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The Tamagawa number conjecture for CM elliptic curves

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Introduction

In this paper we prove the weak form of the Tamagawa number conjecture of Bloch and Kato for elliptic curves with CM by the ring of integers $\mathcal{O}_K$ in a quadratic number field. For this we use a new explicit description of the p-adic specialization of the elliptic polylogarithm. The word "weak" indicates that we have nothing new to say about the rank of the K-groups involved.

The Tamagawa number conjecture describes the special values of the L-function of a variety in terms of the regulator maps of the K-theory of the variety into Deligne and étale cohomology (see 1.1 for the exact formulation). There are only two cases proven so far in the non critical situation, both due to Bloch and Kato [Bl-Ka]: The first is the Riemann zeta function (i.e. the case of $\mathbb{Q}$) and the second the L-value at 2 of a CM elliptic curve defined over $\mathbb{Q}$ for regular primes. In the last case Bloch and Kato use an ad hoc method to describe the p-adic regulator of the K-theory. This does not extend to higher K-groups.

The regulator map to Deligne cohomology was computed by Deninger [Den1] with the help of the Eisenstein symbol. Here, the regulator can be described in terms of real analytic Eisenstein series (whence the name) and leads to a proof of the weak form of the Beilinson conjecture for CM elliptic curves. For the Tamagawa number conjecture one needs an understanding of the p-adic regulator on the subspace of K-theory defined by the Eisenstein symbol. In an earlier paper [Hu-Ki2] we established with A. Huber the relation of the p-adic regulator of the Eisenstein symbol with the specialization of the p-adic elliptic polylogarithm sheaf. The problem remains, to compute these specializations in p-adic cohomology.

The elliptic polylogarithm is one of the most powerful tools in the study of special values of L-functions. All known cases of the Beilinson conjecture are proved or can be proved with specializations of the elliptic polylog. A universal property characterizes the polylog, which simplifies the explicit computations. So far, only the absolute Hodge realization of the elliptic polylog was understood, due to the extensive work of Beilinson and Levin [Be-Le]. Missing was a theory of the p-adic realization, which give manageable étale cohomology classes. In the cyclotomic case, such an explicit realization is known. The approach there, mainly due to Deligne [Del2], uses torsors (Galois coverings) over $\mathbb{G}_m \setminus 1$ which are ramified in 1. This is not transferable to the elliptic case, because such torsors over elliptic curves do not exist (there are no Galois coverings ramified in exactly one point due to compactness). In our approach we allow instead ramification at torsion points, constructing in fact coverings over $E \setminus E[p^n]$. But this is not the only change of point of view compared to the cyclotomic case. The question is also, what is the group whose torsors we have to consider. The right choice is the group of torsion points of the torus with character group the augmentation ideal of the group ring of $E[p^n]$.

It turns out (see Sect. 4.1) that the elliptic polylogarithm is an inverse limit of p^n-torsion points of certain one-motives, which are essentially the generalized Jacobian defined by the divisor of all p^n-torsion points on the elliptic curve E. The cohomology classes of the elliptic polylogarithm sheaf can then be described by classes of sections of certain line bundles. These sections are elliptic units and going carefully through the construction one finds an analog of the elliptic Soulé elements of [So3].

Now enters Iwasawa theory: By an idea of Kato, going back in part to earlier work of Soulé [So3], the étale cohomology groups can be described in terms of Iwasawa modules. Rubin's "main conjecture" [Ru3] allows then to give a bound on the kernel and the cokernel of Soulé's map from elliptic units to the étale cohomology. On the way we also need some of the tools developed by Rubin to prove the main conjecture. Rubin's theory is the second decisive input into the proof of the Tamagawa number conjecture.

Let us finally give a rough sketch of the contents of this paper. More overviews can be found at the beginning of each section. In the first section we recall the statement of the Tamagawa number conjecture and formulate our Main result 1.1.5. After this we recall Deninger's construction of the K-theory elements leading to the Beilinson conjecture for CM elliptic curves. Here we also reduce to the computation of the specialization of the elliptic polylogarithm sheaf by using our earlier work [Hu-Ki2].

The second section reviews the "main conjecture" and relates the Iwasawa modules to étale cohomology.

The next two sections are independent of the rest of the paper. The third section introduces the elliptic polylog and its specializations. The approach follows the important paper [Be-Le] but puts the emphasis on different aspects, which are important for our geometric construction of the elliptic polylog.

The technical heart of the paper is section four. Here we describe the polylog as an inverse limit of torsion points of one motives. The cohomology classes of what we call “geometric polylog” are then computed as the classes of sections of certain line bundles.

The last section puts the various results together and gives the proof of the Main theorem 1.1.5.

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1 The Bloch-Kato conjecture for CM-elliptic curves

This part of the paper contains our first main result, the Bloch-Kato conjecture for CM elliptic curves. For the precise formulation of our result we refer to 1.1.5. This part is organized as follows: First we review the Bloch-Kato conjecture in the case of interest to us. Here we also formulate the main theorem. Then we recall the construction due to Deninger of elements in the K-theory of CM elliptic curves. These elements satisfy the weak Beilinson conjecture for these curves as was shown by Deninger. This is the starting point of our investigations of the Bloch-Kato conjecture.

1.1 The Tamagawa number conjecture of Bloch-Kato and the main theorem

This section recalls the Bloch-Kato conjecture [Bl-Ka] about special values of L-functions in the formulation of Kato [Ka1] and [Ka2]. We review this only for certain weights, which suffices for our purpose. Then we formulate our main result.

1.1.1 The Tamagawa number conjecture in the formulation of Kato. Let X/K be a smooth proper variety over a number field K with ring of integers $\mathcal{O}_K$. Fix integers $m \geq 0$ and r such that $m - 2r \leq -3$ and $r > \inf(m, \dim(X))$. Let p be a prime number not equal to 2. Let S be a set of finite primes of K containing the primes lying over p and the ones where X has bad reduction. Let $\mathcal{O}_S$ be the ring $\mathcal{O}_K[\frac{1}{S}]$, where the primes in S are inverted. Define $\mathrm{Gal}(\overline{K}/K)$-modules

$$V_p := H^m(X \times_K \overline{K}, \mathbb{Q}_p(r))$$
$$T_p := H^m(X \times_K \overline{K}, \mathbb{Z}_p(r))$$

Let $j : \mathrm{Spec}\, K \to \mathrm{Spec}\, \mathcal{O}_S$ and define the p-adic realizations to be

$$H^i_p := H^i(\mathcal{O}_S, j_* T_p).$$

We will omit the j_*, if no confusion is likely. Define

$$H_{h,\mathbb{Z}} := H^m_{\mathrm{sing}}(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+$$

where the $+$ denotes the fixed part under $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of the singular cohomology of X. Here $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}$ and on $(2\pi i)^{r-1}\mathbb{Z}$.

Finally we need the K-theory of X: Let

$$H_{\mathcal{M}} = (K_{2r-m-1}(X) \otimes \mathbb{Q})^{(r)}$$

be the r-th Adams eigenspace of the $2r - m - 1$-th Quillen K-theory of X. There are regulator maps due to Beilinson and Soulé

$$r_{\mathcal{D}} : H_{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{R} \to H_{h,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$r_p : H_{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

called the Deligne regulator (see [Be1]) and the p-adic regulator (see [So3]).

Remark: Note that because of our assumption $r > \inf(m, \dim(X))$, the Deligne cohomology coincides with $H_{h,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [Sch2] sequence (*) on page 9). The same condition (together with $m - 2r \leq -3$) also guarantees that $H^i_{\lim} = H^i_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (cf. [Ka2] 2.2.6 (4)).

Let us define local Euler factors for X. Let for a prime $\mathfrak{p} \nmid p$ in $\mathcal{O}_K$

$$P_{\mathfrak{p}}(V_p, s) := \det_{\mathbb{Q}_p}\bigl(1 - \mathrm{Fr}_{\mathfrak{p}}\, \mathrm{N}\,\mathfrak{p}^{-s} | V_p^{I_{\mathfrak{p}}}\bigr)$$

be the characteristic polynomial of the geometric Frobenius $\mathrm{Fr}_{\mathfrak{p}}$ at $\mathfrak{p}$ on the invariants of V_p under the inertia group $I_{\mathfrak{p}}$ at $\mathfrak{p}$. For $\mathfrak{p}|p$ set

$$P_{\mathfrak{p}}(V_p, s) := \det_{\mathbb{Q}_p}\bigl(1 - \phi_{\mathfrak{p}}^{-1}\, \mathrm{N}\,\mathfrak{p}^{-s} | D_{\mathrm{cris}}(V_p)\bigr)$$

where $D_{\mathrm{cris}}(V_p)) := (V_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ and $\phi_{\mathfrak{p}}$ is the arithmetic Frobenius. Define the L-function of X as

$$L_S(V_p, s) := \prod_{\mathfrak{p} \notin S} P_{\mathfrak{p}}(V_p, s)^{-1}.$$

Let V_p^* be the dual Galois module of V_p. We now give Kato's formulation of the Tamagawa number conjecture. Here and in the rest of the paper the determinants are taken in the sense of Knudsen and Mumford [Kn-Mu].

Conjecture 1.1.1. *(cf. [Ka2] 2.2.7) Let $p \neq 2$ be a prime number, r, m and S be as above. Assume that*

$$P_{\mathfrak{p}}\big(V_p^*(1), 0\big) \neq 0$$

for all $\mathfrak{p} \in S$ and that $L_S(V_p^*(1), s)$ has an analytic continuation to all of $\mathbb{C}$. Then:

a) The maps $r_{\mathfrak{D}}$ and r_p are isomorphisms and H_p^2 is finite.
b) $\dim_{\mathbb{Q}}(H_{h,\mathbb{Z}} \otimes \mathbb{Q}) = \mathrm{ord}_{s=0} L_S(V_p^*(1), s)$.
c) Let $\eta \in \det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ be a $\mathbb{Z}$-basis and let $e := \dim_{\mathbb{Q}}(H_{h,\mathbb{Z}} \otimes \mathbb{Q})$. There is an element $\xi \in \det_{\mathbb{Q}}(H_{\mathcal{M}})$ such that

$$r_{\mathfrak{D}}(\xi) = \Big(\lim_{s\to 0} s^{-e} L_S\big(V_p^*(1), s\big)\Big)\eta.$$

This is the "Beilinson conjecture".

d) Consider $r_p(\xi) \in \det_{\mathbb{Q}_p}(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, then $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice

$$\det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset \det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1]) \cong \det_{\mathbb{Q}_p}\big(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\big),$$

i.e.

$$\Big[\det_{\mathbb{Z}_p}\big(H_p^1\big) : r_p(\xi)\mathbb{Z}_p\Big] = \#\big(H_p^2\big).$$

Remark: a) The assumption in the conjecture is true for abelian varieties with CM.
b) The space H_p^0 is zero for weight reasons.
c) Part b) follows from the expected shape of the functional equation, (see e.g. [Sch2] proposition page 9).

As our knowledge of K-theory is limited, let us also formulate a weak version of the above conjecture.

Conjecture 1.1.2. *(weak form of conjecture 1.1.1) There is a subspace $H_{\mathcal{M}}^{\mathrm{constr}}$ in $H_{\mathcal{M}}$ (the constructible elements of $H_{\mathcal{M}}$) such that*

a') $r_{\mathfrak{D}}$ and r_p restricted to $H_{\mathcal{M}}^{\mathrm{constr}}$ are isomorphisms and H_p^2 is finite.
b') same as b)
c') There is an element $\xi \in \det_{\mathbb{Q}}(H_{\mathcal{M}}^{\mathrm{constr}})$ such that

$$r_{\mathfrak{D}}(\xi) = \Big(\lim_{s\to 0} s^{-e} L_S\big(V_p^*(1), s\big)\Big)\eta.$$

d') The element $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice

$$\det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset \det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1]) \cong \det_{\mathbb{Q}_p}\big(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\big).$$

1.1.2 Elliptic curves with CM. Before we formulate the main theorem, we introduce the elliptic curves we want to consider. We follow the notations and conventions in Deninger [Den1]. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E/K be an elliptic curve with CM by $\mathcal{O}_K$. Note that this implies that the class number of K is one. We fix an isomorphism

$$\vartheta : \mathcal{O}_K \cong \mathrm{End}_K(E_K),$$

such that for $\omega \in \Gamma(E_K, \Omega_{E_K/K})$ and $\alpha \in \mathcal{O}_K$ we have $\vartheta^*(\alpha)\omega = \alpha\omega$. We fix also an embedding of K into $\mathbb{C}$, such that the algebraic j-invariant of E is the same as the corresponding complex analytic j-invariant of $\mathcal{O}_K$. Let us denote by

$$\psi : \mathbb{A}_K^* \to K^* \subset \mathbb{C}^*$$

the CM-character or Serre-Tate character of E_K and let $\mathfrak{f}$ be its conductor. The elliptic curve E has bad reduction precisely at the primes dividing $\mathfrak{f}$. Denote by $\overline{\psi}$ the complex conjugate character. Its conductor is also $\mathfrak{f}$.

Definition 1.1.3. *Fix a prime number p. We let S be the set of primes in K dividing $p\mathfrak{f}$.*

Associated to ψ is an L-series

$$L_S(\psi, s) = \prod_{\mathfrak{p} \nmid p\mathfrak{f}} \frac{1}{1 - \frac{\psi(\mathfrak{p})}{\mathrm{N}\,\mathfrak{p}^s}}.$$

We want to relate this to the L-function $L_S(E, s)$ of E. Recall the fundamental result of Deuring:

Theorem 1.1.4. *(see [Si]II 10.5.) Let $L_S(E/K, s) := L_S(V_p, s)$ be the L-series of the Galois representation $V_p := H^1(E \times_K \overline{K}, \mathbb{Q}_p)$ as defined in Sect. 1.1.1. Then*

$$L_S(E/K, s) = L_S(\psi, s) L_S(\overline{\psi}, s).$$

Let $T_pE = \varprojlim_n E[p^n]$ be the Tate-module of E. This is a $\mathrm{Gal}(\overline{K}/K)$-module. Then $H^1(E \times_K \overline{K}, \mathbb{Z}_p) \cong \mathrm{Hom}(T_pE, \mathbb{Z}_p) \cong T_pE(-1)$, where $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$ acts now conjugate linear on T_pE. There is a canonical isomorphism

$$H^1\big(E \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^r\mathbb{Z}\big)^+ \cong H^1\big(E \times_K \mathbb{C}, (2\pi i)^r\mathbb{Z}\big),$$

where we used the fixed embedding $K \subset \mathbb{C}$. Let

$$H^i_{\mathcal{M}}(E, j) := (K(E)_{2j-i} \otimes \mathbb{Q})^{(j)}$$

be the $2j - i$-th Quillen K-theory of E.

For $m = 1$, $r = k + 2$ with $k \geq 0$ in the notation of 1.1.1 we have

$$H^i_p = H^i(\mathcal{O}_S, T_pE(k+1))$$
$$H_{h,\mathbb{Z}} = H^1\big(E \times_K \mathbb{C}, (2\pi i)^{k+1}\mathbb{Z}\big)$$
$$H_{\mathcal{M}} = H^2_{\mathcal{M}}(E, k+2).$$

Note that on all these spaces we have a canonical $\mathcal{O}_K$-action and that $H_{h,\mathbb{Z}}$ is an $\mathcal{O}_K$-module of rank 1. It is a result of Jannsen ([Ja2] Corollary 1) that if H^2_p is finite then the free part of H^1_p is an $\mathcal{O}_K \otimes \mathbb{Z}_p$-module of rank 1.

1.1.3 The main theorem. Now we can formulate our main result. We let $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$.

Theorem 1.1.5. *Let $p \neq 2, 3$ and $p \nmid \mathrm{N}_{K/\mathbb{Q}}\mathfrak{f}$ and $k \geq 0$. Then, there is an $\mathcal{O}_K$ submodule $\mathcal{R}_\psi \subset H_{\mathcal{M}}$ of rank 1 such that*
a) $\det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\psi)) \cong L^*_S(\overline{\psi}, -k)\det_{\mathcal{O}_K}(H_{h,\mathbb{Z}})$ *in* $\det_{\mathcal{O}_K\otimes\mathbb{R}}(H_{h,\mathbb{Z}} \otimes \mathbb{R})$
and
b) The map r_p induces an isomorphism

$$\det_{\mathcal{O}_p}(\mathcal{R}_\psi) \cong \det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, T_pE(k+1)))^{-1}.$$

Here $L^*(\overline{\psi}, -k) = \lim_{s\to -k} \frac{L(\overline{\psi},s)}{s+k}$ denotes the leading coefficient of the Taylor series of $L(\overline{\psi}, s)$ at $-k$. Moreover, if H^2_p is finite, r_p is injective on $\mathcal{R}_\psi$ and

$$\det_{\mathcal{O}_p}\big(H^1_p/r_p(\mathcal{R}_\psi)\big) \cong \det_{\mathcal{O}_p} H^2_p.$$

Remark: i) Part a) was proven by Deninger in [Den1]. This is the Beilinson conjecture for Hecke characters.

ii) For $k = 0$ and CM elliptic curves defined over $\mathbb{Q}$ with CM by $\mathcal{O}_K$ and p regular, part b) was proven in [Bl-Ka]. They used an ad hoc computation of the p-adic realization of the K-theory elements, which does not generalize.

iii) As P. Colmez pointed out to me, the theory developed by Perrin-Riou in [P-R] leads to the fact that the Soulé elements in H^1_p have the right index for p split in K, if H^2_p is finite. This is explained in [Co] Theorem 3.4.

iv) Let us explain part b) in more detail: The Soulé regulator

$$r_p : \mathcal{R}_\psi \to H^1_p \otimes \mathbb{Q}_p$$

extends to a map to $R\Gamma(\mathcal{O}_S, T_pE(k+1))[-1] \otimes \mathbb{Q}_p$ because H^0_p is zero for weight reasons. The determinant of this complex has as $\mathcal{O}_p$-lattice the determinant of $R\Gamma(\mathcal{O}_S, T_pE(k+1))[-1]$, which is $\det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, T_pE(k+1)))^{-1}$. Part b) means that the determinant of the complex

$$\mathcal{R}_\psi \xrightarrow{r_p} R\Gamma(\mathcal{O}_S, T_pE(k+1))[-1]$$

is trivial. If H^2_p is finite, and $r_p \otimes \mathbb{Q}_p$ an isomorphism, we get from this $\det_{\mathcal{O}_p}(H^1_p/r_p(\mathcal{R}_\psi)) \cong \det_{\mathcal{O}_p} H^2_p$.

v) The theorem is only formulated for the main order $\mathcal{O}_K$ in K and the existence of E/K with CM implies that K has class number one. The result and the method of proof extends to elliptic curves defined over certain abelian extensions F/K as in [Den1], whenever Rubin's "main conjecture" of Iwasawa theory is available (i.e. $[F : K]$ is not divisible by p). We stick to the case $F = K$, because we don't want to overburden this already long paper with purely technical and notational complications.

vi) Note that H^2_p is finite for almost all $k \geq 0$ or for p regular. See the next section for remarks on the finiteness of H^2_p.

vii) From this result for the set of primes S we get it for all other set of primes which contain S. This follows from [Ka2] 4.11.

As a corollary we get a result about $\mathbb{Z}_p$ determinants and the L-function of E/K:

Corollary 1.1.6. *Under the conditions of the theorem*

a) $\det_{\mathbb{Z}}(r_{\mathcal{D}}(\mathcal{R}_\psi)) = L^*_S(E/K, -k)\det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$
and

b) $\det_{\mathbb{Z}_p}(r_p(\mathcal{R}_\psi)) = \det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_pE(k+1)))^{-1}$.
Here $L^*(E/K, -k) = \lim_{s\to -k} \frac{L(E/K,s)}{(s+k)^2}$ *denotes the leading coefficient of the Taylor series of* $L(E/K, s)$ *at* $-k$.

Proof. This follows from the theorem and the remark that if we multiply an $\mathcal{O}_K$-module with an element $L^*(\overline{\psi}, -k)$ from $\mathcal{O}_K \otimes \mathbb{R}$, then the determinant is multiplied by the norm $\mathrm{N}_{\mathcal{O}_K\otimes\mathbb{R}/\mathbb{R}}(L^*(\overline{\psi}, -k))$. But $\mathrm{N}_{\mathcal{O}_K\otimes\mathbb{R}/\mathbb{R}}(L^*(\overline{\psi}, -k))$ $= L^*(E/K, -k)$. Part b) is obvious. □

Here is a short overview of the proof: We start by recalling the Beilinson-Deninger definition of K-theory elements for CM elliptic curves and Deninger's main result about the relation of these to the L-value. Then we use an idea of Soulé to construct a submodule of

$$H^1(\mathcal{O}_S, T_pE(k+1))$$

via elliptic units. Iwasawa theory allows us to compute the index of this submodule. The proof concludes with the comparison of this submodule and $\mathcal{R}_\psi$. This is the main step in the proof which needs the theory developed in part 4 and in particular the explicit description of the elliptic polylogarithm of Theorem 4.2.9. The injectivity of r_p, in the case that H^2_p is finite, will be proved in Sect. 5.2.2.

1.1.4 Some remarks concerning the finiteness of H^2. Note that we do *not* prove that H^2_p is finite. Nevertheless, following an idea of Soulé [So2] (corrected by Jannsen [Ja2] Lemma 8) we have finiteness of H^2_p for almost all twists:

Theorem 1.1.7. *For fixed p the group*

$$H^2(\mathcal{O}_S, T_pE(k+1))$$

is finite for almost all k.

Proof. By [Ru2] Theorem 4.4 and discussions and Theorem 1 in [Mc] the conditions in [Ja2] Lemma 8 b) are satisfied. Hence the result follows from the equivalent conditions given in that lemma. □

For regular p, we have results of Soulé and Wingberg:

Theorem 1.1.8 ([So3] 3.3.2, [Win] Cor. 2). *Let p be a regular prime for E (see e.g.[So3] 3.3.1 for the definition of regular), then*

$$H^2\big(\mathcal{O}_S, E[p^\infty](k+1)\big) = 0.$$

It is easy to see that this vanishing implies the finiteness of H^2_p (cf. [Ja2] Lemma 1).

Remark: In [Ja2] it is conjectured that H^2_p is always finite for $k \geq 0$.

1.2 Review of the Beilinson–Deninger elements for CM elliptic curves over an imaginary quadratic field

Here we describe briefly the construction by Beilinson and Deninger [Den1] of the elements in K-theory, which interpret the L-value up to rational numbers as predicted by Beilinson's conjecture.

We are interested in the L-values $L(\overline{\psi}, k+2)$ with $k \geq 0$. Thus according to 1.1.1 we need an element in $H^2_{\mathcal{M}}(E, k+2)$.

1.2.1 The Beilinson–Deninger construction. We fix an algebraic differential $\omega \in H^0(E, \Omega_{E/K})$ and let Γ be its period lattice. Then we have an isomorphism

$$\begin{aligned} E(\mathbb{C}) &\to \mathbb{C}/\Gamma \\ z &\mapsto \int_0^z \omega \end{aligned}$$

using the fixed embedding $K \subset \mathbb{C}$. This isomorphism is equivariant for the action of complex multiplication and because $j(E) = j(\mathcal{O}_K)$ the lattice is of the form $\Gamma = \Omega\mathcal{O}_K$ for some $\Omega \in \mathbb{C}^*$. Fix an $\mathcal{O}_K$ generator $\gamma \in H_1(E(\mathbb{C}), \mathbb{Z})$, then

$$\Omega = \int_\gamma \omega.$$

Recall that $\mathfrak{f}$ is the conductor of ψ and the locus of bad reduction of E. Let $\mathbb{Z}[E[\mathfrak{f}] \setminus 0]$ be the group of divisors with support in the $\mathfrak{f}$-torsion points without 0 of E, defined over K. Beilinson defines a map:

Theorem 1.2.1 ([Be2]). *There is a non-zero map, a variant of the Eisenstein symbol,*

$$\mathbb{Z}[E[\mathfrak{f}] \setminus 0] \xrightarrow{\mathcal{E}^{2k+1}_{\mathcal{M}}} H^{2k+2}_{\mathcal{M}}(E^{2k+1}, 2k+2),$$

where $E^n := E \times_K \ldots \times_K E$.

Deninger constructs a projector

$$\mathcal{K}_{\mathcal{M}} : H^{2k+2}_{\mathcal{M}}(E^{2k+1}, 2k+2) \to H^2_{\mathcal{M}}(E, k+2) \tag{1}$$

as follows: Let d_K be the discriminant of K and $\sqrt{d_K}$ be a square root of d_K. Complex multiplication gives a map

$$\delta = (\mathrm{id}, \vartheta(\sqrt{d_K})) : E \to E \times_K E$$

and taking this k-times gives $\delta^k \times \mathrm{id} : E^k \times_K E \to E^{2k} \times_K E$. Then

$$\mathcal{K}_{\mathcal{M}} = \mathrm{pr}_* \circ (\delta^k \times \mathrm{id})^*$$

where pr is the projection $E^k \times_K E \to E$ onto the last component. Hence we get a map

$$\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}^{2k+1}_{\mathcal{M}} : \mathbb{Z}[E[\mathfrak{f}] \setminus 0] \to H^2_{\mathcal{M}}(E, k+2).$$

1.2.2 The Beilinson conjecture for CM elliptic curves. Following Deninger we define an element β in $\mathbb{Z}[E[\mathfrak{f}] \setminus 0]$. Let $K(\mathfrak{f})$ be the ray class field associated to $\mathfrak{f}$ and note that $K(\mathfrak{f}) = K(E[\mathfrak{f}])$.

Let f be a generator of $\mathfrak{f}$. Then

$$\Omega f^{-1} \in \mathfrak{f}^{-1}\Gamma \tag{2}$$

defines an element in $E[\mathfrak{f}](K(\mathfrak{f}))$. This gives a divisor (Ωf^{-1}) in $\mathbb{Z}[E[\mathfrak{f}] \setminus 0]$ defined over $K(\mathfrak{f})$ on which the Galois group $\mathrm{Gal}(K(\mathfrak{f})/K)$ acts. We define:

$$\beta := \mathrm{N}_{K(\mathfrak{f})/K}((\Omega f^{-1})).$$

This is a divisor defined over K. Recall that γ is an $\mathcal{O}_K$ generator of $H_1(E(\mathbb{C}), \mathbb{Z})$. By Poincaré duality we have an isomorphism (conjugate linear for the $\mathcal{O}_K$-action)

$$H^1(E(\mathbb{C}), \mathbb{Z}(k+1)) \cong \mathrm{Hom}(H^1(E(\mathbb{C}), \mathbb{Z}), \mathbb{Z}(k)) = H_1(E(\mathbb{C}), \mathbb{Z}(k)).$$

Denote by η the $\mathcal{O}_K$ generator of $H^1(E(\mathbb{C}), \mathbb{Z}(k+1))$ corresponding to $(2\pi i)^k \gamma$ under this isomorphism. We can now formulate the main result of [Den1] in our case:

Theorem 1.2.2 ([Den1] Thm. 11.3.2). *Let β and η be as above and define*

$$\xi := (-1)^{k-1} \frac{(2k+1)!}{2^{k-1}} \frac{L_p(\overline{\psi}, -k)^{-1}}{\psi(f)\, \mathrm{N}_{K/\mathbb{Q}}\, \mathfrak{f}^k} \mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}^{2k+1}(\beta) \in H^2_{\mathcal{M}}(E, k+2),$$

where $L_p(\overline{\psi}, -k)$ is the Euler factor of $\overline{\psi}$ at p, evaluated at $-k$. Then

$$r_{\mathcal{D}}(\xi) = L_S^*(\overline{\psi}, -k)\eta \in H^1\big(E \times_K \mathbb{C}, (2\pi i)^{k+1}\mathbb{R}\big),$$

where $L_S^*(\overline{\psi}, -k) = \lim_{s \to -k} \frac{L_S(\overline{\psi}, s)}{s+k}$.

Note that $L_{\mathfrak{p}}(\overline{\psi}, -k) = 1$, if $\mathfrak{p} | \mathfrak{f}$. We can now define the space $\mathcal{R}_\psi$ of the Main theorem 1.1.5

Definition 1.2.3. *We define*

$$\mathcal{R}_\psi := \xi \mathcal{O}_K \subset H^2_{\mathcal{M}}(E, k+2)$$

to be the $\mathcal{O}_K$-submodule of $H^2_{\mathcal{M}}(E, k+2)$ generated by ξ.

Note that by the above theorem, $\mathcal{R}_\psi$ is an $\mathcal{O}_K$-module of rank 1.

Corollary 1.2.4. *With the above notation*

$$r_{\mathcal{D}}(\det_{\mathbb{Z}}(\mathcal{R}_\psi)) = L_S^*(E/K, -k) \det_{\mathbb{Z}}(H^1(E(\mathbb{C}), \mathbb{Z}(k+1))).$$

where S is the set of primes in K dividing $p\mathfrak{f}$.

Proof. This follows from the theorem and the remark that if we multiply an $\mathcal{O}_K$-module with an element $L_S^*(\overline{\psi}, -k)$ from $\mathcal{O}_K \otimes \mathbb{R}$, then the determinant is multiplied by the norm $\mathrm{N}_{\mathcal{O}_K \otimes \mathbb{R}/\mathbb{R}}(L_S^*(\overline{\psi}, -k))$. But $\mathrm{N}_{\mathcal{O}_K \otimes \mathbb{R}/\mathbb{R}}(L_S^*(\overline{\psi}, -k)) = L_S^*(E/K, -k)$. □

1.2.3 The space $r_p(\mathcal{R}_\psi)$ in terms of the specialization of the elliptic polylog. Recall from Definition 1.2.3 that the space $\mathcal{R}_\psi$ is generated as an $\mathcal{O}_K$-module by the element ξ from Theorem 1.2.2. The element ξ is up to some factors of the form $\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}^{2k+1}(\beta)$. Let us define

$$t := \Omega f^{-1}$$

with the notation from (2). This is an $\mathrm{N}_{K/\mathbb{Q}}\, \mathfrak{f}$-division point. Then we have $\beta = \mathrm{N}_{K(\mathfrak{f})/K}((t))$. Now let

$$(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_p})^{2k+1} \in H^1\big(\mathcal{O}_S, \mathrm{Sym}^{2k+1} T_p E \otimes \mathbb{Q}_p(1)\big)$$

be the specialization of the polylogarithm as defined in 3.5.9. Note that $\mathcal{H}_{\mathbb{Q}_p} = T_p E \otimes \mathbb{Q}_p$ in the notation in loc. cit. We have the following comparison theorem:

Theorem 1.2.5. *There is an equality*

$$r_p\big(\mathcal{E}_{\mathcal{M}}^{2k+1}(\beta)\big) = -\mathrm{N}\,\mathfrak{f}^{4k+2}(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_p})^{2k+1}$$

in $H^1(\mathcal{O}_S, \mathrm{Sym}^{2k+1} T_pE \otimes \mathbb{Q}_p(1))$.

Proof. The formula is the combination of two results: Theorem 2.2.4 in [Hu-Ki2], which states that

$$r_p(\mathcal{E}is_{\mathcal{M}}^{2k+1}(\varrho\beta)) = -\mathrm{N}\,\mathfrak{f}^{2k}(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_p})^{2k+1}$$

where $\mathcal{E}is_{\mathcal{M}}^{2k+1}$ is Beilinson's Eisenstein symbol and ϱ the horospherical map. Note that what is here called $(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_p})^{2k+1}$ is in loc. cit. $(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_p})^{2k+2}$. Furthermore, according to [Den2] Formula 3.35.,

$$\mathcal{E}_{\mathcal{M}}^{2k+1}(\beta) = \mathrm{N}\,\mathfrak{f}^{2k+2}\mathcal{E}is_{\mathcal{M}}^{2k+1}(\varrho\beta).$$

Note that the Formula 3.35. in [Den2] uses an other normalization of the horospherical map and that there is a factor $\mathrm{N}\,\mathfrak{f}$ missing because of a wrong normalization of the residue map (the residue of $\frac{dq}{q}$ in formula (3.7) in loc. cit. is not 1 but N). □

Remark: In fact it is not necessary to use this comparison theorem. The better approach would be to use the K-theoretic polylogarithm classes to show Deninger's result. Of course nothing new happens in this approach and we use this comparison result to avoid the lengthy computations.

We have now two tasks: To compute the specialization of the elliptic polylog and to identify the étale cohomology groups to compute the "index" of $r_p(\mathcal{R}_\psi)$ in $H^1(\mathcal{O}_S, T_pE(k+1))$. The answer to these problems involves elliptic units and we will start to use Iwasawa theory and Rubin's proof of the main conjecture to identify the étale cohomology group $H^1(\mathcal{O}_S, T_pE(k+1))$ or rather the complex $R\Gamma(\mathcal{O}_S, T_pE(k+1))$.

2 Iwasawa theory

This section treats the relation between certain Iwasawa modules and étale cohomology. Rubin's "main conjecture" is used in an essential way. We first review the results of Rubin and then use an idea of Soulé to produce elements in étale cohomology using elliptic units. An idea of Kato, which partly goes back to Soulé as well, allows to compare the elliptic units and étale cohomology. The main result of this part is Theorem 2.2.12.

2.1 Review of the "main conjecture" of Iwasawa theory for CM elliptic curves

In this section we review the "main conjecture" of Iwasawa theory for CM elliptic curves proved by Rubin [Ru3]. This will be used in Sect. 2.2 to reduce the Tamagawa number conjecture to an "index computation".

2.1.1 Definition of the Iwasawa modules. We follow Rubin [Ru3]: Let E/K be as before an elliptic curve with CM by $\mathcal{O}_K$, K an imaginary quadratic field. We fix an embedding of K into $\mathbb{C}$ and view K as a subfield of $\mathbb{C}$. Fix a prime $p \nmid \#\mathcal{O}_K^*$ and a prime $\mathfrak{p}$ of $\mathcal{O}_K$ lying over p and denote by $E[p^n]$ the p^n-torsion points of E. Let $K_n := K(E[p^{n+1}])$ be the extension field defined by these torsion points and $K_\infty := \varinjlim_n K_n$. Denote the ring of integers in these fields by $\mathcal{O}_n$ (resp. $\mathcal{O}_\infty$). Then $\Delta := \mathrm{Gal}(K_0/K)$ has order prime to p and $\Gamma := \mathrm{Gal}(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p^2$. Let $\mathcal{G} := \mathrm{Gal}(K_\infty/K)$ be the Galois group of the extension K_∞/K. Then $\mathcal{G} \cong \Delta \times \Gamma$. Define $\mathcal{A}_n$ to be the p-part of the ideal class group of K_n, $\mathcal{E}_n$ to be the group of global units $\mathcal{O}_n^*$ of K_n and $\mathcal{U}_n^{\mathfrak{p}}$ the local units of $K_n \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 modulo the primes above $\mathfrak{p}$. For every prime v of K_n above $\mathfrak{p}$ there is an exact sequence

$$1 \to \mathcal{U}_{n,v} \to K_{n,v}^* \to \mathbb{Z} \times \kappa_n^* \to 1 \tag{3}$$

and $\mathcal{U}_n^{\mathfrak{p}} = \bigoplus_{v|\mathfrak{p}} \mathcal{U}_{n,v}$. Here $\mathcal{U}_{n,v}$ are the local units congruent to 1 modulo v and κ_n is the residue class field of $K_{n,v}$. Let $\mathcal{C}_n$ be the elliptic units in K_n as defined in [Ru3] paragraph 1. We recall their definition. For every ideal $\mathfrak{a} \in \mathcal{O}_K$ prime to 6 consider the function $\theta_{\mathfrak{a}}(z)$ that will be defined in 4.2.2. The function $\theta_{\mathfrak{a}}(z)$ is a 12-th root of the function in [deSh] II.2.4.

Let $t := \Omega f^{-1}$ and $\mathfrak{a}$ be an ideal prime to $6\mathfrak{f}$.

Definition 2.1.1. *(cf. [Ru4] 11.2) Let C_n be the subgroup of units generated over $\mathbb{Z}[\mathrm{Gal}(K_n/K)]$ by*

$$\prod_{\sigma \in \mathrm{Gal}(K(\mathfrak{f})/K)} \theta_{\mathfrak{a}}(t^\sigma + h_n),$$

where $\mathfrak{a}$ runs through all ideals prime to $6p\mathfrak{f}$, $K(\mathfrak{f})$ is the ray class field defined by $\mathfrak{f}$ and h_n is a primitive p^n-torsion point. Define

$$\mathcal{C}_n := \mu(K_n) C_n,$$

the group of elliptic units *of K_n. Here $\mu(K_n)$ are the roots of unity in K_n.*

Denote by $\overline{\mathcal{E}}_n$ and $\overline{\mathcal{C}}_n$ the closures of $\mathcal{E}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ resp. $\mathcal{C}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ in $\mathcal{U}_n^{\mathfrak{p}}$. Finally define

$$\mathcal{A}_\infty := \varprojlim_n \mathcal{A}_n, \quad \overline{\mathcal{E}}_\infty := \varprojlim_n \overline{\mathcal{E}}_n, \quad \overline{\mathcal{C}}_\infty := \varprojlim_n \overline{\mathcal{C}}_n, \quad \mathcal{U}_\infty^{\mathfrak{p}} := \varprojlim_n \mathcal{U}_n^{\mathfrak{p}}$$

where the limits are taken with respect to the norm maps. Denote by $M_\infty^{\mathfrak{p}}$ the maximal abelian p-extension of K_∞ which is unramified outside of the primes above $\mathfrak{p}$, and write $\mathcal{X}_\infty^{\mathfrak{p}} := \mathrm{Gal}(M_\infty^{\mathfrak{p}}/K_\infty)$. Global class field theory gives an exact sequence

$$0 \to \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{U}_\infty^{\mathfrak{p}}/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty^{\mathfrak{p}} \to \mathcal{A}_\infty \to 0. \tag{4}$$

Define the Iwasawa algebra

$$\mathbb{Z}_p[[\mathcal{G}]] := \varprojlim_n \mathbb{Z}_p[[\mathrm{Gal}(K_n/K)]]$$

this has an action of $\mathbb{Z}_p[\Delta]$. For any irreducible $\mathbb{Z}_p$-representation χ of Δ, let

$$e_\chi := \frac{1}{\#\Delta}\sum_{\tau\in\Delta}\mathrm{Tr}(\chi(\tau))\tau^{-1} \in \mathbb{Z}_p[\Delta]$$

and for every $\mathbb{Z}_p[\Delta]$-module Y let $Y^\chi := e_\chi Y$ be the χ-isotypical component. In particular we define

$$\Lambda^\chi := \mathbb{Z}_p[[\mathcal{G}]]^\chi = R_\chi[[\Gamma]]$$

where R_χ is the ring of integers in the unramified extension of $\mathbb{Z}_p$ of degree $\dim(\chi)$. As we will work with $\mathbb{Z}_p[[\Gamma]] \otimes \mathcal{O}_p$-modules, we let

$$\Lambda := \mathcal{O}_p[[\Gamma]].$$

Then $\overline{\mathcal{E}}^\chi_\infty$, $\overline{\mathcal{C}}^\chi_\infty$, $\mathcal{U}^{\mathfrak{p}\ \chi}_\infty$, $\mathcal{A}^\chi_\infty$ and $\mathcal{X}^\chi_\infty$ are finitely generated Λ^χ-modules (see [Ru3] paragraph 5). The modules $\mathcal{A}^\chi_\infty$ and $\overline{\mathcal{E}}^\chi_\infty/\overline{\mathcal{C}}^\chi_\infty$ are even torsion Λ^χ-modules.

2.1.2 Rubin's "main conjecture" for imaginary quadratic fields. We have the following observation due to Kato:

Lemma 2.1.2 (see [Ka2] Proposition 6.1.). *Let Y be a finitely generated torsion Λ^χ-module. Then*

$$\det_{\Lambda^\chi}(Y) = \mathrm{char}(Y)^{-1},$$

where char(Y) *is the usual characteristic ideal in Iwasawa theory (see e.g. [Ru3] paragraph 4) and the determinant is taken in the sense of [Kn-Mu].*

With this lemma we can formulate the main result of [Ru3] as follows:

Theorem 2.1.3 ([Ru3] theorem 4.1.). *Let $p \nmid \#\mathcal{O}_K^*$.*

i) Suppose that p splits in K, then

$$\det_{\Lambda^\chi}\left(\mathcal{A}^\chi_\infty\right) = \det_{\Lambda^\chi}\left(\overline{\mathcal{E}}^\chi_\infty/\overline{\mathcal{C}}^\chi_\infty\right).$$

ii) Suppose that p remains prime or ramifies in K and that χ is nontrivial on the decomposition group of $\mathfrak{p}$ in Δ, then

$$\det_{\Lambda^\chi}\left(\mathcal{A}^\chi_\infty\right) = \det_{\Lambda^\chi}\left(\overline{\mathcal{E}}^\chi_\infty/\overline{\mathcal{C}}^\chi_\infty\right).$$

Using the theory of the determinant and the exact sequence (4) we get:

Corollary 2.1.4. *In the situation of the Theorem 2.1.3,*

$$\det_{\Lambda^\chi}\big(\mathcal{X}_\infty^{\mathfrak{p}\ \chi}\big) = \det_{\Lambda^\chi}\big(\mathcal{U}_\infty^{\mathfrak{p}\ \chi}/\overline{\mathcal{C}}_\infty^\chi\big).$$

We need a variant of this. Let $\mathcal{X}_\infty$ be the Galois group of the maximal abelian p-extension M_∞^p of K_∞ which is unramified outside of the primes above p. In the case where p is inert or ramified this is the same as $\mathcal{X}_\infty^{\mathfrak{p}}$. Define also

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$$

if $p = \mathfrak{p}\mathfrak{p}^*$ is split, and

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}}$$

if p is inert or ramified. Let similarly $\mathcal{Y}_n$ be the p-adic completion of $(K_n \otimes \mathbb{Q}_p)^*$ and $\mathcal{Y}_\infty := \varprojlim_n \mathcal{Y}_n$. We have an inclusion $\mathcal{U}_\infty \subset \mathcal{Y}_\infty$. Class field theory gives

$$0 \to \overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{U}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty \to \mathcal{A}_\infty \to 0. \tag{5}$$

where $\overline{\mathcal{C}}_\infty$ is diagonally embedded into $\mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$ if p is split. On this sequence acts $\mathcal{G}$ and we get:

Corollary 2.1.5. *In the situation of the Theorem 2.1.3,*

$$\det_{\Lambda^\chi}\big(\mathcal{X}_\infty{}^\chi\big) = \det_{\Lambda^\chi}\big(\mathcal{U}_\infty{}^\chi/\overline{\mathcal{C}}_\infty^\chi\big).$$

Lemma 2.1.6. *Let $p \nmid \mathrm{N}\mathfrak{f}$ be a prime. If p splits in K, the inclusion $\mathcal{U}_\infty \to \mathcal{Y}_\infty$ is an isomorphism and if p is inert or ramified in K, there is an exact sequence*

$$0 \to \mathcal{U}_\infty \to \mathcal{Y}_\infty \to \mathbb{Z}_p[\Delta/\Delta_p] \to 0,$$

where Δ_p is the decomposition group of p in $\Delta = \mathrm{Gal}(K_0/K)$.

Proof. We have exact sequences

$$1 \to \mathcal{U}_{n,v} \to K_{n.v}^* \to \mathbb{Z} \times \kappa_n^* \to 1$$

where κ_n is the residue class field of $K_{n.v}$. By definition $\mathcal{U}_n = \bigoplus_{v|p} \mathcal{U}_{n,v}$.

As the order of the residue class field κ_n^* is prime to p, we have an exact sequence

$$0 \to \varprojlim_n \mathcal{U}_{n,v}/p^n \to \varprojlim_n K_{n,v}^*/p^n \to \mathbb{Z}_p \to 0.$$

As E has good reduction at p, we now how p decomposes in K_n (see [Ru1] Prop. 3.6). If p is split, the ramification degree of v in K_{n+1} is p and the degree of K_{n+1} over K_n is p^2. Hence the norm map induces multiplication by p on $\mathbb{Z}_p$ and the inverse limit over these maps is zero. This gives the first claim. In the case where p is inert of ramified, v is totally ramified in K_{n+1} and the norm map from $K_{n',v}^* \to K_{n,v}^*$ induces the identity on $\mathbb{Z}_p$. Putting these sequences together for all $v|p$ and using $\bigoplus_{v|p} \mathbb{Z}_p = \mathbb{Z}_p[\Delta/\Delta_p]$ gives the result. □

2.2 Reductions via Iwasawa theory

In this section we use Rubin's "main conjecture" of Iwasawa theory to reduce the Bloch-Kato conjecture to a comparison between the space $\mathcal{R}_\psi$ of 1.2.3 and the elliptic units $\overline{\mathcal{C}}_\infty$.

We have a subspace

$$r_p(\mathcal{R}_\psi) \subset H^1(\mathcal{O}_S, V_p)$$

where $\mathcal{O}_S = \mathcal{O}_K[\frac{1}{S}]$ and $V_p = T_pE(k+1) \otimes \mathbb{Q}_p$. Recall that S is the set of primes of K dividing $p\mathfrak{f}$. We want to compute the relation of the submodule $r_p(\mathcal{R}_\psi)$ to $H^1(\mathcal{O}_S, T_pE(k+1))$. Our method, which is inspired by Kato's paper [Ka2], relates $H^1(\mathcal{O}_S, T_pE(k+1))$ to a certain submodule defined by the elliptic units $\overline{\mathcal{C}}_\infty$. This submodule in turn is defined using an idea of Soulé. Our aim is to relate the determinant of $\overline{\mathcal{C}}_\infty \otimes T_pE(k)$ to the determinant of $R\Gamma(\mathcal{O}_S, T_pE(k+1))$ (see Theorem 2.2.12 for the exact formulation).

In this section p is always a prime which does not divide $\#\mathcal{O}_K^*$ and where E has good reduction over the primes above p, i.e. $p \nmid \mathrm{N}\,\mathfrak{f}$.

Denote by abuse of notation by S_p the set of primes over p in the ring $\mathcal{O}_n$ for every n and by $\mathcal{O}_{n,S_p}$ the ring of integers in K_n where the primes above p are inverted. We define $\mathcal{O}_{\infty,S_p} := \varinjlim_n \mathcal{O}_{n,S_p}$. Similarly we define $\mathcal{O}_{n,S}$ and $\mathcal{O}_{\infty,S}$.

2.2.1 Review of the Soulé elements. We keep the notations from the Sect. 2.1.

Recall that $T_pE = \varprojlim_n E[p^n]$ the Tate module of E and let $T_pE(k) := T_pE \otimes \mathbb{Z}_p(k)$ its Tate twist. This is a $\mathcal{O}_p[[\mathcal{G}]]$-module. We start by defining a map in the spirit of Soulé

$$\overline{\mathcal{C}}_\infty \otimes_{\mathbb{Z}_p} T_pE(k) \to H^1(\mathcal{O}_S, T_pE(k+1)).$$

Here $T_pE(k+1)$ is a sheaf on $\mathcal{O}_S$ because it is unramified outside of S. Write

$$H^1(\mathcal{O}_S, T_pE(k+1)) = \varprojlim_r H^1\big(\mathcal{O}_S, E[p^{r+1}](k+1)\big)$$

and let for a norm compatible system of elliptic units $(\theta_r)_r$ and an element $(t_r)_r \in T_pE(k)$

$$e_p((\theta_r \otimes t_{r+1})_r) := (\mathrm{N}_{K_r/K}(\theta_r \otimes t_{r+1}))_r$$

where $\theta_r \otimes t_{r+1}$ is an element in

$$\mathcal{O}_{r,S}^*/(\mathcal{O}_{r,S}^*)^{p^{r+1}} \otimes E[p^{r+1}](k) \subset H^1(\mathcal{O}_{r,S}, E[p^{r+1}](k+1))$$

(the inclusion comes from Kummer theory) and $\mathrm{N}_{K_r/K}$ is the norm map on the cohomology. According to Soulé ([So3] Lemma 1.4) this gives an element in $H^1(\mathcal{O}_S, T_pE(k+1))$. The map e_p factors through the coinvariants under $\mathcal{G}$, so that we can make the following definition:

Definition 2.2.1. *The* Soulé elliptic elements *are defined by the map*

$$e_p : (\overline{\mathcal{C}}_\infty \otimes T_pE(k))_{\mathcal{G}} \to H^1(\mathcal{O}_S, T_pE(k+1)).$$

We want to investigate to what extend this map gives generators for $H^1(\mathcal{O}_S, T_pE(k+1))$, i.e. what is the kernel and cokernel of the map e_p.

2.2.2 The Poitou-Tate localization sequence. Our main tool in describing $H^1(\mathcal{O}_S, T_pE(k+1))$ in terms of elliptic units will be the Poitou-Tate localization sequence. It is convenient for us to write down a derived category version of it.

For technical reasons we have to work first over K_0. The reason for this is that over K_0 the module $T_pE(k+1)$ is unramified outside of the primes above p:

Lemma 2.2.2. *([Ru1]1.3) If $p \nmid \#\mathcal{O}_K^*$, then over K_0 the elliptic curve E has good reduction at all places not dividing p. In particular there exists a model of E over $\mathcal{O}_{0,S_p}$ and $T_pE(k+1)$ is unramified.*

The localization sequence now reads as follows (see [Ka2] (6.3)). Here * is the Pontryagin dual $\mathrm{Hom}_{\mathcal{O}_p}(\_, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$.

$$\begin{aligned} R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)) \to & R\Gamma(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k))^*[-2] \\ & \to R\Gamma(\mathcal{O}_{0,S_p}, E[p^\infty](-k))^*[-2] \to, \end{aligned}$$

where we have used the identification

$$E[p^\infty](-k) = \mathrm{Hom}_{\mathcal{O}_p}(T_pE(k+1), \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \mathcal{O}_p).$$

Our next task is to rewrite this Poitou-Tate sequence in terms of Iwasawa theory.

2.2.3 Identification of some Galois cohomology groups with Iwasawa modules. Let us define

$$\begin{aligned} H^1\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big) :&= \varinjlim_n H^1\big(K_n \otimes \mathbb{Q}_p, E[p^\infty](-k)\big) \\ &= \varinjlim_n \bigoplus_{v|p} H^1\big(K_{n,v}, E[p^\infty](-k)\big). \end{aligned}$$

Note that there are only finitely many primes above p in K_∞.

Proposition 2.2.3. *There are isomorphisms of $\mathcal{O}_p[[\mathcal{G}]]$-modules*

$$\begin{aligned} \mathcal{X}_\infty \otimes_{\mathbb{Z}_p} T_pE(k) &\cong H^1\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)^* \\ \mathcal{Y}_\infty \otimes_{\mathbb{Z}_p} T_pE(k) &\cong H^1\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*, \end{aligned}$$

where * is the Pontryagin dual $\mathrm{Hom}_{\mathcal{O}_p}(\_, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$.

Proof. We have

$$\begin{aligned} H^1\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)^* &= \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K_\infty), E[p^\infty](-k))^* \\ &= \mathrm{Hom}\left(\mathrm{Gal}\left(M_\infty^p/K_\infty\right), E[p^\infty](-k)\right)^* \\ &= \mathcal{X}_\infty \otimes T_pE(k). \end{aligned}$$

In the local case, we have an isomorphism

$$\begin{aligned} H^1\big(K_n \otimes \mathbb{Q}_p, E[p^n](-k)\big) &= \bigoplus_{v|p} H^1\big(K_{n,v}, E[p^n](-k)\big) \\ &= \bigoplus_{v|p} \mathrm{Hom}\left(\mathrm{Gal}(\overline{K}_{n,v}/K)^{\mathrm{ab}}, E[p^n](-k)\right) \end{aligned}$$

By class field theory

$$\mathrm{Hom}\left(\mathrm{Gal}(\overline{K}_{n,v}/K)^{\mathrm{ab}}, E[p^n](-k)\right)^* \cong K_{n,v}^*/p^n \otimes E[p^n](k)$$

so that

$$H^1\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* = \bigoplus_{v|p} \mathcal{Y}_\infty \otimes T_pE(k).$$

□

2.2.4 Rewriting the Poitou-Tate localization sequence in terms of Iwasawa theory. To proceed further, we need the following vanishing result.

Proposition 2.2.4. *The groups*

$$H^2\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big) \quad \textit{and} \quad H^2\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)$$

are zero.

Proof. By local duality we have

$$H^2\big(K_n \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* \cong H^0\big(K_n \otimes \mathbb{Q}_p, T_pE(k+1)\big) = 0.$$

On the other hand it is a result of Soulé [So1] that the cohomology groups $H^2(\mathcal{O}_{n,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(m))$ are zero for $m > 1$. Hence $H^2(\mathcal{O}_{\infty,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(m)) = 0$ and because over K_∞ the module $\mathbb{Q}_p/\mathbb{Z}_p(m)$ is trivial,

$$H^2(\mathcal{O}_{\infty,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(-k)) = 0.$$

Now we can write

$$H^2\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big) = H^2\big(\mathcal{O}_{\infty,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(-k)\big) \otimes_{\mathbb{Z}_p} T_pE$$

which proves our claim. □

This vanishing result implies that we get actually a map from the Iwasawa modules to complexes computing the Galois cohomology.

Corollary 2.2.5. *There are exact triangles*

$$\begin{aligned}\mathcal{Y}_\infty \otimes T_pE(k)[1] &\to R\Gamma\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* \\ &\to H^0\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* \\ \mathcal{X}_\infty \otimes T_pE(k)[1] &\to R\Gamma\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)^* \\ &\to H^0\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)^*\end{aligned}$$

Proof. The Propositions 2.2.3 and 2.2.4 show that we have a canonical map from $\mathcal{Y}_\infty \otimes T_pE(k)[1]$ to $R\Gamma(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k))^*$ because the second cohomology vanishes. The same argument gives the result for $\mathcal{X}_\infty \otimes T_pE(k)[1]$. □

To relate these groups to the cohomology groups of $\mathcal{O}_{S_p}$ we want to take the coinvariants under $\Gamma = \mathrm{Gal}(K_\infty/K_0)$.

Lemma 2.2.6. *Let M be an perfect complex of $\Lambda = \mathcal{O}_p[[\Gamma]]$-modules. Then there are canonical isomorphisms*

$$M^* \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \cong R\Gamma(\Gamma, M)^*$$

where the right hand side is the (continuous) group cohomology of Γ and $M^* = \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathcal{O}_p)$.

Proof. We have

$$\begin{aligned}R\,\mathrm{Hom}_\Lambda(\mathcal{O}_p, M^*) &= R\,\mathrm{Hom}_\Lambda(\mathcal{O}_p, \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathcal{O}_p)) \\ &= R\,\mathrm{Hom}_\Lambda(M \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathcal{O}_p))\end{aligned}$$

which by biduality $M^{**} = M$ proves our claim. □

Corollary 2.2.7. *There are exact triangles*

$$\begin{aligned}(\mathcal{Y}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p &\to R\Gamma\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*[-1] \\ &\to R\Gamma\big(\Gamma, H^0\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)\big)^*[-1]\end{aligned}$$

and

$$\begin{aligned}(\mathcal{X}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p &\to R\Gamma\big(\mathcal{O}_{0,S_p}, E[p^\infty](-k)\big)^*[-1] \\ &\to R\Gamma\big(\Gamma, H^0\big(\mathcal{O}_{\infty,S_p}, [p^\infty](-k)\big)\big)^*[-1]\end{aligned}$$

Proof. Apply Lemma 2.2.6 to the exact triangles in Corollary 2.2.5. □

Now we come back to the Poitou-Tate localization sequences over $\mathcal{O}_{0,S_p}$

$$\begin{aligned}R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)) &\to R\Gamma\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*[-2] \\ &\to R\Gamma\big(\mathcal{O}_{0,S_p}, E[p^\infty](-k)\big)^*[-2] \to\end{aligned}$$

and the map e_p. Recall that

$$e_p : \overline{\mathcal{C}}_\infty \otimes T_pE(k) \to H^1(\mathcal{O}_{S_p}, T_pE(k+1))$$

(see Definition 2.2.1) and taking in the definition of e_p only the norm maps to K_0 we get a map:

$$e_p : \overline{\mathcal{C}}_\infty \otimes T_pE(k) \to H^1(\mathcal{O}_{0,S_p}, T_pE(k+1))$$

As $H^0(\mathcal{O}_{0,S_p}, T_pE(k+1)) = 0$ for weight reasons, we get a map of complexes

$$e_p : (\overline{\mathcal{C}}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \to R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1))[1].$$

This is compatible with the maps defined before:

Lemma 2.2.8. *The following diagram is commutative*

$$\begin{array}{ccc}
(\overline{\mathcal{C}}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \xrightarrow{e_p} & R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1))[1] \\
\downarrow & & \downarrow \\
(\mathcal{Y}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \longrightarrow & R\Gamma\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*[-1] \\
\downarrow{\scriptstyle\alpha} & & \downarrow \\
(\mathcal{X}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \longrightarrow & R\Gamma\big(\mathcal{O}_{0,S_p}, E[p^\infty](-k)\big)^*[-1]
\end{array}$$

Here α is induced by the map $\mathcal{Y}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty$.

Proof. The commutativity of the lower square is clear. Let us treat the upper square. The map

$$(\mathcal{Y}_\infty \otimes T_pE(k))_\Gamma \to H^1\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*$$

is the dual of the corestriction

$$H^1\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big) \to H^1\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^\Gamma.$$

By the local duality theorem the corestriction map is dual to the norm map

$$H^1(K_\infty \otimes \mathbb{Q}_p, T_pE(k+1))_\Gamma \to H^1(K_0 \otimes \mathbb{Q}_p, T_pE(k+1)).$$

This together with the definition of e_p proves our claim. □

2.2.5 The comparison theorem between elliptic units and Galois cohomology. The next step is to relate the determinants of $(\overline{\mathcal{C}}_\infty \otimes T_pE(k)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p$ and $R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1))$ as $\mathcal{O}_p[\Delta]$-modules. For this we need Rubin's "main conjecture". As the "main conjecture" is not proven for characters of Δ, which are trivial on the decomposition group Δ_p of p, we need the following lemma:

Lemma 2.2.9. *Let p be inert or ramified, where p is a prime over which E has good reduction. Then $\Delta_p = \Delta$.*

Proof. As E has good reduction, the prime above p is totally ramified in K_0 ([Ru1] 3.6.) and $\Delta \cong (\mathcal{O}/\mathfrak{p})^*$. Thus $\Delta_p = \Delta$ as claimed. □

Lemma 2.2.10. *Let p be inert or ramified, where p is a prime over which E has good reduction. Let χ be the Δ representation on* $\mathrm{Hom}_{\mathcal{O}_p}(T_pE(k), \mathcal{O}_p)$. *Then χ is non trivial on Δ_p.*

Proof. Let p be an inert or ramified prime and χ' be the Δ representation on T_pE. Then χ' is irreducible ([Ru3] 11.5.) and because $\Delta_p = \Delta$ it is non trivial on δ_p. Now χ' is two dimensional and χ is simply a twist of χ' by a power of $\det \chi'$. Thus χ acts non trivially on $\mathrm{Hom}_{\mathcal{O}_p}(T_pE(k), \mathcal{O}_p)$. □

Corollary 2.2.11. *Let χ be the Δ representation on* $\mathrm{Hom}_{\mathcal{O}_p}(T_pE(k), \mathcal{O}_p)$ *and $p \nmid 6\mathrm{N}\mathfrak{f}$ be a prime. Then*

$$\mathcal{U}^\chi_\infty \cong \mathcal{Y}^\chi_\infty.$$

Proof. If p is split this follows immediately from Lemma 2.1.6 and if p is inert or prime in K this follows from the same lemma and the above Lemma 2.2.10 because the χ-eigenspace of $\mathcal{O}_p[\Delta/\Delta_p]$ is zero. □

We can now formulate the main theorem of this section.

Theorem 2.2.12. *Let χ be the Δ-representation on* $\mathrm{Hom}_{\mathcal{O}_p}(T_pE(k), \mathcal{O}_p)$ *and assume that $p \nmid 6\mathrm{N}\mathfrak{f}$. Then the map e_p induces an isomorphism of $\mathcal{O}_p$-modules*

$$\det_{\mathcal{O}_p}\Big(\big(\overline{\mathcal{C}}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k)\big) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\Big) \cong \det_{\mathcal{O}_p}\big(R\Gamma(\mathcal{O}_S, T_pE(k+1))\big)^{-1}.$$

The rest of this section is concerned with the proof of this theorem. Let us first show:

Proposition 2.2.13. *Let χ and p be as in Theorem 2.2.12, then*

$$\det_{\mathcal{O}_p}\big(R\Gamma\big(\mathcal{G}, H^0\big(K_\infty \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)\big)\big) \cong \mathcal{O}_p$$
$$\det_{\mathcal{O}_p}\big(R\Gamma\big(\mathcal{G}, H^0\big(\mathcal{O}_{\infty,S_p}, E[p^\infty](-k)\big)\big)\big) \cong \mathcal{O}_p$$

Proof. The action of $\mathcal{G}$ on $T_p(k) \cong \mathcal{O}_p$ is via a character $\mathcal{G} \to \mathcal{O}_p^*$. This gives a surjection $\mathcal{O}_p[[\Gamma]] \to T_pE(k)$. As $\Gamma \cong \mathbb{Z}_p^2$ the kernel of this surjection is an ideal with height 2 and hence

$$\det_{\mathcal{O}_p[[\mathcal{G}]]}(T_pE(k)) \cong \mathcal{O}_p[[\mathcal{G}]].$$

This implies $\det_{\mathcal{O}_p}(T_pE(k) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p) \cong \mathcal{O}_p$. Lemma 2.2.6 then implies the claim. □

Recall that by Corollary 2.2.11 we have an isomorphism

$$\mathcal{U}^\chi_\infty \cong \mathcal{Y}^\chi_\infty.$$

Corollary 2.2.14. *The triangles in Corollary 2.2.7 give rise to isomorphisms*

$$\begin{aligned}
&\det_{\mathcal{O}_p}\big(\big(\mathcal{U}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k)\big) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big)\\
&\qquad \cong \det_{\mathcal{O}_p}\big(H^0\big(\Delta, R\Gamma\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*[-1]\big)\big)\\
&\det_{\mathcal{O}_p}\big(\big(\mathcal{X}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k)\big) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big)\\
&\qquad \cong \det_{\mathcal{O}_p}\big(H^0\big(\Delta, R\Gamma\big(\mathcal{O}_{0,S_p}, E[p^\infty](-k)\big)^*[-1]\big)\big)
\end{aligned}$$

Proof. The complexes in the triangle in 2.2.7 are $\mathcal{O}_p[\Delta]$-modules and we apply $R\Gamma(\Delta, \_)$. Then

$$R\Gamma(\Delta, \mathcal{Y}_\infty \otimes_{\mathcal{O}_p} T_pE(k)) \cong \mathcal{Y}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k)$$

by definition of χ. The same holds for $\mathcal{X}_\infty \otimes_{\mathcal{O}_p} T_pE(k)$. The result follows with Proposition 2.2.13. □

Corollary 2.2.15. *There is an isomorphism of determinants*

$$\begin{aligned}
&\det_{\mathcal{O}_p}\big(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)))\big)^{-1}\\
&\quad \cong \det_{\mathcal{O}_p}\big(\big(\mathcal{U}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k)\big) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big) \det_{\mathcal{O}_p}\big(\mathcal{X}^\chi_\infty \otimes_{\mathcal{O}_p} T_pE(k) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big)^{-1}
\end{aligned}$$

Proof. Apply $R\Gamma(\Delta, \_)$ to the triangle

$$\begin{aligned}
R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)) &\to R\Gamma\big(K_0 \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^*[-2]\\
&\to R\Gamma\big(\mathcal{O}_{0,S_p}, E[p^\infty](-k)\big)^*[-2]
\end{aligned}$$

and use the above corollary. □

Finally, we need to investigate the relation of the cohomology of $\mathcal{O}_{0,S_p}$ and $\mathcal{O}_{0,S}$, which is the integral closure of $\mathcal{O}_S$ in K_0.

Lemma 2.2.16. *Let p and χ be as in the Theorem 2.2.12. The restriction map of cohomology of $\mathcal{O}_{0,S_p}$ to $\mathcal{O}_{0,S}$ induces an equality of determinants*

$$\begin{aligned}\det_{\mathcal{O}_p}\big(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)))\big)\\ &\cong \det_{\mathcal{O}_p}\big(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S}, T_pE(k+1)))\big)\\ &\cong \det_{\mathcal{O}_p}\big(R\Gamma(\mathcal{O}_S, T_pE(k+1)))\big).\end{aligned}$$

Proof. There is an exact triangle

$$\begin{aligned}R\Gamma(\mathcal{O}_{0,S_p}, T_pE(k+1)) &\to R\Gamma(\mathcal{O}_{0,S}, T_pE(k+1))\\ &\to \bigoplus_{v_0\in S\smallsetminus S_p} R\Gamma_{\kappa(v_0)}(\mathcal{O}_{v_0}, T_pE(k+1))[1]\end{aligned}$$

where $\mathcal{O}_{v_0}$ is the local ring at v_0. As $T_pE(k+1)$ is unramified at the places v_0 in K_0, which are in $S\smallsetminus S_p$ we have by purity

$$R\Gamma_{\kappa(v_0)}(\mathcal{O}_{v_0}, T_pE(k+1)) \cong R\Gamma(\kappa(v_0), T_pE(k)).$$

Let us prove that

$$H^0(\Delta, \bigoplus_{v_0\in S\smallsetminus S_p} R\Gamma(\kappa(v_0), T_pE(k))) = 0.$$

For this note that $H^1(\kappa(v_0), T_pE(k)) \cong T_pE(k)_{\mathrm{Gal}(\overline{\kappa(v_0)}/\kappa(v_0))}$ are the coinvariants and that $H^0 = 0$. Fix a prime $v \in S\smallsetminus S_p$ of K dividing $\mathfrak{f}$, then the primes $v_0|v$ of K_0 are permuted by Δ. Fix v_0 dividing v and let Δ_{v_0} be the stabilizer of v_0. It suffices to prove that Δ_{v_0} acts non trivially on $T_pE(k)_{\mathrm{Gal}(\overline{\kappa(v_0)}/\kappa(v_0))}$. Let $I_{v_0} \subset \Delta_{v_0}$ be the inertia group of v_0. This group is non trivial because K_0/K is ramified above v by Lemma 2.2.2 and it acts non trivially on $T_pE(k)$ because $v_0|\mathfrak{f}$ and by the Neron-Ogg-Shavarevich criterion. This proves our claim. □

Now we can prove the theorem.

Proof of Theorem 2.2.12. Let χ and p be as in the theorem. By Rubin's "main conjecture" 2.1.5 we have

$$\begin{aligned}\det_{\mathcal{O}_p}\big((\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi) \otimes T_pE(k) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big)\\ &\cong \det_\Lambda\big((\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi) \otimes T_pE(k)\big) \otimes_\Lambda \mathcal{O}_p\\ &\cong \det_\Lambda\big(\mathcal{X}_\infty^\chi \otimes T_pE(k)\big) \otimes_\Lambda \mathcal{O}_p\\ &\cong \det_{\mathcal{O}_p}\big(\mathcal{X}_\infty^\chi \otimes T_pE(k) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\big).\end{aligned}$$

On the other hand,

$$\det_{\Lambda^\chi}\big(\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi\big) \cong \det_{\Lambda^\chi}\big(\mathcal{U}_\infty^\chi\big) \otimes \det_{\Lambda^\chi}\big(\overline{\mathcal{C}}_\infty^\chi\big)^{-1}.$$

This together with the above corollaries gives the result. □

We have now achieved a description of $\det_{\mathcal{O}_p} R\Gamma(\mathcal{O}_S, T_pE(k+1))$ in terms of the elliptic Soulé elements. The next step is to investigate the relation of $r_p(\mathcal{R}_\psi)$ with these elements. According to Theorem 1.2.5, this means that we have to compute the specialization of the p-adic polylog. This is the goal of the next two sections.

3 The elliptic polylogarithm sheaf

We start afresh with the aim of computing the specialization of the elliptic polylog. For this we have to recall the definition of the polylogarithm sheaf and give a geometric interpretation of it. Sections 3 and 4 are independent of the rest of the paper and should be of interest to anybody who wants to study the l-adic properties of the elliptic polylog.

We review here mostly Beilinson and Levin [Be-Le]. Everything that follows will be in the general setting of an elliptic curve over any base S. Because of this we start with fixing the notations. Then we review the unipotent elliptic polylog of Beilinson and Levin. For our geometrical construction we need a different description of this polylogarithm sheaf in terms of the fundamental group of the elliptic curve. This description is given in Sect. 3.3. The comparison of these two approaches will be carried out in Sect. 3.4. Finally we consider the specialization of the polylogarithm sheaf at torsion points. This gives the l-adic Eisenstein classes.

3.1 Notations and conventions

Let S be a scheme, and l be a prime number invertible on S. We fix a base ring $\Lambda := \mathbb{Z}/l^r\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$. In this section we introduce some notations for elliptic curves over S and for pro-Λ-sheaves. Note that in our applications we let $l = p$. The notation l for the l-adic theory is for historical reasons.

3.1.1 Elliptic curves and coverings

Definition 3.1.1. *An* elliptic curve *is a smooth proper morphism* $\overline{\pi} : E \to S$ *together with a section* $e : S \to E$*, such that the geometric fibers* $E_{\overline{s}}$ *of* $\overline{\pi}$ *are connected curves of genus* 1.

We introduce the following *notation*: On E we have the multiplication by N map, which we denote by $[N]$. We let $H_n := \ker[l^n]$ and we denote by E_n the curve E over S considered as a H_n-torsor over E. The l^n-multiplication map will then be denoted by $p_n : E_n \to E$. Let $U_n := E_n \setminus H_n$ and $U := E \setminus e(S)$, so that we have a Cartesian diagram

$$\begin{array}{ccccc} H_n & \xrightarrow{h_n} & E_n & \xleftarrow{j_n} & U_n \\ \downarrow{\scriptstyle p_{H_n}} & & \downarrow{\scriptstyle p_n} & & \downarrow \\ S & \xrightarrow{e} & E & \xleftarrow{j} & U. \end{array}$$

The unit section of E_n will be e_n, if confusion is likely. The map $E_m \to E_n$ for $m \geq n$, which is the multiplication by l^{m-n}, is denoted by $p_{m,n}$ or even p. Let $\overline{\pi}_n : E_n \to S$ and $\pi_n : U_n \to S$ be the structure maps.

3.1.2 Pro-sheaves. The polylogarithm is an extension of pro-sheaves and we will work in the category of pro-sheaves. For convenience of the reader we recall the definition and the main properties of pro-objects in the case we need.

Let $\mathcal{A}$ be an abelian category.

Definition 3.1.2. *The category* pro $-\mathcal{A}$ *of* pro-objects *is the category whose objects are projective systems*

$$A : I^{\mathrm{op}} \to \mathcal{A}$$

denoted by $(A_i)_{i\in I}$, where I is some small filtered index category. The morphisms are

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{A}}((A_i), (B_j)) := \varprojlim{}_j \varinjlim{}_i \mathrm{Hom}_{\mathcal{A}}(A_i, B_j).$$

The category pro $-\mathcal{A}$ is again abelian (see [Ar-Ma] A 4.5). We call an object $(A_i)_I \in$ pro $-\mathcal{A}$ *Mittag-Leffler zero* if for every $i \in I$ there is an $i \to j$ such that $A_j \to A_i$ is the zero map. An element is zero in pro $-\mathcal{A}$ if and only if it is Mittag-Leffler zero (see [Ar-Ma] A 3.5). A functor $F : \mathcal{A} \to \mathcal{B}$ is extended to the pro-categories in the obvious way $F((\mathcal{F}_i)_i) := (F(\mathcal{F}_i))_i$.

Let us specialize to the category $\mathrm{Sh}(X)$ of étale sheaves on a scheme X. We denote by pro $-\,\mathrm{Sh}(X)$ the associated category of pro-sheaves as defined above. Pro-sheaves will usually be written as $(\mathcal{F}_i)_i$ the transition maps understood. For two pro-sheaves $(\mathcal{F}_i)_i$ and $(\mathcal{G}_i)_i$ on a scheme X define $\mathrm{Ext}^j_X((\mathcal{F}_i)_i, (\mathcal{G}_i)_i)$ to be the group of j-th Yoneda extensions of $(\mathcal{F}_i)_i$ by $(\mathcal{G}_i)_i$ in pro $-\,\mathrm{Sh}(X)$.

3.2 *Review of the elliptic polylogarithm*

We first recall the definition of the elliptic logarithm sheaf from [Be-Le]. Then we define the elliptic polylogarithmic sheaf.

3.2.1 The unipotent logarithm sheaf. Recall that Λ is either $\mathbb{Z}/l^r\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$.

Definition 3.2.1. *A lisse Λ-sheaf $\mathcal{F}$ on E is unipotent of length n, if it admits a filtration $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^n \supset 0$ such that* $\mathrm{Gr}^i\,\mathcal{F} = \overline{\pi}^*\mathcal{G}^i$ *for some lisse Λ-sheaf $\mathcal{G}^i$ on S.*

Let $\mathcal{H}_\Lambda := \underline{\mathrm{Hom}}_S(R^1\overline{\pi}_*\Lambda, \Lambda)$, then the boundary map for $R\overline{\pi}_*$ applied to the exact sequence $0 \to \mathrm{Gr}^{i+1}\mathcal{F} \to \mathcal{F}^i/\mathcal{F}^{i+2} \to \mathrm{Gr}^i\mathcal{F} \to 0$ induces by duality a map

$$\mathcal{H}_\Lambda \otimes \mathrm{Gr}^i\mathcal{F} \to \mathrm{Gr}^{i+1}\mathcal{F}.$$

This gives an action of the ring $S^{\leq n} := \bigoplus_{k=0}^n \mathrm{Sym}^k \mathcal{H}_\Lambda$ on $\mathrm{Gr}^\bullet\mathcal{F}$. Beilinson and Levin prove:

Theorem 3.2.2 ([Be-Le] 1.2.6.). *There is a k-unipotent sheaf $\mathcal{L}\mathrm{og}^{(k)}$ together with a section $1^{(k)} : \Lambda \to e^*\mathcal{L}\mathrm{og}^{(k)}$ of the fibre at the unit section e of E, such that for every k-unipotent sheaf $\mathcal{F}$ the map*

$$\begin{aligned}\overline{\pi}_*\underline{\mathrm{Hom}}_E(\mathcal{L}\mathrm{og}^{(k)}, \mathcal{F}) &\to e^*\mathcal{F}\\ f &\mapsto f\circ 1^{(k)}\end{aligned}$$

is an isomorphism. The pair $(\mathcal{L}\mathrm{og}^{(k)}, 1^{(k)})$ is unique up to unique isomorphism.

Recall also from [Be-Le] that this is equivalent to the fact that the map $\nu : S^{\leq k} \to \overline{\pi}_*\mathrm{Gr}^\bullet\mathcal{L}\mathrm{og}^{(k)}$ that sends 1 to $1^{(k)}$ is an isomorphism.

Definition 3.2.3. *The canonical maps $\mathcal{L}\mathrm{og}^{(k+1)} \to \mathcal{L}\mathrm{og}^{(k)}$ that map $1^{(k+1)}$ to $1^{(k)}$ make*

$$\mathcal{L}\mathrm{og} := (\mathcal{L}\mathrm{og}^{(k)})_k$$

a pro-sheaf, which is called the logarithm sheaf. *If it is necessary to indicate Λ we write $\mathcal{L}\mathrm{og}^{(k)}_\Lambda$ and $\mathcal{L}\mathrm{og}_\Lambda$.*

Denote by $\mathcal{R}^{(k)} := e^*\mathcal{L}\mathrm{og}^{(k)}$ the fibre of $\mathcal{L}\mathrm{og}^{(k)}$. This is a ring with identity given by $1^{(k)}$. Moreover $\mathcal{R} := e^*\mathcal{L}\mathrm{og}$ has a Hopf algebra structure. The ring $\overline{\pi}^*\mathcal{R}^{(k)}$ acts on $\mathcal{L}\mathrm{og}^{(k)}$ and for every section $t : S \to E$ the sheaf $t^*\mathcal{L}\mathrm{og}^{(k)}$ is a free module of rank 1 over $\mathcal{R}^{(k)}$. The action of $\overline{\pi}^*\mathcal{R}^{(k)}$ on $\mathcal{L}\mathrm{og}^{(k)}$ induces via the isomorphism $\overline{\pi}_*\underline{\mathrm{Hom}}_E(\mathcal{L}\mathrm{og}^{(k)}, \mathcal{F}) \xrightarrow{\sim} e^*\mathcal{F}$ an action of $\mathcal{R}^{(k)}$ on $e^*\mathcal{F}$. In fact we have:

Proposition 3.2.4 ([Be-Le]1.2.10 v)). *The map $\mathcal{F} \mapsto e^*\mathcal{F}$ is an equivalence of the category of k-unipotent sheaves on E with the category of lisse $\mathcal{R}^{(k)}$-modules on S.*

We remark that the inverse functor is $\mathcal{M} \mapsto \overline{\pi}^*\mathcal{M} \otimes_{\overline{\pi}^*\mathcal{R}^{(k)}} \mathcal{L}\mathrm{og}^{(k)}$.

3.2.2 Higher direct images of the logarithm sheaf. Denote by $\mathcal{I}^{(k)}$ the augmentation ideal of the ring $\mathcal{R}^{(k)}$. The pro-sheaves $(\mathcal{R}^{(k)})_k$ and $(\mathcal{I}^{(k)})_k$ are denoted by $\mathcal{R}$ and $\mathcal{I}$ respectively. The most important step in the definition of the polylogarithm is the computation of the higher direct images of $\mathcal{L}\mathrm{og}^{(k)}$.

Proposition 3.2.5 ([Be-Le] 1.2.7). *The higher direct images of* $\mathcal{L}og_\Lambda^{(k)}$ *are*

$$R^i\overline{\pi}_* \mathcal{L}og_\Lambda^{(k)} = \begin{cases} \mathrm{Sym}^k \mathcal{H}_\Lambda & \text{if } i = 0 \\ \mathrm{Sym}^{k+1} \mathcal{H}_\Lambda(-1) & \text{if } i = 1 \\ \Lambda(-1) & \text{if } i = 2 . \end{cases}$$

The transition maps $R^i\overline{\pi}_* \mathcal{L}og^{(k+1)} \to R^i\overline{\pi}_* \mathcal{L}og^{(k)}$ *are zero for* $i = 0, 1$ *and the identity for* $i = 2$. *In particular* $R^i\overline{\pi}_* \mathcal{L}og = 0$ *for* $i = 0, 1$ *and* $R^2\overline{\pi}_* \mathcal{L}og = \Lambda(-1)$.

For all the properties of the logarithm sheaf we refer to Sect. 1.2. in [Be-Le].

Remark: Note that in the case of $\Lambda = \mathbb{Q}_l$ we have an isomorphism $\mathcal{L}og^{(k)} \cong \mathrm{Sym}^k \mathcal{L}og^{(1)}$ which sends $1^{(k)}$ to $1^{(1)k}/k!$. This approach to the logarithm sheaf is used in [Hu-Ki2].

Recall that $U := E \setminus e$ is the complement of the unit section and $\pi : U \to S$ its structure map.

Proposition 3.2.6. *The pro-sheaves* $(R^i\pi_* \mathcal{L}og^{(k)})_k$ *are Mittag-Leffler zero for* $i \neq 1$ *and the canonical map*

$$R^1\pi_* \mathcal{L}og^{(k)} \to e^* \mathcal{L}og^{(k)}(-1) = \mathcal{R}^{(k)}(-1)$$

induces an isomorphism of pro-sheaves $(R^1\pi_* \mathcal{L}og^{(k)}(1))_k \cong (\mathcal{I}^{(k)})_k$.

Proof. Consider the localization sequence

$$\to R^i\overline{\pi}_* \mathcal{L}og^{(k)} \to R^i\pi_* \mathcal{L}og^{(k)} \to R^{i+1}e^! \mathcal{L}og^{(k)} \to$$

and the purity isomorphism $R^2e^! \mathcal{L}og^{(k)} = e^* \mathcal{L}og^{(k)}(-1)$. Moreover $R^ie^! \mathcal{L}og^{(k)} = 0$ for $i \neq 2$. This together with the above computation of $R^i\overline{\pi}_* \mathcal{L}og$ gives the desired result. □

3.2.3 The polylogarithm sheaf. We are now going to define the (unipotent) elliptic polylogarithm. We will not use the usual approach using an identification of an Ext with a Hom-group but an other direct construction due to Beilinson and Levin [Be-Le] 1.3.6. This has the advantage of giving directly a pro-sheaf and not only an extension class. Moreover this sheaf can be easily compared to the geometric construction we give later.

For every sheaf $\mathcal{F}$ on E we denote by $\mathcal{F}_U$ its restriction to U. Let $\mathcal{F}$ be a lisse Λ-sheaf on E and consider the open immersion

$$g : U \times_S U \setminus \Delta \hookrightarrow U \times_S U$$

where Δ is the diagonal. Define a lisse Λ-sheaf $H_e(\mathcal{F})$ on U as follows:

Definition 3.2.7. *Define a functor from lisse Λ-sheaves on E to lisse Λ-sheaves on U by*

$$H_e(\mathcal{F}) := R^1 \operatorname{pr}_{1*} g_! g^* \operatorname{pr}_2^* \mathcal{F}_U,$$

where pr_2 *is the projection of $U \times_S U$ to the second factor.*

The exact sequence

$$0 \to g_! g^* \operatorname{pr}_2^* \mathcal{F}_U \to \operatorname{pr}_2^* \mathcal{F}_U \to \Delta_* \mathcal{F}_U \to 0$$

induces an exact sequence

$$0 \to \pi^* \overline{\pi}_* \mathcal{F} \to \mathcal{F}_U \xrightarrow{\alpha} H_e(\mathcal{F}) \to \pi^* R^1 \pi_* \mathcal{F}_U \to 0. \tag{6}$$

Obviously this sequence is functorial in $\mathcal{F}$ so that we have the same sequence for pro-sheaves $(\mathcal{F}_k)_k$.

Lemma 3.2.8. *The sequence (6) induces an exact sequence of pro-sheaves*

$$0 \to \mathcal{L}\mathrm{og}(1)_U \to H_e(\mathcal{L}\mathrm{og}(1)) \to \pi^* \mathcal{I} \to 0.$$

Proof. By Proposition 3.2.6 we have $\overline{\pi}_* \mathcal{L}\mathrm{og}(1) = 0$ and $R^1 \pi_* \mathcal{L}\mathrm{og}(1) \cong \mathcal{I}$. This implies the claim. □

Thus $H_e(\mathcal{L}\mathrm{og}(1))$ gives a class in $\operatorname{Ext}^1_U(\pi^*\mathcal{I}, \mathcal{L}\mathrm{og}(1)_U)$ which was defined in Sect. 3.1.2 as the group of Yoneda extensions in the abelian category pro $-$ Sh(U). We have $H_e(\mathcal{L}\mathrm{og}(1) \otimes \pi^*\mathcal{R}) \cong H_e(\mathcal{L}\mathrm{og}(1)) \otimes \pi^*\mathcal{R}$ so that the action of $\pi^*\mathcal{R}$ on $\mathcal{L}\mathrm{og}$ gives a $\pi^*\mathcal{R}$-module structure on $H_e(\mathcal{L}\mathrm{og}(1))$. In particular $H_e(\mathcal{L}\mathrm{og}(1))$ is a class in $\operatorname{Ext}^1_{U,\pi^*\mathcal{R}}(\pi^*\mathcal{I}, \mathcal{L}\mathrm{og}(1)_U)$, i.e. a Yoneda extension of $\pi^*\mathcal{R}$-modules.

Definition 3.2.9. *The pro-sheaf*

$$\mathcal{P}\mathrm{ol} := H_e(\mathcal{L}\mathrm{og}(1))$$

is the (unipotent) elliptic polylogarithm sheaf. *If we need to indicate the dependence on Λ we write $\mathcal{P}\mathrm{ol}_\Lambda$. We also define $\mathcal{P}\mathrm{ol}^{(k)} := H_e(\mathcal{L}\mathrm{og}^{(k)}(1))$.*

3.3 A geometric approach to the elliptic polylog sheaf

We now present a different construction of the logarithm sheaf in the case $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$. This makes explicit the remark in [Be-Le] 1.2.5.

3.3.1 The geometric logarithm sheaf. Recall that $p_n = [l^n] : E_n \to E$ and consider the sheaves

$$\mathcal{L}og_n^{\mathrm{g}} := p_{n*}\Lambda$$

on E. For $m \geq n$ we have the trace map $p_{m*}\Lambda \to p_{n*}\Lambda$ and we define:

Definition 3.3.1. *The* geometric logarithm sheaf *is the pro-sheaf*

$$\mathcal{L}og^{\mathrm{g}} := \left(\mathcal{L}og_n^{\mathrm{g}}\right)_n$$

where the transition maps are the above trace maps. Let

$$\mathcal{R}^{\mathrm{g}} := \left(\mathcal{R}_n^{\mathrm{g}}\right)_n := \left(e^* \mathcal{L}og_n^{\mathrm{g}}\right)_n$$

be the pro-sheaf defined by the pull-back of $\mathcal{L}og_n^{\mathrm{g}}$ along the unit section e. Let $\mathcal{I}^{\mathrm{g}} := (\mathcal{I}_n^{\mathrm{g}})_n := \ker(\mathcal{R}^{\mathrm{g}} \to \Lambda)$ be the augmentation ideal of $\mathcal{R}^{\mathrm{g}}$.

Note that the existence of the section e_n of $H_n = p_n^{-1}(e)$ implies that there is a map $1_n : \Lambda \to e^* \mathcal{L}og_n^{\mathrm{g}} = \mathcal{R}_n^{\mathrm{g}}$. The action of H_n on E_n over E gives an action of H_n on $\mathcal{L}og_n^{\mathrm{g}}$, hence an action of $\overline{\pi}^* \mathcal{R}_n^{\mathrm{g}}$ on $\mathcal{L}og_n^{\mathrm{g}}$.

The sheaf $\mathcal{L}og_n^{\mathrm{g}}$ has the following important property. Recall that $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$.

Proposition 3.3.2. *For every lisse Λ-sheaf $\mathcal{F}$ the map*

$$\begin{aligned} \varinjlim{}_n \overline{\pi}_* \underline{\mathrm{Hom}}_E\left(\mathcal{L}og_n^{\mathrm{g}}, \mathcal{F}\right) &\to e^*\mathcal{F} \\ f &\mapsto f \circ 1_n \end{aligned}$$

is an isomorphism.

Proof. The map p_n is finite étale, so that

$$\underline{\mathrm{Hom}}_E(p_{n*}\Lambda, \mathcal{F}) = \underline{\mathrm{Hom}}_{E_n}(\Lambda, p_n^*\mathcal{F}) = p_n^*\mathcal{F}.$$

As $\mathcal{F}$ is a lisse Λ-sheaf, there is an n such that $p_n^*\mathcal{F}$ comes from S, i.e. $p_n^*\mathcal{F} \cong \overline{\pi}_n^* e_n^* p_n^* \mathcal{F}$. Thus

$$\varinjlim{}_n \overline{\pi}_* \underline{\mathrm{Hom}}_E\left(\mathcal{L}og_n^{\mathrm{g}}, \mathcal{F}\right) = \varinjlim{}_n \overline{\pi}_{n*}\overline{\pi}_n^* e_n^* p_n^* \mathcal{F} = \varinjlim{}_n e_n^* p_n^* \mathcal{F} = e^*\mathcal{F},$$

which proves our claim. □

Let $\mathcal{F}$ be a lisse Λ-sheaf, then the action of $\mathcal{R}_n^{\mathrm{g}}$ on $\mathcal{L}og_n^{\mathrm{g}}$ induces via the above proposition an action of $\mathcal{R}_n^{\mathrm{g}}$ on $e^*\mathcal{F}$ for some n.

Corollary 3.3.3. *The functor $\mathcal{F} \mapsto e^*\mathcal{F}$ induces an equivalence of the category of lisse Λ-modules on E and lisse Λ-modules on S with a continuous action of the pro-sheaf $(\mathcal{R}_n^{\mathrm{g}})_n$ (i.e. the action factors through $\mathcal{R}_m^{\mathrm{g}}$ for some m).*

Proof. The inverse functor is given by

$$\mathcal{M} \mapsto \overline{\pi}^*\mathcal{M} \otimes_{\overline{\pi}^*\mathcal{R}_m^{\mathrm{g}}} \mathcal{L}og_m^{\mathrm{g}}$$

if the action of $(\mathcal{R}_n^{\mathrm{g}})_n$ factors through $\mathcal{R}_m^{\mathrm{g}}$. □

3.3.2 The higher direct images of the geometric logarithm sheaf. As for $\mathcal{L}\mathrm{og}$ we can compute the higher direct images of $\mathcal{L}\mathrm{og}^{\mathrm{g}}$:

Lemma 3.3.4. *The pro-sheaf*

$$\left(R^i\overline{\pi}_* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n\right)_n$$

is Mittag-Leffler zero for $i \neq 2$ and

$$\left(R^2\overline{\pi}_* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n\right)_n \cong \Lambda(-1).$$

Proof. We have to compute the transition maps in

$$\left(R^i\overline{\pi}_* p_{n*}\Lambda\right)_n = \left(R^i\overline{\pi}_{n*}\Lambda\right)_n \cong \left(R^i\overline{\pi}_*\Lambda\right)_n$$

where in the last term the transition maps $R^i\overline{\pi}_*\Lambda \to R^i\overline{\pi}_*\Lambda$ are given by multiplication with $(l^{m-n})^{2-i}$. This map is zero for $m \geq n + r$, because $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$. □

Corollary 3.3.5. *The pro-sheaves $(R^i\pi_* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n)_n$ are Mittag-Leffler zero for $i \neq 1$ and the canonical map*

$$R^1\pi_* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n \to e^* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n(-1) = \mathcal{R}^{\mathrm{g}}_n$$

induces an isomorphism of pro-sheaves $(R^1\pi_* \mathcal{L}\mathrm{og}^{\mathrm{g}}_n)_n \cong (\mathcal{I}^{\mathrm{g}}_n)_n$.

Proof. See the proof of 3.2.6. □

3.3.3 The geometric polylogarithm sheaf. The geometric polylog sheaf can now be defined in the same way as in 3.2.9. Recall the functor H_e from Definition 3.2.7.

Definition 3.3.6. *The* geometric elliptic polylog sheaf *is the pro-sheaf*

$$\mathcal{P}\mathrm{ol}^{\mathrm{g}} := H_e(\mathcal{L}\mathrm{og}^{\mathrm{g}}(1)).$$

To indicate the dependence on Λ we write $\mathcal{P}\mathrm{ol}^{\mathrm{g}}_\Lambda$ and we define

$$\mathcal{P}\mathrm{ol}^{\mathrm{g}}_n := H_e\left(\mathcal{L}\mathrm{og}^{\mathrm{g}}_n(1)\right).$$

As for the (unipotent) elliptic polylog, the pro-sheaf $\mathcal{P}\mathrm{ol}^{\mathrm{g}}$ is a sheaf of $\pi^*\mathcal{R}^{\mathrm{g}}$-modules and defines a Yoneda extension class in

$$\mathrm{Ext}^1_{U,\pi^*\mathcal{R}^{\mathrm{g}}}(\pi^*\mathcal{I}^{\mathrm{g}}, \mathcal{L}\mathrm{og}^{\mathrm{g}}(1)).$$

3.4 The comparison of $\mathcal{P}\mathrm{ol}$ and $\mathcal{P}\mathrm{ol}^{\mathrm{g}}$

In this section we compare $\mathcal{P}\mathrm{ol}$ and $\mathcal{P}\mathrm{ol}^{\mathrm{g}}$. Recall that $\mathcal{P}\mathrm{ol}^{\mathrm{g}}$ is only defined for $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$.

3.4.1 Comparison of the logarithm sheaves. The universal Property 3.3.2 of $\mathcal{L}og^{g}$ and the section $1^{(k)}$ of $e^* \mathcal{L}og_{\Lambda}^{(k)}$ implies:

Lemma 3.4.1. *Let $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$, then there is a unique map*

$$\varrho_{\Lambda}^{(k)} : \mathcal{L}og^{g} \to \mathcal{L}og_{\Lambda}^{(k)} .$$

corresponding to $1^{(k)} : \Lambda \to e^* \mathcal{L}og_{\Lambda}^{(k)}$.

Denote by m the integer depending on k, such that $\varrho_{\Lambda}^{(k)}$ factors through $\mathcal{L}og_{m}^{g}$. Recall that $\mathcal{I}^{g}$ is the augmentation ideal of $\mathcal{R}^{g}$.

Proposition 3.4.2. *Let $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$, then the inverse limit of the $\varrho_{\Lambda}^{(k)}$ induces an isomorphism*

$$\varrho_{\Lambda} : \mathcal{L}og_{\Lambda}^{g} \cong \mathcal{L}og_{\Lambda} .$$

Proof. Consider the canonical surjection

$$\mathcal{L}og^{g} \to \mathcal{L}og^{g} / \mathcal{I}^{g k+1} \mathcal{L}og^{g} .$$

The last sheaf is unipotent of length k, because $\mathcal{L}og^{g}$ is fibrewise isomorphic to the regular ring $\mathcal{R}^{g}$. The universal property of $\mathcal{L}og_{\Lambda}^{(k)}$ induces a map

$$\alpha : \mathcal{L}og_{\Lambda}^{(k)} \to \mathcal{L}og^{g} / \mathcal{I}^{g k+1} \mathcal{L}og^{g} .$$

We get a commutative diagram

$$\begin{array}{ccc} \mathcal{L}og^{g} & \longrightarrow & \mathcal{L}og^{g} / \mathcal{I}^{g k+1} \mathcal{L}og^{g} \\ \downarrow & & \uparrow \alpha \\ \mathcal{L}og_{m}^{g} & \xrightarrow{\varrho_{\Lambda}^{(k)}} & \mathcal{L}og_{\Lambda}^{(k)} . \end{array}$$

We will show that α is bijective, then $\varrho_{\Lambda}^{(k)}$ is surjective. But the diagram implies that α is surjective, hence bijective because $\mathcal{L}og_{\Lambda}^{(k)}$ and $\mathcal{L}og^{g} / \mathcal{I}^{g k+1} \mathcal{L}og^{g}$ have the same cardinality. As $\bigcap_{k \geq 0} \mathcal{I}^{g k} = 0$ the inverse limit of the $\varrho_{\Lambda}^{(k)}$ is bijective. □

3.4.2 Comparison with the $\mathbb{Z}_l$–version. Let $\Lambda_r := \mathbb{Z}/l^r\mathbb{Z}$ and consider the pro-sheaf $\mathcal{L}og_{\Lambda_r}^{g}$. The projection map $\Lambda_{r+1} \to \Lambda_r$ induces $\mathcal{L}og_{\Lambda_{r+1},n}^{g} \to \mathcal{L}og_{\Lambda_r,n}^{g}$, hence in the limit a map $\mathcal{L}og_{\Lambda_{r+1}}^{g} \to \mathcal{L}og_{\Lambda_r}^{g}$. This map is obviously compatible with the isomorphisms $\varrho_{\Lambda_{r+1}}$ and ϱ_{Λ_r} and the reduction map $\mathcal{L}og_{\Lambda_{r+1}} \to \mathcal{L}og_{\Lambda_r}$. We observe:

Lemma 3.4.3. *There is an isomorphism of pro-sheaves*

$$\mathcal{L}og_{\mathbb{Z}_l} \cong (\mathcal{L}og_{\Lambda_r})_r$$

induced by the canonical maps $\mathcal{L}og_{\mathbb{Z}_l} \to \mathcal{L}og_{\Lambda_r}$.

Proof. We have $(\mathcal{L}\mathrm{og}_{\Lambda_r})_r = ((\mathcal{L}\mathrm{og}^{(k)}_{\Lambda_r})_r)_k$. The pro-sheaf $(\mathcal{L}\mathrm{og}^{(k)}_{\Lambda_r})_r$ is by definition of $\mathbb{Z}_l$-sheaves the same as $\mathcal{L}\mathrm{og}^{(k)}_{\mathbb{Z}_l}$ and the claim follows from the definition of $\mathcal{L}\mathrm{og}_{\mathbb{Z}_l}$. □

Proposition 3.4.4. *The inverse limit of the maps ϱ_{Λ_r} induces an isomorphism of pro-sheaves*

$$\varrho : \left(\mathcal{L}\mathrm{og}^{\mathrm{g}}_{\Lambda_r}\right)_r \cong \left(\mathcal{L}\mathrm{og}_{\Lambda_r}\right)_r \cong \mathcal{L}\mathrm{og}_{\mathbb{Z}_l}.$$

Proof. This is immediate from the fact that ϱ_{Λ_r} is an isomorphism. □

3.4.3 Comparison of polylogarithm sheaves. Using the functor H_e from 3.2.7 we can translate the comparison results for the logarithm sheaves to the polylog.

Proposition 3.4.5. *The isomorphism ϱ form 3.4.4 induces an isomorphism of pro-sheaves*

$$H_e(\varrho) : \left(\mathcal{P}\mathrm{ol}^{\mathrm{g}}_{\Lambda_r}\right)_r = \left(H_e\left(\mathcal{L}\mathrm{og}^{\mathrm{g}}_{\Lambda_r}(1)\right)\right)_r \xrightarrow{\cong} \mathcal{P}\mathrm{ol}_{\mathbb{Z}_l}.$$

Proof. Clear from the definition. □

3.5 Specialization of the elliptic polylogarithm sheaf, l-adic Eisenstein classes

The specialization along torsion sections of the elliptic polylog gives interesting cohomology classes. These are the l-adic Eisenstein classes investigated in [Be-Le], [Hu-Ki1] and [Hu-Ki2]. We recall their construction.

3.5.1 Invariance of the logarithm sheaf under translation by torsion sections. Let $N \in \mathbb{Z}$ be invertible on S and $[N] : E \to E$ the N-multiplication. The universal property in Theorem 3.2.2 gives us canonical maps sending $1^{(k)}$ to $[N]^* 1^{(k)}$

$$\mathcal{L}\mathrm{og}^{(k)}_{\Lambda} \to [N]^* \, \mathcal{L}\mathrm{og}^{(k)}_{\Lambda}$$

for $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$. Thus for every N-torsion point $t : S \to E$ we get a map of pro-sheaves

$$\mathrm{pr}^N_t : t^* \, \mathcal{L}\mathrm{og}_{\Lambda} \to e^* \, \mathcal{L}\mathrm{og}_{\Lambda}.$$

Similarly, we have a map

$$\mathrm{pr}^N_t : t^* \, \mathcal{L}\mathrm{og}^{\mathrm{g}}_n \to e^* \, \mathcal{L}\mathrm{og}^{\mathrm{g}}_n$$

for the geometric logarithm sheaf for $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$.

Lemma 3.5.1. *If $l \nmid N$, then the map pr^N_t is an isomorphism.*

Proof. Let us first treat the unipotent case. It suffices to show that $\mathcal{L}og_\Lambda \to [N]^* \mathcal{L}og_\Lambda$ is an isomorphism. Using the equivalence of categories 3.2.4, this can be tested after pull-back with e^*. The resulting map $\mathcal{R}_\Lambda \to \mathcal{R}_\Lambda$ is on the the k-th graded piece $\mathrm{Sym}^k \mathcal{H}_\Lambda$ induced by the map $\mathcal{H}_\Lambda \xrightarrow{[N]^*} \mathcal{H}_\Lambda$, which is just the N-multiplication and thus an isomorphism, because $l \nmid N$. In the geometric case we have $t^* \mathcal{L}og_n^{\mathrm{g}} = \Lambda[p_n^{-1}(t)]$ and $e^* \mathcal{L}og_n^{\mathrm{g}} = \Lambda[H_n]$. The map pr_t^N is by definition induced by $t_n \mapsto [N]t_n$ for $t_n \in p_n^{-1}(t)$. This is obviously an isomorphism if $l \nmid N$. □

To define a morphism pr_t independent of N, we let

$$\mathrm{pr}_t := \mathrm{pr}_t^N \circ \left(\mathrm{pr}_e^N\right)^{-1} : t^* \mathcal{L}og_\Lambda \to e^* \mathcal{L}og_\Lambda . \tag{7}$$

We need an explicit description of the map pr_t on the geometric logarithm sheaf.

We have $t^* \mathcal{L}og_n^{\mathrm{g}} = \Lambda[p_n^{-1}(t)]$ and $e^* \mathcal{L}og_n^{\mathrm{g}} = \Lambda[H_n]$. Here $p_n^{-1}(t) \subset E[Nl^n]$ and if $l \nmid N$ we have $E[Nl^n] = E[N] \oplus E[l^n]$.

Lemma 3.5.2. *The map from (7)*

$$\mathrm{pr}_t : \Lambda\left[p_n^{-1}(t)\right] \to \Lambda[H_n]$$

is induced by the projection of $t_n \in p_n^{-1}(t) \subset E[N] \oplus E[l^n]$ *to* $H_n = E[l^n]$.

Proof. The map $\mathrm{pr}_t^N : \Lambda[p_n^{-1}(t)] \to \Lambda[H_n]$ maps $t_n \in p_n^{-1}(t)$ to $[N]t_n \in H_n$ and pr_e^N is induced by the isomorphism $[N] : H_n \to H_n$. □

Passing to the limit we get a map of pro-sheaves

$$\mathrm{pr}_t : \mathbb{Z}_l[[\mathcal{H}_{\mathbb{Z}_l,t}]] \to \mathbb{Z}_l[[\mathcal{H}_{\mathbb{Z}_l}]]$$

where $\mathcal{H}_{\mathbb{Z}_l,t} := \varprojlim_n p_n^{-1}(t)$. This is induced by the projection $\mathcal{H}_{\mathbb{Z}_l,t} \to \mathcal{H}_{\mathbb{Z}_l}$.

3.5.2 The moment map. The pro-sheaf $e^* \mathcal{L}og_{\mathbb{Q}_l} = \mathcal{R}_{\mathbb{Q}_l}$ is a Hopf algebra (see [Be-Le] 1.2.10 iv)) and we have an isomorphism $\nu : \mathrm{Sym}^{\leq k} \mathcal{H}_{\mathbb{Q}_l} \to \mathrm{Gr}^{\leq k} \mathcal{R}_{\mathbb{Q}_l}^{(k)}$. Let $\hat{\mathfrak{U}}(\mathcal{H}_{\mathbb{Q}_l})$ be the completion of the universal enveloping algebra of the abelian Lie algebra $\mathcal{H}_{\mathbb{Q}_l}$. The canonical filtration makes this a pro-sheaf $(\mathfrak{U}^k(\mathcal{H}_{\mathbb{Q}_l}))_k$.

The structure theorem, [Bour] ch. II, paragraph 1, no. 6, gives:

Lemma 3.5.3. *The map* $\nu : \mathcal{H}_{\mathbb{Q}_l} \to \mathcal{R}_{\mathbb{Q}_l}$ *extends to an isomorphism of Hopf algebra pro-sheaves*

$$\nu : \hat{\mathfrak{U}}(\mathcal{H}_{\mathbb{Q}_l}) \cong \mathcal{R}_{\mathbb{Q}_l},$$

which on the k-th graded piece $\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l}$ *is multiplication by* $k!$.

Note that we can describe the composition

$$\left(\mathcal{R}^{g}_{\Lambda_r}\right)_r \cong \mathcal{R}_{\mathbb{Z}_l} \to \mathcal{R}_{\mathbb{Q}_l} \cong \hat{\mathfrak{U}}(\mathcal{H}_{\mathbb{Q}_l})$$

where the last map is ν^{-1}, as follows (on stalks): Choose a $\mathbb{Z}_l$-basis γ, η of $\mathcal{H}_{\mathbb{Z}_l}$. Identify $(\mathcal{R}^{g}_{\Lambda_r})_r = \mathbb{Z}_l[[\mathcal{H}_{\mathbb{Z}_l}]] := \varprojlim_n \mathbb{Z}_l[H_n]$ and view elements in $\mathbb{Z}_l[[\mathcal{H}_{\mathbb{Z}_l}]]$ as $\mathbb{Z}_l$-valued measures λ on $\mathcal{H}_{\mathbb{Z}_l}$.

Definition 3.5.4. *We define the* k-th moment map μ^k *to be*

$$\begin{aligned}\mu^k : \mathbb{Z}_l[[\mathcal{H}_{\mathbb{Z}_l}]] &\to \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}\\ \lambda &\mapsto \sum_{s+t=k} \int_{\mathcal{H}_{\mathbb{Z}_l}} \binom{k}{s} < \gamma, h >^s < \eta, h >^t d\lambda(h) \frac{\gamma^{\otimes s}\eta^{\otimes t}}{k!}.\end{aligned}$$

Here $<,>$ *is the Weil pairing on* $\mathcal{H}_{\mathbb{Z}_l}$.

With this definition the map

$$\left(\mathcal{R}^{g}_{\Lambda_r}\right)_r \cong \mathcal{R}_{\mathbb{Z}_l} \to \mathcal{R}_{\mathbb{Q}_l} \cong \hat{\mathfrak{U}}(\mathcal{H}_{\mathbb{Q}_l}) = \prod_{k\geq 0} \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}$$

sends the measure λ to $(\mu^k(\lambda))_k$. In particular, suppose that $\lambda = \sum_i n_i \delta_{h_i}$ where δ_{h_i} is the Dirac-measure at h_i. Then

$$\mu^k(\lambda) = \frac{1}{k!}\sum_i n_i h_i^{\otimes k}.$$

For later reference we record an explicit description of the moment map at finite level.

Lemma 3.5.5. *The map* $k!\mu^k$ *is the inverse limit over* r, m *of*

$$\begin{aligned}\mu^k : \Lambda_r[H_m] &\to \operatorname{Sym}^k H_m \otimes \Lambda_r\\ \sum_i n_i(h_i) &\mapsto \sum_i n_i h_i^{\otimes k}\end{aligned}$$

where on the right hand side h_i *is considered as element in* $H_m \otimes \Lambda_r$.

Proof. This follows immediately from the explicit description of the maps μ^k and ϱ. □

3.5.3 A splitting

Lemma 3.5.6. *Let $\Lambda = \mathbb{Z}/l^r\mathbb{Z}$, $\mathbb{Z}_l$ or $\mathbb{Q}_l$. Then the inclusion $\mathcal{I}_\Lambda \hookrightarrow \mathcal{R}_\Lambda$ induces an injective map*

$$\mathcal{R}_\Lambda(1) = \underline{\mathrm{Hom}}_{S,\mathcal{R}_\Lambda}(\mathcal{R}_\Lambda, \mathcal{R}_\Lambda(1)) \to \underline{\mathrm{Hom}}_{S,\mathcal{R}_\Lambda}(\mathcal{I}_\Lambda, \mathcal{R}_\Lambda(1)),$$

which is an isomorphism for $\Lambda = \mathbb{Q}_l$.

Proof. This follows from

$$\underline{\mathrm{Hom}}_{S,\mathcal{R}_\Lambda}(\Lambda, \mathcal{R}_\Lambda(1)) = 0 = \underline{\mathrm{Ext}}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathbb{Q}_l, \mathcal{R}_{\mathbb{Q}_l}(1)),$$

which is a consequence of the Koszul resolution as $\mathcal{R}_{\mathbb{Q}_l}$ is a sheaf of regular rings. □

Corollary 3.5.7. *The map $\mathcal{I}_{\mathbb{Q}_l} \hookrightarrow \mathcal{R}_{\mathbb{Q}_l}$ induces an injection*

$$\mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathbb{Q}_l, \mathcal{R}_{\mathbb{Q}_l}(1)) \cong \mathrm{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathcal{R}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1))) \xrightarrow{a} \mathrm{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathcal{I}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)).$$

Using the isomorphism $\nu : \mathcal{R}_{\mathbb{Q}_l} \cong \hat{\mathfrak{U}}(\mathcal{H}_{\mathbb{Q}_l})$ we have an exact sequence (Koszul resolution for the Lie algebra $\mathcal{H}_{\mathbb{Q}_l}$)

$$0 \to \mathcal{R}_{\mathbb{Q}_l}(1) \to \mathcal{H}_{\mathbb{Q}_l} \otimes_{\mathbb{Q}_l} \mathcal{R}_{\mathbb{Q}_l} \xrightarrow{b} \mathcal{I}_{\mathbb{Q}_l} \to 0.$$

Here we used $\Lambda^2\mathcal{H}_{\mathbb{Q}_l} \cong \mathbb{Q}_l(1)$ induced by the Weil-pairing. We get

$$\begin{array}{ccc}
\mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathbb{Q}_l, \mathcal{R}_{\mathbb{Q}_l}(1)) & \xrightarrow{a} & \mathrm{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathcal{I}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)) \\
 & & \downarrow b^* \\
 & & \mathrm{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathcal{H}_{\mathbb{Q}_l} \otimes_{\mathbb{Q}_l} \mathcal{R}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)) \\
 & & \downarrow = \\
 & & \mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)).
\end{array}$$

Lemma 3.5.8. *The map*

$$\mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathbb{Q}_l, \mathcal{R}_{\mathbb{Q}_l}(1)) \xrightarrow{b^* \circ a} \mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1))$$

has a canonical splitting.

Proof. We have an isomorphism

$$\mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)) \cong \mathrm{Ext}^1_{S,\mathbb{Q}_l}(\mathbb{Q}_l, \underline{\mathrm{Hom}}_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)))$$

and a contraction map

$$\underline{\mathrm{Hom}}_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1))) \to \mathcal{R}_{\mathbb{Q}_l}(1)$$

given on the k-th component of $\mathcal{R}_{\mathbb{Q}_l} = \prod_{k\geq 0} \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}$ by

$$\underline{\operatorname{Hom}}_{S,\mathbb{Q}_l}\big(\mathcal{H}_{\mathbb{Q}_l}, \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big) \cong \underline{\operatorname{Hom}}_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathbb{Q}_l) \otimes \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)$$

composed with

$$\begin{aligned} \underline{\operatorname{Hom}}_{S,\mathbb{Q}_l}(\mathcal{H}_{\mathbb{Q}_l}, \mathbb{Q}_l) \otimes \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1) &\to \operatorname{Sym}^{k-1} \mathcal{H}_{\mathbb{Q}_l}(1) \\ f \otimes h_1 \otimes \ldots \otimes h_k &\mapsto \frac{1}{k+1} \sum_{i=1}^{k} f(h_i) h_1 \otimes \ldots \hat{h}_i \ldots \otimes h_k. \end{aligned} \tag{8}$$

This gives the required map and it is straightforward to check that this is indeed a splitting of $b^* \circ a$. □

3.5.4 The specialization of the polylogarithm, l-adic Eisenstein classes. We now define the specialization of the elliptic polylogarithm. Let $\beta \in \mathbb{Z}[E[N](S) \setminus e]$ be of the form

$$\beta = \sum_{t \in E[N](S) \setminus e} n_t t.$$

We want to define an element

$$\big(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}\big)^k \in H^1\big(S, \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big).$$

First observe that $\mathrm{pr}_t\, t^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ gives an element in

$$\begin{aligned} \operatorname{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}} \big(\mathcal{I}_{\mathbb{Q}_l}, t^* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}(1)\big) \xrightarrow{\mathrm{pr}_t} &\operatorname{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}} \big(\mathcal{I}_{\mathbb{Q}_l}, e^* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}(1)\big) \\ &= \operatorname{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}} \big(\mathcal{I}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)\big). \end{aligned}$$

Define

$$\sigma : \operatorname{Ext}^1_{S,\mathcal{R}_{\mathbb{Q}_l}}(\mathcal{I}_{\mathbb{Q}_l}, \mathcal{R}_{\mathbb{Q}_l}(1)) \to \operatorname{Ext}^1_{S,\mathbb{Q}_l}(\mathbb{Q}_l, \mathcal{R}_{\mathbb{Q}_l}(1))$$

to be the composition of b^* and the splitting of Lemma 3.5.8. This is a splitting of a. Denote by σ^k the projection onto

$$\operatorname{Ext}^1_{S,\mathbb{Q}_l} \big(\mathbb{Q}_l, \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big) = H^1\big(S, \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big).$$

Definition 3.5.9. *For $\beta = \sum_{t \in E[N](S) \setminus e} n_t t \in \mathbb{Z}[E[N](S) \setminus e]$ define*

$$\big(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}\big)^k := \sum_{t \in E[N](S) \setminus e} n_t \big(\sigma^k \,\mathrm{pr}_t\, t^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}\big) \in H^1\big(S, \operatorname{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big)$$

These are the l-adic Eisenstein classes *associated to β.*

Remark: This numbering disagrees with the one of [Hu-Ki2], where we wrote $(\beta^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^{k+1}$ for this class.

Our aim is to compute a variant of the specialization. Namely let $[\mathfrak{a}] : E \to E$ be an isogeny of degree $\mathrm{N}\,\mathfrak{a} := \deg[\mathfrak{a}]$ prime to lN (the notations are chosen to fit the CM case which is our ultimate goal). Define $[\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ as follows: let $[\mathfrak{a}]^{-1}U$ be the preimage of U under $[\mathfrak{a}]$. The pro-sheaf $\mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ is an extension

$$0 \to \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}(1) \to \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l} \to \pi^* \mathcal{I}_{\mathbb{Q}_l} \to 0$$

on U, which we first restrict to $[\mathfrak{a}]^{-1}U$ and then apply $[\mathfrak{a}]_*$

$$0 \to [\mathfrak{a}]_* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}(1) \to [\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l} \to [\mathfrak{a}]_* \pi^* \mathcal{I}_{\mathbb{Q}_l} \to 0.$$

The canonical map $\mathcal{L}\mathrm{og}_{\mathbb{Q}_l} \to [\mathfrak{a}]^* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}$ induces by adjunction $[\mathfrak{a}]_* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l} \to \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}$ and we push-out the above sequence with this map. Finally the pull-back with the adjunction map $\pi^* \mathcal{I}_{\mathbb{Q}_l} \to [\mathfrak{a}]_* \pi^* \mathcal{I}_{\mathbb{Q}_l}$ gives an extension

$$0 \to \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}(1) \to [\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l} \to \pi^* \mathcal{I}_{\mathbb{Q}_l} \to 0$$

which we call $[\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$. Let $t : S \to [\mathfrak{a}]^{-1}U$ be an N-torsion section as above. Then we can define with the same procedure a specialization

$$(([\mathfrak{a}]t)^* [\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k \in H^1\big(S, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1)\big)\ .$$

In the next section we will give an explicit formula for

$$(([\mathfrak{a}]t)^* [\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k - N\mathfrak{a}(t^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k.$$

This suffices our need because of the following result:

Lemma 3.5.10. *The class*

$$(([\mathfrak{a}]t)^* [\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k - N\mathfrak{a}(t^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k$$

equals $N\mathfrak{a}([\mathfrak{a}]^k N\mathfrak{a} - 1)(t^* \mathcal{P}\mathrm{ol}_{Q_l})^k$ *in* $H^1(S, \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l}(1))$.

Proof. First of all we have by [Be-Le] 1.3.13 that $[\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l} = N\mathfrak{a}\, \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ and it suffices to compute $(([\mathfrak{a}]t)^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k = (t^* [\mathfrak{a}]^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k$. The sheaf $[\mathfrak{a}]^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ is the push-out of the sheaf $\mathcal{P}\mathrm{ol}_{\mathbb{Q}_l}$ (restricted to $[\mathfrak{a}]^{-1}U$) with the map $\mathcal{L}\mathrm{og}_{\mathbb{Q}_l} \to [\mathfrak{a}]^* \mathcal{L}\mathrm{og}_{\mathbb{Q}_l}$ and the pull-back with the map $\pi^* \mathcal{I}_{\mathbb{Q}_l} \to \pi^* \mathcal{I}_{\mathbb{Q}_l}$ induced by $\mathfrak{a}$. Both maps are given on the associated graded $\mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_l}$ by $[\mathfrak{a}]^k$. Going through the construction of the specialization, one sees that $(t^* [\mathfrak{a}]^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k$ is just $[\mathfrak{a}]^k N\mathfrak{a}(t^* \mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^k$, because $[\mathfrak{a}]$ acts on the dual of $\mathcal{H}_{\mathbb{Q}_l}$ through $[\mathfrak{a}]^t$ and $[\mathfrak{a}][\mathfrak{a}]^t = N\mathfrak{a}$. □

4 The l-adic realization of the elliptic polylog

This part is concerned with the construction of the polylog on elliptic curves in a geometric way, which allows to compute its specializations explicitly. This is the technical heart of the paper and in our opinion our main contribution to the problem of computing the Tamagawa number for elliptic curves.

4.1 The polylog as a one-motive

From the definition of the polylogarithm it is quite obvious that it is a pro-sheaf consisting of $\mathbb{Z}/l^r\mathbb{Z}$-realizations of one-motives. In this section we make this connection more explicit.

4.1.1 A reformulation. Let $\Lambda_r = \mathbb{Z}/l^r\mathbb{Z}$ and recall from formula (6) in Sect. 3.2.3 that the geometric polylog sits in an exact sequence

$$0 \to \pi^*\overline{\pi}_* \mathcal{L}\mathrm{og}_n^{\mathrm{g}}(1) \to \mathcal{L}\mathrm{og}_n^{\mathrm{g}}(1)_U \to \mathcal{P}\mathrm{ol}_n^{\mathrm{g}} \to \pi^* R^1\pi_* \mathcal{L}\mathrm{og}_{n,U}^{\mathrm{g}}(1) \to 0.$$

By the definition of the geometric logarithm $\mathcal{L}\mathrm{og}_n^{\mathrm{g}} = p_{n*}\Lambda_r$ and hence $\overline{\pi}_* \mathcal{L}\mathrm{og}_n^{\mathrm{g}} = \Lambda_r$. We get

$$0 \to \Lambda_r(1) \to p_{n*}\Lambda_{r,U}(1) \to \mathcal{P}\mathrm{ol}_n^{\mathrm{g}} \to \pi^* R^1\pi_{n*}\Lambda_{r,U}(1) \to 0.$$

For a lisse Λ-sheaf $\mathcal{F}$ on U consider the dual

$$(\mathcal{F})^\vee := \underline{\mathrm{Hom}}_U(\mathcal{F}, \Lambda).$$

Then, using Poincaré duality, we get by dualizing and twisting by 1 an exact sequence

$$0 \to \pi^* R^1\pi_{n!}\Lambda_{r,U}(1) \to \left(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}}\right)^\vee(1) \to p_{n*}\Lambda_{r,U} \to \Lambda_{r,U} \to 0.$$

Denote by I_{n,Λ_r} the kernel of the map $p_{n*}\Lambda_{r,U} \to \Lambda_{r,U}$, then $\left(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}}\right)^\vee(1)$ gives a class in

$$\mathrm{Ext}_U^1\left(I_{n,\Lambda_r}, \pi^* R^1\pi_{n!}\Lambda_{r,U}(1)\right).$$

We want to give a geometric interpretation of this class. For this we will relate $(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ with the l^r-torsion points of a one-motive, which is defined via a generalized Picard scheme.

4.1.2 The generalized Picard scheme. We will give a geometric interpretation of $(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$. For this we need the Picard scheme of line bundles on E_n trivialized along H_n (cf. the article by Raynaud [Ra]).

Definition 4.1.1. *Let P_{H_n} be the* generalized Picard scheme *representing the functor, which associates to $S' \to S$ the isomorphism classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle on $E_n \times_S S'$ and $\alpha : h_n^*\mathcal{L} \cong \mathcal{O}_{H_n}$ is a trivialization of $\mathcal{L}$ along $H_n \times_S S'$.*

That H_n is a rigidificator in the sense of [Ra] follows from the fact that H_n contains the section $e_n : S \to E_n$. Denote by P_n the Picard scheme of E_n. Then we have an exact sequence of group schemes on S

$$0 \to T_{H_n} \to P_{H_n} \to P_n \to 0,$$

where T_{H_n} is the torus with character group $I[H_n] := \ker(p_{n*}\mathbb{Z} \to \mathbb{Z})$ and the map $P_{H_n} \to P_n$ is given by forgetting the trivialization. The l^r-torsion of P_{H_n} can be identified as follows:

Lemma 4.1.2. *There is a canonical isomorphism of lisse sheaves*

$$R^1\pi_{n!}\Lambda_r(1) \cong P_{H_n}[l^r].$$

Proof. Define an fppf-sheaf on E_n by ${}_{H_n}\mathbb{G}_m := \ker(\mathbb{G}_m \to h_{n*}h_n^*\mathbb{G}_m)$, where $h_n : H_n \to E_n$. Then $P_{H_n} = R^1\overline{\pi}_{n*}({}_{H_n}\mathbb{G}_m)$ (see [SGA4,III], Expose XVIII Proposition 1.5.14), where the higher direct image is taken for the flat topology. The sequence on E_n from loc. cit. Lemma 1.6.1

$$0 \to j_{n!}\mu_{l^r} \to {}_{H_n}\mathbb{G}_m \xrightarrow{[l^r]} {}_{H_n}\mathbb{G}_m \to 0$$

gives an isomorphism $R^1\overline{\pi}_{n*}j_{n!}\mu_{l^r} \cong P_{H_n}[l^r]$, because $\overline{\pi}_{n*\,H_n}\mathbb{G}_m = 0$. □

On $E_n \times_S U_n$ we have the line bundle $\mathcal{O}(\Delta_n)$ associated to the Cartier divisor defined by the diagonal Δ_n. By definition $\mathcal{O}(\Delta_n)$ sits in an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta_n) \to \Delta_{n*}\mathcal{O}_{U_n} \to 0,$$

which induces a trivialization of $\mathcal{O}(\Delta_n)$ along $H_n \times_S U_n$. Thus we get a section $\Delta_n : \mathbb{Z} \to \pi_n^* P_{H_n}$ of étale sheaves. Adjunction with respect to p_n gives a map

$$\Delta_n : p_{n*}\mathbb{Z} \to \pi^* P_{H_n} \tag{9}$$

also denoted by Δ_n by abuse of notation.

4.1.3 Comparison with a one-motive. Consider $p_{n*}\mathbb{Z} \to \pi^* P_{H_n}$ as a complex of sheaves in degree 0 and 1. Recall from [Del1] 10.1.10 that a one-motive $M = [X \to G]$ is a complex in degree 0 and 1, consisting of a group scheme X/S, which is étale locally a constant $\mathbb{Z}$-module, free of finite rank and G is an extension of an abelian scheme with a torus over S. We have a morphism of group schemes $X \to G$ over S. Note that $p_{n*}\mathbb{Z} \to \pi^* P_{H_n}$ is not a one-motive because P_{H_n} is not a semi-abelian scheme, as it is not connected. For the connected component of the identity $P^0_{H_n}$, we have an exact sequence $0 \to P^0_{H_n} \to P_{H_n} \to \mathbb{Z} \to 0$ and if we let $I[H_n]$ be the kernel of the composition $p_{n*}\mathbb{Z} \to P_{H_n} \to \mathbb{Z}$ we get a quasi-isomorphism

$$\left[I[H_n] \to P^0_{H_n}\right] \cong \left[p_{n*}\mathbb{Z} \to \pi^* P_{H_n}\right].$$

Here $I[H_n] \to P^0_{H_n}$ is of course a one-motive.

Theorem 4.1.3. *There is a canonical isomorphism of étale sheaves*

$$\underline{H}^0\left(\left[p_{n*}\mathbb{Z} \to \pi^* P_{H_n}\right] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}\right) \cong \left(\mathcal{P}\mathrm{ol}^{\mathrm{g}}_n\right)^{\vee}(1)$$

which is compatible with the morphism $(\mathcal{P}\mathrm{ol}^{\mathrm{g}}_n)^{\vee}(1) \to (\mathcal{P}\mathrm{ol}^{\mathrm{g}}_m)^{\vee}(1)$ *induced by the trace map* $\mathcal{L}\mathrm{og}^{\mathrm{g}}_m \to \mathcal{L}\mathrm{og}^{\mathrm{g}}_n$. *Here* $\underline{H}^0$ *denotes the zeroth cohomology.*

Proof. We prove first a statement for a non zero section $t : S \to E$ and then apply this to the universal section Δ. Let $H_{t,n} := p_n^{-1}(t)$ be the preimage of t in E_n. Let $h_n^t : H_{t,n} \to E_n$ be the embedding and denote by g_n^t the open immersion of the complement

$$g_n^t : E_n \setminus H_{t,n} \hookrightarrow E_n.$$

Denote by $p_{n,t} : H_{t,n} \to S$ the structure map. Recall that $h_n : H_n \to E_n$ is the closed immersion and that $j_n : U_n \to E_n$ is the inclusion of the open complement. There is an isomorphism of complexes $j_{n!}\mu_{l^r} \cong [\mu_{l^r} \to h_{n*}\mu_{l^r}]$ induced by $0 \to j_{n!}\mu_{l^r} \to \mu_{l^r} \to h_{n*}\mu_{l^r} \to 0$. We compute

$$\begin{aligned} R^1\big(\overline{\pi}_n \circ g_n^t\big)_* j_{n!}\mu_{l^r} &\cong \underline{H}^1\big[R\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r} \to R\big(\overline{\pi}_n \circ g_n^t\big)_* h_{n*}\mu_{l^r}\big] \\ &\cong \underline{H}^1\big[R\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r} \to p_{H_n*}\mu_{l^r}\big]. \end{aligned}$$

We have a map

$$R\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r} \to R^1\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r}[-1]$$

because R^2 vanishes and we can write by Poincaré duality

$$R^1\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r}[-1] \cong \operatorname{Hom}\big(P^0_{H_{n,t}}[l^r], \mu_{l^r}\big)[-1].$$

The exact sequence

$$0 \to P^0_{H_{n,t}}[l^r] \to P^0_{H_{n,t}} \xrightarrow{[l^r]} P^0_{H_{n,t}} \to 0$$

shows that $P^0_{H_{n,t}}[l^r] \cong P^0_{H_{n,t}} \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}[-1]$. Also

$$\operatorname{Hom}\big(P^0_{H_{n,t}}[l^r], \mu_{l^r}\big) \cong R\operatorname{Hom}\big(P^0_{H_{n,t}}[l^r], \mu_{l^r}\big),$$

because there are no higher Ext-groups. We get a map

$$R\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r} \to R^1\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r}[-1] \cong R\operatorname{Hom}\big(P^0_{H_{n,t}} \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big).$$

On the other hand we have

$$\begin{aligned} p_{H_n*}\mu_{l^r} &\cong R\operatorname{Hom}\big(p_{H_n*}\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big) \\ &\to R\operatorname{Hom}\big((p_{H_n*}\mathbb{Z})^0 \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big), \end{aligned}$$

where $(p_{H_n*}\mathbb{Z})^0 := \ker(p_{H_n*}\mathbb{Z} \to \mathbb{Z})$. Thus we have a map from

$$\underline{H}^1\big[R\big(\overline{\pi}_n \circ g_n^t\big)_*\mu_{l^r} \to p_{H_n*}\mu_{l^r}\big]$$

to

$$\begin{aligned} &\underline{H}^1\big[R\operatorname{Hom}\big(P^0_{H_{n,t}} \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big) \to R\operatorname{Hom}\big((p_{H_n*}\mathbb{Z})^0 \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big)\big] \\ &\cong \underline{H}^1 R\operatorname{Hom}\big(\big[(p_{H_n*}\mathbb{Z})^0 \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z} \to P^0_{H_{n,t}} \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}\big], \mu_{l^r}\big) \\ &\cong \underline{H}^1 R\operatorname{Hom}\big(\big[(p_{H_n*}\mathbb{Z})^0 \to P^0_{H_{n,t}}\big] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big) \\ &\cong \operatorname{Hom}\big(\big[(p_{H_n*}\mathbb{Z})^0 \to P^0_{H_{n,t}}\big] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\big), \end{aligned}$$

where $[(p_{H_n*}\mathbb{Z})^0 \to P^0_{H_{n,t}}]$ is now a complex in degree 0 and 1. By Poincaré duality for one-motives [Del1] 10.2.11, we have

$$\mathrm{Hom}\left(\left[(p_{H_n*}\mathbb{Z})^0 \to P^0_{H_{n,t}}\right] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}, \mu_{l^r}\right) \cong \underline{H}^0\left[(p_{H_{n,t}*}\mathbb{Z})^0 \to P^0_{H_n}\right] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}.$$

Thus we have constructed a map

$$R^1\left(\overline{\pi}_n \circ g_n^t\right)_* j_{n!}\mu_{l^r} \to \underline{H}^0[p_{H_{n,t}*}\mathbb{Z} \to P_{H_n}] \otimes^{\mathbb{L}} \mathbb{Z}/l^r\mathbb{Z}.$$

Both sides are extensions of $\ker(p_{H_{n,t}*}\mathbb{Z}/l^r\mathbb{Z} \to \mathbb{Z}/l^r\mathbb{Z})$ with $P_{H_n}[l^r]$ and by base change to an algebraic closed field one sees that the above map is the identity on these two groups, hence an isomorphism itself. Finally we apply this to the universal section Δ over U to get the desired result. □

Here is an explicit way to get the extension $(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ from $[p_{n*}\mathbb{Z} \to \pi^* P_{H_n}]$. Consider the exact sequence

$$0 \to \pi^* P_{H_n}[l^r] \to \pi^* P_{H_n} \xrightarrow{l^r} \pi^* P_{H_n} \to \mathbb{Z}/l^r\mathbb{Z} \to 0$$

defined by the l^r-multiplication. The pull-back by the the map $p_{n*}\mathbb{Z} \to \pi^* P_{H_n}$ gives an extension

$$0 \to \pi^* P_{H_n}[l^r] \to \mathscr{E} \to p_{n*}\mathbb{Z} \to \mathbb{Z}/l^r\mathbb{Z} \to 0$$

with some sheaf $\mathscr{E}$. If we tensor this with $\mathbb{Z}/l^r\mathbb{Z}$ over $\mathbb{Z}$, then we get an exact sequence

$$0 \to \pi^* P_{H_n}[l^r] \to \mathscr{E} \otimes_{\mathbb{Z}} \mathbb{Z}/l^r\mathbb{Z} \to p_{n*}\mathbb{Z}/l^r\mathbb{Z} \to \mathbb{Z}/l^r\mathbb{Z} \to 0,$$

because the kernel of $p_{n*}\mathbb{Z} \to \mathbb{Z}/l^r\mathbb{Z}$ is a sheaf of free $\mathbb{Z}$-modules. With the above theorem we conclude that $(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1) \cong \mathscr{E} \otimes_{\mathbb{Z}} \mathbb{Z}/l^r\mathbb{Z}$ (up to sign).

4.1.4 The class of the geometric polylog. Recall from Sect. 4.1.1 that the polylog $(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ defines a class in

$$\mathrm{Ext}^1_U\left(I_{n,\Lambda}, \pi^* R^1\pi_{n!}\Lambda_U(1)\right),$$

where $I_{n,\Lambda}$ is the kernel of the map $p_{n*}\Lambda_U \to \Lambda_U$. With Lemma 4.1.2 we can write

$$\mathrm{cl}\left(\left(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}}\right)^\vee(1)\right) \in \mathrm{Ext}^1_U\left(I_{n,\Lambda}, P_{H_n}[l^r]\right).$$

Let $[\mathfrak{a}]$ be an isogeny of E of degree $\mathrm{N}\,\mathfrak{a}$ prime to lN. We consider as in Lemma 3.5.10 $[\mathfrak{a}]^*[\mathfrak{a}]_*(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ and the restriction of $N\mathfrak{a}(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ to $[\mathfrak{a}]^{-1}U$. We want to compute the specialization of of these two sheaves at an N-torsion point $t : S \to [\mathfrak{a}]^{-1}U$.

Define for t and $[\mathfrak{a}]t$

$$H_{n,t} := p_n^{-1}(t) \text{ and } H_{n,[\mathfrak{a}]t} := p_n^{-1}([\mathfrak{a}]t).$$

We also let $\mathbb{Z}[H_{n,t}] := t^* p_{n*}\mathbb{Z}$ and similarly for $[\mathfrak{a}]t$. Denote by $I_{\mathbb{Z}}[H_{n,t}]$ the kernel of the augmentation $\mathbb{Z}[H_{n,t}] \to \mathbb{Z}$. Recall that we have an exact sequence

$$0 \to \pi^* P_{H_n}[l^r] \to \pi^* P_{H_n} \xrightarrow{l^r} \pi^* P_{H_n} \to \mathbb{Z}/l^r\mathbb{Z} \to 0.$$

Also recall from the end of Sect. 4.1.2 that the line bundle $\mathcal{O}(\Delta_n)$ on $E_n \times_S U_n$ with its canonical trivialization gives a section $\Delta_n : \mathbb{Z} \to \pi_n^* P_{H_n}$.

Lemma 4.1.4. *The sheaf* $[\mathfrak{a}]_*(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ *is given by the pull-back of the above sequence by the map*

$$[\mathfrak{a}]_*\Delta_n : p_{n*}\mathbb{Z} \to p_{n*}[\mathfrak{a}]_*[\mathfrak{a}]^*\mathbb{Z} \to \pi^* P_{H_n}$$

whose adjoint $\mathbb{Z} \to \pi_n^* P_{H_n}$ *maps* 1 *to* $([\mathfrak{a}] \times \mathrm{id})^*\mathcal{O}(\Delta_n)$ *and tensor the resulting sequence with* Λ. *In particular the sheaf* $([\mathfrak{a}]t)^*[\mathfrak{a}]_*(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ *is given by the pull-back via*

$$([\mathfrak{a}]t)^*[\mathfrak{a}]_*\Delta_n : \mathbb{Z}[H_{n,[\mathfrak{a}]t}] \to P_{H_n}$$

which maps a section $D \in \mathbb{Z}[H_{n,[\mathfrak{a}]t}]$ *to* $[\mathfrak{a}]^*\mathcal{O}(D)$.

Proof. By definition of $[\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_n^{\mathrm{g}}$ cf. 3.5.10 we have to push-out $[\mathfrak{a}]_* \mathcal{P}\mathrm{ol}_n^{\mathrm{g}}$ by the dual of $p_{n*}\mathbb{Z} \to p_{n*}[\mathfrak{a}]_*[\mathfrak{a}]^*\mathbb{Z}$. Observing that we get $(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee$ by dualizing everything this gives the first claim. The second follows immediately from this. □

The sheaf $N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ has a similar but easier interpretation. Consider

$$0 \to [\mathfrak{a}]^*\pi^* P_{H_n}[l^r] \to [\mathfrak{a}]^*\pi^* P_{H_n} \xrightarrow{l^r} [\mathfrak{a}]^*\pi^* P_{H_n} \to [\mathfrak{a}]^*\mathbb{Z}/l^r\mathbb{Z} \to 0.$$

Lemma 4.1.5. *The sheaf* $N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ *is given by the pull-back of the above exact sequence by the map*

$$N\mathfrak{a}\Delta_n : p_{n*}\mathbb{Z} \xrightarrow{N\mathfrak{a}} p_{n*}\mathbb{Z} \to [\mathfrak{a}]^*\pi^* P_{H_n}$$

whose adjoint $\mathbb{Z} \to [\mathfrak{a}]^*\pi_n^* P_{H_n}$ *maps* 1 *to* $\mathcal{O}(\Delta_n)^{\otimes N\mathfrak{a}}$ *and then tensor with* Λ. *In particular the sheaf* $t^* N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ *is given by the pull-back via*

$$t^* N\mathfrak{a}\Delta_n : \mathbb{Z}[H_{n,t}] \to P_{H_n}$$

which maps a section $D \in \mathbb{Z}[H_{n,t}]$ *to* $\mathcal{O}(D)^{\otimes N\mathfrak{a}}$.

Proof. Clear. □

As in (7) we have isomorphisms $\mathrm{pr}_t : \mathbb{Z}[H_{n,t}] \cong \mathbb{Z}[H_n]$ and $\mathrm{pr}_{[\mathfrak{a}]t} : \mathbb{Z}[H_{n,[\mathfrak{a}]t}] \cong \mathbb{Z}[H_n]$. Thus via $(\mathrm{pr}_{[\mathfrak{a}]t})^{-1}$ and $(\mathrm{pr}_t)^{-1}$ we can consider the difference

$$(\mathrm{pr}_{[\mathfrak{a}]t})^{-1}([\mathfrak{a}]t)^*[\mathfrak{a}]_*\Delta_n - (\mathrm{pr}_t)^{-1}t^* N\mathfrak{a}\Delta_n : \mathbb{Z}[H_n] \to P_{H_n}.$$

Let us denote this map by $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n$.

Lemma 4.1.6. *The map* $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n$ *factors through* $T_{H_n} \subset P_{H_n}$. *In particular the composition with* $P_{H_n} \to \mathbb{Z}/l^r\mathbb{Z}$ *is zero.*

Proof. For any $h_n \in H_n$ let $t_n \in H_{n,t}$ be the unique point which maps to h_n under pr_t. Then $[\mathfrak{a}]t_n$ is the unique point which maps to h_n under $\mathrm{pr}_{[\mathfrak{a}]t}$. The explicit description in the above lemmas shows that $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n$ maps $(h_n) \in \mathbb{Z}[H_n]$ to

$$[\mathfrak{a}]^*\mathcal{O}([\mathfrak{a}]t_n) \otimes \mathcal{O}(t_n)^{\otimes -N\mathfrak{a}}.$$

We have to show that this line bundle is zero in the Picard group P_n, because then the section $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n$ factors through the kernel of $P_{H_n} \to P_n$, which is T_{H_n}. For this consider T_{-t_n} the translation by $-t_n$. Then

$$[\mathfrak{a}]^*\mathcal{O}([\mathfrak{a}]t_n) \otimes \mathcal{O}(t_n)^{\otimes -N\mathfrak{a}} \cong T^*_{-t_n}\left([\mathfrak{a}]^*\mathcal{O}(e_n) \otimes \mathcal{O}(e_n)^{\otimes -N\mathfrak{a}}\right).$$

As the last bundle is trivial, this implies our claim. □

Corollary 4.1.7. *The difference*

$$\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*\left(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}}\right)^\vee(1) - \mathrm{pr}_t\, t^*N\mathfrak{a}\left(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}}\right)^\vee(1) \in \mathrm{Ext}^1_S(\Lambda[H_n], P_{H_n}[l^r])$$

is is a class in $\mathrm{Ext}^1_S(\Lambda[H_n], T_{H_n}[l^r])$.

By construction the class $\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1) - \mathrm{pr}_t\, t^*N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ in

$$\mathrm{Ext}^1_S(\Lambda[H_n], T_{H_n}[l^r]) = \mathrm{Ext}^1_{H_n}(\Lambda, T_{H_n}[l^r]) = H^1(H_n, T_{H_n}[l^r])$$

is given by the pull-back of the exact sequence

$$0 \to T_{H_n}[l^r] \to T_{H_n} \xrightarrow{[l^r]} T_{H_n} \to 0 \tag{10}$$

by the map $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n : \mathbb{Z}[H_n] \to T_{H_n}$ over S. This is the same as a map $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n : \mathbb{Z} \to T_{H_n}$ over H_n. This map will be computed in the next section.

4.2 Computation of the specialization of the polylog

Let $t : S \to U$ be a non zero N-torsion point. We continue to compute the specialization at t of the polylog.

4.2.1 Specialization of the class of the geometric polylog. We keep the notations from Sect. 4.1.4. The class

$$\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*\big(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}}\big)^\vee(1) - \mathrm{pr}_t\, t^* N\mathfrak{a}\big(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}}\big)^\vee(1) \in H^1(H_n, T_{H_n}[l^r])$$

is given by the pull-back of the sequence (10) by the section $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n : \mathbb{Z} \to T_{H_n}$ over H_n.

To describe this section in terms of functions, we extend our base to a Galois covering S_n of S, where H_n (hence $H_{n,t}$ and $H_{n,[\mathfrak{a}]t}$) is rational and then use descent: Let G_n be the Galois covering group of S_n/S. Consider $\prod_{h_n\in H_n(S_n)} \mathbb{G}_m$ and write a typical element of this product as $\sum_{h_n} g_{h_n}(h_n)$. Similar definitions apply to $\prod_{h_n\in H_n(S_n)} \mu_{l^r}$. Let us write T_{H_n} as a quotient:

Lemma 4.2.1. *Over S_n, the torus T_{H_n} is the quotient of $\prod_{h'_n\in H_n} \mathbb{G}_m$ by the diagonal. In particular, there is a surjection*

$$\prod_{h'_n\in H_n(S_n)} \mathbb{G}_m(S_n) \to T_{H_n}(S_n)$$

For $\sum_{h'_n}(g_{h'_n})(h'_n) \in \prod_{h'_n\in H_n(S_n)} \mathbb{G}_m(S_n)$ write $[\sum_{h'_n}(g_{h'_n})(h'_n)]$ for its class in $T_{H_n}(S_n)$. The elements in $T_{H_N}(S)$ are the G_n-invariant classes.

Proof. Clear, because the character group of T_{H_n} is $\ker(p_{H_n*}\mathbb{Z} \to \mathbb{Z})$. □

We have

$$H^0(H_n \times_S S_n, T_{H_n}) = \prod_{h_n\in H_n(S_n)} T_{H_n}(S_n).$$

With this notation we rewrite the section $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n : \mathbb{Z} \to T_{H_n}$ using the explicit description of 4.1.4 and 4.1.5. Write the section $t^*([\mathfrak{a}]^*[\mathfrak{a}]_* - N\mathfrak{a})\Delta_n : \mathbb{Z} \to T_{H_n}$ as

$$\sum_{h_n\in H_n(S_n)} s(h_n),$$

where $s(h_n) \in T_{H_n}(S_n)$. This section is given by the trivialization of the (trivial) line bundle $[\mathfrak{a}]^*\mathscr{O}([\mathfrak{a}]t_n) \otimes \mathscr{O}(t_n)^{\otimes -\mathrm{N}\,\mathfrak{a}}$ along H_n, where t_n is the unique point in $H_{n,t}$ projecting to h_n. Using Lemma 4.2.1, we write

$$s(h_n) = \Big[\sum_{h'_n} s(h_n)_{h'_n}\big(h'_n\big)\Big],$$

where $s(h_n)_{h'_n} \in \mathbb{G}_m$ is the value of the trivialization of $[\mathfrak{a}]^*\mathscr{O}([\mathfrak{a}]t_n) \otimes \mathscr{O}(t_n)^{\otimes -\mathrm{N}\,\mathfrak{a}}$ at h'_n.

We want to compute $s(h_n)_{h'_n}$ for $h'_n \in H_n(S_n)$. For this we need the elliptic units on E_n.

4.2.2 Elliptic units. Elliptic units were introduced and studied by Robert and Gillard. An algebraic approach to elliptic units was proposed by Kato.

Let us recall the definition of the elliptic units and their characterization in [Ka2] III 1.1.5. (see also [Scho]).

With Kato let us make the definition that $a, b \in \mathrm{End}_{\mathcal{O}_S}(E)$ are relatively prime $(a, b) = 1$, if $\ker(a) \cap \ker(b) = e$ where e is endowed with the reduced subscheme structure and $ab = ba$.

Theorem 4.2.2 ([Ka2] III 1.1.5. and [Scho] Thm. 1.2.1). *Let* $a \in \mathrm{End}_{\mathcal{O}_S}(E)$ *be an endomorphism with* $(a, 6) = 1$. *There is a unique section*

$$\theta_a \in \mathcal{O}^*(E \setminus \ker a)$$

compatible with base change in S, with the following properties:
i) $\mathrm{Div}(\theta_a) = \deg(a)(e) - \ker a$
ii) for any $b \in \mathrm{End}_{\mathcal{O}_S}(E)$ *with* $(a, b) = 1$

$$b_*\theta_a = \theta_a.$$

iii) Moreover, for $b \in \mathrm{End}_{\mathcal{O}_S}(E)$ *with* $(6, b) = 1$ *and* $ab = ba$

$$\frac{\theta_{a} \circ b}{\theta_a^{\deg(b)}} = \frac{\theta_{b} \circ a}{\theta_b^{\deg(a)}}.$$

Definition 4.2.3. *The values of* θ_a *at torsion sections* $t : \mathcal{O}_S \to E \setminus \ker a$ *are called* elliptic units.

4.2.3 The specialization of the polylog in terms of elliptic units. We turn to the computation of $s(h_n) = [\sum_{h'_n} s(h_n)_{h'_n}(h'_n)]$ from the end of Sect. 4.2.1.

Proposition 4.2.4. *Let* $[\mathfrak{a}]$ *be relatively prime to Nl. Then the section* $s(h_n)_{h'_n} \in \mathbb{G}_m(S_n)$ *is given by*

$$s(h_n)_{h'_n} = \theta_{\mathfrak{a}}\big(h'_n - t_n\big)^{-1}$$

for $h'_n \in H_n(S_n)$ *using the identification from Lemma 4.2.1.*

Proof. The section $s(h_n)_{h'_n}$ is given by evaluating the trivialization of

$$[\mathfrak{a}]^*\mathcal{O}([\mathfrak{a}]t_n) \otimes \mathcal{O}(t_n)^{\otimes -\mathrm{N}\,\mathfrak{a}}$$

at h'_n. Here t_n is such that $\mathrm{pr}_t(t_n) = h_n$. Let T_{-t_n} be the translation with $-t_n$ on E_n, then

$$[\mathfrak{a}]^*\mathcal{O}([\mathfrak{a}]t_n) \cong T^*_{-t_n}[\mathfrak{a}]^*\mathcal{O}(e_n) \text{ and } \mathcal{O}(t_n)^{\otimes \mathrm{N}\,\mathfrak{a}} \cong T^*_{-t_n}\mathcal{O}(e_n)^{\otimes \mathrm{N}\,\mathfrak{a}}.$$

The function $\theta_{\mathfrak{a}}$ gives a section of $\mathcal{O}(e_n)^{\otimes \mathrm{N}\,\mathfrak{a}} \otimes [\mathfrak{a}]^*\mathcal{O}(e_n)^{\otimes -1}$ and thus $T^*_{-t_n}\theta_{\mathfrak{a}}$ gives a section of

$$\mathcal{O}(t_n)^{\otimes \mathrm{N}\,\mathfrak{a}} \otimes [\mathfrak{a}]^*\mathcal{O}([\mathfrak{a}]t_n)^{\otimes -1}.$$

This proves the claim. □

Remark: Note that in this proposition we do not really need the elliptic units, because any function with the right divisor would also describe the sections $s(t_n)$. The point is that there is one more problem to solve before we get the specialization of the elliptic polylog. We still have to compute the splitting from 3.5.8 and it is here that the elliptic units will be necessary because of their norm compatibility property.

Consider the section

$$\sum_{\{t_n \mid \mathrm{pr}_t(t_n) \in H_n(S_n)\}} \left[\sum_{h'_n} \theta_{\mathfrak{a}}\big(h'_n - t_n\big)\big(h'_n\big) \right] (\mathrm{pr}_t(t_n))$$

of $\prod_{\mathrm{pr}_t(t_n)\in H_n(S_n)} T_{H_n}(S_n)$. It is invariant under the group G_n and thus defines a section in $H^0(H_n, T_{H_n})$. Denote by

$$\delta : H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[l^r])$$

the boundary map for the exact sequence

$$0 \to T_{H_n}[l^r] \to T_{H_n} \xrightarrow{[l^r]} T_{H_n} \to 0.$$

The main result of this section can now be formulated as follows:

Proposition 4.2.5. *Write* $\widetilde{t}_n$ *for* $\mathrm{pr}_t(t_n)$. *This is an element of* H_n. *Then, the class of*

$$\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*\big(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}}\big)^\vee(1) - \mathrm{pr}_t\, t^* N\mathfrak{a}\big(\mathscr{P}\mathrm{ol}_n^{\mathrm{g}}\big)^\vee(1) \in H^1(H_n, T_{H_n}[l^r])$$

is given by (up to sign)

$$\delta \sum_{\widetilde{t}_n \in H_n(S_n)} \left[\sum_{h'_n} \theta_{\mathfrak{a}}\big(h'_n - t_n\big)\big(h'_n\big) \right] (\widetilde{t}_n) \in H^1(H_n, T_{H_n}[l^r]).$$

4.2.4 The splitting. In this section we show that the element of Proposition 4.2.5 is in fact in the image of the map a from 3.5.6. Recall that the map a is

$$\mathrm{Ext}^1_S\big(\Lambda, \mathcal{R}_n^{\mathrm{g}}(1)\big) \xrightarrow{a} \mathrm{Ext}^1_{S,\mathcal{R}_n^{\mathrm{g}}}\big(\mathcal{I}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)\big).$$

We factor this as

$$\begin{aligned}\mathrm{Ext}^1_S\big(\Lambda, \mathcal{R}_n^{\mathrm{g}}(1)\big) &\to \mathrm{Ext}^1_S\big(\Lambda, \underline{\mathrm{Hom}}_{\mathcal{R}_n^{\mathrm{g}}}\big(\mathcal{R}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)\big)\big)\\ &\to \mathrm{Ext}^1_S\big(\Lambda, \underline{\mathrm{Hom}}_{\mathcal{R}_n^{\mathrm{g}}}\big(\mathcal{I}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)\big)\big),\end{aligned}$$

(where the last map is induced by $\mathcal{I}_n^{\mathrm{g}} \hookrightarrow \mathcal{R}_n^{\mathrm{g}}$) and the canonical map

$$\mathrm{Ext}^1_S\big(\Lambda, \underline{\mathrm{Hom}}_{\mathcal{R}_n^{\mathrm{g}}}\big(\mathcal{I}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)\big)\big) \to \mathrm{Ext}^1_{S,\mathcal{R}_n^{\mathrm{g}}}\big(\mathcal{I}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)\big).$$

Recall that the torus T_{H_n} is defined by the character group

$$I[H_n] := \ker(p_{n*}\mathbb{Z} \to \mathbb{Z}).$$

By definition of $\mathcal{I}_n^{\mathrm{g}}$, we get that $I[H_n] \otimes \Lambda = \mathcal{I}_n^{\mathrm{g}}$ is the augmentation ideal in $\mathcal{R}_n^{\mathrm{g}}$. In particular

$$T_{H_n}[l^r] = \underline{\mathrm{Hom}}_S\big(\mathcal{I}_n^{\mathrm{g}}, \mu_{l^r}\big).$$

This gives $H^1(H_n, T_{H_n}[l^r]) = H^1(S, \underline{\mathrm{Hom}}_S(\mathcal{I}_n^{\mathrm{g}}, \mathcal{R}_n^{\mathrm{g}}(1)))$, because $p_{n*}\mu_{l^r} = \mathcal{R}_n^{\mathrm{g}}(1)$. Write $\mathcal{R}_n^{\mathrm{g}}(1) = p_{n*}\mu_{l^r}$, then we can identify the above maps with

$$H^1(S, p_{n*}\mu_{l^r}) \to H^1(H_n, p_{n*}\mu_{l^r}) \to H^1(H_n, T_{H_n}[l^r]). \tag{11}$$

The first map has the following explicit description: Write

$$H^1(H_n, p_{n*}\mu_{l^r}) = H^1(S, p_{n*}\mathbb{Z}/l^r\mathbb{Z} \otimes p_{n*}\mu_{l^r})$$

then the map on coefficients is

$$\begin{aligned} p_{n*}\mu_{l^r} &\to p_{n*}\mathbb{Z}/l^r\mathbb{Z} \otimes p_{n*}\mu_{l^r} \\ a &\mapsto \sum_{h'_n} \big(h'_n\big) \otimes \big(h'_n\big)a, \end{aligned}$$

where $(h'_n)a$ is the multiplication of $(h'_n) \in p_{n*}\mathbb{Z}/l^r$ with $a \in p_{n*}\mu_{l^r}$. Write $a = \sum_{\widetilde{t}_n} \alpha_{\widetilde{t}_n}(\widetilde{t}_n)$, then

$$\big(h'_n\big)a = \sum_{\widetilde{t}_n} \alpha_{\widetilde{t}_n}\big(h'_n + \widetilde{t}_n\big) = \sum_{\widetilde{t}_n} \alpha_{\widetilde{t}_n - h'_n}(\widetilde{t}_n).$$

In particular an element $\delta \sum_{\widetilde{t}_n} \alpha_{\widetilde{t}_n}(\widetilde{t}_n) \in H^1(S, p_{n*}\mu_{l^r})$ is mapped to

$$\sum_{\widetilde{t}_n} \Big[\delta \sum_{h'_n} \alpha_{\widetilde{t}_n - h'_n}\big(h'_n\big)\Big](\widetilde{t}_n) \in H^1(H_n, T_{H_n}[l^r]).$$

To compute the splitting it suffices to write down a norm compatible element in $H^1(S, p_{n*}\mu_{l^r})$, which maps to the class of $\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1) - \mathrm{pr}_t\, t^* N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$ in $H^1(H_n, T_{H_n}[l^r])$. Define $\widetilde{t}_n := \mathrm{pr}_t(t_n)$, the projection of t_n to H_n.

Proposition 4.2.6. *The element*

$$\delta \sum_{\widetilde{t}_n \in H_n(S_n)} \theta_{\mathfrak{a}}(-t_n)(\widetilde{t}_n) \in H^1(S, p_{n*}\mu_{l^r})$$

maps under (11) to $\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1) - \mathrm{pr}_t\, t^* N\mathfrak{a}(\mathcal{P}\mathrm{ol}_n^{\mathrm{g}})^\vee(1)$.

Proof. From the above computation we get that the image is

$$\sum_{\widetilde{t}_n \in H_n(S_n)} \Big[\delta \sum_{h'_n} \theta_{\mathfrak{a}}\big(h'_n - t_n\big)\big(h'_n\big)\Big](\widetilde{t}_n).$$

This is the element in 4.2.5. □

We show now that the element defined in Proposition 4.2.6 is norm compatible if we vary n. This will imply that we have actually computed the splitting and thus the l-adic Eisenstein classes.

Let $\mathrm{N}_{n,n'}$ be the norm map from E_n to $E_{n'}$ for $n \geq n'$.

Proposition 4.2.7. *In $H^1(H_{n'}, \mu_{l^r})$ the following equality holds:*

$$\mathrm{N}_{n,n'}\, \delta \sum_{\widetilde{t}_n \in H_n(S)} \theta_{\mathfrak{a}}(-t_n)(\widetilde{t}_n) = \delta \sum_{\widetilde{t}_{n'} \in H_{n'}(S)} \theta_{\mathfrak{a}}(-t_{n'})(\widetilde{t}_{n'}),$$

where $[l^n]t_n = t$ and $[l^{n-n'}]t_n = t_{n'}$.

Proof. From Theorem 4.2.2 we know that $\mathrm{N}_{n,n'}\,\theta_{\mathfrak{a}}(-t_n) = \theta_{\mathfrak{a}}(-t_{n'})$ because $\mathrm{N}\,\mathfrak{a}$ is prime to Nl. This proves the claim. □

It is clear that the element $\delta \sum_{\widetilde{t}_n \in H_n(S_n)} \theta_{\mathfrak{a}}(-t_n)(\widetilde{t}_n)$ is compatible with the reduction map $\Lambda_r \to \Lambda_{r'}$ for $r \geq r'$. Hence we get an element

$$\left(\delta \sum_{\widetilde{t}_n \in H_n(S_n)} \theta_{\mathfrak{a}}(-t_n)(\widetilde{t}_n)\right)_n \in H^1\big(S, \mathcal{R}^{\mathrm{g}}_{\mathbb{Z}_l}(1)\big).$$

Recall the map

$$H^1\big(S, \mathcal{R}^{\mathrm{g}}_{\mathbb{Z}_l}(1)\big) = \mathrm{Ext}^1_{S,\mathbb{Z}_l}\big(\mathbb{Z}_l, \mathcal{R}^{\mathrm{g}}_{\mathbb{Z}_l}(1)\big) \xrightarrow{a} \mathrm{Ext}^1_{S,\mathcal{R}^{\mathrm{g}}_{\mathbb{Z}_l}}\big(\mathcal{I}^{\mathrm{g}}_{\mathbb{Z}_l}, \mathcal{R}^{\mathrm{g}}_{\mathbb{Z}_l}(1)\big)$$

from Corollary 3.5.7.

Lemma 4.2.8. *The element $\big(\delta \sum_{\widetilde{t}_n \in H_n(S_n)} \theta_{\mathfrak{a}}(-t_n)(\widetilde{t}_n)\big)_n$ maps to the class of $\mathrm{pr}_{[\mathfrak{a}]t}([\mathfrak{a}]t)^*[\mathfrak{a}]_*\, \mathcal{P}\mathrm{ol}^{\mathrm{g}}_{\mathbb{Z}_l} - \mathrm{pr}_t\, t^*\, \mathrm{N}\,\mathfrak{a}\, \mathcal{P}\mathrm{ol}^{\mathrm{g}}_{\mathbb{Z}_l}$ under the map a.*

Proof. This follows immediately from Proposition 4.2.6. □

4.2.5 The main theorem on the specialization of the elliptic polylog. It remains to compute the moment map to get the l-adic Eisenstein classes explicitly.

Theorem 4.2.9. *Let* $\beta = \sum_{t \in E[N](S) \setminus e} n_t(t)$ *and* $[\mathfrak{a}] : E \to E$ *an isogeny of degree relatively prime to* Nl. *Then for* $k \geq 0$ *the* l*-adic Eisenstein class*

$$\mathrm{N}\,\mathfrak{a}([\mathfrak{a}]^k\,\mathrm{N}\,\mathfrak{a} - 1)\big(\beta^*\,\mathscr{P}\mathrm{ol}_{\mathbb{Q}_l}\big)^k \in H^1\big(S, \mathrm{Sym}^k\,\mathscr{H}_{\mathbb{Q}_l}(1)\big)$$

is given by

$$\pm\frac{1}{k!}\left(\delta \sum_{t \in E[N](S) \setminus e} n_t \sum_{[l^n]t_n = t} \theta_{\mathfrak{a}}(-t_n)\widetilde{t}_n^{\otimes k}\right)_n$$

where $\widetilde{t}_n$ *is the projection of* t_n *to* $E[l^n]$.

Proof. The recipe to compute the moment map from Lemma 3.5.5 combined with Lemma 4.2.8 and Lemma 3.5.10 gives immediately the result. □

5 Proof of the main theorem

In this section we will carry out the actual comparison between the space $r_p(\mathscr{R}_\psi)$ and the Soulé elements $e_p(\overline{\mathcal{C}}_\infty \otimes T_pE(k))$ defined by elliptic units.

5.1 Comparison with the Soulé elements

We first transfer the result from 4.2.9 into the setting of 1.2.2 and then compare these elements with the Soulé map e_p.

5.1.1 The specialization of the elliptic polylog. The Theorem 4.2.9 gives us an explicit description of

$$\mathrm{N}\,\mathfrak{a}([\mathfrak{a}]^{2k+1}\,\mathrm{N}\,\mathfrak{a} - 1)\big(\beta^*\,\mathscr{P}\mathrm{ol}_{\mathbb{Q}_l}\big)^{2k+1} \in H^1\big(\mathcal{O}_S, \mathrm{Sym}^{2k+1}\,\mathscr{H}_{\mathbb{Q}_p}(1)\big),$$

which we now translate in the setting of Sect. 1.2.2. Note first that $p = l$ and $\mathscr{H}_{\mathbb{Q}_p} = T_pE \otimes \mathbb{Q}_p$ and $S = \mathrm{Spec}\,\mathcal{O}_S$. Here the second S denotes of course the set of primes dividing $p\mathfrak{f}$. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal prime to $6p\mathfrak{f}$. The isogeny we are going to consider is multiplication by $\psi(\mathfrak{a}) \in \mathcal{O}$. Let $\theta_{\mathfrak{a}}$ be the function defined in 4.2.2. To have shorter formulas we introduce the following notation: Define for $\widetilde{t}_r \in E[p^r]$

$$\gamma(\widetilde{t}_r)^k := < \widetilde{t}_r, \sqrt{d_K}\widetilde{t}_r >^{\otimes k}$$

where $< \_, \_ >$ is the Weil pairing and $\sqrt{d_K}$ is a root of the discriminant of $K/\mathbb{Q}$. Note that $\gamma(\psi(\mathfrak{p})\widetilde{t}_r)^k = (\mathrm{N}\,\mathfrak{p})^k\gamma(\widetilde{t}_r)^k$. In Sect. 1.2.1 formula (1) we explained the projection map $\mathcal{K}_{\mathcal{M}}$. On the Galois cohomology this is given by

$$H^1\big(\mathcal{O}_S, \operatorname{Sym}^{2k+1}\mathcal{H}_{\mathbb{Q}_p}(1)\big) \to H^1\big(\mathcal{O}_S, \mathcal{H}_{\mathbb{Q}_p}(k+1)\big)$$

induced by the projection $\operatorname{Sym}^{2k+1}\mathcal{H}_{\mathbb{Q}_p}(1) \to \mathcal{H}_{\mathbb{Q}_p}(k+1)$. Moreover

$$\mathcal{K}_{\mathcal{M}}(\psi(\mathfrak{a})^{2k+1}) \operatorname{Sym}^{2k+1}\mathcal{H}_{\mathbb{Q}_p}(1) = \psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^k \mathcal{H}_{\mathbb{Q}_p}(k+1).$$

Theorem 5.1.1. *Let $p \nmid 6\,\mathrm{N}\,\mathfrak{f}$ and denote for a $p^r\,\mathrm{N}\,\mathfrak{f}$-torsion point t_r by $\widetilde{t_r}$ its projection to $E[p^r]$. Then with $t = \Omega f^{-1}$*

$$\begin{aligned}&\mathrm{N}\,\mathfrak{a}(\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}-1)r_p(\xi)\\ &= \pm\frac{\mathrm{N}\,\mathfrak{f}^{3k+2}L_p(\overline{\psi},-k)^{-1}}{2^{k-1}\psi(f)}\left(\delta\,\mathrm{N}_{K(\mathfrak{f})/K}\sum_{p^r t_r=t}\theta_{\mathfrak{a}}(-t_r)\otimes\widetilde{t_r}\otimes\gamma(\widetilde{t_r})^k\right)_r\end{aligned}$$

where $\widetilde{t_r}$ is the projection of t_r to $E[p^r]$.

Proof. We have by definition and and Theorem 1.2.5

$$\begin{aligned}r_p(\xi) &= \frac{(-1)^{k-1}(2k+1)!L_p(\overline{\psi},-k)^{-1}}{2^{k-1}\psi(f)\,\mathrm{N}\,\mathfrak{f}^k}\mathcal{K}_{\mathcal{M}}\, r_p\big(\mathcal{E}^{2k+1}_{\mathcal{M}}(\beta)\big)\\ &= \frac{(-1)^{k-1}(2k+1)!\,\mathrm{N}\,\mathfrak{f}^{3k+2}L_p(\overline{\psi},-k)^{-1}}{2^{k-1}\psi(f)}\mathcal{K}_{\mathcal{M}}(\beta^*\,\mathcal{P}\mathrm{ol}_{\mathbb{Q}_l})^{2k+1}.\end{aligned}$$

With the above notation, we have

$$\mathcal{K}_{\mathcal{M}}\big(\widetilde{t_r}^{\otimes 2k+1}\big) = \widetilde{t_r}\otimes\gamma\,(\widetilde{t_r})^k$$

so that Theorem 4.2.9 gives:

$$\begin{aligned}&\mathrm{N}\,\mathfrak{a}(\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}-1)r_p(\xi)\\ &= \pm\frac{\mathrm{N}\,\mathfrak{f}^{3k+2}L_p(\overline{\psi},-k)^{-1}}{2^{k-1}\psi(f)}\left(\delta\,\mathrm{N}_{K(\mathfrak{f})/K}\sum_{p^r t_r=t}\theta_{\mathfrak{a}}(-t_r)\otimes\widetilde{t_r}\otimes\gamma(\widetilde{t_r})^k\right)_r.\end{aligned}$$

This is the desired result. □

5.1.2 The comparison theorem. We want to rewrite the formula in Theorem 5.1.1 in terms of the norm map for $K_n(\mathfrak{f})/K$.

Fix a prime $\mathfrak{p}$ of K where E has good reduction. Define a uniformizer by $\pi := \psi(\mathfrak{p})$. Denote by

$$H^{\mathfrak{p}}_{r,t} := \left\{t_r \in E[\mathfrak{p}^r\mathfrak{f}] \middle| \pi^r t_r = t\right\}.$$

We write $t_r = (\widetilde{t_r}, \pi^{-r}t) \in E[\mathfrak{p}^r\mathfrak{f}] = E[\mathfrak{p}^r] \oplus E[\mathfrak{f}]$. Denote by $K(\mathfrak{p}^r\mathfrak{f})$ the ray class field for $\mathfrak{p}^r\mathfrak{f}$. This is the field where the $E[\mathfrak{p}^r\mathfrak{f}]$-points are rational. Let $\sigma_{\mathfrak{p}}$ be the Frobenius at $\mathfrak{p}$ in the Galois group of $K(\mathfrak{f})/K$, then $t_r = (\widetilde{t_r}, t^{\sigma_{\mathfrak{p}}^{-r}})$. Recall that $\gamma(\widetilde{t_r})^k := <\widetilde{t_r}, \sqrt{d_K}\,\widetilde{t_r}>^{\otimes k}$. Define a filtration of $H^{\mathfrak{p}}_{r,t}$ as follows:

$$F^i_{r,t} := \left\{t_r = (\widetilde{t_r}, \pi^{-r}t) \in H^{\mathfrak{p}}_{r,t} | \pi^{r-i}\widetilde{t_r} = 0\right\}.$$

Thus

$$H^{\mathfrak{p}}_{r,t} = F^0_{r,t} \supset \ldots \supset F^r_{r,t} = 0.$$

Define $T_{\mathfrak{p}}E := \varprojlim_n E[\mathfrak{p}^n]$.

Theorem 5.1.2. *Let $\mathfrak{p}$ be as above and $t_r = (\widetilde{t_r}, \pi^{-r}t) \in F^0_{r,t} \setminus F^1_{r,t}$. Let $L_{\mathfrak{p}}(\overline{\psi}, -k)$ be the Euler factor for $\overline{\psi}$ at $\mathfrak{p}$ evaluated at $-k$, then*

$$L_{\mathfrak{p}}(\overline{\psi}, -k)^{-1}\left(\mathrm{N}_{K(\mathfrak{f})/K}\sum_{s_r \in H^{\mathfrak{p}}_{r,t}} \theta_{\mathfrak{a}}(-s_r) \otimes \widetilde{s_r} \otimes \gamma(\widetilde{s_r})^k\right)_r = \left(\mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K}(\theta_{\mathfrak{a}}(-t_r) \otimes \widetilde{t_r} \otimes \gamma(\widetilde{t_r})^k)\right)_r$$

in $H^1(\mathcal{O}_S, T_{\mathfrak{p}}E(k+1) \otimes \mathbb{Q}_p)$ for all $\mathfrak{a}$ relatively prime to $\mathfrak{p}\mathfrak{f}$.

Proof. Observe that we identified $\mathrm{Hom}_{\mathcal{O}_p}(T_pE, \mathcal{O}_p) \cong T_pE(-1)$ where T_pE has now the conjugate linear $\mathcal{O}_p$-action. In particular, $\overline{\psi(\mathfrak{p})}^i t_r = t_{r-i}$ for $t_r \in E[\mathfrak{p}^r]$. We compute

$$\begin{aligned}
&\left(\frac{\overline{\psi(\mathfrak{p})}}{\mathrm{N}\,\mathfrak{p}^{-k}}\right)^i \mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}\left(\theta_{\mathfrak{a}}(-t_r) \otimes \widetilde{t_r} \otimes \gamma(\widetilde{t_r})^k\right) = \\
&\quad = \mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}\left(\theta_{\mathfrak{a}}\big(-(\widetilde{t_r}, \pi^{-r}t)\big) \otimes \overline{\psi(\mathfrak{p})}^i\widetilde{t_r} \otimes \gamma\big(\overline{\psi(\mathfrak{p})}^i\widetilde{t_r}\big)^k\right) \\
&\quad = \left(\mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}\,\theta_{\mathfrak{a}}\big(-(\widetilde{t_r}, \pi^{-r}t)\big)\right) \otimes \widetilde{t_{r-i}} \otimes \gamma(\widetilde{t_{r-i}})^k \\
&\quad = \theta_{\mathfrak{a}}\big(-(\widetilde{t_{r-i}}, \pi^{i-r}t)\big) \otimes \widetilde{t_{r-i}} \otimes \gamma(\widetilde{t_{r-i}})^k,
\end{aligned}$$

where we used the distribution relation for $\theta_{\mathfrak{a}}$ (see [deSh] II 2.5)

$$\mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}\,\theta_{\mathfrak{a}}(-t_r) = \theta_{\mathfrak{a}}\big(\pi^i t_r\big)$$

for the last equality.

As the Galois group of $K(\mathfrak{p}^{r-i}\mathfrak{f})/K(\mathfrak{f})$ acts simply transitively on $F^i_{r,t} \setminus F^{i+1}_{r,t}$, we get

$$\left(\frac{\overline{\psi(\mathfrak{p})}}{\mathrm{N}\,\mathfrak{p}^{-k}}\right)^i \mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{f})}\left(\theta_{\mathfrak{a}}(-t_r)\otimes \widetilde{t}_r \otimes \gamma(\widetilde{t}_r)^k\right) = \sum_{t_{r-i}\in F^i_{r,t}\setminus F^{i+1}_{r,t}} \theta_{\mathfrak{a}}\big(-(\widetilde{t}_{r-i}, \pi^{i-r}t)\big)\otimes \widetilde{t}_{r-i}\otimes\gamma(\widetilde{t}_{r-i})^k.$$

We have $\theta_{\mathfrak{a}}(-(\widetilde{t}_{r-i}, \pi^{i-r}t)) = \theta_{\mathfrak{a}}(-(\widetilde{t}_{r-i}, \pi^{-r}t))^{\sigma^i_{\mathfrak{p}}}$ which gives

$$\left(\frac{\overline{\psi(\mathfrak{p})}}{\mathrm{N}\,\mathfrak{p}^{-k}}\right)^i \mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K}\left(\theta_{\mathfrak{a}}(-t_r)\otimes \widetilde{t}_r \otimes \gamma(\widetilde{t}_r)^k\right) = \mathrm{N}_{K(\mathfrak{f})/K}\sum_{t_{r-i}\in F^i_{r,t}\setminus F^{i+1}_{r,t}} \left(\theta_{\mathfrak{a}}\big(-(\widetilde{t}_{r-i}, \pi^{-r}t)\big)\otimes \widetilde{t}_{r-i}\otimes\gamma(\widetilde{t}_{r-i})^k\right)$$

because the norm $\mathrm{N}_{K(\mathfrak{f})/K}$ is the sum over all the Galois translates, which act trivially on $\widetilde{t}_{r-i}$. If we finally take the sum over i and let r get bigger and bigger we get

$$L_{\mathfrak{p}}(\overline{\psi(\mathfrak{p})}, -k)\left(\mathrm{N}_{K(\mathfrak{p}^r\mathfrak{f})/K}\,\theta_{\mathfrak{a}}(-t_r)\otimes \widetilde{t}_r \otimes \gamma(\widetilde{t}_r)^k\right)_r = \left(\mathrm{N}_{K(\mathfrak{f})/K}\sum_{t_r\in H^{\mathfrak{p}}_{r,t}} \theta_{\mathfrak{a}}(-t_r)\otimes \widetilde{t}_r \otimes \gamma(\widetilde{t}_r)^k\right)_r,$$

where we used

$$\sum_{i\geq 0}\left(\frac{\overline{\psi(\mathfrak{p})}}{\mathrm{N}\,\mathfrak{p}^{-k}}\right)^i = \frac{1}{1-\frac{\overline{\psi(\mathfrak{p})}}{\mathrm{N}\,\mathfrak{p}^{-k}}}.$$

This is the desired result. □

With Theorem 5.1.1 we get:

Corollary 5.1.3. *With the notations of Theorem 5.1.1*

$$\mathrm{N}\,\mathfrak{a}(\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}-1)r_p(\xi) = \pm\frac{\mathrm{N}\,\mathfrak{f}^{3k+2}}{2^{k-1}\psi(f)}\delta\left(\mathrm{N}_{K(p^r\mathfrak{f})/K}\,\theta_{\mathfrak{a}}(-t_r)\otimes \widetilde{t}_r \otimes \gamma(\widetilde{t}_r)^k\right)_r,$$

where $p^r t_r = t$ and t_r is a primitive $p^r\mathfrak{f}$-torsion point.

Proof. If p is inert of prime this is just a reformulation of Theorem 5.1.2. If p is split, r_p decomposes into a direct sum for the $\mathfrak{p}$ and the $\mathfrak{p}^*$ part. Putting them together gives the result. □

5.2 End of proof of the main theorem

Here we finish the proof of Theorem 1.1.5 by computing the image of the Soulé map e_p. In the last section we prove that r_p is injective on $\mathcal{R}_\psi$ if H^2_p is finite.

5.2.1 Relation to elliptic units. Our aim is to show that the elements

$$\left(\mathrm{N}_{K(p^r\mathfrak{f})/K}\,\theta_{\mathfrak{a}}(t_r)\otimes\widetilde{t}_r\otimes\gamma(\widetilde{t}_r)^k\right)_r$$

where $\mathfrak{a}$ is prime to $6p\mathfrak{f}$ generate $(\overline{\mathcal{C}}^{\chi}_{\infty}\otimes T_pE(k))_{\Gamma}$, where χ is the representation of Δ on $\mathrm{Hom}_{\mathcal{O}_p}(T_pE(k),\mathcal{O}_p)$.

Proposition 5.2.1. *Let $p\nmid 6\,\mathrm{N}\,\mathfrak{f}$ and $\mathfrak{a}$ be an ideal in $\mathcal{O}_p$, which is prime to $6p\mathfrak{f}$ and such that $\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}\not\equiv 1(\mathrm{mod}\ p)$. (For example take a prime $\mathfrak{q}$ such that the Frobenius at $\mathfrak{q}$ acts non trivially on $K(E[p](k+1))$). Then the $\mathcal{O}_p[[\Gamma]]$-module*

$$\overline{\mathcal{C}}^{\chi}_{\infty}\otimes_{\mathcal{O}_p}T_pE(k)$$

is generated by $\left(\theta_{\mathfrak{a}}(t_r)\otimes\widetilde{t}_r\otimes\gamma(\widetilde{t}_r)^k\right)_r$, where t_r is a primitive $p^r\mathfrak{f}$-division point.

Proof. Let $\mathfrak{b}$ be another ideal prime to $6p\mathfrak{f}$. Let $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ be the elements in Γ associated to $\mathfrak{a}$ and $\mathfrak{b}$ by the reciprocity map. Then by Theorem 4.2.2

$$\begin{aligned}\left(\sigma_{\mathfrak{a}}-\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}\right)&\left(\theta_{\mathfrak{b}}(t_r)\otimes\widetilde{t}_r\otimes\gamma(\widetilde{t}_r)^k\right)\\ &=\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^k\left(\theta_{\mathfrak{b}}(t_r)^{\sigma_{\mathfrak{a}}-\mathrm{N}\,\mathfrak{a}}\otimes\widetilde{t}_r\otimes\gamma(\widetilde{t}_r)^k\right)\\ &=\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^k\left(\theta_{\mathfrak{a}}(t_r)^{\sigma_{\mathfrak{b}}-\mathrm{N}\,\mathfrak{b}}\otimes\widetilde{t}_r\otimes\gamma(\widetilde{t}_r)^k\right).\end{aligned}$$

It is enough to show that $\sigma_{\mathfrak{a}}-\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}$ is invertible in $\Lambda=\mathcal{O}_p[[\Gamma]]$, because $\overline{\mathcal{C}}^{\chi}_{\infty}$ is a torsion free Λ-module. Λ is a local ring if p is inert or prime in K and a product of local rings if p is split. We have $\Lambda/\mathfrak{m}=\mathcal{O}_p/p$, where $\mathfrak{m}$ is either the maximal ideal or the product of the maximal ideals. The element $\sigma_{\mathfrak{a}}$ acts via 1 on $\mathcal{O}_p/p$ and thus $\sigma_{\mathfrak{a}}-\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}$ is invertible in Λ if $\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}\not\equiv 1(\mathrm{mod}\ p)$. It remains to see that $\gamma(\widetilde{t}_r)^k$ generates $\mathbb{Z}_p(k)$. We have

$$\begin{aligned}<\widetilde{t}_r,\sqrt{d_K}\widetilde{t}_r>^{\pm 1}&=\exp\left(p^{-r}|\Omega|^2\left(\overline{\sqrt{d_K}}-\sqrt{d_K}\right)\right)\\ &=\exp\left(-2ip^{-r}|\Omega|^2\sqrt{|d_K|}\right)\\ &=\exp\left(-4\pi ip^{-r}\right),\end{aligned}$$

which is for $p\neq 2$ a primitive root of unity. □

Corollary 5.2.2. *The image $r_p(\mathcal{R}_\psi)$ in $H^1(\mathcal{O}_S, T_pE(k+1)\otimes\mathbb{Q}_p)$ coincides with the image of*

$$e_p\big((\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k)\big)_\Gamma\big).$$

Proof. As $\frac{\mathrm{N}\,\mathfrak{f}^{3k+2}}{2^{k-1}\psi(f)}$ is prime to p, this follows from Corollary 5.1.3, the definition of e_p in 2.2.1 and the fact that $\mathrm{N}\,\mathfrak{a}(\psi(\mathfrak{a})\,\mathrm{N}\,\mathfrak{a}^{k+1}-1)$ is invertible in $\mathcal{O}_p$. □

To conclude the proof of Theorem 1.1.5 it remains to see the following lemma:

Lemma 5.2.3. *The canonical map*

$$\big(\overline{\mathfrak{C}}_\infty \otimes T_pE(k)\big) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p \to \big(\overline{\mathfrak{C}}_\infty \otimes T_pE(k)\big)_{\mathcal{G}} \cong \big(\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k)\big)_\Gamma$$

is an isomorphism and $(\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k))_\Gamma \cong \mathcal{O}_p$.

Proof. By [Ru3] Theorem 7.7 we have an isomorphism

$$\overline{\mathfrak{C}}^\chi_\infty \cong \Lambda^\chi = \mathcal{O}_p[[\Gamma]].$$

This implies that the $\mathcal{O}_p[[\Gamma]]$-module is induced and hence as $\mathcal{O}_p[[\Gamma]]$-module isomorphic to $\mathcal{O}_p[[\Gamma]]$. This implies $(\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k))_\Gamma \cong \mathcal{O}_p$ and the claim of the corollary, because the higher Tor-terms vanish. □

We get as a corollary part b) of Theorem 1.1.5:

Corollary 5.2.4. *The map*

$$\mathcal{R}_\psi \otimes \mathbb{Z}_p \to R\Gamma(\mathcal{O}_S, T_pE(k+1)\otimes\mathbb{Q}_p)[1]$$

induced by r_p, gives an isomorphism

$$\det\nolimits_{\mathcal{O}_p}\mathcal{R}_\psi \cong \det\nolimits_{\mathcal{O}_p} R\Gamma(\mathcal{O}_S, T_pE(k+1))^{-1}.$$

Proof. The complex $\mathcal{R}_\psi \otimes \mathbb{Z}_p \to R\Gamma(\mathcal{O}_S, T_pE(k+1)\otimes\mathbb{Q}_p)[1]$ is isomorphic to

$$\big(\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k)\big)_\Gamma \xrightarrow{e_p} R\Gamma(\mathcal{O}_S, T_pE(k+1)\otimes\mathbb{Q}_p)[1]$$

because by 5.2.2 r_p and e_p have the same image and as $\mathcal{O}_p$-modules $(\overline{\mathfrak{C}}^\chi_\infty \otimes T_pE(k))_\Gamma \cong \mathcal{O}_p$ and $\mathcal{R}_\psi\otimes\mathbb{Z}_p \cong \mathcal{O}_p$. Theorem 2.2.12 implies then the claim. □

5.2.2 Finiteness of H^2_p and injectivity of r_p. Here we prove the claim of Theorem 1.1.5, that the finiteness of H^2_p implies that r_p is injective on $\mathcal{R}_\psi$.

Recall that by Corollary 5.2.2 the image of r_p coincides with the image of e_p. It suffices for the injectivity of r_p to prove that $r_p(\mathcal{R}_\psi)$ is non zero, because $\mathcal{R}_\psi \cong \mathcal{O}_K$.

Proposition 5.2.5. *Let H^2_p be finite, then e_p is injective.*

Proof. We show first that H^2_p finite implies the finiteness of $(\mathcal{A}_\infty \otimes T_pE(k))_{\mathcal{G}}$. Note that this is the cokernel of

$$(\mathcal{U}_\infty \otimes T_pE(k))_{\mathcal{G}} \to (\mathcal{X}_\infty \otimes T_pE(k))_{\mathcal{G}}.$$

Computing up to finite groups we get from Corollary 2.2.7 (using Corollary 2.2.11) that this cokernel is isomorphic (up to finite groups) to the cokernel of

$$H^1\big(K \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* \to H^1\big(\mathcal{O}_{S_p}, E[p^\infty](-k)\big)^*$$

which is contained in $H^2(\mathcal{O}_{S_p}, T_pE(k+1))$. This group is of course finite if $H^2_p = H^2(\mathcal{O}_S, T_pE(k+1))$ is finite. Thus $(\mathcal{A}_\infty \otimes T_pE(k))_{\mathcal{G}}$ is finite. Using Lemma 6.2 from [Ru3] we see that this implies that $(\mathcal{A}_\infty \otimes T_pE(k))^{\mathcal{G}}$ is finite. We will now show that this last group controls the kernel of e_p. It suffices to show that the kernel of e_p on $(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}}$ is finite because by [Ru3] 7.8. both $\overline{\mathcal{E}}_\infty$ and $\overline{\mathcal{C}}_\infty$ are Λ-modules of rank 1 with $\overline{\mathcal{E}}_\infty/\overline{\mathcal{C}}_\infty$ a torsion module. So suppose that the image of $(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}}$ under e_p has not rank 1, i.e. is finite. Then, because $(\mathcal{U}_\infty \otimes T_pE(k))_{\mathcal{G}} \cong H^1(K \otimes \mathbb{Q}_p, E[p^\infty](-k))^*$ the image of $(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}}$ in $(\mathcal{U}_\infty \otimes T_pE(k))_{\mathcal{G}}$ must be finite as well. The kernel of the map

$$(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}} \to (\mathcal{U}_\infty \otimes T_pE(k))_{\mathcal{G}}$$

is $H_1(\mathcal{G}, \mathcal{U}_\infty/\overline{\mathcal{E}}_\infty \otimes T_pE(k))$ (group homology). On the other hand, up to finite groups, Corollary 2.2.7 implies that $H_1(\mathcal{G}, \mathcal{X}_\infty \otimes T_pE(k)) \cong H^2(\mathcal{O}_{S_p}, E[p^\infty](-k))^*$. By Lemma 2.2.8 we get a commutative diagram (up to finite groups)

$$\begin{array}{ccc}
H_1\big(\mathcal{G}, \mathcal{U}_\infty/\overline{\mathcal{E}}_\infty \otimes T_pE(k)\big) & \xrightarrow{\alpha} & H_1(\mathcal{G}, \mathcal{X}_\infty \otimes T_pE(k)) \\
\downarrow & & \downarrow \\
(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}} & \xrightarrow{e_p} & H^1(\mathcal{O}_{S_p}, T_pE(k+1)) \\
\downarrow & & \downarrow \\
(\mathcal{U}_\infty \otimes T_pE(k))_{\mathcal{G}} & \xrightarrow{\cong} & H^1\big(K \otimes \mathbb{Q}_p, E[p^\infty](-k)\big)^* .
\end{array}$$

The kernel of the map α is a quotient of $(\mathcal{A}_\infty \otimes T_pE(k))^{\mathcal{G}}$ which by the above is finite. Thus, we arrive at a contradiction and e_p can not be zero on the free part of $(\overline{\mathcal{E}}_\infty \otimes T_pE(k))_{\mathcal{G}}$. Hence, e_p is non zero on $(\overline{\mathcal{C}}_\infty \otimes T_pE(k))_{\mathcal{G}}$. □

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