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Integral points on subvarieties of semiabelian varieties, I ⋆

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Dedicated to Professor Wolfgang M. Schmidt on the occasion of his sixtieth birthday

Let X be a closed subvariety of an abelian variety A, and assume that both are defined over some number field k. Then a conjecture of Lang [L 1] states that the set of rational points is as small as one might reasonably expect:

Theorem 0.1. *The set $X(k)$ is contained in a finite union $\bigcup B_i(k)$, where each B_i is a translated abelian subvariety of A contained in X.*

In [F 1], Faltings proved this in the special case where $X \times_k \bar{k}$ contains no nontrivial translated abelian subvarieties of $A \times_k \bar{k}$; the conclusion in that case simplifies to the assertion that $X(k)$ is finite. He proved this in general in [F 2]. This proof is also described in detail in [V 4]; we will follow the latter exposition closely here.

In this paper we generalize Theorem 0.1 to cover the corresponding statement for integral points on closed subvarieties of semiabelian varieties:

Theorem 0.2. *Let k be a number field, with ring of integers R. Let S be a finite set of places of k, containing the set of archimedean places, and let R_S be the localization of R away from (non-archimedean) places in S. Let X be a closed subvariety of a semiabelian variety A; assume both are defined over k. Let $\mathscr{X}$ be a model for X over* $\operatorname{Spec} R_S$. *Then the set $\mathscr{X}(R_S)$ of R_S-valued points in $\mathscr{X}$ equals a finite union $\bigcup \mathscr{B}_i(R_S)$, where each $\mathscr{B}_i$ is a subscheme of $\mathscr{X}$ whose generic fiber B_i is a translated semiabelian subvariety of A.*

A future paper will address similar questions for certain *open* subvarieties of A.

Theorem 0.2 partially proves a conjecture of Lang ([L 2], p. 221): Let A be a semiabelian variety, and let Γ be a finitely generated subgroup of A. Let

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$\bar{\Gamma}$ be the division group of Γ; i.e., the group of all $x \in A$ such that $nx \in \Gamma$ for some positive integer n. Then Lang conjectures that the intersection of $\bar{\Gamma}$ with any closed subvariety X of A is contained in the union of finitely many translated semiabelian subvarieties of A contained in X. Theorem 0.2 does not apply to this more general conjecture, but it is equivalent to a similar statement where one does not take the division group. Indeed, the set of integral points on A is a finitely generated group. More recently, M. McQuillan [McQ] has proved Lang's conjecture in full generality, by using methods of M. Hindry to reduce the general statement to the special case proved here.

I doubt that this result can be generalized to a larger class of group varieties: consider, for example, Pell's equation on $\mathbb{A}^2 \cong \mathbb{G}_a^2$.

By a standard result on subvarieties of abelian varieties (Theorem 4.2), Theorem 0.1 gives an affirmative answer, in the case of subvarieties of abelian varieties, to a question posed by Bombieri [N 2]: if the variety X is of general type, then is the set $X(k)$ contained in a proper Zariski-closed subset? Similarly, in the semiabelian case, Theorem 0.2 provides a partial answer to ([V 1], 4.1.2).

Moreover, by the Kawamata structure theorem (Theorem 4.3), the nontrivial B_i occurring in the conclusion of Theorem 0.2 must lie in a proper subvariety which is geometrical in nature. This supports a conjecture of Lang which strengthens the question posed by Bombieri: Lang conjectures in [L 3] that if X is of general type then the higher dimensional components of $X(k)$ must lie in a subvariety which is independent of k.

Section 14 proves a corollary of Theorem 0.2 which generalizes ([V 1], Theorem 2.4.1). The proof essentially reduces to showing that the given variety embeds into a semiabelian variety.

Corollary 0.3. *Let X be a projective variety defined over a number field k, and let ρ denote its Picard number. Let D be an effective divisor on X, also defined over k, which has at least* $\dim X - h^1(X, \mathcal{O}_X) + \rho + 1$ *geometrically irreducible components. Then any set of D-integral points on X is not dense in the Zariski topology.*

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1. Notation

We use the notational conventions of ([V 4], Sect. 5), which for convenience are summarized here.

For places v and absolute values $\|\cdot\|$ on a number field k, we use the conventions of ([V 1], 1.1); in particular, for a place v corresponding to a real or complex embedding $\sigma: k \hookrightarrow \mathbb{C}$, the absolute value $\|x\|_v$ equals $|\sigma(x)|$ or $|\sigma(x)|^2$, respectively, and the product formula reads

$$\prod_v \|x\|_v = 1$$

for all nonzero $x \in k$. Let $\mathbb{C}_v$ denote the completion of the algebraic closure of the completion k_v of k at v; this field is algebraically closed. If v is archimedean then it is isomorphic to $\mathbb{C}$. The absolute values $\|\cdot\|$ extend from k_v to $\mathbb{C}_v$.

For $\rho \geq 0$ and for a given v let

$$\mathbb{D}_\rho = \{z \in \mathbb{C}_v \mid \|z\| < \rho\} \qquad \text{and} \qquad \partial\mathbb{D}_\rho = \{z \in \mathbb{C}_v \mid \|z\| = \rho\} .$$

Note that these differ from the usual notation if v is a complex place, and that $\partial\mathbb{D}_\rho$ is not the topological boundary if v is non-archimedean. Let $\mathbb{D} = \mathbb{D}_1$.

We will also use the notations and conventions of arithmetic schemes, as in for example [V 2] or [V 3], except that complex conjugate pairs of fibers at an archimedean place will be identified, to conform with the above convention on absolute values. This is possible because in the Gillet-Soulé theory, all objects at complex conjugate places are assumed to be taken into each other by conjugating.

In particular, we assume $\mathscr{X}$ is an integral arithmetic scheme which is quasi-projective and flat over $\operatorname{Spec} R$ with generic fiber X. This assumption on $\mathscr{X}$ differs from that used in the statement of Theorem 0.2, but the change does not affect the set $\mathscr{X}(R_S)$. The exact choice of $\mathscr{X}$ is made in the beginning of Sect. 10.

Throughout this paper we will refer to $\mathbb{Q}$-divisors (divisors with rational coefficients) and $\mathbb{Q}$-divisor classes. The latter are taken to be elements of $\operatorname{Div}(X)\otimes\mathbb{Q}$ modulo principal ($\mathbb{Z}$-)divisors, as opposed to $\operatorname{Pic}(X) \otimes \mathbb{Q}$. If D is a $\mathbb{Q}$-divisor, then writing $\mathscr{O}(dD)$ shall implicitly assume that d is sufficiently divisible to cancel all of the denominators in D.

If g is a Green function or Weil function with respect to a divisor D, then we say $D = \operatorname{div}(g)$ and $\operatorname{Supp} g = \operatorname{Supp} \operatorname{div}(g)$.

In this paper, a **variety** is an integral scheme of finite type over a field. All schemes in this paper are assumed to be separated. As in [V 4], we use the notation **line sheaf** and **vector sheaf** to mean invertible sheaf and locally free sheaf, respectively.

We use $\mathbb{N} = \{0, 1, \ldots\}$.

And finally, on any product (such as A^n or X^n), let pr_i denote the projection onto the i^{th} factor.

2. Structure of semiabelian varieties

A semiabelian variety is a group variety A for which there exists an exact sequence

$$0 \to \mathbb{G}_{\mathrm{m}}^{\mu} \to A \xrightarrow{\rho} A_0 \to 0 \;, \tag{2.1}$$

where A_0 is an abelian variety. By ([I 2], Lemma 4), A is commutative. In general the kernel of ρ need not be a split torus, but we may assume this to be the case by enlarging the number field k; this will not weaken Theorem 0.2.

Lemma 2.2. *For fixed μ and A_0, the set of semiabelian varieties (2.1), modulo isomorphisms fixing the factors $\mathbb{G}_m^\mu$ and A_0, is in 1–1 correspondence with the set* $\operatorname{Pic}^0(A_0)^\mu$*, via the function taking a tuple $(\mathscr{M}_1, \dots, \mathscr{M}_\mu)$ to the product*

$$\mathbb{P}'(\mathscr{O}_{A_0} \oplus \mathscr{M}_1) \times_{A_0} \dots \times_{A_0} \mathbb{P}'(\mathscr{O}_{A_0} \oplus \mathscr{M}_\mu) \ , \tag{2.2.1}$$

where $\mathbb{P}'$ denotes the open subset of $\mathbb{P}(\mathscr{O}_{A_0} \oplus \mathscr{M}_m)$ obtained by removing the sections corresponding to the projections onto each factor of $\mathscr{O}_{A_0} \oplus \mathscr{M}_m$. Moreover, $\operatorname{Pic}(A) \cong \operatorname{Pic}(A_0)$.

Proof . When $\mu = 1$ the first assertion follows by ([L 2], Ch. 11, Sect. 6); the general case then follows by ([S 1], Ch. VII, Sect. 1, (10)). The second assertion is then a consequence of ([H 2], II, Ex. 7.9a). See also ([S 1], Ch. VII) for a treatment of this topic in full generality. □

The group law on A can be described in terms of this construction; see the proof of Proposition 2.6.

The fact that the $\mathscr{M}_m$ lie in Pic^0 is vital here: it implies that, although A may not equal a product of A_0 and $\mathbb{G}_m^\mu$, it is close enough to a product that some of the properties of the product still apply.

We will also frequently use a completion of A to a proper variety $\bar{A}$, which will be chosen as the completion

$$\bar{A} := \mathbb{P}(\mathscr{O}_{A_0} \oplus \mathscr{M}_1) \times_{A_0} \dots \times_{A_0} \mathbb{P}(\mathscr{O}_{A_0} \oplus \mathscr{M}_\mu) \ . \tag{2.3}$$

The morphism $\rho \colon \bar{A} \to A_0$ extends in the obvious manner. This completion was originally defined by Serre, ([S 2], 1.3). It has a canonical exact sequence

$$0 \to \operatorname{Pic}(A_0) \to \operatorname{Pic}(\bar{A}) \to \mathbb{Z}^\mu \to 0 \ , \tag{2.4}$$

where $\mathbb{Z}^\mu = \operatorname{Pic}((\mathbb{P}^1)^\mu)$. Also let $[\infty]_m$ and $[0]_m$ denote the divisors corresponding (respectively) to the projections

$$\mathscr{O}_{A_0} \oplus \mathscr{M}_m \to \mathscr{O}_{A_0} \qquad \text{and} \qquad \mathscr{O}_{A_0} \oplus \mathscr{M}_m \to \mathscr{M}_m \ . \tag{2.5}$$

By (2.2.1) the divisor $\bar{A} \setminus A$ is the sum $\sum_{m=1}^{\mu}([0]_m + [\infty]_m)$. Also, note that

$$\mathscr{O}\big([\infty]_m - [0]_m\big) \cong \rho^* \mathscr{M}_m \ ,$$

so that in particular we have the numerical equivalence

$$[0]_m \equiv [\infty]_m \ , \qquad m = 1, \dots, \mu \ .$$

Note also that by ([I 1], Theorem 2), any morphism of semiabelian varieties is the composition of a group homomorphism and a translation. Thus, in the wording "translated semiabelian subvariety," it is not necessary to specifically state that the group law on the subvariety is obtained from the group law on A.

The above completed semiabelian varieties have a natural choice of Green function for the divisors $[0]_m$ and $[\infty]_m$.

Proposition 2.6. *Let $\bar{A}$ be the completion of a semiabelian variety A defined over a local or global field k. Then for $m = 1, \dots, \mu$ there is a unique Weil function λ_m for $[0]_m - [\infty]_m$ satisfying*

$$\lambda_m(P + Q) = \lambda_m(P) + \lambda_m(Q) \tag{2.6.1}$$

for all $P, Q \in A(\mathbb{C}_v)$ and all places v of k. Moreover, if v is archimedean, then λ_m is C^∞.

Proof . We may assume $\mu = m = 1$; the general case follows by pulling back to A. Thus we may assume that $A = \mathbb{P}'(\mathscr{O}_{A_0} \oplus \mathscr{M})$, and omit m from the notation.

It is well known that points on $\mathbb{P}(\mathscr{O}_{A_0} \oplus \mathscr{M})$ correspond bijectively to pairs consisting of a point $P' \in A_0$ and a surjection $(\mathscr{O}_{A_0}\big|_{P'}) \oplus (\mathscr{M}\big|_{P'}) \twoheadrightarrow \mathscr{O}_{P'}$, up to multiplication by a nonzero constant. Let s be a local generator for $\mathscr{M}$ in a neighborhood of P'; then points in $\bar{A}(\mathbb{C}_v)$ lying outside the support of $[\infty]$ correspond to pairs (P', z), where $P' \in A_0(\mathbb{C}_v)$ and $z \in \mathbb{C}_v$, corresponding to a surjection $(f_1, f_2 s) \mapsto z f_1 - f_2$. Regarding z as a rational function on $\bar{A}$, we then have the equality of divisors

$$(z) = [0] - [\infty] - \rho^*(s) \ . \tag{2.6.2}$$

For divisors D on A_0, let λ_D denote a Néron function as in ([L 2], Ch. 11, 1.1 and 1.5). Let $\Gamma = \bigoplus_v \mathbb{R}$ denote the group of M_k-constants; then Néron functions have the properties:

1. $\lambda_{D+D'} = \lambda_D + \lambda'_D \mod \Gamma$;
2. $\lambda_{(f),v} = -\log \|f\|_v \mod \Gamma$; and
3. $\lambda_{\phi^* D} = \lambda_D \circ \phi \mod \Gamma$ for all morphisms ϕ between abelian varieties for which $\phi^* D$ is defined.

Moreover, these functions are unique modulo Γ. Then we may define

$$\lambda(P) = (-\log \|z\|_v + \lambda_{(s)} \circ \rho) - (-\log \|z(0)\|_v + \lambda_{(s)}(0)) \ , \tag{2.6.3}$$

provided s is chosen so as to generate $\mathscr{M}$ at $0 \in A_0$. By (1) and (2) and (2.6.2), this definition does not depend on the choice of s.

Before showing (2.6.1), we first describe the group law on $A(\mathbb{C}_v)$ explicitly. Let P_1 and P_2 be points in $A(\mathbb{C}_v)$, and let s be a rational section of $\mathscr{M}$ which generates $\mathscr{M}$ in neighborhoods of 0, $\rho(P_1)$, $\rho(P_2)$, and $\rho(P_1) + \rho(P_2)$. Then for $i = 1, 2$ there exist $z_i \in \mathbb{C}_v$ such that P_i corresponds to $\rho(P_i) \in A_0(\mathbb{C}_v)$ and the surjection $(f, gs) \mapsto z_i f - g$. Let z_0 be the element of $\mathbb{C}_v$ for which $0 \in A(\mathbb{C}_v)$ corresponds to $0 \in A_0(\mathbb{C}_v)$ and the surjection $(f, gs) \mapsto z_0 f - g$. Finally, for $i = 1, 2$ let $\tau_i : Q \mapsto Q + P_i$ denote translation by P_i. The theorem of the square then gives an isomorphism

$$\tau_1^* \tau_2^* \mathscr{M} \cong \tau_1^* \mathscr{M} \otimes \tau_2^* \mathscr{M} \otimes \mathscr{M}^{-1}$$

which varies algebraically in P_1 and P_2 and which is the obvious isomorphism if $P_1 = 0$ or $P_2 = 0$. Therefore there is a rational function u defined by

$$u = \frac{s \otimes \tau_1^* \tau_2^* s}{\tau_1^* s \otimes \tau_2^* s} \;.$$

We claim that $P_1 + P_2$ corresponds to $\rho(P_1) + \rho(P_2)$ and the surjection

$$(f, gs) \mapsto (z_1 z_2 / z_0 u(0)) f - g \;.$$

Indeed, this defines a rational map $A \times A \dashrightarrow A$. Replacing s by $s' = hs$ changes z to $z' = z/h$ and u to $u' = u \cdot (h\tau_1^*\tau_2^* h)/(\tau_1^* h \cdot \tau_2^* h)$. Therefore this map is independent of the choice of s and hence it extends to a morphism on all of $A \times A$. Checking its value on $0 \times A$ and $A \times 0$ then shows that it must be the group law.

Now by (2.6.3), the identity (2.6.1) is equivalent to

$$\lambda_{(s)}(\rho(P_1) + \rho(P_2)) = \lambda_{(s)}(\rho(P_1)) + \lambda_{(s)}(\rho(P_2)) - \lambda_{(s)}(0) - \log \|u(0)\|_v \;.$$

But the proof of the theorem of the square, viewed in the context of Néron functions, gives exactly this identity. □

At times it will be convenient to use a multiplicative version of λ_m: the function $\alpha_m := e^{-\lambda_m}$ satisfies

$$\alpha_m(P + Q) = \alpha_m(P)\alpha_m(Q) \tag{2.7}$$

for all $P, Q \in A(\mathbb{C}_v)$ and all places v of k.

Therefore, for archimedean v, the functions

$$-\log \frac{\alpha_m}{1 + \alpha_m} \qquad \text{and} \qquad -\log \frac{1}{1 + \alpha_m} \tag{2.8}$$

can be taken as Green functions for $[0]_m$ and $[\infty]_m$, respectively. Since the metrics on $\mathscr{M}_m$ defined by Néron functions are flat,

$$\alpha_m dd^c \alpha_m = d\alpha_m \wedge d^c \alpha_m \;, \tag{2.9}$$

and the curvatures of the above Green functions are both equal to

$$\begin{aligned} dd^c \log(1 + \alpha_m) &= \frac{(1 + \alpha_m) dd^c \alpha_m - d\alpha_m \wedge d^c \alpha_m}{(1 + \alpha_m)^2} \\ &= \frac{dd^c \alpha_m}{(1 + \alpha_m)^2} \;. \end{aligned} \tag{2.10}$$

Also let $\alpha_{m,i} = \mathrm{pr}_i^* \alpha_m$ on $\overline{A}^n$.

3. The divisor

Once and for all, fix an ample symmetric divisor class L_0 on A_0, let

$$L_1 = \sum_{m=1}^{\mu} \bigl([0]_m + [\infty]_m\bigr) \;,$$

and let

$$L = \rho^* L_0 + L_1 \ .$$

The following lemma implies that L is ample on $\overline{A}$.

Lemma 3.1. *Let $\rho\colon \overline{A} \to A_0$ be a morphism of complete schemes, all of whose closed fibers are isomorphic. Let L_1 be a nef divisor class on $\overline{A}$ whose restrictions to closed fibers of ρ are the same under the above isomorphisms, and which is ample on those fibers. Let L_0 be an ample divisor class on A_0. Then $L := \rho^* L_0 + L_1$ is ample.*

Proof . This is a straightforward application of Seshadri's criterion for ampleness ([H 1], Ch. I, Sect. 7). For curves C on $\overline{A}$ and points $P \in C$ let $m_P(C)$ denote the multiplicity of P on C, and let $m(C) = \sup_{P \in C} m_P(C)$. Seshadri's criterion implies that there exists $\delta > 0$ depending only on $\overline{A}$, L_0, and L_1, such that (a) if C lies in a fiber of ρ, then $(L_1 \,.\, C) \geq \delta m(C)$, and (b) otherwise, $(L_0 \,.\, \rho(C)) \geq m(\rho(C))$. Now for any curve C on $\overline{A}$ we have $(L \,.\, C) \geq \delta m(C)$: if C lies in a fiber of ρ then this follows from (a); otherwise

$$(L \,.\, C) \geq (\rho^* L_0 \,.\, C) = (L_0 \,.\, \rho(C)) \geq \delta m(\rho(C)) \geq \delta m(C)$$

since L_1 is nef. □

For this section, $X_1, \ldots, X_n$ will be closed subvarieties of A and $\bar{X}_1, \ldots, \bar{X}_n$ will denote their closures in $\overline{A}$.

Let $\mathbf{s} = (s_1, \ldots, s_n)$ be a tuple of positive integers, and let i and j be integers with $1 \leq i < j \leq n$. We let $s_i \cdot \mathrm{pr}_i - s_j \cdot \mathrm{pr}_j$ denote the morphism from A^n to A (or from A_0^n to A_0 if it is clear from the context) defined using the group law. In the semiabelian case this leads to a problem, because the group law does not extend to a morphism $\overline{A} \times \overline{A} \to \overline{A}$ unless $\mu = 0$.

Therefore we will need a blow-up of $\prod \bar{X}_i$. Let $\psi_{\mathbf{s}}\colon \prod \bar{X}_i \dashrightarrow \overline{A}^{n(n-1)/2}$ be the rational map whose components are the restrictions of $s_i \cdot \mathrm{pr}_i - s_j \cdot \mathrm{pr}_j$ as i and j vary over integers with $1 \leq i < j \leq n$. Let $W_{\mathbf{s}}$ be the closure of the graph of this rational map. We have a proper birational morphism $\pi_{\mathbf{s}}\colon W_{\mathbf{s}} \to \prod \bar{X}_i$.

For n-tuples $\mathbf{s}$ of positive integers and for rational δ we define

$$(3.2)\quad L_{\delta,\mathbf{s}} = \sum_{i<j} (\rho^n)^* (s_i \cdot \mathrm{pr}_i - s_j \cdot \mathrm{pr}_j)^* L_0 + \sum_{i<j} (s_i^2 \cdot \mathrm{pr}_i - s_j^2 \cdot \mathrm{pr}_j)^* L_1 + \delta \sum_{i=1}^n s_i^2 \,\mathrm{pr}_i^* L$$

as a $\mathbb{Q}$-divisor class on $W_{\mathbf{s}}$. Also let

$$(3.3)\quad M_{\delta,\mathbf{s}} = \sum_{i<j} (\rho^n)^* (s_i \cdot \mathrm{pr}_i - s_j \cdot \mathrm{pr}_j)^* L_0 + (n-1) \sum_{i=1}^n s_i^2 \cdot \mathrm{pr}_i^* L_1 + \delta \sum_{i=1}^n s_i^2 \cdot \mathrm{pr}_i^* L \ .$$

It is a $\mathbb{Q}$-divisor class on $\prod \bar{X}_i$.

The major part of the proof of Theorem 0.2 will consist of showing that there exists some $\epsilon > 0$ and $d \in \mathbb{N}$ such that $\mathscr{O}(dL_{-\epsilon,\mathbf{s}})$ has certain properties, uniformly in $\mathbf{s}$. Here d depends on $\mathbf{s}$, but ϵ may not.

Let ℓ be a positive integer. Multiplication by ℓ extends to a morphism $\mu_\ell \colon \overline{A} \to \overline{A}$; moreover, $\mu_\ell^*[0]_m = \ell[0]_m$ and $\mu_\ell^*[\infty]_m = \ell[\infty]_m$ for all m. This, together with the theorem of the cube, implies that $L_{\delta,\mathbf{s}}$ and $M_{\delta,\mathbf{s}}$ are homogeneous of degree 2 in $s_1,\dots,s_n$ (up to pulling back by the obvious morphism $W_{\mathbf{s}} \to W_{\ell\mathbf{s}}$). Thus it is natural to extend these definitions to tuples $\mathbf{s}$ of positive rational numbers: for such $\mathbf{s}$, let ℓ be the lowest common denominator. Then define $W_{\mathbf{s}} = W_{\ell\mathbf{s}}$, $L_{\delta,\mathbf{s}} = \ell^{-2}L_{\delta,\ell\mathbf{s}}$, and $M_{\delta,\mathbf{s}} = \ell^{-2}M_{\delta,\ell\mathbf{s}}$.

Ideally, one would prefer to work entirely with $L_{\delta,\mathbf{s}}$, but for technical reasons it is easier to introduce $M_{\delta,\mathbf{s}}$ (see (10.1)). We now describe the divisor class $\pi_{\mathbf{s}}^*M_{\delta,\mathbf{s}} - L_{\delta,\mathbf{s}}$.

Definition 3.4. *Let* $\mathbf{s}$ *be a tuple of positive integers and let* $1 \le m \le \mu$ *and* $1 \le i,j \le n$ *be integers. We define*

(3.4.1)
$$Q^{ij}_{\mathbf{s},m} = s_i^2\cdot \mathrm{pr}_i^*([0]_m+[\infty]_m)+s_j^2\cdot \mathrm{pr}_j^*([0]_m+[\infty]_m)-(s_i^2\cdot \mathrm{pr}_i - s_j^2\cdot \mathrm{pr}_j)^*([0]_m+[\infty]_m)$$

as a divisor on $W_{\mathbf{s}}$. *This is homogeneous of degree* 1 *in* $\mathbf{s}$ *(up to pulling back, as before). Therefore if* $\mathbf{s}$ *is a tuple of positive rational numbers with lowest common denominator* ℓ, *we define the* $\mathbb{Q}$*-divisor* $Q^{ij}_{\mathbf{s},m} = \ell^{-1}Q^{ij}_{\ell\mathbf{s},m}$.

Proposition 3.5. *The* $\mathbb{Q}$*-divisor* $Q^{ij}_{\mathbf{s},m}$ *is effective. Its support is contained in the exceptional set of* $\pi_{\mathbf{s}}$.

Proof . We may assume that $\mathbf{s}$ is a tuple of integers. To shorten notation, let $a = s_i^2$ and $b = s_j^2$. Using the expression (3.4.1), the divisor can be given the Green function

$$-\log\frac{\alpha^a_{m,i}}{\left(1+\alpha^a_{m,i}\right)^2} - \log\frac{\alpha^b_{m,j}}{\left(1+\alpha^b_{m,j}\right)^2} + \log\frac{\alpha^a_{m,i}/\alpha^b_{m,j}}{\left(1+\alpha^a_{m,i}/\alpha^b_{m,j}\right)^2}$$
$$= -\log\frac{\left(\alpha^a_{m,i}+\alpha^b_{m,j}\right)^2}{\left(1+\alpha^a_{m,i}\right)^2\left(1+\alpha^b_{m,j}\right)^2}\ .$$

The result then follows immediately from the fact that this function is bounded from below and is smooth except near the sets

$$\mathrm{pr}_i^*[0]_m \cap \mathrm{pr}_j^*[0]_m \qquad \text{and} \qquad \mathrm{pr}_i^*[\infty]_m \cap \mathrm{pr}_j^*[\infty]_m\ .$$

These sets have codimension two, so they must come from the exceptional set of $\pi_{\mathbf{s}}$. □

Thus we have

(3.6)
$$L_{\delta,\mathbf{s}} = \pi_{\mathbf{s}}^*M_{\delta,\mathbf{s}} - \sum_{m=1}^{\mu}\sum_{i<j} Q^{ij}_{\mathbf{s},m}\ ,$$

where the last term is effective. For the bulk of the proof it will be convenient to regard $L_{\delta,\mathbf{s}}$ as a subsheaf of $\pi_{\mathbf{s}}^* M_{\delta,\mathbf{s}}$ and replace the notion of section of $L_{\delta,\mathbf{s}}$ with the notion of section of $M_{\delta,\mathbf{s}}$ satisfying certain vanishing conditions.

4. Reductions

First of all, we may assume that X is relatively closed in A and geometrically irreducible. In the latter case this may involve extending the ground field k, but such a change will not weaken the theorem.

The next set of reductions follows from some standard results on subvarieties of abelian varieties defined over $\mathbb{C}$, which carry over directly to the semiabelian case.

Definition 4.1. *Let $B(X)$ be the identity component of the subgroup*

$$\{a \in A \mid X + a = X\}$$

in A. Then the restriction of the quotient map $A \to A/B(X)$ to X exhibits X as a fibering with fiber $B(X)$. This map is called the **Ueno fibration** *associated to X. It is* **trivial** *when $B(X)$ is.*

Theorem 4.2 ([N 1], Sect. 4). *If X has trivial Ueno fibration then it is of logarithmic general type.*

Theorem 4.3 ([N 1], Lemma 4.1). *The union $Z(X)$ of all nontrivial translated semiabelian varieties of A contained in X is a finite union of irreducible subvarieties of X, each of which has nontrivial Ueno fibration.*

By a simple Galois theoretic argument, if X and A are defined over k, then so are $B(X)$ and $Z(X)$.

The general plan, then, is the same as in ([V 4], Sect. 10): we may assume that $B(X)$ is trivial; this implies that $Z(X) \neq X$. It then suffices to show that $\mathscr{X}(R_S) \setminus Z(X)$ is finite. To do so, let

$$n = \dim X + 1$$

and choose points $P_1, \dots, P_n$ in $\mathscr{X}(R_S) \setminus Z(X)$ satisfying conditions $C_P(c_1, c_2, \epsilon_1)$:

4.4.1. $h_L(P_1) \geq c_1$.
4.4.2. $h_L(P_{i+1})/h_L(P_i) \geq c_2 \geq 1$ for all $i = 1, \dots, n-1$.
4.4.3. $P_1, \dots, P_n$ all point in roughly the same direction in $A(R_S) \otimes_{\mathbb{Z}} \mathbb{R}$, up to a factor $1 - \epsilon_1$ (see (13.2) and (13.3)).

The main part of the proof involves closed subvarieties $X_1, \dots, X_n$ of X. We start with $X_1 = \dots = X_n = X$ and successively find collections with $\sum \dim X_i$ strictly smaller. At each stage, $X_1, \dots, X_n$ are assumed to satisfy conditions $C_X(c_3, c_4, P_1, \dots, P_n)$:

4.5.1. Each X_i contains P_i (and hence has trivial Ueno fibration since $X_i \not\subseteq Z(X)$).
4.5.2. Each X_i is geometrically irreducible and defined over k.
4.5.3. The degrees $\deg X_i$ satisfy $\deg X_i \le c_3$.
4.5.4. The heights $h(X_i)$ will be bounded by the formula

$$\sum_{i=1}^{n} \frac{h(X_i)}{h_L(P_i)} \le c_4 \sum_{i=1}^{n} \frac{1}{h_L(P_i)} .$$

Here and throughout the proof, constants c and c_i depend only on $\mathscr{X}$, $\mathscr{A}$, n, k, S, L, and sometimes the tuple $(\dim X_1, \dots, \dim X_n)$, but not on P_i, X_i, or s_i.

Eventually, this inductive process reaches the point where some X_i is zero dimensional; i.e., $X_i = P_i$. As in ([V 4], 10.6), this leads to an upper bound on $h_L(P_1)$, contradicting (4.4.1).

The following gives the rigorous description of the main step of the proof.

$\forall\ c_3, c_4$ and $\forall\ \delta_1, \dots, \delta_n \in \mathbb{N}$
$\exists\ c_1, c_2, \epsilon_1, c_3', c_4'$ such that
$\forall\ P_1, \dots, P_n \in \big(X \setminus Z(X)\big)(k)$ satisfying $C_P(c_1, c_2, \epsilon_1)$ and
$\forall\ X_1, \dots, X_n \subseteq X$ satisfying $C_X(c_3, c_4, P_1, \dots, P_n)$ and $\dim X_i = \delta_i\ \forall\ i$
$\exists\ X_1', \dots, X_n'$ with $X_i' \subseteq X_i\ \forall\ i$ and $X_i' \ne X_i$ for some i,
and satisfying $C_X(c_3', c_4', P_1, \dots, P_n)$.

In Sects. 12 and 13, s_i will be taken to be rational numbers close to $1 \big/ \sqrt{h_L(P_i)}$. The main step of the proof starts by constructing a small section of $\mathscr{O}(dL_{-\epsilon,\mathbf{s}})$ for all large and sufficiently divisible integers d.

5. Self-intersections of $L_{\delta,\mathbf{s}}$

Lemma 5.1. *If $n \ge \dim X + 1$, then the rational map $f : \prod \bar{X}_i \dashrightarrow \bar{A}^{n(n-1)/2}$ given by $(x_1, \dots, x_n) \mapsto (x_i - x_j)_{i<j}$ is generically finite.*

Proof. If X is a closed subvariety of A and $P \in X_{\text{reg}}$, then the tangent space $T_{X,P}$ may be identified with a linear subspace of the tangent space $T_{A,0}$ at the origin of A via translation. Via this identification, the intersection of all such $T_{X,P}$ equals $T_{B(X),0}$. (This fact is proved by passing to the analytic category and using the universal covering space; details are left to the reader.)

Since all X_i have trivial Ueno fibration, there exists a point $Q = (Q_1, \dots, Q_n) \in \prod X_i$ such that f is smooth at Q, such that Q_i lies in $(X_i)_{\text{reg}}$ for all i, and such that

$$\bigcap_{i=1}^{n} T_{X_i, Q_i} = (0) .$$

Then any tangent to the fiber of f at Q must be zero, so f is a finite map there. □

Proposition 5.2. *If $n \ge \dim X + 1$, then* $(L_{0,1}^{\sum \dim X_i}) > 0$.

Proof . Note that $L_{0,1} = \sum_{i<j}(\mathrm{pr}_i - \mathrm{pr}_j)^* L$. Thus it is the pull-back, to some blowing-up of $\prod \bar{X}_i$, of an ample divisor class on $\bar{A}^{n(n-1)/2}$ via a generically finite morphism (see ([Kl], Ch. 1, Sect. 2, Proposition 6)). □

The remainder of this section is devoted to proving a homogeneity result in $\mathbf{s}$.

Lemma 5.3. *Fix an embedding of k into $\mathbb{C}$. Then the cohomology class in $H^{1,1}_{\bar\partial}(A_0^n)$ corresponding to the divisor class*

$$\mathscr{P}_{ij} := (\mathrm{pr}_i + \mathrm{pr}_j)^* L_0 - \mathrm{pr}_i^* L_0 - \mathrm{pr}_j^* L_0$$

is represented over $A_0(\mathbb{C})^n$ by a form in

$$\mathrm{pr}_i^* \mathscr{E}^{1,0}(A_0) \otimes \mathrm{pr}_j^* \mathscr{E}^{0,1}(A_0) + \mathrm{pr}_i^* \mathscr{E}^{0,1}(A_0) \otimes \mathrm{pr}_j^* \mathscr{E}^{1,0}(A_0) \subseteq \mathscr{E}^{1,1}(A_0^n) \ .$$

Proof . See ([V 4], Lemma 11.3). □

By counting degrees we immediately obtain:

Proposition 5.4. *Let $X_1, \dots, X_n$ be closed subvarieties of A_0. Then any intersection product*

$$\prod_{i<j} \mathscr{P}_{ij}^{e_{ij}} \cdot \prod_{i=1}^{n} \mathrm{pr}_i^* L_0^{e_i}$$

of maximal codimension on $\prod X_i$ vanishes unless

$$2e_i + \sum_{j<i} e_{ji} + \sum_{j>i} e_{ij} = 2 \dim X_i \ , \qquad i = 1, \dots, n \ .$$

Consequently, since

(5.4.1) $$(s_i \cdot \mathrm{pr}_i - s_j \cdot \mathrm{pr}_j)^* L_0 = s_i^2 \, \mathrm{pr}_i^* L_0 + s_j^2 \, \mathrm{pr}_j^* L_0 - s_i s_j \mathscr{P}_{ij} \ ,$$

it follows that if $\mu = 0$ then the highest self-intersection number of $L_{\delta,\mathbf{s}}$ is homogeneous of degree $2 \dim X_i$ in each s_i.

The argument in the semiabelian case is not as straightforward.

Theorem 5.5. *The highest self-intersection number of $L_{\delta,\mathbf{s}}$ is homogeneous of degree $2 \dim X_i$ in each s_i.*

Proof . First note that this self-intersection number is independent of the scheme on which $L_{\delta,\mathbf{s}}$ is taken to be defined. Indeed, given any birational morphism, Chow's moving lemma allows us to move the corresponding 0-cycle away from the exceptional set.

As before, we begin by writing the self-intersection number $(L_{\delta,\mathbf{s}}^{\sum \dim X_i})$ as a sum of terms, each of which is a product of either $\mathscr{P}_{ij}$, $\mathrm{pr}_i^* L_0$, $\mathrm{pr}_i^*([0]_m + [\infty]_m)$, or $(s_i^2 \cdot \mathrm{pr}_i - s_j^2 \cdot \mathrm{pr}_j)^*([0]_m + [\infty]_m)$. Such products can be evaluated by integrating suitably chosen Chern-like forms over $\prod X_i$. For $\mathscr{P}_{ij}$ and $\mathrm{pr}_i^* L_0$ we use the same forms as in the proof of Lemma 5.3; i.e., obtained from a translation-invariant metric on $\mathscr{O}(L_0)$. For the terms $\mathrm{pr}_i^*([0]_m + [\infty]_m)$ we use (2.10). Finally, for the terms $(s_i^2 \cdot \mathrm{pr}_i - s_j^2 \cdot \mathrm{pr}_j)^*([0]_m + [\infty]_m)$ we use

$$
2e \cdot \frac{dd^c \left(\alpha_{m,i}^{s_i^2/e} \Big/ \alpha_{m,j}^{s_j^2/e} \right)}{\left(1 + \alpha_{m,i}^{s_i^2/e} \Big/ \alpha_{m,j}^{s_j^2/e} \right)^2} \tag{5.5.1}
$$

for $e \in \mathbb{Z}$, $e > 0$.

Fixing e for the moment, each of these terms now has a $(1,1)$-form attached to it; this defines a $(1,1)$-form Ξ corresponding to $L_{\delta,\mathbf{s}}$. Now Ξ is not necessarily smooth over any scheme birational to $\prod \bar{X}_i$, so in general it is not a Chern form for $L_{\delta,\mathbf{s}}$. However, it is sufficiently close to a Chern form in the sense that the integral of its top exterior power still equals the highest self-intersection number of $L_{\delta,\mathbf{s}}$. This is proved as follows. Let $a = n(n-1)/2$ and recall the rational map $\psi_\mathbf{s} \colon \prod \bar{X}_i \dashrightarrow \bar{A}^a$ used in defining $W_\mathbf{s}$. Let $f : \bar{A}^a \to \bar{A}^a$ be the morphism given on each factor by multiplication by e. Let V be a desingularization of $W_\mathbf{s} \times_{\bar{A}^a} \bar{A}^a$, and let $g \colon V \to W_\mathbf{s}$ be the projection.

$$
\begin{array}{ccc}
V & \longrightarrow & \bar{A}^a \\
\Big\downarrow g & & \Big\downarrow f \\
W_\mathbf{s} & \xrightarrow{\psi_\mathbf{s}} & \bar{A}^a
\end{array}
$$

Then g is generically finite. Moreover the forms (5.5.1) come from forms on $\bar{A}^a$ which pull back via f to smooth forms which are indeed Chern forms associated to Green functions as in (2.10). Thus $g^*\Xi$ is a Chern form representing $g^*L_{\delta,\mathbf{s}}$ and therefore the equality between the integral and the intersection number holds after pulling back to V. By formal properties of intersection theory (see ([Kl], Ch. 1, Sect. 2, Proposition 6)) and integration, the desired property therefore holds on $W_\mathbf{s}$. Note in particular that the integral in question is independent of e; the proof proceeds by breaking the integral into parts and for each part taking the limit as $e \to \infty$.

However, if one breaks this integral further into subterms in the naïve way, one obtains divergent integrals. Therefore some care is needed.

Replacing each $\bar{X}_i$ with a desingularization such that L_1 pulls back to a normal crossings divisor does not affect the integral. Therefore it suffices to work on a bounded open set $\Omega \subseteq \mathbb{C}^N$ such that each $\alpha_{m,i}$ is of the form

$$
\rho_{m,i} \prod_{j=1}^{N} |z_j|^{2f_{mij}} ,
$$

where $f_{mij} \in \mathbb{Z}$ and $\rho_{m,i}$ is a nonzero smooth function on the closure of Ω. We may further assume that for each $I \subseteq \{1, \dots, N\}$, the boundary of Ω intersects the coordinate subspace defined by $z_i = 0 \ \forall\ i \in I$ in a set of measure zero in $\mathbb{C}^{N-\#I}$. We also assume each z_i comes from a function on some $\bar{X}_j$.

To fix notation, let $0 \le p \le N$ and let Ψ be a smooth $(N-p, N-p)$-form, which we may assume to be of the form

$$\text{(smooth function)} \cdot dz_{i_1} \wedge \dots \wedge dz_{i_{N-p}} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_{N-p}} .$$

Let $F = (f_{ij})$ be a $p \times N$ matrix with entries in $\mathbb{Z}$, and for $i = 1, \dots, p$ let

$$\gamma_i = \prod_{j=1}^{N} |z_j|^{2f_{ij}} .$$

For all such i let ρ_i be a positive smooth function, bounded away from zero on the closure of Ω, and let $\beta_i = \rho_i \gamma_i$. We will consider integrals

$$\int_\Omega \Psi \wedge \frac{e\partial\bar\partial\beta_1^{1/e}}{(1+\beta_1^{1/e})^2} \wedge \dots \wedge \frac{e\partial\bar\partial\beta_p^{1/e}}{(1+\beta_p^{1/e})^2} ,$$

where β_i are of the form $\alpha_{j,m}^a / \alpha_{k,m}^b$ (cf. (5.5.1)). Thus, as in (2.9),

$$\frac{e\partial\bar\partial\beta^{1/e}}{(1+\beta^{1/e})^2} = \frac{1}{e} \cdot \frac{\beta^{1/e}}{(1+\beta^{1/e})^2} \cdot \frac{\partial\beta}{\beta} \wedge \frac{\bar\partial\beta}{\beta} ,$$

and therefore the integral can be rewritten as

$$\text{(5.5.2)} \quad e^{-p} \int_\Omega \frac{\beta_1^{1/e}}{(1+\beta_1^{1/e})^2} \cdots \frac{\beta_p^{1/e}}{(1+\beta_p^{1/e})^2} \Psi \wedge \frac{\partial\beta_1}{\beta_1} \wedge \frac{\bar\partial\beta_1}{\beta_1} \wedge \dots \wedge \frac{\partial\beta_p}{\beta_p} \wedge \frac{\bar\partial\beta_p}{\beta_p} .$$

Lemma 5.5.3. *For each positive $e \in \mathbb{Z}$, let $\phi_e \colon \Omega \to \mathbb{C}$ be a function which is measurable and bounded uniformly in e and $z \in \Omega$. Let p, F, γ_i, ρ_i, and β_i be as above, and let*

$$\kappa_i = \prod_{j=1}^{p} |z_j|^{2f_{ij}} .$$

Then for each e the integral

$$\text{(5.5.3.1)} \quad e^{-p} \int_\Omega \phi_e \cdot \frac{\beta_1^{1/e}}{(1+\beta_1^{1/e})^2} \cdots \frac{\beta_p^{1/e}}{(1+\beta_p^{1/e})^2} \cdot \frac{\partial\gamma_1}{\gamma_1} \wedge \dots \wedge \frac{\partial\gamma_p}{\gamma_p} \wedge dz_{p+1} \wedge \dots \wedge dz_N \wedge \frac{d\bar{z}_1}{\bar{z}_1} \wedge \dots \wedge \frac{d\bar{z}_N}{\bar{z}_N}$$

converges to a value bounded uniformly in e. Moreover, let F' be the matrix consisting of the first p columns of F and assume that $\phi := \phi_e$ is independent of e and is C^1 on the closure of Ω. Then as $e \to \infty$ these integrals approach the finite limit

(5.5.3.2)

$$\frac{(-1)^{N(N-1)/2}(2\pi)^N}{(\sqrt{-1})^N}(\det F')\int_{\Omega\cap\{z_1=\ldots=z_p=0\}}\phi\frac{dd^c|z_{p+1}|^2}{|z_{p+1}|}\wedge\ldots\wedge\frac{dd^c|z_N|^2}{|z_N|}$$
$$\cdot\int_{\mathbb{D}^p}\frac{\kappa_1}{(1+\kappa_1)^2}\cdots\frac{\kappa_p}{(1+\kappa_p)^2}\frac{dd^c|z_1|^2}{|z_1|^2}\wedge\ldots\wedge\frac{dd^c|z_p|^2}{|z_p|^2}\ .$$

Proof . Since

$$\frac{\partial\gamma_i}{\gamma_i}=\sum_{j=1}^N f_{ij}\frac{dz_j}{z_j}\ ,$$

it follows that (5.5.3.1) equals

$$e^{-p}\int_\Omega\phi_e\cdot\frac{\beta_1^{1/e}}{(1+\beta_1^{1/e})^2}\cdots\frac{\beta_p^{1/e}}{(1+\beta_p^{1/e})^2}$$
$$\cdot(\det F')\frac{dz_1}{z_1}\wedge\ldots\wedge\frac{dz_p}{z_p}\wedge dz_{p+1}\wedge\ldots\wedge dz_N\wedge\frac{d\bar z_1}{\bar z_1}\wedge\ldots\wedge\frac{d\bar z_N}{\bar z_N}\ .$$

Hence if F' is singular then the integral vanishes.

Otherwise there exist $t_1,\ldots,t_p\in[-1,1]$ such that, letting

$$\epsilon_j=2\sum_{i=1}^p t_i f_{ij}\qquad\text{for } j=1,\ldots,N\ ,$$

we have $\epsilon_j>0$ for $j=1,\ldots,p$. But now note that for $x>0$ and $t\in[-1,1]$,

$$\frac{x}{(1+x)^2}\le x^t\ .$$

In particular, we apply the facts that

$$\frac{\beta_i^{1/e}}{(1+\beta_i^{1/e})^2}\le\beta_i^{t_i/e}\ ,\qquad i=1,\ldots,p$$

to bound the absolute value of the integral by

(5.5.3.3)

$$\frac{(2\pi)^N|\det F'|}{e^p}\left(\prod_{i=1}^N\sup_{z\in\Omega}\rho_i(z)^{t_i/e}\right)$$
$$\cdot\int_\Omega|\phi_e|\prod_{j=1}^p|z_j|^{\epsilon_j/e}\cdot\prod_{j=p+1}^N|z_j|^{1+\epsilon_j/e}\cdot\frac{dd^c|z_1|^2}{|z_1|^2}\wedge\ldots\wedge\frac{dd^c|z_N|^2}{|z_N|^2}\ .$$

The first assertion of the lemma is then clear, since

$$\int_{\mathbb{D}}|z|^{\epsilon/e}\frac{dd^c|z|^2}{|z^2|}\le O(e)\ .$$

The second assertion follows from the fact that for a region $\Omega' \subseteq \mathbb{C}^p$, for positive smooth $\rho_i : \Omega' \to \mathbb{R}$, for smooth $\phi\colon \Omega' \to \mathbb{C}$, and for κ_i as above,

$$\lim_{e\to\infty} \frac{1}{e^p} \int_{\Omega'} \phi(z) \frac{(\rho_1\kappa_1)^{1/e}}{(1+(\rho_1\kappa_1)^{1/e})^2} \cdots \frac{(\rho_p\kappa_p)^{1/e}}{(1+(\rho_p\kappa_p)^{1/e})^2} \frac{dd^c|z_1|^2}{|z_1|^2} \wedge \dots \wedge \frac{dd^c|z_p|^2}{|z_p|^2}$$
$$= \int_{\mathbb{D}^p} \phi(0) \frac{\kappa_1}{(1+\kappa_1)^2} \cdots \frac{\kappa_p}{(1+\kappa_p)^2} \frac{dd^c|z_1|^2}{|z_1|^2} \wedge \dots \wedge \frac{dd^c|z_p|^2}{|z_p|^2} .$$

This is proved by replacing z_i with z_i^e and applying straightforward arguments.

By (5.5.3.3), we may then use Fubini's theorem to reduce the second assertion of the lemma to the above limit. □

Corollary 5.5.4. *The same conclusions hold with (5.5.3.1) replaced by*

(5.5.4.1)
$$e^{-p} \int_\Omega \phi_e \cdot \frac{\beta_1^{1/e}}{(1+\beta_1^{1/e})^2} \cdots \frac{\beta_p^{1/e}}{(1+\beta_p^{1/e})^2} \cdot \frac{\partial\beta_1}{\beta_1} \wedge \dots \wedge \frac{\partial\beta_p}{\beta_p} \wedge dz_{p+1} \wedge \dots \wedge dz_N \wedge \frac{d\bar z_1}{\bar z_1} \wedge \dots \wedge \frac{d\bar z_N}{\bar z_N} .$$

Proof . We use the identity

$$\frac{\partial\beta_i}{\beta_i} = \frac{\partial\rho_i}{\rho_i} + \frac{\partial\gamma_i}{\gamma_i} , \qquad i = 1, \dots, p$$

and expand (5.5.4.1) into a sum of 2^p integrals. Each such integral can be written as a sum of integrals of the form (5.5.3.1) by expanding out any smooth forms $\partial \log \rho_i$ in terms of $dz_1, \dots, dz_p$, incorporating them into ϕ_e, and permuting the indices $1, \dots, p$. This gives the convergence. But now each term involving a $d \log \rho_i$ vanishes as $e \to \infty$, due to the extra factor $1/e$ which appears when p decreases. This proves the assertion on taking a limit. □

Continuing with the proof of Theorem 5.5, consider again the integral (5.5.2). Breaking Ψ into its components, we may assume that

$$\Psi = \psi\, dz_{i_1} \wedge \dots \wedge dz_{i_{N-p}} \wedge d\bar z_{j_1} \wedge \dots \wedge d\bar z_{j_{N-p}} .$$

Permuting coordinates, we may assume $\{i_1, \dots, i_{N-p}\} = \{p+1, \dots, N\}$. Then we may also assume that $\{j_1, \dots, j_{N-p}\} = \{p+1, \dots, N\}$. Indeed, if, for example, $1 \in \{j_1, \dots, j_{N-p}\}$, then ϕ in (5.5.4.1) vanishes along $z_1 = 0$ since Ψ has a term $d\bar z_1$, while (5.5.4.1) only requires $d\bar z_1/\bar z_1$. This causes the limit (5.5.3.2) to vanish. Thus we are reduced to considering

(5.5.5)
$$\int_{\Omega\cap\{z_1=\dots=z_p=0\}} \Psi \cdot \int_{\mathbb{D}^p} \frac{\kappa_1}{(1+\kappa_1)^2} \cdots \frac{\kappa_p}{(1+\kappa_p)^2} \frac{\partial\kappa_1}{\kappa_1} \wedge \dots \wedge \frac{\partial\kappa_p}{\kappa_p} \wedge \frac{\bar\partial\kappa_1}{\kappa_1} \wedge \dots$$
$$\dots \wedge \frac{\bar\partial\kappa_p}{\kappa_p} = (\det F')^2 \int_{\Omega\cap\{z_1=\dots=z_p=0\}} \Psi \cdot \int_{\mathbb{D}^p} \frac{\kappa_1}{(1+\kappa_1)^2} \cdots \frac{\kappa_p}{(1+\kappa_p)^2} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge \frac{d\bar z_1}{\bar z_1} \wedge \dots \wedge \frac{d\bar z_p}{\bar z_p} .$$

Now consider how this expression changes as the matrix F varies. Suppose $f_{ij} = s_j^2 g_{ij}$. Then $\det F'$ is quadratic in each of $s_1, \dots, s_p$. Also, letting

$$\omega_i = \prod_{j=1}^{N} |z_j|^{2g_{ij}} ,$$

we find that the factor

$$\int_{\mathbb{D}^p} \frac{\kappa_1}{(1+\kappa_1)^2} \cdots \frac{\kappa_p}{(1+\kappa_p)^2} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \wedge \dots \wedge \frac{d\bar{z}_p}{\bar{z}_p}$$
$$= \frac{1}{s_1^2 \dots s_p^2} \int_{\mathbb{D}^p} \frac{\omega_1}{(1+\omega_1)^2} \cdots \frac{\omega_p}{(1+\omega_p)^2} \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_p}{z_p} \wedge \frac{d\bar{z}_1}{\bar{z}_1} \wedge \dots \wedge \frac{d\bar{z}_p}{\bar{z}_p}$$

by replacing each z_j with z_j^{1/s_j^2}. Thus the expression (5.5.5) is quadratic in each of $s_1, \dots, s_p$. As in (5.4.1), however, the form Ψ is also quadratic in each of $s_{p+1}, \dots, s_N$. Therefore, keeping track of which X_i each z_j comes from gives the theorem. □

6. A lower bound on h^0

The goal of this section is to prove a lower bound on $h^0(W_{\mathbf{s}}, \mathscr{O}(dL_{-\epsilon,\mathbf{s}}))$ for some fixed $\epsilon > 0$ and sufficiently large (and divisible) $d > 0$.

Lemma 6.1. *For all (rational) $\delta > 0$, the $\mathbb{Q}$-divisor class $L_{\delta,\mathbf{s}}$ is ample.*

Proof . We may regard $W_{\mathbf{s}}$ as a closed subscheme of $\bar{A}^{n+n(n-1)/2}$ in an obvious way, and $L_{\delta,\mathbf{s}}$ extends in an obvious manner. The lemma then follows by applying Lemma 3.1 to the morphism $\bar{A}^{n+n(n-1)/2} \to A_0^{n+n(n-1)/2}$. □

Proposition 6.2. *There exist constants $c > 0$ and $\epsilon > 0$, depending only on X, $\bar{A}$, L, $\dim X_1, \dots, \dim X_n$, and the bounds on $\deg X_i$, such that for all tuples $\mathbf{s} = (s_1, \dots, s_n)$ of positive rational numbers,*

$$h^0(W_{\mathbf{s}}, \mathscr{O}(dL_{-\epsilon,\mathbf{s}})) \ge c d^{\sum \dim X_i} \prod_{i=1}^{n} s_i^{2 \dim X_i}$$

for all sufficiently large d (depending on $\mathbf{s}$).

Proof . By Lemma 6.1, $L_{\delta,\mathbf{s}}$ is ample. Riemann-Roch therefore implies that, as $d \to \infty$,

$$h^0(W_{\mathbf{s}}, \mathscr{O}(dL_{\delta,\mathbf{s}})) = d^{\dim W_{\mathbf{s}}} \frac{(L_{\delta,\mathbf{s}}^{\dim W_{\mathbf{s}}})}{(\dim W_{\mathbf{s}})!} (1 + o(1)) .$$

Choose ℓ such that ℓL is very ample; then for each index i let H_i be the subscheme of $W_{\mathbf{s}}$ cut out by some section of $\Gamma(\bar{X}_i, \mathscr{O}(\ell L))$. As before,

$$h^0(H_i, \mathscr{O}(dL_{\delta,\mathbf{s}})\big|_{H_i}) = d^{\dim W_\mathbf{s}-1}\frac{(H_i \,.\, L_{\delta,\mathbf{s}}^{\dim W_\mathbf{s}-1})}{(\dim W_\mathbf{s}-1)!}(1+o(1))$$
$$= \ell d^{\dim W_\mathbf{s}-1}\frac{(\mathrm{pr}_i^* L \,.\, L_{\delta,\mathbf{s}}^{\dim W_\mathbf{s}-1})}{(\dim W_\mathbf{s}-1)!}(1+o(1))\ .$$

These two estimates replace the first two estimates in the proof of ([V 4], Proposition 11.5); the proof then continues as in that case, with a little extra care because of the variable ℓ. □

7. Generalized Weil functions

This section gives some preliminary results on Weil functions in preparation for Sect. 9.

For a general reference on Weil functions, see [L 2] or [L 4]. Instead of working over $X \times M_k$ (for a scheme X of finite type over k), however, we will work over $\coprod_v X(\mathbb{C}_v)$. This will be denoted by $X(M_k)$. Also, Weil functions will be normalized so that $-\log\|f\|$ is a Weil function for the principal divisor (f). The results of [L 2] carry over into this situation.

Definition 7.1. *A* **generalized Weil function** *on a scheme X of finite type over k is an equivalence class of pairs (U, g). Here U is a dense Zariski-open subset of X and $g\colon U(M_k) \to \mathbb{R}$ is a function such that there exists a scheme $\widetilde{X}$ and a proper birational morphism $\Phi\colon \widetilde{X} \to X$ such that $g \circ \Phi$ extends to a Weil function for some divisor $\widetilde{D}$ on $\widetilde{X}$. Pairs (U, g) and (U', g') are equivalent if $g = g'$ on $(U \cap U')(M_k)$. We say that g is* **effective** *if $\widetilde{D}$ is an effective divisor. The* **support of** *g, written $\operatorname{Supp} g$, is defined as the set $\Phi(\operatorname{Supp}\widetilde{D})$.*

Proposition 7.2. *Generalized Weil functions on a scheme X form an abelian group under addition. If $\phi\colon X \dashrightarrow Y$ is a dominant rational map and g is a generalized Weil function on Y, then $\phi^* g$ (defined in the obvious way) is a generalized Weil function on X.*

Proof . Obvious. □

Proposition 7.3. *Let g_1 and g_2 be generalized Weil functions on a proper scheme X/k. Then $g_3 := \min(g_1, g_2)$ is also a generalized Weil function on X. If g_1 and g_2 are effective, then so is g_3, and $\operatorname{Supp} g_3 \subseteq \operatorname{Supp} g_1 \cap \operatorname{Supp} g_2$.*

Proof . Since $\min(g_1, g_2) = g_2 + \min(g_1 - g_2, 0)$, we may assume that $g_2 = 0$. By blowing up X, we may assume that g_1 is a Weil function. Moreover, we may further blow up X to the point where components occurring with positive multiplicities in $\operatorname{div}(g_1)$ do not intersect those occurring with negative multiplicities. This is accomplished as follows. Let $D = \operatorname{div}(g_1)$, and let $U_1, \dots, U_n$ be a finite cover of X by open affines such that $D = (f_i)$ on U_i for each i. Then we replace

X with the graph of the rational map $X \dashrightarrow (\mathbb{P}^1)^n$ given by $(f_1, \dots, f_n)$. The first assertion then follows by standard properties of Weil functions.

To prove the other assertion, assume g_1 and g_2 are effective. Let $\Phi\colon \widetilde{X} \to X$ be a proper birational morphism such that $\Phi^* g_1$ and $\Phi^* g_2$ are Weil functions. Moreover, writing $\operatorname{div}(\Phi^* g_i) = \sum n_{iD} \cdot D$ for $i = 1, 2$ we may assume that prime divisors D for which $n_{1D} > n_{2D}$ do not meet prime divisors with $n_{1D} < n_{2D}$. Then $\Phi^* g_3$ is also a Weil function, associated to the divisor $\sum \min(n_{1D}, n_{2D}) \cdot D$. This easily gives $\operatorname{Supp} \Phi^* g_3 = \operatorname{Supp} \Phi^* g_1 \cap \operatorname{Supp} \Phi^* g_2$. Pushing it down to X gives the desired inclusion (which may become strict). ☐

Proposition 7.4. *Let $f : X \to Y$ be a morphism of proper schemes over k and let g be a generalized Weil function on X whose restriction to a generic closed fiber of f is effective. Then there exists a generalized Weil function g' on Y such that* $\operatorname{Supp} g'$ *does not contain $f(X)$ and such that $f^* g' \le g$. If f is surjective then we may choose g' such that* $\operatorname{Supp} g' = f(\operatorname{Supp} g)$.

Proof . Replacing X with a suitable nonsingular blowing-up, we may assume that g is a Weil function, relative to a divisor D. The condition then implies that the restriction of D to the generic fiber of f is effective. Then there exists a divisor D' on Y such that $D - f^* D'$ is effective on X. Let g' be a Weil function for D' on Y; then the desired inequality holds up to the addition of an M_k-constant (i.e., an element of $\bigoplus_v \mathbb{R}$). We may then adjust g' by such a constant to obtain the inequality without constants. To prove the last assertion, we may assume after blowing up X and Y that X and Y are nonsingular, g is a Weil function (as above), and that $f(\operatorname{Supp} g)$ is a divisor. Then the above choice of D' may be made such that $\operatorname{Supp} D' = f(\operatorname{Supp} D)$. ☐

Definition 7.5. *Let X be a variety. A* **min-min generalized Weil function on** X *is a generalized Weil function on X which can be written in the form*

(7.5.1)

$$\min(-\log\|\phi_1\|, \dots, -\log\|\phi_n\|) - \min(-\log\|\psi_1\|, \dots, -\log\|\psi_m\|) + c_v$$

for some M_k-constant (c_v), where $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_m$ are nonzero rational functions on X. If g is a min-min generalized Weil function, then we also say that g is **of min-min type**. *A* **min-min Weil function** *is a Weil function of min-min type.*

Definition 7.6. *Let g be a generalized Weil function on a variety X and let $F \subseteq X$ be a finite set. We say that g is* **nicely defined at** F *if* $\operatorname{Supp} g$ *is disjoint from F, if g is of min-min type, and if g can be written as an expression (7.5.1) in which $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_m$ are all regular at all $P \in F$, and in which some fixed ϕ_i and some fixed ψ_j are nonzero at all $P \in F$.*

Proposition 7.7. *Min-min generalized Weil functions on a variety X form a subgroup of the group of generalized Weil functions on X. The same assertion* *holds for generalized Weil functions nicely defined at some fixed finite subset* $F \subseteq X$.

Proof . Additivity of min-min generalized Weil functions follows from the identity

$$\min_i a_i + \min_j b_j = \min_{i,j}(a_i + b_j) \ .$$

The other assertions are trivial. □

Proposition 7.8. *If* $\phi\colon X \dashrightarrow Y$ *is a dominant rational map of varieties and* g *is a min-min generalized Weil function on* Y*, then* $\phi^* g$ *is also of min-min type. If* ϕ *is regular at all* $P \in F$ *for some finite subset* $F \subseteq X$ *and* g *is nicely defined at* $\phi(F)$*, then* $\phi^* g$ *is nicely defined at* F.

Proof . Obvious. □

Proposition 7.9. *If* g_1 *and* g_2 *are min-min generalized Weil functions on a given variety, then so are* $\min(g_1, g_2)$ *and* $\max(g_1, g_2)$*. If* g_1 *and* g_2 *are nicely defined at some finite subset* $F \subseteq X$*, then so are* $\min(g_1, g_2)$ *and* $\max(g_1, g_2)$.

Proof . The assertions regarding $\min(g_1, g_2)$ follow from the identity

$$\begin{aligned} &\min\Big(\min_i a_i - \min_j b_j , \min_k c_k - \min_\ell d_\ell\Big) \\ &\quad = \min\Big(\min_{i,\ell}(a_i + d_\ell), \min_{j,k}(b_j + c_k)\Big) - \min_{j,\ell}(b_j + d_\ell) \ . \end{aligned}$$

The result for $\max(g_1, g_2)$ is similar. □

Proposition 7.10. *Let* X *be a projective variety,* D *a Cartier divisor on* X*, and* $F \subseteq X$ *a finite set disjoint from* $\operatorname{Supp} D$*. Then there is a Weil function* g *with divisor* D *which is nicely defined at* F.

Proof . First assume $X = \mathbb{P}^n$ and D is the hyperplane at infinity. Let $x_1, \dots, x_n$ be the standard coordinate functions on $X \setminus D$. By applying a suitably chosen automorphism of $\mathbb{P}^n$ fixing D, we may assume that x_1 is nonzero at all $P \in F$. Then $g := \max_{1 \le i \le n}(\log \|x_i\|)$ has the required properties. By Proposition 7.8 this extends to the case where X is arbitrary and D is very ample. The general case then follows by writing D as a difference of two very ample divisors not passing through F. □

8. Metrics at non-archimedean places

This section introduces metrics on line sheaves at non-archimedean places, to parallel the theory at infinite places.

Definition 8.1. *Let K be a local field with valuation ring R, let X be a proper scheme over* $\operatorname{Spec} R$*, let $\mathscr{L}$ be a line sheaf on X, let U be a Zariski-open subset of $X \times_R K$, and let $\gamma \in \Gamma(U, \mathscr{L})$. For closed points $P \in U$ we define $\|\gamma(P)\|$ as follows. Let $K_1 = K(P)$ and let R_1 be its valuation ring. The valuative criterion of properness implies that P extends to a section $\sigma\colon \operatorname{Spec} R_1 \to X$ over* $\operatorname{Spec} R$*. Then $\sigma^*\gamma$ is a rational section of $\sigma^*\mathscr{L}$; letting g be a generator of $\sigma^*\mathscr{L}$ we have $\sigma^*\gamma = ag$ for some $a \in K_1$. We then define $\|\gamma(P)\| = \|a\|$; this is independent of the choice of g.*

This defines a metric on $\mathscr{L}$, in the sense that if f is a function that is regular at P then $\|(f\gamma)(P)\| = \|f(P)\| \cdot \|\gamma(P)\|$.

Proposition 8.2. *With notation as above, the function $P \mapsto \|\gamma(P)\|$ is continuous on $U(\bar{K})$ (in the topology induced by the valuation).*

Proof . Fix a point $P_0 \in U(\bar{K})$. Let V be an open neighborhood of the point where P_0 passes through the special fiber of X; we may assume that $\mathscr{L}$ is trivial on V. Then there exists a rational function f on V, regular on $U \cap V$, such that $\|\gamma(P)\| = \|f(P)\|$ for all $P \in (U \cap V)(\bar{K})$ entirely contained in V. Since all P in a sufficiently small neighborhood of P_0 satisfy this condition, it follows that $P \mapsto \|\gamma(P)\|$ is continuous in this neighborhood. This implies continuity. □

Definition 8.3. *If U and γ are as above, then we define $\|\gamma(P)\|$ on $U\left(\widehat{\bar{K}}\right)$ by continuity.*

Lemma 8.4.

(a) The above definition is functorial: if, in addition to the above notation, $f\colon X_2 \to X$ is a morphism of proper schemes over $\operatorname{Spec} R$ *and $P_2 \in f^{-1}(U)\left(\widehat{\bar{K}}\right)$, then $\|f^*\gamma(P_2)\| = \|\gamma(f(P_2))\|$.*
(b) If $\gamma \in \Gamma(X, \mathscr{L})$ then $\|\gamma(P)\| \le 1$ for all $P \in X\left(\widehat{\bar{K}}\right)$.
(c) If $a \in \bar{K}$ then $\|a\gamma(P)\| = \|a\| \cdot \|\gamma(P)\|$.
(d) If $\mathscr{L}_2$ is another line sheaf on X and $\gamma_2 \in \Gamma(U, \mathscr{L}_2)$ then

$$\|(\gamma \otimes \gamma_2)(P)\| = \|\gamma(P)\| \cdot \|\gamma_2(P)\| .$$

Proof . Obvious. □

We also note that the converse of (b) holds if X is normal.

Definition 8.5. *If X is a proper scheme over a localization of the ring of integers of a number field, then we define $\|\gamma(P)\|_v$ for non-archimedean places v by base change to the completed local ring at v.*

Lemma 8.4 holds also in the context of number fields.

Proposition 8.6. *Let R be as in Definition 8.1 (resp. 8.5), let X be a proper scheme over* $\operatorname{Spec} R$, *let $\mathscr{L}$ be a line sheaf over X, and let $\gamma \in \Gamma(U, \mathscr{L})$ be a nonzero global section over an open subset U. Then $-\log\|\gamma\|$ (resp. $-\log\|\gamma\|_v$) defines a Weil function on X relative to the divisor associated to γ.*

Proof . If $\mathscr{L} = \mathscr{O}_X$ then this is trivial. If $\mathscr{L}$ and γ are trivial on the generic fiber then this follows from parts (b) and (d) of Lemma 8.4. By combining these two facts with Chow's lemma and functoriality, we may reduce the problem to the case where $X = \mathbb{P}^n_R$, $\mathscr{L} = \mathscr{O}(1)$, and $\gamma = x_0$. In that case it can be checked by direct computation. □

9. An analytic result

This section proves an analytic result which will be needed in Sects. 10 and 11.

For the latter section, it will be important to establish bounds having a uniformity as v varies over all places of a number field. This uniformity is provided by the formalism of Weil functions.

We start with some lemmas.

Lemma 9.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety of dimension n and let $P_0 \in X$. Then for generic linear subspaces L of codimension $n - s + 1$, the linear projection from L induces a smooth morphism $p\colon U \to \mathbb{P}^{n-s}$ for a Zariski-open $U \subseteq X$ such that $X \setminus U$ meets the fiber $p^{-1}(p(P_0))$ at only finitely many points. Moreover, L meets X transversally. Finally, if $s = 0$ then U contains the entire fiber over $p(P_0)$.*

Proof . By a minor adaption of the proof of Bertini's theorem, one can show that the generic hyperplane passing through P_0 crosses X transversally except at finitely many points. This holds even if X has finitely many singular points. Thus, by induction, the generic linear subspace L_0 of codimension $n - s$ containing P_0 meets X transversally except at finitely many points. If $L \subseteq L_0$ is any linear subspace with $P_0 \notin L$ and $\dim L = \dim L_0 - 1$, then the corresponding projection satisfies the first assertion of the lemma, by ([H 2], III 10.4(iii)). The second assertion is satisfied for a generic choice of L within L_0, by Bertini's theorem.

To prove the last assertion, we first assume that X is a curve (possibly reducible) and $P_0 \in X$, and show that the generic hyperplane H through P_0 crosses X transversally. Indeed, it is sufficient that H is not tangent to X at P_0 and that it avoid the (finitely many) singular points and the points $Q \in X \setminus \{P_0\}$ such that the line P_0Q is tangent to X at Q. Any irreducible component of X containing infinitely many such Q must be a line through P_0, which the generic hyperplane avoids. For such generic projections, p is étale at all of $p^{-1}(p(P_0))$. This proves the last assertion. □

Lemma 9.2. *Let $Y \subseteq X$ be affine schemes of finite type over k and let F be a finite subset of Y such that X and Y are regular at all $P \in F$ and such that $\dim \mathscr{O}_{P,X} - \dim \mathscr{O}_{P,Y}$ is independent of P for $P \in F$. Let r equal this constant. Then there exist $f_1, \dots, f_r \in \mathscr{O}(X)$ which generate the sheaf of ideals $\mathscr{I}$ of Y in X in a neighborhood of F.*

Proof . If $r = 0$ then this is immediate. If $r > 0$ then for each $P \in F$ there exists $g_P \in \mathscr{I}$ which lies in the maximal ideal $\mathfrak{m}_{P,X} \subseteq \mathscr{O}_{P,X}$, but not in $\mathfrak{m}_{P,X}^2$. There exists a suitable linear combination $f_1 := \sum \phi_P g_P$ for $\phi_P \in \mathscr{O}(X)$ which lies in $\mathfrak{m}_{P,X} \setminus \mathfrak{m}_{P,X}^2$ for all P. Let $X' = \operatorname{Spec} \mathscr{O}(X)/(f_1)$; by induction on r there exist $\overline{f_2}, \dots, \overline{f_r} \in \mathscr{O}(X')$ generating the sheaf of ideals of Y in X'. Lifting the $\overline{f_i}$ to $f_i \in \mathscr{O}_X$ for $i = 2, \dots, r$ then gives the required factors. □

For vectors $\mathbf{z} = (z_1, \dots, z_r) \in \mathbb{C}_v^r$, we define $\|\mathbf{z}\|_v = \max \|z_i\|_v$ if v is non-archimedean; otherwise we use the standard definition $\|\mathbf{z}\| = (|z_1|^2 + \dots + |z_r|^2)^{[k_v:\mathbb{R}]/2}$.

Definition 9.3. *Let Y be a projective scheme over k and let g be an effective generalized Weil function on Y. For each place v let*

$$\Lambda_v(g) = \{(P, \mathbf{z}) \in Y(\mathbb{C}_v) \times \mathbb{C}_v^r \mid \|\mathbf{z}\| < e^{-g(P)}\}$$

and

$$\Upsilon_v(g) = \{(P, \mathbf{z}) \in Y(\mathbb{C}_v) \times \mathbb{C}_v^r \mid \|z_i\| < e^{-g(P)},\ i = 1, \dots, r\} .$$

Here, if $P \in \operatorname{Supp} g$ then we take $g(P) = \infty$ so that $e^{-g(P)} = 0$.

Note that these two definitions coincide if v is non-archimedean. Strictly speaking, the value of r should be specified in the notation, but its value will always be clear from the context. Often these sets will be identified with subsets of $\mathbb{P}_Y^r(\mathbb{C}_v)$.

The goal of the rest of this section is to construct certain rigid analytic maps with domain $\Lambda_v(g)$ or $\Upsilon_v(g)$. For our purposes, though, it suffices to regard them as maps such that, for all $P \in Y(\mathbb{C}_v)$ with $P \notin \operatorname{Supp} g$, the restriction to the disc or polydisc $\Lambda_v(g) \cap \{P\} \times \mathbb{C}_v^r$ or $\Upsilon_v(g) \cap \{P\} \times \mathbb{C}_v^r$ is given by a power series.

The following lemma does most of the work that will be needed.

Lemma 9.4. *Let $p \colon \Gamma \to Y$ be a morphism of equidimensional projective k-schemes with a regular section $\sigma \colon Y \to \Gamma$, let $q \colon \Gamma \to \mathbb{P}_Y^r$ be a generically finite morphism such that $q \circ \sigma$ equals the canonical section of the natural map $\pi \colon \mathbb{P}_Y^r \to Y$ with image $Y \times \{[1 : 0 : \dots : 0]\}$, and let $\mathbf{z} = (z_1, \dots, z_r)$ denote the coordinate functions on $\mathbb{A}^r = \mathbb{P}^r \setminus \{x_0 = 0\}$. Let F be a finite subset of the image of σ such that Γ is regular at all $P \in F$ and such that q is étale in a neighborhood of F. Then:*

(a) There is an effective generalized Weil function g_1 on Y with support disjoint from $p(F)$ such that, for each v, the map $\Gamma(\mathbb{C}_v) \to \mathbb{P}_Y^r(\mathbb{C}_v)$ has a rigid analytic partial section $\theta_v \colon \Lambda_v(g_1) \to \Gamma$ whose image contains $\sigma(Y \setminus \operatorname{Supp} g_1)$.

(b) *Let $\phi_1, \dots, \phi_s$ be rational functions on Γ which are regular on F. Then g_1 can be chosen such that there exist effective generalized Weil functions $g_1^*, \dots, g_s^*$ on Y such that for all v, all $(P, \mathbf{z}) \in \Lambda_v(g_1)$, and all i,*

$$\text{(9.4.1)} \qquad \|\phi_i(\theta_v(P, \mathbf{z})) - \phi_i(\theta_v(P, \mathbf{0}))\| \le e^{g_i^*(P)} \cdot \|\mathbf{z}\| \,.$$

Moreover, Supp g_i^* *is disjoint from $p(F)$ for all i.*

Proof . First consider non-archimedean places v. Fix a finite cover of Y by open affines such that $p(F)$ lies in each open set. Let V be any element of this cover, and let $y_1, \dots, y_\ell$ be a generating set for $\mathscr{O}(V)$ over k. After adjusting y_i by constant factors, we may assume that for all non-archimedean v the sets

$$\{Q \in V(\mathbb{C}_v) \mid \|y_i(Q)\| \le 1 \text{ for all } i\}$$

cover $Y(\mathbb{C}_v)$ as V varies over the chosen cover. For each such V fix an open affine $U \subseteq q^{-1}(V)$ containing F and let $x_1, \dots, x_M$ be a generating set for $\mathscr{O}(U)$ over k. We may assume that $x_i = y_i \circ p$ for $i = 1, \dots, \ell$. By Lemma 9.2, there exist polynomials $f_{\ell+1}(X) = f_{\ell+1}(X_1, \dots, X_n), \dots, f_{M-r}(X)$ which generate the sheaf of ideals of Γ in $\mathbb{A}^M$ near all $P \in F$. We may assume that the coefficients of $f_{\ell+1}, \dots, f_{M-r}$ all lie in R (the ring of integers of k). Let $f_{M-r+1}, \dots, f_M$ be polynomials in $R[X_1, \dots, X_M]$ which equal $a_i \cdot z_i \circ q$ on U for some $a_i \in k^*$ and all $i = 1, \dots, r$.

Fix a non-archimedean place v and $Q \in V(\mathbb{C}_v)$ with $\|x_i(Q)\| \le 1$ for all i. For $i = 1, \dots, \ell$ let $f_i(X) = X_i - x_i(Q)$; then all of $f_1, \dots, f_M$ lie in $R_v[X_1, \dots, X_M]$, where R_v is the valuation ring of $\mathbb{C}_v$. Let J denote the matrix $(\partial f_i/\partial X_j)_{1 \le i,j \le M}$. All entries in this matrix lie in $R[X_1, \dots, X_M]$ and are independent of v and Q. The assumption that q is étale near F implies that $\det J \ne 0$ at all $P \in F$.

For an $M \times M$ matrix A with entries in $\mathbb{C}_v$ we let $\|A\| = \inf_{\|\mathbf{b}\|=1} \|A\mathbf{b}\|$. It follows that if A is nonsingular then $\|A\|^{-1}$ equals the largest absolute value of an entry of A^{-1}. If all entries of A lie in R_v, then $\|A\| \ge \|\det A\|$.

By ([V 4], Corollary 15.13a), there is a rigid analytic lifting of q over the subset $\|\mathbf{z}\| < \|J(\sigma(Q))\|^2/\max\|a_i\|$ of $\pi^{-1}(Q)$ which maps $\mathbf{0}$ to $\sigma(Q)$. (Moreover, as Q varies, the lifting varies rigid analytically since the convergents vary algebraically in Q.)

Let $g_V = \max\bigl(0, -\log\bigl(\|\det J \circ \sigma\|^2/\max\|a_i\|\bigr)\bigr)$. This is an effective generalized Weil function on Y. Its support is disjoint from F since $\det J(P) \ne 0$ for all $P \in F$. By construction, for all non-archimedean v there exists a unique lifting of q over

$$\Lambda_v(g_V) \cap \pi^{-1}(\{Q \in V(\mathbb{C}_v) \mid \|y_i(Q)\| \le 1 \text{ for all } i\}) \,.$$

Let g_1 be the maximum of all g_V. Then g_1 satisfies part (a) for all non-archimedean v.

The proof for archimedean places v is essentially the same, except that it requires a little more care due to the archimedean property of v. We will work with $|\cdot|$ instead of $\|\cdot\|$ for consistency with [V 4]. We define $|A|$ for an $M \times M$ matrix A by the same formula as before. We then have

$$|A| \geq \frac{|\det A|}{M! \cdot \max|a_{ij}|^{M-1}} .$$

We will use ([V 4], Lemma 15.8 and Corollary 15.13b), with $c = 1/3$. We will use the same open subsets U and V as before; for each such V let C be a compact subset of $V(\mathbb{C}_v)$ such that the union of all these C cover $Y(\mathbb{C}_v)$. Let

$$B = \max\left(\sup\left|\sum_{i=1}^{M}\sum_{j=1}^{M}\frac{\partial^2\mathbf{f}(\alpha)}{\partial X_i\partial X_j}v_i w_j\right|,\ \frac{c'}{3}\sup_{\alpha_0\in\sigma(C)}\Big|J(\alpha_0)\Big|\right) ;$$

here the first supremum is taken over the set of all points $\alpha \in U(\mathbb{C}_v)$ of distance ≤ 1 from $\sigma(C)$ and the set of all unit vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{C}^M$. In the second term in the above maximum,

$$c' = \sum_{i=0}^{\infty}\left(\frac{4}{3}\right)^i\left(\frac{3}{8}\right)^{2^i-1} . \tag{9.4.2}$$

(This second term ensures that $\rho < 1$ in the statement of ([V 4], Lemma 15.8).) This variant of Hensel's lemma then gives a unique complex analytic map θ_v over the set

$$\left\{(Q,\mathbf{z}) \in C \times \mathbb{C}^r \;\middle|\; |\mathbf{z}| < \frac{|J(\sigma(Q))|^2}{3B}\right\} .$$

with the desired properties. Letting

$$D = \sup_{\alpha_0\in\sigma(C)}\max_{1\leq i,j\leq M}\left|\frac{\partial f_i}{\partial X_j}(\alpha_0)\right| ,$$

we have

$$|J(\sigma(Q))| \geq \frac{|\det\sigma(Q)|}{M!\cdot D^{M-1}}$$

and therefore g_1 will also satisfy part (a) for the archimedean places after adding the constants $[k_v : \mathbb{R}]\log\big(3B(M!\cdot D^{M-1})^2\big)$ to $g_{1,v}$ for $v \mid \infty$.

To prove (b), we may assume that $\phi_1,\ldots,\phi_s$ were included among the generators $x_1,\ldots,x_M$ of $\mathscr{O}(U)$ in the proof of part (a). Then the application of Hensel's lemma bounds the variation in ϕ_i. Indeed, if v is non-archimedean, then ([V 4], (15.5)) implies that

$$\|\phi_i(\theta_v(P,\mathbf{z})) - \phi_i(\theta_v(P,\mathbf{0}))\| \leq \frac{\|\mathbf{z}\|}{\|J(P)\|} ,$$

and in the archimedean case the same inequality holds up to multiplication by c' or c'^2 by ([V 4], (15.11)) and (9.4.2). So we may take $g_i^* = -\log\|\det J(P)\|$, with $[k_v : \mathbb{R}]\log c'$ added at archimedean places. □

Definition 9.5. *Let A be a nonsingular projective variety over k. An* **extended model of** *A is a model $\mathscr{A}$ of A which is proper and flat over* Spec R *plus, for each $v \mid \infty$, a hermitian metric on the tangent bundle of $A(\mathbb{C}_v)$. For a given extended model we define distance functions $d_v(\cdot,\cdot)$ on $A(\mathbb{C}_v)$ for each v as follows. If $v \mid \infty$ then d_v is the distance relative to the chosen metric on the tangent space of $A(\mathbb{C}_v)$. For non-archimedean v, suppose P and Q lie in $A(\mathbb{C}_v)$. Let R_v denote the valuation ring of $\mathbb{C}_v$; then P and Q define sections of $\mathscr{A} \times_R R_v$. If these sections do not meet on the closed fiber, then let $d_v(P,Q) = 1$. Otherwise let B be the local ring of $\mathscr{A} \times_R R_v$ at the point where these sections meet the closed fiber, and let $d_v(P,Q) = \sup_{\phi\in B} \|\phi(P) - \phi(Q)\|$.*

Remark 9.6. If v is non-archimedean and U is an open affine in $\mathscr{A} \times_R R_v$ which contains the images of the sections corresponding to P and Q, then the above local ring B may be replaced with $\mathscr{O}(U)$. This holds even if the corresponding sections do not meet on the closed fiber.

Lemma 9.7. *Let $f : Z \to X$ be a birational projective morphism of integral schemes, quasi-projective and of finite type over k. Then there exists a coherent sheaf of ideals $\mathscr{T}$ on X such that $f : Z \to X$ is isomorphic to the blowing-up of X with respect to $\mathscr{T}$. Moreover, there exists a finite collection $\mathscr{T}_1, \dots, \mathscr{T}_s$ of such sheaves of ideals such that the intersection of the corresponding subschemes of X equals the set over which f fails to be an isomorphism.*

Proof . The first assertion follows from the proof of ([H 2], II 7.17). It is stated there only for quasi-projective varieties over k, but the proof can be adapted to the present situation as follows. First replace ([H 2], II 5.20) with ([H 2], III Remark 8.8.1). Next, when proving that S and T agree in all large enough degrees in Step 2, replace the use of finiteness of the integral closure with an adaptation of the proof of ([H 2], III 5.2).

To prove the second assertion, it suffices to show that given any point $P \in X$ such that f is an isomorphism over a neighborhood of P, one can choose $\mathscr{T}$ such that its corresponding subscheme of X does not contain P. This can be done by choosing the map $\mathscr{O}_X \to \mathscr{J} \otimes \mathscr{M}^n$ to be an isomorphism at P. □

Proposition 9.8. *Let A be a nonsingular projective variety over k, let $\mathscr{A}$ and $\mathscr{A}'$ be two projective models for A over R, and let d_v and d'_v be the corresponding distance functions on $A(\mathbb{C}_v)$ for non-archimedean v. Then there exists an M_k-constant (c_v) such that*

$$d_v(P,Q) \le e^{c_v} d'_v(P,Q)$$

for all non-archimedean v and all $P, Q \in A(\mathbb{C}_v)$.

Proof . It will suffice to show that if there exists a morphism $\psi\colon \mathscr{A} \to \mathscr{A}'$ which restricts to the identity on A, then there exists (c_v) such that

$$d'_v(P,Q) \le d_v(P,Q) \le e^{c_v} d'_v(P,Q)$$

for all v, P, and Q as above.

The first inequality holds because ψ induces a homomorphism of local rings on $\mathscr{A}'$ into their corresponding local rings on $\mathscr{A}$.

To prove the second inequality, Lemma 9.7 implies that there exist sheaves of ideals $\mathscr{I}_1, \dots, \mathscr{I}_r$ on $\mathscr{A}'$ such that for all i, $\mathscr{A}$ is isomorphic to the blowing-up of $\mathscr{A}'$ at $\mathscr{I}_i$, and such that $\bigcap_i Z(\mathscr{I}_i)$ is contained in the special fiber of $\mathscr{A}'$. This latter condition implies that the sheaf of ideals $(\mathscr{I}_1, \dots, \mathscr{I}_r)$ contains some nonzero $a \in R$.

Let $P, Q \in A(\mathbb{C}_v)$ for some non-archimedean v. We will show that

$$d_v(P,Q) \le \frac{d'_v(P,Q)}{\|a\|_v} . \tag{9.8.1}$$

If P and Q correspond to different points on the special fiber of $\mathscr{A}' \times_R R_v$ then this is obvious since $d_v(P,Q) = d'_v(P,Q) = 1$. Otherwise let $\operatorname{Spec} B$ be an open affine such that $\operatorname{Spec} B \otimes_R R_v$ contains this point on the special fiber. Write $B' = B \otimes_R R_v$. For all i let I_i be the ideal in B' corresponding to $\mathscr{I}_i$ and let $I'_i = I_i \otimes_R R_v$. Since R_v is flat over R (it is a torsion free module over the local ring, which is principal), $I'_i \subseteq B'$ for all i, and the restriction of $\psi' \colon \mathscr{A} \times_R R_v \to \mathscr{A}' \times_R R_v$ to $(\psi')^{-1}(\operatorname{Spec} B')$ equals $\operatorname{Proj} \bigoplus_{n \ge 0} (I'_i)^n$ for all i. Since $a \in (I_1, \dots, I_r)$, there exists some i and some $b \in I_i$ such that $\|b(P)\| \ge \|a\|_v$. Let $b_1, \dots, b_s$ be generators for I_i; we may assume that $\|b_1(P)\| \ge \|b_j(P)\|$ for $j = 1, \dots, s$; therefore $\|b_1(P)\| \ge \|a\|_v$. Now if $\|b_1(Q)\| \ne \|b_1(P)\|$ then $\|b_1(P) - b_1(Q)\| \ge \|a\|_v$, and if $\|b_j(Q)\| > \|b_1(P)\|$ for some j then $\|b_j(Q) - b_j(P)\| \ge \|a\|_v$; in either case we have $d'_v(P,Q) \ge \|a\|_v$ which implies (9.8.1) since $d_v \le 1$ always. So we may assume that $\|b_1(Q)\| \ge \|b_j(Q)\|$ for all j and that $\|b_1(P)\| = \|b_1(Q)\|$. In that case the open affine $\operatorname{Spec} B'[b_2/b_1, \dots, b_s/b_1] \subseteq \mathscr{A} \times_R R_v$ contains the liftings of the sections determined by P and Q. Thus

$$\begin{aligned}
d_v(P,Q) &= \sup_{\phi \in B'[b_2/b_1,\dots,b_s/b_1]} \|\phi(P) - \phi(Q)\| \\
&= \max\left(\sup_{\phi \in B'} \|\phi(P) - \phi(Q)\|,\ \max_j \left\| \frac{b_j(P)}{b_1(P)} - \frac{b_j(Q)}{b_1(Q)} \right\| \right) \\
&= \max\Bigg(d'_v(P,Q),\ \frac{1}{\|b_1(P)\|} \max_j \|b_j(P) - b_j(Q)\| , \\
&\qquad \frac{1}{\|b_1(P)\|^2} \max_j \|b_j(Q)\| \|b_1(P) - b_1(Q)\| \Bigg) \\
&\le \frac{d'_v(P,Q)}{\|b_1(P)\|} \\
&\le \frac{d'_v(P,Q)}{\|a\|_v} .
\end{aligned}$$

Thus (9.8.1) holds. □

Lemma 9.9. *Given a nonsingular projective variety A with an extended model, a morphism* $\psi\colon \Gamma \to A$*, and an* M_k*-constant* (c_v)*, one may choose* g_1 *in Lemma 9.4 such that*

$$d_v(\psi(\theta_v(P,\mathbf{z})),\psi(\theta_v(P,\mathbf{0}))) < e^{c_v}\ . \tag{9.9.1}$$

Proof . We may assume that A is embedded into $\mathbb{P}^N$ in such a way that $\psi(F)$ is disjoint from the hyperplane at infinity. Let $x_1,\dots,x_N$ denote the coordinates on $\mathbb{A}^N \subseteq \mathbb{P}^N$. We may assume that $x_1,\dots,x_N$ were included among the ϕ_i of part (b) of Lemma 9.4; let $g_1^*,\dots,g_N^*$ be the corresponding generalized Weil functions.

First consider archimedean v; fix such a v. There exists a constant c_v' such that

$$d_v(P,Q) \le e^{c_v'} \max_i \|x_i(P) - x_i(Q)\|$$

for all $P,Q \in (A\cap\mathbb{A}^N)(\mathbb{C}_v)$. By (9.4.1), inequality (9.9.1) will hold for v if

$$g_1 \ge \max_i g_i^* + c_v' - c_v\ .$$

Now let v be non-archimedean. Taking the closure of A in $\mathbb{P}^N_R$ defines a different model for A; relative to this model we have

$$d_v(P,Q) \le \max_i \|x_i(P) - x_i(Q)\| \tag{9.9.2}$$

for all $P,Q \in (A\cap\mathbb{A}^N)(\mathbb{C}_v)$ such that the right-hand side is strictly less than one. If $\|x_i(P)\| \le 1$ for all i then this holds because the sections on $\mathbb{P}^N_{R_v}$ corresponding to P and Q do not meet the hyperplane at infinity, so the inequality follows by Remark 9.6. Otherwise we may assume $\|x_1(P)\| \ge \|x_i(P)\|$ for all i; preceding assumptions then imply that $\|x_1(P)\| = \|x_1(Q)\| > 1$ and $\|x_1(Q)\| \ge \|x_i(Q)\|$ for all i. Then the sections are contained in the open affine $\operatorname{Spec} R_v[1/x_1, x_2/x_1,\dots,x_N/x_1]$, and (9.9.2) follows from the inequality

$$\left\|\frac{x_i(P)}{x_1(P)} - \frac{x_i(Q)}{x_1(Q)}\right\| \le \frac{\max(\|x_i(P)\|\|x_1(P)-x_1(Q)\|, \|x_1(Q)\|\|x_i(P)-x_i(Q)\|)}{\|x_1(P)\|^2}$$

and from a similar inequality for $1/x_1$. Then (9.9.1) follows from (9.9.2) if

$$g_1 \ge \max_i g_i^* + c_v' - c_v\ ;$$

in this case c_v' comes from Proposition 9.8. □

Lemma 9.10. *Let* $\pi\colon \Gamma \to C$ *be a projective morphism whose generic fiber is smooth, and let* $D_1,\dots,D_r$ *be divisors on* Γ *whose restrictions to the generic fiber of* π *are prime, are smooth, and meet transversally. Let* $g_{D,1},\dots,g_{D,r}$ *be Weil functions for* $D_1,\dots,D_r$*, respectively. Then there exists effective generalized Weil functions* g_1 *and* $g_{2,1},\dots,g_{2,r}$ *on* $D_1\cap\ldots\cap D_r$ *and generalized Weil functions* $g_{3,1},\dots,g_{3,r}$ *on* Γ *whose support does not contain* $D_1\cap\ldots\cap D_r$*. Letting*

$$\Sigma_v = \{(P, \mathbf{z}) \in (D_1 \cap \ldots \cap D_r)(\mathbb{C}_v) \times \mathbb{C}_v^r \mid \|z_i\| < e^{-g_1(P)},\ i = 1, \ldots, r\} ,$$

there is also an injection $\theta_v \colon \Sigma_v \hookrightarrow \Gamma(\mathbb{C}_v)$. These objects have the following properties for all places v of k. From now on we omit v from the notation.

i. There exists an M_k-constant (c_v) such that for all i and all $(P, \mathbf{z}) \in \Sigma$ with $z_i \neq 0$,

(9.10.1) $$\left| g_{D,i}(\theta(P, \mathbf{z})) + \log \|z_i\| - g_{2,i}(P) \right| \le c_v .$$

ii. if $Q \in \Gamma$ satisfies $Q \notin \operatorname{Supp} g_{3,i}$ and $g_{D,i}(Q) > g_{3,i}(Q)$ for all i then $Q = \theta(P, \mathbf{z})$ for some $(P, \mathbf{z}) \in \Sigma$ and $P \notin \operatorname{Supp} g_{2,i}$ for any i.

iii. $\operatorname{Supp} g_{3,i} \supseteq \operatorname{Supp} g_1 \cup \operatorname{Supp} g_{2,i}$.

Moreover:

(a) For any prescribed $P_0 \in D_1 \cap \ldots \cap D_r$ with $\pi(P_0)$ suitably generic, the above choices can be made such that P_0 is not in the support of any of the above generalized Weil functions.

(b) For any prescribed generalized Weil functions $g_1', \ldots, g_m'$ on Γ with $P_0 \notin$ $\operatorname{Supp} g_j'$ *for any j, g_1 may be chosen sufficiently large such that there exists an M_k-constant (c_v) with*

(9.10.2) $$\left| g_j'(\theta(P, \mathbf{z})) - g_j'(P) \right| \le c_v$$

for all j and all $(P, \mathbf{z}) \in \Sigma$.

Proof . We assume ξ is sufficiently generic that C is regular at ξ; $\pi^{-1}(\xi)$ is regular; and $D_1, \ldots, D_r$ meet $\pi^{-1}(\xi)$ transversally, and remain prime, remain smooth, and still meet transversally there.

If v is archimedean and real let $N_v = 1$; if archimedean and complex, let $N_v = 2$; otherwise let $N_v = 0$.

Let $\Gamma \subseteq \mathbb{P}^N_C$ for some N. Let d be the relative dimension of π. Let L be a linear subspace of codimension $d - r + 1$ satisfying the conclusions of Lemma 9.1 for $X = \Gamma$ and $s = r$ and also for $X = D_1 \cap \ldots \cap D_r$ and $s = 0$. After blowing up L in $\mathbb{P}^N_C$ (and replacing Γ with its strict transform in the blow-up), the linear projection from L extends to a morphism $p \colon \Gamma \to Y := \mathbb{P}^{d-r}_C$ whose restriction to $D_1 \cap \ldots \cap D_r$ is étale at $F := p^{-1}(p(P_0)) \cap D_1 \cap \ldots \cap D_r$ and which is smooth on Γ except at a set which meets the fiber containing P_0 at only finitely many points. We also assume that L is chosen such that $\operatorname{Supp} g_i'$ is disjoint from F for all i.

Let $z_1, \ldots, z_r$ be rational functions on Γ whose principal divisors equal $D_1, \ldots, D_r$, respectively, near F. By Bertini's theorem (writing D_i as a difference of two very ample divisors) we may assume that all components of all polar and zero divisors of the z_i are distinct and have multiplicity 1 on the fiber of p containing P_0, and that their union is a normal crossings divisor on that fiber. After further blowing up Γ (but leaving a neighborhood of P_0 unchanged) these functions define a morphism $q \colon \Gamma \to \mathbb{P}^r_Y$, which is étale at $q^{-1}(q(P_0))$. Then Lemma

9.4 gives a partial section of the canonical map $\Gamma \times_Y (D_1 \cap \ldots \cap D_r) \to Y$ defined over $\coprod_v \Lambda_v(g_4)$; this can then be composed with the projection to Γ. Moreover g_4 is effective and $\operatorname{Supp} g_4$ is disjoint from F.

Next consider (b). We may assume that $g'_1, \ldots, g'_m$ are nicely defined at F. It will suffice to consider the case $m = 1$, and (by Proposition 7.10) to assume that g'_1 is of the form

$$g'_1 = \min(-\log \|\phi_1\|, \ldots, -\log \|\phi_s\|) ,$$

with all ϕ_i regular at all $P \in F$ and $\phi_1(P) \neq 0$ for all $P \in F$. We now assume g_4 was constructed so that (9.4.1) holds for $\phi_1, \ldots, \phi_s$. Replace g_4 with the generalized Weil function

$$\max(g_4, \max_i g_i^* - \log \|\phi_1\| + N_v \log 2) .$$

At non-archimedean places, (9.4.1) and the fact that $\|z\| < e^{-g_4}$ imply that the right-hand side of (9.4.1) is strictly less than $\|\phi_1(P)\|$. If $\|\phi_i(P)\| \geq \|\phi_1(P)\|$ then this implies that $\|\phi_i(\theta_v(P, \mathbf{z}))\| = \|\phi_i(P)\|$; otherwise it implies that $\|\phi_i(\theta_v(P, \mathbf{z}))\| \leq \|\phi_1(P)\|$. Thus $g'_1(\theta_v(P, \mathbf{z})) = g'_1(P)$. At archimedean places, a similar argument shows that the right-hand side of (9.4.1) is less than $\|\phi_1(P)\|/2^{N_v}$, which in turn gives

$$|g'_1(\theta_v(P, \mathbf{z})) - g'_1(P)| \leq N_v \log 2 .$$

Thus if g'_1 is nicely defined at F then (9.10.2) holds with $c_v = N_v \log 2$. (For arbitrary g'_1, (c_v) will be different.)

Now let $g_1 = g_4 + (N_v/2) \log r$; this implies that $\Upsilon_v(g_1) \subseteq \Lambda_v(g_4)$ for all v. Note that $\operatorname{Supp} g_1$ is still disjoint from F.

The above also implies (9.10.1), since $g_{D,i} + \log \|z_i\|$ is a generalized Weil function on Γ for each i.

Next consider condition (ii). Let U be an open affine subset of Γ such that $q\big|_U$ is étale, such that U contains $q^{-1}(q(P_0))$, such that $p(U)$ is contained in an open affine $V \subseteq Y$, and such that $D_i = (f_i)$ on U, for some $f_1, \ldots, f_r \in \mathscr{O}(U)$. Let $Y' = q^{-1}(Y \times [1:0:\ldots:0])$. Another application of Lemma 9.4 gives an effective generalized Weil function g_5 on Y' such that the injection $Y' \subseteq \Gamma$ extends for each v to a rigid analytic partial section $\theta'_v \colon \Lambda_v(g_5) \to \Gamma(\mathbb{C}_v)$. Moreover, there exists a generalized Weil function g_6 on Y' such that

$$\|f_1(\theta'_v(P, \mathbf{z})) - f_1(\theta'_v(P, \mathbf{0}))\| \leq e^{g_6(P)} \cdot \|\mathbf{z}\| \tag{9.10.3}$$

holds for all $(P, \mathbf{z}) \in \Lambda_v(g_5)$. Furthermore, the coordinate functions on U are bounded as in (9.4.1). And finally, $\operatorname{Supp} g_5$ and $\operatorname{Supp} g_6$ are disjoint from $q^{-1}(q(P_0))$.

By Proposition 7.4 there exists a generalized Weil function g_7 on Y such that $\left(q\big|_{Y'}\right)^* g_7 \geq g_5$ (identifying Y with $Y \times [1:0:\ldots:0]$), but $q(P_0) \notin \operatorname{Supp} g_7$. We may also assume that $\left(q\big|_{D_1 \cap \ldots \cap D_r}\right)^* g_7 \geq g_1$. Likewise there exists g_8 on Y such that $\left(q\big|_{Y'}\right)^* g_8 \geq g_6$ and $q(P_0) \notin \operatorname{Supp} g_8$. Let Y'' be the closure of

$Y'\setminus(D_1\cap\ldots\cap D_r)$. By the choice of L made previously, no irreducible component of Y'' is contained in D_1. Therefore there is a generalized Weil function g_9 on Y such that $\left(q\big|_{Y''}\right)^* g_9 \ge -\log\|f_1\|$, but $q(P_0)\notin \operatorname{Supp} g_9$.

Let $g_{10} = \max(g_8 + g_9 + N_v\log 2, g_7)$. Then

$$g_{3,1} := \max(g_{D,1}+\log\|f_1\|+p^*g_9+N_v\log 2, g_{D,1}+\log\|z_1\|+p^*g_{10}+(N_v/2)\log r, 0)$$

and

$$g_{3,i} := \max(g_{D,i} + \log\|z_i\| + p^*g_{10} + (N_v/2)\log r, 0)\ , \qquad i = 2,\ldots,r$$

satisfy the requirement of part (ii). To see this, first note that neither D_i nor $\operatorname{Supp}(-\log\|z_i\|)$ contain any fiber of p, so $\operatorname{Supp} g_{3,i} \supseteq p^{-1}(\operatorname{Supp} g_{10})$ for all i. Now suppose $Q\in\Gamma(\mathbb{C}_v)$ for some v, and $Q\notin\operatorname{Supp} g_{3,i}$ for any i. If $q(Q)\notin \Lambda_v(g_{10})$ then $q(Q)\notin\Upsilon_v(g_{10}+(N_v/2)\log r)$ and therefore, trivially,

$$-\log\|z_i(Q)\|_v \le g_{10}(p(Q)) + (N_v/2)\log r$$

for some i; thus $g_{3,i}(Q)\ge g_{D,i}(Q)$. Otherwise $Q=\theta'_v(P,\mathbf{z})$ for some $P\in Y'$ and, by (9.10.3),

$$\|f_1(\theta'_v(P,\mathbf{z})) - f_1(\theta'_v(P,\mathbf{0}))\| \le e^{g_8(q(P))-g_{10}(q(P))} \le e^{-g_9(q(P))-N_v\log 2}$$

and therefore

$$\|f_1(Q)\| < e^{-g_9(p(P))}\Big/ 2^{N_v}$$

if and only if Q lies in the image of the θ_v constructed earlier. In particular, if Q is outside the image of θ_v then $-\log\|f_1(Q)\|\le g_9(p(Q))+N_v\log 2$, which again implies $g_{3,i}(Q)\ge g_{D,i}(Q)$. Thus (ii) holds.

Finally, we may increase $g_{3,i}$ such that $\operatorname{Supp} g_{3,i}\supseteq \operatorname{Supp} g_1\cup\operatorname{Supp} g_{2,i}$ for all i. This gives (iii). □

A divisor with **simple normal crossings** is a divisor whose components are smooth, meeting transversally. Here we allow the components to be multiple.

Proposition 9.11. *Let $\pi\colon\Gamma\to C$ be a projective morphism to a projective variety over k, let D be a Cartier divisor on Γ which is effective on the generic fiber, and let g_D be a Weil function for D. Let $\mathscr{L}_1,\ldots,\mathscr{L}_L$ be line sheaves on Γ and let $c_{1,v},\ldots,c_{L,v}$ and $c'_{1,v},\ldots,c'_{L,v}$ be constants such that for all i: $0<c_{i,v}\le c'_{i,v}$ for all v, $c_{i,v}=c'_{i,v}$ for almost all v, and $c_{i,v}<c'_{i,v}$ if v is archimedean. Then there exists a generalized Weil function g on C with the following properties. For each $Q\in\Gamma(\mathbb{C}_v)$ there exists an integer $r\ge 0$ and a power series map $\phi\colon\mathbb{D}^r\to\pi^{-1}(\xi)(\mathbb{C}_v)$ such that*

i. the image of ϕ contains Q;
ii. there exist positive integers $f_1,\ldots,f_r$ (depending on ϕ) such that ϕ^*D equals the principal divisor $(z_1^{f_1}\ldots z_r^{f_r})$;
iii. for all $\mathbf{z}\in\mathbb{D}^r$ with $z_1\ldots z_r\ne 0$,

$$\left|g_D(\phi(\mathbf{z})) + f_1\log\|z_1\| + \ldots + f_r\log\|z_r\|\right| \le g(\xi)\ ;$$

and

iv. *for all* i *and all* v *there exist sections* $\gamma_i \in \Gamma(\mathbb{D}^r, \phi^*\mathscr{L}_i)$ *whose norms satisfy*

$$c_{i,v} \le \|\gamma_i\| \le c'_{i,v}$$

on all of $\mathbb{D}^r$.

Moreover, as ϕ *varies, the tuple* $(f_1, \ldots, f_r)$ *takes on only finitely many values. If in addition the generic fiber of* π *is smooth, if the restriction of* D *to that generic fiber has simple normal crossings, and if* ξ *is suitably generic, then the image of* ϕ *crosses* D *transversally.*

Proof. First, we immediately reduce to the case where the generic fiber of π is smooth, and the restriction of D to this fiber is a divisor with simple normal crossings.

Let $D_1, \ldots, D_r$ be the components of D. Applying the lemma to $D_1 \cap \ldots \cap D_r$ with various P_0 gives various $g_{1,j}$, $g_{2,i,j}$, and $g_{3,i,j}$ such that $\cap_j \cup_i \operatorname{Supp} g_{3,i,j}$ does not meet $D_1 \cap \ldots \cap D_r$ on the generic fiber of π. Then the generalized Weil function

$$\min_j \max_i \min(g_{3,i,j}, g_{D,1}, g_{D,2}, \ldots, g_{D,r})$$

has no support along the generic fiber, so by Proposition 7.4 it is bounded from above by some generalized Weil function g' coming from C. Therefore if $g_{D,1}, \ldots, g_{D,r}$ are all greater than or equal to g' at some point Q, then there exists some j and some $P \in D_1 \cap \ldots \cap D_r$ as in condition (ii) of Lemma 9.10. By condition (iii) of Lemma 9.10, $g_1 + g_{2,i}$ is also bounded from above (as well as from below) at P for each i; regarding the map ϕ as being a family of polydiscs and dilating the polydisc attached to P then gives the map required for the proposition.

If, however, some of $g_{D,1}, \ldots, g_{D,r}$ is less than g', one can apply Lemma 9.10 with a smaller value of r (taking care by (9.10.2) that the discarded $g_{D,i}$ remain small in that polydisc).

Part (iv) of the proposition can be guaranteed by fixing a local generator for each $\mathscr{L}_i$ at P_0 and applying (9.10.2) to the logarithm of its metric. ☐

Proposition 9.12. *Given a nonsingular projective variety* A *with an extended model, a morphism* $\psi\colon \Gamma \to A$, *and an* M_k*-constant* (c_v), *one may choose* g, r, *and* ϕ *in Proposition 9.11 so that*

$$d_v(\psi(\phi(\mathbf{z})), \psi(\phi(\mathbf{0}))) < e^{c_v}$$

for all $\mathbf{z} \in \mathbb{D}^r$.

Proof. This follows immediately from Lemma 9.9 by choosing g_1 appropriately in Lemma 9.10. ☐

10. Models and complexes

This section defines models over R for A, $\overline{A}$, X, X_i, and the line sheaves $M_{-\epsilon,\mathbf{s}}$ to models over Spec R. Extending $L_{-\epsilon,\mathbf{s}}$ would lead to technical difficulties, however, so instead of doing that we spend the bulk of this section working around that difficulty.

To begin, we choose a model $\overline{\mathscr{A}}$ for $\overline{A}$, as follows. Fix a model $\mathscr{A}_0$ for A_0 such that the line sheaf $\mathscr{O}(L_0)$ and the line sheaves $\mathscr{M}_m$ of Sect. 2 extend as line sheaves to $\mathscr{A}_0$. Then the construction (2.3) gives a model $\overline{\mathscr{A}}$ for $\overline{A}$. Again let $[\infty]_m$ and $[0]_m$ correspond to the projections (2.5); they are the closures in $\overline{\mathscr{A}}$ of the corresponding divisors in $\overline{A}$. Let $\mathscr{A}$ denote the complement of their union; it is therefore a model for A.

Let $\mathscr{X}$ be the closure of X in $\mathscr{A}$; it is a model for X. Of course, this model is not necessarily the same as the model $\mathscr{X}$ chosen in Theorem 0.2. To fix this discrepancy, the extended line sheaves $\mathscr{M}_m$ can be tensored with some fractional ideal in R; then finitely many modified models obtained from these modified $\mathscr{M}_m$ will have the property that the union of their sets of integral points will contain the set of integral points $\mathscr{X}(R_S)$ from Theorem 0.2. Or, one can also fix this discrepancy by enlarging the set S of exceptional primes. In either case, we assume from now on that $\mathscr{X}$ is the closure of X in the model $\mathscr{A}$ constructed above, and that $\overline{\mathscr{X}}$ is the closure of $\bar{X}$ in $\overline{\mathscr{A}}$.

Now let $\mathscr{V}_0$ be the model for A_0^n constructed as in ([V 4], Sect. 13); we may assume $\mathscr{V}_0$ dominates $\mathscr{A}_0^n$. Let

$$\mathscr{V}_1 = (\overline{\mathscr{A}} \times_{\operatorname{Spec} R} \dots \times_{\operatorname{Spec} R} \overline{\mathscr{A}}) \times_{\mathscr{A}_0^n} \mathscr{V}_0 .$$

Let $X_1, \dots, X_n$ be closed subvarieties of X satisfying (4.5.1–4.5.4). Let $\bar{X}_i$ be the closure of X_i in X, and let $\mathscr{X}_i$ and $\overline{\mathscr{X}}_i$ be the closures of X_i and $\bar{X}_i$ in $\mathscr{A}$ and $\overline{\mathscr{A}}$, respectively. Let $\mathscr{V}$ be the closure of $\prod \bar{X}_i$ in $\mathscr{V}_1$.

Since the $\mathscr{M}_m$ extend as line sheaves to $\mathscr{A}_0$, the extended divisors $[0]_m$ and $[\infty]_m$ on $\overline{\mathscr{A}}$ are Cartier. These extensions, together with (5.4.1) and ([V 4], Lemma 13.2), define an extension of $M_{\delta,\mathbf{s}}$ to a line sheaf on $\mathscr{V}$. Let ϵ be as in Proposition 6.2.

A logical next step might then be to define a model $\mathscr{W}_{\mathbf{s}}$ for the scheme $W_{\mathbf{s}}$ defined in Sect. 3. One would then extend $L_{-\epsilon,\mathbf{s}}$ to that model and construct a suitably small section $\gamma \in \Gamma(\mathscr{W}_{\mathbf{s}}, dL_{-\epsilon,\mathbf{s}})$. However, this presents a number of difficulties since the natural extensions of the divisors $Q^{ij}_{\mathbf{s},m}$ to $\mathscr{W}_{\mathbf{s}}$ may contain fiber components over Spec R. Instead, we will identify $\Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ and $\Gamma(W_{\mathbf{s}}, dL_{-\epsilon,\mathbf{s}})$ with R-submodules of $\Gamma(W_{\mathbf{s}}, d\pi_{\mathbf{s}}^* M_{-\epsilon,\mathbf{s}})$ via $\pi_{\mathbf{s}}$ and (3.6), and work with sections

$$\gamma \in \Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}}) \cap \Gamma(W_{\mathbf{s}}, dL_{-\epsilon,\mathbf{s}}) . \tag{10.1}$$

Such a section will be constructed in Sect. 12, for sufficiently large d. This will be done by applying geometry of numbers arguments to various terms in a modified Faltings complex. This section and the next develop the machinery necessary for dealing with the vanishing conditions in that complex.

Let ℓ be an integer such that $\mathscr{O}(\ell L)$ and $\mathscr{O}(\ell L_0)$ are generated by their global sections over $\bar{A}$ and A_0, respectively. Then the sheaves

$$\mathscr{O}\left(\ell\sum_{i<j}(\rho^n)^*(s_i\cdot \mathrm{pr}_i + s_j\cdot \mathrm{pr}_j)^* L_0 + \ell(n-1)\sum s_i^2\,\mathrm{pr}_i^* L\right) \tag{10.2}$$

and

(10.3)

$$\mathscr{O}\left(\ell\sum_{i<j}(\rho^n)^*(s_i\cdot \mathrm{pr}_i - s_j\cdot \mathrm{pr}_j)^* L_0\right)\otimes\mathscr{O}\left(\ell\sum_{i<j}(\rho^n)^*(s_i\cdot \mathrm{pr}_i + s_j\cdot \mathrm{pr}_j)^* L_0\right) \cong \mathscr{O}\left(2\ell(n-1)\sum s_i^2\,\mathrm{pr}_i^*\,\rho^* L_0\right)$$

are generated by their global sections over $\bar{A}^n$, and remain so when pulled back to $W_{\mathbf{s}}$. Letting

$$L' = 2(n-1)\rho^* L_0 + (n-1)L_1 + (n-1-\epsilon)L$$

and

$$L'' = 4(n-1)\rho^* L_0 + (n-1)L_1 + (n-1-\epsilon)L\ ,$$

we can create a Faltings complex as in ([V 4], Sect. 9):

(10.4)

$$0\to\Gamma(\mathscr{V}^{\cdot}, dM_{-\epsilon,\mathbf{s}})\xrightarrow{\alpha}\Gamma\left(\mathscr{V}^{\cdot}, d\sum s_i^2\,\mathrm{pr}_i^* L'\right)^a\xrightarrow{\beta}\Gamma\left(\mathscr{V}^{\cdot}, d\sum s_i^2\,\mathrm{pr}_i^* L''\right)^b\ .$$

Here the maps α and β are defined by tensoring with sections of the sheaves in (10.2) and (10.3), respectively. Also a and b are independent of d and $\mathbf{s}$.

To construct the desired section, then, we use the Faltings complex to construct a suitable set of sections of $\Gamma(\mathscr{V}^{\cdot}, dM_{-\epsilon,\mathbf{s}})$, and then show that one can obtain a section of $\mathscr{O}(dL_{-\epsilon,\mathbf{s}})$ by imposing certain vanishing conditions on the sections of $\mathscr{O}(dM_{-\epsilon,\mathbf{s}})$. These vanishing conditions will be treated in the next section; the remainder of this section compares the modules $\Gamma(\mathscr{V}^{\cdot}, dM_{-\epsilon,\mathbf{s}})$ and $\Gamma(W_{\mathbf{s}}, dL_{-\epsilon,\mathbf{s}})$. First of all, the difference between the two divisor classes in question is an effective divisor, so the module $\Gamma(W_{\mathbf{s}}, dL_{-\epsilon,\mathbf{s}})$ can be regarded as a submodule of $\Gamma(W_{\mathbf{s}}, dM_{-\epsilon,\mathbf{s}})$.

We now compare their metrics. This will be done in the slightly more general context of singular metrics on $\mathscr{O}(dM_{-\epsilon,\mathbf{s}})$. These will be metrics of the form

$$\|\cdot\|' = \frac{\|\cdot\|}{\prod_{j=1}^J\left(\sum_{i\in I_j}\exp(-e_{ij}\,\mathrm{pr}_i^*\,g_{ij})\right)} \tag{10.5}$$

Here each I_j is a nonempty subset of $\{1,\dots,n\}$, each g_{ij} is some effective Green function on $\bar{A}$, and each e_{ij} is a positive integer. Moreover, I_j, g_{ij}, and e_{ij}/ds_i^2

are independent of d and $\mathbf{s}$. In the case at hand, we use the Green functions (2.8) and the inequality

$$\frac{(\alpha+\beta)^2}{(1+\alpha)^2(1+\beta)^2} \gg\ll \left(\frac{1}{1+\alpha}+\frac{1}{1+\beta}\right)^2\left(\frac{\alpha}{1+\alpha}+\frac{\beta}{1+\beta}\right)^2 \tag{10.6}$$

to find a singular metric of the form (10.5) which is equivalent to the metric on $\mathscr{O}(dL_{-\epsilon,\underline{\mathbf{s}}})$. Here all constants are of the form $\exp\bigl(cd\sum s_i^2\bigr)$, with c depending only on A. In particular note that if $\gamma \in \Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ lies in $\Gamma(\mathscr{V}, dL_{-\epsilon,\mathbf{s}})$ then its singular metric is bounded. This immediately gives the following corollary of Proposition 6.2:

Proposition 10.7. *Let $\Gamma'(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ be the subset of $\Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ consisting of sections γ whose norm $\|\cdot\|'$ is bounded. Then there exist c and ϵ as in Proposition 6.2 such that for all tuples* $\mathbf{s}$*,*

$$\dim_k \Gamma'(\mathscr{V}, dM_{-\epsilon,\mathbf{s}}) \geq cd^{\sum \dim X_i} \prod_{i=1}^n s_i^{2\dim X_i}$$

for all sufficiently large d (depending on $\mathbf{s}$*).*

We now compare these singular metrics with the original one.

Lemma 10.8. *Let $v \in S$, let $I_1, \dots, I_J$ be as above, let ℓ_{ijk} ($j = 1, \dots, J$, $i \in I_j$, $k = 1, \dots, r$) be nonnegative integers, let $\ell = \max \ell_{ijk}$, and let*

$$\Psi = \prod_{j=1}^J \sum_{i\in I_j} \prod_{k=1}^r \|z_k\|^{\ell_{ijk}} . \tag{10.8.1}$$

Then every power series $f : \mathbb{D}^r \to \mathbb{C}_v$ such that $\|f(z_1, \dots, z_r)\| \big/ \Psi$ is bounded satisfies

$$\sup_{\mathbb{D}^r} \frac{\|f\|}{\Psi} \leq c^\ell \sup_{\mathbb{D}^r} \|f\|$$

for some constant c depending only on r and $I_1, \dots, I_J$.

Proof . Let $Q \in \mathbb{D}^r$ and fix $\rho \in (1/2, 1)$ with $\rho \geq \max \|z_k(Q)\|$. It will suffice to show that

$$\frac{\|f(Q)\|}{\Psi(Q)} \leq c^\ell \sup_{(\partial\mathbb{D}_\rho)^r} \frac{\|f\|}{\Psi} . \tag{10.8.2}$$

After rearranging coordinates, we may assume that $\|z_k(Q)\| < \rho$ for $k = 1, \dots, s$ and $\|z_k(Q)\| = \rho$ for $k = s+1, \dots, r$. We prove (10.8.2) by induction on s. If $s = 0$ then we are done. Otherwise, find integers $m_k \geq 0$, numbers $\omega_k \in \mathbb{C}_v$ with $\rho \leq \|\omega_k\| \leq 2$, and $\xi \in \mathbb{D}$ such that

$$z_k(Q) = \omega_k \xi^{m_k}$$

for $k = 1, \dots, r$. Note that the restrictions on $\|\omega_k\|$ imply that $m_k > 0$ if and only if $k \le s$. Letting ξ vary defines a curve in $\overline{\mathbb{D}}_\rho^r$ on which

$$\Psi \gg\ll \prod_{j=1}^{J} \# I_j \cdot \|\xi\|^m$$

for some integer $m \ge 0$; moreover the constants are of the form c^ℓ (even if $\ell = 0$). By the maximum principle applied to $f(\omega_1 \xi_1^{m_1}, \dots, \omega_k \xi_k^{m_k})/\xi^m$, we may move Q to a point with one more of its coordinates lying on $\partial\mathbb{D}$, but affecting $\|f\| \big/ \Psi$ by at most a factor of c^ℓ. This then gives (10.8.2) by induction. □

Lemma 10.9. *Let $I_1, \dots, I_J$ and g_{ij} ($j \in I_i$) be as above. For $i = 1, \dots, n$ let $\pi_i : \Gamma_i \to C_i$ be morphisms of projective varieties, and let $\psi_i : \Gamma_i \to \bar{A}$ be morphisms. Then there exist generalized Green functions $g_1, \dots, g_n$ on $C_1, \dots, C_n$, respectively, and an M_k-constant (c_v) with the following properties. For each v, for each $\xi_1, \dots, \xi_n$ in $C_1, \dots, C_n$ with $\xi_i \notin \operatorname{Supp} g_i$ for all i, and for each $P \in \prod \bar{X}_i(\mathbb{C}_v)$ (where $\bar{X}_i := \pi_i^{-1}(\xi_i)$) there exists an integer $r \ge 0$ and a power series map $\phi\colon \mathbb{D}^r \to \prod \bar{X}_i(\mathbb{C}_v)$ such that:*

i. the image of ϕ contains P;
ii. all $\phi^* \operatorname{pr}_i^* \psi_i^* g_{ij}$ are of the form $\tau - \sum_k f_{ijk} \log \|z_k\|$ with $f_{ijk} \in \mathbb{Z}$ and $|\tau| \le g_i(\xi_i)$ on $\mathbb{D}^r$; and
iii. For all rational δ and all tuples $\mathbf{s}$ of positive rational numbers, let $M_{\delta,\mathbf{s}}$ be the $\mathbb{Q}$-divisor class on $\prod \Gamma_i$ defined by $\psi_1, \dots, \psi_n$ and (3.3). Then for sufficiently divisible $d > 0$, $\phi^* \mathscr{O}(dM_{-\epsilon,\mathbf{s}})$ has a power series section on $\mathbb{D}^r$ whose metric is bounded from below by $\exp(-c_v d \sum s_i^2)$ and from above by 1.

Moreover, as ϕ varies, the tuple (f_{ijk}) takes on only finitely many values. If in addition the generic fibers of $\pi_1, \dots, \pi_n$ are smooth, if the restrictions of all $\psi_i^* \operatorname{Supp}\bigl(\sum_j g_{ij}\bigr)$ to those generic fibers have simple normal crossings, and if $\xi_1, \dots, \xi_n$ are suitably generic, then the images of ϕ_i cross the above divisors transversally.

Proof . By Proposition 9.11, for $i = 1, \dots, n$ there exist integers $r_i \ge 0$, power series maps $\phi_i : \mathbb{D}^{r_i} \to \bar{X}_i(\mathbb{C}_v)$, and generalized Weil functions g_i on C_i such that:

(a) the image of ϕ_i contains P_i for all i;
(b) all $\phi_i^* g_{ij}$ are of the form $\tau - \sum_k f_{ijk} \log \|z_k\|$ with $f_{ijk} \in \mathbb{Z}$ and $|\tau| \le g_i(\xi_i)$ on $\mathbb{D}^{r_i}$; and
(c) for all i the sheaves $\phi_i^* \rho^* \mathscr{O}(L_0)$ and $\phi_i^* \mathscr{O}(L_1)$ have sections on $\mathbb{D}^{r_i}$ whose metric lies in the interval $(1/2, 1)$ if v is archimedean or is identically equal to 1 if v is non-archimedean.

Indeed, (a) follows from condition (i) of Proposition 9.11, (b) follows from (iii), and (c) follows from (iv). Moreover, for any prescribed extended model of A_0 and for any prescribed $c > 0$, r_i and ϕ_i can be chosen such that

$$d_v(\rho_i(\psi_i(\phi_i(\mathbf{z}))), \rho_i(\psi_i(\phi_i(\mathbf{0})))) < c$$

for all $\mathbf{z} \in \mathbb{D}^{r_i}$ and all i. This holds by Proposition 9.12.

The above maps ϕ_i combine to give a map $\phi\colon \mathbb{D}^r \to \prod \bar{X}_i(\mathbb{C}_v)$ (with $r = \sum r_i$). This map automatically satisfies (i) and (ii) above. To show (iii), it suffices to show that for all i and j with $1 \le i < j \le n$, the Poincaré sheaf $\phi^*\mathscr{P}_{ij}$ has a section on $\mathbb{D}^r$ whose metric lies in the interval $(1/2, 1)$ if v is archimedean or is identically 1 if v is non-archimedean. To see this, fix any extended model for A_0 and let d_v^n denote the corresponding distance on $A_0(\mathbb{C}_v)^n$. If v is archimedean, then a compactness argument shows that there exists $c > 0$ such that for all $P \in A_0(\mathbb{C}_v)^n$ there exists a rational section of $\mathscr{P}_{ij}$ whose metric on the set

$$\{Q \in A_0(\mathbb{C}_v)^n \mid d_v^n(P, Q) < c\}$$

lies in the range $(1/2, 2)$. If v is non-archimedean, an analogous constant $c > 0$ exists by Proposition 9.8: if we replace d_v^n with the distance associated to the model $\mathscr{W}_0$, then we may take $c = 1$. Now (iii) is immediate, by (3.3).

Note that the tuple $\mathbf{s}$ only occurs in (iii). □

Proposition 10.10. *Let $\gamma \in \Gamma(\mathscr{V}^\cdot, dM_{-\epsilon,\mathbf{s}})$ and let $v \in S$. If $\|\gamma\|'_v$ is bounded, then*

$$-\log \|\gamma\|'_{\sup,v} \ge -\log \|\gamma\|_{\sup,v} + cd \sum s_i^2 h(X_i) + c'd \sum s_i^2 .$$

Here c and c' depend only on A, X, $\dim X_i$ *($i = 1, \dots, n$), and the bounds on* $\deg X_i$ *($i = 1, \dots, n$).*

Proof . For this proof, an **admissible multiplicative constant** is a constant of the form $\exp(cd \sum s_i^2 h(X_i) + c'd \sum s_i^2)$. Here c and c' depend only on A, X, $\dim X_i$ for all i, and the bounds on the degrees of X_i, but not on $\mathbf{s}$.

Let $P = (P_1, \dots, P_n)$ be a point on $\prod \bar{X}_i(\mathbb{C}_v)$ where $\|\gamma\|'$ comes close to its maximum, and let $\phi\colon \mathbb{D}^r \to \prod \bar{X}_i(\mathbb{C}_v)$ be as in Lemma 10.9. Here C_i is an appropriate Chow variety, Γ_i is the corresponding family of varieties, and ξ_i is the point on C_i corresponding to $\bar{X}_i$. If $\xi_i \in \operatorname{Supp} g_i$, then we proceed by Noetherian induction.

Let

$$\Psi = \prod_j \Bigl(\sum_{i \in I_j} \exp(-e_{ij}\,\phi^* \operatorname{pr}_i^* g_{ij})\Bigr) .$$

By condition (iii) of Lemma 10.9, it suffices to show that if f is a power series on $\mathbb{D}^r$ satisfying

$$\|f\| \ll \Psi ,$$

then

$$\sup_{\mathbb{D}^r} \frac{\|f\|}{\Psi} \le \sup_{\mathbb{D}^r} \|f\| \tag{10.10.1}$$

up to an admissible multiplicative constant. By condition (ii), replacing Ψ with the expression (10.8.1) changes it by at most an admissible multiplicative constant; after doing so (10.10.1) follows immediately from Lemma 10.8. □

11. Vanishing conditions

This section describes how to use derivative conditions to define the set $\Gamma'(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ of sections $\gamma \in \Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ whose metric (10.5) is bounded.

For this section let L_2 be the union of the supports of the Green functions $\mathrm{pr}_i^* g_{ij}$ in (10.5). By an embedded resolution of singularities we may replace each $\bar{X}_i$ with a smooth proper $\widetilde{X}_i$ such that the support of L_2 on $\prod \widetilde{X}_i$ is a divisor with simple normal crossings; i.e., components of the support are smooth and cross transversally.

Definition 11.1. *Let $\mathbb{R}_{\ge 0}$ denote the set of nonnegative real numbers. If $\mathbf{e}, \mathbf{f} \in \mathbb{R}^N_{\ge 0}$, then we say $\mathbf{e} \le \mathbf{f}$ if the inequality holds for all components. Then:*

(a) A **leading set** *in $\mathbb{R}^N_{\ge 0}$ is a subset σ such that $\mathbf{e} \in \sigma$ and $\mathbf{f} \le \mathbf{e}$ implies $\mathbf{f} \in \sigma$.*
(b) A leading set is **boundedly generated** *if it can be written as a union of sets $\mathrm{pr}_I^* \sigma_I$, as I varies over subsets of $\{1, \dots, N\}$, where each σ_I is a bounded subset of $\mathbb{R}^{\#I}_{\ge 0}$. (Here pr_I denotes the projection to $\mathbb{R}^{\#I}_{\ge 0}$ obtained by throwing out coordinates not in I.)*
(c) A leading set σ is **boundedly generated of multiweight** $\le (d_1, \dots, d_N)$ *if moreover each set σ_I as above is a subset of $\prod_{i \in I} [0, d_i]$.*

This is a big generalization of the notion of the index. The situation here is complicated by the fact that the sheaf defined by $\|\cdot\|'$ is a tensor product of sheaves defined by the index, but with varying sets of multiplicities and involving different subsets of the variables. This happens because the restrictions of $\mathrm{pr}_i^*[0]_m$ and $\mathrm{pr}_i^*[\infty]_m$ to $\widetilde{X}_i$ can involve several components of L_2 with different multiplicities.

In this section and the next, such leading sets will be used to indicate the required vanishing of the corresponding power series coefficients. The reason for the σ_I is that one can ensure the vanishing of all coefficients in $\mathrm{pr}_I^* \sigma_I$ by considering finitely many partial derivatives along $\bigcap_{i \in I} \{z_i = 0\}$. The orders of these derivatives also need to be bounded; hence the notion of multiweight.

Lemma 11.2. *Fix an embedding of k into $\mathbb{C}$. Then there exists a constant c with the following property. Let $U \subseteq \prod \widetilde{X}_i(\mathbb{C})$ be a coordinated open subset such that the support of L_2 is contained in the union of the coordinate hyperplanes, and such that all coordinates z_k are pull-backs of local coordinates from $\widetilde{X}_{i(k)}$. Let γ_0 be a local generator of $\mathscr{O}(dM_{-\epsilon,\mathbf{s}})\big|_U$. Then there exists a leading subset $\sigma \subseteq \mathbb{R}^N_{\ge 0}$, boundedly generated of multiweight $\le (cds^2_{i(1)}, \dots, cds^2_{i(N)})$, such that a local section $\gamma \in \Gamma(U, dM_{-\epsilon,\mathbf{s}})$ lies in $\Gamma'(U, dM_{-\epsilon,\mathbf{s}})$ if and only if*

$$(11.2.1)\qquad \left(\frac{\partial}{\partial z_1}\right)^{r_1}\cdots\left(\frac{\partial}{\partial z_N}\right)^{r_N}\left(\frac{\gamma}{\gamma_0}\right)(0,\dots,0)=0$$

for all $(r_1,\dots,r_N)\in\sigma\cap\mathbb{N}^N$.

Proof . Up to a bounded function, each g_{ij} in (10.5) satisfies

$$g_{ij}=\sum_{k=1}^{N}-f_{ijk}\log\|z_k\|$$

for some integers f_{ijk}. Let σ be the complement of the set of all $(r_1,\dots,r_N)\in\mathbb{R}^N_{\ge 0}$ such that

$$(11.2.2)\qquad \prod_{k=1}^{N}t_k^{r_k}\ll\prod_j\max_{i\in I_j}\left(\prod_{k=1}^{N}t_k^{e_{ij}f_{ijk}}\right)$$

for all $(t_1,\dots,t_N)\in[0,1]^N$.

With this σ, if γ satisfies (11.2.1) then $\gamma\in\Gamma'(U,dM_{-\epsilon,\mathbf{s}})$. Conversely, if $\mathbf{r}\in\sigma$ then the derivative (11.2.1) must vanish, by an appropriate use of Cauchy's inequalities.

Next consider the question of the boundedness of σ. We show that σ is boundedly generated of multiweight $\le(d_1,\dots,d_N)$, where

$$d_k=\sum_j\max_{i\in I_j}e_{ij}f_{ijk}\ .$$

Indeed, we show that if $\mathbf{r}\in\sigma$ and $r_1\ge d_1$, then $\mathbb{N}\times(r_2,\dots,r_N)\subseteq\sigma$ (and likewise for the other coordinates of $\mathbf{r}$). This follows from the fact that if $r_1\ge d_1$ then (11.2.2) holds for all $(t_1,\dots,t_N)$ if and only if it holds for all $(1,t_2,\dots,t_N)$.

To finish the proof, first note that if $f_{ijk}\ne 0$ then $i=i(k)$. Since e_{ij} is a fixed multiple of ds_i^2 as d and $\mathbf{s}$ vary, it follows that all d_k are fixed multiples of $ds_{i(k)}$. Moreover, as U varies, the integers f_{ijk} have only finitely many possibilities, so this multiple may be taken independent of U. Likewise, it can be taken independent of the $\widetilde{X}_i$ if their degrees are bounded. □

The conditions (11.2.1) are equivalent to the vanishing of certain partial derivatives along the components of L_2. The remainder of this section proves uniform upper bounds on the heights of those vanishing conditions as the X_i vary.

In order to obtain these uniformities, we assume that the $\widetilde{X}_i$ used above have been defined in the following manner. As in ([V 4], Sect. 16) let $\mathscr{C}_i$ be a projective arithmetic scheme over Spec R, let C_i denote its generic fiber over Spec k, and let Γ_i be a family of projective varieties over $\mathscr{C}_i$. We perform an embedded resolution of $L_{2,i}:=\sum_j\operatorname{Supp}g_{ij}$ on the generic fiber of Γ_i over C_i, and extend Γ_i so as to dominate the original Γ_i. We may also assume that each irreducible component D_j^0 of $L_{2,i}$ extends to a Cartier divisor on Γ_i. For each D_j^0 choose metrics at all archimedean places (i.e., choose Green forms for the cycles D_j^0).

For $\xi_i \in C_i$ let $\widetilde{X}_i$ be the fiber in Γ_i and let $\widetilde{\mathscr{X}_i}$ be the closure of $\widetilde{X}_i$ in Γ_i. (Note that $\widetilde{\mathscr{X}_i}$ may be singular.) Then for $\xi_1, \dots, \xi_n$ in nonempty Zariski-open subsets of C_i, $\widetilde{X}_i$ will be irreducible and smooth, and the restriction of $L_{2,i}$ to $\widetilde{X}_i$ will still be a divisor with simple normal crossings with the D_j^0 still irreducible. Outside of these Zariski-open subsets of C_i, the situation can be handled by Noetherian induction.

In this situation we have partial derivatives, as follows. Let γ be a global section on $\prod \widetilde{X}_i$ of $\mathscr{O}(dM_{-\epsilon,\mathbf{s}})$, let $D_1, \dots, D_J$ be some subset of the $\operatorname{pr}_i^* D_j^0$, and let $\ell = (\ell_1, \dots, \ell_J)$ be a tuple of nonnegative integers describing a leading term. Then we have a global section

$$D_\ell \gamma(D_1 \cap \ldots \cap D_J) \in \Gamma\left(\prod \widetilde{X}_i \cap \bigcap D_j, \mathscr{O}(dM_{-\epsilon,\mathbf{s}} - \ell_1 D_1 - \ldots - \ell_J D_J)\big|_{\bigcap D_j}\right) .$$

The chosen metrics on D_j determine metrics $\|\cdot\|_v$ on $\mathscr{O}(D_j)$ for archimedean v; for non-archimedean v assign metrics $\|\cdot\|_v$ by Definition 8.5. Proposition 8.6 then determines Weil functions $g_{D,1}, \dots, g_{D,J}$ for $D_1, \dots, D_J$. Similarly, the extension of $M_{-\epsilon,\mathbf{s}}$ to $\mathscr{V}$ defines metrics $\|\cdot\|_v$ on $\mathscr{O}(dM_{-\epsilon,\mathbf{s}})$ for all v. The product of these metrics then gives a metric on $\mathscr{O}(dM_{-\epsilon,\mathbf{s}} - \ell_1 D_1 - \ldots - \ell_J D_J)$ at all v.

We now determine upper bounds for the heights of these partial derivatives.

Lemma 11.3. *There exist effective generalized Weil functions $g_1, \dots, g_J$ on $\prod C_i$ and an M_k-constant (c_v) such that the inequality*

$$\begin{aligned} &-\log \|D_\ell \gamma(D_1 \cap \ldots \cap D_J)\|_{\sup,v} \\ &\qquad \geq -\log \|\gamma\|_{\sup,v} - \ell_1 g_{1,v}(\xi) - \ldots - \ell_J g_{J,v}(\xi) - c_v d \sum s_i^2 \end{aligned}$$

holds for all tuples $\mathbf{s}$, all $\ell_1, \dots, \ell_J \in \mathbb{N}$, all sufficiently divisible d, and all places v of k. Moreover, if each D_j is the pull-back of a metrized divisor on $\Gamma_{i(j)}$, then each g_j can be taken as the pull-back of a generalized Weil function on $C_{i(j)}$.

Proof . Let v be a place and let $P \in \prod \widetilde{X}_i(\mathbb{C}_v)$ be a point where the supremum on the left is attained. Applying Lemma 10.9 with the g_{ij}'s corresponding to $g_{D,1}, \dots, g_{D,J}$ (appropriately collated) gives an integer $r \geq 0$ and a power series map $\phi \colon \mathbb{D}^r \to \prod \widetilde{X}_i(\mathbb{C}_v)$ satisfying the conditions of the lemma. In particular, divide γ by the section in condition (iii). This gives a power series f on $\mathbb{D}^r$ which satisfies

$$-\log \|f\|_{\sup} \geq -\log \|\gamma\|_{\sup,v} - c_v d \sum s_i^2 \ . \tag{11.3.1}$$

For each j, the Cartier divisor $\phi^* D_j$ is some multiple μ_k of some coordinate hyperplane $z_k = 0$; thus $z_k^{\mu_k}$ is a local generator of $\phi^* \mathscr{O}(-D_j)$. Recall that $g_{D,j}$ is defined as minus the logarithm of the metric of the section $1 \in \mathscr{O} \hookrightarrow \mathscr{O}(D_j)$. Therefore the metric $\|\cdot\|_j$ of $z_k \in \mathscr{O}(-\phi^* D_j)$ equals $-\mu_k \log \|z_k\| - \phi^* g_{D,j}$. By condition (ii) of the lemma it follows that

$$\left| -\log \|z_k\|_j \right| \leq g_j(\xi) \ , \tag{11.3.2}$$

where g_j is a generalized Weil function on $C_1 \times \ldots \times C_n$. Moreover, let i be such that D_j is the pull-back of a divisor on $\widetilde{X}_i$; then g_j is the pull-back of a generalized Weil function on C_i. We may decrease r so that $r = J$; since $P \in D_1 \cap \ldots \cap D_J$, it follows that $\phi^{-1}(P) = \{\mathbf{0}\}$. We also assume that $k = j$ above.

The inequality (11.3.2) holds in particular at $\phi^{-1}(P)$. Writing

$$f(\mathbf{z}) = \sum_{(i)\ge 0} a_{(i)}\mathbf{z}^{(i)}$$

with $a_{(i)} \in \mathbb{C}_v$, it follows that the derivative $D_\ell\gamma(D_1 \cap \ldots \cap D_J)$ corresponds to the term

$$a_\ell \mathbf{z}^\ell \ .$$

By (11.3.1) and (11.3.2) it therefore suffices to show that

$$\|a_\ell\| \le \sup_{\mathbb{D}^r} \|f\| \ .$$

This inequality holds by Cauchy's inequalities ([V 4], Lemma 15.1). □

Summing this inequality over all places v of k and applying Lemma 8.4 (f) then gives the following bound of heights.

Theorem 11.4. *There exist height functions $h_1, \ldots, h_J$ on $\prod C_i$ and a constant c such that, for any tuple ℓ, we may clear denominators in the map $\gamma \mapsto D_\ell\gamma$ to obtain an R-module map D'_ℓ from the appropriate subset of $\Gamma\left(\prod \widetilde{\mathscr{X}_i}, \mathscr{O}(dM_{-\epsilon,\mathbf{s}})\right)$ to the R-module*

$$\left\{\gamma \in \Gamma\left(\prod \widetilde{\mathscr{X}_i} \cap \bigcap D_j, \mathscr{O}(dM_{-\epsilon,\mathbf{s}} - \ell_1 D_1 - \ldots - \ell_J D_J)\big|_{\bigcap D_j}\right) \;\middle|\; \|\gamma\|_{\sup,v} \le 1 \text{ for all } v \notin S_\infty\right\} .$$

For archimedean places v we metrize this module via the supremum norm on the line sheaf $\mathscr{O}(dM_{-\epsilon,\mathbf{s}} - \ell_1 D_1 - \ldots - \ell_J D_J)\big|_{\bigcap D_j}$; then

$$\text{(11.4.1)}\qquad \prod_{v|\infty} \|D'_\ell\gamma\|_v \le \prod_{v|\infty} \|\gamma\|_v \cdot \exp\Big(\ell_1 h_1(\xi) + \ldots + \ell_J h_J(\xi) + cd\sum s_i^2\Big) \ .$$

Moreover, if each D_j is the pull-back of a metrized divisor on $\Gamma_{i(j)}$, then each h_j can be taken as the pull-back of a height function on $C_{i(j)}$. Here c depends only on the families $\Gamma_1, \ldots, \Gamma_n$ over $C_1, \ldots, C_n$, the divisors $D_1, \ldots, D_J$, their Weil functions $g_{D,1}, \ldots, g_{D,J}$, the model $\mathscr{V}$, and the extensions of various line sheaves to $\mathscr{V}$.

In the next section, the above theorem will be combined with the following proposition to show that existence of a small section of $\Gamma(\mathscr{W}, dM_{-\epsilon,\mathbf{s}})$ implies the existence of a small section of $\Gamma(\mathscr{W}, dL_{-\epsilon,\mathbf{s}})$.

Proposition 11.5. *Let $\mathscr{C}_1, \ldots, \mathscr{C}_n$ be projective arithmetic schemes over* Spec R *with generic fibers $C_1, \ldots, C_n$ respectively, over* Spec k. *For $i = 1, \ldots, n$ let*

$\pi_i : \Gamma_i \to \mathscr{C}_i$ be surjective projective morphisms. Let $L_1, \dots, L_n$ and $L_1', \dots, L_n'$ be metrized divisor classes on $\Gamma_1, \dots, \Gamma_n$, respectively. Let A_0 be an abelian variety defined over k, and for $i = 1, \dots, n$ let $\rho_i : \Gamma_i \times_R k \to A_0$ be a morphism. Let P be a Poincaré divisor class on $A_0 \times A_0$; for $i < j$ let M_{ij} be a metrized divisor class on $\Gamma_i \times_R \Gamma_j$ which equals $(\rho_i \times \rho_j)^* P$ on the generic fiber (over k). Let $h()$ denote logarithmic height functions on $C_1, \dots, C_n$ relative to ample divisors.

Then there exist nonempty Zariski-open $U_i \subseteq C_i$, $i = 1, \dots, n$, with the following property. For $\xi_i \in U_i(k)$ let X_i denote the variety $\pi_i^{-1}(\xi_i)$. Let $d_1, \dots, d_n$ be positive integers such that $\sqrt{d_i d_j} \in \mathbb{Z}$ for all pairs (i, j), let $\ell_1, \dots, \ell_n$ be positive integers with $\ell_i \le r d_i$ for all i, and let γ be a nonzero global section in

$$\Gamma\left(\prod X_i, \sum d_i \operatorname{pr}_i^* L_i + \sum \ell_i \operatorname{pr}_i^* L_i' + \sum_{i<j} \sqrt{d_i d_j} (\operatorname{pr}_i \times \operatorname{pr}_j)^* M_{ij}\right) .$$

Then
(11.5.1)
$$\prod_v \|\gamma\|_{\sup, v} \ge \exp\left(-c \sum_{i=1}^n d_i h(\xi_i) - c' \sum_{i<j} \sqrt{d_i d_j h(\xi_i) h(\xi_j)} - c'' \sum_{i=1}^n d_i\right) .$$

Here the constants c, c', and c'' depend only on $\pi_1, \dots, \pi_n$, r, and the metrized divisor classes.

Proof . For all i let $m_i = \dim \Gamma_i - \dim \mathscr{C}_i$ and let $\psi_i : \Gamma_i \to \mathbb{P}^{m_i}_{\mathscr{C}_i}$ be a generically finite rational map which is finite over the generic point of C_i. After expanding Γ_i, we may assume that ψ_i is a morphism. The proof of ([V 4], Corollary 18.3) shows that there exist generically finite surjective morphisms $\phi_i : \Gamma_i^\sharp \to \Gamma_i$ such that the metrized divisor classes L_i and L_i' define norms $L_i^\sharp$ and $(L_i')^\sharp$ on $\Gamma_i^\sharp$ which agree with suitable multiples of $\phi^* \psi^* \mathscr{O}(1)$ up to divisors supported on fiber components over $\operatorname{Spec} R$ and, correspondingly, changes of metrics at archimedean places. This proof also constructs norms $M_{ij}^\sharp$ associated to M_{ij} on $\Gamma_i^\sharp \times_R \Gamma_j^\sharp$; the following lemma characterizes these norms.

Lemma 11.5.2. *For $i = 1, 2$ let $\pi_i^\sharp : \Gamma_i^\sharp \to C_i$ be a surjective morphism of complex projective varieties, let A_i be abelian varieties, and let $\rho_i^\sharp : \Gamma_i^\sharp \to A_i$ be morphisms. Let P be a divisor class on $A_1 \times A_2$ whose restrictions to $\{0\} \times A_2$ and $A_1 \times \{0\}$ are algebraically equivalent to zero, and let $M = (\rho_1^\sharp \times \rho_2^\sharp)^* P$. Assume that C_1 and C_2 are normal, and that the restriction of M to the generic fiber of $\pi_1^\sharp \times \pi_2^\sharp$ is trivial. Then some positive integral multiple of M equals $(\pi_1^\sharp \times \pi_2^\sharp)^* M'$ for some divisor class M' on $C_1 \times C_2$. Moreover, the restrictions of M' to $\{\xi_1\} \times C_2$ and $C_1 \times \{\xi_2\}$ are trivial for $\xi_1 \in C_1$ and $\xi_2 \in C_2$.*

Proof . For $i = 1, 2$ let B_i be the smallest translated abelian subvariety of A_i containing $\rho_i^\sharp(\Gamma_i^\sharp)$. Also for all $\xi_i \in C_i$ let $D_i(\xi_i)$ denote the smallest abelian subvariety of A_i such that $\rho_i^\sharp((\pi_i^\sharp)^{-1}(\xi_i))$ is contained in a translate of $D_i(\xi_i)$. For generic ξ_i, $D(\xi_i)$ equals its maximal value; let this be denoted D_i. Since $M = 0$ on fibers of $\pi_1^\sharp \times \pi_2^\sharp$, it follows that $P = 0$ on $\rho_1^\sharp((\pi_1^\sharp)^{-1}(\xi_1)) \times \rho_2^\sharp((\pi_2^\sharp)^{-1}(\xi_2))$. For generic (ξ_1, ξ_2), $\rho_1^\sharp((\pi_1^\sharp)^{-1}(\xi_1)) \times \rho_2^\sharp((\pi_2^\sharp)^{-1}(\xi_2))$ spans the corresponding translate of $D_1 \times D_2$ as an abelian variety (for any choice of origin); hence some positive multiple mP is trivial on this translate. Moreover, there exists one m which works for all (ξ_1, ξ_2). For $i = 1, 2$ let $\alpha_i : B_i \to E_i$ be the quotient under the action of D_i; then by ([H 2], III Ex. 12.4) there exists a divisor class Q on $E_1 \times E_2$ such that $mP = (\alpha_1 \times \alpha_2)^* Q$. Since fibers of $\pi_i^\sharp$ are collapsed by α_i, there exist morphisms $\beta_i : C_i \to E_i$ making the following diagram commute:

$$\begin{array}{ccccc} \Gamma_i^\sharp & \xrightarrow{\rho_i^\sharp} & B_i & \subseteq & A_i \\ \downarrow{\scriptstyle \pi_i^\sharp} & & \downarrow{\scriptstyle \alpha_i} & & \\ C_i & \xrightarrow{\beta_i} & E_i & & \end{array}$$

Let $M' = (\beta_1 \times \beta_2)^* Q$. It then follows that $mM = (\pi_1 \times \pi_2)^* M'$.

Finally, the restrictions of M to $\{\eta_1\} \times \Gamma_2^\sharp$ and $\Gamma_1^\sharp \times \{\eta_2\}$ (for $\eta_1 \in \Gamma_1^\sharp$ and $\eta_2 \in \Gamma_2^\sharp$) are numerically equivalent to zero; hence a similar statement is true for M'. The last assertion of the lemma then follows from [Ma]. □

The statement of the proposition is unchanged if we multiply all divisor classes by a positive integer and raise γ to that same power. Therefore we may assume that $M_{ij}^\sharp$ is a pull-back of a metrized divisor class on $C_i \times C_j$, as in the lemma, up to changes at fiber components over $\operatorname{Spec} R$ and changes in metrics. When applying the lemma, note that $A_i = A_0^{\deg \psi_i}$.

Corresponding to the section γ there is a nonzero global section

$$\gamma^\sharp \in \Gamma\left(\prod \Gamma_i^\sharp,\ \sum d_i \operatorname{pr}_i^* L_i^\sharp + \sum \ell_i \operatorname{pr}_i^* L_i'^\sharp + \sum_{i<j} \sqrt{d_i d_j} (\operatorname{pr}_i \times \operatorname{pr}_j)^* M_{ij}^\sharp \right) .$$

The above divisor class coincides with a product of multiples of $\operatorname{pr}_i^* \phi_i^* \psi_i^* \mathscr{O}(1)$ up to fiber components over $\operatorname{Spec} R$, changes of metrics, and a divisor class

$$\sum_{i<j} \sqrt{d_i d_j} (\operatorname{pr}_i \times \operatorname{pr}_j)^* (\pi_i^\sharp \times \pi_j^\sharp)^* M'_{ij} ,$$

where M'_{ij} is as in the lemma. However, we note that Poincaré-like divisors M'_{ij} satisfy

$$h_{M'_{ij}}(\xi_i, \xi_j) \le c' \sqrt{h(\xi_i) h(\xi_j)} . \tag{11.5.3}$$

The proof then concludes as in ([V 4], Lemma 13.9) by pulling up a suitable point from $\prod \mathbb{P}_{C_i}^{m_i}$ and applying (11.5.3). □

12. Construction of a global section

Let Γ be a metrized, torsion free finitely generated module of rank δ over the ring of integers R of k. For all archimedean places v of k, let the completion $\Gamma_v := \Gamma \otimes_R k_v$ of Γ at v be given a Haar measure such that the unit ball has measure 1, and let $\operatorname{covol}(\Gamma)$ denote the covolume of Γ in $\prod_{v|\infty} \Gamma_v$. Define a length function $\ell(\gamma) = \prod_v \|\gamma\|_v$, and for $i = 1, \dots, \delta$ define successive minima λ_i to be the minimum λ such that there exist R-linearly independent elements $\gamma_1, \dots, \gamma_i \in \Gamma$ such that $\ell(\gamma_j) \le \lambda$ for all $j = 1, \dots, i$.

Lemma 12.1. *In this situation, there exist constants c_5 and c_6 depending only on k such that*

$$\frac{1}{c_5^\delta \delta^{[k:\mathbb{Q}]\delta/2}} \le \frac{\lambda_1 \dots \lambda_\delta}{\operatorname{covol}(\Gamma)} \le c_6^\delta \delta^{[k:\mathbb{Q}]\delta/2} . \tag{12.1.1}$$

Proof. This follows from ([V 1], Theorem 6.1.11 and Remark 6.1.12). (When $k = \mathbb{Q}$ this is Minkowski's "second theorem.") The factors $\delta^{\pm[k:\mathbb{Q}]\delta/2}$ come from the factorials in *loc. cit.* and from the volume of the unit ball in Euclidean space. □

Lemma 12.2. *Let $\beta\colon \Gamma_1 \to \Gamma_2$ be a homomorphism of metrized R-modules. Let δ_0 and δ_2 be the ranks (over R) of the kernel and image of β, respectively. For all $v \mid \infty$ assume that C_v is a constant such that*

$$\|\beta(\gamma)\|_v \le C_v \|\gamma\|_v \qquad \textit{for all } \gamma \in \Gamma_1, \tag{12.2.1}$$

and let $C = \prod_{v|\infty} C_v$. Then

$$\operatorname{covol}(\Gamma_1) \ge 2^{-[k:\mathbb{Q}]\delta_0} \operatorname{covol}(\operatorname{Ker}\beta) C^{-\delta_2} \operatorname{covol}(\operatorname{Image}\beta) . \tag{12.2.2}$$

Proof. See ([Ko], Lemma 5). □

These lemmas, together with the results of Sect. 10, provide us with the main tools needed to construct a small section. At this point we introduce the assumption that

$$s_i \approx \frac{1}{\sqrt{h_L(P_i)}} . \tag{12.3}$$

Theorem 12.4. *Let ϵ be as in Proposition 10.7, let $M_{-\epsilon,\mathbf{s}}$ be as in (3.6), and let $\|\cdot\|'$ denote a metric as in (10.5). Then for all tuples $\mathbf{s} = (s_1, \dots, s_n)$ of positive rational numbers satisfying (12.3) and for all sufficiently large (and divisible) d (depending on $\mathbf{s}$), there exists a section $\gamma \in \Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$ such that $\|\gamma\|'$ is bounded and such that the inequality*

$$\prod_{v|\infty} \|\gamma\|_{\sup,v} \le \exp\left(cd \sum_{i=1}^n s_i^2\right) \tag{12.4.1}$$

holds. Here the constant c is independent of $\mathbf{s}$ and d.

Proof. This proof is a matter of obtaining bounds for covolumes of various modules in the diagram

$$\begin{array}{lcll} 0 & \longrightarrow & \Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}}) & \\ & \xrightarrow{\alpha} & \Gamma\Big(\mathscr{V}, d\sum s_i^2 \operatorname{pr}_i^* L'\Big)^a & \xrightarrow{\beta} \quad \Gamma\Big(\mathscr{V}, d\sum s_i^2 \operatorname{pr}_i^* L''\Big)^b \\ & & \cup & \qquad\quad \cup \\ & & \Gamma'(\mathscr{V}, dM_{-\epsilon,\mathbf{s}}) & \longrightarrow \quad \Gamma'\Big(\mathscr{V}, d\sum s_i^2 \operatorname{pr}_i^* L'\Big)^a , \end{array}$$

where the top row is the Faltings complex (10.4) and the symbols Γ' in the bottom row denote the submodules of sections γ for which $\|\gamma\|'$ is bounded.

First, note that if F is a divisor class on $\prod \overline{\mathscr{X}}_i$, then $\Gamma\big(\prod \overline{\mathscr{X}}_i, dF\big) \hookrightarrow \Gamma(\mathscr{V}, dF)$ and the cokernel is annihilated by an integer independent of d. Therefore we may pass between the two modules a few times without affecting the estimates.

To shorten notation, let Γ_0, Γ_1, and Γ_2 denote the modules in the top row, and Γ_0' and Γ_1' the modules in the bottom row. For all except Γ_2, we shall use metrics induced by the injections into Γ_1; on Γ_1 and Γ_2 the metrics shall be those induced by the largest of the sup norms on the direct summands. Also let $\delta_1 = \operatorname{rank} \Gamma_1$.

First, by ([V 4], Lemma 13.8),

$$\begin{aligned} \operatorname{covol}(\Gamma_1) &\le \exp\Big(\delta_1 \cdot cd \sum s_i^2\Big) \cdot \delta_1^{[k:\mathbb{Q}]\delta_1/2} \\ &\le \exp\Big(\delta_1 \cdot cd \sum s_i^2\Big) , \end{aligned}$$

where from now on c is a constant which is independent of d and $\mathbf{s}$, but whose value may change from line to line. The extra factor in the first step appears because we are using the first half of (12.1.1); it disappears because δ_1 grows only polynomially in d, so this factor can be absorbed into the other factor if d is sufficiently large. This argument will be used implicitly several times in this proof.

Next we show a similar bound for $\operatorname{covol}(\Gamma_1')$, as follows. Let $\widetilde{X}_i$ be a resolution as in Sect. 11, and regard Γ_1 as a submodule of $\Gamma\big(\prod \widetilde{X}_i, d\sum s_i^2 \operatorname{pr}_i^* L'\big)$. We have

$$\Gamma_1' = \Gamma_{1,m} \subseteq \Gamma_{1,m-1} \subseteq \ldots \subseteq \Gamma_{1,0} = \Gamma_1 ,$$

where each $\Gamma_{1,f}$ is the kernel of a map β_f from $\Gamma_{1,f-1}$ to some module determined by certain partial derivatives, as in Lemma 11.2. We will apply Lemma 12.2 to β_f. First consider (11.4.1). Since each D_j in Theorem 11.4 is the pull-back of a metrized divisor on some $\widetilde{X}_i$, the corresponding height satisfies $h(\xi_j) \le ch(X_{i(j)}) + c'$. Moreover by Lemma 11.2 the factors ℓ_j are bounded by cds_i^2, so by (4.5.4) and the condition (12.3) on $\mathbf{s}$, the bound (11.4.1) is at most $\exp\big(cd\sum s_i^2\big)$. Thus (12.2.1) holds with $C = \exp\big(cd\sum s_i^2\big)$. Similarly, (4.5.4) and (12.3) (with $d_i = ds_i^2$) imply that the right-hand side of (11.5.1) is not smaller than $\exp\big(-cd\sum s_i^2\big)$. This gives

$$\operatorname{covol}(\beta_f(\Gamma_{1,f-1})) \ge \exp\Big(-\operatorname{rank} \beta_f(\Gamma_{1,f-1}) \cdot cd \sum s_i^2\Big) .$$

By (12.2.2), therefore,

$$\frac{\operatorname{covol}(\Gamma_{1,f})}{\operatorname{covol}(\Gamma_{1,f-1})} \le \exp\Bigl(\delta_1 \cdot cd \sum s_i^2\Bigr)$$

and thus, by descending induction on f,

$$\operatorname{covol}(\Gamma_1') \le \exp\Bigl(\delta_1 \cdot cd \sum s_i^2\Bigr) .$$

Here we note that some steps can be done in parallel: in fact, the above bounds on the ℓ_j imply that we can take $m \le cd \sum s_i^2$, so the power of 2 in (12.2.2) does not affect the shape of this estimate.

Next, it is an easy matter to bound $\operatorname{covol}(\Gamma_0')$; this argument is the same as in [F 1]. Indeed, by ([V 4], Lemma 13.9),

$$\operatorname{covol}(\beta(\Gamma_1')) \ge \exp\Bigl(-\operatorname{rank}\beta(\Gamma_1') \cdot cd \sum s_i^2\Bigr) ,$$

and by construction (12.2.1) holds with $C \le \exp\bigl(cd \sum s_i^2\bigr)$. Thus by (12.2.2)

$$\text{(12.4.2)} \qquad \operatorname{covol}(\Gamma_0') \le \exp\Bigl(\delta_1 \cdot cd \sum s_i^2\Bigr) .$$

Now let $\delta_0' = \operatorname{rank}\Gamma_0'$. By Proposition 10.7, δ_1/δ_0' is bounded, so (12.4.2) holds (after adjusting c) with δ_1 replaced by δ_0'. Then, by the second half of (12.1.1), there exists a nonzero section $\gamma \in \Gamma_0'$ with

$$\prod_{v|\infty} \|\gamma\|_v \le \exp\Bigl(cd \sum s_i^2\Bigr) .$$

This formula still refers to the norm on Γ_1; however, by ([V 4], Lemma 13.2b and Corollary 13.7), the same bound holds using the norm on $\Gamma(\mathscr{V}, dM_{-\epsilon,\mathbf{s}})$. □

Applying Proposition 10.10 then gives the following result:

Proposition 12.5. *Let $P \in \prod X_i(k)$ and let γ be a global section as in Theorem 12.4. Then we have*
(12.5.1)

$$\prod_{v\in S} \|\gamma(P)\|_v \le \exp\Bigl(cd \sum_{i=1}^n s_i^2\Bigr) \cdot \prod_{v\in S} \prod_{m=1}^{\mu} \prod_{i<j} \frac{\Bigl(\alpha_{vmi}^{ds_i^2} + \alpha_{vmj}^{ds_j^2}\Bigr)^2}{\Bigl(1+\alpha_{vmi}^{ds_i^2}\Bigr)^2 \Bigl(1+\alpha_{vmj}^{ds_j^2}\Bigr)^2} ,$$

where $\alpha_{vmi} = \operatorname{pr}_i^* \alpha_m$ as defined in (2.7), relative to k_v.

Proof . This is immediate from Proposition 10.10 and (10.6). □

Remark 12.6. Theorem 12.4 can also be used to find a section whose norm is bounded with respect to a given finite set of singular metrics. Indeed, one only needs to repeat the fifth paragraph of the proof for each of the metrics. This fact will be used in a subsequent paper.

13. End of the proof

The next few steps of the proof of Theorem 0.2 are much the same as have appeared previously; therefore they can be described very quickly.

To begin, for places $v \in S$ let $\lambda_{m,v}$ be the Weil function defined in Proposition 2.6. The set $\mathscr{A}(R_S)$ can be embedded into the finite dimensional vector space

$$V := \big(A_0(k) \otimes_{\mathbb{Z}} \mathbb{R}\big) \oplus \mathbb{R}^{\mu \cdot \# S}$$

via the map

$$P \mapsto (\rho(P), (\lambda_{m,v}(P))_{m,v}) .$$

On the first factor, the canonical height defines a length function

$$\ell_0(P) = \sqrt{\hat{h}_{L_0}(\rho(P))} ;$$

on the second factor, let

$$\ell_1(P) = \frac{1}{[k : \mathbb{Q}]} \sum_{v \in S} \sum_{m=1}^{\mu} |\lambda_{m,v}(P)| .$$

Both length functions are nondegenerate. We note that $\ell_1(P)$ is related to the height function $h_{L_1}(P)$. Indeed, the functions $\max(\lambda_{m,v}, 0)$ and $\max(-\lambda_{m,v}, 0)$ are Weil functions for the divisors $[0]_m$ and $[\infty]_m$, respectively, at the place v, although they may differ by a bounded amount from the metrics (2.8). However, this fact still implies that

$$h_{L_1}(P) = \ell_1(P) + O(1) . \tag{13.1}$$

(This equality requires the fact that $P \in \mathscr{A}(R_S)$. It will be needed for (13.7).)

This argument uses the same sphere packing argument as in [V 4], except that now it must take place simultaneously on two spheres, due to the fact that h_{L_0} behaves quadratically in the group law, whereas h_{L_1} behaves linearly. But note that both h_{L_0} and h_{L_1} are bounded from below, and $h_L = h_{L_0} + h_{L_1}$. Then, for points $P \in \mathscr{A}(R_S)$, the points

$$\frac{1}{\sqrt{h_L(P)}} \cdot \rho(P) \in A_0(k) \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$\left(\frac{1}{h_L(P)} \cdot \lambda_{m,v}(P) \right)_{v \in S,\ 1 \le m \le \mu} \in \mathbb{R}^{\# S \cdot \mu} ;$$

both lie in the unit balls relative to the length functions ℓ_0 and ℓ_1, respectively.

Now we assume that for some predetermined $\epsilon_1 > 0$, the points $P_1, \dots, P_n$ have been chosen such that

$$\ell_0 \left(\frac{1}{\sqrt{h_L(P_i)}} \cdot \rho(P_i) - \frac{1}{\sqrt{h_L(P_j)}} \cdot \rho(P_j) \right) \le \epsilon_1 \tag{13.2}$$

and

$$\ell_1\left(\frac{1}{h_L(P_i)}\cdot P_i - \frac{1}{h_L(P_j)}\cdot P_j\right)\le \epsilon_1 \tag{13.3}$$

for all $i<j$. For such a tuple $(P_1,\dots,P_n)$ choose rational numbers $s_1,\dots,s_n$ sufficiently close to $1\big/\sqrt{h_L(P_i)}$, $i=1,\dots,n$, such that

$$\ell_0\bigl(s_i\cdot\rho(P_i)-s_j\cdot\rho(P_j)\bigr)\le\epsilon_1 \tag{13.4}$$

and

$$\ell_1\bigl(s_i^2\cdot P_i - s_j^2\cdot P_j\bigr)\le\epsilon_1 \tag{13.5}$$

for all $i<j$.

Let E be the arithmetic curve on $\mathscr{V}^{\boldsymbol{\cdot}}$ corresponding to $(P_1,\dots,P_n)$. Applying (13.4) to the definition (3.3) of $M_{-\epsilon,\mathbf{s}}$ gives

(13.6)

$$\frac{1}{[k:\mathbb{Q}]}\deg M_{-\epsilon,\mathbf{s}}\big|_E\le\frac{n(n-1)}{2}\epsilon_1-n\epsilon+(n-1)\sum_{i=1}^n s_i^2h_{L_1}(P_i)+O\left(\sum s_i^2\right).$$

This follows as in ([V 4], 17.2). But also (13.5) implies that $\ell_1(ds_i^2\cdot P_i-ds_j^2\cdot P_j)\le d\epsilon_1$; combining this with the inequality

$$\frac{(\alpha+\beta)^2}{(1+\alpha)^2(1+\beta)^2}\ll\min\left(\alpha,\frac{1}{\alpha}\right)\min\left(\beta,\frac{1}{\beta}\right)e^{|\log(\alpha/\beta)|},\qquad\alpha>0,\ \beta>0$$

implies that

(13.7)

$$\begin{aligned}&ds_i^2h_{L_1}(P_i)+ds_j^2h_{L_1}(P_j)\\&\qquad\le\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S}\sum_{m=1}^{\mu}-\log\frac{\left(\alpha_{vmi}^{ds_i^2}+\alpha_{vmj}^{ds_j^2}\right)^2}{\left(1+\alpha_{vmi}^{ds_i^2}\right)^2\left(1+\alpha_{vmj}^{ds_j^2}\right)^2}+d\epsilon_1+O\left(d\sum s_i^2\right).\end{aligned}$$

Therefore

$$\begin{aligned}(n-1)\sum_{i=1}^n ds_i^2h_{L_1}(P_i)\le\frac{1}{[k:\mathbb{Q}]}\sum_{v\in S}\sum_{m=1}^{\mu}\sum_{i<j}-\log\frac{\left(\alpha_{vmi}^{ds_i^2}+\alpha_{vmj}^{ds_j^2}\right)^2}{\left(1+\alpha_{vmi}^{ds_i^2}\right)^2\left(1+\alpha_{vmj}^{ds_j^2}\right)^2}\\+\frac{n(n-1)d}{2}\epsilon_1+O\left(d\sum s_i^2\right).\end{aligned}$$

Combining this with (13.6) and (12.5.1) then gives

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\notin S}-\log\bigl\|\gamma\big|_E\bigr\|_v\le dn((n-1)\epsilon_1-\epsilon)+O\left(d\sum s_i^2\right).$$

If ϵ_1 and $\epsilon_2>0$ are chosen sufficiently small and if the heights $h_L(P_i)$ are sufficiently large, then this bound becomes

$$\frac{1}{[k:\mathbb{Q}]}\sum_{v\notin S} -\log\big\|\gamma\big|_E\big\|_v \le -d\epsilon_2 .$$

These bounds depend only on $\mathscr{X} \subseteq \mathscr{A}$. As in [F 1] and ([V 4], Proposition 16.1), we obtain a positive lower bound for the index of γ along E:

$$t(\gamma,(P_1,\dots,P_n),ds_1^2,\dots,ds_n^2) \ge \epsilon_3 ,$$

where $\epsilon_3 = \epsilon_3(\epsilon_2, \mathscr{X} \subseteq \mathscr{A})$.

The last step of the proof consists of applying Faltings' product theorem, as was done in ([F 1], Sect. 6) or [V 4]. This implies that at least one of $P_1,\dots,P_n$ lies in a strictly smaller X_i, still satisfying (4.5). The inductive step of Sect. 4 may then be carried out.

14. Proof of Corollary 0.3

Let $g = h^1(X, \mathscr{O}_X)$. Then the Albanese variety $\mathrm{Alb}(X)$ has dimension g. By the condition on the number of components of $\operatorname{Supp} D$, there are at least $\dim X - g + 1$ linearly independent divisors E_i such that $\operatorname{Supp} E_i \subseteq \operatorname{Supp} D$, and such that each E_i is algebraically equivalent to zero. The line sheaves $\mathscr{O}(E_i)$ can be used to define a semiabelian variety A, which $X \setminus D$ maps into. But since $\dim A > \dim X$, the image of $X \setminus D$ is a proper subvariety, to which we can apply Theorem 0.2. This implies Corollary 0.3. □

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