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Harmonic measure and expansion on the boundary of the connectedness locus

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Abstract. The paper develops a technique for proving properties that are typical in the boundary of the connectedness locus with respect to the harmonic measure. A typical expansion condition along the critical orbit is proved. This condition implies a number of properties, including the Collet-Eckmann condition, Hausdorff dimension less than 2 for the Julia set, and the radial continuity in the parameter space of the Hausdorff dimensions of totally disconnected Julia sets.

1. Introduction

1.1. Generic properties

Dynamics of unimodal polynomials $f_c = z^d + c$ on the Riemann sphere was a subject of intensive studies in a couple of last decades. The focus was on determining generic systems and explaining their geometric structure. Despite considerable effort, only a limited progress was achieved. The research concentrated mainly on the simplest class of quadratic polynomials.

The notion of a generic map usually requires specification. Topological and metrical pictures of a typical dynamical system are often drastically different. This dichotomy is a staple of the dynamics of real quadratic polynomials. We will see that an analogous phenomenon is present on the

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boundary of the *connectedness locus*

$$\mathcal{M}_d = \left\{c \in \mathbb{C} : \sup_{n>0} \left| f_c^n(c) \right| < \infty \right\} .$$

Generic maps on the connectedness locus. Let $f_c = z^d + c$. Then its filled Julia set $\mathcal{K}_c$ is defined as

$$\mathcal{K}_c = \left\{z \in \mathbb{C} : \sup_{n>0} \left| f_c^n(z) \right| < \infty \right\} .$$

The connectedness locus $\mathcal{M}_d$ is the set of parameters c for which the corresponding Julia set $\mathcal{J}_c = \partial\mathcal{K}_c$ is connected. It is well known that $\mathcal{M}_d$ is a full compact, that is its complement is an open topological disk. For $c \in \mathcal{M}_d$ the critical orbit $\{f^n(c)\}$ belongs to the filled Julia set $\mathcal{K}_c$. When c traverses $\mathcal{M}_d$ in the outward direction, $\mathcal{K}_c$ which is initially connected bifurcates into a Cantor set outside of $\mathcal{M}_d$.

There are at least two intrinsic notions of a generic parameter on the boundary of $\mathcal{M}_d$, one with respect to the induced planar topology and the other with respect to the harmonic measure. A possibility of using the two-dimensional Lebesgue measure remains open since it is not known whether the area of the boundary of $\mathcal{M}_d$ is zero or not. We recall that the harmonic and two-dimensional Lebesgue measure on the boundaries of planar domains are always mutually singular, [14].

The topology and “outside geometry” of $\partial\mathcal{M}_d$ (given by the distribution of the harmonic measure) manifest themselves by very different properties of the corresponding generic dynamics.

The harmonic measure at ∞ of $\partial\mathcal{M}_d$ can be described in terms of the Riemann map

$$\mathcal{R}_d : \mathbb{D} = \{|z| < 1\} \mapsto \hat{\mathbb{C}} \setminus \mathcal{M}_d$$

which fixes ∞. Namely, $\mathcal{R}_d$ extends radially almost everywhere on the unit circle with respect to the normalized Lebesgue measure λ. Therefore,

$$\omega = \mathcal{R}_{d*}(\lambda) .$$

Logistic family. The family $z^2 + c$, $c \in M \cap \mathbb{R}$ is affinely equivalent to the logistic family

$$f_a : x \mapsto ax(1 - x),$$

$0 < a \leq 4$ and $x \in [0, 1]$.

The real Fatou conjecture, see [8], states that a set of parameters $0 < a \leq 4$ for which f_a has an attracting cycle is open and dense. In 1981 Jakobson proved that for a set $A \subset [0, 4]$ of positive Lebesgue measure, f_a has an invariant probability equivalent to the Lebesgue measure, [10].

These two notions of generic dynamics are fundamentally different. By Jakobson’s result, for a set of parameters of positive measure, the polynomial has almost every orbit distributed according to the absolutely continuous invariant measure while a topologically generic polynomial has almost every orbit attracted to a unique attracting cycle.

1.2. Expansion

For maps in the boundary of the connectedness locus one cannot expect hyperbolicity in the usual sense, meaning uniform expansion on the Julia set. The next best condition is uniform expansion on the ω-limit set c, which leads to the Misiurewicz condition about non-recurrence of the critical orbit. This is still too strong condition to be typical in the sense of the harmonic measure, so one is left looking for even weaker properties of expansion along the critical orbit.

Collet-Eckmann polynomials. The Collet-Eckmann condition serves as a natural bridge between topological and metrical aspects of one-dimensional dynamics. For unimodal polynomials $f_c = z^d + c$, the condition is that

$$\liminf_{n\to\infty} \frac{\log\left|D f_c^n(c)\right|}{n} > 0 .$$

Misiurewicz maps are defined by the condition that the critical point is not recurrent and all periodic points are repelling.

It is well known that a real Collet-Eckmann unimodal polynomial has on one hand an invariant absolutely continuous probability measure while on the other hand its dynamics shares some properties with Misiurewicz maps. Benedicks and Carleson's theorem (see [1]) states that in the logistic family the Collet-Eckmann condition holds for a set of positive Lebesgue measure. Being "almost hyperbolic" and abundant, Collet-Eckmann maps on the interval were studied intensively in the eighties. Yet, it took another decade before in the mid-nineties Przytycki advanced the technique of "shrinking neighborhoods" to study complex maps which satisfy the Collet-Eckmann condition.

In [15], Collet-Eckmann rational maps were studied for the first time, mainly from the point of view of ergodic and measure theoretical properties. Another direction was adopted in [6], where an interaction between the Collet-Eckmann condition and regularity problems of the Fatou components was explored.

Definition 1.1 *The boundary of a simply connected domain Ω is called a* Hölder compact (with exponent $\alpha \in (0,1]$) *if the Riemann mapping $\phi : \mathbb{D} \to \Omega$ can be extended to a Hölder continuous (with exponent α) mapping on the closed unit disk.*

The Collet-Eckmann condition for unimodal polynomials implies Hölder regularity of Julia sets, [6]. The immediate consequence, by the work of [11], is that the Hausdorff dimension of the Julia set of a Collet-Eckmann unimodal polynomial is strictly less than 2.

A few definitions. Recall map $\mathcal{R}_d$ which uniformizes the complement of $\mathcal{M}_d$ in the sphere and is tangent to $z \to 1/z$ at 0. Denote

$$r_d(\gamma) := \overline{\{\mathcal{R}_d(te^{2\pi i\gamma}) : 0 < t < 1\}} \setminus \{\infty\}.$$

The *Green function* of the filled-in Julia set $\mathcal{K}_c$ is defined in terms of iterates of f_c as:

$$G_c(z) = \lim_{n\to\infty} \frac{\log\left|f_c^n(z)\right|}{d^n} .$$

Definition 1.2 *For w in the unit disc, let $\log w = x + iy$. Define the set $\hat{Q}(w)$ to be the image by $\exp$ of the square:*

$$\left\{u + iv : |u - x|, |v - y| < \frac{|x|}{2}\right\} .$$

If $G_c(z) \geq 2G_c(0)$, define a Whitney domain $Q(z) := \Gamma_c(\hat{Q}(\Gamma_c^{-1}(z)))$, where Γ_c is the Böttker coordinate on $D(0, \exp(-G_c(0)))$.

Whitney domains are constructed to control distortion of f_c. Since $Q(z)$ is a very simple set in the Böttker coordinate, the action of f_c on $Q(z)$ is simple, i.e. $f_c(Q(z)) = Q(f_c(z))$.

Typical expansion statement. Let us state our main theorem which establishes an expansion along the critical orbit, typically with respect to the harmonic measure.

Theorem 1.1 *Fix $d \geq 2$. There is a set $\mathcal{H} \subset [0, 1)$ of full Lebesgue measure with the following properties. For every $\gamma \in \mathcal{H}$ there exist positive constants ϵ, θ, σ and an infinite increasing sequence of integers m_i, so that the following hold whenever $c \in r_d(\gamma)$ and $G_c(c) < \epsilon$:*

- *if $G_c(f_c^{m_i}(c)) \leq \epsilon$, then there is a neighborhood U_i of c which contains $Q(c)$, is mapped univalently by $f_c^{m_i}$ onto the geometric disc $D(f_c^{m_i}(c), \sigma)$, and $U_{i-1} \setminus \overline{U_i}$ is a ring domain with conformal modulus at least ϵ,*
- $\limsup_{i\to\infty} \frac{m_i}{i} \leq \theta$,
- $\lim_{i\to\infty} \frac{m_{i+1}}{m_i} = 1$.

Theorem 1.1 establishes a form of Tsujii's condition introduced in [19] for almost every $c \in \mathcal{M}_d$ in the sense of the harmonic measure. In our setting, this condition means that we can pass univalently to a large scale from neighborhoods of the critical value with positive density and relative gaps shrinking to 0. As such the condition appears stronger than the Collet-Eckmann condition and in fact implies it easily. For all we know, however, it might still be equivalent. For c not in $\mathcal{M}_d$, the same expansion holds until the orbit of c gets far away from $\mathcal{K}_c$. The constant θ has a purely combinatorial meaning (see Sect. 2.2 and estimate (3)) and is bounded by $9p$, where p is the maximal number of external rays which land at a fixed point of $z^d + c$, $c \in r_d(\gamma)$. In fact, $\theta(\gamma)$ is constant on open sets.

An added feature of Theorem 1.1 is some uniformity of estimates. We claim uniformity along external radii with arguments from $\mathcal{H}$, which will be used in some corollaries.

Genericity of the Collet-Eckmann condition. We will derive this result from Theorem 1.1.

Theorem 1.2 *For every $d \geq 2$ there is a subset $\mathcal{H} \subset [0, 1)$ of full Lebesgue measure such that for every $\gamma \in \mathcal{H}$ there are constants $K > 0$ and $\lambda > 1$ so that for each $c \in r_d(\gamma)$ and every $n > 0$*

$$\left|D f_c^n(c)\right| \geq K\lambda^n \ .$$

In particular, for all $c \in \bigcup_{\gamma\in\mathcal{H}} r_d(\gamma)$ the Collet-Eckmann condition holds. The fact that the Collet-Eckmann condition is satisfied for almost every $c \in \mathcal{M}_d$ in the sense of the harmonic measure was independently proved by S. Smirnov, see [18]. That work also includes estimates of the Hausdorff dimension of maps violating the Collet-Eckmann condition, which we only show to be of measure zero. On the other hand, it is known that the Collet-Eckmann condition cannot be satisfied by infinitely tunable maps, see [9], which form a set with positive harmonic capacity, see [13].

As an immediate corollary to Theorem 1.2 we obtain that

Corollary 1 *For almost every parameter $c \in \partial\mathcal{M}_d$, the corresponding Julia set $\mathcal{J}_c$ is locally connected and has Hausdorff dimension strictly less than* 2*.*

Also,

Corollary 2 *For almost every $c \in \mathcal{M}_d$ in the sense of the harmonic measure the orbit of c is dense in the Julia set.*

Corollary 2 follows since the Julia set is locally connected and the external argument of c in the parameter space is the same as the external angle of the ray from ∞ which converges at c. But almost every external angle is dense in the circle under the action of the map $x \to dx \pmod 1$.

These corollaries about the outside geometry of the Mandelbrot set should be contrasted with Shishikura's work on topologically generic quadratic polynomials, [17]. He proved that for a residual set of parameters from the boundary of the Mandelbrot set the Hausdorff dimension of the corresponding Julia set is equal to 2, in contrast to Corollary 1. Implicitly, he proved that the set of Misiurewicz parameters on the boundary of the Mandelbrot set has the Hausdorff dimension 2. This again is contrasted by Corollary 2.

The harmonic measure also admits a probabilistic interpretation. If we release a Brownian particle $p(t)$ from ∞ and E is a subset of the boundary of $\mathcal{M}_d$, then

$$\omega(E) = \mathrm{Prob}\{p(t_0) \in E\} \ ,$$

where t_0 is the first time the particle $p(t)$ hits $\mathcal{M}_d$. Hence, a generic Brownian particle on its trip to the Mandelbrot set omits a subset of its boundary of the Hausdorff dimension 2 (Shishikura's result), and accumulates on a set of dimension 1 (by Makarov's work [12]) of parameters for which Julia sets have Hausdorff dimension strictly less than 2.

Radial limits of the Hausdorff dimension of Julia sets. For every unicritical Collet-Eckmann polynomial $z^d + c$ with $c \in \partial\mathcal{M}_d$, there exists a sequence $c_n \in \hat{\mathbb{C}} \setminus \mathcal{M}_d$ (Shishikura's theorem) such that

$$\lim_{n\to\infty} \mathrm{HD}(\mathcal{J}_{c_n}) = 2 \ .$$

By [6], $\mathrm{HD}(\mathcal{J}_c) < 2$ and hence the Hausdorff dimension of Julia sets as a function of $c \in \hat{\mathbb{C}} \setminus \mathcal{M}_d$ does not extend continuously to $\partial\mathcal{M}_d$.

Another type of discontinuity of $\mathrm{HD}(\cdot)$ is due to a parabolic implosion. Assume that $z^d + c_0$, $c_0 \in r_d(\gamma)$, has a parabolic cycle. In this setting, the parabolic implosion means that $\mathcal{J}_{c_0}$ is strictly smaller than the Hausdorff limit of $\mathcal{J}_c$ when $r_d(\gamma) \ni c \to c_0$. It was recently shown in [3] that if $d = 2$ and $c > 1/4$ then

$$\mathrm{HD}(\mathcal{J}_{1/4}) < \liminf_{c\mapsto 1/4} \mathrm{HD}(\mathcal{J}_c) \leq \limsup_{c\mapsto 1/4} \mathrm{HD}(\mathcal{J}_c) < 2 \ .$$

Yet, typically with respect to the harmonic measure on $\partial\mathcal{M}_d$, $\mathrm{HD}(\cdot)$ extends radially as a continuous function.

Corollary 3 *For every $d \geq 2$ there is a subset $\mathcal{H}$ of $[0, 1)$ of full Lebesgue measure such that for every $\gamma \in \mathcal{H}$ and $c_0 \in r_d(\gamma) \cap \mathcal{M}_d$,*

$$\lim_{r_d(\gamma)\ni c\to c_0} \mathrm{HD}(\mathcal{J}_c) = \mathrm{HD}(\mathcal{J}_{c_0}) \ .$$

The proof of Corollary 3 is based on Theorem 1.2, in particular its claim regarding uniform expansion along r_d, and the continuity properties of the Hausdorff dimension in the class of rational functions which satisfy the so-called summability condition in a uniform way, [7].

Definition 1.3 *We say that polynomials $f_c(z) = z^d + c$, $c \in \mathcal{I}$, satisfy the uniform summability condition with exponent α if they do not have parabolic orbits and there exists $M > 0$ so that for every $c \in \mathcal{I}$,*

$$\sum_{j=1}^{\infty} \left| D f_c^j(c) \right|^{-\alpha} < M \ .$$

One of the main results of [6] asserts continuity of the Hausdorff (Minkowski) dimension of Julia sets in the class of uniformly summable rational functions. We formulate a weak version of this result for unicritical polynomials.

Fact 1.1 *If polynomials $f_c(z) = z^d + c$ satisfy the uniform summability condition with an exponent $\alpha < \frac{1}{d+1}$ then*

$$\lim_{c\to c_0} \mathrm{HD}(\mathcal{J}_c) = \mathrm{HD}(\mathcal{J}_{c_0}) \ .$$

If $c \in r_d(\gamma)$ then, by Theorem 1.2, $z^d + c$ are uniformly summable with any positive exponent α and Corollary 3 follows.

1.3. Outline of the proof

The main result is Theorem 1.1. Theorem 1.2 will be derived from it.

The paper begins with a review of Yoccoz partitions and their relation with induced dynamics. We recall that to construct a Yoccoz partition one needs to find a repelling fixed point and a periodic ray from ∞ which converges at this point. These rays divide the plane into finitely many pieces which we call ray-sectors. We show how such a point can be found for polynomials *outside* of $\mathcal{M}_d$, and then demonstrate that it persists as c tends to the boundary of $\mathcal{M}_d$ along almost every external radius. This part of the paper basically recapitulates known facts.

The key observation is that itinerary of the critical orbit under f_c through the ray-sectors is easily predictable in terms of the external argument γ of c defined by the Riemann map of the complement of $\mathcal{M}_d$. In fact, it is the same as the itinerary of γ under the map $T(x) := dx \pmod 1$ with respect to the partition of the circle by the external angles of the rays converging to the fixed point.

We interpret the dynamical system defined by T probabilistically as a Bernoulli shift. This allows us to prove a certain property of almost every itinerary purely in terms of the Bernoulli shift, essentially stating that the critical itinerary does not come back too close to itself too often under the shift. This property, however, translates into the same property of the critical itinerary with respect to the ray-sectors which holds almost everywhere with respect to the harmonic measure, and that property implies Theorem 1.1.

The proof is self-contained except for rudiments from complex dynamics.

2. Yoccoz partitions

We consider the dynamics of polynomials $f_c(z) = z^d + c$ with c a complex parameter, z a complex variable, and d an integer greater than 1. The degree d is fixed once for all in our proofs. The parameter c is variable, but sometimes we will still suppress it from the notation by writing f for f_c.

2.1. Rays from ∞

Recall the Green function of the filled-in Julia set $\mathcal{K}_c$:

$$G_c(z) = \lim_{n\to\infty} \frac{\log \left| f_c^n(z) \right|}{d^n} .$$

When $c \notin \mathcal{K}_c$, then the Green function has critical points at points $f_c^{-i}(0)$ for $i = 0, 1, \cdots$. A *smooth ray* in the phase space is a gradient line of the G_c with closure that intersects both ∞ and $\mathcal{K}_c$. It is understood that a gradient line by definition avoids critical points of G_c and is, therefore, smooth. The literature sometimes talks of rays which are not smooth, but we will not discuss those. For some smooth rays, the closure intersects $\mathcal{K}_c$ at precisely one point. We then say that the ray *converges* at that point. Notice that the image of a smooth ray ρ is another smooth ray and if ρ converges at z, then $f_c(\rho)$ converges at $f_c(z)$.

Dynamics near ∞. All gradient lines are well defined on the set $\{z : G_c(z) > G_c(0)\}$. They are labeled by angles at which they enter ∞, the so-called *external angles*. We will follow the tradition and identify the set of external angles with real numbers modulo 1. On the set of angles we have the map $T(x) := dx$ modulo 1. Map f_c sends a gradient line with external angle x to the gradient line with external angle $T(x)$. Of particular importance is the *critical external angle* $\gamma(c)$ which is the angle of the gradient line which passes through c.

Each of these gradient lines near ∞ continues until it meets either a critical point of G_c or converges to $\mathcal{K}_c$. If a line meets a critical point, then some image of it is a gradient line which hits c. We see that a gradient line with external angle x extends to a smooth ray provided that x is not a preimage of $\gamma(c)$ under T.

Riemann map of the complement of $\mathcal{M}_d$. We have the following basic fact, see [2]:

The function $\exp(-G_c(c) + i\gamma(c))$ *considered on the complement of the connectedness locus on the Riemann sphere is univalent and maps onto the unit disk. Its inverse is the Riemann map* $\mathcal{R}_d$.

Any line in the parameter space of the form $\gamma(c) = \gamma_0$ will be termed an *external radius* with angle γ_0.

Rays converging at fixed points. We will now consider what happens when c is outside of the connectedness locus and varies along an external radius. This means that the external argument of the critical value c in the phase plane remains fixed at γ.

Lemma 2.1 *Suppose that the orbit of γ under T is dense. Then for each c in this external radius one can find a repelling fixed point $q(c)$ which depends analytically on c and an external angle λ with the following properties:*

- *λ is periodic under T with period $p > 1$,*
- *the entire orbit of λ under T is contained in a certain arc S, which contains points $T^i(\gamma)$, $0 \leq i \leq p-2$, but no elements of $T^{-1}(\gamma)$,*
- *there is a smooth ray of external argument λ which converges to $q(c)$,*
- *λ only depends on γ, i.e. remains constant along the external radius.*

Proof: Choose c on this external radius and consider the set $L := \{z : G_c(z) < 0.9\, G_c(0)\}$. Set L consists of d disjoint smooth Jordan domains, each of which is mapped by f_c univalently over the disc $f_c(L)$ containing $\overline{L}$. Since the mapping is an expansion in the hyperbolic metric of $f_c(L)$, sets of points characterized by finite itineraries through various components of L shrink exponentially in diameter.

The boundary of each component of L attracts gradient lines from ∞ with external angles contained between two preimages of γ under T, consecutive in the cyclical ordering. At least one of these d sets is free from the fixed points of T. Choose one and call it S. The corresponding component of L is L_0. L_0 contains a fixed point of f_c uniquely defined by the condition that it remains in L_0 forever under iteration by f_c. This is $q(c)$. Observe that if an external angle forever stays in S under iteration, then there is a smooth ray with this angle which converges at $q(c)$. Indeed, the orbit of this angle under T is not dense, so it cannot hit γ whose orbit is dense by the hypothesis, thus the ray is smooth. Also, it intersects the boundaries of all sets defined by finite itineraries $L_0 \cdots L_0$. These sets shrink to $q(c)$ in the Hausdorff metric, and so each such ray indeed converges at $q(c)$.

The set $\Lambda = \bigcap_{i=0}^{\infty} T^{-i}(S)$ is clearly compact, non-empty, nowhere dense and $T(\Lambda) \subset \Lambda$. We will prove that Λ is periodic. Let us first suppose that Λ is infinite.

Consider connected components of the complement of Λ and call them *gaps*. Those which are contained in S will be designated as *inner*. The remaining *outer* gap has length greater than $\frac{d-1}{d}$. Fix an orientation on the circle so we may talk of the beginning and end of each gap. Since Λ is infinite it has infinitely many gaps. Notice that each inner gap is mapped by T onto some gap, possibly the outer one. Indeed, if x is inside an inner gap and $T(x) \in \Lambda$, then the orbit of x forever stays in S and hence $x \in \Lambda$.

We claim that for each $i > 0$, there is an inner gap G_i mapped by T^i onto the outer one. Write G_0 for the outer gap. For some i, consider the set $P_i := \bigcup_{j=0}^{i-1} T^{-j}(G_0)$. Since we assumed that there are infinitely many gaps, the complement of P_i has interior and so cannot be mapped into itself under T. Then there is an inner gap G_i not in P_i, but with the image in P_i. One easily sees that $T^i(G_i) = G_0$.

Gaps G_i, $i > 0$, cannot be all distinct, since the length of G_i is $d^{-i}|G_0|$, and their joint length would be more than $1/d$ in contradiction to the fact that all these gaps are in S. Thus $G_i = G_j$ for some $j > i$, or $T^{j-i}(G_0) = G_0$ which cannot be. Thus Λ is finite, let's say with p elements. Observe that since Λ is contained in an arc with length less than $1/d$, T acts on Λ as a permutation. The shortest gap G_{p-1} is mapped on all other gaps by consecutive iterations of T. Hence, the orbit of an endpoint of G_{p-1} is transitive in Λ, and Λ is a single periodic orbit.

To finish the proof of Lemma 2.1, we still have to look at the second claim. Λ was constructed so that it is contained between two consecutive preimages of γ. The point γ belongs to a gap of Λ. Unless this gap is G_{p-1}, it has a preimage under T which is another inner gap. Then γ has a preimage in that inner gap, contrary to the construction of Λ. Hence $\gamma \in G_{p-1}$ and so $T^i(\gamma)$ for $0 \le i \le p-2$ all belong to inner gaps. □

Passage to the boundary

Lemma 2.2 *Fix some c_0 in the boundary of the connectedness locus and in the closure of an external radius of angle γ. For almost every γ in the sense of the Lebesgue measure, one gets the following picture.*

The orbit of γ under T is dense on the circle, and for every c on the external radius one gets the repelling fixed point $q(c)$ which intercepts a periodic smooth ray with external argument λ, by Lemma 2.1. Furthermore, the function $q(c)$ has a limit q_0 as $c \to c_0$, q_0 is repelling and the ray with angle λ still converges at q_0.

Proof: Consider the subset Γ of the circle consisting of all γ with dense orbits under T. Clearly, Γ is of full measure. By subtracting from Γ another set of zero measure, we can assume that every external radius with angle from Γ converges to some point on the boundary of the connectedness locus. Clearly, if a fixed point is repelling and attracts a smooth ray with external angle α, then this situation is stable under a perturbation, moreover the ray moves continuously in the Hausdorff metric on the sphere.

Let c_n belong to the radius and converge to c_0. The fixed points $q(c_n)$ converge to a fixed point q_0. If q_0 is repelling, then we are done. So it remains to rule out the case of q_0 neutral. The ray with external angle λ, must converge to some periodic orbit Q of f_{c_0} which is either repelling or neutral. Notice that $c_0 \in \mathcal{M}_d$, and in this situation convergence of rays with rational angles goes back to [5]. Point Q cannot be irrationally neutral by the "snail argument", see the proof of Lemma 2.4 on pages 76–77 in [1]. It may be parabolic. However, there are only countably many c_0 for which f_{c_0} has a parabolic orbit, as we prove below. On the other hand, the harmonic measure is non-atomic, see Theorem 17.18 on page 345 in [16]. So, the harmonic measure of c_0 for which f_{c_0} has a parabolic orbit is null. So, by subtracting from Γ a set of zero measure we see that Q is repelling. But then it persists under a perturbation, so a nearby periodic point $Q(c_n)$ still intercepts the ray with external argument λ for almost all c_n. Hence $Q(c_n) = q(c_n)$ and $Q = q_0$ in the limit which proves that q_0 is repelling. □

For completeness, we prove the following.

Fact 2.1 *For every $d > 1$ there are only countably many complex values of c for which the map $f_c(z) = z^d + c$ has a parabolic periodic point.*

Proof: We will prove a stronger statement.

If $k > 0$, and λ is a complex number with absolute value less or equal to 1, then the pair of equations

$$f_c^k(z) = z \text{ and } \frac{d f_c^k}{dz}(z) = \lambda$$

has only finitely many solutions (c, z).

The proof is based on the following theorem about Riemann surfaces of algebraic functions, see [4] Theorem IV.II.4 on pages 231–232,

Fact 2.2 *Consider the equation $P(c, z) = 0$ where P is an irreducible polynomial of two complex variables. Then the set of solutions, compactified by adding points at infinity, has the structure of a compact Riemann surface. Moreover, projections on c and z are meromorphic of this surface.*

This theorem applied to the polynomial $f_c^k(z) - z = 0$ implies that the set of solutions splits into the union of finitely many compact Riemann surfaces. On each of these, the function

$$\frac{d f_c^k}{dz}(z)$$

is meromorphic. If it takes value λ infinitely many times, by the identity principle it must be constant on one of the surfaces, call it S. If a pair (c, z) solves both equations, it means that c must be in the connectedness locus in the parameter space, and z is in the Julia set. Hence, both projections map the finite points of S into a bounded set in the complex plane. The image of S under either projection must be compact, since the projection is continuous. But since the projections are also open mappings or constant, the image of either of them is just a point. Hence, S must be a point, which is impossible. □

2.2. Construction of symbolic dynamics

Ray-sectors and gaps. For this section, we assume that $c \in \mathcal{M}_d$ is in the closure of the external radius with angle γ and γ belongs to the set of full measure on the circle on which the assertions of Lemmas 2.1 and 2.2 hold. In particular, f_c has a fixed point $q(c)$ which attracts a smooth ray with external angle λ. This ray is periodic under T with period p. Rays with external angles λ, .., $T^{p-1}(\lambda)$ divide that plane into p ray-sectors. These sectors correspond to the gaps of the set $\Lambda = \{T^i(\lambda) : \ i = 0, \cdots, p-1\}$ in the parlance of the proof of Lemma 2.1. Namely, each gap consists of the external angles of gradient lines from ∞ which belong to the corresponding ray-sector. As in the proof of Lemma 2.1 we distinguish inner and outer gaps.

The action of f_c on ray-sectors is easy to understand in terms of T acting on gaps. Denote s_1 the gap which contains γ and s_0 the outer gap. Then Lemma 2.1 implies that s_1 needs $p-1$ iterates of T to be mapped onto s_0. This affords a natural labeling $s_1, \cdots, s_{p-1}$ such that $T(s_i) = s_{i+1}$ modulo p. Finally s_0 is mapped over the whole picture several times. We apply the same labeling to ray-sectors S_i. Then f_c is univalent on any S_i, $i > 0$, and maps it onto S_{i+1}, and has a critical point 0 in S_0. S_0 is mapped onto all ray-sectors.

Itineraries. We can now study itineraries of points in the phase plane of f_c with respect to the ray-sectors and points on the circle with respect to the gaps. Consider itineraries $k(c, z) = k_0, \cdots, k_n, \cdots$ defined by the condition $f_c^i(z) \in S_{k_i}$ and $\ell(\gamma) = \ell_0, \cdots, \ell_n, \cdots$ given by $T^i(\gamma) \in s_{\ell_i}$. Since neither the ray-sectors nor gaps form a complete covering, either itinerary may be finite.

Here is a key observation.

Lemma 2.3 *For c typical with respect to the harmonic measure, $k(c, c) = \ell(\gamma)$ and both are infinite.*

Proof: Choose a sequence c_n of points in the external radius with argument γ, in such a way that $c_n \to c$. Typically, by Lemma 2.1, $\ell(\gamma)$ is indeed infinite. Moreover, $k(c_n, c_n) = \ell(\gamma)$ for every n, because c_n belongs to the gradient line from ∞ with external angle γ. The limiting picture as $c_n \to c$ is described by Lemma 2.2 which implies that the sets consisting of $q(c_n)$ and the smooth rays with angles $\lambda, \cdots, T^{p-1}(\lambda)$ in the phase space of f_{c_n} converge in the Hausdorff metric on the sphere to $q(c)$ and the corresponding rays in the phase space of f_c. Thus, one gets $k(c_n, c_n) \to k(c, c)$ in Tychonoff's topology, and so $k(c, c) = \ell(\gamma)$ follows. □

The Yoccoz partition. Let us continue to develop the picture of ray-sectors and corresponding gaps. Consider the topological disc $\Delta := \{z : G_c(z) < L\}$, L is a parameter which will be specified by Fact 2.3. Δ intersects each ray-sector S_i along a "curvilinear triangle" Δ_i. The collection of these Δ_i's is sometimes referred to as the *Yoccoz partition* for f_c. One can consider itineraries $k'(c, z)$ with respect to the Yoccoz partition. Clearly, $k'(c, z)$ is an initial substring of $k(c, z)$, but may be shorter if the orbit of z leaves Δ. For $z \in \mathcal{K}_c$, the itineraries are identical.

Induced maps. The action of f_c on ray-sectors S_i, $i > 0$, is boring: each is mapped univalently onto S_{i+1} modulo p. As a result, codes k and ℓ contain a lot of redundant information, since every non-zero symbol predicts the following one.

Definition 2.1 *An induced map Φ_c is defined on the union of all ray-sectors S_i. On S_i, $i > 0$, we set $\Phi_c := f_c^{p-i+1}$. On S_0, $\Phi_c = f_c$.*

Fact 2.3 *For an appropriate choice of the parameter L, depending on γ, in the construction of the Yoccoz partition, the following statements hold:*

- *Φ_c maps any Δ_i over the union of all Δ_j.*
- *If $K \subset \Delta_j$ is relatively compact in Δ_j, then $\Phi_c^{-1}(K)$ is relatively compact in $\bigcup_{i=0}^{p-1} \Delta_i$.*
- *If $j \neq 1$, then $\overline{\Phi_c^{-1}(\Delta_j)}$ is relatively compact in $\bigcup_{i=0}^{p-1} \Delta_i$.*
- *If the parameter L is chosen sufficiently small, depending only on the set of rays converging at $q(c)$ and if $z \in \Delta_i$, $\Phi_c(z) \in \Delta_j$, $j \neq 1$, then the closure $\overline{Q(z)}$ of the Whitney neighborhood $Q(z)$ is contained in Δ_i.*
- *The only critical value of Φ_c is at c and is a branching point of degree d.*

Proof: To get $\Phi_c(z)$ we first map z to S_0 and then one more time. The properties of Φ_c depend on this last iteration, which can be easily understood in terms of the map T acting the external angles of rays. Proofs are then easy and mostly standard except for the claim concerning $Q(z)$. If $z \in \Delta_i$ and $\Phi_c(z) \in \Delta_j$, $j \neq 1$, then consider a piece of the equipotential curve which passes through $\Phi_c(z)$ and joins the boundaries of Δ_j and two adjacent sectors. This curve has a preimage in Δ_i which is a piece of the equipotential curve of level at most $L/d \leq L/2$ and extends a fixed distance, as measured in external angles, on both sides of z. If $L/2$ is smaller than that angular distance, it means that $\overline{Q(z)} \subset \Delta_i$. □

We can now consider simplified itineraries $\omega(c, z)$ with respect to Φ_c. $\omega(c, z)$ is easy to obtain from $k'(c, z)$: we skip every symbol which has a predecessor and this predecessor is different from 0. The transformation $z \to \omega(c, z)$ semi-conjugates Φ_c to the full shift on p symbols.

2.3. The Bernoulli model

We can construct a mapping ϕ induced by T on the circle which corresponds to Φ_c. Thus, $\phi = T$ on s_0 and $\phi = T^{p-i+1}$ on any other s_i. The critical itinerary $\omega(c) := \omega(c, c)$ equals the itinerary of γ under ϕ according to Lemma 2.3.

Notice that ϕ depends only on Λ, but not directly on γ. If c changes along $\partial\mathcal{M}_d$, γ changes as well but Λ is locally fixed. This justifies an important strategy for proving properties of the critical itinerary of $\omega(c)$ which are valid almost everywhere with respect to the harmonic measure. Namely, one fixes ϕ and proves that the property holds for almost all γ.

As to the structure of ϕ, it maps every gap s_i affinely onto s_0 and then piecewise affinely onto various gaps s_j. Hence, for any $q > 0$ and $0 \leq i, j < p$,

$$P(i, j, q) = \frac{|\phi^{-q}(s_j) \cap s_i|}{|s_i|}$$

depends only on j but not on i or q. Hence the mapping $c \to S(\omega(\gamma))$ where S denotes the shift transports the harmonic measure from the boundary of $\mathcal{M}_d$ to the probability distribution of a Bernoulli shift on p symbols. The shift, as a probability space, is determined by ϕ and hence properties of codes that hold with probability 1 for the shift are typical with respect to the harmonic measure.

Definition of the model. Consider a Bernoulli shift on p symbols denoted $0, 1, \cdots, p-1$, with its natural Tychonoff topology and product probability measure. Let Ω denote the space of the shift and S the shift map. A point $\omega \in \Omega$ is identified with a sequence $x_0(\omega) = 1, x_1(\omega), \cdots, x_n(\omega), \cdots$ with $x_i(\omega) \in \{0, 1, \cdots, p-1\}$ for $i > 0$. The initial 1 reflects the fact that $\gamma \in s_1$.

Let us introduce a metric on Ω which induces the Tychonoff topology: namely if $\omega_1, \omega_2 \in \Omega$ we find the least $i \geq 0$ for which $x_i(\omega_1) \neq x_i(\omega_2)$ and set $d(\omega_1, \omega_2) = 2^{-i}$. Note the improved triangle inequality:

$$d(\omega_1, \omega_2) \leq \max(d(\omega_1, \omega_3), d(\omega_3, \omega_2)) \ .$$

Furthermore, S is 2-Lipschitz with respect to d.

Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Definition 2.2 *Given $\omega \in \Omega$, let us define a function*

$$\rho_\omega : \mathbb{N} \to \mathbb{N}$$

as follows

$$\rho_\omega(k) = \inf\{j > 0 : d(S^j\omega, \omega) \leq 2^{j-k}\} \ .$$

Informally speaking, $\rho_\omega(k) = j$ means that starting from j at least through $k-1$ the code $x_i(\omega)$ repeats its initial sequence starting with $x_0(\omega)$ and j is the smallest positive number with this property. For $j = k$ this requirement becomes vacuous, so $\rho_\omega(k) \leq k$.

The probability of 1. By Lemma 2.1 the symbol 1 corresponds to the smallest gap which is mapped onto s_0 by T^{p-1}. Hence $|s_1| \leq \frac{|s_0|}{d}$. Since T is measure preserving, the preimage of s_1 in s_0 has length no more than $|s_0|/d$ and so the probability of 1 is no more than $1/d \leq 1/2$.

We are now ready to state the result.

Theorem 2.1 *In the Bernoulli model suppose that the probability of* 1 *is no more than* $1/2$. *Then there is a function $\kappa : \Omega \to \mathbb{N} \cup \infty$, finite and continuous almost for sure, and for ω from a set of full measure one can find an increasing infinite sequence of integers $k_i(\omega)$ so that*

- *for every i, $k_i - \rho_\omega(k_i) < \kappa$ and $x_{k_i+1}(\omega) \neq 1$,*
- $\limsup_{i\to\infty} \frac{k_i}{i} \leq 9$,
- $\lim \frac{k_{i+1}}{k_i} = 1$.

As we will later show the properties of $\omega(c)$ given by Theorem 2.1 easily translate to the typical properties stipulated by our main theorem.

3. A theorem in symbolic dynamics

This section is devoted to the proof of Theorem 2.1 which proceeds entirely inside the Bernoulli model.

3.1. Combinatorial considerations

We will prove certain properties of the function ρ_ω. We give formal proofs based on the metric d. Alternative proofs can be constructed by using the interpretation of function ρ_ω in terms of repeating codes.

Lemma 3.1 *If $\omega \in \Omega$ is not periodic under S, then*

$$\lim_{k\to\infty} \rho_\omega(k) = \infty\ .$$

Proof: Function ρ_ω is non-decreasing, so we have to prove that it does not stabilize. If it does for a certain value j, then Definition 2.2 implies that

$$d(S^j\omega, \omega) \leq 2^{j-q}$$

for q arbitrarily large, so $S^j\omega = \omega$ contrary to the hypothesis. □

Lemma 3.2 *Let $\omega \in \Omega$. If $\rho_\omega(k+1) > \rho_\omega(k)$, then*

$$\rho_\omega(k+1) \geq \rho_\omega(k) + \rho_\omega(k - \rho_\omega(k))\ .$$

Proof: Denote $j = \rho_\omega(k)$ and $J = \rho_\omega(k+1)$. Then $d(S^J\omega, \omega) \leq 2^{J-k-1}$. Next,

$$d(S^{J-j}\omega, \omega) \leq \max(d(S^J\omega, \omega), d(S^{J-j}\omega, S^J\omega)) \tag{1}$$

and

$$d(S^J\omega, S^{J-j}\omega) = d(S^{J-j}(S^j\omega), S^{J-j}\omega) \leq 2^{J-j}d(S^j\omega, \omega) \leq 2^{J-k}\ .$$

Now estimate (1) leads to

$$d(S^{J-j}\omega, \omega) \leq \max(2^{J-k-1}, 2^{J-k}) = 2^{J-k} = 2^{(J-j)-(k-j)}\ .$$

In view of Definition 2.2 this means that

$$\rho_\omega(k-j) \leq J - j$$

which is precisely what the lemma claims. □

In the first step of the proof of Theorem 2.1, we can now define function κ. It will still depend on an integer parameter L to be specified later. Define κ_L by the condition that for $k \geq \kappa_L(\omega)$ we have $\rho_\omega(k) \geq L+1$. If this condition is impossible to satisfy, put $\kappa(\omega) = \infty$. In view of Lemma 3.1 function κ_L is finite for all ω not periodic under the shift. Also, wherever $\kappa_L(\omega) < \infty$, κ_L is constant on a neighborhood of ω, which means that the set of its points of continuity has full measure.

3.2. The key argument

Proposition 1 *For a certain choice of a positive integer L the following holds true. Let G_ω be defined as the set of all $k \in \mathbb{N}$ for which $k - \rho_\omega(k) < \kappa_L(\omega)$.*

Then almost surely for $\omega \in \Omega$:

- $\kappa_L(\omega) < \infty$,
- $$\liminf_{n\to\infty} \frac{|G_\omega \cap \{1, \cdots, n\}|}{n} \geq \frac{1}{3} ,$$
- *except for finitely many n, if $n \in G_\omega$, then*
$$G_\omega \cap (n, n+\sqrt{n}) \neq \emptyset .$$

The first property holds whenever ω is not periodic under S, which is true almost everywhere regardless of L. Since κ_L is locally constant, we can restrict the attention to a cylinder Ω_0 on which κ_L is finite and constant. In particular, we will talk of probabilities conditioned onto Ω_0.

Let us start with an elementary lemma.

Lemma 3.3 *Consider a sequence of m independent Bernoulli trials, each with the probability of success at most $P < 1$. For some $M > 0$ let X_i be 1 if the i-th trial is a success, or $-M$ if it is a failure.*

There is a constant M_0 only depending on P so that if $M \geq M_0$, then $\sum_{i=1}^m X_i \leq -m$ with probability at least $1 - \exp(-m\frac{1-P}{4})$.

Proof: Consider a generating function

$$G(t) = E\Big(\exp(t\sum_{i=1}^m X_i)\Big) \leq (Pe^t + (1-P)e^{-Mt})^m$$

where $t > 0$. Choose $M_0(t)$ so as to ensure that $e^{-tM_0(t)} \leq \frac{1-P}{2}$. Then

$$G(t) \leq \left(\frac{1+P}{2}\right)^m \exp tm \leq \exp\left(mt - m\frac{1-P}{2}\right) .$$

If α denotes the probability of the event consisting in $\sum_{i=0}^m X_i > -m$, then

$$\alpha \exp(-mt) \leq G(t)$$

and

$$\alpha \leq \exp\left(2mt - m\frac{1-P}{2}\right) .$$

Fix $t = \frac{1-P}{8}$. Now the Lemma follows with $M_0 = M_0(\frac{1-P}{8})$. □

We now indicate the idea of the proof of the remaining part of Proposition 1. We watch a non-negative function $r_\omega(k) := k - \rho_\omega(k)$. As k grows by 1, then $r_\omega(k)$ may increase at most by 1, and that only happens if $x_{k+1}(\omega) = x_{k-\rho_\omega(k)+1}(\omega)$. Call the event $r_\omega(k+1)(\omega) > r_\omega(k)$ a success at $k+1$. Clearly, this defines independent trials with probabilities of success all bounded by $P < 1$ where P is the maximum of probabilities of any symbol. If $k \notin G_\omega$ meaning that $r_\omega(k) \geq \kappa_L$, then failure at $k+1$ means that we get $\rho_\omega(r_\omega(k)) \geq L + 1$ and $r_\omega(k+1) - r_\omega(k) \leq -L$.

If we count only the trials which follow $k \notin G_\omega \cap [n, n+m)$, then by Lemma 3.3, with overwhelming probability r_ω will jointly drop by the number of such k. Regardless of the outcome of trials following $k \in G_\omega$, $r_\omega(k)$ may grow at most by 1. If $k \notin G_\omega$ are more than $2/3m$, this implies a drop by $m/3$. But on the other hand, $r_\omega(k)$ is non-negative which yields a lower bound on the density of G_ω in $[n, n+m)$.

Let us state this reasoning formally.

Lemma 3.4 *Fix L and consider ω in a cylinder where κ_L takes a constant finite value, choose integers $n > \kappa_L$ and m in such a way that $10 r_\omega(n) \leq m$. Let P denote the maximum of probabilities of any single symbol. There is a constant L_0 which depends only on P so that if $L \geq L_0$, then with probability at least* $1 - \exp(-m\frac{1-P}{6})$

$$|G_\omega \cap [n, n+m)| > \frac{m}{3} \ .$$

Proof: Let βm be the number of integers in the set $B = [n, n+m) \setminus G_\omega$. For any $k \in B$ let us call it a success when $\rho_\omega(k+1) = \rho_\omega(k)$. This defines a sequence of Bernoulli trials with the probability of success at most P. On the other hand, if $k \notin G_\omega$, then $r_\omega(k) \geq \kappa_L$ and by Lemma 3.2 and the definition of κ_L, $r_\omega(k+1) - r_\omega(k) \leq -L$. We apply Lemma 3.3 to this sequence of βm trials. We get that there is $L_0 := M_0$ depending only on P, so that if $L \geq L_0$, then

$$\sum_{k \in B} r_\omega(k+1) - r_\omega(k) \leq -m\beta \tag{2}$$

with probability at least $1 - \exp(-m\beta\frac{1-P}{4})$. For any $k \in [n, n+m)$,

$$r_\omega(k+1) - r_\omega(k) \leq 1 \ .$$

Let $B^c := [n, n+m) \cap G_\omega$. Hence, assuming estimate (2),

$$\begin{aligned} r_\omega(n+m) - r_\omega(n) &= \sum_{k \in B} r_\omega(k+1) - r_\omega(k) + \sum_{k \in B^c} r_\omega(k+1) - r_\omega(k) \\ &\leq -m\beta + m(1-\beta) \leq m(1-2\beta) \ . \end{aligned}$$

But $r_\omega(n+m) \geq 0$ while $r_\omega(n) \leq 0.1m$ by the hypothesis of the lemma, so

$$m(1-2\beta) \geq r_\omega(n+m) - r_\omega(n) \geq -0.1m \ .$$

Thus, estimate (2) implies $\beta \leq 0.55$. Hence, if $\beta \geq 2/3$, then estimate (2) fails, which happens only with probability not exceeding

$$\exp\Big(-m\frac{1-P}{6}\Big) .$$

□

Now Proposition 1 follows easily. The claim regarding the asymptotic density of G_ω follows if we apply Lemma 3.4 with $n = \kappa_{L_0}$ and m expanding to ∞ and invoke Borel-Cantelli's lemma. If we now apply the lemma with $m = [\sqrt{n}]$, we get that almost surely for almost all n either $r_\omega(n) > 0.1[\sqrt{n}]$ or

$$|G_\omega \cap [n, n+\sqrt{n})| \geq [\sqrt{n}]/3 .$$

For all n sufficiently large the first condition implies that $n \notin G_\omega$ and the second one that $G_\omega \cap (n, n+\sqrt{n}) \neq \emptyset$ as needed.

3.3. Avoiding 1*'s in the itinerary*

Choose an $\omega \in \Omega$ for which the properties specified in Proposition 1 hold. Let us order the elements of the set G_ω in an increasing order. This gives us a sequence $k_j(\omega)$. If we try to satisfy Theorem 2.1 with this sequence, it works except for one detail: there is no reason to expect that $x_{k_j(\omega)+1}(\omega) \neq 1$. Thus, we have to win now the sequence which satisfies this condition and make sure that we don't lose the desired density properties.

We can define a sequence of Bernoulli trials in which the j-th trial is considered a success if and only if $x_{k_j(\omega)+1}(\omega) \neq 1$.

Lemma 3.5 *Almost surely for $\omega \in \Omega$ in addition to the properties listed in Proposition 1 there is a set H_ω of positive integers for which the following hold:*

- *for every $j \in H_\omega$, $x_{k_j(\omega)+1}(\omega) \neq 1$,*
- $$\liminf_{n\to\infty} \frac{|H_\omega \cap [1, n]|}{n} \geq \frac{1}{3} ,$$
- *for almost every $n \in \mathbb{N}$, $H_\omega \cap (n, n+\sqrt[4]{n}) \neq \emptyset$.*

Proof: We define the set H_ω by the requirement that the first property hold. Symbol 1 corresponds to the smallest gap in the set Λ and so its probability is no more than $1/2$ by the hypothesis of Theorem 2.1. So for every j, we can consider a "trial" which succeeds when $x_{k_j+1}(\omega) \neq 1$. This defines a sequence of Bernoulli trials with probability of success at least $1/2$. By techniques similar to those used in the proof of Lemma 3.3, one proves that in the sequence of n such trials there are at least $n/3$ successes except for an event with probability exponentially small in n.

Then the remaining two properties of Lemma 3.5 follow easily from Borel-Cantelli's lemma, just as in the proof of Proposition 1. □

We can now form an increasing sequence of elements of the set H_ω, and denote it with $j_i(\omega)$. Ultimately we form a sequence $k_{j_i(\omega)}(\omega)$. From now on we will skip the dependence on ω in order to unclutter the notation. We claim that the sequence k_{j_i} can be used as k_i in Theorem 2.1. The first property of Theorem 2.1 is now satisfied since the sequence was specifically chosen that way. As regards the second one, we compute

$$\limsup_{i\to\infty} \frac{k_{j_i}}{i} \le \limsup_{j\to\infty} \frac{k_j}{j} \cdot \limsup_{i\to\infty} \frac{j_i}{i} \le 9$$

since both factors are bounded by 3: the first by Proposition 1 and the second by Lemma 3.5.

To prove that last claim, we first observe based on Lemma 3.5 that for almost every $j \in \mathbb{N}$ among $k_j, \cdots, k_{j+[\sqrt[4]{j}]}$ we can find an element of the sequence k_{j_i}. It will be enough to show that

$$\lim_{j\to\infty} \frac{k_{j+[\sqrt[4]{j}]}}{k_j} = 1 \; .$$

From Proposition 1, for almost all j

$$\frac{k_{j+1}}{k_j} \le 1 + \frac{1}{\sqrt{k_j}} \; .$$

Hence,

$$\frac{k_{j+[\sqrt[4]{j}]}}{k_j} \le \prod_{s=0}^{[\sqrt[4]{j}]} \left(1 + \frac{1}{\sqrt{k_{j+s}}}\right) \le \left(1 + \frac{1}{\sqrt{k_j}}\right)^{\sqrt[4]{j}} \; .$$

Since obviously $k_j \ge j$, we further estimate

$$\left(1 + \frac{1}{\sqrt{k_j}}\right)^{\sqrt[4]{j}} \le \left(1 + \frac{1}{\sqrt{j}}\right)^{\sqrt[4]{j}} \le \exp \frac{1}{\sqrt[4]{j}}$$

which tends to 1 as j tends to ∞.

Theorem 2.1 has been proved.

4. Proofs of main theorems

As already observed, Theorem 2.1 lists properties of the critical itinerary $\omega(c)$ which are typical with respect to the harmonic measure of $\partial\mathcal{M}_d$. We will now translate these properties into a statement which already clearly shows expansion along the critical orbit and proves Theorem 1.1.

We start with preparatory geometric considerations which go back to the induced mapping Φ_c introduced by Definition 2.1.

4.1. Yoccoz pieces

Consider an itinerary $x_0, \cdots, x_k$. A *Yoccoz piece* of order k following this itinerary is any maximal connected set of points z for which the first $k+1$ symbols of their itineraries $\omega(c, z)$ are the same as $x_0, \cdots, x_k$.

Lemma 4.1 *If a Yoccoz piece D of order k follows the itinerary $x_0, \cdots, x_k$, then Φ_c^k restricted to the piece is a proper holomorphic map onto Δ_{x_k}.*

Proof: Obviously, $F := \Phi_c^k$ is a holomorphic map from the Yoccoz piece into Δ_{x_k}. We have to show that F is proper. The proof proceeds by induction with respect to k. For $k = 0$, F is the identity map. In general, $\Phi_c^k = \Phi_c \circ \Phi_c^{k-1}$, both proper by the induction hypothesis. □

As a consequence of Lemma 4.1, Φ_c^k is onto Δ_{x_k} and a finitely branched cover. In particular, D is a topological disk. Here is another corollary:

Lemma 4.2 *Consider two Yoccoz pieces: D of order k which follows an itinerary $x_0, \cdots, x_k$ and $D' \subset D$ which follows the itinerary $x_0, \cdots, x_k, x_{k+1}$ with $x_{k+1} \neq 1$. Then $\overline{D'}$ is contained in D.*

Proof: Observe that

$$D' \subset \Phi_c^{-k}\left(\Phi_c^{-1}(\Delta_{x_{k+1}})\right) .$$

But by the properties of Φ_c, listed following its definition, $\Phi^{-1}(\Delta_{x_{k+1}})$ is relatively compact in Δ_{x_k} and so the claim follows by Lemma 4.1. □

The next lemma establishes a connection between the dynamics of Φ_c on neighborhoods of c and the combinatorial function $\rho_{\omega(c)}$.

Lemma 4.3 *Suppose that D is a Yoccoz piece of order k which contains c and follows the critical itinerary $\omega(c)$. Then Φ_c^j are univalent on D for all $j < \rho_{\omega(c)}(k)$.*

Proof: Choose the smallest j for which Φ_c^j is not univalent. Then $\Phi_c^j(D) \ni c$. But then the $k - j$ consecutive symbols of $\omega(c)$ starting from the beginning and from the $\omega_j(c)$ are the same, or $d(S^j\omega(c), \omega(c)) \leq 2^{j-k}$ which implies $j \geq \rho_{\omega(c)}(k)$ by Definition 2.2. □

Another observation concerns Whitney domains $Q(z)$.

Lemma 4.4 *Consider a Yoccoz piece D of order k and suppose that $z \in D$ and the itinerary of z has length at least $k+1$ with the $k+1$-st symbol other than 1. Then $\overline{Q(z)} \subset D$.*

Proof: The proof follows by induction with respect to k. For $k = 0$, this is one of the claims of Fact 2.3. Then Φ_c is proper from a piece of order k onto a piece of order $k - 1$ and $\Phi_c(Q(z)) = Q(\Phi_c(z))$ so the claim follows. □

Finally, on a Yoccoz piece of order at least k, Φ_c^k is the same as $f_c^{m_D(k)}$, which defines a function m_D. Obviously, for every D,

$$k \leq m_D(k) \leq pk$$

where p is the number of ray-sectors in the construction.

4.2. Proof of Theorem 1.1

Let us reduce Theorem 1.1 to Theorem 2.1. Set $\mathcal{H}$ is chosen as the set of external angles for which the Yoccoz partition is well-defined and the claim of Theorem 2.1 holds for the critical itinerary $\omega(c)$. For $\gamma \in \mathcal{H}$, choose $c_0 \in \mathcal{M}_d \cap r_d(\gamma)$.

For $c \in r_d(\gamma)$, let $V_n'(c)$ be the Yoccoz piece of order n which follows the itinerary $\omega(c)$. Then $K_n'(c)$ is the union of pieces of order $n+1$ contained in $V_n'(c)$ which follow $\omega(c)$ up until x_n, and then x_{n+1} runs through all values other than 1. Both are well-defined for c close enough to $\mathcal{M}_d$. By Lemma 4.2, the closure of $K_n'(c)$ is contained in $V_n'(c)$.

For every fixed n, the set $V_n'(c) \setminus K_n'(c)$ contains an annulus $A_n'(c)$. Pick some N, which will be specified later. Then the moduli of $A_n'(c)$ for $n = 0, \cdots, N$ can be bounded from below by $\alpha(c) > 0$. If c varies along $r_d(\gamma)$, the sets $K_n'(c)$ and $V_n'(c)$ move continuously in the Hausdorff metric and hence $\alpha(c)$ can be chosen as a continuous function of $c \in r_d(\gamma)$. Hence, for c sufficiently close to $\mathcal{M}_d$, we can extract a common lower bound α. That c is close to $\mathcal{M}_d$ can be guaranteed by choosing parameter ϵ small enough in the hypothesis of Theorem 1.1.

We verify the first claim of Theorem 1.1. Recall the sequence k_i from Theorem 2.1. Suppose that the critical itinerary continues at least through one more symbol beyond $x_{k_i}(\gamma)$. Let U_i be the Yoccoz piece of order k_i which contains c and follows the critical itinerary. Of course, U_i depends on c, but we suppress that in our notation. By Lemma 4.3, $\Phi_c^{\rho_{\omega(c)}(k_i)}$ maps U_i as a branched cover of degree d with the only critical value at c onto a Yoccoz piece of order $k_i - \rho_{\omega(c)}(k_i)$ which follows the critical trajectory. But Theorem 2.1 states that $k_i - \rho_{\omega(c)}(k_i) < N := \kappa$, so that in fact

$$\Phi_c^{\rho_{\omega(c)}(k_i)}(U_i) = V_n'$$

for some $n = 0, .., N-1$. Thus we have specified N used before to choose α. Moreover, Theorem 2.1 also states that the symbol in $\omega(c)$ following the $\rho_{\omega(c)}$-th one is not 1. Thus, $\Phi_c^{\rho_{\omega(c)}(k_i)}(c) \in K_n'$. Annulus A_n' has a preimage A_i by $\Phi_c^{\rho_{\omega(c)}(k_i)}$. Clearly, A_{i-1} contains U_i in the bounded connected component of its complement. Lastly, by Lemma 4.4 applied to $D := U_i$ and $z := c$, $Q(c) \subset U_i$.

To prove Theorem 1.1, we choose $V_n(c) := f_c^{-1}(V_n'(c))$ and $K_n(c) = f_c^{-1}(K_n'(c))$ so that U_i is mapped univalently onto $V_n(c)$ by $f_c^{m_i}$, with $f_c^{m_i}(c) \in K_n(c)$. For a fixed c and $n \leq N$, diam $K_n(c) \geq \eta(c) > 0$. Since $K_n(c)$ move continuously with c in the Hausdorff metric, η can be made independent of c for c close to $\mathcal{M}_d$. Since $V_n(c)$ and $K_n(c)$ are separated by an annulus with modulus at least α/d and the diameter of K_n is at least η, by Teichmüller's module theorem, see [8] page 89, $V_n(c)$ contains a ball centered at $f_c^{m_i}(c)$ with a fixed radius.

Finally, we have to justify our assumption that the critical itinerary has length at least $k_i + 2$. If not, $G_c(\Phi_c^{k_i}(c))$ is bounded from below by a constant which only depends on γ. But $\Phi_c^{k_i} = f_c^{m_i+1}$ and hence $G_c(f_c^{m_i}(c))$ has a similar lower bound. By making the constant ϵ in the statement of Theorem 1.1 small enough, we can then satisfy the first claim in a vacuous way.

It remains to verify the two claims of Theorem 1.1 which have only to do with the sequence $m_i = m_{U_i}(\rho_{\omega(c)}(k_i)) - 1$. Recall that the function m_{U_i} simply recomputes iterates of Φ_c into iterates of f_c. We have that $i \le m_i \le pk_i$ and

$$\frac{i}{m_i} \ge \frac{1}{p}\,\frac{i}{k_i}\ . \tag{3}$$

But $\inf \frac{i}{k_i} \ge 1/9$ by Theorem 2.1, and so we put $\theta := 9p$.

It remains to prove that

$$\lim_{i\to 0} \frac{m_{i+1} - m_i}{m_i} = 0\ .$$

We estimate

$$m_{i+1} - m_i \le m_{U_{i+1}}((k_{i+1} - k_i) + (k_i - \rho_{\omega_c}(k_i)) \le p(k_{i+1} - k_i + N)\ .$$

Similarly,

$$m_i \ge k_i - N \quad \text{and}$$

$$\lim_{i\to\infty} \frac{m_{i+1} - m_i}{m_i} \le \lim_{i\to\infty} p\,\frac{k_{i+1} - k_i + N}{k_i - N} = 0$$

so $\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1$ by Theorem 2.1.

This concludes the proof of Theorem 1.1.

4.3. Exponential derivative growth

We will prove Theorem 1.2. Let us take γ from the set $\mathcal{H}$ of Theorem 1.1 and use notations of that Theorem.

We need a few auxiliary lemmas to start. Our estimates must be independent of $c \in r_d(\gamma)$, but can and will depend on γ.

Lemma 4.5 *There are constants $C_1 > 0$ and $\zeta_1 > 1$, depending only on γ, so that for every $c \in r_d(\gamma)$, $G_c(c) < \epsilon$, whenever $G_c(f_c^{m_i}(c)) < \epsilon$, then $Df_c^{m_i}(c) \ge C_1\,\zeta_1^{m_i}$.*

Proof: U_i is surrounded by a sequence of nesting annuli $U_{j-1} \setminus \overline{U}_j$, $j = 1, \cdots, i$, each with modulus at least ϵ. Hence $\mathrm{mod}\,(U_0 \setminus \overline{U}_j) \ge i\epsilon$. Recall that U_0 is contained within the region delimited by the equipotential curve

$G_c(z) = L$, thus a bounded region of the plane. By Teichmüller's module theorem, see [8] page 89,

$$i\epsilon < 2\log 4 + \log\left(1 + \frac{\text{diam } U_0}{\text{diam } U_i}\right)$$

and hence for i sufficiently large

$$\text{diam } U_i < \text{ diam } U_0 \exp(-\epsilon i/2) \ .$$

Since $\limsup_{i\to\infty} \frac{m_i}{i} \leq \theta$, for all $i \geq i_0$, where i_0 only depends on γ,

$$\text{diam } U_i < \text{ diam } U_0 \exp(-\epsilon m_i/2\theta) \ . \tag{4}$$

But $f_c^{m_i}$ maps U_i univalently over $D(f_c^{m_i}(c), \sigma)$, so the Schwarz lemma implies that $|Df_c^{m_i}(c)| \geq \frac{\sigma}{\text{diam} U_i}$ which ends the proof. □

Lemma 4.6 *Suppose that* $G_c(z) \geq \eta > 0$ *and* $G_c(0) < \eta/4$. *For every* $\eta > 0$ *there is* $r > 0$ *so that the Whitney domain* $Q(z) \supset D(z, r)$.

Proof: Let us look at the preimage of $Q(z)$ by the Böttker coordinate Γ_c, which is well defined on $D(0, \exp(-G_c(0)))$. This is a ring-sector with size determined by η, so contains a ball centered at $\Gamma_c^{-1}(z)$ of radius determined by η. Also, by compactness the derivative of Γ_c at the center of this ball is bounded below depending on η. The lemma now follows from Köbe's one-quarter theorem. □

We will use this lemma to estimate the derivative at the moment when the orbit of c gets out of the region limited by the equipotential curve of level ϵ. To this end, define

$$I(c) := \min\left\{i : \ G_c\big(f_c^i(c)\big) \geq \epsilon\right\} \ .$$

It is understood that $I(c) = +\infty$ if $c \in \mathcal{M}_d$.

Lemma 4.7 *For* $c \notin \mathcal{M}_d$ *suppose that* $I(c) \geq 2$. *Then there are constants* $C_2 > 0$ *and* $\zeta_2 > 1$, *depending only on* γ, *so that*

$$\left|Df_c^{I(c)}(c)\right| \geq C_2 \ \zeta_2^{I(c)} \ .$$

Proof: Given c, choose the largest i so that $G_c(f_c^{m_i}(c)) < \epsilon$. Then $I(c) \leq m_{i+1}$. As a consequence, $I(c) < Mm_i$ where M only depends on the sequence m_i, hence only on γ. By Theorem 1.1, $U_i \supset Q(c)$. Then $z := f_c^{I(c)}(c)$ satisfies the hypothesis of Lemma 4.6 with $\eta := \epsilon$. In particular, since $I(c) \geq 2$, $G_c(f_c^2(0)) < \epsilon$ and $G_c(0) < \epsilon/4$. Hence $f_c^{I(c)}(Q(c))$ contains a ball centered at z of radius $r(\epsilon)$. Taking an inverse branch and using the Schwarz lemma, we see that

$$\left|Df_c^{I(c)}(c)\right| \geq \frac{r(\epsilon)}{\text{diam } Q(c)} \ .$$

But $\operatorname{diam} Q(c) \leq \operatorname{diam} U_i \leq K \exp(-m_i\epsilon/2\theta)$ by estimate (4) in the proof of Lemma 4.5. Recalling $I(c) < Mm_i$, we get

$$|Df_c^{I(c)}(c)| \geq C_2 \exp(I(c)\epsilon/2\theta M)$$

as needed. □

So far we have seen uniform expansion for times $m_1, \cdots, m_i, I(c)$. We will now get it for all times in between.

Lemma 4.8 *Choose γ in the subset $\mathcal{H} \in [0,1)$ specified in Theorem 1.1 and consider the sequence m_i given by the the same Theorem. Suppose that $I(c) \geq 2$, possibly infinite. Then there are constants $C_3 > 0$ and $\zeta_3 > 1$ depending only on γ so that for each $c \in r_d(\gamma)$ and every $0 < n \leq I(c)$,*

$$\left|Df_c^n(c)\right| \geq C_3\, \zeta_3^n\ .$$

Proof: Given $n \leq I(c)$ choose i to be the largest so that m_i does not exceed n, and M to be m_{i+1} or $I(c)$, whichever is less. If ζ and C are the minimum of ζ_1 and C_1 from Lemma 4.5 and ζ_2 and C_2 from Lemma 4.7, respectively, then $|Df_c^M(c)| \geq C\zeta^M$. Points $f_c^n(c), \cdots, f_c^{M-1}(c)$ are in the region bounded by the equipotential of level ϵ. Hence the derivatives at all these points are bounded, independent of c, by some λ. Hence

$$\begin{aligned}\left|Df_c^n(c)\right| &\geq \left|Df_c^M(c)\right|\lambda^{-M+n} \geq C\zeta^M\lambda^{m_i-m_{i+1}}\\ &= C\zeta^M\left(\lambda^{(m_{i+1}-m_i)/M}\right)^{-M}\ .\end{aligned} \tag{5}$$

Since $\lim_{i\to\infty}\frac{m_{i+1}}{m_i} = 1$ by Theorem 1.1, except for finitely many i, hence for all $n \geq n_0$, we have

$$\lambda^{(m_{i+1}-m_i)/M} \geq \sqrt{\zeta}$$

and then estimate (5) yields

$$\left|Df_c^n(c)\right| \geq C\zeta^{n/2}$$

as needed. For $n < n_0$ the estimate gives a positive constant on the right hand-side, which can be absorbed in C. □

To finish the proof of Theorem 1.2, we need to establish expansion for points which are far from $\mathcal{K}_c$, as measured by the Green function.

Lemma 4.9 *Suppose that $G_c(z) \geq \eta$ and $G_c(0) \leq G_c(z)/2$. For every $\eta > 0$ there exist constants $C_4 > 0$ and $\zeta_4 > 1$ so that for all $n > 0$,*

$$\left|Df_c^n(z)\right| \geq C_4\, \zeta_4^n\ .$$

Proof: The filled-in Julia set contains a non-attracting fixed point q whose distance to 0 is at least $1/2$. Points $f_c^n(z)$, $n \geq 0$, are all separated from the Julia set and 0 by an annulus with modulus at least $\eta/2$, hence $|Df_c(f_c^n(c))| \geq r(\eta) > 0$ for all such n by Teichmüller's module theorem. For $n > \frac{\log 100 - \log(\eta/2)}{\log d}$, such an annulus already has modulus at least 100 and by Teichmüller's module theorem $|f_c^n(z)| > e^{50}$ which implies very large derivative. □

Theorem 1.2 is now proven. We split the orbit $f_c^n(c)$ in two parts. For $n \leq I(c)$ if $I(c) \geq 2$ the expansion follows from Lemma 4.7. For subsequent n or if $I(c) < 2$ implying $G_c(c) \geq \epsilon/d$, we rely on Lemma 4.9.

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References

1. Carleson, L., Gamelin, T.: Complex dynamics, Springer-Verlag, New York (1995)
2. Douady, A.: Systemès Dynamiques Holomorphes, Astérisque **105** (1982), 39–63
3. Douady, A.; Sentenac, P.; Zinsmeister, M.: Implosion parabolique et dimension de Hausdorff, C. R. Acad. Sci. Paris Sèr. I Math. **325** (1997), no. 7, 765–772
4. Farkas, H., Kra, I.: Riemann surfaces, Springer-Verlag, New York-Heidelberg-Berlin (1980)
5. Fatou, P.: Sur les équations fonctionelles, Bull. Soc. Math. Fr. **47** (1919), 161–271; **48** (1920), 33–94; 208–314
6. Graczyk, J., Smirnov, S.: Collet, Eckmann, & Hölder, Invent. math. **133** (1998), 69–96
7. Graczyk, J., Smirnov, S.: Weak expansion and geometry of Julia set, preprint (1998)
8. Graczyk, J., Ṡwiątek, G.: Generic hyperbolicity in the logistic family, Ann. of Math **146** (1997), 1–56
9. Graczyk, J., Świątek, G.: SURVEY: Smooth unimodal dynamics in the 1990s, Erg. Th. Dyn. Sys. **19** (1999), 263–287
10. Jakobson, M.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, Commun. Math. Phys. **81** (1981), 39–88
11. Jones, P., Makarov, N.: Density properties of harmonic measure, Ann. of Math. **142** (1995), 427–455
12. Makarov, N.: Distortion of boundary sets under conformal mappings, Proc. London. Math. Soc. **51** (1985), 369–384
13. Manning, A.:Logarithmic capacity and renormalizability for landing on the Mandelbrot set, Bull. London. Math. Soc. **28** (1996), 521–526
14. Øxendal, B.: Brownian motion and sets of harmonic measure zero, Pacific J. Math. **95** (1981), 179–192
15. Przytycki, F.: On measure and Hausdorff dimension of Julia sets for holomorphic Collet-Eckmann maps, International Conference on Dynamical Systems (Montevideo, 1995), 167–181, in Pitman Res. Notes Math. Ser. 362, Longman, Harlow (1996)
16. Rudin, W.: Real and complex analysis, 3rd ed., McGraw-Hill, New York (1987)
17. Shishikura, M.: The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets, Ann. of Math. **147** (1998), 225–267
18. Smirnov, S.: Symbolic Dynamics and the Collet-Eckmann condition, Internat. Math. Res. Notices 2000, no. 7, 333–351
19. Tsujii, M.:Positive Lyapunov exponents in families of one dimensional dynamical systems, Invent. math. **111** (1993), 113–137