Invent. math. 126, 1–45 (1996)

Inventiones mathematicae

Lie algebras graded by finite root systems and intersection matrix algebras

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Oblatum 1-XI-1994 & 22-I-1996

Abstract. This paper classifies the Lie algebras graded by doubly-laced finite root systems and applies this classification to identify the intersection matrix algebras arising from multiply affinized Cartan matrices of types B, C, F, and G. This completes the determination of the Lie algebras graded by finite root systems initiated by Berman and Moody who studied the simply-laced finite root systems of rank $\geqq 2$.

Introduction

0.1. Let F be a field of characteristic zero, and let Δ be a finite irreducible reduced root system. Assume Γ is the integer lattice generated by Δ. In [BM] S. Berman and R. Moody began the study of Lie algebras graded by the root system Δ. Following them we say that a Lie algebra is graded by Δ or is Δ-*graded* if

(i) L has a Γ-gradation $L = \sum_{\gamma \in \Gamma} L_\gamma$ in which $L_\gamma \neq (0)$ if and only if $\gamma \in \Delta \cup \{0\}$;
(ii) the split simple Lie algebra $\mathscr{G}$ whose root system is Δ is a subalgebra of L, and relative to some split Cartan subalgebra $\mathscr{H}$ of $\mathscr{G}$ we have $L_\gamma \supseteq \mathscr{G}_\gamma$ for all $\gamma \in \Delta \cup \{0\}$;
(iii) for all $h \in \mathscr{H}$ the operator $\operatorname{ad} h$ acts diagonally on L_γ with eigenvalue $\gamma(h)$; and
(iv) L is generated by its nonzero root spaces L_γ where $\gamma \in \Delta$.

The conditions for being a Δ-graded Lie algebra imply that L is a direct sum of finite-dimensional irreducible $\mathscr{G}$-modules whose highest weights are

★ Supported in part by National Science Foundation Grant #DMS-9300523.
★★ Supported in part by National Science Foundation Grant #DMS-9212608.

roots, hence are either the highest long or highest short root or are zero. Thus, condition (iii) in the definition of a Δ-graded Lie algebra can be replaced by:

(iii)′ As a $\mathscr{G}$-module L is a direct sum of adjoint modules (modules isomorphic to $\mathscr{G}$), *little adjoint modules* whose highest weight is the highest short root, or one-dimensional $\mathscr{G}$-modules; the latter being contained in L_0.

0.2. It is easy to see that a Δ-graded Lie algebra L is *perfect* ($L = [L, L]$), and in particular, $L_0 = \sum_{\gamma \in \Delta} [L_{-\gamma}, L_\gamma]$. Since conditions (i)–(iv) concern only the root spaces corresponding to nonzero weights, it is appropriate to classify Δ-graded Lie algebras up to *central isogeny*: Any perfect Lie algebra L has a universal central extension which is also perfect, called a *universal covering algebra* (u.c.a.) of L (see [G]). Since any two u.c.a.'s of L are isomorphic, we will refer to *the* universal covering algebra of L.

Definition 0.3. *Two perfect Lie algebras L_1 and L_2 are said to be centrally isogenous if they have the same u.c.a.* (*up to isomorphism*).

0.4. The fact that Δ-graded Lie algebras decompose into $\mathscr{G}$-submodules of the type described in (iii)′ enables us to adopt the same *rational methods* (representation techniques) used by Seligman [Se] in his study of finite-dimensional simple Lie algebras having a split simple subalgebra. Seligman determines the finite-dimensional simple Δ-graded Lie algebras over fields of characteristic zero. As the results of [Se] show, the assumptions of finite-dimensionality and simplicity allow very detailed conclusions to be drawn about the structure of those algebras. The applications which we develop to intersection matrix algebras and to Lie bialgebras require us to deal with infinite-dimensional (and in general nonsimple) Δ-graded Lie algebras and so necessitate classifying these algebras up to central isogeny.

0.5. Berman and Moody [BM] present the following examples of Δ-graded Lie algebras:

(1) Let A be an associative F-algebra with identity. For any positive integer $n > 1$ the algebra of $n \times n$ matrices with coefficients in A forms a Lie algebra $gl_n(A)$ under the commutator product. The subalgebra $e_n(A)$ generated by the elements $ae_{i,j}$ for $a \in A$ and $i \neq j$ is an ideal of $gl_n(A)$ which is perfect. The algebra $e_n(A)$ is graded by the root system A_{n-1}. The u.c.a. $st_n(A)$ and $e_n(A)$ is the Lie algebra analogue of the Steinberg group $St_n(A)$.

(2) Let A be a commutative associative F-algebra and let $\mathscr{G}$ be the split simple Lie algebra with root system Δ. Then $\mathscr{G} \otimes A$ is obviously a Δ-graded Lie algebra. It is perfect, and the u.c.a. of $\mathscr{G} \otimes A$ is a generalization of the affine Kac–Moody algebra determined by $\mathscr{G}$.

0.6. Berman and Moody classify the Lie algebras graded by simply-laced finite root systems of rank $\geqq 2$, i.e. root systems of types A_n, $n \geqq 2$, D_n, $n \geqq 4, E_6$, E_7, E_8.

Recognition Theorem. ([BM]) Let L be a Lie algebra of characteristic zero graded by a simply-laced finite root system Δ of rank $n \geqq 2$.

(i) If $\Delta = D_n,\ n \geqq 4$, or if $\Delta = E_6, E_7, E_8$, then there exists a commutative associative unital F-algebra A such that L is centrally isogenous with $\mathscr{G} \otimes A$, where $\mathscr{G}$ is the split simple Lie algebra with root system Δ.
(ii) If $\Delta = A_n,\ n \geqq 3$, then there exists a unital associative F-algebra A such that L is centrally isogenous with $e_{n+1}(A)$.
(iii) If $\Delta = A_2$, then L is centrally isogenous with $st_3(A)$, where A is a unital alternative F-algebra.

0.7. In this paper we prove a recognition theorem for each of the remaining irreducible reduced root systems i.e. the root systems A_1, B_n, C_n, F_4, and G_2.

0.8. The result for Lie algebras graded by root systems of type C_n is very similar to the theorem for type A. In fact, the problem can be translated into the language of Jordan algebras using the notion of a Jordan system (see Sect. 1), and then the Recognition Theorem for Lie algebras graded by a root system of type C is an immediate consequence of N. Jacobson's Coordinatization Theorem for Jordan algebras (see [Jac1]).

Recognition Theorem for type C_n. *Let L be a Δ-graded Lie algebra.*

(i) *If $\Delta = C_n,\ n \geqq 4$, then there exists a unital, associative algebra A with an involution $* : A \to A$ such that L is centrally isogenous with the algebra $sp_{2n}(A,*)$ of symplectic $(2n) \times (2n)$ matrices over A. (See Sect. 1 for the definition.)*
(ii) *If $\Delta = C_3$, then L is centrally isogenous with the symplectic Steinberg algebra $st\,sp_6(A,*)$, where A is an alternative involutive algebra whose symmetric elements, $\{a \in A | a^* = a\}$, lie in the associative center of A.*
(iii) *If $\Delta = C_2$, then L is centrally isogenous with a Tits–Kantor–Koecher construction of a unital Jordan algebra J which contains the Jordan algebra of symmetric 2×2 matrices, and the identity of J lies in this subalgebra.*
(iv) *If $\Delta = C_1 = A_1$, then L is centrally isogenous with a Tits–Kantor–Koecher construction of a unital Jordan algebra J.*

0.9. The key to the Recognition Theorems for types B_n, F_4, and G_2 lies in the celebrated Tits construction of Lie algebras. Let $\mathscr{C}$ denote the split octonion algebra (Cayley algebra) over the field F, and let J denote the split exceptional 27-dimensional Jordan algebra over F. It is known (see for example [Sc] or [Jac2]) that the derivation algebras $\mathrm{Der}_F\,\mathscr{C}$ and $\mathrm{Der}_F\,J$ are isomorphic to the split exceptional simple Lie algebras G_2 and F_4 respectively. Let $\mathscr{C}_0$ and J_0 denote the subspaces of trace zero elements in $\mathscr{C}$ and J. J. Tits [T2] proved that the space

$$\mathscr{T}(\mathscr{C}/F, J/F) = \mathrm{Der}_F\,\mathscr{C} \oplus (\mathscr{C}_0 \otimes J_0) \oplus \mathrm{Der}_F\,J$$

with a suitable multiplication (see [Sc], [Jac2], or Sect. 3 for a detailed discussion) becomes the exceptional simple Lie algebra of type E_8. Taking other Jordan algebras J such as (I) the field F, (II) the algebra $H(M_3(F))$ of 3×3 symmetric matrices over F, (III) the Jordan algebra $M_3(F)^+$ over F, or (IV) the algebra $H(M_3(\mathcal{Q}))$ of 3×3 Hermitian matrices over a quaternion algebra $\mathcal{Q}$, provides realizations of the exceptional simple Lie algebras of types G_2, F_4, E_6, E_7 respectively. Moreover, if the alternative algebra $\mathcal{C}$ is varied too, this construction produces the entire "magic table" of Freudenthal.

The algebras $\mathcal{T}(\mathcal{C}/F, J/F)$ (with $\mathcal{C}$ the split octonions) have a natural G_2-grading coming from the adjoint action of $\mathrm{Der}_F\,\mathcal{C} = G_2$, and so F_4, E_6, E_7 and E_8 are all G_2-graded. When we specialize the Jordan algebra J to be $J = F1 \oplus J(W)$ where $J(W)$ is the Jordan algebra of a nondegenerate symmetric bilinear form on an m-dimensional vector space W, then $\mathcal{T}(\mathcal{C}/F, J/F)$ is a simple Lie algebra of type B_{n+3} if $m = 2n$ and of type D_{n+4} if $m = 2n + 1$ for $n \geqq 0$. Thus, the Lie algebras of types B and D possess G_2-gradings too.

The formula for $\mathcal{T}(\mathcal{C}/F, J/F)$ also provides an F_4-grading of the algebra of type E_8 coming from the adjoint action of $\mathrm{Der}_F\, J = F_4$. If instead of the octonion algebra we take another alternative algebra $\mathcal{C}$ such as (i) F, (ii) $F \oplus F$, or (iii) the quaternion algebra, then the construction above will yield F_4-gradings of F_4, E_6, and E_7.

0.10. We prove that (up to central isogeny) all the Lie algebras graded by root systems of types F_4 and G_2 arise from this general procedure. In Sect. 3 we introduce what we term the *generalized Tits construction*. Let A and $\mathfrak{A}$ be unital, commutative, associative algebras over F. Assume X is an algebra over A having a trace $t : X \to A$ such that $t(ww') = t(w'w)$ and $t((ww')w'') = t(w(w'w''))$ for all $w, w', w'' \in X$. If the trace is *normalized*, that is if $t(1) = 1$, then the splitting $w = t(w)1 + w - t(w)1$ for all $w \in X$ gives the decomposition $X = A1 \oplus X_0$, where X_0 is the set of elements of trace zero. We say that X satisfies the Cayley–Hamilton trace identity

$$\mathrm{ch}_2(x) = x^2 - 2t(x)x + (2t(x)^2 - t(x^2))1 \quad \text{or}$$
$$\mathrm{ch}_3(x) = x^3 - T(x)x^2 + S(x)x - N(x)1\,,$$

where $T(x) = 3t(x)$, $S(x) = (9/2)t(x)^2 - (3/2)t(x^2)$, and $N(x) = t(x^3) - (9/2)t(x^2)t(x) + (9/2)t(x)^3$, if $\mathrm{ch}_2(x) = 0$ (or $\mathrm{ch}_3(x) = 0$) for all $x \in X$.

Let $\mathcal{D}(X)$ be a Lie subalgebra of the A-derivations of X which map X_0 to X_0, and assume $(Y, \circ)$, and $\mathcal{D}(Y)$ are similarly chosen over $\mathfrak{A}$. Then under suitable restrictions (see Sect. 3),

$$\mathcal{T}(X/A, Y/\mathfrak{A}) = (\mathcal{D}(X) \otimes \mathfrak{A}) \oplus (X_0 \otimes Y_0) \oplus (A \otimes \mathcal{D}(Y))\,,$$

is a Lie algebra. The most important Lie algebras arising from this procedure are the Tits constructions $\mathcal{T}(\mathcal{C}/F, J/F)$ discussed in (0.9) which provide explicit models of the exceptional simple Lie algebras. The generalized construction enables us to prove the following theorems:

Recognition Theorem for type G_2. *Let L be a G_2-graded Lie algebra. Then there exists a unital, commutative, associative algebra A and a Jordan algebra J over A having a normalized trace which satisfies* $\mathrm{ch}_3(x) = 0$ *such that L is centrally isogenous with*

$$\begin{aligned}\mathscr{T}(\mathscr{C}/F, J/A) &= (\mathrm{Der}_F\,\mathscr{C} \otimes A) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \langle J, J\rangle\ ,\\ &= (G_2 \otimes A) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \langle J, J\rangle\ ,\end{aligned}$$

where $\mathscr{C}$ is the split octonion algebra over $F, \mathscr{C}_0$ and J_0 are the trace zero elements of $\mathscr{C}$ and J respectively, and $\langle J, J\rangle$ is the Lie algebra of inner derivations of J.

Recognition Theorem for type F_4. *Let L be an F_4-graded Lie algebra. Then there exists a unital, commutative, associative algebra A and an alternative algebra $\mathscr{C}$ over A having a normalized trace which satisfies* $\mathrm{ch}_2(x) = 0$ *such that L is centrally isogenous with*

$$\begin{aligned}\mathscr{T}(\mathscr{C}/A, J/F) &= \langle \mathscr{C}, \mathscr{C}\rangle \oplus (\mathscr{C}_0 \otimes J_0) \oplus (A \otimes \mathrm{Der}_F\, J)\ ,\\ &= \langle \mathscr{C}, \mathscr{C}\rangle \oplus (\mathscr{C}_0 \otimes J_0) \oplus (A \otimes F_4)\ ,\end{aligned}$$

where J is the split exceptional 27-dimensional Jordan algebra $\mathscr{C}_0$ and J_0 are the trace zero elements of $\mathscr{C}$ and J respectively, and $\langle \mathscr{C}, \mathscr{C}\rangle$ is the Lie subalgebra of inner derivations of $\mathscr{C}$.

0.11. Let V denote a vector space over F having a nondegenerate symmetric bilinear form $(\ ,\ )$. The sum $J(V) = F1 \oplus V$ becomes a Jordan algebra with $u \cdot v = (u, v)1;\ u, v \in V$. Similarly for any unital, commutative, associative F-algebra A, and any unital A-module W having a symmetric A-bilinear form $(\ ,\ ) : W \times W \to A$, the operation $w \cdot x = (w, x)1$ defines the structure of a Jordan algebra on $J(W) = A1 \oplus W$.

Recognition Theorem for type B_n. Let L be a B_n-graded Lie algebra for $n \geqq 3$. Then there exists a unital, commutative, associative F-algebra A and a unital A-module W with a symmetric A-bilinear form $(\ ,\ ) : W \times W \to A$ such that L is centrally isogenous with

$$\begin{aligned}\mathscr{T}(J(V)/F, J(W)/A) &= (\langle V, V\rangle \otimes A) \oplus (V \otimes W) \oplus \langle W, W\rangle\ ,\\ &= (B_n \otimes A) \oplus (V \otimes W) \oplus \langle W, W\rangle\ ,\end{aligned}$$

where V is $(2n+1)$-dimensional F-vector space with a nondegenerate symmetric bilinear form (the defining representation for B_n), $\langle V, V\rangle$ is the set of skew-symmetric transformations on V relative to the form on V, and $\langle W, W\rangle$ is the set of skew-symmetric transformations on W relative to the form on W.

0.12. In the fourth section we show how the generalized Tits construction makes it possible to extend the Freudenthal–Tits magic square to provide constructions of the exceptional Lie superalgebras.

0.13. In the final section we apply these Recognition Theorems to determine the intersection matrix algebras of Slodowy (see [S1]) for types B_n, C_n, F_4, and G_2. The intersection matrix algebras which correspond to the simply-laced root systems of rank $\geqq 2$ have been identified by Berman and Moody [BM]. In each of the cases $\Delta = B_n$, for $n \geqq 3, F_4$, and G_2, a Δ-graded Lie algebra is centrally isogenous to a Tits construction $\mathscr{T}(Q/F, R/\mathfrak{A})$ where Q is a fixed F-algebra which is the Jordan algebra $J(V)$ of a nondegenerate symmetric bilinear form on a $2n+1$-dimensional space when $\Delta = B_n$, the split octonion algebra $\mathscr{C}$ when $\Delta = G_2$, or the split exceptional Jordan algebra of 3×3 Hermitian matrices over $\mathscr{C}$ when $\Delta = F_4$. The algebra R is an alternative algebra (in the F_4 case) and is a Jordan algebra (in the B_n and G_2 cases) over a unital, commutative associative F-algebra $\mathfrak{A}$. Furthermore, R satisfies a trace identity $f = 0$ which is $\mathrm{ch}_2(x) = 0$ when $\Delta = B_n$ or F_4 and $\mathrm{ch}_3(x) = 0$ when $\Delta = G_2$. From this description it follows that the corresponding intersection matrix algebra is isomorphic to the universal covering algebra of $\mathscr{T}(Q/F, R/\mathfrak{A}) \otimes_F F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$ where R is the universal algebra generated by s invertible elements in the variety of alternative or Jordan algebras satisfying the trace identity $f = 0$, and $F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$ is the algebra of Laurent polynomials in l variables. The values of s and l are determined in (5.7) below and the universal algebras R are described more precisely in (5.11).

From the Recognition Theorem in (0.8) above it follows that an intersection algebra of type C_n for $n \geqq 4$ is isomorphic to $st\,sp_{2n}(R, \pi_{l,s})$ where R is the group algebra of a free group on $l + 2s$ free generators x_i, y_j, z_j for $i = 1, \ldots, l$, and $j = 1, \ldots, s$. The involution $\pi_{l,s}$ fixes the x_i's and interchanges the other generators, $\pi_{l,s}(y_j) = z_j$ and $\pi_{l,s}(z_j) = y_j$ for each j. For $n = 3$, the ring R of coefficients is a homomorphic image of the free product $\mathscr{A}_{l+2s} = (\circledast_{i=1}^{l} F[x_i, x_i^{-1}]) \circledast (\circledast_{j=1}^{s} F[y_j, y_j^{-1}]) \circledast (\circledast_{j=1}^{s} F[z_j, z_j^{-1}])$ in the variety of alternative algebras, and the involution $\pi_{l,s}$ behaves exactly as before on the generators. When $\Delta = C_2$, an intersection matrix algebra is isomorphic to the Tits–Kantor–Koecher construction $K((H(M_2(F)) \circledast \mathscr{J}_{l+s})/I)$ obtained by taking a certain homomorphic image of the free product of the Hermitian 2×2 matrices $H(M_2(F))$ with a free Jordan algebra $\mathscr{J}_{l+s}$ on $l + s$ invertible generators. An intersection matrix algebra for $\Delta = C_1 = A_1$ is just the Tits–Kantor–Koecher construction $K(\mathscr{J}_l)$. These constructions and results are described in (5.8) and Theorem 5.9.

0.14. The preprint [Z2] (see also [Z3]) gave the first proof of the Recognition Theorem for type C and treated Lie algebras graded by root systems of types B_n, C_n, F_4, and G_2 in which the algebras decompose into copies of the adjoint and trivial modules (no little adjoint modules allowed). Except in the C_n case where the argument appeals directly to Jacobson's Coordinatization Theorem, the methods we use to establish the results discussed above are the rational methods employed by Seligman [Se]; they allow arbitrary root graded algebras without restrictions on the summands, and they afford easier proofs than those in [Z2]. The C_n case could be addressed similarly, but we have elected the other route because it is more direct. E. Neher's preprint [N] provides an alternate uniform treatment of the algebras graded by root systems of types $A_n, B_n, C_n, D_n, E_6, E_7$ using the theory of 3-graded root systems, which is essentially the theory of certain Jordan pairs.

0.15. The results in this paper were announced in [BZ]. Further applications of them to the study of Lie bialgebras will be explored in a subsequent paper [MZ].

1. Δ-graded Lie algebras and Jordan theory

1.1. In this section we relate the notion of central isogeny to Jordan systems. Then we discuss the Tits–Kantor–Koecher construction and use it to deduce the Recognition Theorem for Lie algebras graded by root systems of type C from Jacobson's Coordinatization Theorem. When $L = \sum_{\gamma \in \Gamma} L_\gamma$ is a Lie algebra graded by Δ, L is generated by its root spaces corresponding to nonzero weights. To focus attention on the subspaces $L_\gamma, \gamma \neq 0$, the following concept was introduced in [Z1]. (See also [Z3].)

1.2. Consider a system of vector spaces $\underline{J} = (J_\gamma, \gamma \in \Delta)$ with transformations $\psi_{\alpha,\beta} : J_\alpha \otimes J_\beta \to J_{\alpha+\beta}$ whenever $\alpha, \beta, \alpha + \beta \in \Delta$; and $\psi_{\alpha,-\alpha} : J_\alpha \otimes J_{-\alpha} \to \bigoplus_{\gamma \in \Delta} \mathrm{End}_F(J_\gamma)$ for all $\alpha \in \Delta$. In other words, we have bilinear operations, $J_\alpha \otimes J_\beta \to J_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta$ and trilinear operations $J_\alpha \otimes J_{-\alpha} \otimes J_\beta \to J_\beta$ for any $\alpha, \beta \in \Delta$. We call this system a *Jordan system* if the direct sum

$$K(\underline{J}) = K(J_\gamma, \gamma \in \Delta) \overset{\mathrm{def}}{=} \bigoplus_{\gamma \in \Delta \cup \{0\}} J_\gamma ,$$

$$\text{where } J_0 \overset{\mathrm{def}}{=} \bigoplus_{\gamma \in \Delta} \psi_{\gamma,-\gamma}(J_\gamma \otimes J_{-\gamma}) ,$$

becomes a Δ-graded Lie algebra with respect to the operation:

$$[x_\alpha, y_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Delta , \\ \psi_{\alpha,\beta}(x_\alpha \otimes y_\beta) & \text{if } \alpha, \beta, \alpha + \beta \in \Delta , \\ \psi_{\alpha,-\alpha}(x_\alpha \otimes y_\beta) & \text{if } \beta = -\alpha \in \Delta , \\ x_\alpha(y_\beta) & \text{if } \alpha = 0, \beta \in \Delta , \\ -y_\beta(x_\alpha) & \text{if } \beta = 0, \alpha \in \Delta , \\ x_\alpha y_\beta - y_\beta x_\alpha & \text{if } \alpha = 0 = \beta . \end{cases}$$

Equivalently, the vector spaces $\underline{J} = (J_\gamma, \gamma \in \Delta)$ form a Jordan system if they are the root subspaces with nonzero weights in some Δ-graded Lie algebra $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$. Jordan systems corresponding to the root system $\Delta = \{-\alpha, \alpha\}$

of type A_1 are the well-known Jordan pairs (see [L]). Of course, such a Lie algebra L is generally speaking not unique. We will show that the category of Δ-graded Lie algebras having a given Jordan system as its subspaces with nonzero weights has a universal object.

1.3. Suppose $\underline{J} = (J_\gamma, \gamma \in \Delta)$ and $\underline{J}' = (J'_\gamma, \gamma \in \Delta)$ are Jordan systems corresponding to Δ. A family of linear mappings $\eta = (\eta_\gamma, \gamma \in \Delta), \eta_\gamma : J_\gamma \to J'_\gamma$, is called a homomorphism if it induces a homomorphism of Lie algebras $K(\underline{J}) \to K(\underline{J}')$ with $J_\gamma \to J'_\gamma$, for all $\gamma \in \Delta \cup \{0\}$.

1.4. Let $\underline{J} = (J_\gamma, \gamma \in \Delta)$ be a fixed Jordan system. For each subspace J_γ we fix an arbitrary basis $\{e_{\gamma,i} | i \in I_\gamma\}$ and structure constants $\psi_{\alpha,\beta}(e_{\alpha,i} \otimes e_{\beta,j}) = \sum_{k \in I_{\alpha+\beta}} c^k_{i,j}(\alpha,\beta) e_{\alpha+\beta,k}$ when $\alpha, \beta, \alpha+\beta \in \Delta$, and $\psi_{\alpha,-\alpha}(e_{\alpha,i} \otimes e_{-\alpha,j})(e_{\beta,q}) = \sum_{k \in I_\beta} c^k_{i,j,q}(\alpha,\beta) e_{\beta,k}$.

Let $L(X_{\underline{J}})$ be the free Lie algebra on the set of generators $X_{\underline{J}} = \{x_{\gamma,i} | \gamma \in \Delta, i \in I_\gamma\}$. To each free generator $x_{\gamma,i}$ we assign the weight γ, thus giving $L(X_{\underline{J}})$ a Γ-grading. Let M be the ideal of $L(X_{\underline{J}})$ generated by all the homogeneous spaces $L(X_{\underline{J}})_\gamma$ where $\gamma \notin \Delta \cup \{0\}$ and by all the relators $[x_{\alpha,i}, x_{\beta,j}] - \sum_{k \in I_{\alpha+\beta}} c^k_{i,j}(\alpha,\beta) x_{\alpha+\beta,k}$ if $\alpha, \beta, \alpha+\beta \in \Delta$, and $[[x_{\alpha,i}, x_{-\alpha,j}], x_{\beta,q}] - \sum_{k \in I_\beta} c^k_{i,j,q}(\alpha,\beta) x_{\beta,k}$ for $\alpha \in \Delta$. Since the ideal M is generated by homogeneous elements, it follows that $U_{\underline{J}} = L(X_{\underline{J}})/M$ is a Γ-graded Lie algebra. Moreover, $(U_{\underline{J}})_\gamma, = (0)$ if $\gamma \notin \Delta \cup \{0\}$. Let μ be the homomorphism of the Jordan system $\underline{J} = (J_\gamma, \gamma \in \Delta)$ into the Jordan system $U_{\underline{J}} = ((U_{\underline{J}})_\gamma, \gamma \in \Delta)$ given by $\mu_\gamma : e_{\gamma,i} \to x_{\gamma,i} + M$.

Suppose now that $L = \bigoplus_{\gamma \in \Gamma} L_\gamma$ is a Γ-graded Lie algebra such that $L_\gamma = (0)$ for $\gamma \notin \Delta \cup \{0\}$. For an arbitrary homomorphism η of the Jordan system $\underline{J} = (J_\gamma, \gamma \in \Delta)$ to the Jordan system $\underline{L} = (L_\gamma, \gamma \in \Delta)$ there exists a unique homomorphism of Lie algebras $\theta : U_{\underline{J}} \to L$ such that the diagram

$$\begin{array}{ccc} (U_{\underline{J}})_\gamma & \overset{\theta_\gamma}{\to} & L_\gamma \\ \mu_\gamma \uparrow & \nearrow \eta_\gamma & \\ J_\gamma & & \end{array}$$

is commutative. From the existence of the Lie algebra $K(\underline{J})$ we conclude that the mapping $\mu_\gamma : J_\gamma \to (U_{\underline{J}})_\gamma$ is bijective for all $\gamma \in \Delta$.

Now let $L = \bigoplus_{\gamma \in \Gamma} L_\gamma$ be a Lie algebra graded by Γ such that $L_\gamma = (0)$ for $\gamma \notin \Delta \cup \{0\}$ and $L_0 = \sum_{\gamma \in \Delta} [L_{-\gamma}, L_\gamma]$. Consider the Jordan system $\underline{L} = (L_\gamma, \gamma \in \Delta)$ and the Lie algebra $U_{\underline{L}}$. Let $\theta : U_{\underline{L}} \to L$ be the Lie algebra homomorphism extending the bijective mappings $(U_{\underline{L}})_\gamma \to L_\gamma$, $\gamma \in \Delta$. Then $\ker \theta \subseteq U_0$ which implies that the ideal $\ker \theta$ lies in the center of $U_{\underline{L}}$. Thus, $\theta : U_{\underline{L}} \to L$ is a central cover. As a consequence, we have the following:

Proposition 1.5. *Let $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$ and $L' = \sum_{\gamma \in \Delta \cup \{0\}} L'_\gamma$ be two Δ-graded Lie algebra. If the Jordan systems $\underline{L} = (L_\gamma, \gamma \in \Delta)$ and $L' = (L'_\gamma, \gamma \in \Delta)$ are isomorphic, then the Lie algebras L and L' are centrally isogenous.*

Proposition 1.6. *If* $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$ *is a* Δ*-graded Lie algebra, then* $U_{\underline{L}}$ *is centrally closed.*

Proof. Let $\phi : \widetilde{U} \to U_{\underline{L}}$ be a central cover. Assume $\mathscr{G} \subseteq U_{\underline{L}}$ is the simple finite-dimensional Lie algebra whose root decomposition $\mathscr{G} = \mathscr{H} \oplus \sum_{\gamma \in \Delta} \mathscr{G}_\gamma$ induces a Δ-gradation on $U_{\underline{L}}$. The preimage $\phi^{-1}(\mathscr{H})$ of $\mathscr{H}$ contains $\ker \phi$, which lies in the center of $\widetilde{U}$, so there is an action of $\mathscr{H}' = \phi^{-1}(\mathscr{H})/\ker \phi$ on $\widetilde{U}$ induced by the adjoint action. If $h' \in \mathscr{H}'$, $\phi(h') = h$, and $\mu(t)$ is the minimal polynomial of $\mathrm{ad}_{U_{\underline{L}}}(h)$, then $\mu(\mathrm{ad}_{\widetilde{U}}(h'))\widetilde{U} \subset \ker \phi$, which implies that $\mathrm{ad}_{\widetilde{U}}(h')$ satisfies the polynomial $t\mu(t)$. Therefore, $\widetilde{U}$ is a sum of root spaces with respect to $\mathrm{ad}_{\widetilde{U}} \mathscr{H}'$, and $\widetilde{U}_\gamma \neq (0)$ if and only if $\gamma \in \Delta \cup \{0\}$. Clearly, ϕ induces an isomorphism of Jordan systems $(\widetilde{U}_\gamma, \gamma \in \Delta) \to ((U_{\underline{L}})_\gamma, \gamma \in \Delta)$. Moreover, $\widetilde{U}_0 = \sum_{\gamma \in \Delta} [\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma] + \ker \phi$, so that $[\widetilde{U}_0, \widetilde{U}_0] \subset \sum_{\gamma \in \Delta} [\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma]$. From $\widetilde{U} = [\widetilde{U}, \widetilde{U}]$ it follows that $\widetilde{U}_0 = [\widetilde{U}_0, \widetilde{U}_0] + \sum_{\gamma \in \Delta} [\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma] = \sum_{\gamma \in \Delta} [\widetilde{U}_{-\gamma}, \widetilde{U}_\gamma]$. Consequently, ϕ is an isomorphism. □

1.7. Our next aim is to show that if $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$ is a Δ-graded Lie algebra where Δ is a root system of rank $\geqq 2$, then the triple operations $\psi_{\alpha, -\alpha}$ in the Jordan system $\underline{L} = (L_\gamma, \gamma \in \Delta)$ are completely determined by the binary operations $\psi_{\alpha, \beta}$.

Proposition 1.8. *Let* Δ *be a root system of rank* $\geqq 2$, *Assume* $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$ *and* $L' = \sum_{\gamma \in \Delta \cup \{0\}} L'_\gamma$ *are two* Δ*-graded Lie algebras. Let* $\eta = (\eta_\gamma, \gamma \in \Delta)$ *be a collection of bijective mappings* $\eta_\gamma : L_\gamma \to L'_\gamma$ *such that for any roots* $\alpha, \beta, \alpha + \beta \in \Delta$, *and arbitrary elements* $a_\alpha \in L_\alpha$, $b_\beta \in L_\beta$, *we have* $\eta_{\alpha+\beta}([a_\alpha, b_\beta]) = [\eta_\alpha(a_\alpha), \eta_\beta(b_\beta)]$. *Then* η *is an isomorphism of the Jordan systems* $\underline{L} = (L_\gamma, \gamma \in \Delta)$ *and* $\underline{L}' = (L'_\gamma, \gamma \in \Delta)$, *and thus, the Lie algebras* L, L' *are centrally isogenous.*

Proof. For ease of notation in this argument we will drop the subscripts on the mappings η_α. In order to establish that η is an isomorphism of Jordan systems we have to prove that for any $\alpha, \beta \in \Delta$ and any elements $a_\alpha \in L_\alpha, a_{-\alpha} \in L_{-\alpha}$ and $b_\beta \in L_\beta$ we have:

$$\eta([a_{-\alpha}, a_\alpha, b_\beta]) = [\eta(a_{-\alpha}), \eta(a_\alpha), \eta(b_\beta)] .$$

(Note that our shorthand is $[x, y, z] = [[x, y], z]$, and similarly for products with more factors.) Suppose first that $\alpha \neq \pm\beta$. Then $[a_{-\alpha}, a_\alpha, b_\beta] = [a_{-\alpha}, [a_\alpha, b_\beta]] + [a_{-\alpha}, b_\beta, a_\alpha]$, where $\alpha + \beta \neq 0$ and $-\alpha + \beta \neq 0$. Hence,

$$\begin{aligned} \eta([a_{-\alpha}, a_\alpha, b_\beta]) &= \eta([a_{-\alpha}, [a_\alpha, b_\beta]]) + \eta([a_{-\alpha}, b_\beta, a_\alpha]) \\ &= [\eta(a_{-\alpha}), \eta([a_\alpha, b_\beta])] + [\eta([a_{-\alpha}, b_\beta]), \eta(a_\alpha)] \\ &= [\eta(a_{-\alpha}), [\eta(a_\alpha), \eta(b_\beta)]] + [\eta(a_{-\alpha}), \eta(b_\beta), \eta(a_\alpha)] \\ &= [\eta(a_{-\alpha}), \eta(a_\alpha), \eta(b_\beta)] \end{aligned}$$

Now suppose that $\alpha = \beta$. Without loss of generality we can assume that a_β lies in a nontrivial irreducible $\mathscr{G}$-submodule of L. In particular, $a_\beta \in [L, \mathscr{G}]$.

Since Δ is a root system of rank $\geqq 2$, an arbitrary root $\beta \in \Delta$ can be represented as $\beta = \beta' + \beta''$, where $\beta', \beta'' \in \Delta \backslash \{\pm\beta\}$. This implies that the algebra $\mathscr{G}$ is generated by root spaces $\mathscr{G}_\gamma, \gamma \in \Delta$, $\gamma \neq \pm\beta$. Hence the element a_β can be represented as

$$a_\beta = \sum [x_\delta, y_\gamma] \, ,$$

where $\gamma \in \Delta$, $\gamma \neq \pm\beta$, $y_\gamma \in \mathscr{G}_\gamma$, $x_\delta \in L_\delta$ and $\delta = \beta - \gamma$. It is easy to see that in each summand the root δ is also distinct from β and $-\beta$. Then we have

$$\begin{aligned} [a_{-\beta}, a_\beta, b_\beta] &= \sum ([a_{-\beta}, x_\delta, y_\gamma, b_\beta] - [a_{-\beta}, y_\gamma, x_\delta, b_\beta]) \\ &= \sum ([[a_{-\beta}, x_\delta], [y_\gamma, b_\beta]] + [a_{-\beta}, x_\delta, b_\beta, y_\gamma] \\ &\quad - [[a_{-\beta}, y_\gamma], [x_\delta, b_\beta]] - [a_{-\beta}, y_\gamma, b_\beta, x_\delta]) \end{aligned}$$

Hence,

$$\begin{aligned} \eta([a_{-\beta}, a_\beta, b_\beta]) &= \sum ([[\eta(a_{-\beta}), \eta(x_\delta)], [\eta(y_\gamma), \eta(b_\beta)]] \\ &\quad + [\eta(a_{-\beta}), \eta(x_\delta), \eta(b_\beta), \eta(y_\gamma)] \\ &\quad - [[\eta(a_{-\beta}), \eta(y_\gamma)], [\eta(x_\delta), \eta(b_\beta)]] \\ &\quad - [\eta(a_{-\beta}), \eta(y_\gamma), \eta(b_\beta), \eta(x_\delta)]) \\ &= [\eta(a_{-\beta}), \sum [\eta(x_\delta), \eta(y_\gamma)], \eta(b_\beta)] \\ &= [\eta(a_{-\beta}), \eta(\sum [x_\delta, y_\gamma]), \eta(b_\beta)] \\ &= [\eta(a_{-\beta}), \eta(a_\beta), \eta(b_\beta)] \, . \end{aligned}$$

Since the case when $\alpha = -\beta$ is completely analogous, the proposition is proved. □

1.9. The classification for Lie algebras graded by a root system of type $C_1 = A_1$ involves the following well-known result.

Theorem 1.10. ((*Tits*[*T*]), (*Kantor* [*Kan*]), (*Koecher* [*Ko*]), *see also* [*Jac*2] *and* [*Sc*].) *Assume L is a Lie algebra containing a subalgebra $\mathscr{G} = \langle e, f, h \rangle$ isomorphic to $sl_2(F)$ where*

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f \, .$$

Suppose further that $L = L_{-2} \oplus L_0 \oplus L_2$ where $L_0 = [L_{-2}, L_2]$ and $L_i = \{x \in L | [h, x] = ix\}$. Then under the product

$$x \circ y = \frac{1}{2}[[x, f], y]$$

$(L_2, \circ)$ is a unital Jordan algebra with identity e.

1.11. Suppose instead we begin with a unital Jordan algebra $(J, \circ)$, and let a_r be the right multiplication operator determined by $a \in J$. The commutators

$D = [b_r, c_r]$ give derivations of J, the so-called *inner derivations*. Then $\mathscr{L}(J) = \{a_r + \sum[(b_i)_r, (c_i)_r] | a, b_i, c_i \in J\}$ is a Lie algebra under the multiplication $[a_r + D, b_r + E] = [a_r, b_r] + (Db)_r - (Ea)_r + [D, E]$, and $\mathscr{L}(J)$ has an automorphism of order two: $a_r + D \to a_r + D = -a_r + D$. If $\overline{J}$ denotes a second copy of J, then $\mathscr{K}(J) = \overline{J} \oplus \mathscr{L}(J) \oplus J$ can be endowed with the structure of a Lie algebra,

(1.12)

$$[a + X + \overline{c}, b + Y + \overline{d}] = Xb - \psi(b \otimes \overline{c}) - Ya + [X, Y] - \overline{Yc} + \psi(a \otimes \overline{d}) + \overline{\overline{X}d}$$

where $\psi : J \otimes \overline{J} \to \mathscr{L}(J)$ is given by $\psi(b \otimes \overline{c}) = (b \circ c)_r + [b_r, c_r]$. The elements $e = 1 \in J$, $f = 2(\overline{1}) \in \overline{J}$, and $h = [e, f] = 2(1)_r$ generate a subalgebra isomorphic to $A_1 = sl_2(F)$, and the algebra $\mathscr{K}(J)$ decomposes into eigenspaces relative to $\operatorname{ad} h : \mathscr{K}_{-2} = \overline{J}, \mathscr{K}_0 = \mathscr{L}(J)$, $\mathscr{K}_2 = J$, corresponding to the eigenvalues $-2, 0, 2$ respectively. Thus, $\mathscr{K}(J)$ is graded by the root system A_1, and $\underline{J} = (J, \overline{J})$ is a Jordan system where the bilinear and trilinear operations can be read from (1.12). The algebra $\mathscr{K}(J)$, now commonly referred to as the *Tits–Kantor–Koecher construction*, was introduced by Tits in [T1] in a more general form, and independently discovered by Kantor [Kan] and Koecher [Ko].

1.13. The Jordan system (L_2, L_{-2}) coming from the Lie algebra L in Theorem 1.10 is completely determined by the Jordan algebra $(L_2, \circ)$ and the Lie algebra $sl_2(F)$. Indeed, suppose we have arbitrary elements $x_2, z_2 \in L_2$ and $y_{-2} \in L_{-2}$. Since L is an $sl_2(F)$-module under the adjoint action, $(\operatorname{ad} f)^2$ provides a bijection between L_2 and L_{-2}. Therefore, there is an element $y_2 \in L_2$ such that $y_{-2} = (1/4)[y_2, f, f]$, and

$$\begin{aligned} 4[[x_2, y_{-2}], z_2] &= [x_2, [y_2, f, f], z_2] \\ &= -2[(x_2 \circ y_2), f, z_2] + [[y_2, f], [x_2, f], z_2] \\ &= -4(x_2 \circ y_2) \circ z_2 - 4(y_2 \circ z_2) \circ x_2 + 4(x_2 \circ y_2) \circ z_2 \,. \end{aligned}$$

For arbitrary elements $x_{-2}, z_{-2} \in L_{-2}$, $y_2 \in L_2$, let $x_2' = (1/4)[x_{-2}, e, e]$ and $z_2' = (1/4)[z_{-2}, e, e]$. Then we have

$$[[x_{-2}, y_2], z_{-2}] = -4[(x_2' \circ y_2) \circ z_2' + (y_2 \circ z_2') \circ x_2' - (x_2' \circ y_2) \circ z_2', f, f]$$

(see [Jac1]).

1.14. Let L' be another Lie algebra with elements e', f', h' which satisfy the same relations as e, f, h. Assume that $L' = L'_{-2} \oplus L'_0 \oplus L'_2$ where $L'_0 = [L'_{-2}, L'_2]$ and $L'_i = \{x' \in L' | [h', x'] = ix'\}$. As above this determines a Jordan algebra structure on L'_2. If the Jordan algebras $(L_2, \circ)$ and $(L'_2, \circ')$ are isomorphic, then the corresponding Jordan systems $\underline{L} = (L_{-2}, L_2)$ and $\underline{L}' = (L'_{-2}, L'_2)$ are isomorphic as well. By Proposition 1.5, the Lie algebras L and L' are centrally isogenous.

1.15. This sets the stage for the Recognition Theorem for type C. Let A be a unital associative F-algebra with an involution $*: A \to A$. Consider the algebra $M_n(A)$ of $n \times n$ matrices over A and the induced involution $\sigma : (a_{i,j}) \to (a^*_{j,i})$ on $M_n(A)$. The space $H(M_n(A), \sigma)$ of all σ-symmetric matrices has the structure of a Jordan algebra relative to the product $x \circ y = (1/2)(xy + yx)$.

Consider also the algebra $M_{2n}(A)$ of $(2n) \times (2n)$ matrices over A. Then the mapping

$$\tau : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d^\sigma & -b^\sigma \\ -c^\sigma & a^\sigma \end{pmatrix} \quad a, b, c, d \in M_n(A) ,$$

is an involution on $M_{2n}(A)$.

The Lie algebra $\widetilde{S}$ of skew-symmetric elements in $M_{2n}(A)$ with respect to τ consists of the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $d = -a^\sigma$ and $b, c \in H(M_n(A), \sigma)$.

Let $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \leqq i, j \leqq n\}$ be the root system of type C_n, and let Γ be the integer lattice generated by Δ. The Lie algebra $\widetilde{S}$ is graded by Γ:

$$\begin{aligned}
\widetilde{S}_{-\varepsilon_i-\varepsilon_j} &= \{ae_{i,j+n} + a^* e_{j,i+n} \,|\, a \in A\} \ 1 \leqq i < j \leqq n \\
\widetilde{S}_{\varepsilon_i+\varepsilon_j} &= \{ae_{i+n,j} + a^* e_{j+n,i} \,|\, a \in A\} \ 1 \leqq i < j \leqq n \\
\widetilde{S}_{-2\varepsilon_i} &= \{ae_{i,i+n} \,|\, a \in A, a^* = a\} \ 1 \leqq i \leqq n \\
\widetilde{S}_{2\varepsilon_i} &= \{ae_{i+n,i} \,|\, a \in A, a^* = a\} \ 1 \leqq i \leqq n \\
\widetilde{S}_{\varepsilon_i-\varepsilon_j} &= \{ae_{i,j} - a^* e_{j+n,i+n} \,|\, a \in A\} \ 1 \leqq i \neq j \leqq n \\
\widetilde{S}_0 &= \sum_i \{ae_{i,i} - a^* e_{i+n,i+n} \,|\, a \in A, a^* = a\} \,.
\end{aligned}$$

Let $S \stackrel{\text{def}}{=} sp_{2n}(A, *)$ denote the subalgebra of $\widetilde{S}$ generated by the subspaces $\widetilde{S}_\gamma, \gamma \neq 0$. Then the Lie algebra S is clearly Δ-graded.

1.16. For $n = 3$ this construction is applicable to more general rings of coefficients. Let A be a unital alternative algebra with an involution $* : A \to A$ such that every symmetric element $a \in A$, $a^* = a$, lies in the associative center $\{a \in A | (ax)y = a(xy) \text{ for any } x, y \in A\}$ of A. Consider the algebra of 3×3 matrices over A and the involution $\sigma : (a_{i,j}) \to (a^*_{j,i})$ on $M_3(A)$. The space $H(M_3(A), \sigma)$ of σ-symmetric matrices is a Jordan algebra with respect to the multiplication $x \circ y = (1/2)(xy + yx)$. (See for example [Jac1].) The Tits–Koecher–Kantor construction $\mathscr{K}(H(M_3(A), \sigma))$ is a Lie algebra graded by the root system C_3. In particular, if A is the split octonion algebra with the standard involution $*$, then $\mathscr{K}(H(M_3(A), \sigma))$ is the split simple Lie algebra of type E_7. Thus, E_7 is C_3-graded. The universal covering algebra of $\mathscr{K}(H(M_3(A), \sigma))$

is an analogue of the Steinberg construction for groups. We will denote it as $st\,sp_6(A,*)$.

1.17. Consider the following elements of the split simple symplectic Lie algebra $\mathcal{G} = sp_{2n}(F)$ of type C_n:

$$e = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} , \quad f = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} , \quad h = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ,$$

where I is the $n \times n$ identity matrix. As in Theorem 1.10 we consider the decomposition of $sp_{2n}(A,*)$ with respect to the action of $\operatorname{ad} h$:

$$sp_{2n}(A,*) = sp_{2n}(A,*)_{-2} \oplus sp_{2n}(A,*)_0 \oplus sp_{2n}(A,*)_2 ,$$

and define a Jordan algebra structure on $sp_{2n}(A,*)_2$. Clearly, the resulting Jordan algebra is isomorphic to the Jordan algebra $H(M_n(A),\sigma)$ of σ-symmetric matrices.

Now let $L = \sum_{\gamma \in \Delta \cup \{0\}} L_\gamma$ be an arbitrary Lie algebra graded by the root system $\Delta = C_n$. The algebra $\mathcal{G} = sp_{2n}(F)$ is imbedded in L so that $\mathcal{G}_\gamma$ lies in L_γ for any $\gamma \in \Delta \cup \{0\}$. Let e, f, h be the same elements of $\mathcal{G}$ as above. Consider the root space decomposition of L with respect to the operator $\operatorname{ad} h$. Then we have

$$L_{-2} = \sum_{1 \leqq i,j \leqq n} L_{-\varepsilon_i - \varepsilon_j}$$

$$L_2 = \sum_{1 \leqq i,j \leqq n} L_{\varepsilon_i + \varepsilon_j}$$

$$[L_{-2}, L_2] = L_0 + \sum_{1 \leqq i \neq j \leqq n} L_{\varepsilon_i - \varepsilon_j} .$$

The Jordan algebra $J = (L_2, \circ)$ contains the unital subalgebra $(\mathcal{G}_2, \circ)$ of symmetric $n \times n$ matrices over F. Such Jordan algebras have been studied by N. Jacobson, who proved the following coordinatization result.

Theorem 1.18. (*Jacobson* [*Jac*1]) *Let J be a Jordan algebra which contains a unital subalgebra of symmetric $n \times n$ matrices for $n \geqq 3$. If $n \geqq 4$, then there exists a unital associative algebra A with an involution $*$ such that J is isomorphic to $H(M_n(A),\sigma)$. If $n = 3$, then J is isomorphic to $H(M_3(A),\sigma)$ where A is an alternative involutive algebra such that every symmetric element $a \in A$ lies in the associative center of A.*

1.19. In view of Proposition 1.5, this theorem applied to the Jordan algebra $(L_2, \circ)$ coming from a C_n-graded Lie algebra immediately implies that Recognition Theorem for types $C_n, n \geqq 3$ in (0.7). If $n = 1$, then all that can be said is that $(L_2, \circ)$ is a unital Jordan algebra and L is centrally isogenous to the Tits–Kantor–Koecher construction coming from it. When $n = 2$, then the Jordan algebra $(L_2, \circ)$ contains a unital subalgebra of symmetric 2×2 matrices, but very little is known about such algebras. This completes the proof of the Recognition Theorem for type C.

2. Coordinatization results

2.1. We derive two general coordinatization results for Lie algebras and apply them to Δ-graded Lie algebras. Interesting consequences for Lie algebras graded by other algebras such as the Witt algebra and for Lie bialgebras can be drawn, and but these directions will not be explored in this paper.

Proposition 2.2. *Let L be a Lie algebra over a field F, and let $\mathscr{G}$ be a perfect subalgebra of L. Suppose under the adjoint action of $\mathscr{G}$ on L that L decomposes into a direct sum of copies of $\mathscr{G}$ and trivial $\mathscr{G}$-modules. Assume that* $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},\mathscr{G})$ *is spanned by the product $f\otimes g\to[f,g]$ and* $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},F)$ *by the bilinear form $\kappa(f,g)$. If there exist $f,g,h\in\mathscr{G}$ so that $f,g,h,[[f,h],g]$ and $[f,[g,h]]$ are linearly independent, then*

$$L\cong(\mathscr{G}\otimes A)\oplus\mathscr{D}$$

where $(A,\cdot\,)$ is a unital, commutative, associative F-algebra and $\mathscr{D}$ is a Lie subalgebra of L. Multiplication in L is given by

$$[f\otimes a,g\otimes b]=[f,g]\otimes a\cdot b+\kappa(f,g)\langle a,b\rangle$$
$$[d,f\otimes a]=f\otimes da=-[f\otimes a,d]$$
$$[d,e]$$

for all $f,g\in\mathscr{G},a,b\in A$, and $d,e\in\mathscr{D}$, where $\langle\,,\rangle:A\otimes A\to\mathscr{D}$ satisfies $[d,\langle a,b\rangle]=\langle da,b\rangle+\langle a,db\rangle$. There is a representation $\phi\colon\mathscr{D}\to\mathrm{Der}_F A$ whose kernel contains $\langle A,A\rangle$, and $\langle a\cdot b,c\rangle+\langle b\cdot c,a\rangle+\langle c\cdot a,b\rangle=0$ for all $a,b,c\in A$. If the form $\kappa(\,,)$ is nonzero, it is symmetric.

Proof. We may assume $L=\mathscr{G}\oplus\sum_{i\in I}\mathscr{G}_i\oplus\sum_{j\in J}Fd_j$ where $\mathscr{G}_i\cong\mathscr{G}$ and where each Fd_j is a trivial $\mathscr{G}$-module. Let A be the F-vector space having basis $\{1,a_i\,|\,i\in I\}$, and let $\mathscr{D}=\sum_{j\in J}Fd_j$. We identify L with $(\mathscr{G}\otimes A)\oplus\mathscr{D}$ by identifying the Lie algebra $\mathscr{G}$ with $\mathscr{G}\otimes 1$ (where $[f,g]$ corresponds to $[f\otimes 1,g\otimes 1]=[f,g]\otimes 1$). The space $\mathscr{G}_i$ is matched with $\mathscr{G}\otimes a_i$ where the action of $\mathscr{G}=\mathscr{G}\otimes 1$ on $\mathscr{G}_i=\mathscr{G}\otimes a_i$ is given by $[f\otimes 1,g\otimes a_i]=[f,g]\otimes a_i$.

Suppose now that $a_0=1$ and that l,m are fixed indices in $I\cup\{0\}$. Then for all $f,g\in\mathscr{G}$, write

$$[f\otimes a_l,g\otimes a_m]=\sum_{i\in I\cup\{0\}}\zeta_i(f,g)\otimes a_i+\sum_{j\in J}\eta_j(f,g)d_j\ ,$$

where $\zeta_i:\mathscr{G}\otimes\mathscr{G}\to\mathscr{G}$ for all $i\in I\cup\{0\}$ and $\eta_j:\mathscr{G}\otimes\mathscr{G}\to F$ for all $j\in J$. Now for $h\in\mathscr{G}$,

$$[h\otimes 1,[f\otimes a_l,g\otimes a_m]]=\sum_{i\in I\cup\{0\}}[h,\zeta_i(f,g)]\otimes a_i$$

and

$$[[h \otimes 1, f \otimes a_l], g \otimes a_m] + [f \otimes a_l, [h \otimes 1, g \otimes a_m]]$$
$$= \sum_{i \in I \cup \{0\}} \zeta_i([h, f], g) \otimes a_i + \sum_{i \in I \cup \{0\}} \zeta_i(f, [h, g]) \otimes a_i$$
$$+ \sum_{j \in J} (\eta_j([h, f], g) + \eta_j(f, [h, g]))d_j .$$

These two expressions must be equal by the Jacobi identity in L. Consequently, ζ_i, η_j are $\mathscr{G}$-module homomorphisms. Since $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes \mathscr{G}, \mathscr{G})$ is one-dimensional and spanned by the commutator product of $\mathscr{G}$, $\zeta_i(f, g) = \zeta_i^{l,m}[f, g]$ for some $\zeta_i^{l,m} \in F$, and similarly, $\eta_j(f, g) = \eta_j^{l,m}\kappa(f, g)$ for all $j \in J$. As a result,

$$[f \otimes a_l, g \otimes a_m] = \sum_{i \in I \cup \{0\}} \zeta_i^{l,m}[f, g] \otimes a_i + \sum_{j \in J} \kappa(f, g)\eta_j^{l,m} d_j$$
$$= [f, g] \otimes \left(\sum_{i \in I \cup \{0\}} \zeta_i^{l,m} a_i \right) + \kappa(f, g) \sum_{j \in J} \eta_j^{l,m} d_j$$

The expressions in A and $\mathscr{D}$ depend only on l and m and not on $f, g \in \mathscr{G}$, and so we define

$$a_l \cdot a_m = \sum_{i \in I \cup \{0\}} \zeta_i^{l,m} a_i \in A$$
$$\langle a_l, a_m \rangle = \sum_{j \in J} \eta_j^{l,m} d_j \in \mathscr{D}$$

for all $l, m \in I \cup \{0\}$. These may be extended to give bilinear mappings $\cdot\ : A \otimes A \to\ A$ and $\langle\,,\rangle A \otimes A \to \mathscr{D}$ such that $[f \otimes a, g \otimes b] = [f, g] \otimes a \cdot b + \kappa(f, g)\langle a, b\rangle$ for all $f, g \in \mathscr{G}, a, b \in A$. Since $[f, g] \otimes a = [f \otimes 1, g \otimes a] = [f, g] \otimes 1 \cdot a$ and $[f, g] \otimes a = -[g \otimes a, f \otimes 1] = -[g, f] \otimes a \cdot 1$, it follows that $1 \cdot a = a = a \cdot 1$ for all $a \in A$.

The anticommutativity of L implies that $a \cdot b = b \cdot a$ for all $a, b \in A$ and $\kappa(g, f)\langle b, a\rangle = -\kappa(f, g)\langle a, b\rangle$. In particular, if $\kappa(f, f) \neq 0$ for some $f \in \mathscr{G}$, then $\langle\,,\rangle$ is skew-symmetric, and $\kappa(\,,)$ is symmetric. Alternately, if $\kappa(f, f) = 0$ for all $f \in \mathscr{G}$, then $\kappa(\,,)$ is skew-symmetric. But when $\kappa(\,,)$ is skew-symmetric, then

$$\kappa([f, g], h) = -\kappa(g, [f, h]) = -\kappa([h, f], g) = \kappa(f, [h, g])$$
$$= \kappa([g, h], f) = -\kappa(h, [g, f]) = -\kappa([f, g], h) \tag{2.3}$$

which implies that $\kappa(\,,) \equiv 0$ because $\mathscr{G}$ is perfect.

Since $\mathscr{D}$ is the centralizer of $\mathscr{G}$, it is a subalgebra of L. The product $[d, \mathscr{G} \otimes a_i] \subseteq \mathscr{G} \otimes A$ for all $d \in \mathscr{D}$. Thus, $[d, f \otimes a_l] = \sum_{i \in I \cup \{0\}} \delta_i(d, f) \otimes a_i$. Then $\delta_i(\,,) \in \mathrm{Hom}_{\mathscr{G}}(Fd \otimes \mathscr{G}, \mathscr{G})$ and so $\delta_i(d, f) = \delta_i f$ for some $\delta_i \in F$ for each i. Thus, $[d, f \otimes a_l] = f \otimes (\sum_i \delta_i a_i)$. Setting $da_l = \sum_i \delta_i a_i$ and extending gives an action of $\mathscr{D}$ on A. The Jacobi identity with $d, f\otimes a, g\otimes b$ and with $d, e, f\otimes a$ shows that

$$d(a\cdot b) = (da)\cdot b + a\cdot(db)$$

$$[d,\langle a,b\rangle] = \langle da,b\rangle + \langle a,db\rangle\ .$$

$$[d,e]a = d(ea) - e(da)\ .$$

Thus, there is a representation $\phi\colon \mathscr{D}\to \mathrm{Der}_F A$, and $\langle A,A\rangle$ is an ideal of $\mathscr{D}$.

Cyclic permutation of the factors in $[[f\otimes a, g\otimes b], h\otimes c]$ produces the relation:

$$\begin{aligned}
&\kappa([f,g],h)\langle a\cdot b,c\rangle + \kappa([g,h],f)\langle b\cdot c,a\rangle + \kappa([h,f],g)\langle c\cdot a,b\rangle\\
&\quad + \kappa(f,g)h\otimes\langle a,b\rangle c + \kappa(g,h)f\otimes\langle b,c\rangle a + \kappa(h,f)g\otimes\langle c,a\rangle b\\
&\quad + [[f,g],h]\otimes((a\cdot b)\cdot c) + [[g,h],f]\otimes((b\cdot c)\cdot a)\\
&\quad + [[h,f],g]\otimes((c\cdot a)\cdot b) = 0\ .
\end{aligned}$$

If we can choose f,g,h so that $f,g,h,[[f,h],g]$ and $[f,[g,h]]$ are linearly independent, then since $[[f,g],h] = [[f,h],g] + [f,[g,h]]$, we obtain

$$0 = (a\cdot b)\cdot c - (b\cdot c)\cdot a = (a\cdot b)\cdot c - a\cdot(b\cdot c)\ .$$

Thus, $(A,\,\cdot\,)$ is a unital, commutative, associative F-algebra, and $[[f,g],h]\otimes((a\cdot b)\cdot c) + [[g,h],f]\otimes((b\cdot c)\cdot a) + [[h,f],g]\otimes((c\cdot a)\cdot b) = ([[f,g],h] + [[g,h],f] + [[h,f],g])\otimes(a\cdot b\cdot c) = 0$ to imply $\kappa(f,g)h\otimes\langle a,b\rangle c + \kappa(g,h)f\otimes\langle b,c\rangle a + \kappa(h,f)g\otimes\langle c,a\rangle b = 0$. Since we can choose f,g,h linearly independent, it follows that $\langle a,b\rangle c = 0$ for all $a,b,c\in A$. Hence, $\langle A,A\rangle$ lies in the kernel of ϕ. Now if $\langle\,,\rangle$ is skew-symmetric, then $\kappa(\,,)$ must be symmetric, and the terms $\kappa([f,g],h) = -\kappa(g,[f,h]) = \kappa([h,f],g) = -\kappa(f,[h,g]) = \kappa([g,h],f)$ are all equal. This forces $\langle a\cdot b,c\rangle + \langle b\cdot c,a\rangle + \langle c\cdot a,b\rangle = 0$ for all $a,b,c\in A$, as claimed. This establishes the result. □

2.4. To apply Proposition 2.2 to the case that $\mathscr{G}$ is a split simple Lie algebra requires knowledge of $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},\mathscr{G})$ and $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},F)$. The explicit decompositions of the tensor products $\mathscr{G}\otimes\mathscr{G}, \mathscr{G}\otimes V$, and $V\otimes V$, where V is the little adjoint module for $\mathscr{G}$ when $\mathscr{G}$ is doubly-laced, are given in the Appendix. Observe from there that $\dim\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},F) = 1 = \dim\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},\mathscr{G})$ for all $\mathscr{G}$ except when $\mathscr{G}$ is of type A_n for $n\geqq 2$, in which case $\dim\mathrm{Hom}_{\mathscr{G}}(\mathscr{G}\otimes\mathscr{G},\mathscr{G}) = 2$. As a result, we have

Corollary 2.5. *Assume L is a Δ-graded Lie algebra whose associated split simple Lie algebra $\mathscr{G}$ is not of type A_n for $n\geqq 1$. Suppose that L decomposes into a direct sum of modules isomorphic to $\mathscr{G}$ and one-dimensional $\mathscr{G}$-modules under the adjoint action of $\mathscr{G}$ on L. Then there is a unital, commutative, associative algebra A such that L is centrally isogenous to $\mathscr{G}\otimes A$.*

Proof. Proposition 2.2 will give the result once we show that there exist $f,g,h\in\mathscr{G}$ such that $f,g,h,[[f,h],g]$ and $[f,[g,h]]$ are linearly independent. Since the rank of $\mathscr{G}$ is at least 2, there must exist two simple roots

α and β which correspond to connected nodes in the Dynkin diagram of $\mathscr{G}$. If $f \in \mathscr{G}_\alpha, g \in \mathscr{G}_\beta$, and $h \in \mathscr{G}_{-\alpha-\beta}$ are nonzero, then $f, g, h, [[f, h], g]$ and $[f, [g, h]]$ are linearly independent. Since L is generated by its root spaces with nonzero weights, we have by Proposition 2.2 that $L \cong (\mathscr{G} \otimes A) + \langle A, A \rangle$, where A is a unital, commutative, associative algebra. But since $\langle A, A \rangle$ is central, it follows that L is centrally isogenous to $\mathscr{G} \otimes A$. □

2.6. In the case that $\mathscr{G}$ is of type D_n, E_6, E_7, or E_8 this result is proven in [BM] by different arguments,

Proposition 2.7. *Let L be a Lie algebra over F, and let $\mathscr{G}$ be a perfect subalgebra on L. Suppose under the adjoint action of $\mathscr{G}$ on L that L decomposes into a direct sum of*

(2.8) (i) *modules isomorphic to the adjoint module $\mathscr{G}$,*
(ii) *modules isomorphic to a nontrivial $\mathscr{G}$-module V,*
(iii) *one-dimensional $\mathscr{G}$-modules.*

Assume that

(2.9)

$$1 = \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes \mathscr{G}, \mathscr{G}) = \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes V, V)$$

$$1 \geqq \dim \mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G}) \quad \textit{and} \quad 1 \geqq \dim \mathrm{Hom}_{\mathscr{G}}(V \otimes V, V)$$

$$1 \geqq \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes \mathscr{G}, F) \quad \textit{and} \quad 1 \geqq \dim \mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$$

$$0 = \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes \mathscr{G}, V) = \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes V, \mathscr{G}) = \dim \mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes V, F)\ .$$

Further suppose that $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G}) = F\pi$ *and* $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, V) = F\rho$ *where π and ρ, if they are nonzero, are symmetric or skew-symmetric, and that* $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes \mathscr{G}, F)$ *and* $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$ *are spanned by $\kappa(\,,\,)$ and $\lambda(\,,\,)$ respectively. When the conditions in* (2.10) *are satisfied*:

(2.10) (i) *there exist $f, g, h \in \mathscr{G}$ such that $f, g, h, [[f, h], g]$ and $[f, [g, h]]$ are linearly independent*;

(ii) *there exist $f, g \in \mathscr{G}$ and $u \in V$ such that $f \cdot (g \cdot u) \neq 0$, $g \cdot (f \cdot u) = 0$, and $\kappa(f, g) = 0$*;

(iii) *there exist $f \in \mathscr{G}$ and $u, v \in V$ such that $\pi(f \cdot u, v) = 0 = \pi(u, f \cdot v)$ and $\lambda(u, v) f \neq 0$*;

(iv) *either $\rho = 0$ or the mappings $\mathscr{G} \otimes V \otimes V \to V$ given by $f \otimes u \otimes v \to \rho(f \cdot u, v)$ and $f \otimes u \otimes v \to \rho(u, f \cdot v)$ are linearly independent in* $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes V \otimes V, V)$. *Similarly, the mappings $\mathscr{G} \otimes V \otimes V \to \mathscr{G}$ given by $f \otimes u \otimes v \to \pi(f \cdot u, v)$ and $f \otimes u \otimes v \to \pi(u, f \cdot v)$ are linearly independent in* $\mathrm{Hom}_{\mathscr{G}}(\mathscr{G} \otimes V \otimes V, \mathscr{G})$.

(v) *there exists a nonzero $\xi \in F$ such that $\xi\kappa(\pi(u, v), f) = \lambda(f \cdot u, v)$ for all $u, v \in V$ and $f \in \mathscr{G}$*;

then the following conclusions can be drawn:

$$L \cong (\mathscr{G} \otimes A) \oplus (V \otimes B) \oplus \mathscr{D}$$

where

(2.11) (i) $(A, \cdot\,)$ *is a unital, commutative, associative* F*-algebra,*
(ii) B *is a unital* A*-module,*
(iii) $\mathscr{D}$ *is a trivial* $\mathscr{G}$*-module.*

Multiplication in L *is given by*

$$(2.12)\quad \begin{aligned}[]
[f \otimes a, g \otimes a'] &= [f,g] \otimes a \cdot a' + \kappa(f,g)\langle a, a'\rangle \\
[f \otimes a, u \otimes b] &= f \cdot u \otimes a \cdot b = -[u \otimes b, f \otimes a] \\
[u \otimes b, v \otimes b'] &= \pi(u,v) \otimes (b,b') + \rho(u,v) \otimes b \cdot b' + \lambda(u,v)\langle b, b'\rangle \\
[d, f \otimes a] &= f \otimes da = -[f \otimes a, d] \\
[d, u \otimes b] &= u \otimes db = -[u \otimes b, d] \\
&[d, d']
\end{aligned}$$

for all $f, g \in \mathscr{G}, a, a' \in A, u, v \in V, b, b' \in B$ *and* $d, d' \in \mathscr{D}$, *where*

(2.13) (i) $(\,,\,)\colon B \otimes B \to A$ *is an* A*-bilinear form which is symmetric (skew-symmetric) if* π *is skew-symmetric (symmetric),*

(ii) $\cdot\, : B \otimes B \to B$ *is an* A*-bilinear map which is symmetric (skew-symmetric) if* ρ *is skew-symmetric (symmetric),*

(iii) $A1 \oplus B$ *is an* A*-algebra under the product* $(a1 + b)\cdot(a'1 + b') = a \cdot a'1 + (b,b')1 + a \cdot b' + a' \cdot b + b \cdot b'$.

(iv) $\mathscr{D}$ *is a Lie subalgebra of* L *and the map* $\langle\,,\rangle\colon (A1 \oplus B) \times (A1 \oplus B) \to \mathscr{D}$ *given by* $\langle a + b, a' + b'\rangle = \langle a, a'\rangle + \langle b, b'\rangle$ *for* $a, a' \in A, b, b' \in B$ *is an* F*-bilinear mapping whose image,* $\langle A, A\rangle + \langle B, B\rangle$, *is an ideal of* $\mathscr{D}$. *There is a representation* $\phi : \mathscr{D} \to \mathrm{Der}_F(A1 \oplus B)$ *whose kernel contains* $\langle A, A\rangle$, *and* $\langle A1 \oplus B, A1 \oplus B\rangle$ *acts trivially on* A.

(v) *If* $\kappa(\,,\,)$ *is nonzero, then* $\kappa(\,,\,)$ *is symmetric, and the relations* $\langle a \cdot a', a''\rangle + \langle a' \cdot a'', a\rangle + \langle a'' \cdot a, a'\rangle = 0$, $\langle a, (b,b')\rangle = \xi(\langle a \cdot b, b'\rangle - \langle b, a \cdot b'\rangle)$ *hold for all* $a, a', a'' \in A$ *and* $b, b' \in B$. *In particular, if* $\langle A, A\rangle = (0)$, *then* $\langle b, a \cdot b'\rangle = \langle a \cdot b, b'\rangle$.

(vi) *If* σ *denotes the permutation* $(1\,2\,3)$ *then*

$$(2.14)\quad \begin{aligned}
0 &= \sum_{j=0}^{2} \lambda(\rho(u_{\sigma^j(1)}, u_{\sigma^j(2)}), u_{\sigma^j(3)})\langle b_{\sigma^j(1)} \cdot b_{\sigma^j(2)}, b_{\sigma^j(3)}\rangle \\
0 &= \sum_{j=0}^{2} \pi(\rho(u_{\sigma^j(1)}, u_{\sigma^j(2)}), u_{\sigma^j(3)}) \otimes (b_{\sigma^j(1)} \cdot b_{\sigma^j(2)}, b_{\sigma^j(3)}) \\
0 &= \sum_{j=0}^{2} \lambda(u_{\sigma^j(1)}, u_{\sigma^j(2)}) u_{\sigma^j(3)} \otimes \langle b_{\sigma^j(1)}, b_{\sigma^j(2)}\rangle b_{\sigma^j(3)} \\
&\quad + \sum_{j=0}^{2} \pi(u_{\sigma^j(1)}, u_{\sigma^j(2)}) \cdot u_{\sigma^j(3)} \otimes (b_{\sigma^j(1)}, b_{\sigma^j(2)}) \cdot b_{\sigma^j(3)} \\
&\quad + \sum_{j=0}^{2} \rho(\rho(u_{\sigma^j(1)}, u_{\sigma^j(2)}), u_{\sigma^j(3)}) \otimes ((b_{\sigma^j(1)} \cdot b_{\sigma^j(2)}) \cdot b_{\sigma^j(3)})\,.
\end{aligned}$$

Conversely, suppose that

(2.15) (i) $\mathcal{G}$ *is an* F*-Lie algebra;* (ii) V *is a nontrivial* $\mathcal{G}$*-module;* (iii) $\pi \in \mathrm{Hom}_{\mathcal{G}}(V \otimes V, \mathcal{G})$, (iv) $\rho \in \mathrm{Hom}_{\mathcal{G}}(V \otimes V, V)$, *and* (v) $\lambda \in \mathrm{Hom}_{\mathcal{G}}(V \otimes V, F)$, *where* π, ρ, *and* λ *are symmetric or skew-symmetric*; (vi) $\kappa \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{G} \otimes \mathcal{G}, F)$ *is symmetric*; (vii) $(A, \cdot)$ *is a unital*, *commutative associative algebra*; (viii) B *is a unital* A*-module*; (ix) $(\,,\,) : B \otimes B \to A$ *is an* A*-bilinear form which is symmetric* (*skew-symmetric*) *if* π *is skew-symmetric* (*symmetric*); (x) $A1 \oplus B$ *is an* F*-algebra under the product* $(a1 + b) \cdot (a'1 + b') = a \cdot a'1 + (b, b')1 + a \cdot b' + a' \cdot b + b \cdot b'$ *where* $(a \cdot a') \cdot b = a \cdot (a' \cdot b) = a' \cdot (a \cdot b)$ *and* $\cdot : B \otimes B \to B$ *is symmetric* (*skew-symmetric*) *if* ρ *is skew-symmetric* (*symmetric*); (xi) $\mathcal{D}$ *is a Lie subalgebra of* $\mathrm{Der}_F(A1 \oplus B)$ *such that* $d(A) = 0$ *for all* $d \in \mathcal{D}$; (xii) $\langle\,,\,\rangle : B \otimes B \to \mathcal{D}$ *is an* F*-bilinear mapping which is symmetric* (*skew-symmetric*) *if* λ *is skew-symmetric* (*symmetric*), *and* $[d, \langle b, b'\rangle] = \langle db, b'\rangle + \langle b, db'\rangle$ *and* $\langle a \cdot b, b'\rangle = \langle b, a \cdot b'\rangle$ *for all* $d \in \mathcal{D}$, $a \in A$, *and* $b, b' \in B$; (xiii) $\langle\,,\,\rangle : A \otimes A \to \mathcal{D}$ *is a skew-symmetric* F*-bilinear mapping which satisfies* $\langle a \cdot a', a''\rangle + \langle a' \cdot a'', a\rangle + \langle a'' \cdot a, a'\rangle = 0 = \langle a, (b, b')\rangle$ *and* $[d, \langle a, a'\rangle] = \langle da, a'\rangle + \langle a, da'\rangle$ *for all* $d \in \mathcal{D}$, $a, a', a'' \in A$ *and* $b, b' \in B$; *and* (xiv) $\langle a, a'\rangle(A1 \oplus B) = (0)$ *for all* $a, a' \in A$;
If (2.14) *and* (2.15) *hold, then* $L = (\mathcal{G} \otimes A) \oplus (V \otimes B) \oplus \mathcal{D}$ *with multiplication given by* (2.12) *is a Lie algebra.*

Proof. The assumptions imply $L = \mathcal{G} \oplus \sum_{i \in I} \mathcal{G}_i \oplus \sum_{j \in J} V_j \oplus \sum_{k \in K} Fd_k$ where $\mathcal{G}_i \cong \mathcal{G}$ and $V_j \cong V$ as $\mathcal{G}$-modules and where each Fd_k is a trivial $\mathcal{G}$-module. Letting A be the F-vector space having basis $\{1, a_i \mid i \in I\}$, B be the F-vector space having basis $\{b_j \mid j \in J\}$, and $\mathcal{D} = \sum_{k \in K} Fd_k$, we may suppose that

$$L = (\mathcal{G} \otimes A) \oplus (V \otimes B) \oplus \mathcal{D}\ .$$

By assumption $\mathrm{Hom}_{\mathcal{G}}(\mathcal{G} \otimes \mathcal{G}, V) = (0)$, so the proof of Proposition 2.2 gives:

$$[f \otimes a, g \otimes a'] = [f, g] \otimes a \cdot a' + \kappa(f, g)\langle a, a'\rangle\ ,$$

for all $f, g \in \mathcal{G}$, $a, a' \in A$, where the all same conclusions hold.

Now $[d, u \otimes b] \in V \otimes B$ for all $d \in \mathcal{D}$, $u \in V$, and $b \in B$. Thus, there is an action of $\mathcal{D}$ on B, $d \times b \to db$ such that $[d, u \otimes b] = u \otimes db$. Suppose that $l \in I$ and $m \in J$ are fixed indices. Then for all $f \in \mathcal{G}$ and $u \in V$,

$$[f \otimes a_l, u \otimes b_m] = \sum_{i \in I \cup \{0\}} \mu_i(f, u) \otimes a_i + \sum_{j \in J} v_j(f, u) \otimes b_j + \sum_{k \in K} \tau_k(f, u) d_k\ ,$$

and applying $\mathrm{ad}(h \otimes 1)$ for $h \in \mathcal{G}$ shows that μ_i, v_j, and τ_k are $\mathcal{G}$-module homomorphisms. By our assumptions in (2.9), $\mu_i = 0 = \tau_k$ for each $i \in I \cup \{0\}$ and $k \in K$, and $v_j(f, u) = v_j^{l,m} f \cdot u$. Setting $a_l \cdot b_m = \sum_{j \in J} v_j^{l,m} b_j$, and extending linearly, gives $[f \otimes a, u \otimes b] = (f \cdot u) \otimes (a \cdot b)$ for all $a \in A$ and $b \in B$. The Jacobi identity with $d, f \otimes a$, and $u \otimes b$ shows that $d(a \cdot b) = (da) \cdot b + a \cdot (db)$. If $f \otimes a, g \otimes a'$, and $u \otimes b$ are used instead, then the relation

$$\begin{aligned} 0 &= ([f, g] \cdot u) \otimes ((a \cdot a') \cdot b) + \kappa(f, g) u \otimes \langle a, a'\rangle b \\ &\quad - (f \cdot (g \cdot u)) \otimes (a \cdot (a' \cdot b)) + g \cdot (f \cdot u)) \otimes (a' \cdot (a \cdot b)) \end{aligned}$$

is produced. Since we can choose f, g, u so that $\kappa(f, g) = 0$ and $g \cdot (f \cdot u) = 0$, but $f \cdot (g \cdot u) \neq 0$, we see that $a \cdot (a' \cdot b) = (a \cdot a') \cdot b = a' \cdot (a \cdot b)$. It follows that B is an A-module and that $\langle a, a' \rangle b = 0$ for all $a, a' \in A$ and $b \in B$.

By arguments similar to the ones used earlier, we can assume that

$$[u \otimes b, v \otimes b'] = \pi(u,v) \otimes (b, b') + \rho(u,v) \otimes (b \cdot b') + \lambda(u,v)\langle b, b' \rangle \ .$$

Then calculating with $d, u \otimes b$ and $v \otimes b'$ and with $d, d', u \otimes b$ shows that

$$\begin{aligned} [d, \langle b, b \rangle] &= \langle db, b' \rangle + \langle b, db' \rangle \\ d(b, b') &= (db, b') + (b, db') \\ d(b \cdot b') &= db \cdot b' + b \cdot db' \\ [d, d']b &= d(d'b) - d'(db) \ . \end{aligned}$$

Hence, $A1 \oplus B$ is an algebra under the product $(a1 + b) \cdot (a1 + b') = a \cdot a'1 + (b, b')1 + a \cdot b' + a' \cdot b' + b \cdot b'$, and there is a representation of $\mathscr{D}$ on $A1 \oplus B$ by derivations such that $\langle A, A \rangle$ is in the kernel. The space $\langle B, B \rangle$ is an ideal of $\mathscr{D}$. The Jacobi identity with $f \otimes a, u \otimes b$ and $v \otimes b'$ gives

$$\lambda(f \cdot u, v)\langle a \cdot b, b' \rangle + \lambda(u, f \cdot v)\langle b, a \cdot b' \rangle + \kappa(\pi(u,v), f)\langle (b, b'), a \rangle = 0$$

$$\lambda(u,v) f \otimes \langle b, b' \rangle a + [\pi(u,v), f] \otimes (b, b') \cdot a$$

$$+ \pi(f \cdot u, v) \otimes (a \cdot b, b') + \pi(u, f \cdot v) \otimes (b, a \cdot b') = 0$$

$$\rho(f \cdot u, v) \otimes (a \cdot b) \cdot b' + \rho(u, f \cdot v) \otimes b \cdot (a \cdot b') - f \cdot \rho(u,v) \otimes a \cdot (b \cdot b') = 0.$$

Since $f \cdot \rho(u,v) = \rho(f \cdot u, v) + \rho(u, f \cdot v)$, and since the mappings $f \otimes u \otimes v \to \rho(f \cdot u, v)$ and $f \otimes u \otimes v \to \rho(u, f \cdot v)$ are assumed to be linearly independent when $\rho \neq 0$, it follows that $(a \cdot b) \cdot b' = a \cdot (b \cdot b') = b \cdot (a \cdot b')$, which shows the product $b \cdot b'$ is A-bilinear, and A is in the associative center of $A1 \oplus B$.

From (2.10) part (v), we find that $\langle a, (b, b') \rangle = \xi(\langle a \cdot b, b' \rangle - \langle b, a \cdot b' \rangle)$, and from (2.10) part (iii) it follows that $\langle b, b' \rangle a = 0$ and $(a \cdot b, b') = a \cdot (b, b') = (b, a \cdot b')$ for all $b, b' \in B$ and $a \in A$. Finally, the Jacobi identity with $u_1 \otimes b_1, u_2 \otimes b_2$ and $u_3 \otimes b_3$ produces (2.14).

Our calculations in the proof show that the conditions in (2.15) along with (2.14) are sufficient to guarantee that L is a Lie algebra. □

2.16. If $\rho = 0$ (as is the case when $\mathscr{G}$ is of type B_n) any product $b \otimes b' \to b \cdot b'$ will work provided the action of any element from $\mathscr{D}$ is still a derivation on $A1 \oplus B$. In particular, $\langle A, A \rangle + \langle B, B \rangle$ must act as derivations on $A1 \oplus B$. Such products on B exist - for instance, the trivial multiplication $b \cdot b' = 0$ works and provides an important example.

2.17. In Sect. 3 we will verify that all the Lie algebras graded by B_n for $n \geqq 3$, F_4, and G_2 satisfy the hypotheses of Proposition 2.7, and so that result may be applied to determine these algebras.

Observe that in the above proposition that V and B are algebras under the products $\rho(u,v)$ and $b \cdot b'$ respectively, and so $V \otimes B$ can be regarded as an algebra under the multiplication $(u \otimes b)(v \otimes b') = \rho(u,v) \otimes b \cdot b'$. If $\langle A, A \rangle = (0)$, then $\mathscr{G} \otimes A$ is a subalgebra of L with $[f \otimes a, g \otimes a'] = [f,g] \otimes a \cdot a'$, and the action $[f \otimes a, u \otimes b] = f \cdot u \otimes a \cdot b$ can be viewed as a derivation on the algebra $V \otimes B$:

$$(f \cdot \rho(u,v)) \otimes a \cdot (b \cdot b') = \rho(f \cdot u, v) \otimes (a \cdot b) \cdot b' + \rho(u, f \cdot v) \otimes b \cdot (a \cdot b') .$$

Similarly, $\mathscr{D}$ is a Lie subalgebra of $\mathrm{Der}_F(V \otimes B)$ where $d(u \otimes b) = u \otimes db$. Thus, Proposition 2.7 can be interpreted as a particular case of more general construction which we introduce in the next section.

3. The generalized Tits construction and Lie algebras graded by B_n, F_4, and G_2

3.1. To provide realizations of the exceptional simple Lie algebras, Tits [T2] developed a construction starting with an alternative algebra $\mathscr{C}$ of degree one or two and a Jordan algebra J of degree one or three. When $\mathscr{C}$ and J are suitably specialized, the resulting Lie algebras form Freudenthal's "magic table" (See [F], [Jac2, Sect. 10], [FF], [M], or (3.27) below.). Certain extensions of the Tits construction are particular cases of a general construction which we present in this section. We prove that all the Lie algebras graded by root systems of types F_4 and G_2 arise from this general procedure. A second specialization of our construction yields all the B_n-graded Lie algebras.

3.2. Let A be a unital, commutative, associative algebra over the field F, and assume X is a unital algebra over A. An A-linear functional $t : X \to A$ is a *trace* on X if $t(ww') = t(w'w)$ and $t((ww')w'') = t(w(w'w''))$ for all $w, w', w'' \in X$. The trace is said to be *normalized* if $t(1) = 1$. The corresponding A-bilinear form $(w, w') = t(ww')$ is symmetric and associative, $(ww', w'') = (w, w'w'')$ for all $w, w', w'' \in X$. The subset X_0 of elements in X of trace zero is an A-module. Moreover, if the trace is normalized, then $X = A1 \oplus X_0$ since each element $w \in X$ decomposes as $w = t(w)1 + w - t(w)1$ where $w - t(w)1 \in X_0$. When $x, x' \in X_0$ we define $x * x' = xx' - t(xx')1 = xx' - (x,x')1$. Thus, X is associated with an A-algebra $(X_0, *)$ having a symmetric A-bilinear form $(\, , \,)$, and multiplication in X is given by

$$(3.3) \qquad (a1 + x)(a'1 + x') = aa'1 + (x,x')1 + ax' + a'x + x * x' ,$$

where $x * x' \in X_0$.

When X is an A-algebra with a normalized trace, let

$$(3.4) \qquad \mathrm{Der}^0_A(X) = \{ D \in \mathrm{Der}_A(X) \mid D(X_0) \subseteq X_0 \}$$

denote the Lie subalgebra of the A-derivations of X which send X_0 to X_0. Every derivation D in $\mathrm{Der}^0_A(X)$ restricts to a derivation in $\mathrm{Der}_A X_0$ such that

$(Dx, x') + (x, Dx') = 0$ for all $x, x' \in X_0$, and conversely every derivation in $\mathrm{Der}_A X_0$ such that $(Dx, x') + (x, Dx') = 0$ for all $x, x' \in X_0$ extends to one in $\mathrm{Der}_A^0 X$ by mapping $A1$ to zero. Let $\mathscr{D}(X)$ be a Lie subalgebra of $\mathrm{Der}_A^0(X)$. Assume there is an A-linear transformation $X_0 \otimes X_0 \to \mathscr{D}(X)$, $x \otimes x' \to D_{x,x'}$, which is skew-symmetric. Suppose $\mathfrak{A}$ is another unital, commutative associative algebra over F (though $A = \mathfrak{A}$ is allowed), and let Y, $(Y_0, \circ)$, and $\mathscr{D}(Y)$ be similarly chosen over $\mathfrak{A}$. Assume $Y_0 \otimes Y_0 \to \mathscr{D}(Y)$, $y \otimes y' \to d_{y,y'}$, is an analogous mapping for Y such that the following relations hold:

$$\text{(3.5)} \qquad \text{(i)}\ [E, D_{x,x'}] = D_{Ex,x'} + D_{x,Ex'}$$
$$\text{(ii)}\ [e, d_{y,y'}] = d_{ey,y'} + d_{y,ey'}$$

for all $E \in \mathscr{D}(X), e \in \mathscr{D}(Y)$. Then

$$\text{(3.6)} \qquad \mathscr{T}(X/A, Y/\mathfrak{A}) \stackrel{\text{def}}{=} (\mathscr{D}(X) \otimes \mathfrak{A}) \oplus (X_0 \otimes Y_0) \oplus (A \otimes \mathscr{D}(Y))$$

is the anticommutative algebra with multiplication defined by

$$\text{(3.7)} \qquad \begin{aligned}
&[D \otimes \alpha, D' \otimes \alpha'] = [D, D'] \otimes \alpha\alpha' \\
&[a \otimes d, a' \otimes d'] = aa' \otimes [d, d'] \\
&[D \otimes \alpha, a \otimes d'] = 0 \\
&[D \otimes \alpha, x \otimes y] = Dx \otimes \alpha y = -[x \otimes y, D \otimes \alpha] \\
&[a \otimes d, x \otimes y] = ax \otimes dy = -[x \otimes y, a \otimes d] \\
&[x \otimes y, x' \otimes y'] = D_{x,x'} \otimes (y, y') + (x * x') \otimes (y \circ y') + (x, x') \otimes d_{y,y'},
\end{aligned}$$

where $x, x' \in X_0$, $y, y' \in Y_0$, $a, a' \in A$, $\alpha, \alpha' \in \mathfrak{A}$, $D, D' \in \mathscr{D}(X)$, and $d, d' \in \mathscr{D}(Y)$.

3.8. We refer to this process of building of the algebra $\mathscr{T}(X/A, Y/\mathfrak{A})$ from X and Y as the *generalized Tits construction*. The symbol specifying the algebra should also include the constituents $\mathscr{D}(X)$ and $\mathscr{D}(Y)$ and the mappings $x \otimes x' \to D_{x,x'}$ and $y \otimes y' \to d_{y,y'}$, but to avoid clumsy notation, we have omitted them.

If X and Y are suitably chosen in the construction above, the resulting algebra $\mathscr{T}(X/A, Y/\mathfrak{A})$ will be a Lie algebra, though that need not always be the case. A criterion for $\mathscr{T}(X/A, Y/\mathfrak{A})$ to be Lie is given by

Proposition 3.9. *The algebra $\mathscr{T}(X/A, Y/\mathfrak{A})$ is a Lie algebra provided for the permutation $\sigma = (1\,2\,3)$ the following relations hold*:

$$\text{(3.10)} \qquad \text{(i)}\ 0 = \textstyle\sum_{j=0}^{2} (x_{\sigma^j(1)} * x_{\sigma^j(2)}, x_{\sigma^j(3)}) \otimes d_{y_{\sigma^j(1)} \circ y_{\sigma^j(2)}, y_{\sigma^j(3)}}$$

$$\text{(ii)}\ 0 = \textstyle\sum_{j=0}^{2} D_{x_{\sigma^j(1)} * x_{\sigma^j(2)}, x_{\sigma^j(3)}} \otimes (y_{\sigma^j(1)} \circ y_{\sigma^j(2)}, y_{\sigma^j(3)})$$

$$\begin{aligned}
\text{(iii)}\ 0 = &\textstyle\sum_{j=0}^{2} D_{x_{\sigma^j(1)}, x_{\sigma^j(2)}} x_{\sigma^j(3)} \otimes (y_{\sigma^j(1)}, y_{\sigma^j(2)}) y_{\sigma^j(3)} \\
&+ \textstyle\sum_{j=0}^{2} ((x_{\sigma^j(1)} * x_{\sigma^j(2)}) * x_{\sigma^j(3)}) \otimes ((y_{\sigma^j(1)} \circ y_{\sigma^j(2)}) \circ y_{\sigma^j(3)}) \\
&+ \textstyle\sum_{j=0}^{2} (x_{\sigma^j(1)}, x_{\sigma^j(2)}) x_{\sigma^j(3)} \otimes d_{y_{\sigma^j(1)}, y_{\sigma^j(2)}} y_{\sigma^j(3)}
\end{aligned}$$

for all $x_1, x_2, x_3 \in X_0$ and $y_1, y_2, y_3 \in Y_0$.

Proof. Since $X_0 \otimes Y_0$ is a Lie module for $(\mathscr{D}(X)\otimes\mathfrak{A})+(A\otimes\mathscr{D}(Y))$, the split null extension $(\mathscr{D}(X)\otimes\mathfrak{A})+(A\otimes\mathscr{D}(Y))+(X_0\otimes Y_0)$ is a Lie algebra. Thus, it suffices to compute the Jacobi sum $\mathscr{J}(z,z',z'') = [[z,z'],z''] + [[z',z''],z] + [[z'',z],z']$ for the triples

$$(z,z',z'') = \begin{cases} (E \otimes \alpha, x \otimes y, x' \otimes y') \\ (a \otimes e, x \otimes y, x' \otimes y') \\ (x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3)\,. \end{cases}$$

Now

$$\begin{aligned} \mathscr{J}(E \otimes \alpha, x \otimes y, x' \otimes y') = {} & D_{Ex,x'} \otimes (\alpha y, y') + (Ex * x') \otimes (\alpha y \circ y') \\ & + (Ex, x') \otimes d_{\alpha y, y'} - [E, D_{x,x'}] \otimes \alpha(y, y') \\ & - E(x * x') \otimes \alpha(y \circ y') + D_{x,Ex'} \otimes (y, \alpha y') \\ & + (x * Ex') \otimes (y \circ \alpha y') + (x, Ex') \otimes d_{y,\alpha y'}\ , \end{aligned}$$

which is zero by the relations in (3.5). The calculations with the second triple are virtually identical, and the third triple just amounts to the relations in (3.10). □

3.11. The most significant special case of above construction is due to J. Tits [T2]. To discuss the Tits construction, we begin with the notion of a trace identity. Let $\mathscr{F}$ denote the absolutely free (nonassociative) A-algebra on the free generators $x_1, x_2, \ldots$, and let $P(\mathscr{F})$ be the polynomial algebra on the A-module $\mathscr{F}$. We call the tensor product $P(\mathscr{F}) \bigotimes_A \mathscr{F}$ the *absolutely free trace algebra*.

Let X be an A-algebra with a trace map $t : X \to A$. An arbitrary mapping $\phi : \{x_1, x_2, \ldots\} \to X$ uniquely extends to an algebra homomorphism $\phi : \mathscr{F} \to X$. Furthermore, there exists a unique A-linear homomorphism $P_\phi : P(\mathscr{F}) \to A$ such that the diagram

$$\begin{array}{ccc} \mathscr{F} & \hookrightarrow & P(\mathscr{F}) \\ {\scriptstyle t\circ\phi} \searrow & & \swarrow {\scriptstyle P_\phi} \\ & A & \end{array}$$

is commutative.

Definition 3.12. *Let X be an A-algebra with a trace t. An element $\sum_i f_i \otimes v_i \in P(\mathscr{F}) \bigotimes_A \mathscr{F}$ is a trace identity for X if for any mapping $\phi : \{x_1, x_2, \ldots\} \to X$, we have $\sum_i P_\phi(f_i)\phi(v_i) = 0$.*

Example 3.13. (Cayley–Hamilton polynomials) Let $X = M_n(A)$ be the algebra of $n \times n$ matrices over A, and let $t(x)$ be the normalized trace map $t(x) = (1/n)\mathrm{Tr}(x)$, where $\mathrm{Tr}(x)$ denotes the usual matrix trace of x. The fact that every matrix is a root of its characteristic polynomial gives a trace identity, which in the cases $n = 2$ and $n = 3$ are the identities

$$\text{(3.14)} \qquad \begin{aligned} \mathrm{ch}_2(x) &= x^2 - 2t(x)x + (2t(x)^2 - t(x^2))1 = 0 \\ \mathrm{ch}_3(x) &= x^3 - T(x)x^2 + S(x)x - N(x)1 = 0 \end{aligned}$$

where

$$T(x) = 3t(x) \quad (3.15)$$
$$S(x) = (9/2)t(x)^2 - (3/2)t(x^2)$$
$$N(x) = t(x^3) - (9/2)t(x^2)t(x) + (9/2)t(x)^3 .$$

Examples 3.16. Let $\mathscr{C}$ denote the alternative algebra of split octonions (Cayley algebra) over the field F with the standard trace $T : C \to F$ (see [Sc, p. 74]). Then $\mathscr{C}$ with the normalized trace $t = (1/2)T$ satisfies the trace identity $\mathrm{ch}_2(x) = 0$.

When J is the 27-dimensional split exceptional Jordan algebra over F with its standard trace $T : J \to F$, then J with the normalized trace $t = (1/3)T$ satisfies the trace identity $\mathrm{ch}_3(x) = 0$, [Sc, p. 109].

3.17. For both alternative and Jordan algebras there are good notions of inner derivations. In particular, when $\mathscr{C}$ is an alternative A-algebra, let b_l and b_r be the left and right multiplication operators determined by $b \in \mathscr{C}$. Then the inner derivation $[b, b']_l - [b, b']_r - 3[b_l, b'_r]$, belongs to $\mathrm{Der}_A \mathscr{C}$. We reserve the notation $D_{b,b'}$ for

$$D_{b,b'} = (1/4)([b, b']_l - [b, b']_r - 3[b_l, b'_r]) . \quad (3.18)$$

Then $D : \mathscr{C} \bigotimes_A \mathscr{C} \to \mathrm{Der}_A(\mathscr{C})$ with $b \otimes b' \to D_{b,b'}$ satisfies:

$$D_{b,b'}(b'') = (1/4)([[b, b'], b''] - 3(b, b', b'')) \quad (3.19)$$
$$D_{b,b'} = -D_{b',b}$$
$$D_{b \cdot b', b''} + D_{b' \cdot b'', b} + D_{b'' \cdot b, b'} = 0$$
$$[E, D_{b,b'}] = D_{Eb,b'} + D_{b,Eb'} ,$$

for all $b, b', b'' \in \mathscr{C}$ and $E \in \mathrm{Der}_A(\mathscr{C})$, where $(b, b', b'') = (bb')b'' - b(b'b'')$, the associator. (See [Sc], pp. 77–78.) Assume $t : \mathscr{C} \to A$ is a normalized trace on $\mathscr{C}$ satisfying $\mathrm{ch}_2(x) = 0$. In particular, $\mathrm{ch}_2(b) = b^2 - t(b^2)1 = 0$ for all $b \in \mathscr{C}_0$. Linearizing that relation gives $0 = b \cdot b' + b \cdot b' - 2t(b \cdot b')1 = b * b' + b' * b$, which implies $b * b' = -b' * b$. It follows that

$$b * b' = (1/2)(b * b' - b' * b) = (1/2)(b \cdot b' - b' \cdot b) = (1/2)[b, b'] . \quad (3.20)$$

The inner derivations $D_{b,b'}$ belong to $\mathrm{Der}^0_A(\mathscr{C})$ for all $b, b' \in \mathscr{C}_0$.

3.21. Let J be a Jordan algebra over $\mathfrak{A}$. Then for arbitrary elements $u, v \in J$, the mapping

$$d_{u,v} = [u_r, v_r] , \quad (3.22)$$

where u_r, v_r are the right multiplication operators, is an $\mathfrak{A}$-derivation of J, and any derivation $e \in \mathrm{Der}_{\mathfrak{A}} J$ satisfies

$$[e, d_{u,v}] = d_{eu,v} + d_{u,ev} . \quad (3.23)$$

Assume J has a normalized trace $t : J \to \mathfrak{A}$ satisfying $\mathrm{ch}_3(x) = 0$. Let $(J_0, \circ)$ denote the trace zero elements where $u \circ v = u \cdot v - t(u \cdot v)1 \in J_0$ for all $u, v \in J_0$. Since the product $u \cdot v$ on J is commutative and the form $(u, v) = t(u \cdot v)$ is symmetric, $u \circ v$ is commutative.

Proposition 3.24. (Tits [T2]) (See also [Jac2, pp. 89–98].) *Assume J is a Jordan algebra over $\mathfrak{A}$ with a normalized trace which satisfies* $\mathrm{ch}_3(x) = 0$. *Let $\mathscr{C}$ denote an alternative algebra over A with a normalized trace satisfying* $\mathrm{ch}_2(x) = 0$. *Assume $\mathscr{D}(\mathscr{C})$ (resp. $\mathscr{D}(J)$) is a Lie subalgebra of* $\mathrm{Der}^0_A\mathscr{C}$ *(resp.* $\mathrm{Der}^0_{\mathfrak{A}}J$*) containing the inner derivations. Then the algebra*

$$\mathscr{T}(\mathscr{C}/A, J/\mathfrak{A}) = (\mathscr{D}(\mathscr{C}) \otimes \mathfrak{A}) \oplus (\mathscr{C}_0 \otimes J_0) \oplus (A \otimes \mathscr{D}(J))$$

with multiplication as in (3.7), *where $D : \mathscr{C} \otimes \mathscr{C} \to \mathscr{D}(\mathscr{C})$ is given by $b \otimes b' \to D_{b,b'}$ as in* (3.18) *and $d : J \otimes J \to \mathscr{D}(J)$ is given by $u \otimes v \to d_{u,v} = [u_r, v_r]$ as in* (3.22), *is a Lie algebra.*

3.25. In Tits' original construction it was assumed that $A = F = \mathfrak{A}$. However, it is clear that the identities needed in the proof hold in the more general context stated above. Several special cases of Proposition 3.24 are noteworthy. First suppose that $\mathfrak{A} = F$ and J is the 27-dimensional split exceptional simple Jordan algebra over F, which as mentioned in (3.16), has a normalized trace $t : J \to F$ such that $\mathrm{ch}_3(x) = 0$ holds. The derivation algebra of J is the split simple Lie algebra $\mathscr{G} = F_4$ and the space $V = J_0$ of trace zero elements is the 26-dimensional little adjoint representation for $\mathscr{G}$. The symmetric product $u \circ v$ gives a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, V)$. The mapping $d_{u,v} = [u_r, v_r]$ affords a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G})$, and we may take the symmetric bilinear form $(u, v) = t(u \cdot v)$ as the basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$. Then for any unital commutative, associative F-algebra A and any alternative A-algebra $\mathscr{C}$ which has a normalized trace t satisfying $\mathrm{ch}_2(x) = 0$.

$$\mathscr{T}(\mathscr{C}/A, J/F) = \mathscr{D}(\mathscr{C}) \oplus (\mathscr{C}_0 \otimes J_0) \oplus (A \otimes \mathscr{G}) = \mathscr{D}(\mathscr{C}) \oplus (\mathscr{C}_0 \otimes J_0) \oplus (A \otimes F_4)$$

with $D_{b,b'}$ as in (3.18) and $d_{u,v} = [u_r, v_r]$ as in (3.22) and with multiplication as in (3.7) is a Lie algebra. When $\mathscr{D}(\mathscr{C}) = \{D_{b,b'} \mid b, b' \in \mathscr{C}_0\}$, then $\mathscr{T}(\mathscr{C}/A, J/F)$ is a Lie algebra graded by F_4.

3.26. A second important example arises by setting $A = F$ and assuming that $\mathscr{C}$ is the algebra of split octonions over F which has a normalized trace t such that $\mathrm{ch}_2(x) = 0$ holds. The derivation algebra of $\mathscr{C}$ is the split simple Lie algebra $\mathscr{G} = G_2$ and the space $V = \mathscr{C}_0$ is the 7-dimensional little adjoint representation for $\mathscr{G}$. The skew-symmetric product $b * b' = (1/2)[b, b']$ provides a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, V)$; the mapping $D_{b,b'}$ gives a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G})$; and the symmetric bilinear form (b, b') is a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$. Then for any unital, commutative, associative F-algebra $\mathfrak{A}$ and any Jordan algebra J over $\mathfrak{A}$ with a normalized trace which satisfies $\mathrm{ch}_3(x) = 0$

$$\mathscr{T}(\mathscr{C}/F, J/\mathfrak{A}) = (\mathscr{G} \otimes \mathfrak{A}) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \mathscr{D}(J) = (G_2 \otimes \mathfrak{A}) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \mathscr{D}(J)$$

with $D_{b,b'}$ as in (3.18) and $d_{u,v} = [u_r, v_r]$ as in (3.22) and with multiplication as in (3.7) is a Lie algebra, which is G_2-graded when $\mathscr{D}(J) = \{[u_r, v_r] \mid u, v \in J_0\}$.

3.27. The construction above provides realizations of all the exceptional simple Lie algebras. Here suppose that $A = F = \mathfrak{A}$ and assume $\mathscr{C}$ is one of the four possibilities: (i) F; (ii) $F \oplus F$; (iii) a quaternion algebra over F; (iv) an octonion algebra over F. Suppose that J is one of the following five Jordan algebras over F : (I) F; (II) $H(M_3(F))$, the 3×3 symmetric matrices; (III) $\mathscr{B}^+$ or $H(\mathscr{B})$ where in the first case $\mathscr{B}$ is a central simple associative algebra of degree 3 and in the second $\mathscr{B}$ is simple of degree three over its center, which is a quadratic extension of F, and $\mathscr{B}$ has an involution of second kind. Here $H(\mathscr{B})$ is the set of symmetric elements under the involution, and in both cases the Jordan product is $u \cdot v = (1/2)(uv + vu)$. (IV) $H(M_3(\mathscr{Q}))$, where $\mathscr{Q}$ is a quaternion algebra over F; (V) an exceptional simple Jordan algebra. Further assume $\mathscr{D}(\mathscr{C}) = \mathrm{Der}_F\mathscr{C}$ and $\mathscr{D}(J) = \mathrm{Der}_F J$. Then for these choices of $\mathscr{C}$ and J the Lie algebra $\mathscr{T}(\mathscr{C}/F, J/F)$ is a simple (or in one case semisimple) Lie algebra given by Freudenthal's magic table:

$\mathscr{C}/J$	(I)	(II)	(III)	(IV)	(V)
(i)	(0)	A_1	A_2	C_3	F_4
(ii)	(0)	A_2	$A_2 \oplus A_2$	A_5	E_6
(iii)	A_1	C_3	A_5	D_6	E_7
(iv)	G_2	F_4	E_6	E_7	E_8

3.28. Other important examples where this construction produces a Lie algebra arise when Jordan algebras of symmetric bilinear forms are used as the ingredients. Suppose A and $\mathfrak{A}$ are two unital, commutative, associative F-algebras. Let $J(V) = A1 \oplus V$, where V is an A-module with a symmetric A-bilinear form $(,)$. We view V as an A-algebra with the trivial multiplication $u * v = 0$, and define $t : J(V) \to A$ by $t(a1 + v) = a$. This provides $J(V)$ with a normalized trace satisfying $\mathrm{ch}_2(x) = 0$. The A-linear derivations of $J(V)$ which map V to V, that is the derivations in $\mathrm{Der}^0_A(J(V))$, are just the A-linear skew-symmetric transformations $E : V \to V$ such that $(Eu, v) + (u, Ev) = 0$ for all $u, v \in V$. Set $\mathscr{D}(J(V)) = \{E \in \mathrm{End}_A(V) \mid (Eu, v) + (u, Ev) = 0 \text{ for all } u, v \in V\}$. The mapping

$$D_{u,v}w = (u, w)v - (v, w)u$$

is in $\mathscr{D}(J(V))$ and satisfies the relations in (3.5). In the special case that $A = F$ and the form $(,)$ is nondegenerate, the Lie algebra $\mathscr{D}(J(V))$ is the special orthogonal Lie algebra $so(V)$, which is a simple Lie algebra of type B_n when $\dim_F V = 2n + 1$ and of type D_n when $\dim_F V = 2n$.

Similarly, we assume $J(W) = \mathfrak{A}1 \oplus W$ where W is an $\mathfrak{A}$-module with a symmetric $\mathfrak{A}$-bilinear form $(,)$. We define $x \circ y = 0$ for all $x, y \in W$, set $\mathscr{D}(J(W)) = \{e \in \mathrm{End}_{\mathfrak{A}}(W) \mid (ex, y) + (x, ey) = 0 \text{ for all } x, y \in W\}$, and let

$$d_{x,y}z = (x,z)y - (y,z)x \in \mathscr{D}(J(W)) .$$

Proposition 3.29. *Assume A and $\mathfrak{A}$ are unital, commutative, associative F-algebras. Let $J(V) = A1 \oplus V$ and $J(W) = \mathfrak{A}1 \oplus W$ be Jordan algebras corresponding to symmetric bilinear forms on the A-module V and $\mathfrak{A}$-module W as in* (3.28)*. Then $\mathscr{T}(J(V)/A, J(W)/\mathfrak{A}) = (D(J(V)) \otimes \mathfrak{A}) \oplus (V \otimes W) \oplus (A \otimes \mathscr{D}(J(W)))$ with $D_{u,v}w = (u,w)v - (v,w)u$ for all $u,v,w \in V$ and $d_{x,y}z = (x,z)y - (y,z)x$ for all $x,y,z \in W$ and with multiplication given by* (3.7) *is a Lie algebra.*

Proof. All that remains to be shown is that (3.10) holds. However, the first two relations are immediate, since the products $u * v$ and $x \circ y$ are zero in these algebras. The third reduces to looking at cyclic permutations of

$$((v_1,v_3)v_2 - (v_2,v_3)v_1) \otimes (x_1,x_2)x_3 + (v_1,v_2)v_3 \otimes ((x_1,x_3)x_2 - (x_2,x_3)x_1)\,,$$

which do indeed sum to zero. □

Corollary 3.30. *Assume $A = F = \mathfrak{A}$ and let V and W be finite dimensional vector spaces over F with nondegenerate symmetric bilinear forms. Extend the forms on V and W to a nondegenerate symmetric bilinear form on $V \oplus W$ by decreeing $(V,W) = 0$. Then*

$$\begin{aligned}\mathscr{T}(J(V)/F, J(W)/F) &\cong so(V \oplus W)\\ &= \{\tau \in \mathrm{End}_F(V \oplus W) \,|\, (\tau y, z)\\ &\qquad + (y, \tau z) = 0 \quad \text{for all } \ y,z \in V \oplus W\}.\end{aligned}$$

Proof. The theorem above endows $\mathscr{T}(J(V)/F, J(W)/F)$ with the structure of a Lie algebra, and

$$\mathscr{T}(J(V)/F, J(W)/F) = \mathscr{D}(J(V)) \oplus (V \otimes W) \oplus \mathscr{D}(J(W)) \tag{3.31}$$

where $\mathscr{D}(J(V)) \cong so(V)$ and $\mathscr{D}(J(W)) \cong so(W)$. Every transformation $\tau : V \to W$ can be extended to an element of $so(V \oplus W)$ by defining its action on W to be the transpose of that on V. Thus, $\mathrm{Hom}_F(V,W)$ embeds in $so(V \oplus W)$. Now $\mathrm{Hom}_F(V,W) \cong V^* \otimes W \cong V \otimes W$ by the identification of V with its dual that comes from the bilinear form. Using this we can see that the right side of (3.31) is isomorphic to $so(V \oplus W)$. □

3.32. The following table displays the Lie algebras $\mathscr{T}(J(V)/F, J(W)/F)$ coming from Corollary 3.30. Here we assume that V has dimension $2m$ or $2m+1$ and W dimension $2n$ or $2n+1$.

V/W	$2n$	$2n+1$
$2m$	D_{m+n}	B_{m+n}
$2m+1$	B_{m+n}	D_{m+n+1}

Note that since the weights of V are not roots of $so(V)$ when V is $2m$-dimensional, this construction produces Lie algebra, which is not however D_m-graded.

Consider the special case that $m = 3$ and $\dim V = 7$. Then for all $n \geqq 0$ we have $B_{n+3} \cong \mathscr{T}(J(V)/F, J(W)/F) = B_3 \oplus (V \otimes W) \oplus so(W)$, where $\dim W = 2n$. The split simple Lie algebra B_3 contains the simple Lie algebra G_2, and B_3 decomposes into the sum of G_2 and its 7-dimensional little adjoint module V relative to the adjoint action of G_2. It is the same 7-dimensional module V in each case since the 7-dimensional B_3-module remains irreducible upon restriction to G_2. Thus, we see that B_{n+3} has a G_2-grading relative to which $B_{n+3} \cong G_2 \oplus (V \otimes (F1 \oplus W)) \oplus so(W)$. Similarly for all $n \geqq 0$, $D_{n+4} \cong G_2 \oplus (V \otimes (F1 \oplus W)) \oplus so(W)$, where $\dim W = 2n + 1$. Theorem 3.47 below characterizes G_2-graded Lie algebras. From there we will see $F1 \oplus W$ is the set of trace zero elements of a Jordan algebra having a normalized trace which satisfies $\mathrm{ch}_3(x) = 0$.

3.33. It is perhaps worth commenting at this juncture that the algebra X in the generalized Tits construction $\mathscr{T}(X/A, Y/\mathfrak{A})$ could be assumed to be associated with an A-algebra $(X_0, *)$ having a skew-symmetric A-bilinear form $(\ ,\ )$ such that multiplication in X is given by (3.3). If the form is skew-symmetric, then the mapping $d : Y \otimes Y \to \mathscr{D}(Y)$ must be symmetric. The algebra Y could be taken to be associated with an $\mathfrak{A}$-algebra $(Y_0, \circ)$ having either a symmetric or skew-symmetric form, provided $D : X \otimes X \to \mathscr{D}(X)$ is assumed to be skew-symmetric when the form is symmetric, and symmetric when the form is skew-symmetric.

For example, the analogue of the construction in (3.28) works when V and W have skew-symmetric bilinear forms, however, the mappings $D_{u,v}w = (u, w)v + (v, w)u$ must be used instead. Then $J(V) = A1 \oplus V$ is not a Jordan algebra; nevertheless

$$\mathscr{T}(J(V)/F, J(W)/F) \cong sp(V \oplus W)\,,$$

so that if $\dim_F V = 2m$ and $\dim_F W = 2n$, then $\mathscr{T}(J(V)/F, /J(W)/F)$ is a split simple Lie algebra of type C_{m+n}. The weights of V are not roots of C_m so the resulting Lie algebra is not C_m-graded.

3.34. We turn our attention now to showing that the above constructions yield all the B_n, F_4, and G_2-graded Lie algebras. Our approach is to appeal to Proposition 2.7. This requires the information in the Appendix, which says that the condition (2.9) on the homomorphisms are satisfied, and the following result, which asserts that (2.10) holds. In proving the next lemma we use certain standard facts about exceptional algebras – all of which can be found in Jacobson's book [Jac2].

Lemma 3.35. *Assume $\mathscr{G}$ is a split simple Lie algebra over F of type B_n for $n \geqq 3$, F_4, or G_2, and let V denote its little adjoint module. Then there exist homomorphisms $\pi \in \mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G})$, $\rho \in \mathrm{Hom}_{\mathscr{G}}(V \otimes V, V)$, and $\lambda \in \mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$, where π, ρ, and λ are symmetric or skew-symmetric such that the conditions in* (2.10) *hold, where $\kappa(\ ,\ )$ may be taken to be any nondegenerate $\mathscr{G}$-invariant form on $\mathscr{G}$.*

Proof. (i) Condition (2.10)(i) can be verified exactly as in the proof of Corollary 2.5.

(ii) Let v_+ denote a nonzero maximal vector in V and let $e_i, f_i, h_i (i = 1, \dots, n)$ be canonical generators for $\mathscr{G}$. Then the fundamental weights ω_i are dual to the h_j's, $\omega_i(h_j) = \delta_{i,j}$. The weight of v_+ is ω_i where $i = 1$ when $\mathscr{G}$ is of type B_n and G_2 and $i = 4$ when F_4. Since $f_j^{\omega_i(h_j)+1} v_+ = 0$ for each j, (see [H], 21.4), then $f_j \cdot v_+ = 0$ for all $j \neq i$. It follows from the representation theory of $\langle e_i, h_i, f_i \rangle \cong sl(2)$ that $f_i \cdot v_+ \neq 0$. In particular, when $\mathscr{G}$ is of type B_n or G_2, we have $f_2 \cdot (f_1 \cdot v_+) \neq 0$, but $f_1 \cdot (f_2 \cdot v_+) = 0$. Since $\kappa(f, g) = 0$ for all root vectors f, g unless they have opposite weights, we have $\kappa(f_1, f_2) = 0$. The same argument works for F_4, the only difference being that the vectors f_3 and f_4 must be used. This shows that (2.10)(ii) holds.

(iii) Suppose first that $\mathscr{G}$ is of type G_2 and identify V with the traceless elements $\mathscr{C}_0$ in a split octonion algebra $\mathscr{C}$. Then $\mathscr{C}$ has two orthogonal idempotents e_1, and e_2 such that $e_1 + e_2 = 1$ and $e_1 - e_2 \in \mathscr{C}_0$. We may assume that $\lambda(x, y) = (x, y)$, the bilinear form coming from the normalized trace t on $\mathscr{C}$, and $\pi(x, y) = D_{x,y}$, the inner derivation in (3.18). Then $\lambda(e_1 - e_2, e_1 - e_2) = t((e_1 - e_2)(e_1 - e_2)) = t(1) = 1$. The subalgebra of G_2 of elements which map e_1 and e_2 to zero is isomorphic to $sl_3(F)$. Taking any nonzero element f from that subalgebra, we have $\pi(f \cdot (e_1 - e_2), e_1 - e_2) = 0 = \pi(e_1 - e_2, f \cdot (e_1 - e_2))$, but $\lambda(e_1 - e_2, e_1 - e_2) f \neq 0$. Thus, (2.10)(iii) is shown. An analogous computation works for F_4. Here we identify V with the traceless elements J_0 of a split exceptional simple Jordan algebra J having normalized trace t. The algebra J has three orthogonal idempotents e_1, e_2, e_3 with $e_1 + e_2 + e_3 = 1$ and $e_1 - e_2, e_2 - e_3 \in J_0$. For the bilinear form we may take $\lambda(x, y) = (x, y) = t(x \cdot y)$ (or any scalar multiple thereof), and we may assume $\pi(x, y) = [x_r, y_r]$. The subalgebra of F_4 of derivations f which map these idempotents to zero is isomorphic to D_4. Then since $(e_1 - e_2, e_2 - e_3) = -t(e_2) \neq 0$, but $\pi(f \cdot (e_1 - e_2), e_2 - e_3) = 0 = \pi(e_1 - e_2, f \cdot (e_2 - e_3))$, the conclusions in (2.10)(iii) hold.

When $\mathscr{G}$ is of type B_n, we may assume V has a basis $v_1, \dots, v_{2n+1}$ and a nondegenerate symmetric bilinear form such that $(v_i, v_j) = 0$ if $j \neq 2n + 2 - i$ and $(v_i, v_{2n+2-i}) = 1$. Then $\mathscr{G}$ is spanned by the matrices $e_{i,j} - e_{2n+2-j,2n+2-i}$, where $e_{i,j}$ is the $(2n+1) \times (2n+1)$ matrix unit. Let $\pi(u, v) = D_{u,v} = (u, w)v - (v, w)u$ as in 3.29. Then for $f = e_{1,2} - e_{2n,2n+1}$ we have $\pi(f \cdot v_3, v_{2n-1}) = 0 = \pi(v_3, f \cdot v_{2n-1})$ but $(v_3, v_{2n-1}) f = f$ (recall $n \geqq 3$). This completes the verification of (2.10)(iii).

(iv) When $\mathscr{G}$ is of type B_n, then $\rho = 0$, and $\pi(x, y) = D_{x,y}$. For f as in (iii), $\pi(f \cdot v_3, v_2) = 0$, but $\pi(v_3, f \cdot v_2) = \pi(v_3, v_1) = D_{v_3,v_1} \neq 0$. Consequently, (iv) holds for algebras of type B_n.

Suppose then that $\mathscr{G}$ is of type F_4. The exceptional Jordan algebra J has a Peirce space decomposition

$$J = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus J_{1,2} \oplus J_{2,3} \oplus J_{1,3}$$

relative to the idempotents discussed above, where $J_{i,j}$ consists of elements $\{b_{i,j} | b \in \mathscr{C}\}$, b ranging over the elements of the split octonion algebra $\mathscr{C}$, and where

$$e_i \cdot b_{i,j} = (1/2)b_{i,j} = b_{i,j} \cdot e_j$$
$$b_{i,j} \cdot c_{j,k} = (1/2)(bc)_{i,k}$$
$$b_{i,j} = \bar{b}_{j,i} \ .$$

The elements f of the D_4 subalgebra of F_4 map the e_i's to 0 and preserve the spaces $J_{i,j}$, $f \cdot b_{i,j} = (fb)_{i,j}$. Thus we can choose f and b so that $fb = c \neq 0$ and $\pi(f \cdot (e_1 - e_2), b_{1,3}) = 0$, but $\pi(e_1 - e_2, f \cdot b_{1,3}) = \pi(e_1 - e_2, c_{1,3})$. Now $\pi(e_1 - e_2, c_{1,3}) = [(e_1 - e_2)_r, (c_{1,3})_r]$, and to see this is nonzero observe:

$$[(e_1 - e_2)_r, (c_{1,3})_r]a_{1,2} = -\{e_1 - e_2, a_{1,2}, c_{1,3}\} = (1/2)(e_1 - e_2)(\bar{a}c)_{2,3}$$
$$= -(1/4)(\bar{a}c)_{2,3} \ .$$

The argument for ρ is virtually identical. For these same elements, $\rho(f \cdot (e_1 - e_2), b_{1,3}) = 0$ but $\rho(e_1 - e_2, f \cdot b_{1,3}) = (e_1 - e_2) \circ (fb)_{1,3} = (e_1 - e_2) \cdot (fb)_{1,3} - t((e_1 - e_2) \cdot (fb)_{1,3}) = (1/2)(fb)_{1,3} \neq 0$. Consequently, the maps $f \otimes u \otimes v \to \rho(f \cdot u, v)$ and $f \otimes u \otimes v \to \rho(u, f \cdot v)$ must be linearly independent, and (iv) of (2.10) is valid in the F_4 case.

The G_2 case is similar in spirit. The idempotents e_1, e_2 used previously decompose the split octonion algebra $\mathscr{C}$ into Peirce spaces

$$\mathscr{C} = Fe_1 \oplus Fe_2 \oplus \mathscr{C}_{1,2} \oplus \mathscr{C}_{2,1}$$

with $\mathscr{C}_{i,j} = \mathscr{Q}_0 e_j$ consisting of elements ae_j where $a \in \mathscr{Q}_0$, the 3-dimensional space of traceless elements in a quaternion algebra $\mathscr{Q}$. Here for $i \neq j$,

$$e_i(be_j) = be_j = (be_j)e_j$$
$$e_j(be_j) = (be_j)e_i \ .$$

The elements f of G_2 which map e_1, e_2 to zero determine a copy of $sl_3(F)$, and the space $\mathscr{C}_{1,2}$ is its natural 3-dimensional representation $f \cdot (ae_2) = (fa)e_2$. Thus we can choose f, a so $fa = b \neq 0$. Then $\pi(f \cdot (e_1 - e_2), ae_2) = 0$, but $\pi(e_1 - e_2, f \cdot ae_2) = \pi(e_1 - e_2, be_2) = D_{e_1 - e_2, be_2}$. Now $D_{e_1 - e_2, be_2} \neq 0$, for example:

$$D_{e_1 - e_2, be_2}(e_1 - e_2) = ([e_1 - e_2, be_2]_r - [e_1 - e_2, be_2]_l - 3[(e_1 - e_2)_r, (be_2)_l])$$
$$\times (e_1 - e_2)$$
$$= (2(be_2)_r - 2(be_2)_l - 3[(e_1 - e_2)_r, (be_2)_l])(e_1 - e_2)$$
$$= 4be_2 \ .$$

Likewise, $\rho(f \cdot (e_1 - e_2), ae_2) = 0$, but $\rho(e_1 - e_2, f \cdot ae_2) = [e_1 - e_2, be_2] = 2be_2 \neq 0$. This concludes the proof of (2.10)(iv).

(v) The mappings $x \otimes y \otimes f \to \kappa(\pi(x, y), f)$ and $x \otimes y \otimes f \to \lambda(f \cdot x, y)$ can be regarded as $\mathscr{G}$-module homomorphisms in $\mathrm{Hom}_{\mathscr{G}}(V \otimes V \otimes \mathscr{G}, F)$. Since

$V \otimes V$ is completely reducible and has a unique summand isomorphic to $\mathscr{G}$, it follows that $\mathrm{Hom}_{\mathscr{G}}(V \otimes V \otimes \mathscr{G}, F)$ is one-dimensional. Thus, the two maps must be scalar multiples of each other. □

Theorem 3.36. *Let L be an F_4-graded Lie algebra over F. Then there exists a unital, commutative, associative algebra A and an alternative algebra $\mathscr{C}$ over A with a normalized trace which satisfies* $\mathrm{ch}_2(x) = 0$ *such that L is centrally isogenous with*

$$\begin{aligned}\mathscr{T}(J/F, \mathscr{C}/A) &= (\mathrm{Der}_F J \otimes A) \oplus (J_0 \otimes \mathscr{C}_0) \oplus \langle \mathscr{C}, \mathscr{C} \rangle \,, \\ &= (\mathscr{G} \otimes A) \oplus (J_0 \otimes \mathscr{C}_0) \oplus \langle \mathscr{C}, \mathscr{C} \rangle \,,\end{aligned}$$

where $\mathscr{G}$ is a split simple Lie algebra of type F_4, which we identify with the derivation algebra of a split exceptional 27*-dimensional Jordan algebra J, and $\langle \mathscr{C}, \mathscr{C} \rangle$ is the Lie subalgebra of inner derivations of $\mathscr{C}$. The multiplication in $\mathscr{T}(J/F, \mathscr{C}/A)$ is given by* (3.7) *where $D_{x,x'}$ is the inner derivation determined by $x, x' \in \mathscr{C}_0$ as in* (3.18) *and $d_{y,y'} = [y_r, y'_r]$ for all $y, y' \in J_0$.*

Proof. By Proposition 2.7 we may assume that L has the structure $(\mathscr{G} \otimes A) \oplus (J_0 \otimes B) \oplus \langle \mathscr{B}, \mathscr{B} \rangle$, where J_0 is the set of trace zero elements of a split, simple exceptional Jordan algebra J over F, and $\mathrm{Der}_F J = \mathscr{G}$, a split simple Lie algebra of type F_4. We take the symmetric bilinear form $\lambda(u, v) = (u, v) = t(u \cdot v)$ as the basis for $\mathrm{Hom}_{\mathscr{G}}(J_0 \otimes J_0, F)$, the symmetric product $\rho(u, v) = u \circ v$ as a basis for $\mathrm{Hom}_{\mathscr{G}}(J_0 \times J_0, J_0)$, and the skew-symmetric mapping $\pi(u, v) = [u_r, v_r]$ as a basis for $\mathrm{Hom}_{\mathscr{G}}(J_0 \otimes J_0, \mathscr{G})$. Thus by Proposition 2.7, L has multiplication given as in (2.12) where A is a unital, commutative, associative F-algebra, B is a unital A-module, the mapping $* : B \otimes B \to B$ is skew-symmetric, and the mapping $\langle \, , \, \rangle : B \otimes B \to \mathrm{Der}(A1 \oplus B), b \otimes b' \to \langle b, b' \rangle$, is skew-symmetric and $\langle b, b' \rangle (A) = 0$ for all $b, b' \in B$. Moreover, (2.14) must hold.

Now setting $u = u_1, \; v = u_2, \; w = u_3, \; b = b_1 = b_3$, and $b' = b_2$, we have from the second equation of (2.14) that

$$[(u \circ v)_r, w_r] \otimes (b * b', b) + [(v \circ w)_r, u_r] \otimes (b' * b, b) = 0 \,.$$

We can find $u, v, w \in J_0$ such that $[(u \circ v)_r, w_r]$ and $[(v \circ w)_r, u_r]$ are linearly independent; for example, we can identify J with the 3×3 Hermitian matrices over the split octonions and set $u = e_{1,1} - e_{2,2}, v = e_{1,2} + e_{2,1}$ and $w = e_{1,1} - e_{3,3}$. Consequently,

$$(b * b', b) = 0 \quad \text{for all } b, b' \in B \,. \tag{3.37}$$

With the same substitution, the third equation in (2.14) becomes

$$\begin{aligned}0 = {} & (u, v) w \otimes \langle b, b' \rangle b + (v, w) u \otimes \langle b', b \rangle b \\ & + [u_r, v_r] w \otimes (b, b') b + [v_r, w_r] u \otimes (b', b) b + [w_r, u_r] v \otimes (b, b) b' \\ & + ((u \circ v) \circ w) \otimes ((b * b') * b) + ((v \circ w) \circ u) \otimes ((b' * b) * b) \,.\end{aligned} \tag{3.38}$$

Now $[u_r, v_r] w = -\{u, w, v\}$ and $[w_r, u_r] v = -[u_r, v_r] w - [v_r, w_r] u = \{u, w, v\} + \{v, u, w\}$, where $\{u, w, v\}$ is the associator in the Jordan algebra J. Moreover, using the fact that $u \circ v = u \cdot v - (u,v)1$, we get $(u \circ v) \circ w = (u \cdot v) \cdot w - (u \cdot v, w)1 - (u,v)w$. Since $(u \cdot v, w) = (v \cdot w, u)$ we may use these results to reduce (3.38) to

$$\begin{aligned} (3.39) \qquad 0 &= ((u,v)w - (v,w)u) \otimes \langle b,b'\rangle b \\ &\quad - (\{u,w,v\} + \{v,u,w\}) \otimes ((b,b')b - (b,b)b') \\ &\quad - (\{w,v,u\} + (u,v)w - (v,w)u) \otimes ((b * b') * b) \\ &= ((u,v)w - (v,w)u) \otimes \langle b,b'\rangle b \\ &\quad + \{w,v,u\} \otimes ((b,b')b - (b,b)b' - ((b * b') * b)) \\ &\quad + ((v,w)u - (u,v)w) \otimes ((b * b') * b). \end{aligned}$$

Since J_0 possesses elements u, v, w such that $u, w, \{u,w,v\}$ are linearly independent and $(u,v)w - (v,w)u \neq 0$ (for example, $u = e_{1,2} + e_{2,1}, v = u + e_{2,2} - e_{3,3}$, and $w = e_{2,3} + e_{3,2}$ where $e_{i,j}$ is the 3×3 matrix unit), we may deduce from (3.39) that

$$\begin{aligned} (3.40) \qquad (b * b') * b &= (b,b')b - (b,b)b' \quad \text{for all } b, b' \in B \\ \langle b,b'\rangle b &= (b * b') * b. \end{aligned}$$

We define a multiplication on $\mathscr{C} \stackrel{\text{def}}{=} A1 \oplus B$ by setting

$$(3.41) \qquad (a1 + b)(a'1 + b') = aa'1 + (b,b')1 + ab' + a'b + b * b' ,$$

for $a, a' \in A$ and $b, b' \in B$. Then under this product, $(a1+b)^2 = 2ab + (b,b)1 + a^2 1$, so that

$$(3.42) \qquad (a1 + b)^2 - 2t(a1 + b)(a1 + b) + n(a1 + b)1 = 0 ,$$

where $t(a1 + b) = a$, and $n(a1 + b) = a^2 - (b,b) = 2t(a1 + b)^2 - t((a1 + b)^2)$. In particular, B is the set $\mathscr{C}_0$ of elements b with trace $t(b) = 0$, and $t(bb') = t(b * b' + (b,b')) = (b,b')$, holds for all $b, b' \in B$. The map $t(a1 + b) = a$ satisfies $t((a1 + b)(a'1 + b')) = aa' + (b,b') = t((a'1 + b')(a1 + b))$ and $t(1) = 1$. It is easy to see that associativity of t amounts to showing $(b * b', b'') = (b, b' * b'')$ for all $b, b', b'' \in B$. But replacing b by $b + b''$ in (3.37) gives $0 = (b * b', b'') + (b'' * b', b) = (b * b', b'') - (b, b' * b'')$, as desired.

Consider the associator (b, b, b') relative to the product in $\mathscr{C}$ for $b, b' \in B$. Then

$$\begin{aligned} (b,b,b') = b^2 b' - b(bb') &= (b,b)b' - b(b * b') - (b,b')b \\ &= (b,b)b' - (b * (b * b')) - (b, b * b') - (b,b')b \\ &= 0 \end{aligned}$$

by (3.40) and (3.42). Similarly,

$$
\begin{aligned}
(b', b, b) = (b'b)b - b'b^2 &= (b' * b)b + (b', b)b - (b, b)b' \\
&= (b' * b) * b + (b' * b, b) + (b, b')b - (b, b)b' \\
&= 0 \,.
\end{aligned}
$$

Since the products $b * b'$ and (b, b') are A-bilinear by Proposition 2.7, the elements of A lie in the associative center of $\mathscr{C}$. Therefore, we can conclude from these calculations that $\mathscr{C}$ is alternative. We have established in (3.42) that normalized trace t satisfies $\mathrm{ch}_2(x) = 0$.

Now to complete the proof we show that

$$
\langle b, b' \rangle b'' = (1/4)([[b, b'], b''] - 3(b, b', b'')) = D_{b,b'}(b'') \,, \tag{3.43}
$$

for all $b, b', b'' \in B$ where $D_{b,b'}$ is the inner derivation given in (3.19). Observe first that (3.41) implies that $b * c = (1/2)[b, c]$ for all $b, c \in B$. Next note that (3.43) holds in the special case that $b'' = b$ by the second relation in (3.40). Having that, we may linearize the relation $\langle b, c \rangle c = (1/4)[[b, c], c]$ to obtain

$$
\begin{aligned}
\langle b, b' \rangle b'' + \langle b, b'' \rangle b' &= \langle b, b' \rangle b'' - \langle b'', b \rangle b' \\
&= (1/4)([[b, b'], b''] + [[b, b''], b']) \,.
\end{aligned} \tag{3.44}
$$

Similarly,

$$
\begin{aligned}
\langle b, b' \rangle b'' + \langle b'', b' \rangle b &= \langle b, b' \rangle b'' - \langle b', b'' \rangle b \\
&= (1/4)([[b, b'], b''] + [[b'', b'], b]) \,.
\end{aligned} \tag{3.45}
$$

Taking $u_1 = u_2 = u_3 = u$ and $b_1 = b$, $b_2 = b'$ and $b_3 = b''$ in (2.14) we get

$$
\begin{aligned}
&(u \circ u) \circ u \otimes ((b * b') * b'' + (b' * b'') * b + (b'' * b) * b') \\
&\quad + (u, u)u \otimes (\langle b, b' \rangle b'' + \langle b', b'' \rangle b + \langle b'', b \rangle b') = 0 \,.
\end{aligned}
$$

Since for all $u \in J_0$, the relation $(u \circ u) \circ u = (1/2)(u, u)u$ holds, it follows that

$$
\begin{aligned}
&\langle b, b' \rangle b'' + \langle b', b'' \rangle b + \langle b'', b \rangle b' \\
&\quad = -(1/2)((b * b') * b'' + (b' * b'') * b + (b'' * b) * b') \\
&\quad = -(1/2)(1/4)([[b, b'], b''] + [[b', b''], b] + [[b'', b], b']) \\
&\quad = -(1/2)(1/4)6(b, b', b'') = -(3/4)(b, b', b'') \,,
\end{aligned} \tag{3.46}
$$

as the identity $[[b, b'], b''] + [[b', b''], b] + [[b'', b], b'] = 6(b, b', b'')$ is valid in any alternative algebra (see [Sc, p. 125]). Combining (3.44), (3.45) and (3.46) we have then

$$
3\langle b, b' \rangle b'' = (1/4)(2[[b, b'], b''] + [[b, b''], b'] + [[b'', b'], b] - 3(b, b', b''))
$$

which together with $[[b, b''], b'] + [[b'', b'], b] = -[[b', b], b''] + 6(b, b'', b')$ yields

$$
3\langle b, b' \rangle b'' = (1/4)(3[[b, b'], b''] - 9(b, b', b'')) \,,
$$

which is equivalent to (3.43). Consequently, all the assertions in Theorem 3.36 hold. □

Theorem 3.47. *Let L be a G_2-graded Lie algebra. Then there exists a unital, commutative, associative algebra A and a Jordan algebra J over A with a normalized trace which satisfies* $\mathrm{ch}_3(x) = 0$ *such that L is centrally isogenous with*

$$\mathscr{T}(\mathscr{C}/F, J/A) = (\mathrm{Der}_F\mathscr{C} \otimes A) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \langle J,J\rangle ,$$
$$= (\mathscr{G} \otimes A) \oplus (\mathscr{C}_0 \otimes J_0) \oplus \langle J,J\rangle ,$$

where $\mathscr{G}$ is a split simple Lie algebra of type G_2, which we identify with the derivations of a split octonion algebra $\mathscr{C}$ over F, and $\langle J,J\rangle$ is the Lie subalgebra of inner derivations of J. Multiplication is given by (3.7) *where $D_{x,x'}$ is the inner derivation in* (3.18) *for all $x,x' \in \mathscr{C}_0$ and $d_{y,y'} = [y_r, y'_r]$ for $y, y' \in J_0$.*

Proof. By Proposition 2.7 we may assume L is centrally isogenous to an algebra $(\mathscr{G}\otimes A)\oplus(\mathscr{C}_0\otimes B)\oplus\langle B,B\rangle$ with multiplication given by that proposition, where $\mathscr{C}$ is the split octonions with its normalized trace, $\mathscr{G} = \mathrm{Der}_F(\mathscr{C}), \mathscr{C}_0$ is the set of trace zero elements in $\mathscr{C}$, $\rho(x, y) = x * y = (1/2)[x, y]$, $\lambda(x, y) = (x, y) = t(xy)$, and $\pi(x, y) = D_{x, y}$ as in (3.18) for all $x, y \in \mathscr{C}_0$. Consider $J \stackrel{\mathrm{def}}{=} A1 \oplus B$ with the multiplication

$$(a1 + b) \cdot (a'1 + b') = aa'1 + (b, b')1 + ab' + a'b + b \circ b' ,$$

and let $t : J \to A$ be the A-linear functional $t(a1+b) = a$ for $a \in A$ and $b \in B$. We claim that J is a Jordan algebra such that for any element $q \in J$,

$$\mathrm{ch}_3(q) = q^3 - T(q)q^2 + S(q)q - N(q)1 = 0 ,$$

where $T(q) = 3t(q), S(q) = (9/2)t(q)^2 - (3/2)t(q^2)$, and $N(q) = t(q^3) - (9/2)t(q^2)t(q) + (9/2)t(q)^3$ as in (3.15). To see this, consider the third of the identities in (2.14) with $b = b_1 = b_2 = b_3$. Then for arbitrary elements $x_1, x_2, x_3 \in \mathscr{C}_0$ we have

$$0 = \left(\sum_{j=0}^{2} \pi(x_{\sigma^j(1)}, x_{\sigma^j(2)}) \cdot x_{\sigma^j(3)}\right) \otimes (b, b)b$$
$$+ \left(\sum_{j=0}^{2} \rho(\rho(x_{\sigma^j(1)}, x_{\sigma^j(2)}), x_{\sigma^j(3)})\right) \otimes ((b \circ b) \circ b) .$$

Since $[[x, y], z] + [[y, z], x] + [[z, x], y] = 6(x, y, z)$ in an alternative algebra and since $D_{x, y}(z) = (1/4)([[x, y], z] - 3(x, y, z))$ as in (3.19), the above implies that for any element of $b \in B$ that

$$(b \circ b) \circ b = (1/2)(b, b)b = (1/2)t(b^2)b .$$

For arbitrary elements $b, b' \in B$ we have

$$b \circ b' = b \cdot b' - (b, b')1 = b \cdot b' - t(b \cdot b')1 .$$

Hence,

$$\begin{aligned}(1/2)t(b^2)b &= (1/2)(b,b)b = (b \circ b) \circ b \\ &= (b^2 - t(b^2)1) \circ b \\ &= (b^2 - t(b^2)1) \cdot b - t((b^2 - t(b^2)1) \cdot b)1 \\ &= b^3 - t(b^2)b - t(b^3)\,.\end{aligned}$$

Consequently,

$$b^3 - (3/2)t(b^2)b - t(b^3)1 = 0\,. \tag{3.48}$$

For an arbitrary element $q = a1 + b$, substituting $b = q - t(q)1$ into equation (3.48) gives

$$q^3 - T(q)q^2 + S(q)q - N(q)1 = 0$$

where $S(q)$ and $N(q)$ are as above. As a result, J satisfies $\mathrm{ch}_3(x) = 0$. The mapping $t(a1 + b) = a$ clearly satisfies $t(q \cdot q') = t(q' \cdot q)$ for all $q, q' \in J$, and $t(1) = 1$. To see that it is associative, hence a normalized trace, substitute $b = b_1,\ b' = b_2,\ b'' = b_3,\ x = u_1 = u_3$ and $y = u_2$ into the second equation of (2.14) to obtain

$$D_{[x,y],x} \otimes ((b \circ b', b'') - (b' \circ b'', b)) = 0\,.$$

Since the algebra $\mathscr{C}$ is split, it contains a copy of the 2×2 matrices. Taking $x = e_{1,2}$ and $y = e_{2,1}$ in that matrix subalgebra of $\mathscr{C}$, where $e_{i,j}$ is the matrix unit, we see that $D_{[x,y],x} \neq 0$. Consequently, $(b \circ b', b'') = (b, b' \circ b'')$ must hold, which can be seen to be equivalent to the associativity of t. Now Proposition 1.4 of [GMW] can be applied to deduce that J is a Jordan algebra.

To conclude the proof, we establish that

$$\langle b, b' \rangle b'' = (b'' \cdot b') \cdot b - (b'' \cdot b) \cdot b' = [b_r, b'_r](b'') \tag{3.49}$$

for all $b, b', b'' \in B$. Consider elements u_1, u_2 and u_3 from the subalgebra of 2×2 matrices in $\mathscr{C}$. Then $\pi(u_i, u_j)u_k = D_{u_i,u_j}(u_k) = (1/4)[[u_i, u_j], u_k] = \rho(\rho(u_i, u_j), u_k)$. Let $b_1 = b_3 = b$ and $b_2 = b'$. Then the third equation of (2.14) reduces to

$$\begin{aligned}(1/4)[[u_3, u_1], u_2] \otimes ((b,b)b' - (b,b')b + (b * b) * b' + (b * b') * b) \\ + ((u_1, u_2)u_3 - (u_2, u_3)u_1) \otimes \langle b, b' \rangle b = 0\,.\end{aligned} \tag{3.50}$$

Observe that

$$\begin{aligned}(b,b)b' - (b,b')b + (b * b) * b' - (b * b') * b &= (b \cdot b) * b' - (b \cdot b') * b \\ &= (b \cdot b) \cdot b' - (b \cdot b') \cdot b\end{aligned} \tag{3.51}$$

because $(b \cdot b, b') = (b, b \cdot b')$ by the associativity of the form. Thus, if we specialize the elements in the matrix subalgebra to be $u_1 = e_{1,2} + e_{2,1}$,

$u_2 = e_{1,2}$, $u_3 = e_{2,1}$, then $(1/4)[[u_3, u_1], u_2] = -(1/2)e_{1,2} = (u_1, u_2)u_3 - (u_2, u_3)u_1$, and (3.50) and (3.51) together show that

$$\langle b, b' \rangle b = (b \cdot b') \cdot b - (b \cdot b) \cdot b' .$$

This is just (3.49) with two of the elements set equal, and we may linearize it to obtain

$$\begin{aligned} \langle b, b' \rangle b'' + \langle b'', b' \rangle b &= \langle b, b' \rangle b'' - \langle b', b'' \rangle b \\ &= (b \cdot b') \cdot b'' + (b'' \cdot b') \cdot b - 2(b \cdot b'') \cdot b' . \end{aligned}$$

This relation implies

$$\begin{aligned} \langle b, b' \rangle b'' + \langle b, b'' \rangle b' &= \langle b, b' \rangle b'' - \langle b'', b \rangle b' \\ &= 2(b' \cdot b'') \cdot b - (b' \cdot b) \cdot b'' - (b'' \cdot b) \cdot b' . \end{aligned}$$

Now setting $u_1 = u_2 = u_3 = u$, where $(u, u) \neq 0$, in the third part of (2.14) allows us to deduce that

$$\langle b, b' \rangle b'' + \langle b', b'' \rangle b + \langle b'', b \rangle b' = 0 ,$$

and these last three equations combine to give the desired conclusion

$$3\langle b, b' \rangle b'' = 3(b' \cdot b'') \cdot b - 3(b'' \cdot b) \cdot b' = 3[b_r, \ b'_r](b'') . \qquad \square$$

3.52. The Δ-graded Lie algebras for all root systems Δ except those of type B_n are now determined. Here we treat this final case, which is comparatively easy.

Theorem 3.53. *Assume L is a Lie algebra over F which is B_n-graded for $n \geqq 3$. Then there exists a unital, commutative associative algebra A and a Jordan algebra $J(W) = A1 \oplus W$ associated with an A-module W having a symmetric bilinear form such that L is centrally isogenous to*

$$\mathscr{T}(J(V)/F, J(W)/A) = (\mathscr{G} \otimes A) \oplus (V \otimes W) \oplus \langle W, W \rangle ,$$

where $\mathscr{G}$ is the split simple Lie algebra of type B_n, V is its natural representation on a $(2n+1)$-dimensional vector space having a nondegenerate symmetric bilinear form $(\ ,\ )$ relative to which $\mathscr{G}$ is the space of skew-symmetric transformations. The multiplication in $\mathscr{T}(J(V)/F,\ J(W)/A)$ is given by (3.7) where $D_{x,x'}v = (x, v)x' - (x', v)x$ for all $v, x, x' \in V$, $d_{y,y'}w = (y, w)y' - (y', w)y$ for all $w, y, y' \in W$ and $\langle W, W \rangle = \{d_{y,y'} \mid y, y' \in W\}$.

Proof. From Proposition 2.7 it follows that L is centrally isogenous to an algebra $(\mathscr{G} \otimes A) \oplus (V \otimes W) \oplus \langle W, W \rangle$. The form (u, v) gives a basis for $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, F)$ in this case, and the mapping $\pi(u, v) = D_{u,v}$ likewise spans $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, \mathscr{G})$. Since the mapping ρ belongs to $\mathrm{Hom}_{\mathscr{G}}(V \otimes V, V)$, which is zero for algebras of type B_n, $n \geqq 3$, the multiplication in $A1 \oplus W$ is not determined uniquely by the multiplication in L. The product in L determines only the form $(\ ,\ ) : W \times W \to A$; whereas the multiplication $\cdot : W \times W \to W$ in

(2.12) can be chosen arbitrarily. In particular, it can be taken to be zero, and the resulting algebra $J(W) = A1 \oplus W$ with $(a1+w)(a'1+y) = aa'+(w,y)+ay+a'w$ is then just the Jordan algebra of the symmetric A-bilinear form $(\,,)$ on W. Equation (2.14) reduces to the single equation in this case:

$$\begin{aligned} 0 = &(u_1,u_2)u_3 \otimes \langle w_1,w_2\rangle w_3 + (u_2,u_3)u_1 \otimes \langle w_2,w_3\rangle w_1 + (u_3,u_1)u_2 \otimes \langle w_3,w_1\rangle w_2 \\ &+ ((u_1,u_3)u_2 - (u_2,u_3)u_1) \otimes (w_1,w_2)w_3 \\ &+ ((u_2,u_1)u_3 - (u_3,u_1)u_2) \otimes (w_2,w_3)w_1 \\ &+ ((u_3,u_2)u_1 - (u_1,u_2)u_3) \otimes (w_3,w_1)w_2 \,. \end{aligned}$$

Assuming u_1, u_2, u_3 in V are such that $(u_1,u_3) = 0 = (u_2,u_3)$ and $(u_1,u_2) = 1$, we determine from this equation that

$$\langle w_1,w_2\rangle w_3 = (w_1,w_3)w_2 - (w_2,w_3)w_1 = d_{w_1,w_2}w_3 \,.$$

Thus, Theorem 3.53 holds. □

4. The generalized Tits construction and Lie superalgebras

4.1. The generalized Tits construction introduced in Sect. 3 enables us to extend the Freudenthal–Tits magic square to Lie superalgebras. In this section we briefly describe how this can be done.

4.2. By a *superalgebra* we mean a $\mathbb{Z}_2$-graded algebra over a field F. For example the Grassman or exterior algebra G presented by generators $e_1, e_2, \ldots$ and relations $e_ie_j + e_je_i = 0$, $e_i^2 = 0$ has a $\mathbb{Z}_2$-grading $G = G^0 \oplus G^1$, where $e_{i_1} \cdots e_{i_r}$ belongs to G^0 or G^1 depending on whether r is even or odd. Let $A = A^0 \oplus A^1$ be a superalgebra over F. The subalgebra $G(A) = A^0 \otimes G^0 + A^1 \otimes G^1$ of the tensor product $A \otimes G$ is called the *Grassman envelope* of A. When $\mathscr{V}$ is a variety of algebras, the superalgebra A is said to be a $\mathscr{V}$-superalgebra if $G(A) \in \mathscr{V}$. Thus, A is a Lie superalgebra if and only if $G(A)$ is a Lie algebra, and A is a Jordan superalgebra if and only if $G(A)$ is a Jordan algebra.

4.3. Let $J = J^0 \oplus J^1$ be a Jordan superalgebra. For $a \in J^0 \cup J^1$, let a_r denote the multiplication operator determined by a. If $a \in J^i, b \in J^j$ are homogeneous elements, then the supercommutator $d_{a,b} = a_rb_r - (-1)^{ij}b_ra_r$ is a superderivation of J. Thus if $d = d_{a,b}$, then $d(xy) = d(x)y + (-1)^{ijk}xd(y)$ whenever $x \in J^k$. The linear span $\langle J,J\rangle$ of all such superderivations is known to be a Lie superalgebra ([Kac2]).

4.4. A linear functional $t : J \to F$ on a Jordan superalgebra J is a *trace* if $t(J^1) = 0$ and t restricted to the Jordan algebra J^0 is a trace in the sense of Sect. 3. It is normalized if $t(1) = 1$. Since the Grassman envelope $G(J)$ of J can be viewed as an algebra over its even part G^0, any trace $t : J \to F$ on J gives rise to a trace $\tilde{t} : G(J) \to G^0$ where $\tilde{t}(J^1 \otimes G^1) = 0$, and $\widetilde{t}(x \otimes a) = t(x)a$ for $x \in J^0$ and $a \in G^0$. Let J_0 denote the subspace of J consisting of the trace zero elements. Then $J_0 = J_0^0 \oplus J^1$, and $G(J)_0 = J_0^0 \otimes G^0 + J^1 \otimes G^1$ are the elements of trace zero relative to t and $\widetilde{t}$ respectively.

4.5. Assume $J = J^0 \oplus J^1$ is a Jordan superalgebra with trace t. Let $\mathscr{C}$ be the algebra of split octonions over F with its normalized trace, and let $\mathscr{C}_0$ be its elements of trace zero. The Tits construction

$$\mathscr{T}(\mathscr{C}/F, J/F) = \mathrm{Der}_F \mathscr{C} \oplus (\mathscr{C}_0 \otimes J_0) \oplus \langle J, J \rangle$$

with multiplication specified by (3.7) is a superalgebra. The Grassman envelope $G(\mathscr{T}(\mathscr{C}/F, J/F))$ is isomorphic to the algebra $\mathscr{T}(\mathscr{C}/F, G(J)/G^0)$ where $G(J)$ is viewed as an algebra over G^0 having the trace $\widetilde{t}$. If $G(J)$ satisfies the trace identity $\mathrm{ch}_3(x) = 0$ relative to $\widetilde{t}$, then it follows from Tits' theorem that $\mathscr{T}(\mathscr{C}/F, G(J)/G^0)$, and hence the Grassman envelope $G(\mathscr{T}(\mathscr{C}/F, J/F))$, are Lie algebras. As a consequence, the superalgebra $\mathscr{T}(\mathscr{C}/F, J/F)$ is a Lie superalgebra. This is indeed the case in the following two examples:

Example 4.6. Consider the 3-dimensional Jordan superalgebra $J = J^0 \oplus J^1$ where $J^0 = F1$, $J^1 = Fx + Fy$, and $xy = 1$, having trace given by $t(1) = 1$, $t(x) = 0 = t(y)$. Then $\mathscr{T}(\mathscr{C}/F, J/F)$ is isomorphic to $G(3)$, the simple exceptional 31-dimensional Lie superalgebra, (see [Kac1] for a description of the simple Lie superalgebras).

Example 4.7. Assume J is the 4-dimensional Jordan superalgebra $J = J^0 \oplus J^1$, where $J^0 = Fe + Ff, J^1 = Fx + Fy$, $e^2 = e, f^2 = f$, $ef = 0 = fe, ex = 1/2x = fx, ey = 1/2y = fy, xy = e + 2f$, with trace given by $t(e) = 2, t(f) = 1$, $t(x) = 0 = t(y)$. Then $\mathscr{T}(\mathscr{C}/F, J/F) \cong F(4)$, the simple exceptional 40-dimensional Lie superalgebra.

5. Intersection matrix algebras

5.1. The Recognition Theorem (0.5) was applied in [BM] to identify the intersection matrix algebras of Slodowy for simply-laced root systems. In this section we establish corresponding results for the doubly-laced root systems.

5.2. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots for the finite irreducible reduced root system Δ. Assume C is the Cartan matrix associated with Δ so that $C_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$. For an n tuple $\underline{m} = (m_1, m_2, \ldots, m_n)$ of natural numbers $m_i \geqq 1$ let $\widetilde{im}(\Delta, \underline{m})$ be the Lie algebra presented by the generators $h_1, \ldots, h_n$, $e_{i,1}, \ldots, e_{i,m_i}$, $f_{i,1}, \ldots, f_{i,m_i}$, $i = 1, \ldots, n$, and the relations

$$(5.3) \qquad [h_i, h_j] = 0, \quad [e_{i,j}, f_{k,l}] = \delta_{i,k} h_i$$
$$[h_i, e_{j,k}] = C_{i,j} e_{j,k}, \quad [h_i, f_{j,k}] = -C_{i,j} f_{j,k} \; .$$

The algebra $\widetilde{im}(\Delta, \underline{m})$ is graded by the lattice Γ generated by Δ where degrees are assigned according to $\deg h_j = 0$, $\deg e_{j,k} = \alpha_j = -\deg f_{j,k}$ for $1 \leqq j \leqq n$ and $1 \leqq k \leqq m_i$.

5.4. The radical $r(\Delta, \underline{m})$ of $\widetilde{im}(\Delta, \underline{m})$ is the ideal of $\widetilde{im}(\Delta, m)$ generated by root spaces $\widetilde{im}(\Delta, \underline{m})_\alpha$ where $\alpha \notin \Delta \cup \{0\}$. Then the *intersection matrix algebra* of Slodowy (see [Sl1], [Sl2]) is the Lie algebra

$$im(\Delta, \underline{m}) \overset{\text{def}}{=} \widetilde{im}(\Delta, \underline{m})/r(\Delta, \underline{m})\ .$$

For every subset $e_1 = e_{1,k_1}, e_2 = e_{2,k_2}, \ldots, e_n = e_{n,k_n}, f_1 = f_{1,k_1}, f_2 = f_{2,k_2}, \ldots, f_n = f_{n,k_n}$ of generators the Serre relations

$$(\mathrm{ad} e_i)^{-C_{i,j}+1} e_j \quad i \neq j$$
$$(\mathrm{ad} f_i)^{-C_{i,j}+1} f_j \quad i \neq j$$

are in the radical. As a consequence, the images of the elements $h_1, \ldots h_n, e_1, \ldots e_n, f_1, \ldots, f_n$ in $\widetilde{im}(\Delta, \underline{m})$ generate a subalgebra isomorphic to the simple Lie algebra $\mathcal{G}$ having Cartan matrix C. Hence, the algebra $im(\Delta, \underline{m})$ is Δ-graded. From its very construction, $im(\Delta, \underline{m}) \cong U_{\underline{J}}$, where $\underline{J} = (im(\Delta, \underline{m})_\gamma, \gamma \in \Delta)$ is the corresponding Jordan system. Therefore by Proposition 1.6, $im(\Delta, \underline{m})$ is centrally closed.

5.5. The definition of intersection matrix algebra we have presented here is a natural extension and interpretation of Slodowy's definition for the algebras which arise from multiply affinized Cartan matrices, Slodowy's definition is both more restrictive (because the matrix is assumed to symmetric) and less (because there is no requirement that the form be positive semi-definite or that the grading be by a finite root system).

5.6. Using the Recognition Theorem, Berman and Moody showed that the intersection matrix algebras of type A_n for $n \geqq 3$ are isomorphic to Steinberg Lie algebras over group algebras of free groups, and for $n = 2$ are Steinberg Lie algebras over certain alternative rings (see [AF], [BM], [F]). When Δ is of type D or E, then $im(\Delta, \underline{m})$ is the u.c.a. of the *toroidal Lie algebra* $\mathcal{G} \otimes F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$, where $l = \sum_{i=1}^{n}(m_i - 1)$ and $F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$, is the algebra of Laurent polynomials in l indeterminates. The universal covering algebras of the toroidal Lie algebras are natural generalizations of the affine Kac–Moody algebras, which are just the $l = 1$ case.

5.7. In this section we apply our Recognition Theorems (0.8, 0.10, and 0.11) to describe the intersection matrix algebras $im(\Delta, \underline{m})$ for $C_1 = A_1$ and for the doubly-laced root systems B_n, C_n, F_4, and G_2. It is convenient in doing this to adopt the following notation:

$$l = \sum_{\substack{i \\ \alpha_i \text{ a long root}}} (m_i - 1) \quad s = \sum_{\substack{j \\ \alpha_j \text{ a short root}}} (m_j - 1)$$

5.8. ($\Delta = C_n$ for $n \geqq 1$) To begin we describe the intersection matrix algebras that correspond to the root systems C_n for $n \geqq 4$. Let G be the free group in $l + 2s$ generators $x_1, \ldots, x_l, y_1, \ldots, y_s, z_1, \ldots, z_s$. The group algebra FG has an involution $\pi_{l,s}$ such that $\pi_{l,s}(x_i) = x_i$ for $i = 1, \ldots, l$; $\pi_{l,s}(y_j) = z_j$, and $\pi_{l,s}(z_j) = y_j$ for $j = 1, \ldots, s$. It follows from the Recognition Theorem for C_n root systems (see (0.8)) that $im(C_n, \underline{m}) \cong st\, sp_{2n}(FG, \pi_{l,s})$ for all $n \geqq 4$.

Handling the case $n = 3$ requires the free product

$$A_{l+2s} = \left(\mathop{\circledast}_{i=1}^{l} F[x_i, x_i^{-1}] \right) \circledast \left(\mathop{\circledast}_{j=1}^{s} F[y_j, y_j^{-1}] \right) \circledast \left(\mathop{\circledast}_{j=1}^{s} F[z_j, z_j^{-1}] \right)$$

in the variety of alternative algebras. The involution $\pi_{l,s}$ is defined the same way as before, $\pi_{l,s}(x_i) = x_i, \pi_{l,s}(y_j) = z_j$, and $\pi_{l,s}(z_j) = y_j$. Now let I be the ideal of $\mathscr{A}_{l+2s}$ generated by all the elements $((a + \pi_{l,s}(a))b)c - (a + \pi_{l,s}(a))(bc), a, b, c \in \mathscr{A}_{l+2s}$. Since $\pi_{l,s}(I) = I$, the involution $\pi_{l,s}$ induces an involution $\overline{\pi_{l,s}}$ on $\mathscr{A}_{l+2s}/I$. It then follows from (0.8) that $im(C_3, \underline{m}) \cong st\, sp_6(\mathscr{A}_{l+2s}/I, \overline{\pi_{l,s}})$.

The intersection matrix algebra $im(C_2, \underline{m})$ is isomorphic to the Tits–Kantor–Koecher construction of the universal Jordan algebra $\mathscr{J}$ containing the algebra $H(M_2(F))$ of 2×2 symmetric matrices and generated by $l + s$ invertible elements $x_1, \ldots, x_l, y_1, \ldots, y_s$ so that the sum $e_{1,1} + e_{2,2}$ of the diagonal matrix units in $H(M_2(F))$ is the identity of $\mathscr{J}$ and $x_i \in \{e_{1,1}, \mathscr{J}, e_{1,1}\} + \{e_{2,2}, \mathscr{J}, e_{2,2}\}, 1 \leqq i \leqq l$; $y_j \in \{e_{1,1}, \mathscr{J}, e_{2,2}\}, 1 \leqq j \leqq s$. Assume $\mathscr{J}_{l+s}$ denotes the free product

$$\mathscr{J}_{l+s} = \left(\mathop{\circledast}_{i=1}^{l} F[x_i, x_i^{-1}] \right) \circledast \left(\mathop{\circledast}_{j=1}^{s} F[y_j, y_j^{-1}] \right)$$

in the variety of Jordan algebras. Consider the free product $H(M_2(F)) \circledast \mathscr{J}_{l+s}$ and the ideal I of it generated by the elements $\{e_{1,1}, x_i, e_{2,2}\}$, $\{e_{1,1}, y_j, e_{1,1}\}$, $\{e_{2,2}, y_j, e_{2,2}\}$, $1 \leqq i \leqq l, 1 \leqq j \leqq s$. Then $\mathscr{J} \cong (H(M_2(F)) \circledast \mathscr{J}_{l+s})/I$ and $im(C_2, \underline{m}) \cong K((H(M_2(F)) \circledast \mathscr{J}_{l+s})/I)$.

For the root system C_1 the result is particularly simple, $im(C_1, \underline{m}) \cong K(\mathscr{J}_l)$. To summarize the various C cases we state:

Theorem 5.9. (i) *$im(C_n, \underline{m}) \cong st\, sp_{2n}(FG, \pi_{l,s})$ for all $n \geqq 4$.*
(ii) *$im(C_3, \underline{m}) \cong st\, sp_6(\mathscr{A}_{l+2s}/I, \overline{\pi_{l,s}})$.*
(iii) *$im(C_2, \underline{m}) \cong K((H(M_2(F)) \circledast \mathscr{J}_{l+s})/I)$.*
(iv) *$im(C_1, \underline{m}) \cong K(\mathscr{J}_l)$.*

5.10. In each of the cases $\Delta = B_n, F_4$, and G_2, a Δ-graded Lie algebra is centrally isogenous to a Tits construction $\mathscr{T}(Q/F, R/\mathfrak{A})$ where Q is a fixed F-algebra, R is an alternative or Jordan algebra over a unital, commutative associative F-algebra $\mathfrak{A}$, and R satisfies a trace identity $f = 0$. From this description it follows that the corresponding $im(\Delta, \underline{m})$ is isomorphic to the universal covering algebra of $\mathscr{T}(Q/F, R/\mathfrak{A}) \bigotimes_F F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$ where R is the universal algebra generated by s invertible elements in the variety of alternative or Jordan algebras satisfying the trace identity $f = 0$ and $F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]$ is the algebra of Laurent polynomials in l variables.

Let us be more precise. Let X be a (nonassociative) F-algebra, and let f denote an element from the universal trace algebra $P(\mathscr{F})\bigotimes_{\mathscr{F}}\mathscr{F}$, where $\mathscr{F}$ is the free nonassociative algebra on generators $\{x_1, x_2, \ldots\}$, (see (3.11)). Assume $P(X)$ is the polynomial algebra on the space X. Then $P(X)\bigotimes_F X$ is the $P(X)$-algebra with normalized trace $t : P(X)\bigotimes_F X \to P(X)$ such that $t(1\otimes a) = a$, where a is to be viewed as an element of $P(X)$.

An arbitrary mapping $\phi : \{x_1, x_2, \ldots\} \to X$ gives rise to the homomorphism $P(\mathscr{F})\bigotimes_F \mathscr{F} \to P(X)\bigotimes_F X$. Assume (f) is the ideal of $P(X)\bigotimes_F X$ generated over $P(X)$ by the images of the element f under all such homomorphisms for all such mappings ϕ. Assume $(P(X)\bigotimes_F X | f = 0)$ is the quotient algebra of $P(X)\bigotimes_F X$ by the ideal (f). If $\overline{P}$ is the image of $P(X)\otimes 1$ in $(P(X)\bigotimes_F X | f = 0)$, then the latter is a $\overline{P}$-algebra.

Let $\mathscr{A}_s$ and $\mathscr{J}_s$ denote the free product $F[x_1, x_1^{\pm 1}] \circledast \cdots \circledast F[x_s, x_s^{\pm 1}]$ (with the joint identity element) in the varieties of alternative and Jordan algebras respectively. The universal alternative (Jordan) algebra generated by s invertible elements and satisfying the trace identity $f = 0$ is $(P(\mathscr{A}_s)\bigotimes_F \mathscr{A}_s | f = 0)$ (resp. $(P(\mathscr{J}_s)\bigotimes_F \mathscr{J}_s | f = 0)$).

5.11. ($\Delta = B_n$ for $n \geqq 3$) Let $\mathscr{G}$ be the split simple Lie algebra of type B_n for $n \geqq 3$, and let V be its natural representation on a $2n+1$-dimensional vector space having a nondegenerate symmetric bilinear form relative to which $\mathscr{G}$ is the space of skew-symmetric transformations. As above, $J(V) = F1 \oplus V$ is the Jordan algebra of the symmetric bilinear form.

Theorem 5.12. *The algebra* $im(B_n, \underline{m})$ *for* $n \geqq 3$ *is isomorphic to the universal covering algebra of the algebra*

$$\mathscr{T}\left(J(V)/F, \left(P(\mathscr{J}_s)\bigotimes_F \mathscr{J}_S | \mathrm{ch}_2(x) = 0\right)/\overline{P}\right) \bigotimes_F F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]\,.$$

5.13. Arguing as in [A] and [Sh] we can show that the algebra $(P(\mathscr{J}_s)\bigotimes_F \mathscr{J}_s | \mathrm{ch}_2(x) = 0)$ is a Jordan domain with a nonzero (associative) center Z. The central localization with respect to $Z^* = Z \setminus \{0\}$ is a Jordan division algebra of a symmetric bilinear form on an s-dimensional vector space over the field $(Z^*)^{-1}Z$. Thus, the algebra $im(B_n, \underline{m})$ is centrally isogenous with a form of a simple algebra of one of the B or D types, (see the table in (3.32).)

5.14. ($\Delta = F_4, G_2$) Recall that by $\mathscr{C}$ and J we denote the split octonion F-algebra and the 27-dimensional Jordan algebra of Hermitian 3×3 matrices over $\mathscr{C}$ respectively.

Theorem 5.15. *The intersection matrix algebra* $im(F_4, \underline{m})$ *is isomorphic to the universal covering algebra*

$$\mathscr{T}\left(J/F, \left(P(\mathscr{A}_s)\bigotimes_F \mathscr{A}_s | \mathrm{ch}_2(x) = 0\right)/\overline{P}\right) \bigotimes_F F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}]\,.$$

5.16. **Conjecture.** *All trace identities of the octonion algebra* $\mathscr{C}$ *follow from the trace identity* $\mathrm{ch}_2(x) = 0$ *and the alternative identities* $(x, x, y) = 0 = (y, x, x)$.

5.17. If this conjecture is true, then the algebra $(P(\mathscr{A}_s) \bigotimes_F \mathscr{A}_s | \mathrm{ch}_2(x) = 0)$ is a domain with a nonzero center Z whose central localization with respect to $Z^* = Z \backslash \{0\}$ is an alternative division algebra. In this case $im(F_4, \underline{m})$ is centrally isogenous with a form of an algebra of type E_8.

Theorem 5.18. *The intersection matrix algebra* $im(G_2, \underline{m})$ *is isomorphic to the universal covering algebra of*

$$\mathscr{T}\left(\mathscr{C}/F, \left(P(\mathscr{J}_s) \bigotimes_F \mathscr{J}_s | \mathrm{ch}_3(x) = 0\right) / \bar{P}\right) \bigotimes_F F[t_1^{\pm 1}, \ldots, t_l^{\pm 1}] \, .$$

Appendix

In applying Proposition 2.7 to determine Δ-graded Lie algebras we required knowledge of various homomorphism spaces. This information, which we summarize below, is well-known and can be found in [BO], [S], [MPR] or can be deduced from [Kas]. In stating these results we adopt the Bourbaki numbering [Bo] of the fundamental weights in all cases except G_2.

Assume $\mathscr{G}$ is a finite-dimensional split simple Lie algebra of characteristic zero, and let $\omega_1, \ldots, \omega_n$ denote the fundamental weights of $\mathscr{G}$ relative to a split Cartan subalgebra $\mathscr{H}$. For ω a dominant weight, let $L(\omega)$ denote the finite-dimensional irreducible $\mathscr{G}$-module having highest weight ω. The two-fold tensor product of the adjoint representation of a split simple Lie algebra $\mathscr{G}$ decomposes according to:

$\mathscr{G}$	
A_1	$L(2\omega_1) \otimes L(2\omega_1) = L(4\omega_1) \oplus L(2\omega_1) \oplus L(0)$
A_2	$L(\omega_1+\omega_2) \otimes L(\omega_1+\omega_2) = L(2\omega_1+2\omega_2) \oplus L(3\omega_1) \oplus L(3\omega_2) \oplus 2L(\omega_1+\omega_2) \oplus L(0)$
A_n $(n \geqq 3)$	$L(\omega_1+\omega_n) \otimes L(\omega_1+\omega_n) = L(2\omega_1+2\omega_n) \oplus L(2\omega_1+\omega_{n-1}) \oplus L(\omega_2+2\omega_n) \oplus 2L(\omega_1+\omega_n) \oplus L(\omega_2+\omega_{n-1}) \oplus L(0)$
B_3	$L(\omega_2) \otimes L(\omega_2) = L(2\omega_2) \oplus L(\omega_1+2\omega_3) \oplus L(2\omega_3) \oplus L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$
B_4	$L(\omega_2) \otimes L(\omega_2) = L(2\omega_2) \oplus L(\omega_1+\omega_3) \oplus L(2\omega_4) \oplus L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$
B_n $(n \geqq 5)$	$L(\omega_2) \otimes L(\omega_2) = L(2\omega_2) \oplus L(\omega_1+\omega_3) \oplus L(\omega_4) \oplus L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$
C_n $(n \geqq 2)$	$L(2\omega_1) \otimes L(2\omega_1) = L(4\omega_1) \oplus L(2\omega_1 + \omega_2) \oplus L(2\omega_1) \oplus L(2\omega_2) \oplus L(\omega_2) \oplus L(0)$
D_4	$L(\omega_2) \otimes L(\omega_2) = L(2\omega_2) \oplus L(\omega_1 + \omega_3 + \omega_4) \oplus L(2\omega_3) \oplus L(2\omega_4) \oplus L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$

D_5	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(\omega_1+\omega_3)\oplus L(\omega_4+\omega_5)$ $\oplus L(2\omega_1)\oplus L(\omega_2)\oplus L(0)$
$D_n\ (n \geqq 6)$	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(\omega_1+\omega_3)\oplus L(\omega_4)$ $\oplus L(2\omega_1)\oplus L(\omega_2)\oplus L(0)$
E_6	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(\omega_1+\omega_6)\oplus L(\omega_2)$ $\oplus L(\omega_4)\oplus L(0)$
E_7	$L(\omega_1)\otimes L(\omega_1) = L(2\omega_1)\oplus L(\omega_1)\oplus L(\omega_3)$ $\oplus L(\omega_6)\oplus L(0)$
E_8	$L(\omega_8)\otimes L(\omega_8) = L(2\omega_8)\oplus L(\omega_1)\oplus L(\omega_7)$ $\oplus L(\omega_8)\oplus L(0)$
F_4	$L(\omega_1)\otimes L(\omega_1) = L(2\omega_1)\oplus L(2\omega_4)\oplus L(\omega_1)$ $\oplus L(\omega_2)\oplus L(0)$
G_2	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(3\omega_1)\oplus L(2\omega_1)$ $\oplus L(\omega_2)\oplus L(0)$

The two-fold tensor product of the little adjoint representation whose highest weight is the highest short root for the Lie algebras $B_n(n \geqq 3)$, $C_n(n \geqq 2)$, F_4, and G_2 decomposes in the following way:

$\mathscr{G}$	
$B_n\ (n \geqq 3)$	$L(\omega_1)\otimes L(\omega_1) = L(2\omega_1)\oplus L(\omega_2)\oplus L(0)$
C_2	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(2\omega_1)\oplus L(0)$
C_3	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(\omega_1+\omega_3)\oplus L(2\omega_1)$ $\oplus L(\omega_2)\oplus L(0)$
$C_n\ (n \geqq 4)$	$L(\omega_2)\otimes L(\omega_2) = L(2\omega_2)\oplus L(\omega_1+\omega_3)\oplus L(\omega_4)$ $\oplus L(2\omega_1)\oplus L(\omega_2)\oplus L(0)$
F_4	$L(\omega_4)\otimes L(\omega_4) = L(2\omega_4)\oplus L(\omega_1)\oplus L(\omega_3)\oplus L(\omega_4)\oplus L(0)$
G_2	$L(\omega_1)\otimes L(\omega_1) = L(2\omega_1)\oplus L(\omega_1)\oplus L(\omega_2)\oplus L(0)$

The tensor product of the adjoint and little adjoint representations for the Lie algebras $B_n(n \geqq 2)$, $C_n(n \geqq 3)$, F_4, and G_2 decomposes in the following way:

$\mathscr{G}$	
B_3	$L(\omega_2)\otimes L(\omega_1) = L(\omega_1+\omega_2)\oplus L(2\omega_3)\oplus L(\omega_1)$
$B_n\ (n \geqq 4)$	$L(\omega_2)\otimes L(\omega_1) = L(\omega_1+\omega_2)\oplus L(\omega_3)\oplus L(\omega_1)$
C_2	$L(2\omega_1)\otimes L(\omega_2) = L(2\omega_1+\omega_2)\oplus L(2\omega_1)\oplus L(\omega_2)$
$C_n\ (n \geqq 3)$	$L(2\omega_1)\otimes L(\omega_2) = L(2\omega_1+\omega_2)\oplus L(\omega_1+\omega_3)$ $\oplus L(2\omega_1)\oplus L(\omega_2)$
F_4	$L(\omega_1)\otimes L(\omega_4) = L(\omega_1+\omega_4)\oplus L(\omega_4)\oplus L(\omega_3)$
G_2	$L(\omega_2)\otimes L(\omega_1) = L(\omega_1+\omega_2)\oplus L(2\omega_1)\oplus L(\omega_1)$.

The results stated above apply for fields of characteristic $p > 0$ provided p is suitably large. This can be verified directly by showing that the tensor products decompose as above for p sufficiently big, or alternately, a uniform argument can be given as follows: Let G be a split semisimple simply connected algebraic group with Borel subgroup B, maximal torus T, and root system Δ. Assume $H^0(\lambda) = \mathrm{ind}_B^G \lambda$ is the G-module induced from the one dimensional B-module associated to the weight λ. Let $L(\lambda)$ be the irreducible G-module with highest weight λ. When $\omega \in C = \{\nu \in X(T) | 0 \leqq (\nu + \varrho, \alpha^\vee) < p$ for all $\alpha \in \Delta^+\}$, where ϱ is the half-sum of the positive roots, then $H^0(\omega) = L(\omega)$ (see [Jan, Cor. 5.6, p. 248]). Let G_1 denote the first Frobenius kernel of G. Suppose λ, μ, and $\lambda + \mu \in C$. Then the irreducible G_1-modules $L(\lambda)$, $L(\mu)$ lift to G-modules with $H^0(\lambda) = L(\lambda)$ and $H^0(\mu) = L(\mu)$. All composition factors of $L(\lambda) \otimes L(\mu)$ have weights in C. Therefore, since $\mathrm{Ext}^1(L(\tau), L(\omega)) = \mathrm{Ext}^1(L(\omega), L(\tau)) = 0$ for all $\tau, \omega \in C$ by [Jan, p. 206–207], we have that the module $L(\lambda) \otimes L(\mu)$ is completely reducible, and the summands must be simple as G-modules and hence isomorphic to some $L(\omega)$ for some $\omega \in C$. The decomposition is characteristic independent because $L(\omega) = H^0(\omega)$ for each $L(\omega)$ which occurs. Since the representation theory of G_1 is equivalent to that of the Lie algebra $\mathscr{G}$ of G as a Lie p-algebra, these results apply when the modules are viewed as $\mathscr{G}$-modules. The assumption $\lambda, \mu, \lambda + \mu \in C$ holds for all the weights considered in the tables above provided p is sufficiently large. That is, $(\nu + \varrho, \alpha^\vee) < p$ must hold for all positive roots α and for ν equal to the highest short root, highest long root, or their sums. But that is equivalent to saying that $(\nu + \varrho, \eta^\vee) < p$ where $\eta^\vee$ is the highest root in the dual system. Thus, this argument will work when $p > n + 2 (A_n)$; $p > 2n + 3 (B_n, C_n)$; $p > 2n - 1 (D_n)$; $p > 13$ (E_6, F_4, G_2); $p > 19$ (E_7); $p > 31 (E_8)$.

Acknowledgements. We would like to take this opportunity to thank H.H. Andersen and D. Nakano for helpful conversations concerning extensions of these results to fields of arbitrary characteristic (see the Appendix).

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