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Quantum cohomology of a product

(with Appendix by R. Kaufmann[2])

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0. Introduction

0.1. Quantum cohomology. The quantum cohomology of a projective algebraic manifold V is a formal deformation of its cohomology ring. The parameters of this deformation are coordinates on the space $H^*(V)$, and the structure constants are the third derivatives of a formal series Φ^V (potential, or free energy) whose coefficients count the number of parametrized rational curves on V with appropriate incidence conditions (see [KM] for details).

A natural problem arises how to calculate $\Phi^{V\times W}$ in terms of Φ^V and Φ^W. In [KM] it was suggested that this operation corresponds to that of tensor multiplication of Cohomological Field Theories, or equivalently, algebras over the moduli operad $\{H_*(\overline{M}_{0,n+1})\}$. The definition of the tensor product depends on a theorem on the structure of $H_*(\overline{M}_{0,n+1})$ whose proof was only sketched in [KM] (Theorem 7.3). One of the main goals of this note is to present this proof and related calculations in full detail. (For another proof, see [G]).

We also discuss the rank one CohFT's and the respective twisting operation. An interesting geometric example of such a theory is furnished by Weil-Petersson forms. The potential of this theory is a characteristic function involving Weil-Petersson volumes calculated in [Z]. We show that a generalization of WP-forms allows one to construct a canonical coordinate system on the group of invertible CohFT's.

We start with a brief review of the relevant structures from [KM]. Let H be a finite-dimensional $\mathbf{Z_2}$-graded linear space over a field K of characteristic zero, endowed with a non-degenerate even symmetric scalar product g. The main fact is the equivalence of two notions:

i) A formal solution Φ of associativity, or WDVV, equations on (H, g).

ii) A structure of CohFT on (H, g).

0.2. Associativity equations. Let $\{\Delta_a\}$ be a basis of H, $g_{ab} = g(\Delta_a, \Delta_b)$, $(g^{ab}) = (g_{ab})^{-1}$. Denote by $\gamma = \sum_a x^a \Delta_a$ a generic element of H, where x^a is a formal variable of the same $\mathbf{Z_2}$-degree as Δ_a. Put $\partial_a = \partial/\partial x^a$. A formal series $\Phi \in K[[x^a]]$ is called a solution of the associativity equations on (H, g) iff for all a, b, c, d

$$\sum_{ef} \partial_a \partial_b \partial_c \Phi \cdot g^{ef} \partial_f \partial_c \partial_d \Phi = (-1)^{a(b+c)} \sum_{ef} \partial_b \partial_c \partial_e \Phi \cdot g^{ef} \partial_f \partial_a \partial_d \Phi. \qquad (0.1)$$

Here we use the simplified notation $(-1)^{a(b+c)}$ for $(-1)^{\tilde{x}_a(\tilde{x}_b+\tilde{x}_c)}$ where $\tilde{x}$ is the $\mathbf{Z_2}$-degree of x.

We usually assume that Φ starts with terms of degree $\geqq 3$, or identify Φ and Φ' differing by a polynomial of degree $\leqq 2$. An extensive geometric treatment of the associativity equations is given in [D]. For the next definition, recall that $\overline{M}_{0n}$ denotes the moduli space of stable curves of genus zero with n labelled pairwise distinct points: see [Ke].

0.3. Cohomological field theories. The structure of a tree level Cohomological Field Theory on (H, g) is given by a sequence of S_n-covariant K-linear maps

$$I_n : H^{\otimes n} \to H^*(\overline{M}_{0n}, K), \quad n \geqq 3 \qquad (0.2)$$

satisfying the following set of identities (0.3). The values of I_n are generally not homogeneous.

Consider an unordered partition σ : $\{1, \ldots, n\} = S_1 \coprod S_2$, $|S_i| = n_i \geqq 2$. It defines an embedding of the boundary divisor $\varphi_\sigma : \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0,n}$. Over the generic point of this divisor the universal curve consists of two components, and the labelled points are distributed between them according to σ. The maps (0.2) must satisfy for all $n \geqq 3$ the relations:

$$\varphi_\sigma^*(I_n(\gamma_1 \otimes \ldots \otimes \gamma_n)) = \varepsilon(\sigma)(I_{n_1+1} \otimes I_{n_2+1})\left(\bigotimes_{j \in S_1} \gamma_j \otimes \Delta \otimes \left(\bigotimes_{k \in S_2} \gamma_k\right)\right) \qquad (0.3)$$

where $\Delta = \Sigma \Delta_a \otimes \Delta_b g^{ab}$ is the Casimir element, and $\varepsilon(\sigma)$ is the sign of the permutation induced on the odd arguments $\gamma_1, \ldots, \gamma_n$. There are two other useful reformulations of CohFT. First, dualizing (0.2) we get a series of maps

$$I_{n+1}^t : H_*(\overline{M}_{0,n+1}, K) \to Hom(H^{\otimes n}, H), \quad n \geqq 2. \qquad (0.4)$$

Thus any homology class in $\overline{M}_{0,n+1}$ is interpreted as an n-ary opertaion on H. The relations (0.3) become identities between these operations whose totality means that H is given a structure of an algebra over the cyclic operad $\{H_*(\overline{M}_{0,n+1}, K)\}$ (see [GK]).

Second, we can iterate the maps φ_σ in order to study the restrictions of the classes $I_n(\gamma_1 \otimes \ldots \otimes \gamma_n)$ to all boundary strata. These strata are naturally indexed by the (dual) trees of stable curves, which form a category. Both sides of (0.2) extend to functors on this category, and (0.3) says that $\{I_n\}$ becomes

a functor morphism. Below we will extensively use the combinatorial side of this picture explained in [KM].

0.4. From $\{I_n\}$ to Φ. Every CohFT $\{H, g, I_n\}$ defines a sequence of symmetric polynomials $Y_n : H^{\otimes n} \to K$:

$$Y_n(\gamma_1 \otimes \ldots \otimes \gamma_n) := \int\limits_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \ldots \otimes \gamma_n) \tag{0.5}$$

Put $x = (x^a)$ and

$$\Phi(x) = \sum_{n \geqq 3} \frac{1}{n!} Y_n\left(\left(\sum_a x^a \Delta_a \right)^{\otimes n} \right). \tag{0.6}$$

From (0.3) and Keel's linear relations between boundary divisors in $\overline{M}_{0n}$ one can formally deduce (0.1).

0.5. From Φ to $\{I_n\}$. This transition is markedly more difficult. If I_n with the properties (0.5), (0.6) exist at all, they are defined uniquely, because iterating (0.3) one can calculate integrals of I_n over all boundary strata of $\overline{M}_{0n}$. Namely, let τ be the dual tree of a stable curve C of genus zero with n marked points. Let us recall that the set of vertices V_τ consists of irreducible components of C, the edges E_τ are (in a bijection with) double points of C, tails T_τ (one vertex edges) are marked points. Incidence relations between V_τ, E_τ, T_τ reflect those in C. A flag of τ is a pair (vertex, incident edge or tail); the set of flags is denoted F_τ. Let $\overline{M}_\tau \subset \overline{M}_{0n}$ be the submanifold parametrizing curves of type τ and their specializations. We have $\overline{M}_\tau \cong \prod_{v \in V_\tau} \overline{M}_{|v|}$. Here $|v| = |F_\tau(v)|$, $F_\tau(v)$ is the set of flags incident to v. The homology classes of all $\overline{M}_\tau$ generate $H_*(\overline{M}_{0n}, K)$, and we have from (0.3):

$$\int\limits_{\overline{M}_\tau} I_n(\gamma_1 \otimes \ldots \otimes \gamma_n) = \left(\bigotimes_{v \in V_\tau} Y_{|v|} \right) (\gamma_1 \otimes \ldots \otimes \gamma_n \otimes \Delta^{\otimes E_\tau}) \in K. \tag{0.7}$$

We use here the formalism of tensor products indexed by arbitrary finite sets and interpret the argument of the r.h.s. (0.7) as an element of $H^{\otimes F_\tau}$ and $\bigotimes_{v \in V_\tau} Y_{|v|}$ as a function on $H^{\otimes F_\tau}$. (0.3) is a particular case of (0.7) for an one-edge tree.

In order to use (0.7) for construction of $\{I_n\}$ it remains to check that the r.h.s. of (0.7) satisfies all the linear relations between the classes of $\overline{M}_\tau$ in $H_*(\overline{M}_{0n})$. This was done in Sect. 8 of [KM] modulo the theorem, describing these relations and proved in Sect. 2 of the present paper.

0.6. The tensor product. Let $\{H', g', I'_n\}$ and $\{H'', g'', I''_n\}$ be two CohFT's. Put $H = H' \otimes H''$ and $g = g' \otimes g''$. We can define a CohFT on (H, g) by

$$I_n(\gamma'_1 \otimes \gamma''_1 \otimes \ldots \otimes \gamma'_n \otimes \gamma''_n) := \varepsilon(\gamma', \gamma'') I'_n(\gamma'_1 \otimes \ldots \otimes \gamma'_n) \wedge I''_n(\gamma''_1 \otimes \ldots \otimes \gamma''_n) \tag{0.8}$$

where $\varepsilon(\gamma', \gamma'')$ is the standard sign in superalgebra, and $\wedge$ is the cup product in $H^*(\overline{M}_{0n}, K)$. One can easily check (0.3). Although (0.8) looks very simple on the level of full CohFT's, it cannot be trivially restricted to calculate the

potential. In fact, for the potential we must know only the top dimensional component of I_n (see (0.5)), but it involves components of all degrees of I'_n, I''_n (see (0.8)), which are given by (0.7) as functionals on the boundary homology classes.

The only remaining obstruction to calculating Φ of $\{I_n\}$ is thus our incomplete understanding of the cup product in terms of dual boundary classes. Ralph Kaufmann obtained a fairly simple formula for the intersection indices of strata of complementary dimensions. With his permission, we reproduce it in Sect. 2. But at the moment we are unable to invert this Gram matrix (of redundant size).

0.7. Application to quantum cohomology. In the application of this formalism to the quantum cohomology of V we have: $H = H^*(V,K)$, $g =$ Poincaré pairing, and $\int_{\overline{M}_{0n}} I_n(\gamma_1 \otimes \ldots \otimes \gamma_n)$ is the (appropriately defined) number of "stable maps" $(C, x_1, \ldots, x_n; \varphi)$ where C is a curve of genus 0 with n marked points $x_1, \ldots, x_n$, $\varphi : C \to V$ is a morphism, such that any connected component contracted by φ (together with its special points) is stable. These Gromov-Witten invariants are of primary interest in enumerative geometry, and we expect that the tensor product of CohFT's described above furnishes an algorithm for calculating them on $V \times W$ from those on V and W. Strictly speaking, this must be proved starting with a geometric construction of GW–invariants. In concrete examples it suffices to simply check the coincidence of a finite set of coefficients of the two potentials using The First Reconstruction Theorem 3.1 of [KM]. For example, $\Phi^{\mathbf{P}^1 \times \mathbf{P}^1}$ actually coincides with the tensor square of the simple potential $\Phi^{\mathbf{P}^1}$. Namely, if $\Delta_0 \in H^0(\mathbf{P}^1)$ and $\Delta_1 \in H^2(\mathbf{P}^1)$ are respectively the fundamental class and the dual class of a point, we have

$$\Phi^{\mathbf{P}^1}(x\Delta_0 + z\Delta_1) = \frac{1}{2}x^2 z + e^z - 1 - z - \frac{z^2}{2}, \tag{0.9}$$

$$\begin{aligned}\Phi^{\mathbf{P}^1 \times \mathbf{P}^1}&(x\Delta_0^{\otimes 2} + y^1\Delta_1 \otimes \Delta_0 + y^2\Delta_0 \otimes \Delta_1 + z\Delta_1^{\otimes 2}) \\ &= \frac{1}{2}x^2 z + xy^1y^2 + \sum_{a+b \geqq 1} N(a,b)\frac{z^{2a+2b-1}}{(2a+2b-1)!}\, e^{ay^1+by^2}\end{aligned} \tag{0.10}$$

where $N(a,b)$ is the number of rational curves of bidegree (a,b) on $\mathbf{P}^1 \times \mathbf{P}^1$ passing through $2a+2b-1$ points in general position. As is remarked in [DFI], the structure of $N(a,b)$ which can be derived from the associativity equations looks simpler than that of GW-numbers for $\mathbf{P}^2$, probably because of this tensor product property.

In the remaining part of this introduction, we summarize our notation and conventions about the combinatorics of trees and the (co)homology of $\overline{M}_{0n}$. The next section is devoted to the multiplicative properties of strata classes. In Sect. 2 we prove the completeness of the standard linear relations between them. The last Sect. 3 discusses rank one CohFT's.

0.8. Partitions and trees. As in [KM], a tree τ for us is a system of finite sets (V_τ, E_τ, T_τ), $V_\tau \neq \emptyset$, with appropriate incidence relations defining F_τ (see 0.5 above for notation). A structure of S-tree on τ (where S is a finite set) is given by a bijection $T_\tau \to S$. Sometimes we identify T_τ with S using this bijection. A tree τ is *stable* if $|v| \geqq 3$ for all $v \in V_\tau$. Most of our trees are stable.

A stable S-tree τ corresponds to a (family of) stable curve(s) of genus 0 with points labelled by elements of S. One-edge S-trees are in a bijection with unordered partitions of S into two subsets $\sigma: \ S = S_1 \coprod S_2$; stability means that $|S_i| \geqq 2$; the tails marked by S_i belong to the vertex v_i, $i = 1, 2$. We will systematically identify such partitions with the corresponding trees. For such a σ and, say, four elements $i, j, k, l \in S$ we use a notation like $ij\sigma kl$ to imply that $\{i, j\}$ and $\{k, l\}$ belong to different parts of σ.

For two unordered stable partitions $\sigma = \{S_1, S_2\}$ and $\tau = \{T_1, T_2\}$ of S put

$$a(\sigma, \tau) := \text{ the number of non-empty pairwise distinct sets among } \ S_i \cap T_j, \ i, j = 1, 2.$$

Clearly, $a(\sigma, \tau) = 2, 3$, or 4. Moreover, $a(\sigma, \tau) = 2$ iff $\sigma = \tau$, and $a(\sigma, \tau) = 4$ iff there exist pairwise distinct $i, j, k, l \in S$ such that simultaneously $ij\sigma kl$ and $ik\tau jl$. If $a(\sigma, \tau) = 3$, we sometimes call σ and τ compatible. A family of 2-partitions $\{\sigma_1, \ldots, \sigma_m\}$ is called *good*, if for all $i \neq j$, σ_i and σ_j are compatible. S-trees form objects of several categories differing by the size of their morphism sets. The most useful morphisms $f: \tau \to \sigma$ contract several edges and tails of τ: f induces a surjection $V_\tau \to V_\sigma$ and injections $E_\sigma \to E_\tau$, $T_\sigma \to T_\tau$; labelling sets for τ and σ may differ. We will mostly consider morphisms of S-trees identical on S (pure contractions of edges, or S-morphisms). The one-vertex tree is a final object in the category of S-trees and S-morphisms. If a direct product $\sigma \times \tau$ of two S-trees in this category exists, it comes equipped with two contractions $\sigma \times \tau \to \sigma$ and $\sigma \times \tau \to \tau$. A geometrically nice case is when $|E_{\sigma \times \tau}| = |E_\sigma| + |E_\tau|$. E.g. for one-edge trees this is the case when $a(\sigma, \tau) = 3$. For $a(\sigma, \tau) = 4$, $\sigma \times \tau$ does not exist, and for $\sigma = \tau$ we have $\sigma \times \sigma = \sigma$.

A few more words about the geometry of an individual tree τ. Any flag $f = (v, e)$, $v \in V_\tau$, $e \in E_\tau$ or $e \in T_\tau$ defines a complete subgraph $\beta(f)$ of τ which we will call *the branch* of f. If $e \in T_\tau$, $\beta(f)$ consist of the vertex v and tail e. Generally, $\beta(f)$ includes v, e, and all edges, vertices and tails that can be reached from v by a no-return path starting with f. We denote by $T_\tau(f)$ the tails belonging to $\beta(f)$, and by $S(f)$ their labels (if τ is a labelled tree).

0.9. Moduli spaces. For a finite set S, $|S| \geqq 3$, $\overline{M}_{0S}$ parametrizes stable curves of genus zero with a family of pairwise distinct points labelled by S. More generally, for a stable S-tree τ, $\overline{M}_\tau$ parametrizes such curves with dual graph (isomorphic to) τ; $\overline{M}_{0S}$ corresponds to the one-vertex S-tree. Any pure contraction $\tau \to \sigma$ bijective (but not necessarily identical) on S induces a morphism $\overline{M}_\tau \to \overline{M}_\sigma$. In this way $\{\overline{M}_\sigma\}$ form a topological cyclic operad (see [GK]), and $\{H_*(\overline{M}_{0S})\}$ form a linear cyclic operad.

If an S-morphism of S-trees $\tau \to \sigma$ exists, it is unique, and $\overline{M}_\tau \to \overline{M}_\sigma$ is a closed embedding whose image is called a (closed) stratum of $\overline{M}_\sigma$. In particular, all $\overline{M}_\tau$ "are" closed strata in $\overline{M}_{0S}$. We have $\overline{M}_\tau \cong \prod_{v \in V_\tau} \overline{M}_{0F_\tau(v)}$ (canonically), $\overline{M}_{0,F_\tau(v)} \cong \overline{M}_{0|v|}$ (non-canonically). The codimension of the stratum $\overline{M}_\tau$ in $\overline{M}_\tau$ is $|E_\tau|$. In particular, stable one-edge S-trees (and stable 2-partitions) σ bijectively correspond to the boundary divisors.

0.10. Keel's presentation. Fixing S, $|S| \geqq 3$, we denote by $\{D_\sigma | \sigma$ stable 2-partitions of $S\}$ a family of commuting independent variables. Put $F_S = K[D_\sigma]$ ($F_S = K$ for $|S| = 3$). We consider F_S as a graded polynomial ring, $\deg D_\sigma = 1$. Define the ideal $I_S \subset F_S$ by means of the following generators:

a) For each pairwise distinct foursome $i, j, k, l \in S$:

$$R_{ijkl} := \sum_{ij\sigma kl} D_\sigma - \sum_{kj\tau il} D_\tau \in I_S. \tag{0.11}$$

b) For each pair σ, τ with $a(\sigma, \tau) = 4$:

$$D_\sigma D_\tau \in I_S. \tag{0.12}$$

Finally, put $H_S^* = K[D_\sigma]/I_S$.

0.10.1. Theorem (Keel [Ke]) *The map*

$$D_\sigma \longmapsto \quad \textit{dual cohomology class of the boundary divisor in } \overline{M}_{0S} \textit{ corresponding to the partition } \sigma$$

induces an isomorphism of rings (doubling the degrees)

$$H_S^* \xrightarrow{\sim} H^*(\overline{M}_{0S}, K). \tag{0.13}$$

Since $\overline{M}_{0S}$ is a smooth manifold whose homology and cohomology is generated by algebraic classes, on which homological and rational equivalence coincide, (0.13) describes the homology and the Chow ring as well. In addition, Keel's presentation is very convenient for describing the operadic structure maps. E.g. bijections $S' \to S''$ (relabelling of points) translate simply by the respective relabelling of D_σ's. For the remaining morphisms, see the next section.

1. Boundary strata and the multiplicative structure of $H^*(\overline{M}_{0S})$

1.1. Good monomials. The monomial $D_{\sigma_1} \ldots D_{\sigma_a} \in F_S$ is called good, if the family of 2-partitions $\{\sigma_1, \ldots, \sigma_a\}$ is good, i.e. $a(\sigma_i, \sigma_j) = 3$ for $i \neq j$. In particular, D_σ and 1 are good.

1.2. Lemma. *Let τ be a stable S-tree with $|E_\tau| \geqq 1$. For each $e \in E_\tau$, denote by $\sigma(e)$ the 2-partition of S corresponding to the one edge S-tree obtained*

by contracting all edges except for e. Then

$$m(\tau) := \prod_{e \in E_\tau} D_{\sigma(e)}$$

is a good monomial.

Proof. Let $e \neq e' \in E_\tau$. There exists a sequence of pairwise distinct edges $e = e'_0, \quad e'_1, \ldots, e'_r, e'_{r+1} = e', r \geqq 0$, such that e'_j and e'_{j+1} have a common vertex v_j.

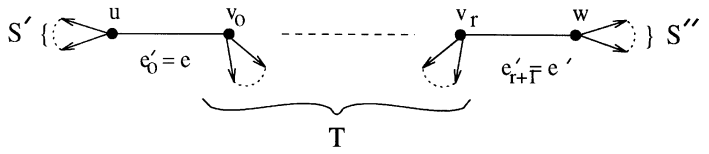

Fig. 1. Arrows symoblize branches

Let u be the remaining vertex of e, w that of e'. Let S' be the set of all tails of τ belonging to the branches starting at u but not with a flag belonging to e; similarly, let S'' be the set of all tails of τ belonging to the branches that start at w but not with a flag belonging to e'. Finally, let T be the set of all tails on the branches at $v_0, \ldots, v_r$ not starting with the flags in $e'_0, \ldots, e'_{r+1}$ (we identify tails with their labels). Since τ is stable, all three sets S', S'' and T are non-empty. Finally

$$\sigma(e) = \{S', S'' \coprod T\}, \sigma(e') = \{S' \coprod T, S''\}.$$

It follows that $a(\sigma(e), \sigma(e')) = 3$ so that $m(\tau)$ is a good monomial.

We put $m(\tau) = 1$, if $|E_\tau| = 0$.

1.3. Proposition. *For any $1 \leqq r \leqq |S| - 3$, the map $\tau \longmapsto m(\tau)$ establishes a bijection between the set of good monomials of degree r in F_S and stable S-trees τ with $|E_\tau| = r$ modulo S-isomorphism. There are no good monomials of degree $> |S| - 3$.*

Proof. For $r = 0, 1$ the assertion is clear. Assume that for some $r \geqq 1$ the map $\tau \longmapsto m(\tau)$ is surjective on good monomials of degree r. We will prove then that it is surjective in the degree $r + 1$.

Let $\deg m' = r + 1$. Choose a divisor D_σ of m' which is *extremal* in the following sense: one element, say S_1, of the partition $\sigma = \{S_1, S_2\}$ is minimal in the set of all elements of all 2-partitions σ' such that $D_{\sigma'}$ divides m'. Put $m' = D_\sigma m$. Since m is good of degree r, we have $m = m(\tau)$ for some stable S-tree τ. We will show that $m' = m(\tau')$ where τ' is obtained from τ by inserting a new edge with tails marked by S_1 at an appropriate vertex $v \in V_\tau$. (This means that vice versa, there exists a contraction $\tau' \to \tau$ of one edge to the vertex having incident flags $S_1$and (half of) this edge).

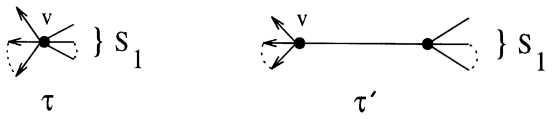

Fig. 2. Inserting edge at a vertex

First we must find v in τ. To this end, consider any edge $e \in E_\tau$ and the respective partition $\sigma(e) : \{S'_e, S''_e\}$ (obtained by contracting all edges except for e). Since $a(\{S_1, S_2\}, \{S'_e, S''_e\}) = 3$ and S_1 is minimal, one sees that exactly one of the sets $\{S'_e, S''_e\}$ strictly contains S_1. Let it be S''_e. Orient e by declaring that the direction from the vertex (corresponding to) S'_e to S''_e is positive. We claim that with this orientation, for any $w \in V_\tau$ there can be at most one edge outgoing from w. In fact, if τ contains a vertex w with two positively oriented flags $f_1$and f_2, then S_1 must be contained in the two subsets of S, $S(f_1)$ and $S(f_2)$. But their intersection is empty.

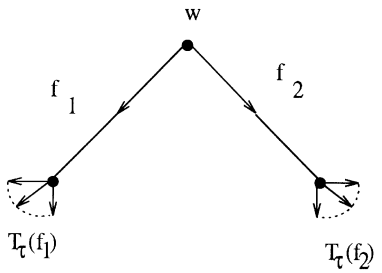

Fig. 3.

It follows that there exists exactly one vertex $v \in V_\tau$ having no outgoing edges. Moreover, S_1 is contained in the set of labels of the tails at v by construction. If we now define τ' by inserting a new edge e' at v so that $\sigma(e') = \sigma$, we will clearly have $m' = m(\tau')$. If $r \leqq |S| - 4$, the tree τ' cannot be unstable because, first, $|S_1| \geqq 2$, and second, at least two more flags converge at $|v|$: otherwise the unique incoming edge would produce the partition $\{S_1, S_2\} = \sigma$ which would mean that D_σ divides already $m(\tau)$.

For $r = |S| - 3$, this argument shows that m' cannot exist because all the vertices of τ have valency three.

It remains to check that if $m(\tau_1) = m(\tau_2)$, then τ_1 and τ_2 are isomorphic.

Assume that this has been checked in degree $\leqq r$ and that deg $\tau_1 =$ deg $\tau_2 = r + 1$. Choose an extremal divisor D_σ of $m(\tau_1) = m(\tau_2)$ as above and contract the respective edges of τ_1, τ_2 getting the trees τ'_1, τ'_2. Since $m(\tau'_1) = m(\tau'_2) = m(\tau_i)/D_\sigma$, τ'_1 and τ'_2 are isomorphic by the inductive assumption. This isomorphism respects the marked vertices v'_1, v'_2 corresponding to the

contracted edges because as we have seen they are uniquely defined. Hence it extends to an S-isomorphism $\tau_1 \to \tau_2$. □

1.3.1. Remark. Proposition 1.3 and Keel's theorem 0.10.1, together with the fact that the boundary divisors have transversal intersections, show that the image of a good monomial $m(\tau)$ in $H^*(\overline{M}_{0S})$ is the dual class of the stratum $\overline{M}_\tau$.

1.4. Multiplication formulas I. Let now σ, τ be two stable S-trees, $|E_\sigma| = 1$. We have the following three possibilities

a) $D_\sigma m(\tau)$ *is a good monomial*. Then

$$D_\sigma m(\tau) = m(\tau') \tag{1.1}$$

where $\tau' \to \tau$ is the unique S-morphism contracting the edge in $E_{\tau'}$, whose 2-partition coincides with that of σ.

More generally, if $m(\sigma)m(\tau)$ is a good monomial, then

$$m(\sigma)m(\tau) = m(\sigma \times \tau) \tag{1.2}$$

where the direct product is the categorical one in the category of S-trees and S-morphisms. We can identify $E_{\sigma\times\tau}$ with $E_\sigma \coprod E_\tau$, and $p_1 : \sigma \times \tau \to \sigma$ (resp. $p_2 : \rho \times \tau \to \tau$) contracts edges of the second factor (resp. of the first one).

b) *There exists a divisor* $D_{\sigma'}$ *of* $m(\tau)$, $|E_{\sigma'}| = 1$, *such that* $a(\sigma, \sigma') = 4$. Then

$$D_\sigma m(\tau) \equiv 0 \bmod I_S, \tag{1.3}$$

where $I_S \subset F_S$ is the ideal of Keel's relations.

c) D_σ *divides* $m(\tau)$. Then let $e \in E_\tau$ be the edge corresponding to σ; v_1, v_2 its vertices, (v_i, e) the corresponding flags.

We will write several different expressions for $D_\sigma m(\tau) \bmod I_S$, corresponding to various possible choices of unordered pairs of distinct flags $\{\bar{i}, \bar{j}\} \subset F_\tau(v_1) \setminus \{(v_1, e)\}$, $\{\bar{k}, \bar{l}\} \subset F_\tau(v_2) \setminus \{(v_2, e)\}$. For each choice, put

$$T_1 = F_\tau(v_1) \setminus \{\bar{i}, \bar{j}, (v_1, e)\},$$
$$T_2 = F_\tau(v_2) \setminus \{\bar{k}, \bar{l}, (v_2, e)\}.$$

Notice that because of stability the set of such choices is non-empty.

1.4.1. Proposition. *For every such choice we have*

$$D_\sigma m(\tau) \equiv -\sum_{\substack{T \subset T_1 \\ |T| \geqq 1}} m(tr_{T,e}(\tau)) - \sum_{\substack{T \subset T_2 \\ |T| \geqq 1}} m(tr_{T,e}(\tau)) \bmod I_S \tag{1.4}$$

where $tr_{T,e}(\tau)$ *is the tree obtained from* τ *by "transplanting all branches starting in* T *to the middle point of the edge* e*."* (*An empty sum is zero*).

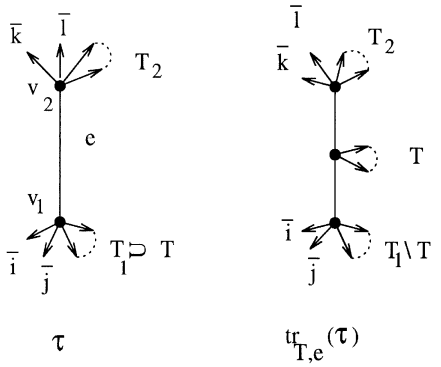

Fig. 4. Transplants: arrows symbolize branches

Remark. We can also describe $tr_{T,e}(\tau)$ as a result of inserting an extra edge instead of the vertex v_1 (resp. v_2) and putting the branches T to the common vertex of the new edge and e, similarly to what we have done in the proof of Proposition 1.3. There exists a unique S-morphism $tr_{T,e}(\tau) \to \tau$ contracting one edge.

Proof. We choose pairwise distinct labels on the chosen branches $i \in S(\bar{i})$, $j \in S(\bar{j})$, $k \in S(\bar{k})$, $l \in S(\bar{\ell})$ and then calculate the element (see (0.11))

$$R_{ijkl} \cdot m(\tau) = \left(\sum_{ij\rho kl} D_\rho - \sum_{kj\rho il} D_\rho \right) m(\tau) \equiv 0 \mod I_S. \tag{1.5}$$

Since $ij\sigma kl$, for all terms D_ρ of the second sum in (1.5) we have $a(\sigma, \rho) = 4$ so that $D_\rho m(\tau) \in I_S$. Among the terms of the first sum, there is one D_σ. If $ij\rho kl$ and $\rho \neq \sigma$, then D_ρ cannot divide $m(\tau)$. Otherwise ρ would correspond to an edge $e' \neq e$, but the 2-partition of such an edge cannot break $\{i, j, k, l\}$ into $\{i, j\}$ and $\{k, l\}$ as a glance to a picture of τ shows. It follows that $D_\rho m(\tau) = m(\rho \times \tau)$ as in (1.2). The projection $\rho \times \tau \to \tau$ contracts the extra edge onto a vertex that can be only one of the ends of e, otherwise, as above, the condition $ij\rho kl$ cannot hold. It should be clear by now that $\rho \times \tau$ must be one of the trees $tr_{T,e}(\tau)$, and that each tree of this kind can be uniquely represented as $\rho \times \tau$ for some ρ with $ij\rho kl$. But from (1.5) it follows that

$$D_\sigma m(\tau) \equiv - \sum_{\substack{ij\rho kl \\ \rho \neq \sigma}} D_\rho m(\tau) \mod I_S$$

which is (1.4). □

1.5. Corollary. *Classes of good monomials linearly generate* $F_S/I_S = H^*_S$.

This follows from (1.1), (1.3) and (1.4) by induction on the degree.

1.5.1. Remark. Formulas (1.1)–(1.4) (and (1.7) below) can be rewritten as expressing operadic morphisms $\varphi^*_\sigma : H^*(\overline{M}_{0S}) \to H^*(\overline{M}_\sigma) = H^*(\overline{M}_{0, S_1 \amalg \{\cdot\}}) \otimes$

$H^*(\overline{M}_{0,S_2 \amalg \{\cdot\}})$, $\sigma = \{S_1, S_2\}$ in terms of classes of boundary divisors. E.g. (1.4) means that

$$\varphi_\sigma^*(D_\sigma) \equiv -\sum_{j\tau\{\cdot\}} D_\tau \otimes 1 - \sum_{\{\cdot\}\rho\{kl\}} 1 \otimes D_\rho \ , \tag{1.6}$$

where τ (resp. ρ) runs over stable 2-partitions of $S_1 \amalg \{.\}$ (resp. $S_1 \amalg \{.\}$).

1.6. Multiplication formulas II. It may be more convenient to have formulas independent of arbitrary choice of $\{\bar{i}, \bar{j}, \bar{k}, \bar{l}\}$. One way to achieve this is to average (1.4) over all possible choices. We will illustrate this procedure by calculating D_σ^2 and some of the "tautological classes."

Let σ be a stable 2-partition $\{T_1, T_2\}$ of S.

1.6.1. Proposition it We have

$$\begin{aligned} D_\sigma^2 = &- \sum_{\substack{T \subset T_1 \\ 1 \leqq |T| \leqq |T_1|-2}} D_\sigma D_{\{T_1 \setminus T, T_2 \amalg T\}} \frac{|T_1 \setminus T|(|T_1 \setminus T| - 1)}{|T_1|(|T_1| - 1)} \\ &- \sum_{\substack{T \subset T_2 \\ 1 \leqq |T| \leqq |T_2|-2}} D_\sigma D_{\{T_1 \amalg T, T_2 \setminus T\}} \frac{|T_2 \setminus T|(|T_2 \setminus T| - 1)}{|T_2|(|T_2| - 1)} \bmod I_S \end{aligned} \tag{1.7}$$

Proof. We first write $\binom{|T_1|}{2}\binom{|T_2|}{2}$ identities (1.4) for all possible choices of $i, j \in T_1$, $k, l \in T_2$, then sum them up and change the summation order by first choosing subsets $T \subset T_1$ or T_2, and then i, j or k, l in the complement. □

1.6.2. Tautological classes. These classes $\tau_d^{(i)} \in H^*(\overline{M}_{0n})$ are defined as $c_1(T_{x_i}^*(C))^d$ where $C \to \overline{M}_{0n}$ is the universal curve, $x_i : \overline{M}_{0n} \to C$ is the i-th section, and $T_{x_i}^*$ is the relative cotangent sheaf to C at x_i.

In order to calculate $\tau_1^{(i)}$, identify $C \to \overline{M}_{0n}$ with the morphism $\overline{M}_{0,n+1} \to \overline{M}_{0n}$ forgetting the $(n+1)$-th section. Then the section $x_i(\overline{M}_{0n})$ becomes the boundary divisor $D_i := D_{\{i,n+1\},\{1,\ldots,\hat{i},\ldots,n\}}$ in $\overline{M}_{0,n+1}$, and $\tau_1^{(i)}$ becomes the pull back of $-D_i^2$. Applying (1.7) to this situation we get:

1.6.3. Proposition.

$$\tau_1^{(i)} = \sum_{\substack{i \in S \subset \{1,\ldots,n\} \\ |S| \geqq 2,\, n-|S| \geqq 2}} \frac{(n-|S|)(n-|S|-1)}{(n-1)(n-2)} D_{s,\{1,\ldots,n\}\setminus S} \bmod I_S \tag{1.8}$$

1.7. Multiplication formulas III. The functional $\int_{\overline{M}_{0,s}} : H^*(\overline{M}_{0,S}) \to K$ is given by

$$m(\tau) \longmapsto \begin{cases} 1, & \text{if} \quad \deg m(\tau) = |S| - 3, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\deg m(\tau) = |S| - 3$ iff $|v| = 3$ for all $v \in V_\tau$, and $\overline{M}_\tau$ is a point in this case. We put $\langle \sigma_1, \sigma_2 \rangle = \int_{\overline{M}_{0S}} m(\sigma_1) m(\sigma_2)$ and set to calculate this intersection index for the case when $\deg m(\sigma_1) + \deg m(\sigma_2) = |S| - 3$. Generally,

we will write $\langle m \rangle$ instead of $\int_{\overline{M}_{0S}} m$. The following notions and results are due to Ralph Kaufmann. We can assume that all pairs of different divisors of $m(\sigma_1)$ and $m(\sigma_2)$ are compatible, otherwise $\langle \sigma_1, \sigma_2 \rangle = 0$. Put $\tau = \sigma_1 \times \sigma_2$ in the category of S-morphisms. This is a tree with a marked subset of edges E corresponding to D_σ's whose squares divide $m(\sigma_1)m(\sigma_2)$. We denote by δ the subgraph of τ consisting of E and its vertices.

Consider an orientation of all edges of δ. Call it *good* if for all vertices v of τ, the number of ingoing edges equals $|v| - 3$, where $|v|$ means the valency in τ. Notice that for $v \notin V_\delta$ we interpret this as $|v| = 3$.

1.7.1. Proposition (R. Kaufmann). *There cannot exist more than one good orientation of δ. If there is none, we have $\langle \sigma_1, \sigma_2 \rangle = 0$. If there is one, we have*

$$\langle \sigma_1, \sigma_2 \rangle = \prod_{v \in V_\tau} (-1)^{|v|-3}(|v| - 3)! \tag{1.9}$$

For a proof, see Appendix.

Remark. Notice that (1.9) depends only on τ whereas E influences only the existence of the good orientation. Curiously, (1.9) coincides with the virtual Euler characteristics of the *non-compact* moduli space $M_\tau = \prod_{v \in V_\tau} M_{0,|v|}$. We do not know why this is so.

2. Boundary strata and the additive structure of $H^*(\overline{M}_{0S})$

2.1. Basic linear relations. Let $|S| \geqq 4$. Consider a system $(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l})$ where τ is an S-tree, $v \in V_\tau$ is a vertex with $|v| \geqq 4$ and $\bar{i}, \bar{j}, \bar{k}, \bar{l} \in F_\tau(v)$ are pairwise distinct flags (taken in this order). Put $T = F_\tau(v) \setminus \{\bar{i}, \bar{j}, \bar{k}, \bar{l}\}$. For any ordered 2-partition of T, $\alpha = \{T_1, T_2\}$, (one or both T_i can be empty) we can define two trees $\tau'(\alpha)$ and $\tau''(\alpha)$. The first one is obtained by inserting a new edge e at $v \in V$ with branches $\{\bar{i}, \bar{j}, T_1\}$ and $\{\bar{k}, \bar{l}, T_2\}$ at its edges. The second one corresponds similarly to $\{\bar{k}, \bar{j}, T_1\}$ and $\{\bar{i}, \bar{l}, T_2\}$. We remind that $S(\bar{i})$ is the set of labels of tails belonging to the branch of $\bar{i}$.

2.1.1. Proposition. *We have*

$$R(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) := \sum_\alpha [m(\tau'(\alpha)) - m(\tau''(\alpha))] \equiv 0 \bmod I_S \tag{2.1}$$

Proof. Choose $i \in S(\bar{i})$, $j \in S(\bar{j})$, $k \in S(\bar{k})$, $l \in S(\bar{l})$, and calculate $R_{ijkl}m(\tau) \equiv 0 \bmod I_S$, where R_{ijkl} is defined by (0.1). Consider e.g. the summands $D_\sigma m(\tau)$ for $ij\sigma kl$.

From the picture of τ it is clear that D_σ does not divide $m(\tau)$. If $D_\sigma m(\tau)$ does not vanish modulo I_S, we must have $D_\sigma m(\tau) = m(\sigma \times \tau)$, and $\sigma \times \tau$ is of the type $\tau'(\alpha)$. Similarly, the summands of $D_\sigma m(\tau)$ with $kj\sigma il$ are of the type $m(\tau''(\alpha))$.

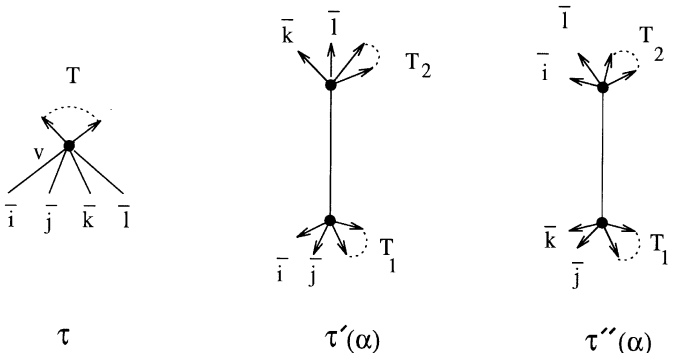

Fig. 5.

2.2. Theorem. *All linear relations modulo I_S between good monomials of degree $r+1$ are spanned by the relations* (2.1) *for* $|E_\tau| = r$.

Proof. For $r = 0$ this holds by definition of I_S. Generally, denote by H_{*S} the linear space, generated by the symbols $\mu(\tau)$ for all classes τ of stable S-trees modulo isomorphisms, satisfying the analog of the relations (2.1)

$$r(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) := \sum_{\alpha} [\mu(\tau'(\alpha)) - \mu(\tau''(\alpha))] = 0 \tag{2.2}$$

Denote by 1 the symbol $\mu(\rho)$ where ρ is one-vertex tree.

2.2.1. Main Lemma. *There exists on H_{*S} a structure of H_S^*-module given by the following multiplication formulas reproducing* (1.1), (1.3) *and* (1.4):

$$D_\sigma \mu(\tau) = \mu(\sigma \times \tau) \tag{2.3}$$

if $D_\sigma m(\tau)$ is a good monomial;

$$D_\sigma \mu(\tau) = 0, \tag{2.4}$$

if there exists a divisor $D_{\sigma'}$ of $m(\tau)$ such that $a(\sigma, \sigma') = 4$;

$$D_\sigma \mu(\tau) = -\sum_{\substack{T \subset T_1 \\ |T| \geqq 1}} b\mu(tr_{T,e}(\tau)) - \sum_{\substack{T \subset T_2 \\ |T| \geqq 1}} \mu(tr_{T,e}(T)) \tag{2.5}$$

if D_σ divides $M(\tau)$, and e corresponds to σ. The notation in (2.5) *is the same as in* (1.4).

Deduction of Theorem 2.2 from the Main Lemma. Since the monomials $m(\tau)$ satisfy (2.1), there exists a surjective linear map $a : H_{*S} \to H_S^* : \mu(\tau) \mapsto m(\tau)$. On the other hand, from (2.3) it follows that $m(\sigma)\mu(\tau) = \mu(\sigma \times \tau)$ if $m(\sigma)m(\tau)$ is a good monomial. Hence we have a linear map $b : H_S^* \to H_{*S} : m(\tau) \mapsto \mu(\tau) = m(\tau)1$ inverse to a. Therefore dim $H_{*S} =$ dim H_S^* so that the Theorem 2.2 follows.

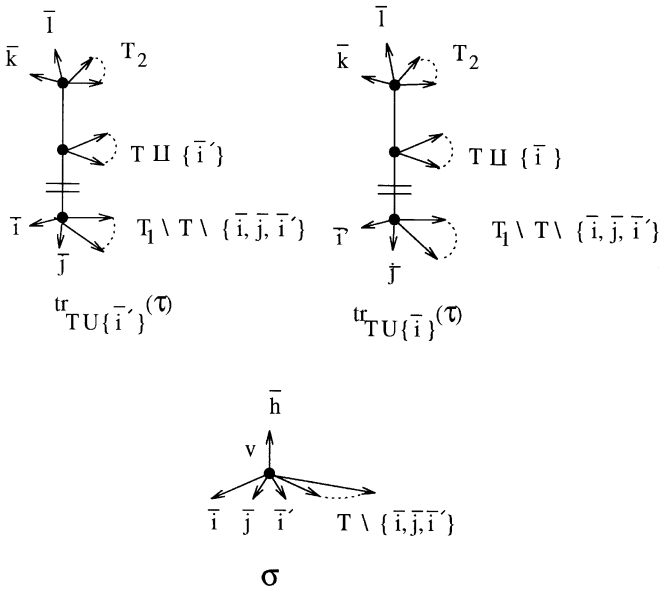

Fig. 6. The edge $\neq$ is contracted to v

We now start proving the Main Lemma.

2.2.3. (*2.5*) *is well defined.* The r.h.s. of (2.5) formally depends on the choice of $\bar{i}, \bar{j}, \bar{k}, \bar{l}$. We first check that different choices coincide modulo (2.2). It is possible to pass from one choice to another by replacing one flag at a time. So let us consider $\bar{i}' \neq \bar{i}, \bar{j}, \bar{k}, \bar{l}$ and write the difference of the right hand sides of the relations (2.5) written for $(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l})$ and $(\tau, v, \bar{i}', \bar{j}, \bar{k}, \bar{l})$. The terms corresponding to those T that do not contain $\{\bar{i}, \bar{i}'\}$ cancel. This includes all terms with $T \subset T_2$. The remaining sum can be rewritten as

$$-\sum_{T \subset T_1 \setminus \{\bar{i}, \bar{i}', \bar{j}\}} [\mu(tr_{T \cup \{\bar{i}'\}}(\tau)) - \mu(tr_{T \cup \{i\}}(\tau))] \tag{2.6}$$

where now T can be empty.

We contend that (2.6) is of the type (2.2). More precisely, consider any of the trees $tr_{T \cup \{\bar{i}'\}}(\tau)$, $tr_{T \cup \{\bar{i}\}}(\tau)$ and contract the edge whose vertices are incident to the flags $\bar{i}, \bar{j}, \bar{i}'$. We will get a tree σ and its vertex $v \in V_\tau$. The pair (σ, v) up to a canonical isomorphism does not depend on the transplants we started with. In $F_\sigma(v)$ there are flags $\bar{i}, \bar{j}, \bar{i}'$ and one more flag whose branch contains both k and l and which we denote $\bar{h}$. Then (2.6) is $-r(\sigma, v, \bar{i}, \bar{j}, \bar{i}', \bar{h})$ (see (2.2)). This is illustrated by the Fig. 6.

2.2.4. Operators D_σ on H_{*S} pairwise commute. We have to prove the identities

$$D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = D_{\sigma_2}(D_{\sigma_1}\mu(\tau)). \tag{2.7}$$

Consider several possibilities separately.

i) *There exists a divisor* D_σ *of* $m(\tau)$ *such that* $a(\sigma_1, \sigma) = 4$, *so that* $D_{\sigma_1}\mu(\tau) = 0$.

If $D_{\sigma_2}\mu(\tau) = 0$ as well, (2.7) is true. If $D_{\sigma_2}\mu(\tau) = \mu(\sigma_2 \times \tau)$, then D_σ divides $m(\sigma_2 \times \tau)$, and (2.7) is again true. Finally, if D_{σ_2} divides $m(\tau)$, then $\sigma_2 \neq \sigma$ (otherwise $m(\tau)$ would not be a good monomial). Hence the transplants $tr_{T,e}(\tau)$ entering the formula of the type (2.5) which we can use to calculate $D_{\sigma_2}\mu(\tau)$ will all contain an edge corresponding to σ so that $D_{\sigma_1}(tr_{T,e}(\tau)) = 0$, and (2.7) again holds.

The same argument applies to the case when $D_{\sigma_2}\mu(\tau) = 0$.

From now on we may and will assume that for any divisor D_σ of $m(\tau)$ we have $a(\sigma, \sigma_1) \leqq 3$, $a(\sigma, \sigma_2) \leqq 3$, and that $\sigma_1 \neq \sigma_2$.

ii) $a(\sigma_1, \sigma_2) = 4$ *and* D_{σ_2} *divides* $m(\tau)$.

Then D_{σ_1} does not divide $m(\tau)$, so that $D_{\sigma_1}\mu(\tau) = \mu(\sigma_1 \times \tau)$, and $D_{\sigma_2}(D_{\sigma_1}\mu(\tau)) = 0$. On the other hand, $D_{\sigma_2}\mu(\tau)$ is a sum of transplants to the midpoint of the edge, corresponding to σ_2. Each such transplant has an edge giving the 2-partition σ_2, so that $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = 0$.

The case $a(\sigma_1, \sigma_2) = 4$ and $D_{\sigma_1}/m(\tau)$ is treated in the same way.

Hence from this point on we can and will in addition assume that $a(\sigma_1, \sigma_2) = 3$.

iii) D_{σ_1} *does not divide* $m(\tau)$.

If D_{σ_2} does not divide $m(\tau)$ as well, then $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = D_{\sigma_1}\mu(\sigma_2 \times \tau) = \mu(\sigma_1 \times \sigma_2 \times \tau) = D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$. If D_{σ_2} divides $m(\tau)$, we will use carefully chosen formulas of the type (2.5) for the calculation of $D_{\sigma_2}\mu(\tau)$. Namely, let v_1 be the (unique) vertex of τ which gets replaced by an edge in $\sigma_1 \times \tau$, and let e_2 be the edge of τ corresponding to D_{σ_2}. Let u_2, u_1 be the vertices of e_2 such that u_1 can be joined to v_1 by a path not passing by e_2.

Consider first the subcase $u_1 \neq v_1$. Choose some $\bar{i}, \bar{j} \in F_\tau(u_2)$ and $\bar{k}, \bar{l} \in F_\tau(u_1)$ in such a way that $\bar{l}$ starts a path leading from u_1 to v_1. Use these $\bar{i}, \bar{j}, \bar{k}, \bar{l}$ in a formula of the type (2.5) to calculate $D_{\sigma_2}\mu(\tau)$ and then $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$, that will insert an edge instead of the vertex v_1 which survives in all the transplants entering $D_{\sigma_2}\mu(\tau)$. Then calculate $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ by first inserting the edge at v_1, and then constructing the transplants not moving $\bar{i}, \bar{j}, \bar{k}, \bar{l}$. Since by our choice of $\bar{l}$ we never transplant the branch containing v_1, the two calculations will give the same result.

Now let $v_1 = u_1$. Let $\{S_1, S_2\}$ be the 2-partition of S corresponding to σ_1. Since $\sigma_1 \times \tau$ exists, $\{S_1, S_2\}$ is induced by a partition of $F_\tau(v_1) = \bar{S}_1 \coprod \bar{S}_2$. We denote by $\bar{S}_2$ the part to which the flag $(v_1 = u_1, e_2)$ belongs. Let $\bar{T} = \bar{S}_2 \setminus (\{(v_1 = u_1, e_2)\} \coprod F_\tau(u_2)$. This set is non-empty because otherwise e_2 would correspond to $\{S_1, S_2\}$ and we would have $\sigma_1 = \sigma_2$. Take $\bar{i}, \bar{j} \in F_\tau(u_2)$, $\bar{k} \in \bar{S}_1$ and $\bar{l} \in T$: see Fig. 7.

Now consider $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ and $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$. To calculate the first expression we form a sum of transplants of $\sigma_1 \times \tau$. To calculate the second one, we form transplants of τ, and then insert an edge at $v_1 = u_1$.

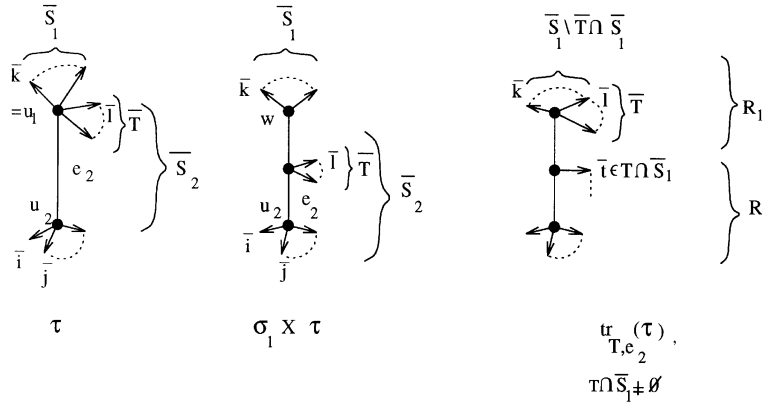

Fig. 7.

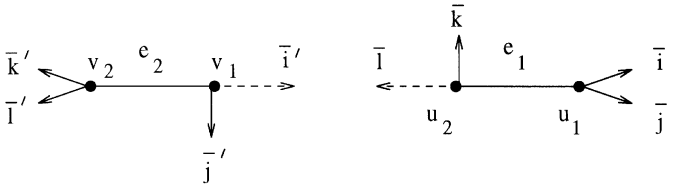

Fig. 8.

The transplants corresponding to the branches at u_2 will be the same in both expressions. The transplants corresponding to the subsets $T \subset \bar{T} \setminus \{\bar{l}\}$ will also be the same. In addition, the second expression will contain the terms $-D_{\sigma_1}(\mu(tr_{T,e_2}(\tau)))$ where $T \cap \bar{S}_1 \neq \emptyset$. But each such term vanishes. In fact, consider the 2-partition $\rho = \{R_1, R_2\}$ of S corresponding to the edge of $tr_{T,e_2}(\tau)$ containing the flag $(v_1 = u_1, e_2)$, and let $k, l \in R_1$. A glance to the third tree of the Fig. 7 shows that $a(\rho, \sigma_1) = 4$, because if $\bar{t} \in T \cap \bar{S}_1$, $t \in S(\bar{t})$, then $kt\sigma_1 il$ and $kl\rho it$. Hence the extra terms are irrelevant.

The case when D_{σ_2} does not divide $m(\tau)$ is treated in the same way. It remains to consider the last possibility.

iv) *D_{σ_1} and D_{σ_2} divide $m(\tau)$, $a(\sigma_1, \sigma_2) = 3$.*

Denote by e_1 (resp. e_2) the edge corresponding to σ_1 (resp. σ_2). Let u_1, u_2 (resp. v_1, v_2) be the vertices of e_1 (resp. e_2) numbered in such a way that there is a path from u_2 to v_1 not passing through e_1, e_2 (the case $u_2 = v_1$ is allowed). To calculate the multiplication by D_{σ_1} choose $\bar{i}, \bar{j} \in F_{u_1}(\tau) \setminus \{(u_1, e_1)\}$, $\bar{l}$ on the path from u_2 to v_1 if $u_2 \neq v_1$, and $\bar{l} = (v_1, e_2)$ if $u_2 = v_1$; $\bar{k} \in F_\tau(v_2) \setminus \{\bar{l}\}$. To calculate the product by D_{σ_2}, choose similarly $\bar{k}', \bar{l}' \in F_\tau(v_2) \setminus \{(v_2, e_2)\}$, $\bar{i}' \in F_\tau(v_1)$ on the path from v_1 to u_2, if $v_1 \neq u_2$, and $\bar{i}' = (u_2, e_1)$ if $v_1 = u_2$, $\bar{j}' \in F_\tau(v_1)$ (see Fig. 8).

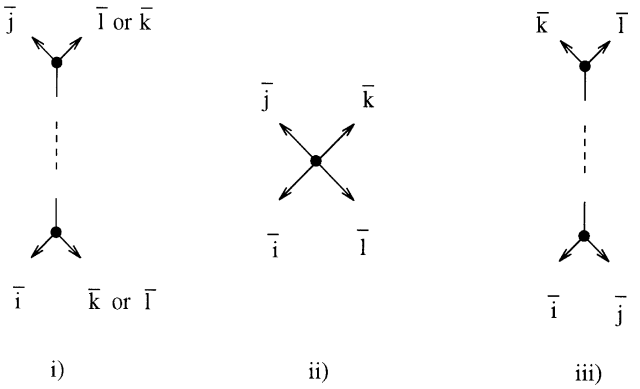

Fig. 9.

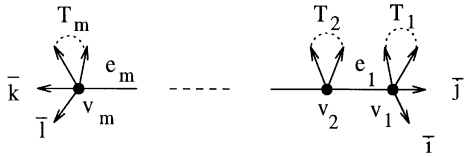

Fig. 10.

The critical choice here is that of $\bar{l}$ and $\bar{i}'$. It ensures that calculating $D_{\sigma_1}(D_{\sigma_2}\mu(\tau))$ and $D_{\sigma_2}(D_{\sigma_1}\mu(\tau))$ we will get the same sum of transplanted trees. This ends the proof of (2.7).

2.2.5. Compatibility with I_S-generating relations. If $D_{\sigma_1} D_{\sigma_2} = 0$ because $a(\sigma_1, \sigma_2) = 4$, one sees that $D_{\sigma_1}(D_{\sigma_2}\mu(\tau)) = 0$ looking through various subcases in 2.2.4. It remains to show that $R_{ijkl}\mu(\tau) = 0$ where R_{ijkl} is defined by (0.11).

Consider the smallest connected subgraph in τ containing the flags i, j, k, l. The Fig. 9 gives the following exhaustive list of alternatives. Paths from i to j and from k to l: i) have at least one common edge; ii) have exactly one common vertex; iii) do not intersect.

Consider them in turn.

i) Let e be an edge common to the paths ij and kl. Denote by ρ the respective 2-partition. Then $ik\rho jl$ or $il\rho kj$. Therefore any summand of R_{ijkl} annihilates D_ρ so that $R_{ijkl}\mu(\tau) = 0$ in view of (2.4).

ii) Let v be the vertex common to the paths ij and kl. Then exactly the same calculation as in the proof of the Proposition 2.1.1 shows that

$$R_{ijkl}\mu(\tau) = \sum_{\alpha}[\mu(\tau'(\alpha)) - \mu(\tau''(\alpha))] = 0$$

(notation as in (2.1) and (2.2)).

iii) This is the most complex case. Let us draw a more detailed picture of τ in the neighborhood of the subgraph we are considering (Fig. 10).

Let v_1 be the vertex on the path ij which is connected by a sequence of edges $e_1, \dots, e_m$ ($m \geqq 1$) with the vertex v_m on the path kl so that e_a has vertices (v_a, v_{a+1}) in this order. Let T_a be the set of flags at v_a which do not coincide with $\bar{i}, \bar{j}, \bar{k}, \bar{l}$, and do not belong to e_{a-1}, e_a.

Consider any summand D_σ of R_{ijkl}. If $jk\sigma il$, then $D_\sigma \mu(\tau) = 0$ because each edge e_a determines a partition ρ of S such that $ij\rho kl$. From now on we assume that $ij\sigma kl$. Then $D_\sigma \mu(\tau)$ can be nonzero if one of the two alternatives holds:

a) For some v_a, there exists a partition $T_a = T'_a \coprod T''_a$,(with $|T'_a| \geqq 1, |T''_a| \geqq 1$, except for the case $a = 1$ where T'_1 can be empty, and $a = m$ where T'_m can be empty) such that the following two sets

$$S_1 = S(\bar{i}) \coprod S(\bar{j}) \coprod S(T'_1) \coprod \dots \coprod S(T'_a),$$

$$S_2 = S(T''_a) \coprod S(T_{a+1}) \coprod \dots \coprod S(T_m) \coprod S(\bar{k}) \coprod S(\bar{l})$$

form the 2-partition corresponding to σ. In this case

$$D_\sigma \mu(\tau) = \mu(\sigma \times \tau),$$

and $\sigma \times \tau$ is obtained by inserting a new edge at v_a and by distributing T'_a and T''_a at different vertices of this edge.

b) For some e_a, the two sets

$$S_1 = S(\bar{i}) \coprod S(\bar{j}) \coprod \left(\coprod_{i \leqq a} S(T_i) \right),$$

$$S_2 = \left(\coprod_{i \geqq a+1} S(T_i) \right) \coprod S(\bar{k}) \coprod S(\bar{l})$$

form the 2-partition corresponding to σ.

In this case D_σ divides $m(\tau)$, and in order to calculate $D_\sigma \mu(\tau)$ using a formula of the type (2.5) we must first choose two pairs of flags at two vertices of v_a.

Contributions from a) and b) come with opposite signs, and we contend that they completely cancel each other.

To see the pattern of the cancellation look first at the case a) at v_1. It brings (with positive sign) the contributions corresponding to the following trees. Form all the partitions $T_1 = T'_1 \coprod T''_1$ such that $T''_1 \neq \emptyset$, where $T_1 = F_\tau(v_1) \setminus \{\bar{i}, \bar{j}, (v_1, e_1)\}$. Transplant all T''_1-branches to the midpoint of e_1. Denote the new vertex v'_1. The result is drawn as Fig. 11.

Now consider the terms of the type b) for the edge e_1. If $m = 2$, we choose for the calculation of $D_{\sigma_1} \mu(\tau)$ (where σ_1 corresponds to e_1) the flags $\bar{i}, \bar{j}, \bar{k}, \bar{l}$. If $m > 2$, we choose the flags $\bar{i}, \bar{j}, (v_2, e_1), t \in T_2$. Then we get the sum of two contributions. One will consist of the trees obtained by transplanting branches at v_1. They come with negative signs and exactly cancel the previously considered terms of the type a). If $m = 2$, the second group will cancel the terms of the type a) coming from v_2.

Consider a somewhat more difficult case $m > 2$. Then this second group of terms comes from the trees indexed by the partitions $T_2 = T'_2 \coprod T''_2, t \in T''_2, T'_2 \neq \emptyset$. Branches corresponding to T'_2 are transplanted to the midpoint v'_1 of the edge e_1. These terms come with negative signs: see Fig. 12.

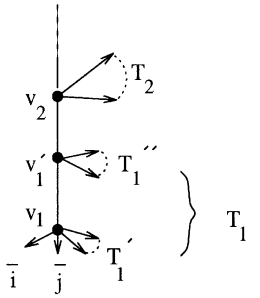

Fig. 11.

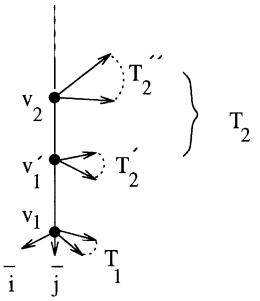

Fig. 12.

These trees in turn cancel with those coming from the terms of the type a) at the vertex v_2 with positive sign. However, there will be additional terms of the type a) for which $t \in T_2'$. They will cancel with one group of transplants contributing to $D_{\sigma_2}\mu(\tau)$ where σ_2 corresponds to the edge e_2 of the Fig. 10, if for the calculation of $D_{\sigma_2}\mu(\tau)$ one uses (2.5) with the following choice of flags: $(v_2, e_1), t$ at one end, (v_3, e_1), some $t' \in T_3$ at the other end (this last choice must be replaced by $\bar{k}, \bar{l}$, if $m = 3$).

The same pattern continues until all the terms cancel.

2.2.6. Compatibility with relations (*2.2*) By this time we have checked that the action of any element of F_S/I_S on the individual generators $\mu(\tau)$ of H_{*S} is well defined modulo the span I_{*S} of relations (2.2). It remains to show that the subspace in $\oplus_\tau K\mu(\tau)$ spanned by these relations is stable with respect to this action. But the calculation in the proof of the Proposition 2.2.1 shows that

$$r(\tau, v, \bar{i}, \bar{j}, \bar{k}, \bar{l}) \equiv m(\tau) r_{ijkl} \mod I_{*S},$$

where r_{ijkl} is obtained from R_{ijkl} by replacing $m(\sigma)$ with $\mu(\sigma)$. To multiply this by any element of H_S^* we can first multiply it by $m(\tau)$, then represent the result as a linear combination of good monomials, and finally multiply each good monomial by r_{ijkl}. The result will lie in I_{*S}.

This finishes the proof of the Main Lemma and the Theorem 2.2.

3. Cohomological Field Theories of rank 1

3.1. Notation. Rank of a CohFT on (H, g) is the (super)dimension of H. In this section we consider the case dim $H = 1$. To slightly simplify notation let us assume that all square roots exist in K. Then $H = K\Delta_0,\ g(\Delta_0, \Delta_0) = 1, \Delta = \Delta_0^{\otimes 2} \in H^{\otimes 2}$. The basic vector Δ_0 is defined up to a sign. We will consider its choice as a rigidification and without further ado call such rigidified theories simply CohFT's of rank one.

A structure of CohFT on (H, Δ_0) boils down to a sequence of cohomology classes (generally non-homogeneous)

$$c_n := I_n(\Delta_0^{\otimes n}) \in H^*(\overline{M}_{0n}, K)^{S_n},\ n \geqq 3, \tag{3.1}$$

satisfying the identities

$$\varphi_\sigma^*(c_n) = c_{n_1+1} \otimes c_{n_2+1}, \tag{3.2}$$

where $\phi_\sigma :\ \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0n}$ is the embedding of the boundary divisor corresponding to a partition σ (see (0.3)). Put $c_n = \sum_{i=0}^{n-3} c_n^{(i)},\ c_n^{(i)} \in H^{2i}(\overline{M}_{0n})$. Changing sign of Δ_0 leads to $c_n \mapsto (-1)c_n$.

The tensor product formula (0.8) becomes

$$\{c_n'\} \otimes \{c_n''\} = \{c_n' \wedge c_n''\}, \tag{3.3}$$

if we agree that $(H', \Delta_0') \otimes (H'', \Delta_0'') = (H' \otimes H'', \Delta_0' \otimes \Delta_0'')$.

Here are some simple consequences of (3.2) and (3.3).

3.1.1 *The theory $c_n = c_n^0 = [\overline{M}_{0n}]$ for $n \geqq 3$ is the identity with respect to the tensor product.*

3.1.2 *The theories $c_n(t) = t^{n-2}[\overline{M}_{0n}]$, $t \in K^*$, form a group isomorphic to K^*.*

3.1.3 *The theory $\{c_n\}$ is invertible iff $c_3^{(0)} \neq 0$. Any invertible theory is a tensor product of one of the type $\{c_n(t)\}$ and one with $c_n^{(0)} = 1$ for all n, and this decomposition is unique.*

3.1.4 *Assume that $c_n^{(0)} = 1$ and put $\lambda_n = \log\ c_n \in H^*(\overline{M}_{0n}, K)^{S_n}$. Then* (3.2) *becomes*

$$\varphi_\sigma^*(\lambda_n) = \lambda_{n_1+1} \otimes 1 + 1 \otimes \lambda_{n_2+1}, \tag{3.4}$$

and (3.3) *becomes*

$$\{\lambda_n'\} \otimes \{\lambda_n''\} = \{\lambda_n' + \lambda_n''\}. \tag{3.5}$$

Vice versa, any sequence of classes $\lambda_n \in H^*(\overline{M}_{0n}, K)^{S_n}$ satisfying (3.4) *gives rise to a CohFT of rank* 1, *$c_n = \exp \lambda_n$. We can say that $\{\lambda_n\}$ forms a logarithmic CohFT of rank* 1.

3.1.5 *There is a canonical bijection between the set of isomorphism classes of CohFT's of rank* 1 *and the set of infinite sequences $(C_3, C_4, \ldots) \in K^\infty$ given by*

$$C_n = \int\limits_{\overline{M}_{0n}} c_n = \int\limits_{\overline{M}_{0n}} c_n^{(n-3)}.$$

In fact, this is a particular case of the equivalence stated in 0.4 and 0.5, because any formal series in one variable $\Phi(x) = \sum \frac{C_n}{n!} x^n$ satisfies the associativity equations.

Formula (0.7) reconstructing $c_n^{(i)}$ from C_m becomes

$$\forall\, \tau, |E_\tau| = i: \quad \int\limits_{\overline{M}_{0n}} [m(\tau)] \wedge c_n^{(i)} = \prod_{v \in V_\tau} C_{|v|}, \tag{3.6}$$

where $[m(\tau)] \in H^*(\overline{M}_{0n})$ is the image of the good monomial $m(\tau)$. We do not know nice formulas for the tensor product in terms of coordinates (C_i). The main goal of this section is to show that there are natural coordinates defined geometrically that are simply additive with respect to the tensor multiplication of invertible theories. This is a reformulation of certain identities from [AC].

3.2. Mumford classes. Consider the universal curve $p_n: X_n \to \overline{M}_{0n}$ and its structure sections $s_i: \overline{M}_{0n} \to X_n,\ i = 1, \ldots, n$. Let $x_i \subset X_n$ be the image of s_i, ω the relative dualizing sheaf on X_n. For $a = 1, 2, \ldots$ put

$$\omega_n(a) := p_{n*}\left(c_1\left(\omega\left(\sum_{i=1}^n x_i \right) \right)^{a+1} \right) \in H^{2a}(\overline{M}_{0n}, \mathbf{Q})^{\mathbf{S_n}}. \tag{3.7}$$

It is proved in [AC] that for any $a \geqq 1$ (in fact, $a = 0$ as well) $\{\omega_n(a) \mid n \geqq 3\}$ satisfy (3.4) i.e., form a logarithmic field theory. Hence we can construct an infinite-dimensional family of invertible theories of rank one:

$$\omega_n[s_1, s_2, \ldots] := \exp\ \left(\sum_{a=1}^{\infty} s_a \omega_n(a) \right),\ n \geqq 3. \tag{3.8}$$

3.2.1. Theorem *(s_a) form a coordinate system on the space of isomorphism classes of theories with $c_3^{(0)} = 1$, defining its group isomorphism with K_+^∞.*

Proof. The sum in the r.h.s. effectively stops at $a = n - 3$ (cf. (3.7)). The C_n-coordinate of the theory (3.8) is therefore

$$\int\limits_{\overline{M}_{0n}} \omega_n[s_1, s_2, \ldots] = s_{n-3} \int\limits_{\overline{M}_{0n}} \omega_n(n-3) + P_n(s_1, \ldots, s_{n-4}),\ n \geqq 4,$$

where P_n is a universal polynomial. Hence it remains to check that the coefficient at s_{n-3} does not vanish. But this follows from the well known fact that $\omega(\sum_{i=1}^n x_i)$ is an ample sheaf on X_n.

3.2.2. Remark. The theories we are considering here are tree level ones in the terminology of [KM]. The general definition of a CohFT given there involves maps $I_{g,n}: H^{\otimes n} \to H^*(\overline{M}_{g,n})$ for all stable pairs (g, n). The classes $\omega_n(a)$ given by (3.7) can be automatically defined in this larger generality, and the extension of the property (3.4) is proved in [AC] for all (g, n). Therefore formulas (3.8) in fact define full (any genus) rank one theories.

However we do not know whether Theorem 3.2.3 extends to the general case because it is unclear whether the functions $C_{g,n} := \int_{\overline{M}_{g,n}} c_{g,n}$ form a coordinate system on the space of full rank 1 theories. In fact, they probably do

not, because of the presence of non–trivial cusp classes in $H^*(\overline{M}_{g,n})$ having vanishing restrictions to the boundary.

3.3. Potential of rank 1 theories. Denote by $\Phi(x; s_1, s_2, \ldots)$ the potential of $\omega_n[s_1, s_2, \ldots]$ at $x\Delta$ (see (0.6)). We have from (3.8):

$$\Phi(x; s_1, s_2, \ldots) = \sum_{n=3}^{\infty} \frac{x^n}{n} \sum_{(m_a): \sum a m_a = n-3} \int_{\overline{M}_{0n}} \prod_a \omega_n(a)^{m_a} \prod_a \frac{s_a^{m_a}}{m_a}. \tag{3.9}$$

We expect that (3.9) satisfies some interesting differential equations encoding recursive relations between the numbers

$$\int_{\overline{M}_{0n}} \prod_a \omega_n(a)^{m_a}, \sum m_a = n-3.$$

Some partial results are given below.

It is even possible that such equations for arbitrary genus are implicit in the relations (conjectured by Witten and proved by Di Francesco, Itzykson, and Zuber) between the numbers denoted in [AC]

$$\int_{W_{(m_a),n}} \prod_a \psi_i^{n_a},$$

where $W_{(m_a),n}$ are certain combinatorial classes defined in terms of ribbon graphs. In fact, it is conjectured in [AC] that the dual cohomology classes of $W_{(m_a),n}$ can be expressed as

$$\prod_{a \geqq 2} \omega_n(a-1)^{m_a} \frac{(2^a(2a-1)!!)^{m_a}}{m_a!}$$

plus terms of lower order and boundary terms (our $\omega_n(a)$ are denoted k_a in [AC]).

We hope to return to this problem elsewhere. Here we will treat the case when only one of the coordinates s_a is non-zero.

3.4. Weil-Petersson theory The noncompact moduli spaces M_{0n} possess a canonical Weil-Petersson hermitian metric. It is singular on the boundary, but its Kähler form extends to a closed L^2-current on $\overline{M}_{0n}$ thus defining a real cohomology class $\omega_n^{WP} \in H^2(\overline{M}_{0n})^{S_n}$ (see [W] and [Z]).

In [AC] this class is identified as

$$\omega_n^{WP} = 2\pi^2 \omega_n(1). \tag{3.10}$$

The additivity property (3.4) for ω_n^{WP} was used by P. Zograf ([Z]) in order to calculate the WP-volumes of $\overline{M}_{0n}$. In our framework, he calculated the coefficients of the potential of the theory $\{\exp(\pi^{-2}\omega_n^{WP})\}$.

3.5. Weil-Petersson potential. The C_n–coordinate of $\{\exp(\pi^{-2}\omega_n^{WP})\}$ is

$$\frac{1}{\pi^{2(n-3)}} \int_{\overline{M}_{0n}} \frac{(\omega_n^{WP})^{n-3}}{(n-3)!} = \frac{v_n}{(n-3)!}$$

in the notation of [Z]. P. Zograf proved that $v_4 = 1, v_5 = 5, v_6 = 61, v_7 = 1379$, and generally

$$v_n = \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2} v_{n-i}, \quad n \geqq 4. \tag{3.11}$$

The potential (0.6) of the theory is therefore

$$\Phi^{WP}(x) := \sum_{n=3}^{\infty} \frac{v_n}{n!(n-3)!} x^n.$$

We can rewrite (3.11) as a differential equation for $\Phi^{WP}(x)$. Following [M], put

$$g(x) = x^2 \frac{d}{dx} \left(x^{-1} \frac{d}{dx} \Phi^{WP}(x) \right).$$

Then we have

$$x(x-g)g'' = x(g')^2 + (x-g)g'.$$

3.6. A generalization of Zograf's recursive relations. Put

$$z_n := \int\limits_{\overline{M}_{0n}} \omega_n(n-3), \qquad n \geqq 3. \tag{3.12}$$

Define for each $n \geqq 3$ an S_n-invariant function A_n on the set of isomorphism classes of n-trees σ with the following property:

$$\left\langle \sum_{\sigma: |E_\sigma| = a} A_n(\sigma)\sigma, \tau \right\rangle = \begin{cases} 1 \text{ if } |E_\tau| = n-3-a \text{ and } \exists v \in V_\tau, \ |v| = a+3, \\ 0 \text{ otherwise,} \end{cases} \tag{3.13}$$

where $\langle \sigma, \tau \rangle$ is defined by (1.9). Presumably, (3.12) can be calculated inductively using the definition (3.7). The solvability of (3.13) will be shown below. To find an explicit solution one has to invert the Poincaré pairing matrix restricted to the S_n-invariant part of $H^*(\overline{M}_{0n})$. Hopefully, a version of the Proposition 1.7.1 can be used to do this.

Put

$$\Omega_n(a) = \begin{cases} \int\limits_{\overline{M}_{0n}} \omega_n(a)^{\frac{n-3}{a}}, \text{ if } a/(n-3), \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.14}$$

This is a part of the coefficients in (3.9).

3.6.1. Theorem *For a fixed $a \geqq 1$, the sequence $\{\Omega_n(a)\}$, $n \geqq 3$ satisfies the recursive relations*

$$\Omega_n(a) = z_{a+3} \sum_{\substack{n\text{-trees } \sigma: \\ |E_\sigma| = a}} A_n(\sigma) \frac{\left(\frac{n-3}{a} - 1\right)!}{\prod\limits_{v \in V_\sigma} \left(\frac{|v|-3}{a}\right)!} \prod_{v \in V_\sigma} \Omega_{|v|}(a). \tag{3.15}$$

Proof. First of all, we can calculate $\omega_n(a)$ as a functional on the homology classes of the strata $\overline{M}_\tau$ where τ runs over stable n–trees with $|E_\tau| = n-3-a$. Namely, since $\{\omega_n(a)\}$ form a logarithmic field theory, we have

$$\int\limits_{\overline{M}_\tau} \omega_n(a) := \langle \varphi_\tau^*(\omega_n(a))\rangle = \left\langle \sum_{v\in V_\tau} \mathrm{pr}_v^*(\omega_{|v|}(a)) \right\rangle, \tag{3.16}$$

where $\mathrm{pr}_v^*: \overline{M}_\tau \to \overline{M}_{0,F_\tau(v)}$ is the canonical projection.

In (3.16), only the summands with $|v| = a+3$ can be non-vanishing, and there can exist at most one such summand, because $|V_\tau| = |E_\tau| + 1 = n-2-a$, so that

$$\sum_{v\in V_\tau} (|v|-3) = |T_\tau| + 2|E_\tau| - 3|V_\tau| = a.$$

It follows that $\omega_n(a)$ is dual to the class $z_{a+3}\sum_{\sigma:|E_\sigma|=a} A_n(\sigma)[\overline{M}_\sigma]$, in the notation of (3.13).

Similarly, one can calculate $\omega_n(a)^{\frac{n-3}{a}-1}$ as a functional on the classes of the tree strata $[\overline{M}_\sigma]$ with $|E_\sigma| = n-3-a(\frac{n-3}{a}-1) = a$. We have, putting $m = \frac{n-3}{a} - 1$:

$$\begin{aligned}
\int\limits_{\overline{M}_\sigma} \varphi_\sigma^*(\omega_n(a)^m) &= \left\langle \left(\sum_{v\in V_\sigma} \mathrm{pr}_v^*(\omega_{|v|}(a)) \right)^m \right\rangle \\
&= \sum_{\substack{(m_v|v\in V_\sigma):\\ \sum m_v = m}} \frac{m!}{\prod m_v}! \prod_{v\in V_\sigma} \int (\omega_{|v|}(a)))^{m_v} \\
&= \frac{m!}{\prod\limits_v \left(\frac{|v|-3}{a}\right)!} \prod_{v\in V_\sigma} \Omega_{|v|}(a),
\end{aligned}$$

because only one summand, with $m_v = \frac{|v|-3}{a}$ for all v, can be non-vanishing.

In view of (3.13), this is equivalent to (3.15), because

$$\Omega_n(a) = \int\limits_{\overline{M}_{0n}} \omega_n(a)^{\frac{n-3}{a}} = \int\limits_{\overline{M}_{0n}} \omega_n(a)^{\frac{n-3}{a}-1} \wedge \omega_n(a),$$

which is $\omega_n(a)^{\frac{n-3}{a}-1}$ integrated along

$$z_{a+3} \sum_{\sigma:|E_\sigma|=a} A_n(\sigma)[\overline{M}_\sigma].$$

3.6.2. Remark. Zograf's argument essentially coincides with our reasoning for the case $a = 1$.

It can be directly generalized to obtain more general recursive relations for all coefficients in (3.9). However, their usefulness depends on the understanding of $\{A_n(\sigma)\}$.

3.7. Twisting. For any CohFT on (H, g), we can define a new theory by tensor multiplying it by $\omega_\cdot[s_1, s_2, \ldots]$. It would be interesting to study the dependence of its potential on $s_1, s_2, \ldots$. This could clarify the analytic properties of the initial theory.

If the initial theory corresponds to a system of GW-classes, as in [KM], it satisfies a number of additional axioms. In particular, it has a scaling group related to the grading of H, and an identity in the quantum cohomology ring. Twisting generally destroys these additional structures.

Appendix: Proof of the Proposition 1.7.1.

R. Kaufmann

We keep notation of Sect. 1.7.

Consider the canonical embedding $\varphi_\tau : \overline{M}_\tau \to \overline{M}_{0S}$. We start with the formula

$$\langle \sigma_1, \sigma_2 \rangle = \left\langle \prod_{e \in E} \varphi_\tau^*(D_{\sigma(e)}) \right\rangle, \tag{A.1}$$

where the cup product in the r.h.s. is taken in $H^*(\overline{M}_\tau) \cong \otimes_{v \in V_\tau} H^*(\overline{M}_{0,F_\tau(v)})$. Applying an appropriate version of the formulas (1.4) and (1.6) we can write for any $e \in E$ with vertices v_1, v_2:

$$\varphi_\tau^*(D_{\sigma(e)}) = -\Sigma_{v_1,e} - \Sigma_{v_2,e} \tag{A.2}$$

where

$$\Sigma_{v_i,e} \in H^*(\overline{M}_{0,F_\tau(v_i)}) \otimes \prod_{v \neq v_i} [\overline{M}_{0,F_\tau(v)}] \tag{A.3}$$

and $[\overline{M}_{0,F_\tau(v)}]$ is the fundamental class. Later we will choose an expression for $\Sigma_{v_i,e}$ depending on the choice of flags denoted $\bar{i}, \bar{j}$ or $\bar{k}, \bar{l}$ in (1.4).

Inserting (A.2) into (A.1), we get

$$\langle \sigma_1, \sigma_2 \rangle = \sum_h \left\langle \prod_{\substack{(v,e) \in F_\delta \\ h(e)=v}} (-\Sigma_{v,e}) \right\rangle, \tag{A.4}$$

where h runs over all orientations of E considered as a choice, for every $e \in E$, of a vertex $h(e)$ of e.

The summand of (A.4) corresponding to a given h can be non-zero only if for every $v \in V_\delta$ the number of factors (v, e) with $h(e) = v$ equals $\dim \overline{M}_{0,F_\tau(v)} = |v| - 3$. This is what was called a good orientation.

Assume that there are two good orientations h, h' of δ. Consider the union of all closed edges on which $h \neq h'$. Each connected component of this union is a tree. Choose an end edge e of this tree and an end vertex v of e. At v, the number of h-incoming and h'-incoming edges must coincide, but on e these

orientations differ. Hence there must exist an edge $e' \neq e$ incident to v upon which h and h' differ. But this contradicts to the choice of v and e.

Now assume that one good orientation h exists. We can rewrite (A.4) as

$$\langle \sigma_1, \sigma_2 \rangle = \left\langle \prod_{v \in V_\tau} \prod_{e: h(e)=v} (-\Sigma_{v,e}) \right\rangle. \tag{A.5}$$

In view of (A.3), this expression splits into a product of terms computed in all $H^*(\overline{M}_{0,F_\tau(v)}), v \in V_\tau$ separately. Each such term depends only on $|v|$, and we want to demostrate that it equals $(-1)^{|v|-3}(|v|-3)!$. Put $|v| = m+3$. We may and will assume that $m \geqq 1$, the case $m = 0$ being trivial.

Let us identify F_τ with $\{1, \ldots, m+3\}$ in such a way that flags $1, \ldots, m$ belong to the edges e with $h(e) = v$. Denote by $D_\rho^{(m+3)}$ the class of a boundary divisor in $H^*(\overline{M}_{0,m+3})$ corresponding to a stable partition ρ of $\{1, \ldots, m+3\}$. We will choose flags $m+1, m+2$ to play the role of $\bar{i}, \bar{j}$ in (1.4) for any $i \in \{1, \ldots, m\}$ corresponding to an edge e in (A.5) (v being now fixed), so that the contribution of v in (A.5) becomes

$$\prod_{i=1}^{m} \left(\sum_{\rho:\ i\rho\{m+1,m+2\}} -D_\rho^{(m+3)} \right) := g(m). \tag{A.6}$$

We will calculate (A.6) inductively. Consider the projection map (forgetting the $(m+3)$-th point) $p: \overline{M}_{0,m+3} \to \overline{M}_{0,m+2}$ and the i-th section map $x_i: \overline{M}_{0,m+2} \to \overline{M}_{0,m+3}$ obtained via the identification of $\overline{M}_{0,m+3}$ with the universal curve. We have $p \circ x_i = \mathrm{id}$, and x_i identifies $\overline{M}_{0,m+2}$ with $D_{\sigma_i}^{(m+3)}$ where

$$\sigma_i = \{\{i, m+3\}, \{1, \ldots, \widehat{i}, \ldots, m+2\}\}.$$

Therefore

$$\sum_{\rho:\ i\rho\{m+1,m+2\}} -D_\rho^{(m+3)} = -p^* \left(\sum_{\rho':\ i\rho'\{m+1,m+2\}} D_{\rho'}^{(m+2)} \right) - x_{i*}([\overline{M}_{0,m+2}]), \tag{A.7}$$

where ρ' runs over stable partitions of $\{1, \ldots, m+2\}$. We now insert (A.7) into (A.6) and represent the resulting expression as a sum of products consisting of several p^*-terms and several x_{i*}-terms each. If such a product contains $\geqq 2$ x_{i*}-terms, it vanishes because the structure sections pairwise do not intersect. The product containing no x_{i*}-terms vanishes because $\dim \overline{M}_{0,m+2} = m-1$. Finally, there are m products containing one x_{i*}-term each. Using the projection formula

$$\langle p^*(X) x_{i*}([\overline{M}_{0,m+2}]) \rangle = \langle X \rangle$$

one sees that each such term equals $-g(m-1)$ (cf. (A.6)). So $g(m) = -mg(m-1) = (-1)^m m!$ because $g(1) = -1$.

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