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The versal deformation of an isolated toric Gorenstein singularity

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Abstract. Given a lattice polytope $Q \subseteq \mathbb{R}^n$, we define an affine scheme $\mathcal{M}$ that reflects the possibilities of splitting Q into a Minkowski sum. Denoting by Y the toric Gorenstein singularity induced by Q, we construct a flat family over $\bar{\mathcal{M}}$ with Y as special fiber. In case Y has an isolated singularity, this family is versal.

1 Introduction

(1.1) The whole deformation theory of an isolated singularity is encoded in its so-called versal deformation. For complete intersection singularities this is a family over a smooth base space obtained by certain perturbations of the defining equations.

As soon as we leave this class of singularities, the structure of the family, and sometimes even the base space, will be more complicated. It is well known that the base space may consist of several components or may be non-reduced. In (9.2) we will present a (three-dimensional) example of a singularity admitting a fat point as base space of its versal deformation.

For two-dimensional cyclic quotient singularities (coinciding with the two-dimensional affine toric varieties), the computations of Arndt, Christophersen, Kollár/ Shepherd-Barron, Riemenschneider, and Stevens provide a description of the versal family - in particular, the number and dimension of the components of the reduced base (these components are smooth) are computed.

Christophersen observed that the total spaces over these components are toric varieties again (cf. [Ch]). This suggests that the entire deformation theory of affine toric varieties might remain inside this category. It should be a challenge to find the versal deformation, its base space, or the total spaces over the components

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by purely combinatorial methods.

(1.2) Affine toric varieties are constructed from rational, polyhedral cones $\sigma \subseteq \mathbb{R}^{n+1}$: One takes the dual cone

$$\sigma^{\vee} := \{r \in (\mathbb{R}^{n+1})^* \mid \langle a, r \rangle \geq 0 \text{ for each } a \in \sigma\},$$

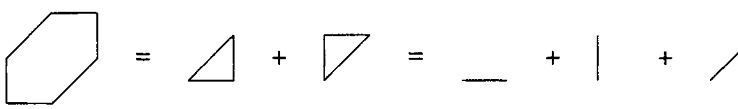
and Y_σ is defined as the spectrum of the semigroup algebra $\mathbb{C}[\sigma^{\vee} \cap (\mathbb{Z}^{n+1})^*]$. In particular, equations of Y_σ are induced from affine relations between lattice points of $\sigma^{\vee} \subseteq (\mathbb{R}^{n+1})^*$. In the following we will no longer differentiate between $\mathbb{R}^{n+1}$ and its dual space; however, for vectors, we will try to use parentheses and brackets for primal and dual ones, respectively. See [Ke] or [Od] for an introduction into the subject of toric varieties.

For investigating versal deformation spaces, Gorenstein singularities could serve as the first class to study beyond complete intersections. Ishida gave a nice description of this class inside toric varieties (cf. [Ish], Theorem 7.7): Y_σ is Gorenstein if and only if σ equals the cone over some lattice polytope $Q \subseteq \mathbb{R}^n$ (i.e. its vertices are lattice points) embedded into height one.

Therefore, our point of view will be the following: Given a lattice polytope $Q \subseteq \mathbb{R}^n$, we want to study the deformation theory of the affine, toric variety Y_σ with $\sigma := \operatorname{Cone}(Q) \subseteq \mathbb{R}^{n+1}$. Examples of these singularities are Del Pezzo surfaces of degree ≥ 6 (cf. (9.1)).

(1.3) The main tool to describe our results is the notion of Minkowski sums:

Definition. *For two polytopes $P, P' \subseteq \mathbb{R}^n$ we define their Minkowski sum as the polytope $P + P' := \{p + p' \mid p \in P, p' \in P'\}$. Obviously, this notion also makes sense for translation classes of polytopes.*



(See (9.3) for another illustration of this notion.) Each Minkowski summand of a given polytope $Q \subseteq \mathbb{R}^n$ (or some multiple of Q) contains, up to the length, the same edges as Q itself. This fact enables us to handle the "moduli space" $C(Q)$ of Minkowski summands which is a polyhedral cone (cf. (2.2)).

Attaching each Minkowski summand at the point that represents it in $C(Q)$ yields the so-called tautological cone $\tilde{C}(Q)$ together with a projection onto $C(Q)$. Its construction is very similar to that of a universal bundle, and indeed, applying the functor that makes toric varieties from cones will provide the main step toward constructing the versal base space of Y_σ (cf. Sect. 4).

(1.4) For a given lattice polytope $Q \subseteq \mathbb{R}^n$ with primitive edges, i.e. they do not contain any interior lattice points, we begin in Sect. 2 with describing an affine scheme $\bar{\mathcal{M}}$ which seems to be interesting independently from the toric or deformation context. It describes the possibilities of splitting Q into Minkowski summands. The underlying reduced space is an arrangement of planes corresponding to those Minkowski decompositions involving summands that are lattice polytopes themselves. Since all the proofs are based on quite the same method, we have collected them in a separate section. Each theorem of Sect. 2 can be translated into an easier language and corresponds to a certain lemma of Sect. 3.

In Sect. 4 we study the tautological cone $\tilde{C}(Q)$. This leads in Sect. 5 to the construction of a flat family over $\bar{\mathcal{M}}$ with Y_σ (σ = Cone(Q)) as special fiber. Note that for Y_σ the assumption of Q having primitive edges means smoothness in codimension two. Computing the Kodaira-Spencer map (in Sect. 6) as well as the obstruction map (in Sect. 7) shows that for isolated singularities the family is versal (nevertheless trivial for $\dim Q \geq 3$, cf. (6.3)). Its components are described in Sect. 8.

In the general case, the Kodaira-Spencer map is an isomorphism onto the homogeneous part of T^1_Y with the most interesting multidegree (cf. Theorem (6.2)), and the obstruction map is still injective (cf. Theorem (7.2)).

Throughout the paper, an example accompanies the general theory. Further examples can be found in Sect. 9.

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2 The Minkowski scheme of a lattice polytope

(2.1) Let $Q \subseteq \mathbb{R}^n$ be a lattice polytope, i.e. the vertices are contained in $\mathbb{Z}^n$. We will always assume that the edges do not contain any interior lattice points. Hence, after choosing orientations they are given by primitive vectors $d^1, \dots, d^N \in \mathbb{Z}^n$.

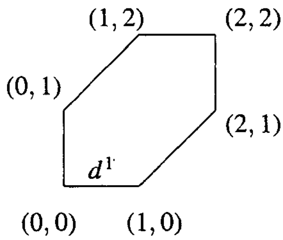
Definition. *For every 2-face $\varepsilon < Q$ we define its sign vector $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N) \in \{0, \pm 1\}^N$ by*

$$\varepsilon_i := \begin{cases} \pm 1 & \textit{if } d^i \textit{ is an edge of } \varepsilon \\ 0 & \textit{otherwise} \end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ fit into a cycle along the boundary of ε. This determines $\underline{\varepsilon}$ up to sign, and we choose one of both possibilities. In particular, $\sum_i \varepsilon_i d^i = 0$.

Example. Let us introduce the following example, which will be continued throughout the paper: For Q we take the hexagon

$$Q_6 := \operatorname{Conv}\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\} \subseteq \mathbb{R}^2.$$



hexagon Q_6

Starting with $d^1 := \overline{(0,0)(1,0)}$, the anticlockwise oriented edges are denoted by $d^1, \ldots, d^6$. As vectors they equal

$$d^1 = (1,0)\,;\; d^2 = (1,1)\,;\; d^3 = (0,1)\,;$$
$$d^4 = (-1,0)\,;\; d^5 = (-1,-1)\,;\; d^6 = (0,-1)\,.$$

Q_6 is 2-dimensional, hence, it is its own unique 2-face $\varepsilon = Q$. For $\underline{Q}$ we take $\underline{Q} = (1, \ldots, 1)$.

(**2.2**) We define the vector space $V \subseteq \mathbb{R}^N$ by

$$V := V(Q) := \{(t_1, \ldots, t_N) \mid \sum_i t_i\, \varepsilon_i\, d^i = 0 \text{ for every 2-face } \varepsilon < Q\}.$$

Then, $C(Q) := V \cap \mathbb{R}^N_{\geq 0}$ is obviously a rational, polyhedral cone in V.

Lemma. *The points of $C(Q)$ correspond to the Minkowski summands of positive multiples of Q.*

Proof. For an element $(t_1, \ldots, t_N) \in C(Q)$, the corresponding summand $Q_{\underline{t}}$ is built by the edges $t_i \cdot d^i$ $(i = 1, \ldots, N)$ as follows: Assume that $0 \in \mathbb{R}^n$ coincides with some vertex of the lattice polytope Q. Then, each vertex a of Q can be reached from 0 by some walk along the edges of Q. We obtain

$$a = \sum_{i=1}^{N} \lambda_i\, d^i \text{ for some } \underline{\lambda} = (\lambda_1, \ldots, \lambda_N)\,,\; \lambda_i \in \mathbb{Z}.$$

Now, given an element $\underline{t} \in C(Q)$, we may define the corresponding vertex $a_{\underline{t}}$ by

$$a_{\underline{t}} := \sum_{i=1}^{N} t_i\, \lambda_i\, d^i\,.$$

The linear equations defining $V = \operatorname{span} C(Q)$ ensure that this definition does not depend on the particular path from 0 to a through the 1-skeleton of Q. The polytope $Q_{\underline{t}}$ is defined as the convex hull of all the $a_{\underline{t}}$. Finally, it is clear that all Minkowski summands arise in this way. □

For a particular Minkowski summand Q' we will denote the corresponding point in the cone by $\varrho(Q') \in C(Q)$.

Example. 1) Applying ϱ, the two splittings of Q_6 drawn in (1.3) become

$$\begin{aligned}(1,1,1,1,1,1) &= (1,0,1,0,1,0)+(0,1,0,1,0,1)\\ &= (1,0,0,1,0,0)+(0,0,1,0,0,1)+(0,1,0,0,1,0).\end{aligned}$$

2) In any case we have $\varrho(t \cdot Q) = (t, \ldots, t) \in C(Q) \subseteq V \subseteq \mathbb{R}^N$.

(2.3) For each 2-face $\varepsilon < Q$ and for each integer $k \geq 1$ we define the (vector valued) polynomial

$$g_{\varepsilon,k}(\underline{t}) := \sum_{i=1}^{N} t_i^k \, \varepsilon_i \, d^i \, .$$

Using coordinates of $\mathbb{R}^n$, the $g_{\varepsilon,k}(\underline{t})$ become regular polynomials; for each pair (ε, k) we will get two linearly independent ones. We obtain an ideal

$$\mathcal{J} := \left(g_{\varepsilon,k} \mid \varepsilon < Q,\ k \geq 1\right) \subseteq \mathbb{C}[t_1, \ldots, t_N]$$

which defines an affine closed subscheme

$$\mathcal{M} := \operatorname{Spec} \mathbb{C}[\underline{t}] \big/ \mathcal{J} \subseteq V_{\mathbb{C}} \subseteq \mathbb{C}^N .$$

Example. For our hexagon Q_6 introduced in (2.1) we obtain

$$\mathcal{J} = \left(t_1^k + t_2^k - t_4^k - t_5^k,\ t_2^k + t_3^k - t_5^k - t_6^k \mid k \geq 1\right).$$

Of course, finitely many polynomials are sufficient to generate the ideal $\mathcal{J}$ - but we can even give an effective criterion to see which equations may be dropped:

Proposition. *Let $\varepsilon < Q$ be a 2-face. Then ε is contained in a two-dimensional subspace of $\mathbb{R}^n$, and this vector space comes with a natural lattice (the restriction of the big lattice $\mathbb{Z}^n$).*

If ε is contained in two different strips defined by pairs of parallel lines of lattice-distance $\leq k_0$ each, then the equations $g_{\varepsilon,k}$ $(k > k_0)$ are contained in the ideal generated by $g_{\varepsilon,1}, \ldots, g_{\varepsilon,k_0}$.

Proof. cf. (3.3).

Example. Obviously, Q_6 is contained in at least three different strips of thickness 2. Hence, $\mathcal{J}$ is generated by polynomials of degree ≤ 2:

$$\mathcal{J} = \left(t_1 + t_2 - t_4 - t_5,\ t_2 + t_3 - t_5 - t_6,\ \ t_1^2 + t_2^2 - t_4^2 - t_5^2,\ t_2^2 + t_3^2 - t_5^2 - t_6^2\right).$$

(2.4) Denote by ℓ the canonical projection

$$\ell : \mathbb{C}^N \longrightarrow \mathbb{C}^N / \mathbb{C}\cdot(1,\dots,1) = \mathbb{C}^N / \mathbb{C}\cdot\varrho(Q)\,.$$

On the level of regular functions this corresponds to the inclusion $\mathbb{C}[t_i - t_j \mid 1 \le i,j \le N] \subseteq \mathbb{C}[t_1,\dots,t_N]$.

Theorem. (See also Remark (4.4))

(1) $\mathcal{J}$ *is generated by polynomials from* $\mathbb{C}[t_i - t_j]$, *i.e.* $\mathcal{M} = \ell^{-1}(\bar{\mathcal{M}})$ *for some affine closed subscheme* $\bar{\mathcal{M}} \subseteq V_{\mathbb{C}}/\mathbb{C}\cdot\varrho(Q) \subseteq \mathbb{C}^N/\mathbb{C}\cdot\varrho(Q)$. $\bar{\mathcal{M}}$ *is defined by the ideal* $\mathcal{J} \cap \mathbb{C}[t_i - t_j]$.

(2) $\mathcal{J} \subseteq \mathbb{C}[t_1,\dots,t_N]$ *is the smallest ideal that meets property (1) and, on the other hand, contains the "toric" equations*

$$\prod_{i=1}^{N} t_i^{d_i^+} - \prod_{i=1}^{N} t_i^{d_i^-} \quad \textit{with}$$

$\underline{d} \in \mathbb{Z}^N \cap \operatorname{span}\{[\langle\varepsilon_1 d^1, c\rangle,\dots,\langle\varepsilon_N d^N, c\rangle] \mid \varepsilon < Q \text{ 2-face}, c \in \mathbb{R}^n\}$. (For an integer h we denote

$$h^+ := \begin{cases} h & \text{if } h \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad ; \quad h^- := \begin{cases} 0 & \text{if } h \ge 0 \\ -h & \text{otherwise} \end{cases} .)$$

Proof. cf. (3.4).

Example. Toric equations for Q_6 are for instance $t_1 t_2 - t_4 t_5$, $t_2 t_3 - t_5 t_6$, and $t_1 t_6 - t_3 t_4$.

(2.5) We want to describe the structure of the underlying reduced spaces of $\mathcal{M}$ or $\bar{\mathcal{M}}$. Let $Q = R_0 + \dots + R_m$ be a decomposition of Q into a Minkowski sum of $m+1$ lattice polytopes. Then, the N-tuples $\varrho(R_0),\dots,\varrho(R_m)$ have entries 0 and 1 only, and they sum up to $(1,\dots,1)$. In particular, the $(m+1)$-plane $\mathbb{C}\cdot\varrho(R_0) + \dots + \mathbb{C}\cdot\varrho(R_m) \subseteq \mathbb{C}^N$ is contained in $\mathcal{M}$. It is given by the linear equations $t_i - t_j = 0$ if d^i, d^j belong to a common summand R_v.

Refinements of Minkowski decompositions (they form a partially ordered set) correspond to inclusions of the associated planes.

Theorem. $\mathcal{M}_{\text{red}}$ *equals the union of those flats corresponding to maximal Minkowski decompositions of* Q *into lattice summands.* $\bar{\mathcal{M}}_{\text{red}}$ *consists of their images via* ℓ.

Proof. cf. (3.5).

Example. $\mathcal{M}(Q_6)$ and $\bar{\mathcal{M}}(Q_6)$ are reduced schemes (for non-reduced examples cf. Sect. 9). Let us study them directly:

- The linear equations allow the following substitution:

$$\begin{array}{rcl} t & := & t_1 \\ s_1 & := & t_1 - t_3 \\ s_2 & := & t_4 - t_2 \\ s_3 & := & t_1 - t_4 \end{array} \qquad \begin{array}{rcl} t_1 & = & t \\ t_2 & = & t - s_2 - s_3 \\ t_3 & = & t - s_1 \\ t_4 & = & t - s_3 \\ t_5 & = & t - s_2 \\ t_6 & = & t - s_1 - s_3 \,. \end{array}$$

- The two quadratic equations transform into $s_1 s_3 = s_2 s_3 = 0$.

In particular, $\bar{\mathcal{M}}$ is the union of a line and a 2-plane - corresponding to the Minkowski decompositions

$$\begin{array}{rcl} Q_6 & = & \operatorname{Conv}\{(0,0),(1,0),(1,1)\} + \operatorname{Conv}\{(0,0),(0,1),(1,1)\} \text{ and} \\ Q_6 & = & \operatorname{Conv}\{(0,0),(1,0)\} + \operatorname{Conv}\{(0,0),(0,1)\} + \operatorname{Conv}\{(0,0),(1,1)\} \end{array}$$

already mentioned in (2.2) and depicted in (1.3).

(2.6) $\bar{\mathcal{M}}$ (or $\mathcal{M} = \ell^{-1}(\bar{\mathcal{M}})$) reflects the possibilities of Minkowski decompositions of Q:

- The underlying reduced space encodes the decompositions of Q into lattice summands.
- Extremal decompositions into rational summands are hidden in the scheme structure of $\bar{\mathcal{M}}$.
  Its tangent space in 0 (the smallest affine space containing $\bar{\mathcal{M}}$) equals $V_{\mathbb{C}}/\mathbb{C} \cdot \varrho(Q)$ - it is the vector space arising from the cone $C(Q)$ of Minkowski summands by killing the summands homothetic to Q.

Therefore, we will call $\bar{\mathcal{M}}$ the (affine) *Minkowski scheme* of Q.

3 Proofs of the statements of Sect. 2

(3.1) Using vectors $c \in \mathbb{Z}^N$ (or certain $c \in \mathbb{R}^N$) we can evaluate the edges $d^1, \ldots, d^N$ to get integers

$$d_1 := \langle \varepsilon_1 d^1, c \rangle, \ldots, d_N := \langle \varepsilon_N d^N, c \rangle$$

for every given 2-face $\varepsilon < Q$. Doing so, the statements of Sect. 2 can be reduced to much simpler lemmas which will be presented here. Then, all these lemmas are proved using the following recipe:

(i) Assume $d_i = \pm 1$ - then the lemmas reduce to well known facts concerning symmetric functions.
(ii) Move to the general case by specialization of variables.

(3.2) For the whole Sect. 3 we use the following notation:

Let $d_1, \dots, d_N \in \mathbb{Z}$ such that $d_1, \dots, d_M \geq 0$, $d_{M+1}, \dots, d_N \leq 0$, and $\sum_{i=1}^N d_i = 0$.

$$\begin{array}{lllll} g_k(\underline{t}) & := & g_{\underline{d},k}(\underline{t}) & := & \sum_{i=1}^N d_i\, t_i^k\,, \\ p(\underline{t}) & := & p_{\underline{d}}(\underline{t}) & := & t_1^{d_1}\cdot\ldots\cdot t_M^{d_M} - t_{M+1}^{d_{M+1}}\cdot\ldots\cdot t_N^{d_N}\,. \end{array}$$

Denote by σ_k and s_k the k-th elementary symmetric polynomial and the sum of the k-th powers of a given set of variables, respectively.

Remark. For $1 \leq i,j \leq M$ or $M+1 \leq i,j \leq N$, identifying the two variables t_i and t_j (i.e. switching from $\mathbb{C}[\underline{t}]$ to $\mathbb{C}[\underline{t}]/_{t_i - t_j}$) yields the following situation:

- t_i, t_j are replaced by a common new variable $\tilde{t}$ (i.e. N is replaced by $N-1$),
- d_i, d_j are replaced by $\tilde{d} := d_i + d_j$, but
- $g_k(\underline{t}), p(\underline{t})$ keep their shapes in the new set up.

In particular, the general situation can always be obtained via factorization from the special case $d_1 = \ldots = d_M = 1$; $d_{M+1} = \ldots = d_N = -1$ (and $N = 2M$). Renaming $t_i = x_i$, $t_{M+i} = y_i$ $(i \leq M)$ it looks like

$$\begin{array}{lll} g_k(\underline{x},\underline{y}) & = & \left(\sum_{i=1}^M x_i^k\right) - \left(\sum_{i=1}^M y_i^k\right) = s_k(\underline{x}) - s_k(\underline{y})\,, \\ p(\underline{x},\underline{y}) & = & (x_1\cdot\ldots\cdot x_M) - (y_1\cdot\ldots\cdot y_M) = \sigma_M(\underline{x}) - \sigma_M(\underline{y}). \end{array}$$

(3.3) **Lemma.** *If* $k_0 := \sum_{i=1}^M d_i = -\sum_{i=M+1}^N d_i$, *then the polynomials* g_k $(k > k_0)$ *are* $\mathbb{C}[\underline{t}]$*-linear combinations of* $g_1, \dots, g_{k_0}$. (This implies Proposition (2.3).)

Proof. As previously discussed, we may regard the special case $d_i = \pm 1$. In particular, $k_0 = M$. Now, for an arbitrary $k\,(> M)$, the expression $s_k(\underline{x})$ is a polynomial in $s_1(\underline{x}), \dots, s_M(\underline{x})$, say

$$s_k(\underline{x}) = P_k\left(s_1(\underline{x}), \dots, s_M(\underline{x})\right).$$

Then,

$$g_k(\underline{x},\underline{y}) = s_k(\underline{x}) - s_k(\underline{y}) = P_k\left(s_1(\underline{x}), \dots, s_M(\underline{x})\right) - P_k\left(s_1(\underline{y}), \dots, s_M(\underline{y})\right),$$

and for each monomial $s_1^{r_1}\, s_2^{r_2} \dots s_M^{r_M}$ occurring in P_k, we have

$$\begin{aligned} s_1(\underline{x})^{r_1} \cdot\ldots\cdot s_M(\underline{x})^{r_M} - s_1(\underline{y})^{r_1}\cdot\ldots\cdot s_M(\underline{y})^{r_M} = \\ = \sum_{v=1}^{M}\sum_{i=1}^{r_v} [s_v(\underline{x}) - s_v(\underline{y})]\cdot s_1(\underline{x})^{r_1}\dots s_{v-1}(\underline{x})^{r_{v-1}}\, s_v(\underline{x})^{i-1} \\ \cdot s_v(\underline{y})^{r_v - i}\, s_{v+1}(\underline{y})^{r_{v+1}}\dots s_M(\underline{y})^{r_M} \\ = \sum_{v=1}^{M} g_v(\underline{x},\underline{y})\cdot\Big(\sum_{i=1}^{r_v} s_1(\underline{x})^{r_1}\dots s_{v-1}(\underline{x})^{r_{v-1}}\, s_v(\underline{x})^{i-1} \\ \cdot s_v(\underline{y})^{r_v - i}\, s_{v+1}(\underline{y})^{r_{v+1}}\dots s_M(\underline{y})^{r_M}\Big)\,, \end{aligned}$$

proving the lemma. □

(3.4) Lemma. (implying Theorem (2.4))

(1) The ideal $\mathcal{J} := (g_k \mid k \geq 1) \subseteq \mathbb{C}[t_1, \ldots, t_N]$ *is generated by polynomials in* $t_i - t_1$ $(i = 2, \ldots, N)$ *only.*

(2) $\mathcal{J}$ *is the smallest ideal generated by polynomials in* $t_i - t_1$*, which additionally contains* p*.*

Proof. (1) Replacing t_i by $t_i - t_1$ as arguments in g_k yields

$$\begin{aligned} g_k(t_1 - t_1, \ldots, t_N - t_1) &= \sum_{i=1}^{N} d_i\,(t_i - t_1)^k = \sum_{i=1}^{N} d_i \cdot \left(\sum_{v=0}^{k} (-1)^v\, t_1^v\, t_i^{k-v} \right) \\ &= \sum_{v=0}^{k} (-1)^v\, t_1^v \cdot \left(\sum_{i=1}^{N} d_i\, t_i^{k-v} \right) = \sum_{v=0}^{k} (-1)^v\, t_1^v\, g_{k-v}(\underline{t}). \end{aligned}$$

In particular, $\big(g_k(\underline{t}) \mid k \geq 1\big)$ and $\big(g_k(\underline{t} - t_1) \mid k \geq 1\big)$ are the same ideals in $\mathbb{C}[\underline{t}]$.

(2) Each polynomial $q(\underline{t})$ can be written uniquely as

$$q(\underline{t}) = \sum_{v \geq 0} q_v(t_2 - t_1, \ldots, t_N - t_1) \cdot t_1^v.$$

If $J \subseteq \mathbb{C}[\underline{t}]$ is an ideal generated by polynomials in $\underline{t} - t_1$ only, then for each $q(\underline{t}) \in J$ the components q_v are automatically contained in J, too. Hence, we should look for the components of the polynomial p. In the polynomial ring $\mathbb{C}[\underline{X}, \underline{Y}, T]$ we know that

$$p(T + \underline{X}, T + \underline{Y}) = (T + X_1) \cdot \ldots \cdot (T + X_M) - (T + Y_1) \cdot \ldots \cdot (T + Y_M)$$

has $\sigma_k(\underline{X}) - \sigma_k(\underline{Y})$ as coefficient of T^{M-k} $(k = 1, \ldots, M)$. On the other hand, there is a polynomial P_k and a non-vanishing rational number c_k (not depending on M) such that

$$\sigma_k(\underline{X}) = P_k(s_1(\underline{X}), \ldots, s_{k-1}(\underline{X})) + c_k \cdot s_k(\underline{X}).$$

As in the proof of the previous lemma we obtain

$$\begin{aligned} \sigma_k(\underline{X}) - \sigma_k(\underline{Y}) &= P_k(s_1(\underline{X}), \ldots, s_{k-1}(\underline{X})) - P_k(s_1(\underline{Y}), \ldots, s_{k-1}(\underline{Y})) \\ &\quad + c_k \cdot s_k(\underline{X}) - c_k \cdot s_k(\underline{Y}) \\ &= \sum_{v=1}^{k-1} q_v(\underline{X}, \underline{Y}) \cdot g_v(\underline{X}, \underline{Y}) + c_k \cdot g_k(\underline{X}, \underline{Y}) \end{aligned}$$

for some coefficients q_v. Specialization (first by $T \mapsto x_1$, $X_i \mapsto x_i - x_1$, $Y_i \mapsto y_i - x_1$, then followed by the usual one) shows that the ideal generated by the components $p_v(\underline{t} - t_1)$ of p equals $\mathscr{J}$. □

(3.5) Lemma. *Let $\underline{c} = (c_1, \ldots, c_N) \in \mathbb{C}^N$ be a point such that $g_k(\underline{c}) = 0$ for each $k \geq 1$. Then, for every fixed $c \in \mathbb{C}$, we have $\sum_{c_i = c} d_i = 0$.* (This implies Theorem (2.5).)

Proof. The equations $\sum_{i=1}^{N} d_i\, c_i^k = 0$ present 0 as a linear combination of the vectors $(c_i, c_i^2, c_i^3, \ldots)$. On the other hand, the Vandermonde tells us that this linear combination has to be a trivial one, i.e. the sum of the coefficients d_i belonging to equal variables vanishes. □

4 The tautological cone over $C(Q)$

(4.1) In (2.2) we have introduced the cone $C(Q)$ of Minkowski summands of $\mathbb{R}_{\geq 0} \cdot Q$. For an element $(t_1, \ldots, t_N) \in C(Q)$ the corresponding summand $Q_{\underline{t}}$ was built by the edges $t_i \cdot d^i$ $(i = 1, \ldots, N)$. Now, we paste the summands at the points they are assigned to:

Definition. *The tautological cone $\tilde{C}(Q) \subseteq \mathbb{R}^n \times V \subseteq \mathbb{R}^{n+N}$ is defined as*

$$\tilde{C}(Q) := \{(a, \underline{t}) \mid \underline{t} \in C(Q);\ a \in Q_{\underline{t}}\}.$$

It comes with a natural projection $\tilde{C}(Q) \to C(Q)$.

$\tilde{C}(Q)$ is (as $C(Q)$) a rational, polyhedral cone. It is generated by the pairs $(a^i_{\underline{t}^j}, \underline{t}^j)$ with

- a^i a vertex of Q and
- $\underline{t}^j$ a fundamental generator of $C(Q)$.

This follows from the simple rule $(a_{\underline{t}+\underline{t}'}, \underline{t} + \underline{t}') = (a_{\underline{t}}, \underline{t}) + (a_{\underline{t}'}, \underline{t}')$ for vertices $a \in Q$ and $\underline{t}, \underline{t}' \in C(Q)$. Defining $\sigma := \text{Cone}(Q) \subseteq \mathbb{R}^{n+1}$ by putting Q into the hyperplane $(t = 1)$, we obtain a fiber product diagram of rational polyhedral cones:

$$\begin{array}{ccccc}
[\tilde{C}(Q) \subseteq \mathbb{R}^n \times V] & \xrightarrow{\text{pr}_V} & [C(Q) \subseteq V] & \longrightarrow & \mathbb{R}^N_{\geq 0} \\
\uparrow i & & \uparrow \varrho(Q) & \nearrow \text{diagonal} & \\
[\sigma \subseteq \mathbb{R}^{n+1}] & \xrightarrow{\text{pr}_{n+1}} & \mathbb{R}_{\geq 0} & &
\end{array}$$

(The horizontal maps are projections onto the V and the $(n+1)$-th component, respectively. The inclusion i is given by $(t \cdot a;\ t) \mapsto (t \cdot a;\ t, \ldots, t)$.)

(4.2) Assigning toric varieties to polyhedral cones is functorial, i.e. we can proceed so with the whole diagram. We obtain affine toric varieties Y, X, and

S with coordinate rings $A(Y) = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^{n+1}]$, $A(X) = \mathbb{C}[\tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V_{\mathbb{Z}}^*)]$, and $A(S) = \mathbb{C}[C(Q)^\vee \cap V_{\mathbb{Z}}^*]$, respectively. (Recall that $Y = Y_\sigma$ is the toric Gorenstein singularity we want to deform.) The varieties are arranged in the following commutative diagram:

$$\begin{array}{ccccccc}
X_{\tilde{C}(Q)} & \xrightarrow{\pi} & S_{C(Q)} & \xrightarrow{p} & \mathbb{C}^N & \xrightarrow{\ell} & \mathbb{C}^N/(1,\ldots,1) \\
\uparrow i & & \uparrow & \nearrow \text{diagonal} & & & \uparrow \\
Y_\sigma & \longrightarrow & \mathbb{C} & & \longrightarrow & & \{0\}
\end{array}$$

In (4.4) and (4.7) we will see that $Y \hookrightarrow X$ is the pull back of the closed embedding $\mathbb{C} \hookrightarrow S$. Notice that $p : S \to \mathbb{C}^N$ defines functions $t_1, \ldots, t_N$ on S.

(4.3) To study the toric varieties Y, X, and S it is important to *understand the dual cones* of σ, $\tilde{C}(Q)$, and $C(Q)$, respectively. Beginning with the dual cone of σ, to each non-trivial $c \in \mathbb{Z}^n$ we associate a vertex $a(c)$ of Q and a non-negative integer $\eta_0(c)$ meeting the properties

$$\langle Q, -c\rangle \le \eta_0(c) \quad \text{and} \quad \langle a(c), -c\rangle = \eta_0(c).$$

With respect to Q, $c \neq 0$ is the inner normal vector of the affine supporting hyperplane $\langle \bullet, -c\rangle = \eta_0(c)$ through $a(c)$. In particular, $\eta_0(c)$ is uniquely determined, while $a(c)$ is not. For $c = 0$ we define $a(0) := 0 \in \mathbb{R}^n$ and $\eta_0(0) := 0 \in \mathbb{Z}$. Recall that the dual cone of σ is defined as $\sigma^\vee := \{r \in \mathbb{R}^{n+1} \mid \langle \sigma, r\rangle \ge 0\}$. Hence, by the definition of η_0, we have

$$\partial\sigma^\vee \cap \mathbb{Z}^{n+1} = \{[c, \eta_0(c)] \mid c \in \mathbb{Z}^n\}.$$

Moreover, if $c^1, \ldots, c^w \in \mathbb{Z}^n \setminus 0$ are those elements producing irreducible pairs $[c, \eta_0(c)]$ (i.e. not allowing any non-trivial lattice decomposition $[c, \eta_0(c)] = [c', \eta_0(c')] + [c'', \eta_0(c'')]$), then the elements

$$[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [\underline{0}, 1]$$

form the minimal generator set for $\sigma^\vee \cap \mathbb{Z}^{n+1}$ as a semigroup. Among them are all pairs $[c, \eta_0(c)]$ corresponding to facets (i.e. top dimensional faces) of Q. We obtain a closed embedding $Y \hookrightarrow \mathbb{C}^{w+1}$. The coordinate functions of $\mathbb{C}^{w+1}$ will be denoted by $z_1, \ldots, z_w, t$ corresponding to $[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [\underline{0}, 1]$, respectively.

Example. We continue our example Q_6 from Sect. 2. Here, the facets of Q_6 equal its edges $d^1, \ldots, d^6$, and they are sufficient for producing all irreducible pairs $[c^1, \eta_0(c^1)], \ldots, [c^6, \eta_0(c^6)]$. We have

$$c^1 = [0,1],\ c^2 = [-1,1],\ c^3 = [-1,0],\ c^4 = [0,-1],\ c^5 = [1,-1],\ c^6 = [1,0].$$

The corresponding vertices are (for instance)

$$a(c^6) = a(c^1) = (0,0), \quad a(c^2) = a(c^3) = (2,1), \quad a(c^4) = a(c^5) = (1,2),$$

and we obtain

$$\eta_0(c^1) = 0,\ \eta_0(c^2) = 1,\ \eta_0(c^3) = 2,\ \eta_0(c^4) = 2,\ \eta_0(c^5) = 1,\ \eta_0(c^6) = 0\,.$$

(4.4) Thinking of $C(Q)$ as a cone in $\mathbb{R}^N$ instead of V allows dualizing the equation $C(Q) = \mathbb{R}^N_{\geq 0} \cap V$ to get $C(Q)^\vee = \mathbb{R}^N_{\geq 0} + V^\perp$. Hence, for $C(Q)$ as a cone in V we obtain

$$C(Q)^\vee = {\mathbb{R}^N_{\geq 0} + V^\perp} \big/ {V^\perp} = \operatorname{im}[\mathbb{R}^N_{\geq 0} \longrightarrow V^*].$$

(As with $\mathbb{R}^n$, we do not use different notation for $\mathbb{R}^N$ and its dual space. Let $e_1, \dots, e_N$ be the canonical basis of the latter one.) The surjection $\mathbb{R}^N_{\geq 0} \longrightarrow$ $\longrightarrow C(Q)^\vee$ induces a map $\mathbb{N}^N \longrightarrow C(Q)^\vee \cap V^*_{\mathbb{Z}}$, which does not need to be surjective at all. This leads to the following definition:

Definition. *On $V^*_{\mathbb{Z}}$ we introduce a partial ordering "$\succeq$" by*

$$\underline{\eta} \succeq \underline{\eta}' \iff \underline{\eta} - \underline{\eta}' \in \operatorname{im}[\mathbb{N}^N \to V^*_{\mathbb{Z}}] \subseteq C(Q)^\vee \cap V^*_{\mathbb{Z}}.$$

On the geometric level, the non-saturated semigroup $\operatorname{im}[\mathbb{N}^N \to V^*_{\mathbb{Z}}] \subseteq C(Q)^\vee \cap V^*_{\mathbb{Z}}$ corresponds to the scheme theoretical image $\bar{S}$ of $p : S \to \mathbb{C}^N$, and $S \to \bar{S}$ is its normalization (cf. (5.2)). The equations of $\bar{S} \subseteq \mathbb{C}^N$ are collected in the kernel of

$$\mathbb{C}[t_1, \dots, t_N] = \mathbb{C}[\mathbb{N}^N] \xrightarrow{\varphi} \mathbb{C}[C(Q)^\vee \cap V^*_{\mathbb{Z}}] \subseteq \mathbb{C}[V^*_{\mathbb{Z}}],$$

and it is easy to see that

$$\begin{aligned} \ker \varphi &= \left(\prod_{i=1}^N t_i^{d_i^+} - \prod_{i=1}^N t_i^{d_i^-} \;\middle|\; \underline{d} \in \mathbb{Z}^N \cap V^\perp \right) \qquad \text{with} \\ V^\perp &= \operatorname{span}\left\{ [\langle \varepsilon_1 d^1, c\rangle, \dots, \langle \varepsilon_N d^N, c\rangle] \;\middle|\; \varepsilon < Q \text{ is a 2-face, } c \in \mathbb{R}^n \right\}. \end{aligned}$$

Remark. Using our new notation, we can reformulate Theorem (2.4) as: $\mathcal{M} \subseteq \mathbb{C}^N$ is the largest closed subscheme that is contained in $\bar{S}$ and, additionally, comes from $\mathbb{C}^N / \varrho(Q)$ via ℓ^{-1}.

On the other hand, dualizing the embedding $\mathbb{R}_{\geq 0} \hookrightarrow C(Q)$ yields

$$\begin{array}{ccc} C(Q)^\vee \cap V^*_{\mathbb{Z}} & \longrightarrow & \mathbb{N} \\ \underline{\eta} & \mapsto & \sum_i \eta_i \end{array}$$

at the level of semigroups. This map is surjective, even after restricting to the subset $\operatorname{im}[\mathbb{N}^N \to V^*_{\mathbb{Z}}]$: All vectors e_i corresponding to the functions t_i map onto

$1 \in \mathbb{N}$. Geometrically this means that both maps $\mathbb{C} \to S$ and $\mathbb{C} \to \tilde{S}$ are closed embeddings, and the corresponding ideals are $\left(x^{\underline{\eta}} - x^{\underline{\eta}'} \mid \underline{\eta}, \underline{\eta}' \in C(Q) \cap V_{\mathbb{Z}}^* \text{ with } \sum_i \eta_i = \sum_i \eta_i'\right)$ and $(t_i - t_j \mid 1 \le i,j \le N)$, respectively. In particular, we have made a first step towards proving the claims made in (4.2).

(4.5) In the next two sections we take a closer look at the dualized cone $\tilde{C}(Q)^{\vee}$.

Definition. *For $c \in \mathbb{Z}^n$ let $\underline{\lambda}^c = (\lambda_1^c, \ldots, \lambda_N^c) \in \mathbb{Z}^N$ describe some path from $0 \in Q$ to $a(c) \in Q$ through the 1-skeleton of Q (similar to that in (2.2)). Then,*

$$\underline{\eta}(c) := \left[-\lambda_1^c \langle d^1, c\rangle, \ldots, -\lambda_N^c \langle d^N, c\rangle\right] \in \mathbb{Z}^N$$

defines an element $\underline{\eta}(c) \in V_{\mathbb{Z}}^*$ not depending on the choice of the particular path $\underline{\lambda}^c$.

(Let $\tilde{\underline{\lambda}}^c$ be a different path from 0 to $a(c)$. It will differ from $\underline{\lambda}^c$ by some linear combination $\sum_{\varepsilon < Q} g_\varepsilon \underline{\varepsilon}$ ($g_\varepsilon \in \mathbb{Z}$ for 2-faces $\varepsilon < Q$) only. In particular, $\tilde{\lambda}_i^c \langle d^i, c\rangle - \lambda_i^c \langle d^i, c\rangle = \sum_{\varepsilon < Q} g_\varepsilon \langle \varepsilon_i\, d^i,\, c\rangle$, and we obtain $\underline{\eta}(c)_{\tilde{\lambda}} - \underline{\eta}(c)_{\lambda} \in V^{\perp}$.)

Lemma. *(i)* $\underline{\eta}(0) = 0 \in V_{\mathbb{Z}}^*$.

(ii) For all $c \in \mathbb{Z}^n$ we have $\underline{\eta}(c) \succeq 0$ (in the sense of Definition (4.4)).

(iii) $\underline{\eta}$ is convex: $\sum_v g_v\, \underline{\eta}(c^v) \succeq \underline{\eta}(\sum_v g_v\, c^v)$ for natural numbers $g_v \in \mathbb{N}$.

(iv) $\sum_{i=1}^N \eta_i(c) = \eta_0(c)$ for arbitrary $c \in \mathbb{Z}^n$.

Proof. (ii) $a(c)$ is a vertex of Q providing the minimal value of the linear function $\langle \bullet, c\rangle$. In particular, we can choose a path $\underline{\lambda}^c$ from $0 \in Q$ to $a(c)$ such that this function decreases in each step, i.e. $\lambda_i^c \langle d^i, c\rangle \le 0$ ($i = 1, \ldots, N$).

(iii) We define the following paths through the 1-skeleton of Q:

- $\underline{\lambda}$:= path from $0 \in Q$ to $a(\sum_v g_v\, c^v) \in Q$,
- $\underline{\mu}^v$:= path from $a(\sum_v g_v\, c^v) \in Q$ to $a(c^v) \in Q$ such that $\mu_i^v \langle d^i, c^v\rangle \le 0$ for each $i = 1, \ldots, N$.

Then, $\underline{\lambda}^v := \underline{\lambda} + \underline{\mu}^v$ is a path from $0 \in Q$ to $a(c^v)$, and for $i = 1, \ldots, N$ we obtain

$$\begin{aligned} \sum_v g_v\, \eta_i(c^v) - \eta_i\Big(\sum_v g_v\, c^v\Big) &= -\sum_v g_v\, (\lambda_i + \mu_i^v)\, \langle d^i, c^v\rangle \\ &\qquad + \lambda_i \Big\langle d^i, \sum_v g_v\, c^v \Big\rangle \\ &= -\sum_v g_v\, \mu_i^v\, \langle d^i, c^v\rangle \ge 0\,. \end{aligned}$$

(iv) By definition of $\underline{\lambda}^c$ we have $\sum_{i=1}^N \lambda_i^c\, d^i = a(c)$. In particular,

$$\sum_{i=1}^N \eta_i(c) = -\sum_{i=1}^N \langle \lambda_i^c\, d^i,\, c\rangle = -\langle a(c),\, c\rangle = \eta_0(c).$$

□

Example. In our hexagon Q_6 we choose the following paths from $(0,0)$ to the vertices $a(c^1), \dots, a(c^6)$, respectively:

$$\underline{\lambda}^6 = \underline{\lambda}^1 := \underline{0}, \quad \underline{\lambda}^2 = \underline{\lambda}^3 := [1,1,0,0,0,0], \quad \underline{\lambda}^4 = \underline{\lambda}^5 := [1,1,1,1,0,0].$$

They provide

$$\begin{array}{rclrcl} \underline{\eta}(c^1) &=& [0,0,0,0,0,0], & \underline{\eta}(c^2) &=& [1,0,0,0,0,0], \\ \underline{\eta}(c^3) &=& [1,1,0,0,0,0], & \underline{\eta}(c^4) &=& [0,1,1,0,0,0], \\ \underline{\eta}(c^5) &=& [-1,0,1,1,0,0], & \underline{\eta}(c^6) &=& [0,0,0,0,0,0]. \end{array}$$

Since $[1,0,-1,-1,0,1] = [\langle d^1,[1,-1]\rangle, \dots, \langle d^6,[1,-1]\rangle] \in V^\perp$, the vector $\underline{\eta}(c^5)$ can be transformed into $[0,0,0,0,0,1]$.

Remark. The definitions of $a(c), \eta_0(c)$, and $\underline{\eta}(c)$ also make sense for general $c \in \mathbb{R}^n$. However, $\eta_0(c) \in \mathbb{R}$ and $\underline{\eta}(c) \in V^*$ no longer need to be contained in the lattices. The previous lemma will remain valid (even for $g_v \in \mathbb{R}_{\geq 0}$ in (iii)), if the relation "$\succeq 0$" is replaced by the weaker version "$\in C(Q)^\vee$".

(4.6) Proposition. *(1)* $\tilde{C}(Q)^\vee = \left\{ [c,\underline{\eta}] \in \mathbb{R}^n \times V^* \;\middle|\; \underline{\eta} - \underline{\eta}(c) \in C(Q)^\vee \right\}$

(2) In particular, $[c,\underline{\eta}(c)] \in \tilde{C}(Q)^\vee$, *and moreover, it is the only preimage of* $[c,\eta_0(c)] \in \sigma^\vee$ *via the surjection* $i^\vee : \tilde{C}(Q)^\vee \twoheadrightarrow \sigma^\vee$.

(3) $[c^1,\underline{\eta}(c^1)], \dots, [c^w,\underline{\eta}(c^w)]$ *together with* $C(Q)^\vee \cap V^*_{\mathbb{Z}}$ *(embedded as* $[0, C(Q)^\vee]$*) generate the semigroup* $\tilde{C}(Q)^\vee \cap \left(\mathbb{Z}^n \times V^*_{\mathbb{Z}}\right)$. *(For the definition of* $c^1, \dots, c^w$, *cf. (4.3).)*

Proof. (1) Let $[c,\underline{\eta}] \in \mathbb{R}^n \times V^*$ be given; if some representative of $\underline{\eta}$ in $\mathbb{R}^N$ is needed, then it will be denoted by the same name. We have the following equivalences:

$$\begin{array}{rcll} [c,\underline{\eta}] \in \tilde{C}(Q)^\vee & \Longleftrightarrow & \langle (Q_{\underline{t}}, \underline{t}), [c,\underline{\eta}] \rangle \geq 0 & \text{for each } \underline{t} \in C(Q) \\ & \Longleftrightarrow & \langle Q_{\underline{t}}, c\rangle + \langle \underline{t}, \underline{\eta}\rangle \geq 0 & \text{for each } \underline{t} \in C(Q) \\ & \Longleftrightarrow & \langle a(c)_{\underline{t}}, c\rangle + \langle \underline{t}, \underline{\eta}\rangle \geq 0 & \text{for each } \underline{t} \in C(Q). \end{array}$$

Using some path $\underline{\lambda}^c$ we obtain:

$$\begin{array}{rcll} [c,\underline{\eta}] \in \tilde{C}(Q)^\vee & \Longleftrightarrow & \sum_{i=1}^N t_i \, \lambda_i^c \langle d^i, c\rangle + \langle \underline{t}, \underline{\eta}\rangle \geq 0 & \text{for each } \underline{t} \in C(Q) \\ & \Longleftrightarrow & \sum_{i=1}^N t_i \cdot \left(\lambda_i^c \langle d^i, c\rangle + \eta_i\right) \geq 0 & \text{for each } \underline{t} \in C(Q) \\ & \Longleftrightarrow & \left[\lambda_1^c \langle d^1, c\rangle + \eta_1, \dots, \lambda_N^c \langle d^N, c\rangle + \eta_N\right] \in C(Q)^\vee. & \end{array}$$

(2) By part (1) we know that for each $[c,\underline{\eta}] \in \tilde{C}(Q)^\vee$ it is possible to choose $\mathbb{R}^N$-representatives of $\underline{\eta}, \underline{\eta}(c)$ such that $\eta_i \geq \eta_i(c)$ for $i = 1, \dots, N$. On the other hand, the two equalities $\sum_i \eta_i(c) = \eta_0(c)$ (cf. (iv) of the previous lemma) and $\sum_i \eta_i = \eta_0(c)$ (corresponding to the fact $[c,\underline{\eta}] \mapsto [c,\eta_0(c)]$) imply $\underline{\eta} = \underline{\eta}(c)$.

(3) Let $[c,\underline{\eta}] \in \tilde{C}(Q)^\vee$. Then, $[c,\eta_0(c)]$ is representable as a non-negative linear combination $[c,\eta_0(c)] = \sum_{v=1}^{w} p_v\, [c^v, \eta_0(c^v)]$ ($p_v \in \mathbb{N}$ if $c \in \mathbb{Z}^n$). Since both elements $[c,\underline{\eta}(c)]$ and $\sum_v p_v [c^v, \underline{\eta}(c^v)]$ are preimages of $[c,\eta_0(c)]$ via $i^\vee$, they must be equal by (2), and we obtain

$$[c,\, \underline{\eta}] = [c,\, \underline{\eta}(c)] + [0, \underline{\eta} - \underline{\eta}(c)] = \sum_v p_v\, [c^v, \underline{\eta}(c^v)] + [0, \underline{\eta} - \underline{\eta}(c)]\,.$$

□

(4.7) Finally, we will take a short look at the geometrical situation reached at this point. The linear map

$$\begin{array}{ccc} \tilde{C}(Q)^\vee \cap \left(\mathbb{Z}^n \times V_{\mathbb{Z}}^*\right) & \twoheadrightarrow & \sigma^\vee \cap \mathbb{Z}^{n+1} \\ [c,\, \underline{\eta}] & \mapsto & [c,\, \sum_i \eta_i] \end{array}$$

is surjective ($[c,\underline{\eta}(c)] \mapsto [c,\eta_0(c)]$; $[0,e_i] \mapsto [0,1]$). Since $x^{[c,\underline{\eta}]} - x^{[c,\underline{\eta}']} = x^{[c,\underline{\eta}(c)]} \cdot (x^{[0,\underline{\eta}-\underline{\eta}(c)]} - x^{[0,\underline{\eta}'-\underline{\eta}(c)]})$, the kernel of the corresponding homomorphism between the semigroup algebras equals the ideal

$$\left(x^{[0,\underline{\eta}]} - x^{[0,\underline{\eta}']} \,\Big|\, \sum_i \eta_i = \sum_i \eta_i'\right).$$

In particular, $Y \hookrightarrow X$ is a closed embedding. Moreover, looking at the similar statement concerning $C(Q)^\vee$ and $\mathbb{N}$ at the end of (4.4), we see that this map equals the pull back of $\mathbb{C} \hookrightarrow S$ as claimed in (4.2).

The elements $[c^1, \underline{\eta}(c^1)], \dots, [c^w, \underline{\eta}(c^w)] \in \tilde{C}(Q)$ induce some regular functions $Z_1, \dots, Z_w$ on X. They define a closed embedding $X \hookrightarrow \mathbb{C}^w \times S$ lifting the embedding $Y \hookrightarrow \mathbb{C}^{w+1}$ of (4.3).

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{C}^w \times S \\ \uparrow & & \uparrow \\ Y & \hookrightarrow & \mathbb{C}^w \times \mathbb{C} \end{array}$$

Moreover, for $i = 1, \dots, N$, Z_i is the only monomial function lifting z_i from Y to X.

5 A flat family over $\bar{\mathcal{M}}$

(5.1) Theorem. *Denote by $\bar{X}$ and $\bar{S}$ the scheme theoretical images of X and S in $\mathbb{C}^w \times \mathbb{C}^N$ and $\mathbb{C}^N$, respectively. Then,*

(1) $X \to \bar{X}$ and $S \to \bar{S}$ are the normalization maps.

(2) $\pi : X \longrightarrow S$ *induces a map* $\bar{\pi} : \bar{X} \longrightarrow \bar{S}$, *and* π *can be recovered from* $\bar{\pi}$ *via base change* $S \longrightarrow \bar{S}$.

(3) Restricting to $\mathscr{M} \subseteq \bar{S}$ *and composing with* ℓ *turns* $\bar{\pi}$ *into a family* $\bar{X} \times_{\bar{S}} \mathscr{M} \xrightarrow{\bar{\pi}} \mathscr{M} \xrightarrow{\ell} \bar{\mathscr{M}}$. *It is flat in* $0 \in \bar{\mathscr{M}} \subseteq \mathbb{C}^{N-1}$, *and the special fiber equals* Y.

The proof of this theorem will fill Sect. 5.

(5.2) The ring of regular functions $A(\bar{S})$ is given as the image of the map $\mathbb{C}[t_1, \dots, t_N] \to A(S)$. Since $\mathbb{Z}^N \longrightarrow V_{\mathbb{Z}}^*$ is surjective, the rings $A(\bar{S}) \subseteq A(S) \subseteq \mathbb{C}[V_{\mathbb{Z}}^*]$ have the same field of fractions. On the other hand, while t-monomials with negative exponents might be involved in $A(S)$, the surjectivity of $\mathbb{R}^N_{\geq 0} \longrightarrow C(Q)^\vee$ tells us that sufficiently high powers of these monomials always come from $A(\bar{S})$. In particular, $A(S)$ is normal over $A(\bar{S})$.

$A(\bar{X})$ is given as the image $A(\bar{X}) = \operatorname{im}(\mathbb{C}[Z_1, \dots, Z_w, t_1, \dots, t_N] \to A(X))$. Since $A(X)$ is generated by $Z_1, \dots, Z_w$ over its subring $A(S)$ (cf. Proposition (4.6)(3)), the same arguments as for S and $\bar{S}$ apply. Hence, Part (1) of the previous theorem is proved.

(5.3) Recalling that $z_1, \dots, z_w, t \in A(Y)$ stand for the monomials with exponents $[c^1, \eta_0(c^1)], \dots, [c^w, \eta_0(c^w)], [0, 1] \in \sigma^\vee \cap \mathbb{Z}^{n+1}$, respectively, we obtain the following equations describing Y as a subset of $\mathbb{C}^{w+1}$:

$$\begin{aligned} f_{(a,b,\alpha,\beta)}(\underline{z}, t) \;&:=\; t^\alpha \prod_{v=1}^{w} z_v^{a_v} - t^\beta \prod_{v=1}^{w} z_v^{b_v} \\ &\text{with } \; a, b \in \mathbb{N}^w : \textstyle\sum_v a_v\, c^v = \sum_v b_v\, c^v \quad \text{and} \\ &\phantom{\text{with }} \; \alpha, \beta \in \mathbb{N} : \textstyle\sum_v a_v\, \eta_0(c^v) + \alpha = \sum_v b_v\, \eta_0(c^v) + \beta\,. \end{aligned}$$

Defining $c := \sum_v a_v\, c^v = \sum_v b_v\, c^v$ we can lift $f_{(a,b,\alpha,\beta)}$ to the following element of $A(\bar{S})[Z_1, \dots, Z_w]$ (described via the map $\mathbb{C}[Z_1, \dots, Z_w, t_1, \dots, t_N] \longrightarrow A(\bar{S})[Z_1, \dots, Z_w]$):

$$\begin{aligned} F_{(a,b,\alpha,\beta)}(\underline{Z}, \underline{t}) := f_{(a,b,\alpha,\beta)}(\underline{Z}, t_1) - \underline{Z}^{[c, \underline{\eta}(c)]} \cdot \Big(& \underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v)} \\ & - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v)} \Big) \cdot \underline{t}^{-\underline{\eta}(c)} \end{aligned}$$

Remark. (1) The symbol $\underline{Z}^{[c, \underline{\eta}(c)]}$ means $\prod_{v=1}^{w} Z_v^{p_v}$ with natural numbers $p_v \in \mathbb{N}$ such that $[c, \underline{\eta}(c)] = \sum_v p_v\, [c^v, \underline{\eta}(c^v)]$ or equivalently $[c, \eta_0(c)] = \sum_v p_v\, [c^v, \eta_0(c^v)]$. This condition does not determine the coefficients p_v uniquely. Any choice satisfying the equation will do. Choosing other coefficients q_v with the same property yields $Z_1^{p_1} \cdot \ldots \cdot Z_w^{p_w} - Z_1^{q_1} \cdot \ldots \cdot Z_w^{q_w} = F_{(p,q,0,0)}(\underline{Z}, \underline{t}) = f_{(p,q,0,0)}(\underline{Z}, t)$, anyway.

(2) By part (iii) of Lemma (4.5), we have $\sum_v a_v \underline{\eta}(c^v), \sum_v b_v \underline{\eta}(c^v) \succeq \underline{\eta}(c)$. In particular, representatives of the $\underline{\eta}$'s can be chosen such that all t-exponents

occurring in monomials of F are non-negative, i.e. F indeed defines an element of $A(\bar{S})[Z_1, \ldots, Z_w]$.

Lemma. *The polynomials* $F_{(a,b,\alpha,\beta)}$ *generate* $\ker(A(\bar{S})[\underline{Z}] \to A(X))$, *i.e. they can be used as equations for* $\bar{X} \subseteq \mathbb{C}^w \times \bar{S}$.

Proof. Recall first that the map from $A(\bar{S})[\underline{Z}]$ to $A(X) = \oplus_{[c,\underline{\eta}]} \mathbb{C}x^{[c,\underline{\eta}]}$, where $[c,\underline{\eta}]$ runs through all elements of $\tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V_{\mathbb{Z}}^*)$, sends $Z_v \mapsto x^{[c^v,\underline{\eta}(c^v)]}$ and $t_i \mapsto x^{[0,e_i]}$. Hence,

$$
\begin{aligned}
F_{(a,b,\alpha,\beta)} \quad &= \quad \left(t_1^\alpha \prod_v Z_v^{a_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) - \\
&\qquad - \left(t_1^\beta \prod_v Z_v^{b_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \\
&\mapsto \quad \left(x^{\alpha[0,e_1] + \sum_v a_v [c^v,\underline{\eta}(c^v)]} - x^{[c,\underline{\eta}(c)] + \alpha[0,e_1] + \sum_v a_v [0,\underline{\eta}(c^v)] - [0,\underline{\eta}(c)]}\right) - \\
&\qquad - \left(x^{\beta[0,e_1] + \sum_v b_v [c^v,\underline{\eta}(c^v)]} - x^{[c,\underline{\eta}(c)] + \beta[0,e_1] + \sum_v b_v [0,\underline{\eta}(c^v)] - [0,\underline{\eta}(c)]}\right) \\
&= \quad 0 - 0 = 0\,.
\end{aligned}
$$

On the other hand, $\ker(A(\bar{S})[\underline{Z}] \to A(X))$ is obviously generated by the binomials

$$
\underline{t}^{\underline{\eta}} Z_1^{a_1} \cdot \ldots \cdot Z_w^{a_w} - \underline{t}^{\underline{\mu}} Z_1^{b_1} \cdot \ldots \cdot Z_w^{b_w} \quad \text{such that}
$$

$$
\sum_v a_v [c^v, \underline{\eta}(c^v)] + [0, \underline{\eta}] = \sum_v b_v [c^v, \underline{\eta}(c^v)] + [0, \underline{\mu}]\,,
$$

i.e.
- $c := \sum_v a_v\, c^v = \sum_v b_v\, c^v$
- $\sum_v a_v\, \underline{\eta}(c^v) + \underline{\eta} = \sum_v b_v\, \underline{\eta}(c^v) + \underline{\mu}\,.$

However,

$$
\begin{aligned}
\underline{t}^{\underline{\eta}} \underline{Z}^a - \underline{t}^{\underline{\mu}} \underline{Z}^b \quad &= \quad \underline{t}^{\underline{\eta}} \cdot \left(\prod_v Z_v^{a_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) - \\
&\qquad - \underline{t}^{\underline{\mu}} \cdot \left(\prod_v Z_v^{b_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \\
&= \quad \underline{t}^{\underline{\eta}} \cdot F_{(a,p,0,\alpha)} - \underline{t}^{\underline{\mu}} \cdot F_{(b,p,0,\beta)}
\end{aligned}
$$

with $p \in \mathbb{N}^w$ such that $\sum_v p_v [c^v, \underline{\eta}(c^v)] = [c, \underline{\eta}(c)]$, $\alpha = \sum_v a_v \eta_0(c^v) - \eta_0(c)$, and $\beta = \sum_v b_v \eta_0(c^v) - \eta_0(c)$. □

(5.4) Using exponents $\underline{\eta}, \underline{\mu} \in \mathbb{Z}^N$ (instead of $\mathbb{N}^N$), the binomials $\underline{t}^{\underline{\eta}} \underline{Z}^a - \underline{t}^{\underline{\mu}} \underline{Z}^b$ generate the kernel of the map

$$
A(S)[\underline{Z}] = A(\bar{S})[\underline{Z}] \otimes_{A(\bar{S})} A(S) \longrightarrow\!\!\!\!\!\rightarrow A(\bar{X}) \otimes_{A(\bar{S})} A(S) \longrightarrow\!\!\!\!\!\rightarrow A(X)\,.
$$

Since $\underline{Z}^a \otimes \underline{t}^{\underline{\eta}} - \underline{Z}^b \otimes \underline{t}^{\underline{\mu}} = \underline{Z}^{[c,\underline{\eta}(c)]} \otimes \left(\underline{t}^{\sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c) + \underline{\eta}} - \underline{t}^{\sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c) + \underline{\mu}}\right) = 0$ in $A(\bar{X}) \otimes_{A(\bar{S})} A(S)$, this implies that the surjection $A(\bar{X}) \otimes_{A(\bar{S})} A(S) \longrightarrow\!\!\!\!\!\rightarrow A(X)$ is injective, too. In particular, part (2) of our theorem is proved.

(5.5) We are going to use the following well known criterion of flatness:

Theorem. ([Ma], (20.C), Theorem 49) *Let $\tilde{\pi} : \tilde{X} \hookrightarrow \mathbb{C}^{w+1} \times \tilde{\mathcal{M}} \twoheadrightarrow \tilde{\mathcal{M}}$ be a map with special fiber $Y = \tilde{\pi}^{-1}(0)$; in particular, $Y \subseteq \mathbb{C}^{w+1}$ is defined by the restrictions of the equations defining $\tilde{X} \subseteq \mathbb{C}^{w+1} \times \tilde{\mathcal{M}}$ to $0 \in \tilde{\mathcal{M}}$. Then, $\tilde{\pi}$ is flat if and only if each linear relation between the (restricted) equations for Y lifts to some linear relation between the original equations for $\tilde{X}$.*

For our special situation take $\tilde{X} := \bar{X} \times_{\bar{S}} \mathcal{M}$ (and $\tilde{\mathcal{M}} := \bar{\mathcal{M}}$, $Y := Y$); in (5.3) we have seen how the equations defining $Y \hookrightarrow \mathbb{C}^w \times \mathbb{C}$ can be lifted to those defining $\bar{X} \hookrightarrow \mathbb{C}^w \times \bar{S}$, hence $\bar{X} \times_{\bar{S}} \mathcal{M} \hookrightarrow \mathbb{C}^w \times \mathcal{M} \xrightarrow{\sim} \mathbb{C}^w \times \mathbb{C} \times \bar{\mathcal{M}}$. In particular, to show (3) of Theorem (5.1), we just have to determine the linear relations between the $f_{(a,b,\alpha,\beta)}$'s and lift them to relations between the $F_{(a,b,\alpha,\beta)}$'s. There are three types of relations between the $f_{(a,b,\alpha,\beta)}$'s:

(i) $f_{(a,r,\alpha,\gamma)} + f_{(r,b,\gamma,\beta)} = f_{(a,b,\alpha,\beta)}$
with • $\sum_v a_v c^v = \sum_v r_v c^v = \sum b_v c^v$ and
• $\sum_v a_v \eta_0(c^v) + \alpha = \sum_v r_v \eta_0(c^v) + \gamma = \sum_v b_v \eta_0(c^v) + \beta$.
For this relation, the same equation between the F's is true.

(ii) $t \cdot f_{(a,b,\alpha,\beta)} = f_{(a,b,\alpha+1,\beta+1)}$ lifts to $t_1 \cdot F_{(a,b,\alpha,\beta)} = F_{(a,b,\alpha+1,\beta+1)}$.

(iii) $\underline{z}^r \cdot f_{(a,b,\alpha,\beta)} = f_{(a+r,b+r,\alpha,\beta)}$.
With $c := \sum_v a_v c^v = \sum_v b_v c^v$, $\tilde{c} := c + \sum_v r_v c^v$ we obtain

$$\begin{aligned}
&\underline{Z}^r \cdot F_{(a,b,\alpha,\beta)} - F_{(a+r,b+r,\alpha,\beta)} = \\
&= \underline{Z}^{[\tilde{c},\underline{\eta}(\tilde{c})]} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) + \sum_v r_v \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) + \sum_v r_v \underline{\eta}(c^v)}\right) \\
&\quad \cdot \underline{t}^{-\underline{\eta}(\tilde{c})} - -\underline{Z}^{[c,\underline{\eta}(c)]}\,\underline{Z}^r \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v)}\right) \cdot \underline{t}^{-\underline{\eta}(c)} \\
&= \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \cdot \\
&\qquad \left(\underline{t}^{\underline{\eta}(c) + \sum_v r_v \underline{\eta}(c^v) - \underline{\eta}(\tilde{c})} \underline{Z}^{[\tilde{c},\underline{\eta}(\tilde{c})]} - \underline{Z}^{[c,\underline{\eta}(c)]} \underline{Z}^r\right).
\end{aligned}$$

Now, the inequalities

$$\sum_v a_v \underline{\eta}(c^v),\ \sum_v b_v \underline{\eta}(c^v) \succeq \underline{\eta}(c) \quad \text{and} \quad \underline{\eta}(c) + \sum_v r_v \underline{\eta}(c^v) - \underline{\eta}(\tilde{c}) \succeq 0$$

imply that
- the first factor is contained in the ideal defining $0 \in \bar{\mathcal{M}}$, and
- the second factor is an equation of $\bar{X} \subseteq \mathbb{C}^w \times \bar{S}$ (called $F_{(\bar{p},p+r,\xi,0)}$ in (7.4)).

In particular, we have found a lift for the third relation, too.

The proof of Theorem (5.1) is complete.

(5.6) *Example.* The singularity Y_6 induced by the hexagon Q_6 equals the cone over the Del Pezzo surface of degree 6 obtained by blowing up three points of $(\mathbb{P}^2, \mathcal{O}(3))$. As a closed subset of $\mathbb{C}^7$, it is given by the following 9 equations:

$$f_{(e_1,e_6+e_2,1,0)} = z_1\,t - z_6\,z_2, \quad f_{(e_2,e_1+e_3,1,0)} = z_2\,t - z_1\,z_3,$$
$$f_{(e_3,e_2+e_4,1,0)} = z_3\,t - z_2\,z_4, \quad f_{(e_4,e_3+e_5,1,0)} = z_4\,t - z_3\,z_5,$$
$$f_{(e_5,e_4+e_6,1,0)} = z_5\,t - z_4\,z_6, \quad f_{(e_6,e_5+e_1,1,0)} = z_6\,t - z_5\,z_1,$$
$$f_{(\underline{0},e_1+e_4,2,0)} = t^2 - z_1\,z_4, \quad f_{(\underline{0},e_2+e_5,2,0)} = t^2 - z_2\,z_5,$$
$$f_{(\underline{0},e_3+e_6,2,0)} = t^2 - z_3\,z_6\,.$$

Then, the construction described in (5.3) yields the liftings

$$\begin{aligned}
F_{(e_1,e_6+e_2,1,0)} &= (Z_1\,t_1 - Z_6\,Z_2) - Z_1(t_1 - t_1)\\
&= Z_1\,t_1 - Z_6\,Z_2,\\
F_{(e_2,e_1+e_3,1,0)} &= (Z_2\,t_1 - Z_1\,Z_3) - Z_2(t_1^2 - t_1\,t_2)\,t_1^{-1}\\
&= Z_2\,t_2 - Z_1\,Z_3,\\
F_{(e_3,e_2+e_4,1,0)} &= (Z_3\,t_1 - Z_2\,Z_4) - Z_3(t_1^2 t_2 - t_1\,t_2\,t_3)\,t_1^{-1}t_2^{-1}\\
&= Z_3\,t_3 - Z_2\,Z_4,\\
F_{(e_4,e_3+e_5,1,0)} &= (Z_4\,t_1 - Z_3\,Z_5) - Z_4(t_1\,t_2\,t_3 - t_2\,t_3\,t_4)\,t_2^{-1}t_3^{-1}\\
&= Z_4\,t_4 - Z_3\,Z_5,\\
F_{(e_5,e_4+e_6,1,0)} &= (Z_5\,t_1 - Z_4\,Z_6) - Z_5(t_1\,t_6 - t_2\,t_3)\,t_6^{-1}\\
&= Z_5\,t_5 - Z_4\,Z_6,\\
F_{(e_6,e_5+e_1,1,0)} &= (Z_6\,t_1 - Z_5\,Z_1) - Z_6(t_1 - t_6)\\
&= Z_6\,t_6 - Z_5\,Z_1,\\
F_{(\underline{0},e_1+e_4,2,0)} &= (t_1^2 - Z_1\,Z_4) - (t_1^2 - t_2\,t_3) = t_2\,t_3 - Z_1\,Z_4\\
&= t_5\,t_6 - Z_1\,Z_4,\\
F_{(\underline{0},e_2+e_5,2,0)} &= (t_1^2 - Z_2\,Z_5) - (t_1^2 - t_3\,t_4)\\
&= t_3\,t_4 - Z_2\,Z_5,\\
F_{(\underline{0},e_3+e_6,2,0)} &= (t_1^2 - Z_3\,Z_6) - (t_1^2 - t_1\,t_2)\\
&= t_1\,t_2 - Z_3\,Z_6\,.
\end{aligned}$$

Together with the four equations mentioned at the end of (2.3), they describe a family contained in $\mathbb{C}^6 \times \mathbb{C}^6 \xrightarrow{\mathrm{pr}_2} \mathbb{C}^6/_{\mathbb{C}\cdot(1,\ldots,1)}$.

6 The Kodaira-Spencer map

(6.1) Denote by $E \subseteq \sigma^\vee \cap \mathbb{Z}^{n+1}$ the minimal generating set

$$E := \{[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [\underline{0}, 1]\}$$

mentioned in (4.3). To each vertex $a^j \in Q$ (or identically named fundamental generator $a^j := (a^j, 1) \in \sigma$) and each element $R \in \mathbb{Z}^{n+1}$ we associate the subset

$$E_j^R := E_{a^j}^R := \{r \in E \mid \langle a^j, r\rangle < \langle a^j, R\rangle\}\,.$$

Theorem. (cf. [Al 1]) *The vector space T_Y^1 of infinitesimal deformations of Y is $\mathbb{Z}^{n+1}$-graded, and in degree $-R$ it equals*

$$T_Y^1(-R) = \left(L_{\mathbb{C}}\left(\cup_j E_j^R\right) \Big/ \sum_j L_{\mathbb{C}}(E_j^R)\right)^*,$$

where $L(\dots)$ denotes the vector space of linear relations.

(6.2) There is a special degree $R^* = [\underline{0}, 1] \in \mathbb{Z}^{n+1}$ corresponding to the affine hyperplane containing Q. The associated subsets of E equal

$$E_j^{R^*} = E \cap (a^j)^\perp = \{[c^v, \eta_0(c^v)] \mid \langle a^j, -c^v\rangle = \eta_0(c^v)\}\,.$$

In (4.5), for each $c \in \mathbb{Z}^n$, we have defined the linear form $\underline{\eta}(c) \in V_{\mathbb{Z}}^*$. Restricted to the cone $C(Q)$, it maps $\underline{t}$ to $\mathrm{Max}\langle Q_{\underline{t}}, -c\rangle = \langle a(c)_{\underline{t}}, -c\rangle$. This induces the following bilinear map:

$$\begin{array}{rccccl} \Phi: & V_{\mathbb{Z}}/_{(1,\dots,1)} & \times & L_{\mathbb{Z}}(E \cap \partial\sigma^\vee) & \longrightarrow & \mathbb{Z} \\ & \underline{t} & , & q & \mapsto & \sum_{v,i} t_i\, q_v\, \eta_i(c^v)\,. \end{array}$$

(Indeed, for $\underline{t} := \underline{1}$ we obtain $\sum_{v,i} q_v\, \eta_i(c^v) = \sum_v q_v\, \eta_0(c^v) = 0$ since $q \in L_{\mathbb{Z}}(E \cap \partial\sigma^\vee)$.) Moreover, if q comes from one of the submodules $L_{\mathbb{Z}}(E_j^{R^*}) \subseteq L_{\mathbb{Z}}(E \cap \partial\sigma^\vee)$, we obtain

$$\begin{aligned} \Phi(\underline{t}, q) &= \sum_v q_v \cdot \mathrm{Max}\langle Q_{\underline{t}}, -c^v\rangle = \sum_v q_v \cdot \langle a^j_{\underline{t}}, -c^v\rangle \\ &= \langle a^j_{\underline{t}}, -\sum_v q_v c^v\rangle = 0\,. \end{aligned}$$

Theorem. *The Kodaira-Spencer map of the family $\bar{X} \times_{\bar{S}} \mathscr{M} \to \bar{\mathscr{M}}$ of Sect. 5 equals the map*

$$T_0\bar{\mathscr{M}} = {}^{V_{\mathbb{C}}}/_{(1,\dots,1)} \longrightarrow \left({}^{L_{\mathbb{C}}(E\cap\partial\sigma^\vee)}\Big/_{\sum_j L_{\mathbb{C}}(E_j^{R^*})} \right)^* = T_Y^1(-R^*)$$

induced by the previous pairing. Moreover, this map is an isomorphism.

Proof. Using the same symbol $\mathscr{J}$ for the ideal $\mathscr{J} \subseteq \mathbb{C}[t_1, \dots, t_N]$ and the intersection $\mathscr{J} \cap \mathbb{C}[t_i - t_j \mid 1 \le i,j \le N]$ (cf. (2.4)), our family corresponds to the flat $\mathbb{C}[t_i - t_j]/_{\mathscr{J}}$-module $\mathbb{C}[\underline{Z}, \underline{t}]/_{(\mathscr{J}, F_\bullet(\underline{Z}, \underline{t}))}$. Now, we fix a non-trivial tangent vector $\underline{t}^0 \in V_{\mathbb{C}}$. Via $t_i \mapsto t + t_i^0\,\varepsilon$ it induces the infinitesimal family given by the flat $\mathbb{C}[\varepsilon]/_{\varepsilon^2}$-module

$$A_{\underline{t}^0} := {}^{\mathbb{C}[\underline{z}, t, \varepsilon]}\Big/_{(\varepsilon^2, F_\bullet(\underline{z}, t + \underline{t}^0\,\varepsilon))}\,.$$

To obtain the associated $A(Y)$-linear map $I/_{I^2} \to A(Y)$, where $I := (f_\bullet(\underline{z}, t))$ denotes the ideal of Y in $\mathbb{C}^{w+1}$, we have to compute the images of $f_\bullet(\underline{z}, t)$ in $\varepsilon A(Y) \subseteq A_{\underline{t}^0}$ and divide them by ε: Using the notation of (5.3), in $A_{\underline{t}^0}$

$$\begin{aligned} 0 &= F_{(a,b,\alpha,\beta)}(\underline{z}, t + \underline{t}^0\,\varepsilon) \\ &= f_{(a,b,\alpha,\beta)}(\underline{z}, t + t_1^0\,\varepsilon) - \\ &\quad -\underline{z}^{[c,\underline{\eta}(c)]} \cdot \left((t + \underline{t}^0\,\varepsilon)^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - (t + \underline{t}^0\,\varepsilon)^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)} \right)\,. \end{aligned}$$

The relation $\varepsilon^2 = 0$ yields

$$f_{(a,b,\alpha,\beta)}(\underline{z}, t + t_1^0\varepsilon) = f_{(a,b,\alpha,\beta)}(\underline{z}, t) + \varepsilon \cdot (\alpha\, t^{\alpha-1}\, t_1^0\, \underline{z}^a - \beta\, t^{\beta-1}\, t_1^0\, \underline{z}^b)\,,$$

and similarly we can expand the other terms. Eventually, we obtain

$$\begin{aligned} f_{(a,b,\alpha,\beta)}(\underline{z}, t) &= -\varepsilon\, t_1^0\, (\alpha\, t^{\alpha-1}\, \underline{z}^a - \beta\, t^{\beta-1}\, \underline{z}^b) + \varepsilon\, \underline{z}^{[c,\underline{\eta}(c)]} \\ & \quad t^{\alpha+\sum_v a_v \eta_0(c^v) - \eta_0(c) - 1} \cdot \left[t_1^0\, (\alpha - \beta) + \textstyle\sum_i t_i^0\, \left(\sum_v (a_v - b_v)\eta_i(c^v)\right)\right] \\ &= \varepsilon \cdot x^{\sum_v a_v [c^v, \eta_0(c^v)] + [0, \alpha-1]} \cdot \left(\textstyle\sum_i t_i^0 \left(\sum_v (a_v - b_v)\eta_i(c^v)\right)\right). \end{aligned}$$

(In $\varepsilon A(Y)$ we were able to replace the variables t and z_i by $x^{[\underline{0},1]}$ and $x^{[c^v, \eta_0(c^v)]}$, respectively.)

On the other hand, we use Theorem (3.4) of [Al 3]: Fixing $R^* \in \mathbb{Z}^{n+1}$, the element of $L_{\mathbb{C}}(E \cap \partial\sigma^\vee)^*$ given by $q \mapsto \sum_{i,v} t_i^0\, q_v\, \eta_i(c^v)$ corresponds to the infinitesimal deformation of $T_Y^1(-R^*)$ defined by the map

$$\begin{array}{ccc} I/I^2 & \longrightarrow & A(Y) \\ t^\alpha\, \underline{z}^a - t^\beta\, \underline{z}^b & \mapsto & \left(\sum_{i,v} t_i^0\, (a_v - b_v)\eta_i(c^v)\right) \cdot x^{\sum_v a_v [c^v, \eta_0(c^v)] + [0, \alpha-1]}\,. \end{array}$$

□

(6.3) To discuss the meaning of the homogeneous part $T_Y^1(-R^*)$ inside the whole vector space T_Y^1, we have to look at the results of [Al 2], (6.5): If $\dim T_Y^1 < \infty$ (for instance, if Y has an isolated singularity), then

(1) $T_Y^1 = T_Y^1(-R^*)$, but
(2) $T_Y^1 = 0$ for $\dim Y \geq 4$.

In particular, the interesting cases arise from 2-dimensional lattice polygons Q with primitive edges only. The corresponding 3-dimensional toric varieties Y have an isolated singularity, and the Kodaira-Spencer map $T_0\mathcal{M} \to T_Y^1$ is an isomorphism.

If T_Y^1 has *infinite dimension*, then this comes from the existence of infinitely many non-trivial homogeneous pieces $T_Y^1(-R)$. Whenever $\langle a^j, R\rangle \leq 1$ holds for all vertices $a^j \in Q$, we have

$$T_Y^1(-R) = V_{\mathbb{C}}(\operatorname{conv}\{a^j \mid \langle a^j, R\rangle = 1\})\,,$$

i.e. $T_Y^1(-R)$ equals the vector space of Minkowski summands of some face of Q, whereas $T_Y^1(-R) = 0$ for all other $R \in \mathbb{Z}^{n+1}$. In particular, $T_Y^1(-R^*)$ is a typical, but nevertheless extremal and perhaps the most interesting part of T_Y^1.

7 The obstruction map

(7.1) Dealing with obstructions in the deformation theory of Y involves the $A(Y)$-module T^2_Y. Usually, it is defined in the following way:

$$\text{Let } m := \{([a,\alpha],[b,\beta]) \in \mathbb{N}^{w+1} \times \mathbb{N}^{w+1} \mid \sum_v a_v\, c^v = \sum_v b_v\, c^v; \quad \sum_v a_v\, \eta_0(c^v) + \alpha = \sum_v b_v\, \eta_0(c^v) + \beta\}$$

denote the set parametrizing the equations $f_{(a,b,\alpha,\beta)}$ generating the ideal $I \subseteq \mathbb{C}[\underline{z},t]$ of Y. Then,

$$\mathscr{R} := \ker(\mathbb{C}[\underline{z},t]^m \twoheadrightarrow I)$$

is the module of linear relations between these equations; it contains the submodule $\mathscr{R}_0$ of the so-called Koszul relations.

Definition. $T^2_Y := \operatorname{Hom}\left(\mathscr{R}/_{\mathscr{R}_0}, A(Y)\right) \Big/ \operatorname{Hom}(\mathbb{C}[\underline{z},t]^m, A(Y))$.

Now, we have a similar theorem for T^2_Y as we had in (6.1) for T^1_Y; in particular, we use the notation introduced there.

Theorem. (cf. [Al 3]) *The vector space T^2_Y is $\mathbb{Z}^{n+1}$-graded, and in degree $-R$ it equals*

$$T^2_Y(-R) = \left(\frac{\ker\left(\oplus_j L_{\mathbb{C}}(E^R_j) \longrightarrow L_{\mathbb{C}}(E)\right)}{\operatorname{im}\left(\oplus_{\langle a^i,a^j\rangle < Q} L_{\mathbb{C}}(E^R_i \cap E^R_j) \to \oplus_i L_{\mathbb{C}}(E^R_i)\right)} \right)^* .$$

(7.2) In this section we build up the so-called obstruction map. It detects all possible infinitesimal extensions of our family over $\bar{\mathscr{M}}$ to a flat family over some larger base space. We follow the explanation given in Sect. 4 of [JS]. As before,

$$\begin{aligned} \mathscr{J} &= (g_{\varepsilon,k}(\underline{t}-t_1) \mid \varepsilon < Q,\, k \geq 1) \\ &= (g_{\underline{d},k}(\underline{t}-t_1) \mid \underline{d} \in V^{\perp} \cap \mathbb{Z}^N,\, k \geq 1) \subseteq \mathbb{C}[t_i - t_j] \end{aligned}$$

denotes the homogeneous ideal of the base space $\bar{\mathscr{M}}$. Let

$$\tilde{\mathscr{J}} := (t_i - t_j)_{i,j} \cdot \mathscr{J} + \mathscr{J}_1 \cdot \mathbb{C}[t_i - t_j] \subseteq \mathbb{C}[t_i - t_j \mid 1 \leq i,j \leq N] .$$

Then, $W := \mathscr{J} \big/ \tilde{\mathscr{J}}$ is a finite-dimensional, $\mathbb{Z}$-graded vector space ($W = \oplus_{k \geq 2} W_k$, and W_k is generated by the polynomials $g_{\underline{d},k}(\underline{t}-t_1)$). It is the kernel of the exact sequence

$$0 \to W \longrightarrow \mathbb{C}[t_i - t_j] \big/ \tilde{\mathcal{J}} \longrightarrow \mathbb{C}[t_i - t_j] \big/ \mathcal{J} \to 0\,.$$

Identifying t with t_1 and $\underline{z}$ with $\underline{Z}$, the tensor product with $\mathbb{C}[\underline{z}, t]$ (over $\mathbb{C}$) yields the important, exact sequence

$$0 \to W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t] \longrightarrow \mathbb{C}[\underline{Z}, \underline{t}] \big/ \tilde{\mathcal{J}} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \longrightarrow \mathbb{C}[\underline{Z}, \underline{t}] \big/ \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \to 0\,.$$

Now, let s be any relation with coefficients in $\mathbb{C}[\underline{z}, t]$ between the equations $f_{(a,b,\alpha,\beta)}$, i.e.

$$\sum s_{(a,b,\alpha,\beta)} f_{(a,b,\alpha,\beta)} = 0 \quad \text{in } \mathbb{C}[\underline{z}, t]\,.$$

By flatness of our family (cf. (5.5)), the components of s can be lifted to $\mathbb{C}[\underline{Z}, \underline{t}]$ obtaining an $\tilde{s}$ such that

$$\lambda(s) := \sum \tilde{s}_{(a,b,\alpha,\beta)} F_{(a,b,\alpha,\beta)} \mapsto 0 \quad \text{in } \mathbb{C}[\underline{Z}, \underline{t}] \big/ \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}]\,.$$

In particular, each relation $s \in \mathcal{R}$ induces some element $\lambda(s) \in W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$, which is well defined after the additional projection to $W \otimes_{\mathbb{C}} A(Y)$. This procedure describes a certain element $\lambda \in T^2_Y \otimes_{\mathbb{C}} W = \operatorname{Hom}(W^*, T^2_Y)$ called the obstruction map.

Theorem. *The obstruction map $\lambda : W^* \to T^2_Y$ is injective.*

Corollary. *If $\dim T^1_Y < \infty$, our family equals the versal deformation of Y. In general, we could say that it is "versal in degree $-R^*$".*

Proof. In (6.2) we have proved that the Kodaira-Spencer map is an isomorphism (at least onto the homogeneous piece $T^1_Y(-R^*)$). By a criterion also described in [JS], this fact combined with injectivity of the obstruction map implies versality. □

The remaining part of Sect. 7 contains the proof of the previous theorem.

(7.3) We have to improve the notation of Sects. 4 and 5. Since $\mathcal{M} \subseteq \bar{S} \subseteq \mathbb{C}^N$, we were able to use the toric equations (cf. (2.4)) during computations modulo $\mathcal{J}$. In particular, the exponents $\underline{\eta} \in \mathbb{Z}^N$ of $\underline{t}$ only needed to be known modulo $V^\perp$; it was enough to define $\underline{\eta}(c)$ as elements of $V^*_{\mathbb{Z}}$. However, to compute the obstruction map, we have to deal with the smaller ideal $\tilde{\mathcal{J}} \subseteq \mathcal{J}$. Let us start by refining the definitions of (4.5):

(i) For each vertex $a \in Q$ we choose the following paths through the 1-skeleton of Q:
   - $\underline{\lambda}(a) :=$ path from $0 \in Q$ to $a \in Q$.
   - $\underline{\mu}^v(a) :=$ path from $a \in Q$ to $a(c^v) \in Q$ such that $\mu^v_i(a)\langle d^i, c^v\rangle \le 0$ for each $i = 1, \dots, N$.

- $\underline{\lambda}^v(a) := \underline{\lambda}(a) + \underline{\mu}^v(a)$ is then a path from $0 \in Q$ to $a(c^v)$, which depends on a.

(ii) For each $c \in \mathbb{Z}^n$ we use the vertex $a(c)$ to define

$$\underline{\eta}^c(c) := \left[-\lambda_1(a(c))\langle d^1, c\rangle, \ldots, -\lambda_N(a(c))\langle d^N, c\rangle\right] \in \mathbb{Z}^N$$

and

$$\underline{\eta}^c(c^v) := \left[-\lambda_1^v(a(c))\langle d^1, c^v\rangle, \ldots, -\lambda_N^v(a(c))\langle d^N, c^v\rangle\right] \in \mathbb{Z}^N .$$

(iii) For each $c \in \mathbb{Z}^n$ we fix a representation $c = \sum_v p_v^c c^v$ ($p_v^c \in \mathbb{N}$) such that $\eta_0(c) = \sum_v p_v^c \eta_0(c^v)$. (That means, c is represented only by those generators c^v that define faces of Q containing the face defined by c itself.)

Remark. Let $a \in \mathbb{N}^w$. Denoting $c := \sum_v a_v c^v$ we obtain $\sum_v a_v \underline{\eta}^c(c^v) - \underline{\eta}^c(c) \in \mathbb{N}^N$ by arguments as in Lemma (4.5). Moreover, for the special representation $c = \sum_v p_v^c c^v$, the equation $\sum_v p_v^c \underline{\eta}^c(c^v) = \underline{\eta}^c(c)$ is true.

Now, we improve the definition of the polynomials $F_\bullet(\underline{Z}, \underline{t})$ given in (5.3). Let $a, b \in \mathbb{N}^w, \alpha, \beta \in \mathbb{N}$ such that

$$c := \sum_v a_v c^v = \sum_v b_v c^v \quad \text{and} \quad \sum_v a_v \eta_0(c^v) + \alpha = \sum_v b_v \eta_0(c^v) + \beta .$$

Then,

$$F_{(a,b,\alpha,\beta)}(\underline{Z}, \underline{t}) := f_{(a,b,\alpha,\beta)}(\underline{Z}, t_1) - \underline{Z}^{p^c} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}^c(c^v) - \underline{\eta}^c(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}^c(c^v) - \underline{\eta}^c(c)}\right) .$$

(7.4) We have to discuss the same three types of relations as we did in (5.5). Since there is only one single element $c \in \mathbb{Z}^n$ involved in the relations (i) and (ii), calculating modulo $\tilde{\mathcal{J}}$ instead of $\mathcal{J}$ makes no difference in these cases - we always obtain $\lambda(s) = 0$.
Let us regard the relation $s := \left[\underline{z}^r \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha,\beta)} = 0\right]$ $(r \in \mathbb{N}^w)$. We will use the following notation:

- $c := \sum_v a_v c^v = \sum_v b_v c^v$; $\underline{p} := \underline{p}^c$; $\underline{\eta} := \underline{\eta}^c$;
- $\tilde{c} := \sum_v (a_v + r_v) c^v = \sum_v (b_v + r_v) c^v = \sum_v (p_v + r_v) c^v$; $\underline{\tilde{p}} := \underline{p}^{\tilde{c}}$; $\underline{\tilde{\eta}} := \underline{\eta}^{\tilde{c}}$;
- $\xi := \sum_i \left(\left(\sum_v (p_v + r_v)\tilde{\eta}_i(c^v)\right) - \tilde{\eta}_i(\tilde{c})\right) = \sum_v (p_v + r_v)\eta_0(c^v) - \eta_0(\tilde{c})$.

Using the same lifting of s to $\tilde{s}$ as in (5.5) yields

$$\begin{aligned}
\lambda(s) &= \underline{Z}^r \cdot F_{(a,b,\alpha,\beta)} - F_{(a+r,b+r,\alpha,\beta)} - \\
&\quad - \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \cdot F_{(\tilde{p}, p+r, \xi, 0)} \\
&= -\underline{Z}^{p+r} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v (a_v - p_v)\underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (b_v - p_v)\underline{\eta}(c^v)}\right) + \\
&\quad + \underline{Z}^{\tilde{p}} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v (a_v + r_v - \tilde{p}_v)\underline{\tilde{\eta}}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (b_v + r_v - \tilde{p}_v)\underline{\tilde{\eta}}(c^v)}\right) -
\end{aligned}$$

$$- \left(\underline{t}^{\alpha e_1+\sum_v (a_v-p_v)\underline{\eta}(c^v)} - \underline{t}^{\beta e_1+\sum_v (b_v-p_v)\underline{\eta}(c^v)}\right)$$
$$\cdot \left(\underline{Z}^{\tilde{p}}\,\underline{t}^{\sum_v (p_v+r_v-\tilde{p}_v)\underline{\tilde{\eta}}(c^v)} - \underline{Z}^{p+r}\right)$$
$$= \underline{Z}^{\tilde{p}} \cdot \left(\underline{t}^{\alpha e_1+\sum_v (a_v+r_v-\tilde{p}_v)\underline{\tilde{\eta}}(c^v)} - \underline{t}^{\alpha e_1+\sum_v (p_v+r_v-\tilde{p}_v)\underline{\tilde{\eta}}(c^v)+\sum_v (a_v-p_v)\underline{\eta}(c^v)}\right)$$
$$-\underline{Z}^{\tilde{p}} \cdot \left(\underline{t}^{\beta e_1+\sum_v (b_v+r_v-\tilde{p}_v)\underline{\tilde{\eta}}(c^v)} - \underline{t}^{\beta e_1+\sum_v (p_v+r_v-\tilde{p}_v)\underline{\tilde{\eta}}(c^v)+\sum_v (b_v-p_v)\underline{\eta}(c^v)}\right)$$

As in (5.5)(iii), we can see that $\lambda(s)$ vanishes modulo $\mathcal{J}$ (or even in $A(\bar{S})$) merely by identifying $\underline{\eta}$ and $\underline{\tilde{\eta}}$.

(7.5) In (7.2) we have already mentioned the isomorphism

$$W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t] \xrightarrow{\sim} \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \Big/ \tilde{\mathcal{J}} \cdot \mathbb{C}[\underline{Z}, \underline{t}]$$

obtained by identifying t with t_1 and $\underline{z}$ with $\underline{Z}$. Now, with $\lambda(s)$, we have obtained an element of the right hand side, which has to be interpreted as an element of $W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$.

Lemma. *Let $A, B \in \mathbb{N}^N$ such that $\underline{d} := A - B \in V^{\perp}$, i.e. $\underline{t}^A - \underline{t}^B \in \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}]$. Then, via the previously mentioned isomorphism, $\underline{t}^A - \underline{t}^B$ corresponds to the element*

$$\sum_{k\geq 1} c_k \cdot g_{\underline{d},k}(\underline{t} - t_1) \cdot t^{k_0-k} \in W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]\,,$$

where $k_0 := \sum_i A_i$, and c_k are the constants occurred in (3.4). In particular, the coefficients from W_k vanish for $k > k_0$.

Proof. First, we remark that we may assume that $A = \underline{d}^+$, $B = \underline{d}^-$, i.e. $\underline{t}^A - \underline{t}^B = p_{\underline{d}}(\underline{t})$ (cf. (3.2)). Otherwise we could write this binomial as

$$\underline{t}^A - \underline{t}^B = \underline{t}^C \cdot \left(\underline{t}^{\underline{d}^+} - \underline{t}^{\underline{d}^-}\right) \quad (C \in \mathbb{N}^N),$$

and since

$$\underline{t}^C = (t_1 + [\underline{t} - t_1])^C \equiv t_1^{\sum_i C_i} \quad (\mathrm{mod}\ (t_i - t_j))\,,$$

we would obtain

$$\underline{t}^A - \underline{t}^B \equiv t_1^{\sum_i C_i} \cdot \left(\underline{t}^{\underline{d}^+} - \underline{t}^{\underline{d}^-}\right) \quad (\mathrm{mod}\ \tilde{\mathcal{J}})\,.$$

In (3.4) we have seen that

$$p_{\underline{d}}(\underline{t}) = \sum_{k=1}^{k_0} t_1^{k_0-k} \cdot \left(\sum_{v=1}^{k-1} q_{v,k}(\underline{t} - t_1) \cdot g_{\underline{d},v}(\underline{t} - t_1) + c_k \cdot g_{\underline{d},k}(\underline{t} - t_1)\right)$$

with $k_0 := \sum_i d_i^+$. Since $q_{v,k}(\underline{t} - t_1) \in (t_i - t_j) \cdot \mathbb{C}[t_i - t_j]$, this implies

$$p_{\underline{d}}(\underline{t}) \equiv \sum_{k=1}^{k_0} t_1^{k_0-k} \cdot c_k \cdot g_{\underline{d},k}(\underline{t}-t_1) \quad (\mathrm{mod}\ \tilde{\mathcal{J}})\,.$$

On the other hand, for $k > k_0$, Lemma (3.3) tells us that $g_{\underline{d},k}(\underline{t}-t_1)$ is a $\mathbb{C}[t_i - t_j]$-linear combination of the elements $g_{\underline{d},1}(\underline{t}-t_1), \ldots, g_{\underline{d},k_0}(\underline{t}-t_1)$. Then, the degree k part of the corresponding equation shows $g_{\underline{d},k}(\underline{t}-t_1) \in \tilde{\mathcal{J}}$. □

Corollary. *Transferred to* $W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$, *the element* $\lambda(s)$ *equals*

$$\sum_{k\geq 1} c_k \cdot g_{\underline{d},k}(\underline{t}-t_1)\cdot \underline{z}^{\tilde{p}} \cdot t^{k_0-k} \quad \textit{with} \quad \begin{array}{rcl} \underline{d} & := & \sum_v (a_v - b_v)\cdot \left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right)\,, \\ k_0 & := & \alpha + \sum_v (a_v + r_v)\,\eta_0(c^v) - \eta_0(\tilde{c})\,. \end{array}$$

The coefficients vanish for $k > k_0$.

Proof. We apply the previous lemma to both summands of the $\lambda(s)$-formula of (7.4). For the first one we obtain

$$\begin{array}{rcl} \underline{d}^a & = & \left[\alpha e_1 + \sum_v (a_v + r_v - \tilde{p}_v)\,\underline{\tilde{\eta}}(c^v)\right] \\ & & -\left[\alpha e_1 + \sum_v (p_v + r_v - \tilde{p}_v)\,\underline{\tilde{\eta}}(c^v) + \sum_v (a_v - p_v)\,\underline{\eta}(c^v)\right] \\ & = & \sum_v (a_v - p_v)\cdot\left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right) \qquad \text{and} \\ k_0 & = & \sum_i \left(\alpha e_1 + \sum_v (a_v + r_v - \tilde{p}_v)\,\underline{\tilde{\eta}}(c^v)\right)_i \\ & = & \alpha + \sum_v (a_v + r_v - \tilde{p}_v)\,\eta_0(c^v) \\ & = & \alpha + \sum_v (a_v + r_v)\,\eta_0(c^v) - \eta_0(\tilde{c})\,. \end{array}$$

k_0 has the same value for both the a- and b-summand, and

$$\begin{array}{rcl} \underline{d} = \underline{d}^a - \underline{d}^b & = & \sum_v (a_v - p_v)\cdot\left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right) \\ & & -\sum_v (b_v - p_v)\cdot\left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right) \\ & = & \sum_v (a_v - b_v)\cdot\left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right)\,. \end{array}$$ □

(7.6) *Now, we try to approach the obstruction map* λ *from the opposite direction. Using the description of* T_Y^2 *given in (7.1), we construct an element of* $T_Y^2 \otimes_{\mathbb{C}} W$ *that afterwards will turn out to equal* λ.

For a path $\varrho \in \mathbb{Z}^N$ along the edges of Q, we denote by

$$\underline{d}(\varrho, c) := [\langle \varrho_1\, d^1,\, c\rangle, \ldots, \langle \varrho_N\, d^N,\, c\rangle] \in \mathbb{Z}^N$$

the vector measuring the behavior of $c \in \mathbb{Z}^n$ passing each particular edge. If, moreover, ϱ comes from a closed path, $\underline{d}(\varrho, c)$ is also contained in $V^\perp$. On the other hand, for each $k \geq 1$, we can use the $\underline{d}$'s from $V^\perp$ to get elements $g_{\underline{d},k}(\underline{t} - t_1) \in W_k$ generating this vector space. Composing both procedures we obtain, for each closed path $\varrho \in \mathbb{Z}^N$, a map

$$\begin{array}{llllll} g^{(k)}(\varrho, \bullet): & \mathbb{R}^n & \longrightarrow & V^\perp & \longrightarrow & W_k \\ & c & & \mapsto & & g_{\underline{d}(\varrho,c),k}(\underline{t} - t_1)\,. \end{array}$$

Lemma. *(1) Taking the sum over all 2-faces we get a surjective map*

$$\sum_{\varepsilon<Q} g^{(k)}(\underline{\varepsilon}, \bullet) : \oplus_{\varepsilon<Q}\mathbb{C}^n \longrightarrow\!\!\!\!\!\rightarrow W_k\,.$$

(2) Let $c \in \mathbb{Z}^n$ (having integer coordinates is very important here). If $\varrho^1, \varrho^2 \in \mathbb{Z}^N$ are two paths each connecting vertices $a, b \in Q$ such that

- $|\langle a, c\rangle - \langle b, c\rangle| \leq k-1$ *and*
- *c is monotone along both paths, i.e. $\langle \varrho_i^1\, d^i, c\rangle$; $\langle \varrho_i^2\, d^i, c\rangle \geq 0$ for $i = 1, \ldots, N$,*

then $\varrho^1 - \varrho^2 \in \mathbb{Z}^N$ will be a closed path yielding $g^{(k)}(\varrho^1 - \varrho^2,\, c) = 0$ in W_k.

Proof. (1) is a consequence of the fact that the elements $\underline{d}(\varepsilon, c)$ ($\varepsilon < Q$ 2-face; $c \in \mathbb{Z}^n$) generate $V^\perp$ as a vector space. For the proof of (2), we look at $\underline{d} := \underline{d}(\varrho^1 - \varrho^2,\, c)$. Since $d_i = \langle \varrho_i^1\, d^i,\, c\rangle - \langle \varrho_i^2\, d^i,\, c\rangle$ is the difference of two non-negative integers, we obtain $d_i^+ \leq \langle \varrho_i^1\, d^i,\, c\rangle$. Hence,

$$\sum_i d_i^+ \leq \sum_i \langle \varrho_i^1\, d^i,\, c\rangle = \langle b, c\rangle - \langle a, c\rangle \leq k-1\,,$$

and as in (7.5) we obtain $g_{\underline{d},k}(\underline{t} - t_1) \in \tilde{\mathcal{J}}$ by Lemma (3.3). □

(7.7) Using the notation introduced in (6.1) we obtain for $R := k\,R^*$, $k \geq 2$

$$E_j^{kR^*} = \{[c^v, \eta_0(c^v)] \mid \langle a^j, c^v\rangle + \eta_0(c^v) \leq k-1\} \cup \{R^*\} \subseteq \sigma^\vee \cap \mathbb{Z}^{n+1}\,.$$

Then, we can define the following linear maps :

$$\begin{array}{llll} \psi_j^{(k)}: & L(E_j^{kR^*}) & \longrightarrow & W_k \\ & q & \mapsto & \sum_v q_v \cdot g^{(k)}\left(\underline{\lambda}(a^j) + \underline{\mu}^v(a^j) - \underline{\lambda}(a(c^v)),\, c^v\right)\,. \end{array}$$

(The q-coordinate corresponding to $R^* \in E_j^{kR^*}$ is not used in the definition of $\psi_j^{(k)}$.)

Lemma. *Let $\langle a^i, a^j\rangle < Q$ be an edge of the polyhedron Q. Then, on $L(E_i^{kR^*} \cap E_j^{kR^*}) = L(E_i^{kR^*}) \cap L(E_j^{kR^*})$, the maps $\psi_i^{(k)}$ and $\psi_j^{(k)}$ coincide. In particular (cf. Theorem (7.1)), the $\psi_j^{(k)}$'s induce a linear map $\psi^{(k)} : T_Y^2(-kR^*)^* \to W_k$.*

Proof. Let $q \in L(E_i^{kR^*} \cap E_j^{kR^*})$, and denote by $\varrho^{ij} \in \mathbb{Z}^N$ the path consisting of the single edge running from a^i to a^j. Then,

$$\begin{aligned}\psi_i^{(k)}(q) - \psi_j^{(k)}(q) &= \textstyle\sum_v q_v \cdot g^{(k)}\left(\underline{\lambda}(a^i) + \underline{\mu}^v(a^i) - \underline{\lambda}(a^j) - \underline{\mu}^v(a^j),\, c^v\right) \\ &= g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^j) + \varrho^{ij},\, \textstyle\sum_v q_v\, c^v\right) + \\ &\quad + \textstyle\sum_v q_v \cdot g^{(k)}\left(\underline{\mu}^v(a^i) - \underline{\mu}^v(a^j) - \varrho^{ij},\, c^v\right),\end{aligned}$$

and both summands vanish for several reasons. The first one is killed simply by the equality $\sum_v q_v\, c^v = 0$. For the second one we can use (2) of the previous lemma: If $q_v \neq 0$, then the assumption about q implies the inequalities

$$0 \leq \langle a^i, c^v\rangle - \langle a(c^v), c^v\rangle\,;\; \langle a^j, c^v\rangle - \langle a(c^v), c^v\rangle \leq k-1\,.$$

Hence, assuming w.l.o.g. $\langle a^i, c^v\rangle \geq \langle a^j, c^v\rangle$, we can take $\varrho^1 := -\underline{\mu}^v(a^j) - \varrho^{ij}$ and $\varrho^2 := -\underline{\mu}^v(a^i)$ to see that $g^{(k)}\left(\underline{\mu}^v(a^i) - \underline{\mu}^v(a^j) - \varrho^{ij},\, c^v\right) = 0$. □

(7.8) Proposition. *$\sum_{k\geq 1} c_k\, \psi^{(k)}$ equals λ^*, the adjoint of the obstruction map.*

Proof. In Theorem (3.5) of [Al 3] we gave a dictionary between the two T^2-formulas mentioned in (7.1). Using this result we can find an element of $\operatorname{Hom}({}^{\mathscr{R}}\!/_{\mathscr{R}_0}, W_k \otimes A(Y))$ representing $\psi^{(k)} \in T_Y^2 \otimes W_k$. It sends relations of type (i) (cf. (5.5)) to 0 and deals with relations of type (ii) and (iii) in the following way:

$$\begin{aligned}&[\underline{z}^r\, t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \\ &\qquad\qquad \mapsto \psi_j^{(k)}(a-b) \cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)] + (\alpha+\gamma-k)R^*}\end{aligned}$$

if

$$\langle (Q,1),\, \sum_v (a_v + r_v)\,[c^v, \eta_0(c^v)] + (\alpha+\gamma-k)R^*\rangle \geq 0$$

and j is such that

$$\langle (a^j,1),\, \sum_v a_v\,[c^v, \eta_0(c^v)] + (\alpha-k)R^*\rangle < 0\,;$$

otherwise the relation is sent to 0 (in particular, if there is not any j meeting the desired property).

On Q, the linear forms $c := \sum_v a_v\, c^v$ and $\tilde{c} = \sum_v (a_v + r_v) c^v$ admit their minimal values at the vertices $a(c)$ and $a(\tilde{c})$, respectively. Hence, we can transform the previous formula into

$$\begin{aligned}&[\underline{z}^r\, t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \\ &\qquad \mapsto \psi_{a(c)}^{(k)}(a-b) \cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)] + (\alpha+\gamma-k)R^*} \\ &\text{if} \qquad \sum_v (a_v + r_v)\eta_0(c^v) - \eta_0(\tilde{c}) + (\alpha+\gamma-k) =\end{aligned}$$

$$= \left\langle (a(\tilde{c}),1),\, \sum_v (a_v + r_v)\,[c^v, \eta_0(c^v)] + (\alpha+\gamma-k)R^* \right\rangle \geq 0\,,$$

$$\sum_v a_v\, \eta_0(c^v) - \eta_0(c) + (\alpha - k) =$$

$$= \left\langle (a(c),1),\, \sum_v a_v\,[c^v, \eta_0(c^v)] + (\alpha-k)R^* \right\rangle < 0$$

(or mapping to 0 otherwise). Adding the coboundary $h \in \operatorname{Hom}(\mathbb{C}[\underline{z},t]^m,\, W_k \otimes A(Y))$

$$h_{(a,\alpha),(b,\beta)} := \begin{cases} \psi^{(k)}_{a(c)}(a-b)\cdot x^{\sum_v a_v[c^v,\eta_0(c^v)]+(\alpha-k)R^*} \\ \qquad\qquad \text{for } \sum_v a_v\,\eta_0(c^v) - \eta_0(c) + \alpha \geq k\,, \\ 0 \qquad \text{otherwise} \end{cases}$$

does not change the class in $T^2_Y(-kR^*)$, which still represents $\psi^{(k)}$, but it does improve the representative from $\operatorname{Hom}({}^{\mathscr{R}}/_{\mathscr{R}_0},\, W_k \otimes A(Y))$. It still maps type-(i)-relations to 0, and moreover

$$[\underline{z}^r\, t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \mapsto$$

$$\mapsto \begin{cases} \left(\psi^{(k)}_{a(c)}(a-b) - \psi^{(k)}_{a(\tilde{c})}(a-b)\right)\cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)]+(\alpha+\gamma-k)R^*} \\ \qquad\qquad \text{for } k_0 + \gamma \geq k \\ 0 \qquad \text{otherwise} \end{cases}$$

with $k_0 = \alpha + \sum_v (a_v + r_v)\,\eta_0(c^v) - \eta_0(\tilde{c})$. By definition of $\psi^{(k)}_j$ and $g^{(k)}$ we obtain

$$\begin{aligned} \psi^{(k)}_{a(c)}(a-b) \; - \psi^{(k)}_{a(\tilde{c})}(a-b) &= \\ &= \sum_v (a_v - b_v)\cdot g^{(k)}\left(\underline{\lambda}(a(c)) + \underline{\mu}^v(a(c)) - \underline{\lambda}(a(\tilde{c})) - \underline{\mu}^v(a(\tilde{c})),\; c^v\right) \\ &= \sum_v (a_v - b_v)\cdot g^{(k)}\left(\underline{\lambda}^v(a(c)) - \underline{\lambda}^v(a(\tilde{c})),\; c^v\right) \\ &= g_{\underline{d},\,k}(\underline{t} - t_1) \\ \text{with} \quad \underline{d} \; &= \sum_v (a_v - b_v)\cdot \underline{d}\left(\underline{\lambda}^v(a(c)) - \underline{\lambda}^v(a(\tilde{c})),\; c^v\right) \\ &= \sum_v (a_v - b_v)\cdot \left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)\right)\,, \end{aligned}$$

and this completes our proof. Indeed,

- for relations of type (ii) (i.e. $r = 0$; $\gamma = 1$) we know $c = \tilde{c}$, hence, these relations map onto 0;

- for relations of type (iii) (i.e. $\gamma = 0$) we compare the previous formula with the result obtained in Corollary (7.5) - the coefficients coincide, and the monomial $\underline{z}^{\bar{p}}\, t^{k_0-k} \in \mathbb{C}[\underline{z}, t]$ maps onto $x^{\sum_v (a_v + r_v)[c^v, \eta_0(c^v)] + (\alpha+\gamma-k)R^*} \in A(Y)$. □

(7.9) It remains to show that the summands $\psi^{(k)}$ of λ^* are indeed surjective maps from $T^2_Y(-kR^*)^*$ to W_k. We will do so by composing them with auxiliary surjective maps $p^k : \oplus_{\varepsilon<Q}\mathbb{C}^n \twoheadrightarrow T^2_Y(-kR^*)^*$ yielding $\psi^{(k)} \circ p^k = \sum_{\varepsilon<Q} g^{(k)}(\underline{\varepsilon}, \bullet)$. Then, the result follows from the first part of Lemma (7.6).

In Sect. 6 of [Al 3] we used a short exact sequence of complexes called

$$0 \to L_{\mathbb{C}}(E^R)_\bullet \longrightarrow (\mathbb{C}^{E^R})_\bullet \longrightarrow \operatorname{span}_{\mathbb{C}}(E^R)_\bullet \to 0$$

to obtain from Theorem (7.1) an isomorphism

$$T^2_Y(-R) \cong \left(\frac{\operatorname{im}[\oplus_{\varepsilon<Q}\mathbb{C}^{n+1} \to \oplus_{\langle a^i, a^j\rangle<Q}\mathbb{C}^{n+1}]}{\operatorname{im}[\oplus_{\varepsilon<Q}\operatorname{span}_{\mathbb{C}}(\cap_{a^j\in\varepsilon}E^R_j) \to \oplus_{\langle a^i,a^j\rangle<Q}\mathbb{C}^{n+1}]} \right)^* .$$

Since $R^* = [\underline{0}, 1] \in E_j^{kR^*}$ for $k \geq 2$, the induced surjective map $\oplus_{\varepsilon<Q}\mathbb{C}^{n+1} \longrightarrow$ $\longrightarrow T^2_Y(-kR^*)^*$ factorizes through $\oplus_{\varepsilon<Q}\mathbb{C}^{n+1}/_{\mathbb{C}\cdot R^*} = \oplus_{\varepsilon<Q}\mathbb{C}^n$ yielding the auxiliary map p^k just mentioned. Taking a closer look at the construction of [Al 3] Sect. 6, we can give an explicit description of p^k; eventually we will be able to compute $\psi^{(k)} \circ p^k$.

Let us fix some 2-face $\varepsilon < Q$. Assume that $d^1, \dots, d^M$ are its counterclockwise oriented edges, i.e. the sign vector $\underline{\varepsilon}$ looks like $\varepsilon_i = 1$ for $i = 1, \dots, M$ and $\varepsilon_j = 0$ otherwise. Moreover, we denote the vertices of $\varepsilon < Q$ by $a^1, \dots, a^M$ such that d^i runs from a^i to a^{i+1} ($M + 1 := 1$). Starting with a $[c, \eta_0] \in \mathbb{C}^{n+1}$ (and, as just mentioned, only the $c \in \mathbb{C}^n$ is essential) we have to proceed as follows:

(i) For $i = 1, \dots, M$ we represent $[c, \eta_0]$ as a linear combination of elements of $E_i^{kR^*} \cap E_{i+1}^{kR^*}$, which corresponds to the lifting from $\operatorname{span}_{\mathbb{C}}(E^R)_\bullet$ to $(\mathbb{C}^{E^R})_\bullet$.

$$[c, \eta_0] = \sum_v q_{iv}\, [c^v, \eta_0(c^v)] + q_i\, [\underline{0}, 1]\,,$$

and $q_{iv} \neq 0$ implies $[c^v, \eta_0(c^v)] \in E_i^{kR*} \cap E_{i+1}^{kR^*}$, i.e.

$$\langle a^i, c^v\rangle + \eta_0(c^v) \leq k - 1\,; \quad \langle a^{i+1}, c^v\rangle + \eta_0(c^v) \leq k - 1\,.$$

(ii) We map the result to $\oplus_{i=1}^M \mathbb{C}^{E_i^{kR^*}}$ by taking successive differences, corresponding to the application of the differential in the complex $(\mathbb{C}^{E^R})_\bullet$. The result is automatically contained in $\ker\left(\oplus_i L(E_i^{kR^*}) \to L(E)\right)$, and its i-th summand is the linear relation

$$\sum_v (q_{i,v} - q_{i-1,v}) \cdot [c^v, \eta_0(c^v)] + (q_i - q_{i-1}) \cdot [\underline{0}, 1] = 0\,.$$

(iii) Finally, we apply $\psi^{(k)}$ to obtain

$$\begin{aligned}\psi^{(k)}(p^k(c)) &= \sum_{i=1}^M \sum_v (q_{i,v} - q_{i-1,v}) \cdot g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^i),\ c^v\right) \\ &= \sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^i),\ q_{i,v}\, c^v\right) - \\ &\qquad - \sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^{i+1}) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^{i+1}),\ q_{i,v}\, c^v\right) \\ &= \sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \underline{\mu}^v(a^i) - \underline{\mu}^v(a^{i+1}),\ q_{i,v}\, c^v\right).\end{aligned}$$

Similar to the proof of Lemma (7.7) we introduce the path ϱ^i consisting of the single edge d^i only. Then, if $q_{iv} \neq 0$ and w.l.o.g. $\langle a^i, c^v\rangle \geq \langle a^{i+1}, c^v\rangle$, the pair of paths $\underline{\mu}^v(a^i)$ and $\underline{\mu}^v(a^{i+1}) + \varrho^i$ meets the assumption of Lemma (7.6)(2) (cf. (i)). Hence, we can proceed as follows:

$$\begin{aligned}\psi^{(k)}(p^k(c)) &= \sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \varrho^i,\ q_{iv}\, c^v\right) + \\ &\qquad + \sum_{i,v} g^{(k)}\left(\underline{\mu}^v(a^i) - \underline{\mu}^v(a^{i+1}) - \varrho^i,\ q_{iv}\, c^v\right) \\ &= \sum_{i=1}^M g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \varrho^i,\ \sum_v q_{iv}\, c^v\right) \\ &= \sum_{i=1}^M g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \varrho^i,\ c\right) \\ &= g^{(k)}\left(\sum_{i=1}^M \varrho^i,\ c\right) \\ &= g^{(k)}(\underline{\varepsilon},\ c).\end{aligned}$$

Hence, Theorem (7.2) is proven.

8 The components of the reduced versal family

(8.1) The components of the reduced base space $\bar{\mathcal{M}}_{red}$ correspond to maximal decompositions of Q into a Minkowski sum $Q = R_0 + \ldots + R_m$ with lattice polytopes $R_k \subseteq \mathbb{R}^n$ as summands. Intersections of components are obtained by the finest Minkowski decompositions of Q that are coarser than all the maximal ones involved.

Theorem. *Fix such a Minkowski decomposition. Then, the corresponding component (or intersection of components) $\bar{\mathcal{M}}_0$ is isomorphic to $\mathbb{C}^{m+1}/\mathbb{C}\cdot(1,\ldots,1)$, and the restriction $X_0 \to \mathbb{C}^m$ of the versal family can be described as follows:*

(i) Defining the cone

$$\tilde{\sigma} := \operatorname{Cone}\left(\bigcup_{k=0}^{m}(R_k \times \{e^k\})\right) \subseteq \mathbb{R}^{n+m+1},$$

it contains $\sigma = \operatorname{Cone}(Q \times \{1\}) \subseteq \mathbb{R}^{n+1}$ via the diagonal embedding $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+m+1}$ $((a,1) \mapsto (a;1,\ldots,1))$. The inclusion $\sigma \subseteq \tilde{\sigma}$ induces a closed embedding of the affine toric varieties defined by these cones, giving $Y \hookrightarrow X_0$.

(ii) *The projection* $\mathbb{R}^{n+m+1} \twoheadrightarrow \mathbb{R}^{m+1}$ *provides* $m+1$ *regular functions on* X_0, *i.e. we obtain a map* $X_0 \to \mathbb{C}^{m+1}$. *Composing this map with* $\ell : \mathbb{C}^{m+1} \twoheadrightarrow \mathbb{C}^{m+1} / \mathbb{C} \cdot (1, \ldots, 1)$ *yields the family.*

The theorem is a straight consequence of knowing the versal deformation. Hence, we omit the proof here.

(8.2) *Example.* At the end of (2.5) we presented two decompositions of Q_6 into a Minkowski sum of lattice summands. Let us describe now the restrictions of the versal family to the associated components of $\bar{\mathcal{M}}$:

(i) Putting the two triangles R_0, R_1 into two parallel planes contained in $\mathbb{R}^3$ yields an octahedron as the convex hull of the whole configuration. Then, $\tilde{\sigma}$ is the (4-dimensional) cone over this octahedron

$$\tilde{\sigma} = \langle (0,0;1,0),\ (1,0;1,0),\ (1,1;1,0),\ (0,0;0,1),\ (0,1;0,1),\ (1,1;0,1) \rangle .$$

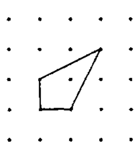
(ii) Looking at the second decomposition, we have to put three line segments R_0, R_1, R_2 on three parallel 2-planes in general position inside the affine space $\mathbb{R}^4$. Taking the convex hull of this configuration yields a 4-dimensional polytope that is dual to (triangle)$\times$(triangle). Again, $\tilde{\sigma}$ is the (5-dimensional) cone over this polytope

$$\begin{aligned} \tilde{\sigma} \;=\; & \langle (0,0;1,0,0),\ (1,0;1,0,0),\ (0,0;0,1,0), \\ & \quad (0,1;0,1,0),\ (0,0;0,0,1),\ (1,1;0,0,1) \rangle . \end{aligned}$$

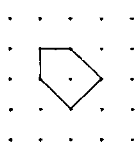
The total spaces over the components arise as the toric varieties defined by $\tilde{\sigma}$. In our example, they equal the cones over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$, respectively.

9 Further examples

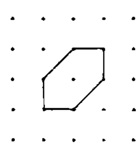
(9.1) Three examples of toric Gorenstein singularities arise as cones over the Del Pezzo surfaces obtained by blowing up $(\mathbb{P}^2, \mathcal{O}(3))$ in one, two, or three points, respectively. They correspond to the following polygons:



Polygon Q_4



Polygon Q_5



Polygon Q_6

Let us discuss these three examples:
(iv) The edges equal

$$d^1 = (1,0),\ d^2 = (1,2),\ d^3 = (-2,-1),\ d^4 = (0,-1),$$

and they imply the following equations of the versal base space as closed subscheme of $\mathbb{C}^4/_{\mathbb{C}\cdot(1,1,1,1)}$:

$$t_1 + t_2 = 2t_3, \quad t_3 + t_4 = 2t_2, \quad t_1^2 + t_2^2 = 2t_3^2, \quad t_3^2 + t_4^2 = 2t_2^2 .$$

Using the two linear equations, only two coordinates $t := t_1$, $\varepsilon := t_1 - t_3$ are sufficient. (We get the t_i's back by $t_1 = t$, $t_2 = t - 2\varepsilon$, $t_3 = t - \varepsilon$, $t_4 = t - 3\varepsilon$.) Then, the two quadratic equations transform into $2\varepsilon^2 = 0$, i.e. the versal base space is a fat point.
On the other hand, Q_4 does not allow any splitting into a Minkowski sum involving lattice summands only. This reflects the triviality of the underlying reduced space. (Cf. (9.2).)

(v) The polygon Q_5 allows the decomposition into the sum of a triangle and a line segment. In particular, the reduced base space of the versal deformation of Y_5 has to be a line. We compute the true base space: $d^1 = (1,1)$, $d^2 = (-1,1)$, $d^3 = (-1,0)$, $d^4 = (0,-1)$, $d^5 = (1,-1)$ yield the equations

$$t_1 - t_3 = t_2 - t_5 = t_4 - t_1 \quad \text{and} \quad t_1^2 - t_3^2 = t_2^2 - t_5^2 = t_4^2 - t_1^2 .$$

With $t := t_1$, $s_1 := t_1 - t_3$, $s_2 := t_1 - t_2$ and $t_1 = t$, $t_2 = t - s_2$, $t_3 = t - s_1$, $t_4 = t + s_1$, $t_5 = t - s_1 - s_2$, they turn into

$$s_1^2 = 2 s_1 s_2 = 0 .$$

(vi) This example was spread throughout the paper.

(9.2) We will use the polygon $Q_4 := \operatorname{Conv}\{(0,0), (1,0), (2,2), (0,1)\}$ of (9.1)(iv) for a more detailed demonstration of how the theory works. In particular, we will describe the versal family of Y_4 over $\operatorname{Spec} \mathbb{C}[\varepsilon]/_{\varepsilon^2}$:

(1) The (t,ε)-coordinates of V correspond to the linear map

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & -1 \\ 1 & -3 \end{pmatrix} : \mathbb{R}^2 \xrightarrow{\sim} V \hookrightarrow \mathbb{R}^4 .$$

We obtain

$$\begin{aligned} C(Q_4) &= \{(a,b) \in \mathbb{R}^2 \mid a \geq 0,\ a - 2b \geq 0,\ a - b \geq 0,\ a - 3b \geq 0\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid a \geq 0,\ a - 3b \geq 0\} \\ &= \langle [1,0],\ [1,-3] \rangle^\vee = \langle (0,-1),\ (3,1) \rangle \subseteq \mathbb{R}^2 , \end{aligned}$$

and the map $\mathbb{N}^4 \to C(Q_4)^\vee \cap V_{\mathbb{Z}}^*$ sends e_1, e_2, e_3, e_4 to $[1,0]$, $[1,-2]$, $[1,-1]$, $[1,-3]$, respectively. In particular, this map is surjective, i.e. $S_4 = \bar{S}_4$ and $X_4 = \bar{X}_4$.

(2) To compute the tautological cone $\tilde{C}(Q_4)$, we need the Minkowski summands associated to the two fundamental generators of $C(Q_4)$:

$$\begin{aligned}(Q_4)_{(0,-1)} &= \operatorname{Conv}\{(0,0),\ (2,4),\ (0,3)\},\\ (Q_4)_{(3,1)} &= \operatorname{Conv}\{(0,0),\ (3,0),\ (4,2)\}.\end{aligned}$$

Hence,

$$\tilde{C}(Q_4) = \Big\langle (0,0;0,-1);\ (2,4;0,-1);\ (0,3;0,-1);\ (0,0;3,1);\ (3,0;3,1);\ (4,2;3,1)\Big\rangle .$$

(3) Now, we have all the information needed to obtain the versal family of Y_4:

– Restrict the family $\operatorname{Spec}\mathbb{C}[\tilde{C}(Q_4)^\vee \cap \mathbb{Z}^4] \to \operatorname{Spec}\mathbb{C}[C(Q_4)^\vee \cap \mathbb{Z}^2] \subseteq \mathbb{C}^4$ to the subspace $\mathbb{C}^2 \simeq V_{\mathbb{C}} \subseteq \mathbb{C}^4$, i.e. use the (t,ε)-coordinates instead of (t_1,t_2,t_3,t_4).

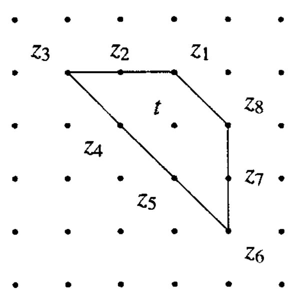
– Compose the result with the projection $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^1$ $((t,\varepsilon) \mapsto \varepsilon)$. That means we no longer regard t as a coordinate of the base space.

– Finally, we restrict our family to the closed subscheme defined by the equation $\varepsilon^2 = 0$.

(4) To obtain equations, we could either take a closer look to the family constructed so far, or we can proceed more directly as described in (4.5) and (5.3):

– Computing the minimal generator set of the semigroup $\operatorname{Cone}(Q_4)^\vee \cap \mathbb{Z}^3$, we get the elements $[c^v;\eta_0(c^v)]$:

$$\begin{aligned}&[c^1;\eta_0^1] = [0,1;0],\ [c^2;\eta_0^2] = [-1,1;1],\ [c^3;\eta_0^3] = [-2,1;2],\\ &[c^4;\eta_0^4] = [-1,0;2],\ [c^5;\eta_0^5] = [0,-1;2],\ [c^6;\eta_0^6] = [1,-2;2],\\ &[c^7;\eta_0^7] = [1,-1;1],\ [c^8;\eta_0^8] = [1,0;0].\end{aligned}$$



Polygon $Q_4^\vee$

Together with $[0,0;1]$, they induce coordinates $z_1,\ldots,z_8,t$ on Y_4, i.e. we have obtained an embedding $Y_4 \hookrightarrow \mathbb{C}^9$. (The sums of the three components of the vectors are always 1. In the figure we have drawn the first two coordinates.)

– $Y_4 \subseteq \mathbb{C}^9$ is defined by the following 20 polynomials:

$$\begin{aligned}&t^2 - z_4z_8,\quad t^2 - z_1z_5,\quad t^2 - z_2z_7,\quad z_1t - z_2z_8,\quad z_2t - z_3z_8,\\ &z_2t - z_1z_4,\quad z_3t - z_2z_4,\quad z_4t - z_3z_7,\quad z_4t - z_2z_5,\quad z_5t - z_4z_7,\\ &z_5t - z_2z_6,\quad z_6t - z_5z_7,\quad z_7t - z_5z_8,\quad z_7t - z_1z_6,\quad z_8t - z_1z_7,\\ &z_1z_3 - z_2^2,\quad z_3z_5 - z_4^2,\quad z_4z_6 - z_5^2,\quad z_6z_8 - z_7^2,\quad z_3z_6 - z_4z_5.\end{aligned}$$

– Choosing paths from $(0,0) \in Q_4$ to the other vertices, we obtain the list

$$\begin{array}{lll} \underline{\eta}^1 = [0,0,0,0], & \underline{\eta}^2 = [1,0,0,0], & \underline{\eta}^3 = [2,0,0,0], \\ \underline{\eta}^4 = [1,1,0,0] = [0,0,2,0], & \underline{\eta}^5 = [0,2,0,0] = [0,0,1,1]. & \\ \underline{\eta}^6 = [0,0,0,2], & \underline{\eta}^7 = [0,0,0,1], & \underline{\eta}^8 = [0,0,0,0], \end{array}$$

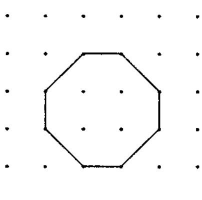
– Now, we can lift our 20 polynomials to the ring $\mathbb{C}[Z_1, \ldots, Z_8, t_1, \ldots, t_4]$:

$$\begin{array}{lllll} t_1t_2 - Z_4Z_8, & t_2^2 - Z_1Z_5, & t_1t_4 - Z_2Z_7, & Z_1t_1 - Z_2Z_8, & Z_2t_1 - Z_3Z_8, \\ Z_2t_2 - Z_1Z_4, & Z_3t_2 - Z_2Z_4, & Z_4t_3 - Z_3Z_7, & Z_4t_2 - Z_2Z_5, & Z_5t_3 - Z_4Z_7, \\ Z_5t_2 - Z_2Z_6, & Z_6t_3 - Z_5Z_7, & Z_7t_3 - Z_5Z_8, & Z_7t_4 - Z_1Z_6, & Z_8t_4 - Z_1Z_7, \\ Z_1Z_3 - Z_2^2, & Z_3Z_5 - Z_4^2, & Z_4Z_6 - Z_5^2, & Z_6Z_8 - Z_7^2, & Z_3Z_6 - Z_4Z_5. \end{array}$$

– Finally, we restrict the family to the versal base space by switching to the (t, ε)-coordinates and obeying the equation $\varepsilon^2 = 0$. Moreover, t is no longer a coordinate of the base space:

$$\begin{array}{lll} t(t-2\varepsilon) - z_4z_8, & t(t-4\varepsilon) - z_1z_5, & t(t-3\varepsilon) - z_2z_7, \\ z_1t - z_2z_8, & z_2t - z_3z_8, & z_2(t-2\varepsilon) - z_1z_4, \\ z_3(t-2\varepsilon) - z_2z_4, & z_4(t-\varepsilon) - z_3z_7, & z_4(t-2\varepsilon) - z_2z_5, \\ z_5(t-\varepsilon) - z_4z_7, & z_5(t-2\varepsilon) - z_2z_6, & z_6(t-\varepsilon) - z_5z_7, \\ z_7(t-\varepsilon) - z_5z_8, & z_7(t-3\varepsilon) - z_1z_6, & z_8(t-3\varepsilon) - z_1z_7, \\ z_1z_3 - z_2^2, & z_3z_5 - z_4^2, & z_4z_6 - z_5^2, \\ z_6z_8 - z_7^2, & z_3z_6 - z_4z_5. & \end{array}$$

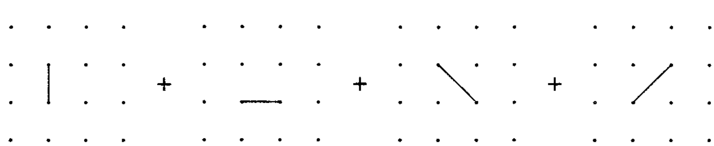
(9.3) At last we want to present an example involving more than only quadratic equations for the versal base space. Let Q_8 be the "regular" lattice 8-gon; it is contained in two strips of lattice thickness 3.



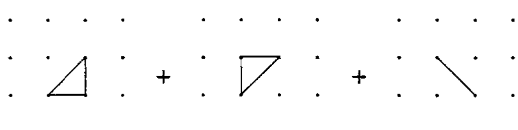
Polygon Q_8

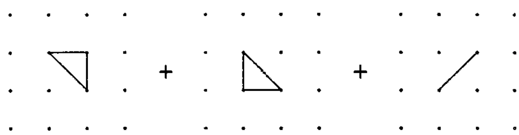
Q_8 admits three maximal Minkowski decompositions into a sum of lattice summands:

(i) Q_8 =

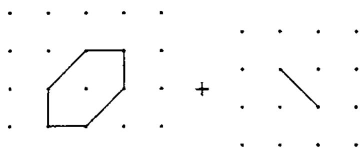


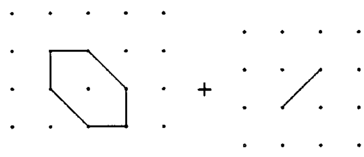
(ii) Q_8 =



(iii) Q_8 = 

The decompositions (i), (ii) and (i), (iii) are refinements of the coarser decompositions

Q_8 = 

and Q_8 =  ,

respectively. These facts translate directly into the geometry of the reduced base space of the versal deformation of Q_8:

- It is embedded in some affine space $\mathbb{C}^5$ and equals the union of a 3-plane with two 2-planes (through $0 \in \mathbb{C}^5$).
- The two 2-planes each have a common line with the 3-dimensional component. However, they intersect each other in $0 \in \mathbb{C}^5$ only.

On the other hand, we can write down the equations of the true versal base space (as a closed subscheme of $\mathbb{C}^8/_{\mathbb{C}\cdot(1,\ldots,1)}$):

$$t_1^k + t_2^k + t_8^k = t_4^k + t_5^k + t_6^k, \quad t_2^k + t_3^k + t_4^k = t_6^k + t_7^k + t_8^k \quad (k = 1,2,3).$$

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