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Harmonic functions on planar and almost planar graphs and manifolds, via circle packings

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Abstract. The circle packing theorem is used to show that on any bounded valence transient planar graph there exists a non constant, harmonic, bounded, Dirichlet function. If P is a bounded circle packing in $\mathbb{R}^2$ whose contacts graph is a bounded valence triangulation of a disk, then, with probability 1, the simple random walk on P converges to a limit point. Moreover, in this situation any continuous function on the limit set of P extends to a continuous harmonic function on the closure of the contacts graph of P; that is, this Dirichlet problem is solvable. We define the notions of almost planar graphs and manifolds, and show that under the assumptions of transience and bounded local geometry these possess non constant, harmonic, bounded, Dirichlet functions. Let us stress that an almost planar graph is not necessarily roughly isometric to a planar graph.

1. Introduction

In this paper we study several aspects of harmonic functions on planar graphs and generalizations. Our main results are concerned with the solution of the Dirichlet problem for infinite transient planar graphs via circle packings. We also prove the existence of non constant, bounded, harmonic functions with finite Dirichlet energy on an even wider familiy of almost planar graphs and manifolds.

In the study of a non-compact Riemannian manifold M, several “type problems” are natural and widely studied. M is said to be *transient*, if Brownian motion is transient on M (or, equivalently, M admits a Greens function). If M is not transient, then it is *recurrent*. Let O_G denote the collection of all recurrent manifolds. M is said to be *weak-Liouville*, if every bounded harmonic function on M is constant, and we then write $M \in O_{HB}$. Let O_{HD} denote the class of manifolds M such that every harmonic function on M with finite Dirichlet integral is a constant.

The following inclusions are valid

$$O_G \subset O_{HB} \subset O_{HD}.$$

Moreover, if there is a non-constant harmonic function on M with finite Dirichlet integral, then there is also such a function which is bounded. (The proofs of these and the following statements can be found in the book [24], which is also the source of our above notations.) Here are some examples. The Euclidean plane $\mathbb{R}^2$ is in O_G, Euclidean n-space $\mathbb{R}^n$ is in $O_{HB} - O_G$ when $n > 2$, hyperbolic n-space H^n is in $O_{HD} - O_{HB}$ when $n > 2$, and the hyperbolic plane is not contained in O_{HD}; that is, H^2 admits non-constant, harmonic functions with finite Dirichlet integral.

Suppose now that S is a Riemannian surface, which is topologically planar; that is, it is homeomorphic to a domain in $\mathbb{R}^2$. Then S is also conformaly equivalent to a domain in the complex plane $\mathbb{C}$. Recall that in dimension 2 the composition of a comformal mapping and a harmonic function is harmonic. Hence, using Riemann's mapping theorem it is not hard to see that $S \in O_G$ iff $S \in O_{HD}$. In other words, a transient topologically planar surface admits non-constant, bounded, harmonic functions with finite Dirichlet integral.

These problems on manifolds are very closely related to analogous problems on graphs. Let $G = (V, E)$ be an infinite connected, locally finite graph. A basic question in discrete potential theory is to decide what classes of harmonic functions on G are non-trivial. In order to study that question for a transient planar graph G, we will use a bounded disk packing P in $\mathbb{R}^2$ whose contacts graph is G. The latter means that the disks of P are indexed by the vertices of G and two disks in P are tangent iff the corresponding vertices neighbor. The existence of such a disk packing in $\mathbb{R}^2$ follows from the Circle Packing Theorem (4.2), first proved by Koebe [16], and the existence of such a **bounded** packing follows from [12].

Suppose that G is the contacts graph of a bounded disk packing $P \subset \mathbb{R}^2$. We may then embed G in the plane by mapping each vertex to the center of the corresponding disk, and drawing each edge as the straight line segment between the corresponding centers. This embedding is called the *geometric nerve* of P. We can consider the closure of V in this embedding, and thus get a compactification of G.

1.1. Theorem *Let $G = (V, E)$ be a planar, bounded valence, transient graph.*

(1) There are non-constant, bounded, harmonic functions on G with finite Dirichlet energy.

(2) Suppose further that G is (the 1-skeleton of) a triangulation of a disk. Let P be a bounded disk packing with contacts graph G, and identify G with the geometric nerve of P. Let $\overline{V}$ be the closure of V in $\mathbb{R}^2$. Then with probability 1 the simple random walk on G converges to a point in $\overline{V} - V$.

(3) Under the assumptions of (2), any continuous function on $\overline{V} - V$ can be continuously extended to a harmonic function on G.

Remarks. 1. The definitions of most of the above terms appear below in Sect. 2. Woess's [29] is a useful survey of results on random walks on graphs. Soardi's recent [28] deals with the potential theory of infinite networks. Doyle and Snell's book [8] also gives a light introduction to random walks and electrical networks.
2. S. Northshield [22] proved the existence of non constant bounded harmonic functions on bounded valence planar graphs with rapidly decaying Green's function, and studied their boundaries. From D. R. DeBaun's work [7] it follows that a bounded valence triangulation of a disk has non constant harmonic functions with finite Dirichlet energy, if it is transient. Cartwright and Woess [5] have shown that a graph which satisfies a strong isoperimetric inequality (positive Cheeger constant), and is uniformly embedded in the hyperbolic plane, admits nonconstant harmonic functions with finite Dirichlet energy. They also prove the existence of solutions to certain Dirichlet problems on the boundary at infinity.
3. The bounded valence assumption is necessary. For example, consider the graph on the natural numbers where 2^n edges connect each n with $n+1$. Though this graph is transient and planar, any bounded harmonic function on it is a constant. This example can easily be modified to have no multiple edges.

1.2. Definition [14] *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces, and let* $\kappa < \infty$. *A* κ-**rough isometry** f *from* X *to* Y *is a (not necessarily continuous) map* $f : X \to Y$ *such that*

$$\kappa^{-1} d_X(x_1, x_2) - \kappa \leq d_Y\big(f(x_1), f(x_2)\big) \leq \kappa d_X(x_1, x_2) + \kappa$$

holds for all $x_1, x_2 \in X$, *and for every* $y_1 \in Y$ *there is some* $x_1 \in X$ *such that*

$$d_Y(y_1, f(x_1)) \leq \kappa.$$

If such an f *exists, we say that* X *and* Y *are roughly isometric.*

M. Gromov [11] uses the term quasi-isometry for Kanai's rough isometry.

1.3. Definition *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces, and let* $\kappa < \infty$. *A (not necessarily continuous) map* $f : X \to Y$ *is a* κ-**quasimonomorphism** *if the following two conditions are satisfied.*

1. $d_Y(f(x_0), f(x_1)) \leq \kappa(1 + d_X(x_0, x_1))$ *holds for every* $x_0, x_1 \in X$, *and*
2. *for every open ball* $B = B(y_0, 1)$ *with radius* 1 *in* Y, *the inverse image* $f^{-1}(B)$ *can be covered by* κ *open balls of radius* 1 *in* X.

A quasimonomorphism is a map which is a κ*-quasimonomorphism for some finite* κ.

It may seem somewhat unnatural that we have given special status to balls of radius 1. But we shall use this definition only in the context of metric spaces that have the following property.

1.4. Definition *A metric space X is said to have the $C(R,r)$ property if for any two positive numbers R, r there is some finite $C = C(R,r)$ such that any ball of radius R in X can be covered by C balls of radius r.*

For example, a connected graph with its natural metric has the $C(R,r)$ property iff there is a global bound on the degrees of its vertices.

1.5. Definition *A metric space X satisfying the $C(R,r)$ property is* **almost planar***, if there is a bounded valence planar graph $G = (V, E)$ and a quasimonomorphism $f : X \to V$, where V is equipped with the distance metric in G. A graph $G^\circ = (V^\circ, E^\circ)$ is* **almost planar***, if V° with the distance metric of G° is almost planar.*

We wish to stress that almost planarity is a much weaker property than being roughly isometric to a planar graph. For example, take two copies of $\mathbb{Z}^2$ and identify them along the x-axis $\{(n,0)\}$. The resulting graph is easily seen to be almost planar, but is not roughly isometric to a planar graph.

As explained below, if G_1 is a planar graph, and G_2 is finite, then any subgraph of $G_1 \times G_2$ is almost planar. This can be used to construct many examples of almost planar graphs.

The following theorem gives an equivalent definition for almost planarity.

1.6. Theorem *Let X be a metric space that satisfies the $C(R,r)$ property. Then X is almost planar iff there exists a topologically planar, complete, Riemannian surface S with bounded curvature, and a quasimonomorphism $f : X \to S$.*

An easy, but useful, observation is:

1.7. Fact *In the category of metric spaces satisfying the $C(R,r)$ property, a rough isometry is a quasimonomorphism, and the composition of quasimonomorphisms is a quasimonomorphism. Hence, for such spaces, almost planarity is invariant under rough isometries.* □

This, in particular, shows that when G_1 is an almost planar graph and G_2 is a finite graph, then any subgraph of $G_1 \times G_2$ is almost planar.

Regarding almost planar graphs, we have the following geometric criterion.

1.8. Theorem *Let G be a connected, bounded degree graph. Then G is almost planar iff there is some finite κ such that G can be drawn in the plane in such a way that every edge has at most κ crossings.*

We shall now clarify the statement of the theorem, by defining what a crossing is. Given a graph G, we let $|G|$ denote the metric space constructed as follows. Start with V, and for every edge $e \in E$, with vertices v, u, say, glue the endpoints of an isometric copy I_e of the interval $[0,1]$ to the two vertices v, u. Let $|G|$ be the union $\cup_e I_e \cup V$ modulo the identifictions, with the path metric. A *drawing* of

G in the plane is a continuous map $f : |G| \to \mathbb{R}^2$. A *crossing* in such a drawing is a pair (p, q) of distinct points in $|G|$ that are mapped by the drawing to a single point: $f(p) = f(q)$. We say that the crossing is in the edge $e \in E$ if one of the points p, q is in I_e.

1.9. Theorem *Let $G^\circ = (V^\circ, E^\circ)$ be a transient, connected, almost planar graph with bounded vertex degree. Then there are non-constant bounded harmonic functions on G° with finite Dirichlet energy.*

This implies, for example, that $\mathbb{Z}^3$ is not almost planar, since it is transient and weak-Liouville.

1.10. Theorem *Let M be a connected, transient, n-dimensional, almost planar, Riemannian manifold with bounded local geometry. Then there are non-constant bounded harmonic functions with finite Dirichlet energy on M.*

The condition 'M has bounded local geometry' means that it is complete, the injectivity radius of M positive and the Ricci curvature is bounded from below. Note that a Riemannian manifold with bounded local geometry satisfies the $C(R, r)$ property [14].

A consequence of Theorem 1.10 is that hyperbolic 3 space H^3 and Euclidean 3 space $\mathbb{R}^3$ are not almost planar. We do have a more direct proof for $\mathbb{R}^3$, but not for H^3. Conjecturally, every simply connected 3-manifold with non positive sectional curvature (Cartan-Hadamard manifold) is not almost planar, but we cannot provide a proof.

For the reader familiar with Gromov-hyperbolicity, we would like to end the introduction with a conjecture, replacing planarity by hyperbolicity. That is,

1.11. Conjecture *Let M be a connected, transient, Gromov hyperbolic, Riemannian manifold with bounded local geometry, with the property that the union of all bi-infinite geodesics meets every ball of sufficiently large radius. Then M admits non constant bounded harmonic functions. Similarly, a Gromov hyperbolic bounded valence, transient graph, with C-dense bi-infinite geodesics has non constant bounded harmonic functions.*

Since every Dirichlet-finite, harmonic function on H^3 is constant, this would seem to make the conjecture harder to prove.

The example of a horoball in H^4 shows that the requirement that geodesics are C-dense is necessary.

A. Ancona [1] proved the conjecture for graphs, but with a strong isoperimetric condition replacing the assumption on the density of geodesics. A strong isoperimetric inequality, even for simply connected manifolds with bounded local geometry, is not sufficient to imply the existence of non constant bounded harmonic functions. This was shown by Benjamini and Cao [2].

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2. Notations and terminology

Let $G = (V, E)$ be a graph. For convenience, we usually only consider graphs with no loops or multiple edges (but the results do apply to multigraphs). The set of vertices incident with an edge e will be denoted ∂e; this is always a subset of V that contains two vertices. We sometimes use $\{v, u\}$ to denote the edge with endpoints v, u.

Initially the graph G is unoriented, but for notational reasons we also consider directed edges. When $\{v, u\} \in E$, we use $[v, u]$ to denote the directed edge from v to u. The set of all direceted edges will usually be denoted $\overrightarrow{E}$; $\overrightarrow{E} = \{[v, u] : \{v, u\} \in E\}$.

The graphs we shall consider will be connected and locally finite. The latter means that the number of edges incident with any particular vertex is finite.

Given any vertex $v \in V$, the collection of all edges of the form $[v, u]$ which are in $\overrightarrow{E}$ will be denoted $\overrightarrow{E}(v)$. The *valence*, or *degree*, of a vertex v is just the cardinality of $\overrightarrow{E}(v)$. G has *bounded valence*, if there is a finite upper bound for the degrees of its vertices.

Let $f : V \to \mathbb{R}$ be some function. Then df is the function $df : \overrightarrow{E} \to \mathbb{R}$ defined by

$$df([v, u]) = f(u) - f(v).$$

We also define the gradient of f to be equal to df,

$$\nabla f(e) = df(e).$$

(The reason for the multiplicity of notation should become clear when we introduce the gradient with respect to a metric on G.)

A function $j : \overrightarrow{E} \to \mathbb{R}$ is a *flow* on G if it satisfies

$$j([u, v]) = -j([v, u])$$

for every $\{v, u\} \in E$. For example, for any $f : V \to \mathbb{R}$, df is a flow. The *divergence* of a flow j is the function $\operatorname{div} j : V \to \mathbb{R}$, defined by

$$\operatorname{div} j(v) = \sum_{e \in \overrightarrow{E}(v)} j(e).$$

If $\operatorname{div} j = 0$, then j is *divergence free*.

For an $f : V \to \mathbb{R}$ we set

$$\triangle f = \operatorname{div} \nabla f,$$

then $\triangle f : V \to \mathbb{R}$ is known as the *discrete laplacian* of f. If $\triangle f = 0$, then f is *harmonic*, while if $\triangle f = 0$ on a subset $V' \subset V$, we say that f is harmonic in V'. Equivalently, f is harmonic iff its value at any $v \in V$ is equal to the average of the values at the neighbors of v.

For a flow j and an $e \in E$ we let $|j(e)|$ denote $|j(v) - j(u)|$, where $\partial e = \{v, u\}$. The norm of a flow j is defined by

$$\|j\|^2 = \frac{1}{2} \sum_{e \in \overrightarrow{E}} j(e)^2 = \sum_{e \in E} |j(e)|^2.$$

The collection of all flows with finite norm is then a Hilbert space with this norm. The *Dirichlet energy* of a function $f : V \to \mathbb{R}$ is defined by

$$\mathcal{D}(f) = \|df\|^2.$$

A *Dirichlet function* is an $f : V \to \mathbb{R}$ with $\mathcal{D}(f) < \infty$. The space of all Dirichlet functions on G is denoted $D(G)$.

The *simple random walk* on a locally finite graph $G = (V, E)$ starting at a vertex v_0 is the Markov process $(v(1), v(2), \ldots)$ on V such that $v(1) = v_0$ and the transition probability from a vertex v to a vertex u is the inverse of the cardinality of $\overrightarrow{E}(v)$. A connected graph G is said to be *transient*, if there is a positive probability that a random walk that starts at a vertex v_0 will never visit v_0 again. It is easy to see that this does not depend on the initial vertex v_0. A non-transient graph is *recurrent*.

A *metric* m on a graph $G = (V, E)$ is a positive function $m : E \to (0, \infty)$. The *random walk on* (G, m) is the Markov process where the transition probability from v to u is equal to $c(v, u)/c(v)$, where $c(v, u) = m(\{v, u\})^{-1}$, and $c(v)$ is the sum of $c(v, u)$ over all neighbors u of v.

The *gradient* of a function $f : V \to \mathbb{R}$ with respect to a metric m is defined by

$$\nabla_m f(e) = df(e)/m(e).$$

f is said to be harmonic on (G, m) if $\triangle_m f = \operatorname{div} \nabla_m f$ is zero. It is clear that f is harmonic on (G, m) iff for every $v \in V$, $f(v)$ is equal to the expected value of $f(u)$, where u is the state of the random walk on (G, m) that starts at v after one step.

The *natural metric* on G is the metric where each edge get weight 1. In the absence of another metric, all metric related notions are assumed to be with respect to the natural metric. It is easy to check that if m is the natural metric then a random walk on (G, m) is the same as a simple random walk on G, and the harmonic functions on (G, m) are the harmonic functions on G.

Two metrics m, m' are *mutually bilipschitz*, if the ratios m/m' and m'/m are bounded.

Let $G = (V, E)$ be a connected, locally finite graph, and let m be a metric on G. The m-length of a path γ in G is the sum of $m(e)$ over all edges in γ,

$$\text{length}_m(\gamma) = \sum_{e \in \gamma} m(e).$$

We define the m-distance $d_m(v, u)$ between any two vertices $v, u \in V$ to be the infimum of the m-lengths of paths connecting v and u. Then (V, d_m) is a metric space.

3. Stability and instability of harmonic functions on graphs

A graph G is said to have the *weak-Liouville property* if every bounded harmonic function on G is constant.

M. Kanai [15] and Markvorsen et al [21] have shown that recurrence on a bounded valence graph is invariant under rough isometries. The weak-Liouville property is not: T. Lyons [19] constructed two mutually bilipschitz metrics m, m' on a graph G, such that (G, m) is weak-Liouville, while (G, m') is not. (Replacing the edges by tubes produces a Riemannian example). We will describe below a relatively easy recipe for making such examples.

While the weak-Liouville property is not stable under rough isometries, Soardi [27] has shown that the existence of non-constant, harmonic functions with finite Dirichlet energy is invariant. Below, we introduce the notion of a resolvable graph, and will see that a transient resolvable graph has non-constant, bounded harmonic functions with finite Dirichlet energy. Moreover, the property of being resolvable is very stable: if $f : G^\circ \to G$ is a quasimonomorphism and G is resolvable, then so is G°.

Definitions. *Let $G = (V, E)$ be some graph, and let Γ be a collection of (infinite) paths in G. Then Γ is* **null** *if there is an $L^2(E)$ metric on G such that* $\text{length}_m(\gamma) = \infty$ *for every $\gamma \in \Gamma$. It is easy to see that Γ is null iff its* **extremal length**

$$\text{EL}(\Gamma) = \sup_m \inf_{\gamma\in\Gamma} \frac{\text{length}_m(\gamma)^2}{\|m\|^2},$$

is infinite. (Extremal length was imported to the discrete setting by Duffin [9]. See [28] for more about extremal length on graphs.) When Γ is a collection of paths and a property holds for every $\gamma \in \Gamma$, except for a null family, we shall say that the property holds for **almost every** $\gamma \in \Gamma$.

*Let m be a metric on G, and recall that d_m is the associated distance function. Let $C_m(G)$ denote the completion of (V, d_m), and let the m-***boundary** *of G be $\partial_m G = C_m(G) - V$. We use d_m to also denote the metric of the completion $C_m(G)$.*

The metric m will be called **resolving** *if it is in $L^2(E)$ and for every $x \in \partial_m G$ the collection of half infinite paths in G that converge to x in $C_m(G)$ is null. G is* **resolvable** *if it has a resolving metric.*

Note that if m is a resolving metric and m' is another L^2 metric satisfying $m' \geq m$, then m' is also resolving.

Theorem 3.2 below shows that any recurrent graph is resolvable, in fact, there is a metric m with $\partial_m G = \emptyset$. On the other hand, the next theorem shows that a transient graph with no non-constant, harmonic functions in $D(G)$ is not resolvable, for example, $\mathbb{Z}^3$ or any lattice in hyperbolic n-space $n > 2$ is not resolvable. We shall see that any bounded valence planar graph is resolvable.

3.1. Theorem *Let $G = (V, E)$ be an infinite, connected, locally finite, resolvable graph.*

1. *If G is transient, then there are non-constant, bounded, harmonic functions on G with finite Dirichlet energy.*
2. *If $f : G^\circ \to G$ is a quasimonomorphism, where G° is a connected, bounded valence graph, then G° is resolvable.*

We shall need the following results.

3.2. Theorem (Yamasaki) *Let G be a locally finite connected graph, and let Γ be the collection of all infinite paths in G. Then G is recurrent if and only if Γ is null.*

This is proven in Yamasaki's [30], [31]; see also [29, Theorem 4.8]. (There they consider only paths that start at a fixed base vertex, but this is equivalent.)

The following result is the discrete version of [10, Corollary 8].

3.3. Theorem (Yamasaki [32, Sect. 3]) *There are non-constant, harmonic, Dirichlet functions on G if and only if there is an $f \in D(G)$ such that for every $c \in \mathbb{R}$ the collection of all one-sided-infinite paths γ in G with $\lim_n f(\gamma(n)) \neq c$ is not null.*

Proof (of 3.1). Assume that G is transient, and m is a resolving metric on G. Let Γ be the collection of all infinite paths $\gamma = (\gamma(0), \gamma(1), \ldots)$ in G. Almost all paths γ in Γ have a limit $\lim_n \gamma(n)$ in $\partial_m G$, since the m-length of those that do not is infinite. (The limit is in the metric d_m, of course.)

We now define $\mathrm{supp}(\Gamma)$, the *support* of Γ in $\partial_m G$, as the intersection of all closed sets $Q \subset \partial_m G$ such that for almost every $\gamma \in \Gamma$ the limit $\lim_n \gamma(n)$ is in Q. Because there is a countable basis for the topology of $\partial_m G$, and a countable union of null collections of curves is null, almost every $\gamma \in \Gamma$ satisfies $\lim_n \gamma(n) \in \mathrm{supp}(\Gamma)$.

Since G is transient, we know from 3.2 that the extremal length of Γ is finite. Consequently, $\mathrm{supp}(\Gamma)$ is not empty. Moreover, the assumption that m is resolving shows that $\mathrm{supp}(\Gamma)$ consists of more than a single point. Let x_0 be an arbitrary point in $\mathrm{supp}(\Gamma)$. Define $f : V \to \mathbb{R}$ by setting $f(p) = d_m(x_0, p)$. It is clear that $|df(e)| \leq m(e)$ holds for $e \in E$. Consequently, $f \in D(G)$.

Pick some $\delta > 0$ that is smaller than the d_m-diameter of $\mathrm{supp}(\Gamma)$. Consider the set $A_\delta = \{x \in \mathrm{supp}(\Gamma) : d(x_0, x) < \delta\}$, and let Γ_δ be the set of $\gamma \in \Gamma$ such that $\lim_n \gamma(n) \in A_\delta$. Since $\mathrm{supp}(\Gamma)$ is not contained in $\overline{A}_\delta$ or in $\partial_m G - A_\delta$, from the definition of $\mathrm{supp}(\Gamma)$ it follows that both Γ_δ and $\Gamma - \Gamma_\delta$ are not null. For every $\gamma \in \Gamma_\delta$, we have $\lim_n f(\gamma(n)) < \delta$, while for almost every $\gamma \in \Gamma - \Gamma_\delta$, we have $\lim_n f(\gamma(n)) \geq \delta$. Since both Γ_δ and $\Gamma - \Gamma_\delta$ are non null, it follows that for every constant $c \in \mathbb{R}$ the set of $\gamma \in \Gamma$ with $\lim_n f(\gamma(n)) \neq c$ is not null.

Now 3.3 implies that there are non-constant, harmonic, Dirichlet functions on G. This then shows that there are bounded, non-constant, harmonic, Dirichlet functions, by [31] (see also [28, Theorem 3.73]).

For the proof of (2), the following lemma will be needed.

3.4. Lemma *Let $f : G^\circ \to G$ be a κ-quasimonomorphism between bounded valence graphs $G^\circ = (V^\circ, E^\circ)$ and $G = (V, E)$. Let $m : E \to (0,\infty)$ be a metric on G, and let $m^\circ : E^\circ \to [0,\infty)$ be defined by*

$$m^\circ\big(\{v^\circ, u^\circ\}\big) = d_m\big(f(v^\circ), f(u^\circ)\big),$$

for any $\{v^\circ, u^\circ\} \in E^\circ$ Then $\|m^\circ\| \le C\|m\|$, where C is a constant that depends only on κ and the maximal valence in G and G°.

Proof. For every edge $e^\circ = \{v^\circ, u^\circ\}$ in G°, let γ_{e° be some path of combinatorial length at most 2κ in G from $f(v^\circ)$ to $f(u^\circ)$. Then

$$m^\circ(e^\circ)^2 \le \text{length}_m(\gamma_{e^\circ})^2 = \left(\sum_{e\in\gamma_{e^\circ}} m(e)\right)^2 \le 4\kappa^2 \sum_{e\in\gamma_{e^\circ}} m(e)^2.$$

Therefore,

$$\|m^\circ\|^2 \le 4\kappa^2 \sum_{e\in E} m(e)^2 \left|\{e^\circ \in E^\circ : e \in \gamma_{e^\circ}\}\right|.$$

Since each of the paths γ_{e° has length at most 2κ and $\left|f^{-1}(v)\right| \le \kappa$ for every $v \in V$, it is clear that for any $e \in E$ the cardinality of $\{e^\circ \in E^\circ : e \in \gamma_{e^\circ}\}$ is bounded by a number that depends only on κ and the maximal valence in G°. Hence, the lemma follows. □

Proof of 3.1(2). Let $m^\circ : E \to [0,\infty)$ be as in Lemma 3.4, and let $m_1^\circ : E \to (0,\infty)$ satisfy $m_1^\circ(e) > m^\circ(e)$ for every $e \in E$, while still $m_1^\circ \in L^2(E)$. (The reason for using m_1° rather than m°, is that, strictly speaking, m° may not be a metric; it may happen that $m^\circ(e) = 0$ for some $e \in E$.) Note that f is a contraction from the metric space $(V^\circ, d_{m_1^\circ})$ to the metric space (V, d_m). The straightforward proof that m_1° is a resolving metric for G° is again based on 3.4, and will be omitted. □

Instability of the weak-Liouville property

As promised, we shall now provide a simple example showing that the weak-Liouville property is not invariant under rough isometries.

Let $G = (V, E)$ be a countable transient graph, let $A \subset V$, let m be the natural metric on G, and let m' be a metric bilipschitz to m. Suppose that with probability 1 the random walk on (G, m) hits A infinitely often, but with probability 1 the random walk on (G, m') hits A only finitely many times. T. Lyons [19] observed that one can find such A and m' when G is an infinite regular tree (which is not $\mathbb{Z}$). For example, on the binary tree consisting of all finite sequences $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of 0's and 1's, where an edge appears between each $(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1})$ and $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, one can let A be the set of all $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$

such that $\sum_{j=1}^{n} \epsilon_j > n/3$, and let $m'\big((\epsilon_1, \ldots, \epsilon_{n-1}), (\epsilon_1, \ldots, \epsilon_n)\big) = 1 + c\epsilon_n$, where $c > 0$ is a sufficiently large constant.

We now construct a new graph H whose vertex set is the disjoint union of V and $\mathbb{Z}^4$. Let X be the set $\{(x, 0, 0, 0) : x \in \mathbb{Z}\} \subset \mathbb{Z}^4$, and let $\phi : A \to \mathbb{Z}^4$ be any injective map from A into X. Let the edges of H consist of the edges in G, the edges in $\mathbb{Z}^4$, and all edges of the form $[a, \phi(a)]$ with $a \in A$. Extend the metrics m, m' to H by letting $m(e) = m'(e) = 1$ for any edge of H that is not in E.

3.5. Theorem (H, m) *is weak-Liouville, but* (H, m') *carries non-constant, bounded harmonic functions.*

Sketch of proof. With probability 1, the random walk on (H, m) will be in $\mathbb{Z}^4$ infinitely often. But there is some constant $c > 0$, such that the probability that a random walk on $\mathbb{Z}^4$ that starts at any vertex in X will never reach X again is c. Hence the random walk on (H, m) will be absorbed in $\mathbb{Z}^4$; that is, it will be in V only finitely many times. Since we may couple [17] the random walk on (H, m) with the random walk on $\mathbb{Z}^4$, it follows that (H, m) is weak-Liouville.

On the other hand, let $f(v)$ be the probability that the random walk on (H, m') that starts at a vertex v will be absorbed in $\mathbb{Z}^4$. Then f is a non-constant, bounded harmonic function on (H, m').

4. Harmonic functions on planar and almost planar graphs

4.1. Theorem *Any planar bounded degree graph* $G = (V, E)$ *has a resolving metric.*

The proof will use the circle packing theorem, which we state shortly. Suppose that $P = (P_v : v \in V)$ is an indexed packing of disks in the plane. This just means that V is some set, to each v corresponds a closed disk $P_v \subset \mathbb{R}^2$, and the interiors of the disks are disjoint. Let G be the graph with vertices V such that there is an edge joining v and u iff P_v and P_u are tangent. Then G is the *contacts graph* of P. (There are no multiple edges in G). It is easy to see that G is planar, the circle packing theorem provides a converse:

4.2. Circle Packing Theorem *Let* $G = (V, E)$ *be a finite planar graph with no loops or multiple edges, then there is a disk packing* $P = (P_v : v \in V)$ *in* $\mathbb{R}^2$ *with contacts graph* G*.*

This theorem was first proved by Koebe [16]. Recently, at least 7 other proofs have been found; some of the more accesible ones can be found in [20], [6], [4].

We shall also need the following lemma.

4.3. Lemma *Let G be a connected, planar, bounded valence graph, with no multiple edges. Then there is a triangulation T of a domain in $\mathbb{R}^2$, which has bounded valence, and such that G is isomorphic to a subgraph of the 1-skeleton of T.*

The lemma is surely known (but we have not located a reference), and not hard to prove, so we leave it as an exercise to the reader. (Hint: for the finite case, surround the graph by cycles, and triangulate the annular regions formed. Continue the process in the untriangulated disks bounded by the cycles.)

Proof of (4.1). We assume that G is the 1-skeleton of a triangulation T. Because of 4.3 and 3.1.(2), there is no loss of generality. The claim is trivial if G is finite, so assume that it is not. Let v_0, v_1, v_2 be the three vertices of a triangle in T. Let $V^1 \subset V^2 \subset V^3 \subset \ldots$ be subsets of V such that $v_0, v_1, v_2 \in V^1$ and $\cup_n V^n = V$. For each n let G^n be the graph spanned by V^n, and let E^n be its set of edges.

Let B_1, B_2, B_3 be any three mutually tangent disks in $\mathbb{R}^2$, and let D be the bounded component of $\mathbb{R}^2 - B_1 \cup B_2 \cup B_3$. The circle packing theorem tells us that for each n there is a packing of disks $P^n = (P^n_v : v \in V^n)$ in the plane $\mathbb{R}^2$ with contacts graph G^n. By normalizing with a Möbius transformation, we assume with no loss of generality that $P^n_{v_j} = B_j$ for each $j = 1, 2, 3$ and $n = 1, 2, \ldots$, and that all the other disks in the packings P^n are contained in $\overline{D}$.[1] Take some subsequence of the packings P^n, so that each of the the disks P^n_v, $v \in V$ has a (Hausdorff) limit, and call the limit P_v. Set $P = (P_v : v \in V)$. Clearly, each P_v is either a disk or a point, each of the sets in P is disjoint from the interior of the others, and P_v intersects P_u when v and u neighbor. We want to show that each P_v is really a disk, not a single point. (Compare [12], [26].)

Let V' be the set of all $v \in V$ such that P_v is a single point. Clearly, $v_0, v_1, v_2 \notin V'$. Suppose that V' is not empty, and let V'' be a connected component of V'. Then all the sets P_v, $v \in V''$, are the same point, say p. A triangulation of any surface is 3-vertex connected (this is an easy and well known fact), so the removal of any 2 vertices from G does not disconnect G. Since $v_0, v_1, v_2 \notin V''$, it follows that there are at least three vertices outside of V' that neighbor with some vertex in V''; suppose these are a, b, c. Then P_a, P_b, P_c are three disks, whose interiors are disjoint, and all must contain the point p. This is clearly impossible, and this contradiction tells us that $V' = \emptyset$. So the P_v are true disks.

Take any $e \in E$, and let its vertices be u, v. We set $m(e) = \text{diameter}(P_u) + \text{diameter}(P_v)$. This defines a metric $m : E \to (0, \infty)$, Because the packing P is contained in $B_1 \cup B_2 \cup B_3 \cup D$, its total area is finite, and this implies that $m \in L^2(E)$.

[1] Here, a Möbius transformation is a composition of inversions in circles; that is, orientation reversing transformations are included. The fact that is used here is that for any three mutually tanget disks B'_1, B'_2, B'_3 and any component D' of the two components of $\mathbb{R}^2 - B'_1 \cup B'_2 \cup B'_3$, there is an (actually unique) Möbius transformation taking each B'_j to B_j and taking D' to D. If $p_{i,j}$ denotes the intersection point of B_i and B_j, $i \neq j$, and similarly for $p'_{i,j}$, then the transformation is the one that takes each $p'_{i,j}$ to $p_{i,j}$, pre-composed, if necessary, by the inversion in the circle passing through the three points $p'_{i,j}$.

We shall now show that m is resolving. For any $v \in V$ we let $z(v)$ denote the center of the disk P_v. Let p be any point in $\partial_m G$. Let $v_1, v_2, \ldots$ be a sequence in V that converges to p in $C_m(G)$. Then $\lim_{n,k\to\infty} d_m(v_n, v_k) = 0$. This easily implies that $\lim_{n,k\to\infty} |z(v_n) - z(v_k)| = 0$. We therefore conclude that the limit $\lim_n z(v_n)$ exists, and denote this limit by $z(p)$. If $w_1, w_2, \ldots$ is another sequence in V that converges to p, then the limit $\lim_n z(w_n)$ will still be $z(p)$. This follows from the fact that any ordering of the union $\{v_n\} \cup \{w_n\}$ as a sequence which will still converge to p. Hence $z(p)$ does not depend on the sequence chosen.

Let Γ_p be the collection of all half-infinite paths in G $\gamma = (\gamma(0), \gamma(1), \ldots)$ that converge to p in $C_m(G)$. We need to show that Γ_p is null. This will be done by producing an $L^2(E)$ metric m_p such that $\text{length}_{m_p}(\gamma) = \infty$ for every $\gamma \in \Gamma_p$. The argument will be similar to an argument in [25] and [12].

Our next objective is to show that $z(p)$ does not belong to any of the disks $P_v, v \in V$. Consider a triangle in T with vertices u_1, u_2, u_3. Let $a(u_1, u_2, u_3)$ denote the boundary of the triangle whose vertices are $z(u_1), z(u_2), z(u_3)$. Because T is a triangulation of a surface, it follows that the removal of $\{u_1, u_2, u_3\}$ from G does not disconnect G. This implies that all the sets in $\big(P_v : v \in V - \{u_1, u_2, u_3\}\big)$ are in the same connected component of $\mathbb{R}^2 - a(u_1, u_2, u_3)$. Let $\tilde{a}(u_1, u_2, u_3)$ denote the union of $a(u_1, u_2, u_3)$ with the connected component of $\mathbb{R}^2 - a(u_1, u_2, u_3)$ that is disjoint from every $P_v, v \neq u_1, u_2, u_3$. Suppose now that v is any vertex in V. It is easy to see that the union of the sets $\tilde{a}(v, u, w)$ for all consecutive neighbors u, w of v contains the disk P_v in its interior. Each one of these sets intersects only three sets in the packing P. Hence P_v does not contain any accumulation point of centers of disks in P. This shows $z(p) \notin P_v$, as required.

We now inductively construct a sequence of positive numbers $r_1 > r_2 > r_3 > \ldots$. For $r > 0$ let $B(r)$ denote the disk $\{z \in \mathbb{R}^2 : |z - z(p)| < r\}$. Take $r_1 = 1$. Suppose that $r_1, r_2, \ldots, r_{n-1}$ have been chosen. Let r_n be in the range $0 < r_n < r_{n-1}/2$ and be sufficiently small so that the two sets of vertices $\{v \in V : z(v) \in B(2r_n)\}$ and $\{v \in V : z(v) \notin B(r_{n-1})\}$ are disjoint and there is no edge in G connecting them. To see that this can be done, observe that for any $r > 0$ there are finitely many vertices $v \in V$ such that $\text{diameter}(P_v) \geq r$. Since $z(p) \notin \cup_v P_v$, for every $r > 0$ there is a $\rho(r) \in (0, r/2]$ such that the closure of $B(\rho(r))$ does not intersect any of the sets P_v satisfying $\text{diameter}(P_v) \geq r/2$. This implies that there will be no P_v that intersects both circles $\partial B(r)$ and $\partial B(\rho(r))$. Hence we may take $r_n = \rho(\rho(r_{n-1}))/2$.

For $r \in (0, \infty)$ let $\psi_r : \mathbb{R}^2 \to [0, r]$ be defined by

$$\psi_r(z) = \begin{cases} r & \text{if } |z - z(p)| \leq r, \\ 2r - |z - z(p)| & \text{if } r \leq |z - z(p)| \leq 2r, \\ 0 & \text{if } |z - z(p)| \geq 2r. \end{cases}$$

In other words, ψ_r is equal to r on $B(r)$, equal to 0 outside $B(2r)$, and in the annulus $B(2r) - B(r)$ it is linear in the distance from its center $z(p)$. For each $n = 1, 2, \ldots$ and $v \in V$, we define

$$\phi_n(v) = \psi_{r_n}(z(v)).$$

The construction of the sequence $r_1, r_2, \ldots$ insures that the supports of $d\phi_n$ and $d\phi_{n'}$ are disjoint when $n \neq n'$. It is easy to see that the definition of ϕ_n shows that there is a finite constant C such that

$$|d\phi_n(e)|^2 \leq C \operatorname{area}((P_u \cup P_v) \cap B(3r_n)), \tag{4.1}$$

where $\{u, v\} = \partial e$. Let Ω be an upper bound on the degrees of the vertices in G. Since the interiors of the sets in P are disjoint, (4.1) implies

$$\|d\phi_n\|^2 = \sum_{e \in E} \phi_n(e)^2 \leq 9\pi C \Omega r_n^2.$$

Now set

$$\phi = \sum_{n=1}^{\infty} \frac{\phi_n}{n r_n}.$$

As we have noted, the supports of the different $d\phi_n$ are disjoint, and therefore,

$$\|d\phi\|^2 = \sum_{n=1}^{\infty} \frac{\|d\phi_n\|^2}{n^2 r_n^2},$$

and the above estimate for $\|d\phi_n\|^2$ shows that $|d\phi| \in L^2(E)$. Therefore, there is some metric $m_p \in L^2(E)$ such that $m_p(e) \geq |d\phi_n(e)|$ for every $e \in E$. (Technically, we cannot take $m_p = |d\phi|$, since $|d\phi|$ is not positive, and hence not a metric.)

Now let $\gamma = (\gamma(1), \gamma(2), \ldots)$ be any path in Γ_p, and let $E(\gamma)$ denote its edges. We have $\lim_n z(\gamma(n)) = z(p)$. Therefore,

$$\lim_n \phi(\gamma(n)) = \lim_{z \to z(p)} \sum_j \frac{\psi_{r_j}(z)}{j r_j} = \sum_j \frac{1}{j} = \infty.$$

This gives

$$\operatorname{length}_{m_p}(\gamma) = \sum_{e \in E(\gamma)} m_p(e) \geq \sum_{e \in E(\gamma)} |d\phi(e)| = \infty.$$

So Γ_p is null, and m is resolving. □

Proof of 1.9 and 1.1.(1). These follow immediately from 4.1 and 3.1. □

5. The Dirichlet problem for circle packing graphs

Let $G = (V, E)$ be the 1-skeleton of a triangulation of an open disk, and suppose that G has bounded valence. Suppose that $P = (P_v : v \in V)$ is any disk packing in the Riemann sphere $\hat{\mathbb{C}}$ whose contacts graph is G. A point $z \in \hat{\mathbb{C}}$ is a *limit point* of P if every open neighborhood of z intersects infinitely many disks of P, and the set of all limit points of P is denoted $\Lambda(P)$. The *carrier* of P, $\operatorname{carr}(P)$, is the connected component of $\hat{\mathbb{C}} - \Lambda(P)$ that contains P. It is easy to see that $\operatorname{carr}(P)$ is homeomorphic to a disk. From [12] we know that $\partial \operatorname{carr}(P)$ is a single point iff G is recurrent. Moreover, if G is transient and D is any simply connected proper subset of $\mathbb{C}$, then there is a disk packing P with contacts graph G that has carrier D.

For each $v \in V$ we let $z(v)$ denote any point in P_v. The following theorem includes 1.1.(2).

5.1. Theorem *Assume that G is transient, and has bounded valence. Let v_0 be an arbitrary vertex in G, and let $(v(0), v(1), v(2), \ldots)$ be a simple random walk starting at $v_0 = v(0)$. Then, with probability 1, the limit $z_\infty = \lim_n z(v(n))$ exists, and is a point in $\Lambda(P)$.*

Moreover, suppose that $z \in W \cap \Lambda(P)$, where W is open. Let $u_1, u_2, \ldots$ be a sequence in V such that $\lim_j z(u_j) = z$. Let p_j be the probability that for the simple walk $(v(0), v(1), v(2), \ldots)$ starting at $v(0) = u_j$ the limit $z_\infty = \lim_n z(v(n))$ is in W. Then $p_j \to 1$ as $j \to \infty$.

Note that the topological notions in the theorem are induced by the topology of $\hat{\mathbb{C}}$. In particular, $\Lambda(P)$ is compact even when it includes ∞.

The main corrolary is 1.1.(3), which we now restate, as follows.

5.2. Corollary *The Dirichlet problem on $V \cup \Lambda(P)$ has a solution for every continuous function $f : \Lambda(P) \to \mathbb{R}$. That is, there exists a harmonic $\tilde{f} : V \to \mathbb{R}$ such that $\lim_n \tilde{f}(v_n) = f(z)$ whenever v_n is a sequence in G such that $\lim_n z(v_n) = z$.*

Proof of (4.1). Let $v \in V$. Let μ_v be the probability measure that assigns to every measurable $H \subset \Lambda(P)$ the probability that the simple random walk $(v(0), v(1), v(2), \ldots)$ starting at $v(0) = v$ will satisfy $\lim_n z(v(n)) \in H$. Then define $\tilde{f}(v) = \int f \, d\mu_v$. It is clear that this gives the required solution. □

The theorem will easily follow from the following lemma.

5.3. Convergence Lemma *Suppose that $\infty \notin \operatorname{carr}(P)$. Let v_0 be some vertex in V, and let δ be the distance from $z(v_0)$ to $\partial \operatorname{carr}(P)$. Let $t > 1$. Then the probability that the simple random walk on G starting at v_0 will ever get to a vertex $v \in V$ satisfying $|z(v) - z(v_0)| > t\delta$ is less than $C / \log t$, where C is a constant that depends only on the maximal degree Ω in G.*

Proof. There is nothing to prove if $\max\{|z(v)| : v \in V\} \le t\delta$, so assume that this is not the case. By applying a similarity, if necessary, assume that $\delta = 1$, $z(v_0) = 0$ and the point 1 is in $\partial \operatorname{carr}(P)$. The Ring Lemma of [23] tells us that the ratio between the radii of any two touching disks in P is bounded by a constant $C_1 = C_1(\Omega)$. It follows that there is a constant $C_2 = C_2(\Omega)$ such that the radius of any disk in P is less than C_2 times its distance from 1. From this we conclude that the following inequality is valid for every $v \in V$:

$$(5.1) \qquad \max\{|z| : z \in P_v\} \le (4C_2 + 2) \max\left(1, \min\{|z| : z \in P_v\}\right).$$

Let $T = \mathbb{R}/2\pi\mathbb{Z}$, the circle of length 2π, and consider

$$\log : \mathbb{C} - \{0\} \to \mathbb{R} \times T,$$

it maps the punctured plane onto a cylinder. For $x_1 < x_2$, let $\mathrm{proj}_{[x_1,x_2]}$ be the projection $\mathbb{R} \to [x_1, x_2]$; that is,

$$\mathrm{proj}_{[x_1,x_2]}(s) = \max\big(x_1, \min(x_2, s)\big).$$

Also let $\mathrm{Proj}_{[x_1,x_2]} : \mathbb{R} \times T \to [x_1, x_2] \times T$ be defined by

$$\mathrm{Proj}_{[x_1,x_2]}(s, \theta) = \big(\mathrm{proj}_{[x_1,x_2]}(s), \theta\big).$$

Set $\tau = \log t$. Finally, for $v \in V$ let

$$\phi(v) = \mathrm{proj}_{[\tau/2,\tau]} \log |z(v)| = \mathrm{Proj}_{[\tau/2,\tau]} \log z(v) .$$

We now estimate $\mathcal{D}(\phi)$. Consider first some P_v such that $\sqrt{t} \le |w| \le t$ for every $w \in P_v$. Since the derivative of log is $1/z$, (5.1) implies that there is a constant $C_3 = C_3(\Omega)$ such that $|\log' w_1|/|\log' w_2| < C_3$ for every $w_1, w_2 \in P_v$. It follows that there is a $C_4 = C_4(\Omega)$ such that

$$C_4 \,\mathrm{area}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log P_v\right) \ge \mathrm{diameter}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log P_v\right)^2,$$

which implies

$$\begin{aligned} C_4 \,\mathrm{area}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log P_v\right) \quad \ge \quad & \Big(\max\{\mathrm{proj}_{[\tau/2,\tau]} \log |w| : w \in P_v\} \\ & - \min\{\mathrm{proj}_{[\tau/2,\tau]} \log |w| : w \in P_v\}\Big)^2. \end{aligned} \tag{5.2}$$

The latter is true also for an arbitrary P_v, since in every disk B that intersects the interior of the annulus $A = \{w : \sqrt{t} \le |w| \le t\}$, there is a disk $B_1 \subset A \cap B$ with $\{|w| : w \in B_1\} = \{|w| : w \in B\} \cap [\sqrt{t}, t]$.

If v_1, v_2 are neighbors in G, then the two disks P_{v_1}, P_{v_2} intersect. Hence, from (5.2) it follows that

$$|\phi(v_1) - \phi(v_2)|^2 \le 4C_4 \max\left\{ \mathrm{area}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log P_{v_1}\right), \mathrm{area}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log P_{v_2}\right) \right\}.$$

We sum this inequality over all edges $\{v_1, v_2\}$ in G. Since the interiors of the sets $\mathrm{Proj}_{[\tau/2,\tau]} \log P_v$ are disjoint, we get,

$$\mathcal{D}(\phi) \le 4\Omega C_4 \,\mathrm{area}\left(\mathrm{Proj}_{[\tau/2,\tau]} \log(\mathbb{C} - \{0\})\right) = 4\pi \Omega C_4 \tau. \tag{5.3}$$

Now let $K \subset V$ be a finite collection of vertices. Let $\phi_K : V \to R$ be the function that is equal to ϕ outside of K and is harmonic in K. Clearly,

$$\mathcal{D}(\phi_K) \le \mathcal{D}(\phi). \tag{5.4}$$

We want to estimate $\phi_K(v_0)$ from above. Let ρ be in the range $1 < \rho < \sqrt{t}$. For all $v \in V - K$ such that $|z(v)| < \sqrt{t}$, we have $\phi_K(v) = \tau/2$. Therefore, it follows from the maximum principle that there is some $v_\rho \in V$ with P_{v_ρ} intersecting the circle $\{|w| = \rho\}$ and $\phi_K(v_\rho) \ge \phi_K(v_0)$. Since the circle $\{|w| = \rho\}$

separates 1 and ∞, it intersects infinitely many of the disks in P. Consequently, there is a finite path γ_ρ in G connecting v_ρ to some vertex outside of K, such that P_v intersects the circle $\{|w| = \rho\}$ for every $v \in \gamma_\rho$. (Here we are using the fact that G is a triangulation to conclude that if there is an arc of the circle $\{|w| = \rho\}$ whose endpoints are in P_{v_1} and P_{v_2} and the arc does not intersect any other disk in P, then v_1 and v_2 neighbor in G.)

Let u_ρ be a vertex of γ_ρ that is outside of K. Then $\phi_K(u_\rho) = \tau/2$. So we get,

$$\phi_K(v_0) - \tau/2 \le \phi_K(v_\rho) - \phi_K(u_\rho) \le \sum_{e \in E(\gamma_\rho)} |d\phi_K(e)|,$$

where $E(\gamma_\rho)$ denotes the set of edges in γ_ρ. This implies,

$$\phi_K(v_0) - \tau/2 \le \sum_{e \in E_\rho} |d\phi_K(e)|,$$

where E_ρ denotes the set of all edges $\{v_1, v_2\} \in E$ such that P_{v_1} and P_{v_2} both intersect the circle $\{|w| = \rho\}$. We multiply the above inequality by $1/\rho$, and integrate from 1 to $\sqrt{t}$. This gives,

$$(\phi_K(v_0) - \tau/2)\tau/2 \le \sum_{e \in E} |d\phi_K(e)| m(e), \tag{5.5}$$

where

$$m(e) = \max\{\mathrm{proj}_{[0,\tau/2]} \log|w| : w \in P_v\} - \min\{\mathrm{proj}_{[0,\tau/2]} \log|w| : w \in P_v\}$$

for v any of the two vertices in e. Following the argument in the previous paragraphs, it is easily seen that

$$\sum_{e \in E} m(e)^2 \le \pi C_4 \Omega \tau$$

Using this and the Cauchy-Schwarz inequality in (5.5), we get

$$(\phi_K(v_0) - \tau/2)\tau/2 \le \sqrt{\pi C_4 \Omega \tau \mathcal{D}(\phi_K)}.$$

Together with (5.3) and (5.4), this gives

$$\phi_K(v_0) \le 4\pi C_4 \Omega + \tau/2. \tag{5.6}$$

This is the estimated we wanted for $\phi_K(v_0)$.

Now let $K_1 \subset K_2 \subset K_3 \subset \ldots$ be a sequence of finite subsets of V, such that $\cup_n K_n$ is the set of all vertices $v \in V$ with $z(v) \le t$. Let p be the probability that the simple random walk that starts at v_0 will ever reach a vertex v with $|z(v)| > t$. It is easy to see that

$$\limsup_n \phi_{K_n}(v_0) \ge (1-p)\tau/2 + p\tau.$$

Using (5.6), this gives

$$p \le 8\pi C_4 \Omega/\tau,$$

as required. $\square$

Proof (of 5.1). The statement of the theorem is invariant under Möbius transformations, hence we may normalize so that $\infty \notin \text{carr}(P)$. To prove that the limit $\lim_n z(v(n))$ exists almost surely, it is enough to show that the diameter of $\{z(v(k)), z(v(k+1)), \ldots\}$ tends to zero with probability 1 as $k \to \infty$. Let $\epsilon > 0$. The lemma implies that there is a $t_1 < \infty$ so that the probability that $|z(v(k))| > t_1$ for any k is less than ϵ. There are finitely many $v \in V$ with $|z(v)| \leq t_1$ and with $d(z(v), \Lambda(P)) > \epsilon$. Since G is transient, this implies that there is some n_0 so that $d(z(v(n_0)), \Lambda(P)) \leq \epsilon$ with probability at least $1 - 2\epsilon$. Then the lemma shows that with probability at most $2\epsilon + C/\log t$ the diameter of $\{z(v(n_0)), z(v(n_0+1)), \ldots\}$ is greater than $t\epsilon$. Choosing $t = \epsilon^{-1/2}$, then shows that the limit $\lim_n z(v(n))$ exists almost surely.

The second part of the theorem follows immediately. □

Remarks. 1. The methods of this section are really not particular to circles, they would apply to a large class of other, well behaved, packings.
2. It turns out that for some purposes it is better to consider a square tiling associated with the graph, rather than a circle packing. In particular, one can get similar results for an especially constructed square tiling when G is not assumed to be a triangulation. We intend to study this in a forthcoming paper [3].
3. The Convergence Lemma gives information about the hitting measure. Let the situation be as in the lemma. Suppose that p is some point on $\partial\,\text{carr}(P)$, and $r > 0$. Let $d = |p - z(v_0)|$ be the distance from $z(v_0)$ to p. The probability that the random walk starting at v_0 will ever reach a vertex v satisfying $|z(v) - p| < r$ is less than $C/\log(d/r)$. This follows from the lemma by inverting the packing in the circle $\{|z - p| = r\}$.

5.4. Problems *Consider the random walk* $(v(1), v(2), \ldots)$ *starting at some vertex* $v_0 \in V$. *Then with probability* 1 *the limit* $z_\infty = \lim_n z(v(n))$ *exists and is a point in* $\partial\,\text{carr}(P)$. *Let* μ *denote the hitting measure; that is,* $\mu(A)$ *is the probability that* $z_\infty \in A$. *Suppose for example that* $\text{carr}(P) = U$, *the unit disk. When is* μ *absolutely continuous with respect to Lebesgue measure on* ∂U? *(By further triangulating some of the faces of the triangulation, one can insure that the hitting measure is singular.) In general, when* $\text{carr}(P) \neq U$, *is it true that* μ *is supported by a set of Hausdorff dimension* 1, *as for the harmonic measure for Brownian motion.*

6. Harmonic functions on almost planar manifolds

In this section we shall prove Theorem 1.10. Namely, we shall show that a transient, bounded local geometry, almost planar manifold admits non constant Dirichlet functions. This will be an easy corollary of Theorem 1.9 and the following very recent theorem.

6.1. Theorem (Holopainen-Soardi) [13] *Let* X_1 *be a connected, bounded local geometry, Riemannian manifold, and suppose that* X_1 *is roughly isometric to a* *bounded valence graph X_2. Then there are non-constant harmonic functions with finite Dirichlet energy on X_1 iff there are non-constant harmonic functions with finite Dirichlet energy on X_2.*

Proof (of 1.10). By Kanai's [14], [15], M is roughly isometric to some transient bounded valence graph G (e.g., a net in M), and 1.7 implies that G is almost planar. Theorem 1.9 tells us that there are non-constant harmonic functions on G with finite Dirichlet energy, and Theorem 6.1 implies that the same is true for M. It then follows that there are also such functions that are bounded. □

Remark. By applying Theorem 6.1, one can get yet another proof for the existence of non constant Dirichlet harmonic functions on planar bounded valence transient graphs (1.1.(1)). Following is a sketch of the argument. To each vertex v in G associate a copy of the regular Ω-gon with side length 1, where Ω is the maximal degree of any vertex in G. To each edge of G associate a copy of the unit square. Glue two opposite sides of every such square to sides of the Ω-gons corresponding to the vertices of the edge. If the gluings are preformed correctly, the resulting Riemman surface S will be topologically planar. Let S' be the subset of all points $p \in S$ so that the open ball of radius $1/5$ about p in S is isometric to a Euclidean ball. Then S' is a Riemann surface with boundary. By Koebe's uniformization theorem, there is a conformal embedding $f : S \to \hat{C}$ of S into the Riemann sphere. Since G is transient, S' is also transient, and hence $f(\overline{S'}) - f(S')$ contains more than a single point. It then follows that there are non constant harmonic Dirichlet functions on $f(S')$ with Neumann boundary conditions on $f(\partial S')$. The same then applies to S', and Theorem 6.1 shows that the same is true for G.

7. Characterizations of almost planarity

Proof (of 1.8). Suppose that $G = (V, E)$ is almost planar. Then there is a bounded degree planar graph $G^\circ = (V^\circ, E^\circ)$, and a κ-quasimonomorphism $f : V \to V^\circ$ with some $\kappa < \infty$. Using Lemma 4.3, we assume without loss of generality that G° is the 1-skeleton of a triangulation T° of a domain D in the plane. Recall that $|G|$ is the union of intervals $I_e, e \in E$, with the natural identifications, and a drawing of G in the plane is a continuous map $g : |G| \to \mathbb{R}^2$. For every edge $e \in E$, with enpoints v, w, say, there is a path, say $\tilde{f}(e)$, of length at most 2κ, joining $f(v)$ and $f(w)$ in G°. Because G° has bounded valence and f is a quasimonomorphism, it follows that there is a finite C such that every edge $e^\circ \in E^\circ$ appears in the path $\tilde{f}(e)$ for at most C edges e in E.

In the domain D one can replace each edge e° in the triangulation T° by C 'parallel' edges joining the same endpoints, such that there are no intersection between edges, except at the endpoints. Let $G^* = (V^*, E^*)$ denote the resulting graph. Since in G^* there are C edges for every edge in G°, it is possible to find for every edge $e \in E$, with endpoints v, w, say, a path $f^*(e)$ in G^* of length at most 2κ, such that all these paths are edge-disjoint. Define now the drawing

$g : |G| \to D$, by letting g map each I_e into the path $f^*(e)$. It is easy to verify that the number of crossings for each edge in the drawing is bounded by C times $2\kappa + 1$ times the maximal degree of G°. Hence there is a drawing of G in the plane in which the number of crossings in any edge is bounded.

Now suppose that there is a drawing $g : |G| \to \mathbb{R}^2$ of G in the plane so that every edge $e \in E$ has at most $\kappa < \infty$ crossings. Let X be the set of all crossing points in $|G|$; that is, the set of all $x \in |G|$ such that there is some $y \in |G| - \{x\}$ with $g(x) = g(y)$. Define $V^\circ = g(V) \cup g(X)$. It is clear that $G^\circ = g\left(|G|\right)$ is a connected, bounded valence graph embedded in the plane, with vertex set V°. Moreover, g induces an obvious quasimonomorphism from G to G°. Hence G is almost planar. □

Proof (of 1.6). Note that a bounded curvature complete Riemannian surface satisfies the $C(R, r)$ property. Hence, using 1.7, we see that it is enough to show that any planar, connected, bounded valence graph admits a quasimonomorphism into a topologically planar, complete, Riemannian surface with bounded curvature, and conversely, every topologically planar, complete, Riemannian surface with bounded curvature admits a quasimonomorphism into an almost planar, connected, bounded valence graph.

Let $G = (V, E)$ be a planar, connected, bounded valence graph. From 4.3 we know that G is isomorphic to a subgraph of the 1-skeleton of a bounded valence triangulation T of a domain D in $\mathbb{R}^2$. Take a metric m on D in which every triangle of T is isometric to a euclidean triangle with edges of length 1. This metric is complete, and locally is a flat Riemannian metric, except for singularities near the vertices. Since the 1-skeleton of T has bounded valence, it is clear that one can modify m in the $(1/3)$-neighborhood of the vertices, to get a complete Riemannian metric m' with bounded curvature. Take S to be D with the metric m'. The isomorphism from G to a subgraph of the 1-skeleton of T induces a quasimonomorphism from G to S, as required.

Now suppose that S is a topologically planar, Riemannian surface, with bounded curvature. Let V be a maximal set of points in S with the property that the distance between any two points in V is at least 1. Let $G = (V, E)$ be the graph on V where two vertices are connected if the distance between them is at most 3. Let V be equipped with the distance metric in G. From [14] we know that V is roughly isometric to S, and the fact that the curvature in S is bounded implies that G has bounded valence. To see that G is almost planar, draw G in S by connecting the vertices v, u of an edge by any of the shortest paths between them. It is clear that there will be at most one crossing between any two edges, unless the edges overlap in nontrivial intervals. The latter possibility is avoided by a generic choice of the points V, hence we assume that this does not happen. If there is a crossing between two edges e_1, e_2, then the distance in S from the vertices of e_1 to the vertices of e_2 is less than 6. Hence we have an upper bound on the number of crossings of any edge in this drawing. Hence Theorem 1.8 implies that G is almost planar. □

7.1. Theorem *Let $G = (V, E)$ be a finite genus, connected, locally finite graph. Then G can be drawn in the plane with finitely many crossings. In particular, if G has bounded valence, then it is almost planar.*

Proof. Let S be a closed (compact, without boundary) surface such that G embedds in S. We assume, with no loss of generality, that G is a triangulation of a domain $D \subset S$. Recall that the genus of any boundaryless surface X is the maximum cardinality of a collection $\{\gamma_1, \dots, \gamma_g\}$ of disjoint simple closed curves in X such that $X - \cup_j \gamma_j$ is connected. Since the genus of D is bounded by the genus of S, it is finite. Hence there is a finite collection of edges $E_0 \subset E$ such that any simple closed path in $G - E_0$ separates D. Since G is a triangulation of D, this implies that there is no simple closed non-separating path in $D - E_1$, where E_1 is the set of all edges in G that share some vertex with some edge in E_0. Therefore every component of $D - E_1$ has zero genus, and is planar. In particular, $G - E_1$ is planar. Let $E' \subset E_1$ be a minimal set of edges subject to the property that $G - E'$ is planar. (Here, $G - E'$ contains all the vertices of G.) Clearly, $G - E'$ is connected, and E' is finite.

Let f be an embedding of $G - E'$ in the plane so that for every edge or vertex j of $G - E'$ there is an open set A containing $f(j)$ that intersects at most finitely many of the vertices and edges of $f(G - E')$. Now suppose that e is some edge in E', and let v, w be its vertices. Since $G - E'$ is connected, there is a path γ in $f(G - E')$ joining $f(v)$ and $f(w)$. We may define $f(e)$ to be a simple path that follows alongside γ and intersects $f(G - E')$ in finitely many points. Similarly, all the edges in E' may be drawn in such a way that there are finitely many crossings. □

Almost planar graphs and separation properties

It is known [18] that any planar graph $G = (V, E)$ has the $\sqrt{n}$ *separation property*. That is, there are $c_0 < \infty$, $c_1 < 1$, so that for any finite set $W \subset V$ of n vertices, there is a subset $W' \subset W$ containing at most $c_0\sqrt{n}$ vertices, such that any connected subset of $W - W'$ has at most $c_1 n$ vertices. Clearly almost planar graphs have this $\sqrt{n}$ separation property too. A natural question is whether $\sqrt{n}$ separation is equivalent to almost planarity. And if not, do transient graphs with the $\sqrt{n}$ separtion property always admit non constant bounded harmonic functions?

Examples. Let T be a tree, and denote by $T \times \mathbb{Z}$ the product of T by $\mathbb{Z}$. In Lemma 7.2 below, we show that $T \times \mathbb{Z}$ has the $\sqrt{n}$ separation property. Coupling shows that any bounded harmonic function h on $T \times \mathbb{Z}$ must be constant on every fibre of the form $\{t\} \times \mathbb{Z}$. Hence the only harmonic Dirichlet functions on $T \times \mathbb{Z}$ are the constants. By Theorem 1.9 it then follows that $T \times \mathbb{Z}$ is not almost planar, if it is transient. In particular, when T is the binary tree, $T \times \mathbb{Z}$ is a graph having the $\sqrt{n}$ separation property, which is not almost planar. Hence, $\sqrt{n}$ separation does not imply almost planarity.

Let C be the following tree. Start with an infinite ray, then to every vertex on that ray add another infinite ray, rooted at that vertex. One can show [21] that $C \times \mathbb{Z}$ is a transient graph. Yet $C \times \mathbb{Z}$ admits no non constant bounded harmonic functions. Hence, $\sqrt{n}$ separation and transience do not together imply the existence of bounded non constant harmonic functions.

To date, we did not manage to prove that $T \times \mathbb{Z}$ is not almost planar whenever T is a tree with infinitely many ends.

7.2. Lemma *Let T be a tree, then $T \times \mathbb{Z}$ has the $\sqrt{n}$ separation property.*

Proof. Let W be a set of n vertices in $T \times \mathbb{Z}$. Let $j_0 \in \mathbb{Z}$ be such that there are at least $n/2$ vertices in W of the form (t, i) with $i \leq j_0$ and at least $n/2$ vertices with $i \geq j_0$. Let j_+ be the least $j \geq j_0$ such that there are less than $\sqrt{n}$ vertices in $W \cap T \times \{j\}$, and let j_- be the largest $j \leq j_0$ such that there are less than $\sqrt{n}$ vertices in $W \cap T \times \{j\}$. Clearly, $0 \leq j_+ - j_- \leq \sqrt{n}$.

It is easy to see that there is a $t_0 \in T$ so that for every component T_1 of $T - \{t_0\}$ the number of vertices in $W \cap (T_1 \times \mathbb{Z})$ is at most $n/2$. (Just choose $t_0 \in T$ to minimize the maximum of $|W \cap (T_1 \times \mathbb{Z})|$ over all components T_1 of $T - \{t_0\}$.) Now set

$$W' = \{(t_0, j) \in W : j_- < j < j_+\} \cup \{(t, j) \in W : j = j_- \text{ or } j = j_+\}.$$

Then W' contains at most $3\sqrt{n}$ vertices, and any component of $W - W'$ contains at most $n/2$ vertices. □

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