action of G_0 on X lifts to X_1 (if $x_1 = (x, z_1) \in X_1$ and $g \in G_0$ then $(g(x), g(z_1)) \in X_1$ as well).

We will now perform a canonical nodal reduction for $Z_1 \to P$ using the Kontsevich space of stable maps. The generic fiber of $Z_1 \to P$ is a normal curve, and therefore smooth. Choose a projective embedding $Z_1 \subset \mathbb{P}^N$. Let d be the degree of the generic fiber of $Z_1 \to P$ and let g be its genus. By [B-M], Theorem 3.14, there exists a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,0}(Z_1, d)$ parametrizing stable maps $C \to Z_1$ of curves of genus g and degree d. By [F-P] this stack admits a projective coarse moduli space. In particular, this implies that there is a finite cover $\rho : M \to \overline{\mathcal{M}}_{g,0}(Z_1, d)$ where M is a projective scheme admitting a stable map $(C \to M, f : C \to Z_1)$ whose moduli map is ρ.

Let $\eta \in P$ be the generic point. The pair $((Z_1)_\eta \to \eta, (Z_1)_\eta \hookrightarrow Z_1)$ is a stable map of genus g and degree d, therefore we have a rational map $P \dashrightarrow \overline{\mathcal{M}}_{g,0}(Z_1, d)$. We can choose a normal resolution of indeterminacies

$$\begin{array}{ccc} P_2 & \to & M \\ \downarrow & & \\ P & & \end{array}$$

such that there is a finite group $G_1 \subset \mathrm{Aut}_P P_2$ with $P_2/G_1 \to P$ birational. Let $Z_2 = C \times_M P_2$. We have an induced stable map $(Z_2 \to P_2, f_2 : Z_2 \to Z_1)$, in particular $Z_2 \to P_2$ is nodal. Over the generic point of P_2 this coincides with $Z_1 \times_P P_2$.

Since stable reduction over a normal base is unique when it exists (see [dJ-O], 2.3), the action of G_1 lifts to Z_2, and it lifts to $X_2 = X_1 \times_{Z_1} Z_2$ as well by pulling back as before. Let $P_2 \to B_2 \to B$ be the Stein factorization. Since the Stein factorization is unique we have canonically an action of G_1 on B_2. Let $G_2 \subset G_1$ be the subgroup acting trivially on B_2. Then $G = G_0 \times G_2 \subset \mathrm{Aut}_{B_2}(X_2 \to P_2)$. We have $X_2 \to P_2$ pluri-nodal, and $X_2/G_2 \to X \tilde{\times}_B B_2$ birational. If we denote $X' := X_2$, $Z'' := Z_2$, $Z' := P_2$, $B' := B_2$ and $G'' := G_2$ we have obtained the goal of the induction step. □

4.2 Mild singularities

We want to show that pluri-nodal families are mild. This seems to be well known (see [Has], Sect. 4), but in our case we can give a proof which is sufficiently short to include here. The following lemma is well known (see [V2], Lemma 3.6):

Lemma 4.3. *Let $Y \to B$ be a nodal family such that B is smooth and the discriminant locus is a divisor of normal crossings. Then Y is rational - Gorenstein.*

(The proof is by taking formal coordinates near a singular point of the form $xy = t_1^{k_1} \cdots t_r^{k_r}$, and either resolving singularities explicitly or noting that this is a toroidal singularity.)

Proposition 4.4. *Let $Y \to B$ be a nodal family such that B is rational - Gorenstein. Then Y is rational - Gorenstein.*

Proof. Let $r : B_1 \to B$ be a resolution of singularities, $Y_1 \to B_1$ the pullback, and assume that the discriminant locus of $Y_1 \to B_1$ is a divisor of normal crossings. Let $f : Y_1 \to Y$ be the induced map. Then $r_*\omega_{B_1} = \omega_B$ and $f^*\omega_{Y/B} = \omega_{Y_1/B_1}$, and by the projection formula we obtain that $f_*\omega_{Y_1} = \omega_Y$. □

By induction we obtain:

Corollary 4.5. *If $\pi : Y \to B$ is a pluri-nodal family where B is rational - Gorenstein, then Y is rational - Gorenstein. In particular the n-th fibered power Y_B^n is rational - Gorenstein.*

Thus pluri-nodal families are mild.

5 Proof of the theorem

Let $X_0 \to B_0$ be a smooth projective family of varieties of general type of maximal variation. Choose a model $X \to B$ where both X and B are projective nonsingular. By 4.1 we may assume, after an alteration $B_1 \to B$, that we have a birational morphism $g_0 : Y/G = X_1 \to X\tilde{\times}_B B_1$ where $\pi_Y : Y \to B_1$ is a pluri-nodal family and $G \subset \mathrm{Aut}_{B_1} Y$ a finite group. Choose a resolution of singularities $r : X_2 \to X_1$ and denote by $\pi_2 : X_2 \to B_1$ the projection. We have a diagram:

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow q & & \\ X_2 & \xrightarrow{r} & X_1 & \xrightarrow{g_0} & X \\ & \searrow^{\pi_2} & \downarrow & & \downarrow \\ & & B_1 & \to & B \end{array} \tag{1}$$

According to 2.7 (where we set $V = Y$ and $W = X_1$) there is an ideal $\mathcal{I}_F \subset \mathcal{O}_{B_1}$ such that $(\mathcal{I}_F^{e(G)\cdot n}) \cdot \mathcal{J}_{m,n}$ gives sufficient conditions for invariant n-differentials to descend from $Y_{B_1}^m$. For arbitrary integer $m > 0$ let $\mathcal{X}_m \to X_B^m$ be a resolution of singularities of the main component, and let $\mathcal{W}_m \to (X_1)_{B_1}^m$ be a resolution of singularities, dominating $\mathcal{X}_m$. According to 2.8 (applied to $B_1 \to B$) there is an ideal $\mathcal{I} \subset \mathcal{O}_{B_1}$, such that for any m, if $\omega_{\mathcal{W}_m/B_1} \otimes \mathcal{I}$ is big then $\mathcal{X}_m$ (and therefore $(X_0)_{B_0}^m$) is of general type.

By the Kollár-Viehweg Theorem 3.5, ω_{π_2} is big. Therefore $q^*r_*\omega_{\pi_2}/\text{torsion}$ is big. We have by definition that $\omega_{\pi_Y} \otimes \mathcal{J}_1(G, Y)$ is big. Therefore, for sufficiently large n we have that $\omega_{\pi_Y}^n \otimes \mathcal{J}_n \mathcal{I} \mathcal{I}_F^{e(G)}$ is big. Pulling back along all the projections $p_i : Y_{B_1}^m \to Y$ we have that $\omega_{Y_{B_1}^m/B_1}^n \otimes \mathcal{J}_{m,n} \mathcal{I}^m \mathcal{I}_F^{m\cdot e(G)}$ is big. In particular, if $m > n$, we have that $\omega_{Y_{B_1}^m/B_1}^n \otimes \mathcal{J}_{m,n} \mathcal{I}^n \mathcal{I}_F^{n\cdot e(G)}$ is big. By 2.7, taking invariants we have that $\omega_{\mathcal{W}_m/B_1} \otimes \mathcal{I}$ is big, and by 2.8 we have that $(X_0)_{B_0}^m$ is of general type for large m.

5.1 An alternative approach

The following argument gives a variation on the proof which is more in line with [Ko] and [V2]. Having chosen the diagram (1), we can alter it as follows:

using semistable reduction in codimension 1 (see [KKMS] II, and [Ka], theorem 17), we can find a nonsingular alteration $B_1' \to B_1$, a variety $X_2' \to B_1'$, and a birational morphism $X_2' \to X_2 \tilde{\times}_{B_1} B_1'$ satisfying the following conditions
1. The discriminant locus Δ of $X_2' \to B_1'$ is a divisor of normal crossings.

2. There is a closed subset $F \subset \Delta$ with $\operatorname{codim}(F, B_1') \geq 2$, and $U = B_1' \setminus F \overset{i}{\hookrightarrow} B_1'$, such that the restriction $X_2'|_U \to U$ is semistable. In particular, this restriction is mild (see [V2], Lemma 3.6).

In fact, using the techniques introduces by de Jong, one should be able to show that $F \subset \operatorname{Sing}(\Delta)$.

Let $X_1' = X_1 \times_{B_1} B_1'$ and $Y' = Y \times_{B_1} B_1'$. We can replace B_1, X_1, X_2, Y by B_1', X_1', X_2', Y' and assume that conditions (1) and (2) are satisfied.

Let $\pi_{X_{(m)}} : X_{(m)} \to B_1$ be the main component of $(X_1)^m_B$. Choose a resolution of singularities $W_m \to X_{(m)}$, and let $\pi_{W_m} : W_m \to B_1$ be the associated projection. Denote $\mathscr{F}_{m,n} = \pi_{W_m *} \omega_{\pi_{W_m}}$ and $\mathscr{G}_{m,n} = (\mathscr{F}_{m,n})^{**}$. Since the restriction of W_m to U is mild, we have that $\mathscr{G}_{m,n} = i_* i^* \mathscr{F}_{m,n}$. Applying 2.6, we obtain:
1. We have natural morphisms

$$\mathscr{G}_{m_1,n} \otimes \mathscr{G}_{m_2,n} \to \mathscr{G}_{m_1+m_2,n}$$

(by pulling back sections to $W_{m_1+m_2}$ over U, multiplying and extending).
2. We have natural morphisms

$$\mathscr{G}_{m,n_1} \otimes \mathscr{G}_{m,n_2} \to \mathscr{G}_{m,n_1+n_2}$$

(by multiplying sections).
3. We have

$$\mathscr{G}_{m,n} \otimes \mathscr{I}_F^{n(e(G)-1)} \subset \mathscr{F}_{m,n} \subset \mathscr{G}_{m,n}$$

(by 2.7. Notice that the remark after 2.7 shows that $\mathscr{I}_F^n$ suffices).

By Kollár's theorem $\mathscr{G}_{1,n}$ is big for sufficiently large n. We can choose an ideal $\mathscr{I}$ as in 2.8. By (2) above, for sufficiently large n we have that $\mathscr{G}_{1,n} \otimes \mathscr{I}\mathscr{I}_F^{e(G)}$ is big, and using (1) above we have that for sufficiently large m, $\mathscr{G}_{m,n} \otimes \mathscr{I}^m \mathscr{I}_F^{n \cdot e(G)}$ is big, therefore by (3) $\mathscr{F}_{m,n} \otimes \mathscr{I}^m$ is big, which is what we need.

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