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Regularity of CR mappings between algebraic hypersurfaces

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0 Introduction

Let M and M' be real analytic hypersurfaces in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ respectively and $H : M \to M'$ a sufficiently smooth CR mapping. Under what conditions does H extend holomorphically to a neighborhood of M in $\mathbb{C}^N$? In this paper we prove that if M and M' are algebraic hypersurfaces in $\mathbb{C}^N$, i.e. both defined by the vanishing of real polynomials, then any sufficiently smooth CR mapping with Jacobian not identically zero extends holomorphically provided the hypersurfaces are holomorphically nondegenerate (see definition below). Conversely, we prove that holomorphic nondegeneracy is necessary for this property of CR mappings to hold. For the case of unequal dimensions, we also prove that if $N' = N + 1$, M' is the sphere, and M is an algebraic hypersurface which does not contain any complex variety of positive codimension, extendability holds for all CR mappings with certain minimal a priori regularity.

Our approach uses the work of Webster [W1, W2], on holomorphic mappings between algebraic hypersurfaces, and the recent generalizations in [H1, H2] and [BR6]. The question of holomorphic extendability of CR mappings between real analytic hypersurfaces has attracted considerable attention since the work of Lewy [Lw] and Pincuk [P]. For more recent work in the case $N = N'$, see Diederich–Webster [DW], Jacobowitz, Treves, and the first author [BJT], Bell and the first and third authors [BBR, BR1, BR2], Diederich–Fornaess [DF, BR3], and the references therein, as well as the survey paper Forstnerič [Fo2]. We note here that the results for $N \geqq 3$ cited above require nonvanishing conditions on the normal component of the mapping and require the first hypersurface to be essentially finite. (See [BR3] for more general results for the case $N = 2$ and Meylan [Me1, Me2] for some extensions of this to

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higher dimensions.) In the algebraic case, studied in this paper, we are able to omit these assumptions. A recent example given by Ebenfelt [E] shows that holomorphic extendibility may fail if the hypersurfaces are not assumed to be algebraic. The authors know of no other example of real analytic hypoellipticity which holds in the algebraic category but not in the real analytic category.

For the case where $N \neq N'$ an important first result was given by Webster [W2], who proved that any CR map of class C^3 from a strongly pseudoconvex real analytic hypersurface in $\mathbb{C}^N$ to the sphere in $\mathbb{C}^{N+1}$ admits a holomorphic extension on a dense open subset. Generalizations were later given by Faran [Fa1, Fa2], Cima–Suffridge [CS1, CS2], Cima–Krantz–Suffridge [CKS], Forstnerič [Fo1], and [H2]. Recently, the second author in [H1, H2] proved that any CR mapping of class $C^{N'-N+1}$ between two strictly pseudoconvex real analytic hypersurfaces in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ $(N' \geqq N > 1)$ respectively, is real analytic on a dense open subset of M, and is algebraic if both M and M' are algebraic. In Theorem 5 below we prove that holomorphic extension holds everywhere under weaker differentiability assumptions than those given in [H2].

We now introduce some notation and definitions which are needed to state precisely our main results. By a germ at p_0 of a holomorphic vector field in $\mathbb{C}^N$, we shall mean a complex vector field of the form $\sum_1^N a_j(Z) \frac{\partial}{\partial Z_j}$, where the $a_j(Z)$ are germs at p_0 of holomorphic functions. Let M be a real analytic hypersurface in $\mathbb{C}^N$. For $p_0 \in M$ we say that M is *holomorphically degenerate* at p_0 if there exists a nonzero germ of a holomorphic vector field tangent to M in a neighborhood of p_0 (see Stanton [Sta, BR6]). We say that M is *holomorphically nondegenerate* if it is not holomorphically degenerate at any p_0 in M. Recall that by Theorem 1 of [BR6], a connected real analytic hypersurface is holomorphically nondegenerate if and only if there is a point p_1 at which it is not holomorphically degenerate. A CR function on M is a function which is annihilated by the tangential Cauchy–Riemann operators; a mapping from M into $\mathbb{C}^{N'}$ is CR if its components are CR functions.

Theorem 1. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ and assume that M is connected and holomorphically nondegenerate. If H is a smooth CR mapping from M to M' with* $\operatorname{Jac} H \not\equiv 0$, *where* $\operatorname{Jac} H$ *is the Jacobian determinant of H, then H extends holomorphically in an open neighborhood of M in $\mathbb{C}^N$.*

The fact that M and M' are algebraic plays an important role. Indeed, as mentioned before, a recent example given by Ebenfelt [E] shows that the conclusion of Theorem 1 need not hold if M is real analytic, but not algebraic. (See Example 2.10 below.)

The following is a refinement of Theorem 1 in which H is assumed to have only a previously prescribed number of derivatives, depending only on M and M'. The *degree* of an algebraic hypersurface is the total degree of the irreducible real polynomial defining M.

Theorem 2. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ of degrees d and d' respectively, and assume that M is connected and holomorphically nondegenerate. Then there exists a positive integer $k = k(d, d', N)$ (depending only on d, d' and N) such that if H is a CR mapping from M to M' of class C^k with $\operatorname{Jac} H \not\equiv 0$, then H extends holomorphically in an open neighborhood of M in $\mathbb{C}^N$ and is algebraic.*

Note that if the Jacobian of a nontrivial CR map is 0, then M' must contain a complex variety (see [BR5]). Therefore we obtain the following corollary of Theorem 1.

Corollary 1. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^N$ and assume that M is connected and holomorphically nondegenerate. If H is a smooth CR mapping from M to M' and if M' contains no complex analytic variety of positive dimension, then H extends holomorphically in an open neighborhood of M in $\mathbb{C}^N$.*

If f is a function defined on M we shall say that f is *algebraic* if there exist holomorphic polynomials $q_j(Z), j = 0, \ldots, k$, not all identically 0, such that $q_k(Z)f(Z)^k + \cdots + q_0(Z) \equiv 0$, for $Z \in M$. Similarly, we say that f is *locally algebraic* if for any point p on M there is a neighborhood of p such that the restriction of f to that neighborhood is algebraic. A mapping is algebraic (resp. locally algebraic) if each of its components is.

Theorem 3. *Let M be a connected, holomorphically nondegenerate, algebraic hypersurface in $\mathbb{C}^N$. Then there exists a positive integer ℓ with $1 \leqq \ell \leqq N-1$ such that if H is a CR mapping of class C^ℓ from M to another algebraic hypersurface M' in $\mathbb{C}^N$ with $\operatorname{Jac} H \not\equiv 0$ on any open subset of M, then H is locally algebraic. Moreover, if the Levi form of M is nondegenerate at some point, then one can take $\ell = 1$.*

In fact in Sect. 1 we define a new invariant ℓ for any connected real analytic hypersurface M which satisfies the conditions of Theorem 3 if M is algebraic.

Since a connected real analytic hypersurface in $\mathbb{C}^2$ is holomorphically nondegenerate if and only if it is not Levi flat, the following is an immediate corollary of Theorems 1 and 3.

Corollary 2. *Let M and M' be two algebraic hypersurfaces in $\mathbb{C}^2$ and assume that M is connected and not Levi flat.*

(i) *If H is a smooth CR mapping from M to M' with $\operatorname{Jac} H \not\equiv 0$, then H extends holomorphically in an open neighborhood of M in $\mathbb{C}^2$.*

(ii) *If H is a CR mapping from M to M' of class C^1 with $\operatorname{Jac} H \not\equiv 0$ on any open subset of M, then H is locally algebraic.*

The following shows that the condition of holomorphic nondegeneracy is necessary for the holomorphic extendability of CR mappings to hold.

Theorem 4. *Let M be a connected real analytic hypersurface in $\mathbb{C}^N$ which is holomorphically degenerate at some point p_1. Let $p_0 \in M$ and suppose there exists a germ at p_0 of a smooth CR function on M which does not extend* *holomorphically to any full neighborhood of* p_0 *in* $\mathbb{C}^N$. *Then there exists a germ at* p_0 *of a smooth CR diffeomorphism from* M *into itself, fixing* p_0, *which does not extend holomorphically to any neighborhood of* p_0 *in* $\mathbb{C}^N$.

Our final result deals with analytic extendability of CR mappings between hypersurfaces in complex spaces of different dimensions. Let M be a real analytic hypersurface, $p \in M$, and ρ a defining function of M in a neighborhood of p. Recall [D1, D2, Le] that if M does not contain a complex analytic variety of positive dimension through p then there exists $C > 0$ such that for any complex analytic curve parametrized by $Z = \gamma(t)$ with $\gamma(0) = p$,

$$\text{ord}(\rho(\gamma(t), \overline{\gamma(t)})) \leqq C \ \text{ord}(\gamma(t)) \,, \tag{0.1}$$

where $\text{ord}(\rho(\gamma(t), \overline{\gamma(t)}))$ and $\text{ord}(\gamma(t))$ denote the orders of vanishing of $\rho(\gamma(t), \overline{\gamma(t)})$ and $\gamma(t)$, respectively, at $t = 0$. In this case we let m_p be the smallest integer for which (0.1) is satisfied with $C = m_p$.

Theorem 5. *Let* $M \subset \mathbb{C}^N$ *be an algebraic hypersurface. Assume that there is no nontrivial complex analytic variety contained in* M *through* $p_0 \in M$, *and let* $m = m_{p_0}$ *be the integer defined as above. If* $H : M \to S^{2N+1} \subset \mathbb{C}^{N+1}$ *is a CR map of class* C^m, *where* S^{2N+1} *denotes the boundary of the unit ball in* $\mathbb{C}^{N+1}$, *then* H *admits a holomorphic extension in a neighborhood of* p_0.

The paper is organized as follows. In Sect. 1 we introduce a new invariant for real analytic hypersurfaces, which will be used in the proofs of Theorems 1 and 2 and could also be of independent interest. The proofs of Theorems 1 and 3 are given in Sects. 2 and 3 respectively. Sect. 4 is devoted to the proof of Theorem 2. In Sect. 5, we study properties of families of CR automorphisms for holomorphically degenerate hypersurfaces and give a proof of Theorem 4. The proof of Theorem 5 is given in Sects. 6 and 7.

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1 A new invariant for real analytic hypersurfaces

Let M be a real analytic hypersurface in $\mathbb{C}^N$ through 0 and $p_0 \in M$ close to 0. If $\rho(Z, \bar{Z})$ is a defining function for M near 0, with $\rho(p_0, \bar{p}_0) = 0$ and $d\rho(p_0, \bar{p}_0) \neq 0$, we define the *Segre surface* through $\bar{p}_0$ by

$$\Sigma_{p_0} = \{\zeta \in \mathbb{C}^N : \rho(p_0, \zeta) = 0\} \,.$$

Note that Σ_{p_0} is a germ of a smooth holomorphic hypersurface in $\mathbb{C}^N$ through $\bar{p}_0$. Let $L_1, \ldots, L_n$, $n = N - 1$, given by $L_j = \sum_{k=1}^N a_{jk}(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_k}$ be a basis of the CR vector fields on M near 0 with the a_{jk} real analytic. If $X_1, \ldots, X_n$, are the complex vector fields given by $X_j = \sum_{k=1}^N a_{jk}(p_0, \zeta) \frac{\partial}{\partial \zeta_k}$, $j = 1, \ldots, n$, then X_j is tangent to Σ_{p_0} and the X_j span the tangent space to Σ_{p_0} for ζ in a neighborhood of $\bar{p}_0$, with $(p_0, \zeta) \mapsto a_{jk}(p_0, \zeta)$ holomorphic near 0 in $\mathbb{C}^{2N}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we define $c_\alpha(Z, p_0, \zeta)$ in $\mathbb{C}\{Z, p_0, \zeta\}$, the ring of convergent power series in $3N$ complex variables, by

$$c_\alpha(Z, p_0, \zeta) = X^\alpha \rho(Z + p_0, \zeta)\,, \tag{1.1}$$

where $X^\alpha = X_1^{\alpha_1}, \ldots, X_n^{\alpha_n}$.

Note that since the X_j are tangent to Σ_{p_0}, we have $c_\alpha(0, p_0, \zeta) = 0$ for all $p_0 \in M$ and $\zeta \in \Sigma_{p_0}$ in a neighborhood of 0. In particular, $c_\alpha(0, p_0, \bar{p}_0) = 0$ for $p_0 \in M$ close to 0. We say that M is *essentially finite* at p_0 if the ideal $(c_\alpha(Z, p_0, \bar{p}_0))$ generated by the $c_\alpha(Z, p_0, \bar{p}_0)$, $\alpha \in \mathbb{Z}_+^n$, in the ring $\mathbb{C}\{Z\}$ is of finite codimension. (By the Nullstellensatz, this is equivalent to the condition that the functions $Z \mapsto c_\alpha(Z, p_0, \bar{p}_0)$ have only 0 as a common zero near the origin for p_0 fixed and $\alpha \in \mathbb{Z}_+^n$.) This definition of essential finiteness, which does not depend on either the choice of holomorphic coordinates or that of the defining function, coincides with that given in [BJT] and that given in [BR2] in a slightly different form. The present definition has the advantage of avoiding the use of the implicit function theorem, thus making explicit calculations easier.

We introduce here a new invariant which will give us a bound on the number of derivatives needed in Theorem 3. If M is essentially finite at $p_0 \in M$ fixed as above, let $\ell(p_0)$ be the minimum positive integer for which the ideal generated by $\{c_\alpha(Z, p_0, \bar{p}_0) \colon |\alpha| \leqq \ell(p_0)\}$ is of finite codimension in $\mathbb{C}\{Z\}$. It follows from the definition of essential finiteness and the fact that $\mathbb{C}\{Z\}$ is a Noetherian ring that $\ell(p_0)$ is finite. It is clear that $\ell(p_0) \geqq 1$.

Proposition 1.2. *Let M be a connected real analytic hypersurface which is holomorphically nondegenerate. Then there is an integer $\ell(M)$, with $1 \leqq \ell(M) \leqq N - 1$ such that $\ell(p) = \ell(M)$ for all p in an open, dense subset of M. Moreover, $\ell(M) = 1$ if and only if M is generically Levi nondegenerate.*

Proof. We need to introduce the following vector-valued functions. For a multi-index α, let V_α be the real analytic function defined near 0 in $\mathbb{C}^N$ by

$$V_\alpha(Z, \bar{Z}) = L^\alpha \rho_Z(Z, \bar{Z})\,, \tag{1.3}$$

where ρ_Z denotes the gradient of ρ with respect to Z. In the rest of the proof we shall say that a property holds *generically* on M (or an open neighborhood of p_0 in M) if it holds in an open, dense subset of M.

Lemma 1.4. *Let M be a connected real analytic hypersurface in $\mathbb{C}^N$. Then M is holomorphically nondegenerate if and only if $\{V_\alpha(Z, \bar{Z}), \alpha \in \mathbb{Z}_+^n\}$ span $\mathbb{C}^N$ generically in a neighborhood of 0 in M.*

Proof. We note first that the condition that the V_α span $\mathbb{C}^N$ is independent of the choice of coordinates and defining function. We introduce here normal coordinates near 0, $Z = (z, w)$, $z \in \mathbb{C}^n$, $w \in \mathbb{C}$, such that M is given there by

$$w = Q(z, \bar{z}, \overline{w}), \quad \text{with } Q(z, 0, w) \equiv w\,,$$

(or equivalently by $\bar{w} = \bar{Q}(\bar{z}, z, w)$). We put $p_0 = (z^0, w^0)$; we may then take

$$X_j = \frac{\partial}{\partial \chi_j} + \bar{Q}_{\chi_j}(\chi, z^0, w^0)\frac{\partial}{\partial \tau}\ , \tag{1.5}$$

where $\zeta = (\chi, \tau)$, $\chi = (\chi_1, \ldots, \chi_n)$. Hence $c_\alpha(Z, p_0, \zeta)$ of (1.1) is given by

$$c_\alpha(Z, p_0, \bar{p}_0) = -\bar{Q}_{\chi^\alpha}(\bar{z}^0, z^0, w^0) + \bar{Q}_{\chi^\alpha}(\bar{z}^0, z + z^0, w + w^0)\ . \tag{1.6}$$

Similarly, we have by using (1.3),

$$V_\alpha(p_0, \bar{p}_0) = -\bar{Q}_{\bar{z}^\alpha, Z}(\bar{z}^0, z^0, w^0)\ . \tag{1.7}$$

Hence $c_{\alpha,Z}(0, p_0, \bar{p}_0) = -V_\alpha(p_0, \bar{p}_0)$. If the rank of the $V_\alpha(Z, \bar{Z})$ is less than N generically, then at any point of maximal rank p_0 near 0 in M, by the implicit function theorem, there is a complex curve $Z(t)$ such that $c_\alpha(Z(t), p_0, \bar{p}_0) = 0$ for all small t and all α. Hence M is not essentially finite at p_0. Since the set of essentially finite points is open [BR2] (and the set of points of maximal rank is open and dense), there exists an essentially finite point if and only if the generic rank of the $V_\alpha(Z, \bar{Z})$ is N. By [BR6], the existence of an essentially finite point is equivalent to holomorphic nondegeneracy of M. This completes the proof of the lemma. □

We shall also need the following lemma, whose simple proof is left to the reader.

Lemma 1.8. *Let f be a holomorphic function defined in an open set Ω in $\mathbb{C}^p$, valued in $\mathbb{C}^N$. If the $\partial^\alpha f(\chi)$, $\alpha \in \mathbb{Z}_+^p$ span $\mathbb{C}^N$ generically in Ω, then the $\partial^\alpha f(\chi)$, $|\alpha| \leqq N-1$ also span $\mathbb{C}^N$ generically in Ω.*

Lemma 1.9. *Let M be a holomorphically nondegenerate real analytic hypersurface of $\mathbb{C}^N$ through 0. There exists an integer k, with $1 \leqq k \leqq N-1$ so that $\{V_\alpha(Z, \bar{Z}), |\alpha| \leqq k\}$ span $\mathbb{C}^N$ generically in a neighborhood of 0 in M.*

Proof of Lemma 1.9. By Lemma 1.4, we may find $p_0 = (z^0, w^0) \in M$ so that the vectors $V_\alpha(p_0, \bar{p}_0)$ span $\mathbb{C}^N$ as α varies over all multi-indices. We put $f(\chi) = \bar{Q}_Z(\chi, z^0, w^0)$. Now by (1.7) and Lemma 1.8, we conclude that there exists k, $1 \leqq k \leqq N-1$, such that the vector-valued functions $\bar{Q}_{\chi^\alpha, Z}(\chi, z^0, w^0)$, $|\alpha| \leqq k$ span $\mathbb{C}^N$ generically for χ in a neighborhood of 0 in $\mathbb{C}^{N-1}$. This is equivalent to the nonvanishing of an $N \times N$ determinant $\Delta(\chi, p_0)$. We claim that the functions $\bar{Q}_{\chi^\alpha, Z}(\bar{z}, z, w)$, $|\alpha| \leqq k$ also span $\mathbb{C}^N$ generically for $(z, w) \in M$ near 0. For this, it suffices to show that $\Delta(\bar{z}, Z)$ does not vanish identically for $Z \in M$ near 0. Indeed, if $\Delta(\bar{z}, Z) \equiv 0$ on M, then by complexifying the variables, we would also have $\Delta(\chi, Z) \equiv 0$ for χ near 0 in $\mathbb{C}^{N-1}$ and Z near 0 in $\mathbb{C}^N$, contradicting $\Delta(\chi, p_0) \not\equiv 0$. □

We return now to the proof of Proposition 1.2. We first note that the function $\ell(p)$ is upper semi-continuous on M, i.e. $\ell(p) \leqq \ell(p_0)$ for p near p_0. By (1.6), (1.7) and the implicit function theorem, it follows that if $\{V_\alpha(p_0, \bar{p}_0), |\alpha| \leqq k\}$ span $\mathbb{C}^N$, then $\ell(p_0) \leqq k$. Conversely, if k is the smallest integer for which $\{V_\alpha(p_0,\bar{p}_0), |\alpha| \leqq k\}$ span $\mathbb{C}^N$ generically for p_0 in a neighborhood of 0, then it cannot happen that $\ell(p_0) < k$ for any p_0 near 0. For if so, by going to an arbitrarily close point p where the rank of $\{V_\alpha(p,\bar{p}), |\alpha| \leqq \ell(p_0)\}$ is maximal and applying the implicit function theorem, we would obtain a complex curve of common zeroes for the functions $\{c_\alpha(Z,p,\bar{p})\colon |\alpha| \leqq \ell(p_0)\}$. This would be a contradiction, since, by the above, $\ell(p) \leqq \ell(p_0)$. This proves that the minimum of $\ell(p)$ in a neighborhood of 0 is the same as the smallest integer k satisfying the conclusion of Lemma 1.9. Since M is connected and real analytic, it suffices to take $\ell(M)$ to be the smallest integer k of Lemma 1.9.

To complete the proof of Proposition 1.2, it remains to show that $\ell(M) = 1$ is equivalent to M being generically Levi nondegenerate. For this note that an easy row and column manipulation show that

$$\det\,[\rho_Z(Z,\bar{Z}), L_1\rho_Z(Z,\bar{Z}), \ldots, L_{N-1}\rho_Z(Z,\bar{Z})]$$

is a nonvanishing multiple of the usual Levi determinant of M at Z. (See e.g. [W1].) □

Proposition 2.1 and its proof suggest the definition of a new invariant which refines the notion of holomorphic nondegeneracy. Let M be a real analytic hypersurface in $\mathbb{C}^N$ and ρ a local defining function. We say that M is *k-holomorphically nondegenerate* at $Z \in M$ if the $L^\alpha\rho_Z(Z,\bar{Z})$, $|\alpha| \leqq k$, span $\mathbb{C}^N$ with k minimal. It follows from the proof of Proposition 1.2 that if M is holomorphically nondegenerate then generically $1 \leqq k \leqq N-1$. Also, M is 1-holomorphically nondegenerate at Z if and only if the Levi form of M is nondegenerate at Z. Note that if M is connected and holomorphically nondegenerate then there exists $\ell = \ell(M)$, $1 \leqq \ell(M) \leqq N-1$, such that M is ℓ-holomorphically nondegenerate at every point in an open dense subset of M. This number $\ell(M)$ is given by Proposition 1.2; we shall call $\ell(M)$ the *Levi type* of M.

2 Proof of Theorem 1

First recall from [BR6, Theorem 2] that since M is connected and holomorphically nondegenerate, the set of points at which M is essentially finite (see [BJT] and [BR1] for definition) is not empty. On the other hand, if M is essentially finite at p, then M is also of finite type (in the sense of Bloom–Graham [BG]) at p. Let

$$U = \{p \in M : M \text{ is of finite type at } p\}\,. \tag{2.1}$$

Since $M\backslash U$ is a real analytic subset of M, it follows from the above that $M\backslash U$ is a proper (possibly empty) real analytic subset of M, and hence U is an open, dense subset of M. More precisely, $\partial U = M\backslash U$ is a smooth complex hypersurface in $\mathbb{C}^N$. Indeed, this can be seen by using a theorem of Nagano [N]; if $M\backslash U$ is nonempty, then it is given locally as the real analytic manifold whose complexified tangent space is spanned by the CR tangent vectors and their complex conjugates. Note that if ∂U is nonempty then it is of codimension 1 in M.

Proposition 2.2. *Let M, M', and H be as in Theorem* 1. *Then* $\operatorname{Jac} H$ *does not vanish identically on any open set in M.*

We shall need the following in the proof of the proposition.

Lemma 2.3. *Let M be a real analytic hypersurface in $\mathbb{C}^N$, and assume that U defined by* (2.1) *is nonempty. If f is a continuous CR function on M and f vanishes identically in some neighborhood of p_0 in U, then f vanishes identically in the whole connected component of p_0 in U.*

Proof. Let U_0 be the connected component of p_0 in U, and let

$$S = \{Z \in U_0 \colon f|_V \not\equiv 0 \text{ for any neighborhood } V \text{ of } Z \text{ in } U_0\}\,.$$

We claim that S is open and closed in U_0. Indeed, it is immediate from the definition that S is closed. To show that S is open, we let $q \in S$ and choose a connected neighborhood $W \subset U_0$ of q sufficiently small such that f extends to one side of M with boundary W. (The extendability of CR functions at q to one side of M follows from the fact that M is of finite type at q [BT].) If f were to vanish on an open subset of W, then the holomorphic function extending f would vanish identically, and hence f would vanish on W, contradicting the assumption that q is in S. This shows $W \subset S$ and completes the proof of Lemma 2.3. □

In the following we shall write $J(Z)$ for $\operatorname{Jac} H(Z)$ for $Z \in M$.

Lemma 2.4. *Under the assumptions of Theorem* 1, *if $J|_{U_0} \not\equiv 0$, where U_0 is a connected component of U, then $J|_{U_0}$ is algebraic.*

Proof. Since M is holomorphically nondegenerate, as noted before by Theorem 2 of [BR6] the set where M is essentially finite is nonempty. Hence by Proposition 1.12 of [BR2], the set of essentially finite points in M is open and dense. By the continuity of J we may find $p_0 \in U_0$ such that M is essentially finite at p_0 and $J(p_0) \neq 0$. By a result in [BJT] we conclude that H extends holomorphically to an open neighborhood $\mathcal{O}$ of p_0 in $\mathbb{C}^N$. Denote by $\mathscr{H}$ this holomorphic extension. We may now use Theorem 1 in [BR6] to conclude that $\mathscr{H}$ is algebraic in $\mathcal{O}$. Since the derivatives and products of algebraic functions are again algebraic, the holomorphic extension $\mathscr{J}$ of J to $\mathcal{O}$ is also algebraic in $\mathcal{O}$. Let $P(Z,X)$ be the polynomial, with polynomial coefficients, such that $P(Z,\mathscr{J}(Z)) \equiv 0$ in $\mathcal{O}$. On the other hand, since J is a CR function on M, we conclude that $f(Z) = P(Z,J(Z))$, $Z \in M$, is also CR on M. By Lemma 2.3, $f(Z)$ vanishes identically on U_0, since it vanishes on $\mathcal{O} \cap M$. This shows that $J|_{U_0}$ is algebraic. □

Lemma 2.5. *Let g be a smooth CR function on M and assume that $g|_{U_0}$ is algebraic, where U_0 is a connected component of U* (*given by* (2.1)). *If*

$g|_{U_0} \not\equiv 0$, then g cannot vanish to infinite order at any point in the closure of U_0.

Proof. Let $P(Z,X)$ be a polynomial such that

$$P(Z,g(Z))|_{U_0} \equiv 0\,. \tag{2.6}$$

Suppose $P(Z,X) = P_1(Z,X)P_2(Z,X)$, where $P_j(Z,X)$ are polynomials of positive degree in X. Then since $P_j(Z,g(Z))$ is CR on M, $j = 1,2$, we may use Lemma 2.3 to conclude that either $P_1(Z,g(Z))|_{U_0} \equiv 0$ or $P_2(Z,g(Z))|_{U_0} \equiv 0$. Hence we may assume that the polynomial $P(Z,X) = \sum_0^k a_j(Z)X^j$ in (2.6) is irreducible, and, in particular, $a_0(Z) \not\equiv 0$. If $g(Z)$ vanishes of infinite order in the closure of U_0, it would follow from (2.6) that the restriction of $a_0(Z)$ to M also vanishes of infinite order at that point. Since $a_0(Z)$ is a polynomial, and M is real analytic, this would imply $a_0(Z)$ vanishes identically, contradicting the irreducibility of P. □

Proof of Proposition 2.2. Define E by

$$E = \{Z \in M : J|_V \not\equiv 0 \text{ for any neighborhood } V \text{ of } Z \text{ in } M\}.$$

Since E is nonempty by assumption, the proposition will follow from the connectedness of M if we prove that E is open and closed. The closedness of E is immediate from the definition. We shall show that E is open. First, if $p_0 \in E \cap U$, and U_0 is the connected component of U containing p_0, then by Lemma 2.1 we have $U_0 \subset E$. If $p_0 \in E \cap \partial U$, and V is a sufficiently small neighborhood of p_0, V will intersect at most two connected components of U, say U_1 and U_2. (For this, recall that ∂U is a smooth submanifold of M of codimension 1.) By definition of E, either $J|_{U_1 \cap V} \not\equiv 0$ or $J|_{U_2 \cap V} \not\equiv 0$. It then follows from Lemmas 2.4 and 2.5 that J cannot vanish to infinite order at p_0; therefore $J|_{U_j \cap V} \not\equiv 0$, $j = 1,2$. By Lemma 2.1, this shows $V \subset E$, which completes the proof that E is open and the proposition follows. □

We shall need the following result, which is probably known. (See also Lemma 6.1 of [BJT] for a special case of this result.)

Lemma 2.7. *Let $G(z,w)$ be a holomorphic function in a neighborhood of 0 in $\mathbb{C}^{p+1}$ with $G_w(z,w) \not\equiv 0$. Let f be a smooth function defined in a neighborhood of 0 in $\mathbb{R}^p$ satisfying*

$$G(x,f(x)) \equiv 0\,, \tag{2.8}$$

for x in a neighborhood of 0 in $\mathbb{R}^p$. Then f is real analytic in a neighborhood of 0.

Proof. We wish to apply the following theorem due to Malgrange [M]: Let Y be the germ of a real analytic set in $\mathbb{R}^q$ through 0 containing a germ Σ of a smooth manifold through 0. If $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} \Sigma$, then Σ is real analytic.

We write $G(x,s) = \sum_{j=0}^{\infty} a_j(x)s^j$, where each a_j is a convergent power series. We may assume that the $a_j(x)$, $j = 0,1,\ldots,$ have no common factors (as convergent power series in p variables). We define the real analytic set $Y \subset \mathbb{R}^{p+2}$ as follows. Let $Q_1(x,y), Q_2(x,y)$ be the real valued functions determined by

$$\sum_0^\infty a_j(x)(y_1 + iy_2)^j = Q_1(x,y) + iQ_2(x,y)$$

with $y = (y_1, y_2) \in \mathbb{R}^2$, and let Y be the germ of the real analytic set at 0 defined by

$$Y = \{(x,y) \in \mathbb{R}^{p+2} : Q_1(x,y) = Q_2(x,y) = 0\} .$$

Let Σ be the germ at 0 of the smooth submanifold of Y given by the parametrization

$$\Sigma = \{(x, g_1(x), g_2(x)) : x \in \mathbb{R}^p, \text{ where } f(x) = g_1(x) + ig_2(x)\} .$$

Clearly $\dim_{\mathbb{R}} \Sigma = p$; the desired real analyticity of f will follow from Malgrange's result above if $\dim_{\mathbb{R}} Y = p$. To prove this last equality, note that if $(x,y) \in Y$, then each y_j is determined by x up to finitely many values unless all the a_j vanish at x. However, since the $a_j(x)$ have no common factors, we claim that the dimension of their common zeros is less than or equal to $p-2$. For this, note first by the Noetherian theorem, there exists k such that $\{x : a_j(x) = 0, j = 0, 1, \ldots, \} = \{x : a_j(x) = 0, j = 0, 1, \ldots, k\}$. Now the claim can be seen by expressing the a_j as Weierstrass polynomials with respect to the same variable and applying an elimination method as e.g. in Lemma 5.1 of [BJT]. □

Proof of Theorem 1. Let U be given by (2.1). Since by Proposition 2.2 J, the Jacobian of H, does not vanish identically on any open subset of M, we conclude as in the proof of Lemma 2.4 that H is algebraic on each connected component of U. In order to show that H extends holomorphically to a neighborhood of M, by standard arguments it suffices to show that H is real analytic in a neighborhood of each point in M. Let $p_0 \in M$. We claim that H is algebraic in some neighborhood of p_0. Indeed, if $p_0 \in U$, then by the above, H is algebraic in the component of p_0. If $p_0 \in \partial U$, then p_0 is in the boundary of at most two components, say U_1 and U_2. Hence for $j = 1, \ldots, N$ there exist polynomials $p_{1j}(Z,X)$ and $p_{2j}(Z,X)$ such that

$$p_{kj}(Z, H_j(Z))|_{U_k} \equiv 0, \quad k = 1, 2, \quad j = 1, \ldots, N . \tag{2.9}$$

Let $p_j(Z,X) = p_{1j}(Z,X)p_{2j}(Z,X)$. It follows from (2.9) that $p_j(Z,H_j(Z))|_{U_1 \cup U_2} \equiv 0$, which proves the claim. By taking a real analytic parametrization of M, we may apply Lemma 2.7 to conclude that H is real analytic at every point. This completes the proof of Theorem 1. □

The following example shows that the assumption that M is algebraic cannot be dropped.

Example 2.10. The following example was given by Ebenfelt in [E]. Let $t = \theta(\xi, s)$ be the unique solution of the algebraic equation $\xi(t^2 + s^2) - t = 0$, with $\theta(0,0) = 0$. Let M and M' be the hypersurfaces through 0 in $\mathbb{C}^2$ given respectively by

$$\Im w = \theta(\arctan |z|^2, \Re w)\ , \qquad \Im w' = (\Re w')|z'|^2\ .$$

(Note that M and M' are both of infinite type at 0.) Let $H = (f, g)$ be the mapping given by $f(z,w) = z$, $g(z,w) = e^{-(1/w)}$ for $\Re w > 0$ and $g(z,w) \equiv 0$ for $\Re w \leqq 0$. It is shown in [E] that H is a smooth CR mapping defined in a neighborhood of 0 which maps M to M'. However, it is clear that H does not extend holomorphically to a neighborhood of 0 in $\mathbb{C}^2$. Note that M' is algebraic, but M is not.

3 Proof of Theorem 3

To prove Theorem 3, we take $\ell = \ell(M)$ to be the Levi type of M defined at the end of Sect. 1. Let U be the open set in M defined in (2.1) and ∂U its boundary. Since $\operatorname{Jac} H$ does not vanish identically on any open set, it does not vanish identically on any connected component of U. We fix such a component U_0 and choose $p_0 \in U_0$ such that $\operatorname{Jac} H(p_0) \neq 0$, and $\ell(p_0) = \ell(M)$; the latter is possible by Proposition 1.2. Note that in particular M is essentially finite at p_0. As before, we assume that local normal coordinates on M and M' are chosen so that $p_0 = 0$ and $H(p_0) = 0$. In these coordinates we write $H = (f, g), f = (f_1, \ldots, f_n)$, $n = N - 1$.

Using the methods of proof of Lemma 6.1 of [BR1] and Proposition 2.5 of [BR6], we obtain the following. For each j, $1 \leqq j \leqq n$, there exists a positive integer N_j and algebraic functions $a_{jk}(u_p^\gamma, v^\beta)$, $|\gamma|, |\beta| \leqq l$, $0 \leqq k \leqq N_{j-1}$, $1 \leqq p \leqq n$, holomorphic near $u_{p,0}^\gamma = L^\gamma \bar{f}_p(0)$ and $v_0^\beta = 0$, such that we have in a neighborhood of 0 in M:

$$f_j^{N_j} + \sum_{k=0}^{N_j - 1} a_{jk}(L^\gamma \bar{f}_p, L^\beta \bar{g}) f_j^k \equiv 0, \quad j = 1, \ldots, n\ . \tag{3.1}$$

(Although in [BR1] and [BR6] the mapping H was assumed to be smooth, the proof of (3.1) uses derivatives only up to length ℓ; hence (3.1) holds also when H is only of class C^ℓ.) We may now move to a point p_1, arbitrarily close to 0 near which the roots of the polynomials (3.1) are analytic functions of the coefficents. We conclude that near that point p_1

$$f_j = \Psi_j(L^\gamma \bar{f}_p, L^\beta \bar{g}), \quad j = 1, \ldots, n\ , \tag{3.2}$$

with Ψ_j analytic. By the standard use of the reflection principle, as for instance in [BR1] (see also Proposition 7.1 below), we conclude that the f_j extend holomorphically near p_1. It then follows easily that the same holds for g near p_1. We continue to denote by $H = (H_1, \ldots, H_N)$ the original CR map as well as its holomorphic extension in a neighborhood of p_1. We may now apply Theorem 1 of [BR6] to conclude that H is algebraic in a neighborhood in $\mathbb{C}^N$. That is, for $j = 1, \ldots, N$, there exist polynomials $P_j(Z, X)$ with holomorphic polynomial coefficients such that $P_j(Z,H_j(Z)) \equiv 0$ is a neighborhood of p_1. Let $k_j(Z) = P_j(Z,H_j(Z))|_M$; each k_j is a CR function on M which vanishes on a neighborhood of p_1 in M. Hence, by the argument of the proof of Theorem 1, k_j must vanish identically in the connected component U_0 of p_1 in U. This shows that the restriction of H_j to each connected component of U is algebraic. It remains to show the same holds near a point $p_0 \in \partial U$. Since ∂U is a smooth hypersurface of M, p_0 is in the closure of two connected components of U. For each j we take the product of the two polynomials corresponding to the two connected components to obtain an algebraic equation satisfied by H_j in the closure of the union of these components. This completes the proof of Theorem 3. □

Example 3.3. In Theorem 1 it was sufficient to assume that $\mathrm{Jac}\, H$ does not vanish identically on M. The following example shows that for the conclusion of Theorem 3 to hold, unlike that of Theorem 1, we must assume the stronger condition that $\mathrm{Jac}\, H \not\equiv 0$ on each component of U. In the smooth case (i.e. Theorem 1), we show in the course of the proof that the two conditions are actually equivalent. For this, let M and M' be the hypersurfaces through 0 in $\mathbb{C}^2$ given respectively by

$$\Im w = (\Re w)^3 |z|^2, \qquad \Im w' = 2\frac{(\Re w')\theta(|z'|^2)}{1 - [\theta(|z'|^2)]^2},$$

where $\theta(t)$ is the unique real solution vanishing for $t = 0$, of the polynomial equation $X^3 + X - t = 0$. Note that both M and M' are algebraic and generically Levi nondegenerate. Consider the CR mapping H defined on M by $H = (f,g)$ with

$$\left\{ \begin{array}{ll} f(z,w) = zw, & \Re w \geqq 0 \\ f(z,w) = zw \exp w, & \Re w \leqq 0 \end{array} \right\}, \qquad \left\{ \begin{array}{ll} g(z,w) = w^2, & \Re w \geqq 0 \\ g(z,w) = 0, & \Re w \leqq 0 \end{array} \right\}.$$

The reader can easily check that H is of class C^1 on M and that H maps a neighborhood of 0 in M into M'. Note that H is not algebraic, and $\mathrm{Jac}\, H(z,w) \not\equiv 0$ for $(z,w) \in M$, with $\Re w > 0$, but $\mathrm{Jac}\, H(z,w) \equiv 0$ for $(z,w) \in M$, with $\Re w < 0$. Note that in this example the number ℓ given by Theorem 3 is 1.

4 Proof of Theorem 2

In this section we shall indicate the modifications to the proof of Theorem 1 needed to give Theorem 2. For a polynomial

$$P(Z,X) = a_J(Z)X^J + a_{J-1}(Z)X^{J-1} + \cdots a_0(Z) \tag{4.1}$$

with polynomial coefficients $a_j(Z)$, where $a_J(Z) \not\equiv 0$, by the *total degree* of P we shall mean the total degree of P as a polynomial in the variables (X,Z). If $f(Z)$ is an algebraic function, by the *total degree* of f we shall mean the minimum of the total degrees among polynomials $P(Z,X)$ for which

$P(Z, f(Z)) \equiv 0$. We need the following lemma, whose proof is based on the Artin Approximation Theorem and was suggested to us by Leonard Lipshitz.

Lemma 4.2. *For any positive integers d and n there exists a positive integer $k = k(d, n)$ such that if f is a function of class C^k defined in a neighborhood of 0 in $\mathbb{R}^n$ and satisfies a nontrivial polynomial equation $p(x, f(x)) \equiv 0$, where $p(x, Y)$ is a polynomial of $n+1$ variables of total degree $\leqq d$, then f is real analytic in a neighborhood of 0.*

Proof. Let $\Delta(x)$ be the discriminant of $p(x, Y)$ regarded as a polynomial in the indeterminate Y. By eliminating repeated factors in the factorization of $p(x, Y)$, we may assume that $\Delta(x) \not\equiv 0$. We use the following consequence of the Artin Approximation Theorem [A1, A2, BDLV]:

Let $p(x, Y)$, $x \in \mathbb{R}^n$, be a polynomial in Y with polynomial coefficients. Then for any positive integer r there exists a positive integer k (depending only on r, n and the total degree of p) such that if $f_1(x)$ is a formal series for which $p(x, f_1(x)) = O(|x|^k)$ there is a convergent series $g(x)$ such that $p(x, g(x)) \equiv 0$ and $g(x) - f_1(x) = O(|x|^r)$.

In fact, the statement above is a special case of Theorem 6.1 of [A2] or Theorem 3.2 of [BDLV]. We shall apply the above with $r = (d_1 + d_2(J(J-1)) + 1)/2$, where d_1 is the degree of the polynomial $\Delta(x)$ defined above, d_2 is the degree of $a_J(x)$. Note that r is bounded by an expression depending only on the total degree of p. If k is given by the statement above, then we claim that the conclusion of Lemma 4.2 holds with this choice of k. For this, let $f_1(x)$ be the truncated Taylor series of $f(x)$ up to degree k and let $g(x)$ be the convergent series given by the statement above. If $\rho(x)$ is a root of $p(x, Y)$, then $a_J(x)\rho(x)$ is a root of the monic polynomial

$$q(x, Y) = Y^J + a_{J-1}(x)Y^{J-1} + \cdots + a_J^{J-1}(x)a_0(x)\,. \tag{4.3}$$

In particular, $a_J(x)f(x)$ and $a_J(x)g(x)$ are both roots of $q(x, Y)$. If $g(x) \not\equiv f(x)$, let $\tau_1(x) = a_J(x)f(x)$, $\tau_2(x) = a_J(x)g(x)$, and let $\tau_3(x), \ldots, \tau_J(x)$ be the rest of the roots of $q(x, Y)$ (counted with multiplicity). Then the discriminant of $q(x, Y)$ is

$$a_J^{J(J-1)}(x)\Delta(x) = -(a_J(x)f(x) - a_J(x)g(x))^2 \Pi(\tau_j(x) - \tau_k(x))\,, \tag{4.4}$$

where the indices on the right hand side run over $j \neq k$ and either j or k is not equal to 1 or 2. Since $q(x, Y)$ is a monic polynomial, the τ_k are bounded. Hence the right hand side of (4.4) vanishes to order at least $2r$. On the other hand, since the left hand side of (4.4) is of degree $\leqq d_1 + d_2(J(J-1))$, both sides must vanish identically, by the choice of r, contradicting the assumption that $\Delta(x) \not\equiv 0$. This contradiction shows that $g(x) \equiv f(x)$, which completes the proof of the lemma. □

We shall now prove Theorem 2. We start with the following analogue of Proposition 2.2.

Lemma 4.5. *If k is sufficiently large, and $\operatorname{Jac} H \not\equiv 0$, then $\operatorname{Jac} H$ does not vanish identically on any open set in M.*

Proof. By the connectedness of M, and using Lemma 2.3, it suffices to show that if $\operatorname{Jac} H \not\equiv 0$ in some connected component U_0 of U, then $\operatorname{Jac} H \not\equiv 0$ on any component of U which is contiguous to U_0. As in Sect. 2 we may find $p \in U_0$ at which M is essentially finite and $\operatorname{Jac} H(p) \neq 0$. Hence, the components of H extend holomorphically in a neighborhood V of p in $\mathbb{C}^N$. By [BR6], there are polynomials $P_j(Z,X)$ of the form (4.1) such that the jth component of H satisfies $P_j(Z,H_j(Z)) \equiv 0$ for $Z \in V$. By the proof given in [BR6], one can see that the total degree of the $P_j(Z,X)$ is bounded by a number which depends only on the total degrees of the defining functions of M and M'. Hence $\operatorname{Jac} H$ is the root of a polynomial $P(Z,X)$ whose total degree is bounded by a number depending only on the total degrees of the defining functions of M and M'.

By propagation of the zeroes of CR functions in U_0,

$$P(Z, \operatorname{Jac} H(Z)) \equiv 0 \tag{4.6}$$

for $Z \in U_0$, and, as in Sect. 2, we may assume that $P(Z,X)$ is irreducible. If $\operatorname{Jac} H$ were to vanish on a component U_1 contiguous to U_0, then it vanishes to order at least $k-1$ on the boundary between U_0 and U_1. Hence the constant coefficient $a_0(Z)$ of P must also vanish to order $k-1$ there. Since the degree of $a_0(Z)$ is bounded by the total degree of P, we would have $a_0(Z) \equiv 0$ if $k-1$ is greater than the total degree of P. Since this would contradict the irreducibility of P, we conclude as before that $\operatorname{Jac} H$ does not vanish on any open set in M. □

We shall now show that each component H_j of H is algebraic with total degree bounded by a number depending only on the total degrees of the defining functions of M and M'. By Lemma 4.2, this will complete the proof of Theorem 2. As in the argument above, for a fixed component U_0 of U, there exists polynomials $P_j(Z,X)$, with total degree bounded by number depending only on the total degrees of the defining functions of M and M', such that $P_j(Z,H_j(Z)) \equiv 0$, $j = 1,\ldots,N$ for $Z \in U_0$. By the connectedness of M, it suffices to show that $P_j(Z,H_j(Z)) \equiv 0$, in any component U_1 of U adjacent to U_0. By Lemma 4.5, $\operatorname{Jac} H \not\equiv 0$ on U_1, so that one can find polynomials $\tilde{P}_j(Z,X)$, with total degree bounded by a number depending only on the total degrees of the defining functions of M and M', such that $\tilde{P}_j(Z,H_j(Z)) \equiv 0$, $j = 1,\ldots,N$ for $Z \in U_1$. We now have that

$$P_j(Z,H_j(Z))\tilde{P}_j(Z,H_j(Z)) \equiv 0, \quad j = 1,\ldots,N\ , \tag{4.7}$$

for $Z \in U_0 \cup U_1$. Hence if k is sufficiently large (depending only on the total degrees of M and M', we may apply Lemma 4.2 to conclude that H is real analytic in the interior of the closure of $U_0 \cup U_1$. By unique continuation of analytic functions, it follows that $P_j(Z,H_j(Z)) \equiv 0$ for $Z \in U_0 \cup U_1$. This completes the proof of Theorem 2. □

It should be noted that in general the integer k in Theorem 2 could be much larger that $\ell(M)$, the Levi type of M defined at the end of Sect. 1. The following example shows that in Theorem 2, the integer k cannot be taken to be $\ell(M)$.

Example 4.8. Let M and M' be given as in Example 3.3. Consider the CR mapping H defined on M by $H = (f, g)$ with $f(z,w) = zw$ and $g(z,w) = w^2$ for $\Re w \geqq 0$, and $g(z,w) = -w^2$ for $\Re w \leqq 0$. As observed by Peter Ebenfelt, H is of class C^1 and $\operatorname{Jac} H$ does not vanish identically on any open subset of M. However, H is algebraic but does not extend holomorphically in any neighborhood of 0 in $\mathbb{C}^2$. Note that $\ell(M) = 1$ here.

5 Families of CR automorphisms; Proof of Theorem 4

We shall prove Theorem 4 in this section. By Proposition 4.2 of [BR6], since M is holomorphically degenerate at p_1, it is holomorphically degenerate at p_0 also. We choose local coordinates near p_0 for which p_0 is the origin, and let $X = \sum_1^N a_j(Z)\frac{\partial}{\partial Z_j}$, where the $a_j(Z)$ are germs at 0 of holomorphic functions, be a nontrivial holomorphic vector field tangent to M near 0. Let $h(Z)$ be a smooth CR function defined on M near 0 which does not extend holomorphically to any neighborhood of 0 in $\mathbb{C}^N$. We may choose h so that $h(0) = 0$. Let $\phi(t,Z)$ be the flow of X for $t \in \mathbb{C}$, $|t|$ small, i.e. $\phi(t,Z)$ satisfies the holomorphic ordinary differential equation

$$\frac{\partial}{\partial t}\phi(t,Z) = A(\phi(t,Z)), \quad \phi(0,Z) = Z\,, \tag{5.1}$$

where $A = (a_1, \ldots, a_N)$. Let Y be the complex vector field on M near 0 obtained from X by multiplication of the coefficients by h i.e.,

$$Y = hX = \sum_1^N h(Z)a_j(Z)\frac{\partial}{\partial Z_j}\ . \tag{5.2}$$

Lemma 5.3. *The differential equation*

$$\frac{\partial}{\partial t}K(t,Z) = h(\phi(K(t,Z),Z)), \qquad K(0,Z) \equiv 0\,, \tag{5.4}$$

has a smooth solution $K(t,Z)$ *defined for* (t,Z) *in a neighborhood of* $(0,0)$ *in* $\mathbb{C} \times M$ *with* $t \mapsto K(t,Z)$ *holomorphic in* t *for fixed* Z*, and* $Z \mapsto K(t,Z)$ *CR for* t *fixed.*

Proof. Let $F(t,Z) = h(\phi(t,Z))$. Note that since $h(0) = 0$, it follows that $F(0,0) = 0$. Since $\phi(t,Z)$ is holomorphic in Z and h is CR, $Z \mapsto F(t,Z)$ is also CR. We claim that $t \mapsto F(t,Z)$ is holomorphic. Indeed, let $\tilde{h}$ be any smooth extension of h to a neighborhood of 0 in $\mathbb{C}^N$. By the chain rule,

$$\frac{\partial}{\partial \bar{t}}(\tilde{h}(\phi(t,Z),Z)) = \tilde{h}_Z \cdot \frac{\partial}{\partial \bar{t}}\phi(t,Z) + \tilde{h}_{\bar{Z}} \cdot \overline{\frac{\partial}{\partial t}\phi(t,Z)}\,. \tag{5.6}$$

The first term on the right in (5.6) is zero since ϕ is holomorphic in t. By (5.1), the second term is $\tilde{h}_{\bar{Z}} \cdot \bar{A}$, which equals $(\bar{X}h)(\phi(t,Z),Z)$, since $\tilde{h} = h$ on M

and $\bar{X}$ is tangent to M. This term also vanishes since $\bar{X}$ is a CR vector field and h is a CR function. The existence of a smooth solution $K(t,Z)$, holomorphic in t for $Z \in M$ is then given by the holomorphic theory of ordinary differential equations.

To see that $K(t,Z)$ is CR, let L be a CR vector field near 0. Then using (5.4), a simple calculation shows that $LK(t,Z)$ satisfies the ODE

$$\frac{\partial}{\partial t} LK(t,Z) = L \frac{\partial}{\partial t} K(t,Z) = (Xh)(\phi(K(t,Z),Z)) LK(t,Z) \,, \tag{5.7}$$

with $LK(0,Z) \equiv 0$. By uniqueness, $LK(t,Z) \equiv 0$ for all t. □

Lemma 5.8. *Let* $\psi(t,Z) = \phi(K(t,Z),Z)$, *with* $K(t,Z)$ *given by Lemma* 5.3, *be defined for* (t,Z) *in a neighborhood of* $(0,0)$ *in* $\mathbb{C} \times M$. *Then* ψ *defines a complex CR flow for the vector field* Y *defined by* (5.2). *That is*,

$$\frac{\partial}{\partial t} \psi(t,Z) = h(\psi(t,Z)) A(\psi(t,Z)) \tag{5.9}$$

with $\psi(0,Z) = Z$ *and* $t \mapsto \psi(t,Z)$ *is holomorphic for fixed* Z. *Furthermore*, $Z \mapsto \psi(t,Z)$ *is CR for each* t.

Proof. Since $\frac{\partial}{\partial t}\psi(t,Z) = (\frac{\partial}{\partial t}\phi)(K(t,Z),Z) \cdot \frac{\partial}{\partial t}K(t,Z)$, (5.9) and the holomorphy of $\psi(t,Z)$ with respect to t are immediate from the properties of $K(t,Z)$ given in Lemma 5.3. The fact that $\psi(t,Z)$ is CR for fixed t follows easily since the same is true of $K(t,Z)$. □

Lemma 5.10. *Let* $R(t,Z)$ *be a smooth function defined in* $\Delta_\varepsilon \times V$ *with* $\Delta_\varepsilon = \{t \in \mathbb{C} : |t| < \varepsilon\}$ *and* V *a neighborhood of* 0 *in* M. *Assume that* R *is holomorphic in t for fixed* Z, $R(t,0) \equiv 0$, *and for each* $t \in \Delta_\varepsilon$ *there exists* $\mathcal{O}_t$, *a neighborhood of* 0 *in* $\mathbb{C}^N$ *such that* $Z \mapsto R(t,Z)$ *extends holomorphically to* $\mathcal{O}_t$. *Then there exist* $\eta > 0$, $t_0 \in \Delta_\varepsilon$, *and* $\mathcal{O}$ *an open neighborhood of* 0 *in* $\mathbb{C}^N$, *such that* $R(t,Z)$ *extends holomorphically to* $\{t : |t - t_0| < \eta\} \times \mathcal{O}$.

Proof. For a positive integer q let $E_q \subset \Delta_\varepsilon$ be given by

$$E_q = \left\{ t \in \Delta_\varepsilon : |D^\alpha R(t,Z)| \leqq \alpha! q^{|\alpha|} \text{ for all } Z \in M,\ |Z| < \frac{1}{q}, \text{ and all } \alpha \right\} ,$$

where D^α denotes differentiation on M in some fixed local parametrization of M near 0. Since by assumption $\bigcup_q E_q = \Delta_\varepsilon$, and the E_p are closed, we may apply the Baire Category Theorem to find q_0 such that E_{q_0} has nonempty interior. That is, there exist $t_0 \in \Delta_\varepsilon$, $\eta > 0$, and $C > 0$ such that for all nonzero α

$$|D^\alpha R(t,Z)| \leqq \alpha! C^{|\alpha|} \quad \text{for all } Z \in M,\ |Z| < \frac{1}{C},\ |t - t_0| < \eta \,.$$

It follows that $R(t,Z)$ extends continuously to a neighborhood of the form $\{t : |t - t_0| < \eta\} \times \mathcal{O}$, separately holomorphic in Z and t. The lemma is then a consequence of Hartog's Theorem. □

Proof of Theorem 4. We shall prove the theorem by contradiction. Suppose that for every germ of a smooth self CR map of M fixing 0 there exists a neighborhood of 0 in $\mathbb{C}^N$ to which the map extends holomorphically. In particular, this would imply that for each $t \in \mathbb{C}$ small, the CR map $Z \mapsto \psi(t,Z)$ extends holomorphically to a neighborhood of 0, where ψ is given by Lemma 5.8. Since $\psi(t,Z) = \phi(K(t,Z),Z)$, we may apply Lemma 2.7 to conclude that for each $t \in \mathbb{C}$ sufficiently small, $Z \mapsto K(t,Z)$ extends holomorphically to a neighborhood of 0 in $\mathbb{C}^N$. We may now apply Lemma 5.10, to conclude that $K(t,Z)$ extends holomorphically to $\{|t-t_0| < \eta\}\times\mathcal{O}$. In particular, we conclude, by differentiating in t and using (5.4), the function $Z \mapsto h(\phi(K(t_0,Z),Z))$ extends holomorphically near the origin. Since $K(t,0) \equiv 0$ (by uniqueness in (5.4)) and the map $Z \mapsto \phi(K(t_0,Z),Z)$ is a local biholomorphism near the origin, we conclude that $h(Z)$ extends holomorphically in a neighborhood of 0 in $\mathbb{C}^N$. This contradicts the assumption on h and completes the proof of Theorem 4. □

Example 5.11. Let $M \subset \mathbb{C}^3$ be given by $\{Z : \Im Z_3 = |Z_1Z_2|^2\}$. Since the holomorphic vector field $Z_1\frac{\partial}{\partial Z_1} - Z_2\frac{\partial}{\partial Z_2}$ is tangent to M, it follows that M is holomorphically degenerate. It is easy to check that the CR function $h(Z) = \exp(-Z_3^{-1/3})$ (restricted to M with the appropriate determination of the argument) is smooth, but does not extend holomorphically to a full neighborhood of 0 in $\mathbb{C}^3$. The conclusion of Theorem 4 then holds for this example.

6 Properties of mappings into the sphere

We will prove Theorem 5 in this section and the next. We begin with some notation. We write $H = (H_1,\ldots,H_{N+1})$, and, as before, $N = n+1$. After a local holomorphic change of variables, we may assume that S^{2N+1} is given by the defining function

$$Z'_{n+2} + \bar{Z}'_{n+2} + \sum_{j=1}^{n+1}|Z'_j|^2 \,. \tag{6.1}$$

Lemma 6.2. *Under the assumptions of Theorem* 5, *M is pseudoconvex in a neighborhood of p_0 and there exist points of strict pseudoconvexity arbitrarily close to p_0.*

Proof. It follows from the hypothesis that M is of Bloom–Graham finite type near p_0; hence H extends holomorphically to one side of M near p_0 [BT, T]. Let

$$\rho^*(Z,\bar{Z}) = H_{n+2}(Z) + \overline{H_{n+2}(Z)} + \sum_{j=1}^{n+1}|H_j(Z)|^2 \,. \tag{6.3}$$

Hence ρ^* is defined on one side of M, of class C^m up to M, and is plurisubharmonic. For any small analytic disc $A(\zeta)$ attached to M near p_0, the function

$\zeta \mapsto \rho^*(A(\zeta),\overline{A(\zeta)})$ is subharmonic in the unit disc and vanishes on its boundary. By the maximum principle, the interior of the disc maps to the unit ball in $\mathbb{C}^{N+1}$. Since the discs cover one side of M, we can apply the Hopf Lemma to conclude $\frac{\partial}{\partial v}\rho^*(Z,\bar{Z})|_{p_0} \neq 0$, where v is the normal direction to M at p_0. Hence ρ^* is a plurisubharmonic defining function for M, proving the pseudoconvexity of M.

Suppose that there is no strongly pseudoconvex point near p_0. Then, by the semi-continuity of the counting function of the positive eigenvalues of the Levi form, we can assume that for some point p_1 close to p_0 the number of the non-zero eigenvalues of the Levi form of M there attains a local maximum value $r < n$. Now, by using a local change of coordinates near p_1, we may assume that $p_1 = 0$ and M can be given near this point by the following equation

$$Z_{n+1} + \bar{Z}_{n+1} = \sum_{j=1}^{r} |Z_j|^2 + h(Z_1, \dots, Z_{n+1}) \, ,$$

where $h(Z) = O(|Z|^3)$. Consider the hypersurface $M^* \subset \mathbb{C}^{n+1-r}$ defined by $Z_{n+1} + \bar{Z}_{n+1} = h(0,\dots,0,Z_{r+1},\dots,Z_{n+1})$. Since it does not contain any non-trivial analytic variety inside (for, otherwise, M cannot be of D'Angelo finite type), one sees that it cannot be Levi-flat and thus its Levi form has a positive eigenvalue at a point w^* near 0 in M^*. Then the Levi form of M has at least $r+1$ positive eigenvalues at $(0,w^*)$. This contradicts the definition of r and completes the proof of Lemma 6.2. □

An immediate consequence of Lemma 6.2 with Theorem 2 in [H2] is that H is an algebraic map. This fact will be useful later.

As in Sect. 1, we choose normal coordinates, $Z = (z,w)$, for M vanishing at p_0 so that M is given by an equation of the form:

$$t = \phi(z,\bar{z},s) \tag{6.4}$$

where $w = s + it$ and $\phi(0,\bar{z},s) \equiv \phi(z,0,s) \equiv 0$. Let $L_1,\dots,L_n$ be a basis for the CR vector fields on M near 0 given by

$$L_j = \frac{\partial}{\partial \bar{z}_j} - 2i \frac{\phi_{\bar{z}_j}}{1 + i\phi_s} \frac{\partial}{\partial \bar{w}} \, , \quad j = 1,\dots,n \, . \tag{6.5}$$

Note that if $L^\alpha = L_1^{\alpha_1},\dots,L_n^{\alpha_n}$, then

$$L^\alpha|_0 = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}|_0 \, . \tag{6.6}$$

We will also assume $H(0) = 0$.

Lemma 6.7. *After a rotation of the vector $(H_1,\dots,H_{n+1})$, one can find a sequence of n multi-indices $(\beta^1,\dots,\beta^n)$ with $|\beta^j| \leqq m$ such that for $j = 1,\dots,n$,*

$$L^{\beta^j}\bar{H}_j|_0 \neq 0, \quad \text{but } L^{\beta^j}\bar{H}_l|_0 = 0 \quad \text{for } 1 \leqq j < l \leqq n \, . \tag{6.8}$$

Proof. We note first that by applying L^β to (6.3) we have

$$L^\beta \bar{H}_{n+2}(0) = 0, \quad \text{for all } |\beta| \leqq m\,. \tag{6.9}$$

We first show that there exists a multi-index β, $|\beta| \leqq m$, and an integer $k \leqq n+1$ such that $L^\beta \bar{H}_k(0) \neq 0$. We argue by contradiction. If no such β exists, then by (6.6) we have $\bar{H}_j(Z) = O(|w|) + o(|z|^m)$, $j = 1, \ldots, n+1$. However, since (6.3) is a defining function for M, $t - \phi(z, \bar{z}, s) = \rho^*(Z, \bar{Z}) h(Z, \bar{Z})$ for some nonvanishing h. Using (6.9), we have $H_{n+2} = aw + o(|w| + |z|^m)$, with $a \neq 0$. Combining these gives

$$\rho^*(Z, \bar{Z}) = aw + \bar{a}\bar{w} + o(|w| + |z|^m)\,. \tag{6.10}$$

Then the complex analytic variety $V^* = \{Z = (\zeta, \ldots, \zeta, 0) : \zeta \in \Delta\}$ has order of contact at least $m+1$ with M at 0, contradicting the definition of m given by (0.1).

Without loss of generality we may assume that there exists β^1, $|\beta^1| \leqq m$ such that $L^{\beta^1} \bar{H}_1(0) \neq 0$ and $L^{\beta^1} \bar{H}_j(0) = 0$, $1 < j \leqq n+1$. Next, we show that there exists a multiple index β^2 with $|\beta^2| \leqq m$ so that $L^{\beta^2} \bar{H}_j(0) \neq 0$ for some $1 < j \leqq n+1$. Indeed, if this is not the case, then $H_j = O(|w|) + o(|z|^m)$, $j = 2, \ldots, n+1$. Write $H_1(Z) = P_1(z) + O(|w|) + o(|z|^m)$ with $P_1(z)$ a polynomial in z of degree $\leqq m$ and $P_1(0) = 0$. Then using again (6.9) we obtain

$$\rho^*(Z, \bar{Z}) = aw + \bar{a}\bar{w} + |P_1(z)|^2 + o(|w| + |z|^m)\,.$$

Let V be the complex analytic variety defined by $w = 0$ and $P_1(z) = 0$. Without loss of generality, we may assume that $P_1(b_1\tau, \ldots, b_{n-1}\tau, z_n) = \sum_{j=0}^{N} a_j(\tau) z_n^j$ with $a_j(\tau) \not\equiv 0$ for some $j > 0$ and an $(n-1)$-tuple $(b_1, \ldots, b_{n-1})$. By the Puiseux expansion, there exists an $N^* \gg 1$ such that $z_n(\tau^{N^*})$ is a holomorphic function in τ, $z_n(0) = 0$, and $P_1(b_1\tau^{N^*}, \ldots, b_{n-1}\tau^{N^*}, z_n(\tau^{N^*})) \equiv 0$. Thus, we obtain a holomorphic curve $\gamma(\tau)$ defined by

$$z_1 = b_1\tau^{N^*}, \ldots, z_{n-1} = b_{n-1}\tau^{N^*}, z_n = z_n(\tau^{N^*}), w = 0\,.$$

If $N' = \operatorname{ord}(\gamma(\tau))$, the order of vanishing of $\gamma(\tau)$ at 0, then since $\rho^*(\gamma(\tau), \overline{\gamma(\tau)}) = o(|\gamma(\tau)|^m)$ we have

$$\operatorname{ord}(\rho^*(\gamma(\tau), \overline{\gamma(\tau)}) > m \operatorname{ord}(\gamma(\tau))\,,$$

which again would contradict the definition of m. Hence β^2 must exist. Now, applying a suitable rotation to $(H_2, \ldots, H_n)$, we may assume that $L^{\beta^2} \overline{H}_2|_0 \neq 0$ for some $|\beta^2| \leqq m$ but $L^{\beta^2} \overline{H}_l|_0 = 0$ for each $l > 2$. Arguing inductively, we obtain the proof of Lemma 6.7. □

We shall use Lemma 6.7 to obtain equations for the components H_i of the mapping H.

Proposition 6.11. *Let* β^j *and* H_i *be as in Lemma* 6.7, *and let*

$$V(Z,\bar Z) = \begin{pmatrix} L^{\beta^1}\bar H_1 & L^{\beta^1}\bar H_2 & \dots & L^{\beta^1}\bar H_n \\ \vdots & \vdots & \vdots & \\ L^{\beta^n}\bar H_1 & L^{\beta^n}\bar H_2 & \dots & L^{\beta^n}\bar H_n \end{pmatrix} .$$

Then $V(0,0)$ *is invertible, and if*

$$\xi = V^{-1}\begin{pmatrix} L^{\beta^1}\bar H_{n+2} \\ \vdots \\ L^{\beta^n}\bar H_{n+2}\end{pmatrix}, \quad \eta = V^{-1}\begin{pmatrix} L^{\beta^1}\bar H_{n+1} \\ \vdots \\ L^{\beta^n}\bar H_{n+1}\end{pmatrix}, \quad F = \begin{pmatrix} H_1 \\ \vdots \\ H_n\end{pmatrix}, \tag{6.12}$$

then the following holds on M:

$$F = -\xi - H_{n+1}\eta . \tag{6.13}$$

Proof. Apply L^{β^j}, $j = 1,\dots,n$, to (6.3), and then solve the resulting system of linear equations for F. □

7 Proof of Theorem 5

We shall complete the proof of Theorem 5 in this section. The main step will be to prove that H is meromorphic. Then the result in Chiappari [Ch] will give the desired holomorphic extension. It will be convenient to have the following criterion.

Proposition 7.1. *Let* M *be a minimal algebraic hypersurface in* $\mathbb{C}^N$ $(N > 1)$ *defined near* 0, *and let* $k(Z,\bar Z)$ *be a vector-valued algebraic continuous CR function defined on* M *near* 0. *Assume that* $h(Z,\bar Z)$ *is also a continuous algebraic CR function on* M *near* 0 *such that* $Q_2(Z,\overline{Z},\overline{k(Z,\bar Z)})h(Z,\bar Z) = Q_1(Z,\overline{Z},\overline{k(Z,\bar Z)})$ *for* $Z \in M$ *near* 0, *where* $Q_j\,(j = 1,2)$ *are holomorphic algebraic functions near* $(0,0,\overline{k(0,0)})$ *such that* $Q_2(Z,\overline{Z},\overline{k(Z,\bar Z)}) \not\equiv 0$ *near the origin in* M. *Then* $h(Z,\bar Z)$ *admits a meromorphic extension near* 0 *in* $\mathbb{C}^N$. *Moreover, when* $Q_2 \equiv 1$, *then* h *admits a holomorphic extension near* 0.

Proof. We will use the edge of wedge theorem for the proof. Assume that M is given in normal coordinates $Z = (z,w)$ by equation (6.4) and that each CR function defined near $0 \in M$ can be extended to the side D^+ given by $t > \phi(z,\bar z,s)$. For each nonzero vector $v \in \mathbb{C}^n$, let $M_v = \{(xv, s + i\phi(xv, x\bar v, s)) : x \in \mathbb{R}^n,\ s \in \mathbb{R}\}$. Since $Q_2(Z,\overline{Z},\overline{k(Z,\bar Z)})$ cannot vanish identically in an open set of M near the origin, we can assume for some fixed v_0, $Q_2(Z,\overline{Z},\overline{k(Z,\bar Z)}) \not\equiv 0$ on M_{v_0} in any neighborhood of 0. Now we define $G : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, a local biholomorphism with $G(0) = 0$ by

$$G(z,w) = (zv_0, w + \phi(zv_0, z\overline{v_0}, w)) .$$

Note that $G(\mathbb{R}^{n+1}) \subset M_{v_0}$ near 0. Furthermore, G maps a standard straight wedge W^+ with edge $\mathbb{R}^{n+1}$ near 0 to D^+. Write $W^- = \overline{W^+}$, the complex conjugate of W^+. Denoting the holomorphic extensions of h and k to D^+ by the same letters, let

$$\Phi^+(Z) = h \circ G(Z), \quad Z \in W^+ ,$$

$$\Phi^-(Z) = \frac{\overline{Q_1(G(Z), \overline{G(\bar{Z})}, \overline{k(G(\bar{Z}), \overline{G(Z)})})}}{\overline{Q_2(G(Z), \overline{G(\bar{Z})}, \overline{k(G(\bar{Z}), \overline{G(Z)})})}}, \quad Z \in W^- .$$

Then, by our hypothesis, one sees that Φ^- is a meromorphic algebraic function in W^-. Thus, $\sum_{j=1}^{N} a_j(Z)\Phi^-(Z)^j \equiv 0$, where a_j's are polynomials. Notice that $a_N\Phi^-$ satisfies a polynomial equation with leading coefficient 1. We conclude that $a_N\Phi^-$ is bounded in W^-. By the Riemann removable singularity theorem, it holds that $a_N\Phi^-$ extends holomorphically to W^-.

Away from a proper real analytic subset of $\mathbb{R}^{n+1}$, we have $a_N(X)\Phi^+(X) = a_N(X)\Phi^-(X)$ for X near 0 in $\mathbb{R}^{n+1}$, by our construction. From the classical edge of the wedge theorem, it follows that $a_N(Z)\Phi(Z)$ can be extended holomorphically to an open subset of 0. Thus Φ^+ has a meromorphic extension near 0. Now, since G is a local biholomorphic map, we conclude that h extends meromorphically across 0.

Finally, if $Q_2 \equiv 1$, then $\Phi^-(Z)$ is bounded near 0 for $Z \in W^-$. The above argument shows that h is holomorphic near 0 in this case. This completes the proof of the proposition. □

We now begin the proof of Theorem 5. We proceed according to the following two cases:

Case I: $L_j(\eta(Z,\bar{Z})) \equiv 0$, $j = 1, \ldots, n$ for Z in M near 0, i.e. the components of the vector η are all CR functions.
Case II: At least one of the components of $\eta(Z,\overline{Z})$ is not a CR function in any neighborhood of O in M.

We first assume the hypothesis of Case I and prove that H extends meromorphically near 0. Since F and H_{n+1} are CR functions, we conclude from (6.13) that ξ is also a CR function. Applying the last part of Proposition 7.1 to the first two equations of (6.12), we conclude that ξ and η both extend holomorphically to a full neighborhood of 0 in $\mathbb{C}^{n+1}$ and hence are both real analytic near 0 in M.

Rewriting (6.3) we obtain on M,

$$H_{n+2} + \overline{H}_{n+2} + F \cdot \overline{F} + H_{n+1}\overline{H}_{n+1} = 0 . \tag{7.2}$$

Replacing F in (7.2) by using (6.13), we have

$$H_{n+2} + \overline{H}_{n+2} + H_{n+1}a + \overline{H}_{n+1}\overline{a} + |H_{n+1}|^2 + c = 0 , \tag{7.3}$$

where

$$a = \eta \cdot \overline{\eta}, \quad b = 1 + |\eta|^2, \quad c = |\xi|^2 \tag{7.4}$$

Rewriting (7.3) we have

$$H_{n+2} + \overline{H}_{n+2} + H_{n+1}(a + \overline{H}_{n+1}b) + \overline{H}_{n+1}\overline{a} + c = 0\ . \tag{7.5}$$

Applying L_j, $j = 1, \ldots, n$ to (7.5) we obtain

$$L_j\overline{H}_{n+2} + H_{n+1}L_j(a + \overline{H}_{n+1}b) + L_j(\overline{H}_{n+1}\overline{a}) + L_jc = 0\ . \tag{7.6}$$

We consider now two cases.

Case Ia: $L_j(a + \overline{H}_{n+1}b) \equiv 0$, $j = 1, \ldots, n$
Case Ib: For some j, $L_j(a + \overline{H}_{n+1}b) \not\equiv 0$.

In Case Ia, we conclude that $H_{n+1} = -a/b$ and hence H_{n+1} extends holomorphically by Proposition 7.1. (Note that b is nowhere vanishing.) Hence from (6.13) and then (7.2) it follows that H_j, $j = 1, \ldots, n+2$ also extend holomorphically.

In Case Ib we apply Proposition 7.1 to equation (7.6) to conclude that H_{n+1} extends meromorphically, and hence again by (6.13) and (7.2) we conclude that all the H_j extend meromorphically.

We now consider Case II. Then choose $j, \ell \in \{1, 2, \ldots, n\}$ so that $L_j\eta_\ell \not\equiv 0$ in any neighborhood of 0 in M. Applying L_j to the lth component of (6.13), we obtain

$$L_j\xi_\ell + H_{n+1}L_j\eta_l = 0 \tag{7.7}$$

We then use Proposition 7.1 to conclude that H_{n+1} has a meromorphic extension. Making use first of (6.13) and then of (7.5), we conclude that all the components of H extend meromorphically.

We have now proved that H extends to a meromorphic mapping in a neighborhood of 0. Since H maps one side of M to the ball and maps M to the sphere, we may now use Theorem 1 of Chiappari [Ch], (generalizing the result of Cima–Suffridge [CS1]) to conclude that H extends holomorphically. This completes the proof of Theorem 5.

Remark 7.8. If in Theorem 5 the source manifold M is assumed to be a sphere, the proof shows that in fact the mapping H is rational. (See [W2, CS1, CS2].)

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