Invent. math. 140, 399–422 (2000)
Digital Object Identifier (DOI) 10.1007/s002220000059

Inventiones mathematicae

The Bernstein problem for affine maximal hypersurfaces*

Neil S. Trudinger, **Xu-Jia Wang**

Centre for Mathematics and its Applications, Australian National University, Canberra, ACT 0200, Australia

Oblatum 16-IV-1999 & 4-XI-1999
Published online: 21 February 2000 –

Abstract. In this paper, we prove the validity of the Chern conjecture in affine geometry [18], namely that an affine maximal graph of a smooth, locally uniformly convex function on two dimensional Euclidean space, $\mathbf{R}^2$, must be a paraboloid. More generally, we shall consider the n-dimensional case, $\mathbf{R}^n$, showing that the corresponding result holds in higher dimensions provided that a uniform, "strict convexity" condition holds. We also extend the notion of "affine maximal" to non-smooth convex graphs and produce a counterexample showing that the Bernstein result does not hold in this generality for dimension $n \geq 10$.

1. Introduction

In this paper, we prove the validity of the Chern conjecture in affine geometry [18], namely that an affine maximal graph of a smooth, locally uniformly convex function on two dimensional Euclidean space, $\mathbf{R}^2$, must be a paraboloid. More generally, we shall consider the n-dimensional case, $\mathbf{R}^n$, showing that the corresponding result holds in higher dimensions provided that a uniform, "strict convexity" condition holds. We also extend the notion of "affine maximal" to non-smooth convex graphs and produce a counterexample showing that the Bernstein result does not hold in this generality for dimension $n \geq 10$.

The Bernstein problem has been a core problem in the study of minimal submanifolds, ever since Bernstein proved that an entire, two dimensional, minimal graph must be a hyperplane [3]. The question of whether the Bernstein theorem carried over to higher dimensions provided a great impetus

* Research supported by Australian Research Council

Mathematics Subject Classification (1991): 53A15, 53A10, 53C45, 35J60

in the development of the higher dimensional theory of minimal surfaces. It was eventually shown to be true by De Giorgi [20] for $n = 3$, Almgren [1] for $n = 4$, and Simons [36] for $n \leq 7$. Finally Bombieri, De Giorgi, and Giusti [6] gave an example showing that the result fails for $n \geq 8$. The analogous spherical Bernstein problem, proposed by Chern [17], asks whether an $(n-1)$-sphere is an equator if it is embedded as a minimal hypersurfaces in the n-sphere, S^n. This is true for $n = 3$ [1, 11] but for all even $n \geq 4$ and some odd n, the result was shown false [25, 37]. The Bernstein problem for maximal space-like hypersurfaces in Minkowski space is completely understood as an entire maximal graph must be a hyperplane in all dimensions [15].

The Bernstein problem for affine maximal hypersurfaces was proposed by Chern [18] and subsequently Calabi [13]. To formulate it, we let $\mathcal{M}$ be a hypersurface immersed in the real affine $(n+1)$-space A^{n+1}. In this paper we restrict attention to locally convex C^2 hypersurfaces, namely C^2 hypersurfaces with local supporting hyperplanes, and there is no loss of generality in introducing a Euclidean structure in A^{n+1} so that we may work directly in $\mathbf{R}^{n+1}$. It is known that a complete, locally uniformly convex hypersurface must be globally convex, and hence the boundary of a convex domain [24, 35]. Consequently if $\mathcal{M}$ is open, it can be represented as the graph of a convex function over a domain in $\mathbf{R}^n$. The normal of the hypersurface will be chosen on its convex side. We remark here that a locally convex C^2 hypersurface is locally uniformly convex if its principal curvatures are positive and was called strongly convex in [13].

Suppose $\mathcal{M}$ is given by

$$x_{n+1} = u(x), \quad x = (x_1, \cdots, x_n), \tag{1.1}$$

where $u \in C^2(\Omega)$ is convex. On $\mathcal{M}$ we can introduce a metric, called the affine metric, given by

$$g_{ij} = \frac{u_{ij}}{[\det D^2 u]^{1/(n+2)}}, \tag{1.2}$$

where $D^2 u = [u_{ij}]$ is the Hessian matrix of the second derivatives of u. If u is locally uniformly convex in Ω, then $\det D^2 u > 0$ and g is well defined. From the metric, we introduce the affine area A by defining

$$\begin{aligned} A(u, \Omega) &= \int_\Omega [\det D^2 u]^{1/(n+2)} \\ &= \int_{\mathcal{M}_\Omega} K^{1/(n+2)}, \end{aligned} \tag{1.3}$$

where K is the Gauss curvature of $\mathcal{M}$ and $\mathcal{M}_\Omega = \{(x, u(x)) \in \mathcal{M} \mid x \in \Omega\}$. The metric g and the area A are invariant under unimodular affine transformations, that is linear transformations (in $\mathbf{R}^{n+1}$) preserving Euclidean volume and orientation, see [13].

A hypersurface $\mathcal{M}$, given by (1.1), is called *affine maximal* if the function u is a critical point of the affine area functional A. Calabi [13] proved that if $u \in C^4(\Omega)$ is a critical point of the functional A, the second variation of A at u is non-positive, that is, the affine area of $\mathcal{M}$ reaches a maximum under smooth interior perturbations. Accordingly he proposed that $\mathcal{M}$ be called an affine maximal hypersurface. The Euler equation of the functional A is a fourth order, nonlinear partial differential equation, given by

$$H_A[\mathcal{M}] =: D_{ij}(U^{ij}w) = 0, \tag{1.4}$$

where

$$w = [\det D^2 u]^{-(n+1)/(n+2)}, \tag{1.5}$$

and $[U^{ij}]$ denotes the cofactor matrix of $[u_{ij}]$. Noting that

$$D_j U^{ij} = 0, \tag{1.6}$$

we see that the above equation may also be written as

$$H_A[\mathcal{M}] = U^{ij} D_{ij} w = 0. \tag{1.7}$$

The quantity $H_A[\mathcal{M}]$ on the left hand side of equations (1.4) and (1.7) represents the affine mean curvature of the hypersurface $\mathcal{M}$.

Denoting

$$\begin{aligned} h = g^{1/2} &= \left(\det[g_{ij}]\right)^{1/2} \\ &= \left(\det D^2 u\right)^{1/(n+2)}, \end{aligned} \tag{1.8}$$

equation (1.4) can also be written as

$$\Delta_{\mathcal{M}}\left(\frac{1}{h}\right) = 0, \tag{1.9}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator with respect to the affine metric (1.2), given by

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{g}} D_i\left(\sqrt{g} g^{ij} D_j\right) = \frac{1}{h} D_i\left(h^2 u^{ij} D_j\right), \tag{1.10}$$

and $[g^{ij}]$, $[u^{ij}]$ are the inverses of $[g_{ij}]$, $[u_{ij}]$. Therefore the hypersurface $\mathcal{M}$ is affine maximal if and only if $1/h$ is harmonic on $\mathcal{M}$.

In [18], Chern conjectured that, in the two dimensional case, any entire solution to (1.7) must be quadratic. From Bernstein [3], if the function $w = o(|x|)$, as $x \to \infty$, then w is constant and Chern's conjecture follows from Jörgens' theorem [26], that an entire convex solution of the Monge-Ampère equation

$$\det D^2 u = \text{constant} \tag{1.11}$$

is a quadratic function, (which is true in all dimensions). Calabi [13] verified the Chern conjecture under the hypothesis that the affine metric of the graph of the solution, defined by (1.2) is complete. For if $n = 2$, the Ricci tensor under the affine metric is non-negative definite, and by a result of Blanc and Fiala [4], (see [38] for the higher dimensional case), that a positive harmonic function on a complete Riemannian manifold with non-negative Ricci curvature is constant, the result follows again from Jörgens' theorem. Different conditions were imposed by Calabi [14]. However, the above conditions represent fairly strong restrictions on the asymptotic behaviour of the second derivatives of the function u. Locally uniformly convex, Euclidean complete hypersurfaces are not generally affine complete, as is the case with the graphs of the functions, given by

$$u = \frac{1}{x_1} + x_2^2, \quad (x_1 > 0), \tag{1.12}$$

and

$$u = (1 + |x|^k)^{1/k}, \quad k > 2. \tag{1.13}$$

The example (1.13) also violates the Bernstein condition, $w = o(|x|)$, for $k \geq 2$. Li [28] proved that if all the affine principal curvatures are bounded, then Euclidean completeness implies affine completeness, so that in the two dimensional case, the Chern conjecture is valid if the affine Gauss curvature is bounded from below; (see also [29]).

This paper is set out as follows. In the next section, we show how the Bernstein problem can be reduced to a problem of *a priori* estimates for solutions of the affine maximal surface equation (1.7), (Theorem 2.1). As a byproduct of our argument, we deduce an extension of Jörgen's theorem in all dimensions. Sect. 3 is concerned with upper and lower bounds for the Hessian determinant of solutions of (1.7). The upper bound, (Lemma 3.1), is derived by typical nonlinear second order PDE techniques while for the lower bound, (Lemma 3.2), we invoke the Legendre transformation, thereby bringing in to play the modulus of convexity. In Sect. 4, we apply the recent Hölder estimate of Caffarelli and Gutiérrez, (Theorem 4.1), [9], for the linearized Monge-Ampère equation, to conclude interior higher order estimates, in terms of the modulus of convexity, and regularity, (Theorem 4.2). The combination of Theorem 4.2 and our reduction in Sect. 2 yields the Bernstein property under a restriction of *uniform strict convexity*, (Corollary 4.3). In Sect. 5, we establish a modulus of convexity estimate for solutions of equation (1.4), (1.7) in *two* dimensions, (Lemma 5.1), thereby completing the proof of the Chern conjecture, (Theorem 5.2). The arguments here cannot be extended to higher dimensions. In Sect. 6, we take up the issue of reduced smoothness. First we prove that our preceding estimates in Theorem 4.2 and the two dimensional Bernstein property, Theorem 5.2, extend to C^2 weak solutions of equation (1.4), (Theorem 6.2). Next we consider the extension of the affine area functional (1.3) to arbitrary convex graphs, proving an approximation result, (Lemma 6.3), and upper semi-continuity, (Lemma 6.4). Finally in Sect. 7, we provide an example to illustrate the scope of our investigations. This example, in dimension ten, violates the uniformly strict convexity condition but unfortunately has a singularity (albeit mild) at one point.

To complete this introduction, we state the two dimensional Bernstein property in its fully generality, taking account of the fact that a compact surface cannot be affine maximal [18].

Theorem 1.1. *A Euclidean complete, affine maximal, locally uniformly convex C^2 hypersurface in $\mathbf{R}^3$ must be an elliptic paraboloid.*

2. Reduction to interior estimates

In this section, we show the Bernstein property can be reduced to the establishment of interior estimates for solutions of equations (1.4) in arbitrary normalized convex domains. We will make use of the fact [21] that for any bounded convex domain Ω in $\mathbf{R}^n$, there exists a unique ellipsoid E, called the *minimum ellipsoid* of Ω, which attains the minimum volume among all ellipsoids concentric with and containing Ω, and a positive constant α_n, depending only on n, such that

$$\alpha_n E \subset \Omega \subset E, \tag{2.1}$$

where $\alpha_n E$ is the α_n dilation of E with respect to its centre. Let T be a dilation mapping E onto the unit ball B. From (2.1),

$$\alpha_n B \subset T(\Omega) \subset B, \tag{2.2}$$

and we call $T(\Omega)$ the *normalized domain* of Ω, and Ω *normalized* if $T(\Omega) = \Omega$, that is $E = B$.

Letting $L = U^{ij} D_{ij}$, we write equation (1.4) in the form

$$Lw = U^{ij} D_{ij} w = 0, \tag{2.3}$$

where

$$w = \left(\det D^2 u\right)^{\theta - 1}, \quad \theta = \frac{1}{n+2}. \tag{2.4}$$

Hypothesis H_n: For all normalized convex domains $\Omega \subset \mathbf{R}^n$ and locally uniformly convex solutions $u \in C^4(\overline{\Omega})$ of equation (2.3) satisfying $u = 0$ on $\partial\Omega$, $\inf_\Omega u = -1$, we have the estimates

$$D^2 u \geq C_1 I, \quad |D^3 u| \leq C_2 \tag{2.5}$$

in the ball γB, where C_1, C_2 and γ are positive constants, depending only on n.

Theorem 2.1. *Suppose that H_n is valid. Then if $u \in C^4(\Omega)$ is a locally uniformly convex solution of equation (2.3) in a convex domain $\Omega \subset \mathbf{R}^n$ satisfying*

$$\lim_{x \to \partial\Omega} u(x) = +\infty, \tag{2.6}$$

it follows that $\Omega = \mathbf{R}^n$ and u is a quadratic function.

Proof. By subtracting a linear function, we may suppose

$$u(0) = D_i u(0) = 0, \quad i = 1, \cdots, n. \tag{2.7}$$

Let $T_t = [a_t^{ij}]$ be a linear transformation which normalizes the *section*

$$S_t = \{x \in \Omega \mid u < t\}, \quad (t > 0), \tag{2.8}$$

and define u_t and Ω_t by

$$u_t(x) = \frac{1}{t} u(T^{-1}(x)), \quad \Omega_t = \{x \mid u_t < 1\} = T_t(S_t). \tag{2.9}$$

By the assumption of Theorem 2.1, $u_t \in C^4(\overline{\Omega}_t)$ is uniformly convex and satisfies the affine invariant equation (2.3) in Ω_t. Furthermore by (2.5), we have

$$D^2 u_t(x) \geq C_1 I \tag{2.10}$$

for any $t \geq 1$ and $x \in \gamma B$. Let Λ_t denote the maximum eigenvalue of T_t. We claim there exists a positive constant Λ_0 such that

$$\overline{\lim}_{t \to \infty} t \Lambda_t^2 \leq \Lambda_0. \tag{2.11}$$

To prove (2.11), we observe from (2.10),

$$u(x) = t u_t(T_t(x)) \geq C_1 t |T_t(x)|^2$$

and hence

$$\sup_{x \in rB} u(x) \geq \sup_{x \in rB} C_1 t |T_t(x)|^2 = C_1 r^2 t \Lambda_t^2, \tag{2.12}$$

where r is chosen small enough to ensure $rB \subset \Omega$. Next for $x \in \Omega$, we estimate

$$\begin{aligned} |D^3 u(x)| &\leq C \Lambda_t^3 t |D^3 u_t(T_t(x))| \\ &\leq C \Lambda_0^{3/2} t^{-1/2} \end{aligned}$$

for $T_t(x) \in \gamma B$, by (2.5) and (2.11). Hence letting $t \to \infty$, we conclude $D^3 u = 0$, whence u is quadratic and $\Omega = \mathbf{R}^n$. □

From the proof of Theorem 2.1, we can also deduce the following extension of the Bernstein property for the Monge-Ampère equation (1.11), which was proved by Jörgens [26] for $n = 2$, Calabi [10] for $2 \leq n \leq 5$, and Pogorelov [32] for $n \geq 2$.

Corollary 2.2. *Let Ω be a convex domain in $\mathbf{R}^n$ and u a convex solution of the equation*

$$\det D^2 u = 1 \quad in \;\; \Omega \tag{2.13}$$

with $\lim_{x\to\partial\Omega} u(x) = \infty$. *Then $\Omega = \mathbf{R}^n$ and u is quadratic.*

To prove Corollary 2.2, we observe that equations of the form (1.11) can be used in the Hypothesis H_n, with the appropriate estimates (2.5) guaranteed by the regularity theory for the Monge-Ampère equation, as in [23] or [33]. We note also that the concept of solutions in Corollary 2.2 may be understood in the generalized sense of Aleksandrov.

The third derivative estimate in (2.5) is stronger than necessary. By inspection of the proof of Theorem 2.1, it can be replaced by a modulus of continuity estimate for the second derivatives in the ball γB.

3. Bounds for the Hessian determinant

In this section, we derive upper and lower bounds for the Hessian determinant, $\det D^2 u$, of solutions u of equation (2.3).

Lemma 3.1. *Let Ω be a bounded convex domain in $\mathbf{R}^n$ and $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ a locally uniformly convex solution of equation (2.3) in Ω, satisfying*

$$u = 0 \quad on \;\; \partial\Omega \qquad \inf_{\Omega} u = -1. \tag{3.1}$$

Then, for $y \in \Omega$,

$$\det D^2 u(y) \leq C, \tag{3.2}$$

where C depends on n, $dist(y, \partial\Omega)$, and $\sup_\Omega |Du|$.

Proof. Let

$$z = \log \frac{w}{(-u)^\beta} - A|Du|^2, \tag{3.3}$$

where β and A are positive constants to be specified later. Since $z \to \infty$ on $\partial\Omega$, it attains a minimum at some point $x_0 \in \Omega$. At x_0, we then have

$$0 = z_i = \frac{w_i}{w} - \beta\frac{u_i}{u} - 2Au_k u_{ki}, \tag{3.4}$$

and

(3.5)
$$0 \le [z_{ij}] = \Big[\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta u_{ij}}{u} + \frac{\beta u_i u_j}{u^2} - 2Au_{ki}u_{kj} - 2Au_k u_{kij}\Big].$$

Recalling $w = \det^{\theta-1} D^2u$, $\theta = \frac{1}{n+2}$, we have

$$u^{ij}u_{kij} = (\log \det D^2u)_k = -\frac{1}{1-\theta}\frac{w_k}{w}, \tag{3.6}$$

where $[u^{ij}] = (\det D^2u)^{-1}[U^{ij}]$ is the inverse of D^2u. From (3.4),

$$\frac{w_i w_j}{w^2} = \beta^2\frac{u_i u_j}{u^2} + \frac{2\beta A}{u}(u_i u_k u_{kj} + u_j u_k u_{ki}) + 4A^2 u_k u_l u_{ki} u_{lj},$$

and hence, at x_0, we have

(3.7)
$$\begin{aligned} 0 &\le u^{ij}z_{ij} \\ &= -\frac{\beta n}{u} - \frac{u^{ij}w_i w_j}{w^2} + \frac{\beta u^{ij}u_i u_j}{u^2} - 2Au^{ij}u_{kj}u_{kj} + \frac{2A}{1-\theta}\frac{u_k w_k}{w} \\ &= -\frac{\beta n}{u} - \beta(\beta-1)\frac{u^{ij}u_i u_j}{u^2} - 2A\Delta u + \frac{4A^2\theta}{1-\theta}u_{ij}u_i u_j \\ &\quad - 2\beta A\frac{1-2\theta}{1-\theta}\frac{|Du|^2}{u} \\ &\le -A\Delta u - \frac{\beta n}{u} + 2\beta A\frac{|Du|^2}{|u|}, \end{aligned}$$

with the choice

$$A = \frac{1-\theta}{4\theta \sup_\Omega |Du|^2}. \tag{3.8}$$

Consequently, we obtain

$$-u\Delta u(x_0) \le C(n,\beta)\sup_\Omega |Du|^2. \tag{3.9}$$

Setting $\beta = (1-\theta)n = n(n+1)/(n+2)$, we obtain

$$\begin{aligned} z(x) &\ge z(x_0) \\ &= (\theta-1)\log |u|^n \det D^2u(x_0) - A|Du|^2(x_0) \\ &\ge (\theta-1)n\log |u|\Delta u(x_0) - A|Du|^2(x_0) \\ &\ge -C(n, M_1), \end{aligned}$$

where $M_1 = \sup_\Omega |Du|$. Accordingly we estimate, for any $y \in \Omega$,

$$\det D^2u \le \frac{C(n, M_1)}{|u(y)|^n} \le \frac{C(n, M_1)(\operatorname{diam}\Omega)^n}{(\operatorname{dist}(y, \partial\Omega))^n}$$

by (3.1) and the convexity of u, and hence Lemma 3.1 is proved. □

Remark. It is clear that Lemma 3.1 will hold for any $\theta \in (0,1)$ in (2.4), with constant C in (3.2) depending also on θ.

We next derive a lower bound for $\det D^2 u$ in terms of a modulus of strict convexity for the function u. We first note that a locally uniformly convex function u on a convex domain Ω will be *strictly convex*, that is at each point P, of the graph $\mathcal{M}$ of u, there exists a supporting plane, lying below $\mathcal{M}$, and intersecting $\mathcal{M}$ only at P. For any $y \in \Omega$, $h > 0$, we define the *section* $S(y,h)$ by

$$S(y,h) = \left\{x \in \Omega \;\middle|\; u(x) < u(y) + Du(y)(y-x) + h\right\}. \tag{3.10}$$

We then define the *modulus of convexity* of u at y, by

$$h_{u,y}(r) = \sup\left\{h \geq 0 \;\middle|\; S(y,h) \subset B_r(y)\right\}, \quad r > 0 \tag{3.11}$$

and the *modulus of convexity* of u on Ω, by

$$h(r) = h_{u,\Omega}(r) = \inf_{y\in\Omega} h_{u,y}(r), \quad r > 0. \tag{3.12}$$

Observe that a function u is strictly convex in Ω if and only if $h(r) > 0$ for all $r > 0$.

Lemma 3.2. *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of equation (2.3) in a domain $\Omega \subset \mathbf{R}^n$, satisfying $-1 \leq u \leq 0$ in Ω. Then, for $y \in \Omega$, there exists a positive constant C depending on n, $dist(y, \partial\Omega)$, $diam(\Omega)$, and $h_{u,\Omega}$, such that*

$$C^{-1} \leq \det D^2 u(x) \leq C. \tag{3.13}$$

Proof. Since $u \in C^4(\Omega)$ is locally uniformly convex, so also is its Legendre transform, u^*, defined by

$$u^*(x) = \sup_{y\in\Omega}\big(x\cdot y - u(y)\big), \quad x \in \Omega^* = Du(\Omega), \tag{3.14}$$

with

$$Du^*(x) = y, \quad \det D^2 u^*(x) = \big(\det D^2 u(y)\big)^{-1}, \tag{3.15}$$

whenever $x = Du(y)$, $y \in \Omega$, [34]. Since u is maximal with respect to the functional A, given by (1.3), it follows that u^* is maximal with respect to the functional A^* given by

$$A^*[u,\Omega] = \int_{\Omega^*} [\det D^2 u]^{(n+1)/(n+2)}. \tag{3.16}$$

Therefore, if u satisfies (2.3), we see that u^* satisfies a similar equation

$$(U^*)^{ij}(w^*)_{ij} = 0, \tag{3.17}$$

where $[(U^*)^{ij}]$ is the cofactor matrix of $(u^*)_{ij}$ and

$$w^* = [\det D^2 u^*]^{-1/(n+2)}. \tag{3.18}$$

We cannot apply Lemma 3.1, with $\theta = \frac{n+1}{n+2}$, directly as the function u^* is not necessarily constant on $\partial\Omega^*$. However, for any point $y \in \Omega$, and $x = Du(y) \in \Omega^*$, we can infer from (3.11) and (3.12) that the section $S^*(x, \delta)$ of the Legendre transform u^* lies in Ω^* for $\delta = h(\frac{1}{2}\text{dist}(y, \partial\Omega))$. Furthermore, we have, (for $0 \in \Omega$),

$$|Du^*| \le \text{diam}(\Omega) \tag{3.19}$$

and hence the ball $B_R(x) \subset S^*(x, \delta)$ for $R \le \delta/\text{diam}\Omega$. Accordingly, we may apply Lemma 3.1, with $\theta = \frac{n+1}{n+2}$, to the function u^* in the domain $S^*(x, \delta)$ to deduce the lower bound in (3.13). The upper bound follows by applying Lemma 3.1 in the section $S(y, \delta)$ where we would have the gradient bound $|Du| \le 2/\text{dist}(y, \partial\Omega)$. □

4. Application of the Caffarelli-Gutiérrez theory

In Sect. 3, we established bounds for the Hessian determinant of solutions of equation (1.7) in bounded convex domains $\Omega \subset \mathbf{R}^n$, namely, for any subdomain $\Omega' \subset\subset \Omega$,

$$0 < \lambda \le \det D^2 u \le \Lambda \tag{4.1}$$

in Ω', where λ and Λ are positive constants depending only on n, diamΩ, dist$(\Omega', \partial\Omega)$, and the modulus of convexity of u, h. The function $u \in C^4(\Omega)$ was also assumed to be locally uniformly convex in Ω, and normalized by $-1 \le u \le 0$ in Ω. Caffarelli and Gutiérrez [9] have recently developed a theory of *linear* operators of the form

$$Lv = U^{ij} D_{ij} v = D_i(U^{ij} D_j v) \tag{4.2}$$

with coefficient matrix U^{ij} given as the cofactor matrix of a convex function u, which is analogous to the De Giorgi, Nash, Moser theory of uniformly elliptic divergence form linear operators and the Krylov-Safonov theory of general form linear operators [23]. In their theory, Euclidean balls are replaced by sections of the convex function u and Lebesgue measure by the Monge-Ampère measure associated with u. We will make use of the following Hölder estimate from [9].

Theorem 4.1. *Let $u \in C^2(\Omega)$ be a convex function in a domain $\Omega \subset \mathbf{R}^n$, satisfying (4.1) and $v \in C^2(\Omega)$ a solution of the equation*

$$Lv = U^{ij} D_{ij} v = 0 \tag{4.3}$$

in Ω. Then for any sections $S(y,r) \subset S(y,R) \subset \Omega$, we have the estimate

$$osc_{S(y,r)} v \le C\left(\frac{r}{R}\right)^{\alpha} osc_{S(y,R)} v, \tag{4.4}$$

where C and α are positive constants depending only on n and Λ/λ.

In order to apply Theorem 4.1 we observe that

$$B_r(y) \subset S(y, kr) \tag{4.5}$$

for $k = 2\sup_{\Omega'} |Du|$, and

$$S(y,R) \subset \Omega' \subset\subset \Omega \tag{4.6}$$

whenever $y \subset \Omega'$, $R < h(\text{dist}(y, \partial\Omega'))$. Consequently we infer from Lemmas 3.1 and 3.2 a Hölder estimate for the function w, namely,

$$[w]_{\alpha,\Omega'} \le C \tag{4.7}$$

for any $\Omega' \subset\subset \Omega$, where α and C are positive constants depending only on n, $\text{dist}(\Omega', \partial\Omega)$, $\text{diam}\Omega$, and h. By the Caffarelli-Schauder estimate for the Monge-Ampère equation [7], we then conclude local $C^{2,\alpha}$ estimates for the function u,

$$|u|_{2,\alpha,\Omega'} \le C, \tag{4.8}$$

where again α and C are positive constants depending only on n, $\text{dist}(\Omega', \partial\Omega)$, $\text{diam}\Omega$, and h. Bootstrapping, via the classical Schauder estimates [23], we thus obtain our desired estimates. Analyticity of solutions follows from [30].

Theorem 4.2. *Let Ω be a bounded convex domain in $\mathbf{R}^n$ and $u \in C^4(\Omega)$ a locally uniformly convex solution of equation (1.4) in Ω satisfying $-1 \le u \le 0$ in Ω. Then $u \in C^\infty(\Omega)$ and for any subdomain $\Omega' \subset\subset \Omega$, $k \ge 2$, we have the estimates*

$$D^2 u \ge C_1 I, \quad |D^k u| \le C_2, \tag{4.9}$$

where C_1 depends on n, $\text{dist}(\Omega', \partial\Omega)$, $\text{diam}\Omega$, and the modulus of convexity $h_{u,\Omega}$, and C_2 depends additionally on k. Moreover, u is also analytic in Ω.

In order to apply Theorem 4.2 to the Bernstein problem, we say that a convex function u in a domain Ω satisfies the *uniform strict convexity* condition if there exists a positive constant $\gamma \le 1$ and a positive function h on $(0,\infty)$ such that for some $y \in \Omega$ and all $r, t > 0$, $x \in \gamma S(y,t)$, we have

$$S(x, th(r)) \subset x + rS(y,t). \tag{4.10}$$

By combining Theorems 2.1 and 4.2, we then obtain

Corollary 4.3. *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of equation (1.4) in a convex domain $\Omega \subset \mathbf{R}^n$, satisfying (2.6). Then, if u satisfies the uniform strict convexity condition in Ω, it follows that $\Omega = \mathbf{R}^n$ and u is a quadratic function.*

5. The two-dimensional case

In this section we show that in the two dimensional case ($n = 2$) a solution to (1.4) with zero boundary condition satisfies a modulus of convexity estimate. We say $p \in \partial D$ is an extreme point of D if there is a supporting hyperplane of D such that D lies on one side of the plane and D touches the plane only at p. If D is convex, then any point in D can be represented as a linear combination of extreme points of D [34].

Lemma 5.1. *Let Ω be a normalized convex domain in $\mathbf{R}^n$ and $u \in C^4(\Omega)$ be a locally uniformly convex solution of (1.4), satisfying (3.1). Then there exists a nondecreasing positive function h on $(0, \infty)$, independent of u, such that*

$$h_{u,x}(r) \geq h(r) \quad \textit{for} \;\; x = (x_1, x_2) \in \frac{1}{2}\alpha_n B, \;\; r > 0, \tag{5.1}$$

where $h_{u,x}(r)$ is defined in (3.11).

Proof. It suffices to prove (5.1) for $x = 0$. If the lemma is not true, then there exist a sequence of functions $\{u_k\}$, and a sequence of normalized domains $\{\Omega_k\}$, satisfying (1.4), (3.1) with $\Omega = \Omega_k$, and a positive number $r_0 > 0$ such that

$$h_{u_k,0}(r_0) \to 0 \quad \text{as} \;\; k \to \infty. \tag{5.2}$$

Noticing that $h_{u,0}$ is non-decreasing, we have $h_{u_k,0}(r) \to 0$ as $k \to \infty$ uniformly for $r \in (0, r_0)$. Since Ω_k and u_k are convex, we may suppose by taking subsequences that $\{\Omega_k\}$ converges to a convex domain Ω and $\{u_k\}$ converges to a convex function u, locally uniformly in Ω. Let

$$\begin{aligned} D_k &= \{(x, x_3) \in \mathbf{R}^3 \;\big|\; u_k(x) < x_3 < 0\}, \\ D &= \{(x, x_3) \in \mathbf{R}^3 \;\big|\; u(x) < x_3 < 0\}. \end{aligned} \tag{5.3}$$

Then the sequence of convex domains $\{D_k\}$ converges to D. The graph of u is understood (only in this section) as $\partial D \backslash \{x_{n+1} = 0\}$, so that the sequence of graphs of $\{u_k\}$ converges to the graph of u as convex surfaces under Hausdorff distance. For $x \in \partial\Omega$, we define $u(x) = \underline{\lim}_{y \to x, y \in \Omega} u(y)$.

By (5.2) we see that u is not strictly convex near the origin. Let $\mathcal{L} = \{x_3 = \ell(x)\}$ be a supporting plane of the graph of u at the point $(0, u(0))$ such that the contact set $\omega_0 =: \{x \in \Omega \;\big|\; u(x) = \ell(x)\}$ is a convex set (possibly a line segment, but not a single point). Let ω be the closure of ω_0.

We claim that if $p = (p_1, p_2) \in \partial\omega$ and $p_3 = u(p) < 0$, then p is an interior point of Ω. Since Ω is convex, it suffices to show that if p is an extreme point of ω such that $u(p) < 0$, then $p \in \Omega$.

To prove this claim, we suppose to the contrary that $p \in \partial\Omega$, so that the line segment $\{(p_1, p_2, t) \;\big|\; p_3 < t < 0\}$ lies on the graph of u. We may suppose that $x = 0$ is an interior point of ω (or the midpoint of ω if ω is a line segment). Since equation (1.4) is affine invariant, we may suppose $p = (-1, 0)$ and

$$\Omega \subset \{x_1 > -1\}. \tag{5.4}$$

Adding a linear function to u and u_k, we may also suppose that

$$u(0) = 0, \quad u(p) = -\frac{\sqrt{3}}{3}, \quad u \geq \frac{\sqrt{3}}{3} x_1, \tag{5.5}$$

Then the line segments

$$\chi = \left\{ \left(t, 0, \frac{\sqrt{3}}{3} t\right) \,\Big|\, -1 < t < 0 \right\}, \tag{5.6}$$
$$\chi^* = \left\{ (-1, 0, t) \,\Big|\, -\frac{\sqrt{3}}{3} < t < 0 \right\}$$

lie on $\mathcal{M}$, the graph of u, and χ and χ^* form an angle of $\frac{\pi}{3}$ at the point $(-1, 0, -\frac{\sqrt{3}}{3})$. Here $\mathcal{M}$ and $\mathcal{M}_k$ denote the graphs of u and u_k after adding the linear function.

We introduce a new coordinate system (y_1, y_2, y_3) by letting

$$\begin{cases} y_1 = \frac{\sqrt{3}}{2}(x_1 + 1) - \frac{1}{2}(x_3 + \frac{\sqrt{3}}{3}), \\ y_2 = x_2, \\ y_3 = \frac{1}{2}(x_1 + 1) + \frac{\sqrt{3}}{2}(x_3 + \frac{\sqrt{3}}{3}) \end{cases} \tag{5.7}$$

Then near the origin of the new coordinates, $\mathcal{M}$ can be represented as the graph of a convex function v, which is nonnegative by (5.4), (5.5). Since p is an extreme point of ω, we see that v is strictly convex at $y = 0$, namely, $v > 0$ except at the origin. Hence Dv is bounded on the set $\{v < h\}$ for $h > 0$ small enough.

Since $\mathcal{M}_k$ is convex and $\{\mathcal{M}_k\}$ converges to $\mathcal{M}$ locally uniformly under Hausdorff distance, we see that $\mathcal{M}_k$ can also be represented as a graph of a convex function v_k in the new coordinates y for sufficiently large k. Obviously $v_k \to v$ near the origin. Hence the functions $|Dv_k|$ are uniformly bounded in the sets $\{v_k < h/2\}$ for k large enough. Since $\mathcal{M}_k$ is affine maximal, it is affine maximal under any coordinate system. We can therefore apply the argument of Lemma 3.1 to v_k and conclude that the functions $\det D^2 v_k$ are locally uniformly bounded. Since $v_k \to v$ uniformly near the origin, the sequence $\{\det D^2 v_k\}$ converges to the Monge-Ampère measure associated with v, $\mu_n[v]$, [2].

For $\varepsilon > 0$ small enough, let

$$G_\varepsilon = \left\{(y_1, y_2\} \,\big|\, v(y_1, y_2) < \varepsilon\right\}. \tag{5.8}$$

By (5.4) and (5.5) we have $v(y) \geq \sqrt{3}|y_1|$ and $v(y_1, 0) = \sqrt{3}|y_1|$ for y_1 small. Hence $G_\varepsilon \subset \{-\frac{\sqrt{3}}{3}\varepsilon < y_1 < \frac{\sqrt{3}}{3}\varepsilon\}$ and $(\pm\frac{\sqrt{3}}{3}\varepsilon, 0) \in \partial G_\varepsilon$. Let

$$\delta_\varepsilon = \sup\{|y_2| \,\big|\, (y_1, y_2) \in G_\varepsilon\}, \tag{5.9}$$

and let N_v denote the normal mapping of v. One easily verifies that

$$|N_v(G_\varepsilon)| \geq C\varepsilon/\delta_\varepsilon, \tag{5.10}$$

by comparing $N_v(G_\varepsilon)$ with the image of the normal mapping of the convex cone with vertex at the origin and base ∂G_ε. On the other hand, by the boundedness of $\det D^2 v$ we have

$$\begin{aligned} |N_v(G_\varepsilon)| &= \mu_n[v](G_\varepsilon) \\ &\leq \liminf_{k\to\infty} \int_{G_\varepsilon} \det D^2 v_k \\ &\leq C|G_\varepsilon| \\ &\leq C\varepsilon\delta_\varepsilon. \end{aligned} \tag{5.11}$$

Hence $\delta_\varepsilon > C > 0$ for some $C > 0$ independent of ε. On the other hand, we have $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ since v is strictly convex at 0. The contradiction shows that p must lie in the interior of Ω. The claim is proved.

Next we show that there is no extreme point of ω in Ω. Suppose $p \in \Omega$ is an extreme point of ω. By adding a linear function to u we may suppose that

$$u(p) = \inf\{u(x) \,\big|\, x \in \omega\}, \tag{5.12}$$

and $u = 0$ on $\partial\Omega$ by replacing Ω by a subdomain if necessary. By the affine invariance of the equation (1.4) we may also suppose $p = (-1, 0)$, $u(p) = -1$, and $\omega \subset \{x_1 \geq -1\}$ such that $\ell(x) = u(0) + x_1$ is a tangent plane of u at p. Let $x = 0$ be an interior point of ω or the midpoint of ω if ω is a line segment. Then for $\delta > 0$ small, $\ell(x) - \delta x_1 < 0$ in Ω. For $\varepsilon > 0$ small, let

$$\Omega_\varepsilon = \left\{(x_1, x_2) \in \Omega \,\big|\, u(x) < \ell(x) - \varepsilon x_1\right\}. \tag{5.13}$$

Let T_ε be a dilation which normalizes the domain Ω_ε and let

$$u_\varepsilon(T_\varepsilon(x)) = \frac{1}{\varepsilon}(u(x) - (\ell(x) - \varepsilon x_1)). \tag{5.14}$$

By taking a subsequence we may suppose that $T_\varepsilon(\Omega_\varepsilon)$ converges to a normalized domain Ω^* and $p_\varepsilon \to p^*$, where $p_\varepsilon = T_\varepsilon(p)$.

Let

$$D_\varepsilon = \left\{(x, x_3) \in \mathbf{R}^3 \,\big|\, u_\varepsilon(x) < x_3 < 0\right\}. \tag{5.15}$$

By (5.12) we have for $x \in \Omega_\varepsilon$, $0 \geq u_\varepsilon \geq -1 - o(1)$ as $\varepsilon \to 0$. Hence we may suppose $D_\varepsilon \to D^*$ as convex bodies and $u_\varepsilon \to u^*$ locally uniformly in Ω^*. Then the graph of u_ε converges to that of u^* as convex surfaces under Hausdorff distance, where the graph of u^* is defined as above by $\partial D^* \backslash \{x_3 = 0\}$.

Obviously u^* is not strictly convex. Indeed, since the line segment $\chi = \{(t, 0) \mid -1 \leq t \leq 0\}$ lies in Ω_ε, we have $T_\varepsilon(\chi) \to \chi^*$ by taking a subsequence. It follows u^* is linear on the line segment χ^*. Moreover, $p^* \in \chi^*$. Next observing that $\Omega_\varepsilon \subset \{x_1 > -1 - \delta_\varepsilon\}$ for some δ_ε, with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, we have by (2.2) that

$$\frac{\alpha_n}{2} - \sigma_\varepsilon \leq T_\varepsilon(e_1) \leq \frac{2}{\alpha_n} + \sigma_\varepsilon \tag{5.16}$$

for some $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$, where $e_1 = (1, 0)$ is the unit vector on x_1 axis. It follows that $\text{dist}(p_\varepsilon, \partial(T_\varepsilon(\Omega_\varepsilon))) \to 0$ and that $p^* \in \partial\Omega^*$.

Finally, since u is the limit of the sequence of affine maximal functions $\{u_k\}$, we see that u_ε and u^* are also the limits of sequences of affine maximal functions. Therefore by the claim above we conclude that p^* is an interior point of Ω^*. This is in contradiction with our construction since $p^* \in \partial\Omega^*$. This completes the proof. □

Combining Lemma 5.1 with Theorems 2.1 and 4.2 we obtain

Theorem 5.2. *An entire, affine maximal, locally uniformly convex C^4 graph in $\mathbf{R}^3$ must be an elliptic paraboloid.*

6. Reduction of smoothness

In this section, we extend the affine area functional (1.3) to general convex graphs and prove a regularity result for weak solutions of the affine maximal surface equation (1.4). We recall that the affine area functional A is defined for convex $u \in C^2(\Omega)$ by

$$A(u) = A(u, \Omega) = \int_\Omega [\det D^2 u]^{1/(n+2)}. \tag{6.1}$$

Since the function, $r \to (\det r)^{1/(n+2)}$ is strictly concave on the cone of positive symmetric $n \times n$ matrices, so also is the functional A, that is

$$A(tu + (1-t)v) \geq tA(u) + (1-t)A(v) \tag{6.2}$$

for all convex $u, v \in C^2(\Omega)$, $0 \leq t \leq 1$, with equality holding if and only if $D^2u = D^2v$ a.e.. Moreover if $u, \eta \in C^2(\Omega)$ are such that $u + t\eta$ is convex for sufficiently small $t \geq 0$, we have, by calculation,

$$\frac{d}{dt} A(u + t\eta)\Big|_{t=0} = \frac{1}{n+2} \int_\Omega w U^{ij} D_{ij}\eta, \tag{6.3}$$

where $w = (\det D^2u)^{-(n+1)/(n+2)}$, and as previously $[U^{ij}]$ is the cofactor matrix of D^2u. Note that if w is not integrable, the right hand side of (6.3) may be infinite. If u is locally uniformly convex, we thus obtain from (6.2) and (6.3) that the graph of u is affine maximal if and only if

$$\int_\Omega wU^{ij} D_{ij}\eta = 0 \tag{6.4}$$

for all $\eta \in C_0^2(\Omega)$, that is the weak form of equation (1.4) is satisfied. The following lemma then enables us to show that $u \in C^4(\Omega)$ and equation (1.4) is satisfied in the classical sense.

Lemma 6.1. *Let $L = a^{ij} D_{ij}$ be a linear elliptic operator on a bounded domain $\Omega \subset \mathbf{R}^n$, with boundary $\partial\Omega \in C^{1,1}$, and coefficients $[a^{ij}] \in C^0(\overline{\Omega})$, $i, j = 1, \cdots, n$, satisfying $[a_{ij}] \geq \lambda I$, for some positive constant λ. Then, if $w \in C^0(\overline{\Omega})$ vanishes on $\partial\Omega$ and satisfies*

$$\int_\Omega w\, L\eta = 0 \tag{6.5}$$

for all $\eta \in C_0^2(\Omega)$, we have $w = 0$ in Ω.

Proof. Let $v \in W^{2,q}(\Omega)$, $1 \leq q < \infty$, be the unique solution of the Dirichlet problem

$$\begin{aligned} Lv &= \operatorname{sign} w \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.6}$$

and take, as a test function in (6.5),

$$\eta = \chi v, \tag{6.7}$$

where $\chi = \chi_\varepsilon \in C_0^2(\Omega)$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ for $d(x) = \operatorname{dist}(x, \partial\Omega) \geq \varepsilon$, $|D\chi| \leq \frac{C}{\varepsilon}$, $|D^2\chi| \leq \frac{C}{\varepsilon^2}$ for some small $\varepsilon > 0$. It is readily seen by approximation that η is a valid test function in (6.5). Then, using the estimate, $v \leq Cd$, we obtain

$$\begin{aligned} \int_\Omega \chi|w| &= -\int_{\{d(x)<\varepsilon\}} w(2a^{ij} D_i\chi D_j v + va^{ij} D_{ij}\chi) \\ &\leq \frac{C}{\varepsilon} \int_{\{d(x)<\varepsilon\}} |w| \to 0, \end{aligned} \tag{6.8}$$

so that $w = 0$ in Ω as required. □

To apply Lemma 6.1, we fix some ball $B \subset\subset \Omega$ and solve the Dirichlet problem

$$Lv = U^{ij} D_{ij} v = 0 \quad \text{in } B, \qquad v = w \quad \text{on } \partial\Omega, \tag{6.9}$$

for unique solution $v \in W^{2,q}(B) \cap C^0(\overline{B})$, $1 \leq q < \infty$. By approximation, using the identity, $D_i U^{ij} = 0$, for smooth u, we see that v also satisfies the weak form of (6.9), that is,

$$\int_B v U^{ij} D_{ij} \eta = 0 \tag{6.10}$$

for all $\eta \in C_0^2(B)$, and hence we infer $w = v$, by replacing w by $w - v$ in Lemma 6.1. Consequently $\det D^2 u \in W^{2,q}(\Omega)$ for all $1 \leq q < \infty$, and further regularity follows, as in Sect. 4. Accordingly we have the following regularity theorem, complementing Theorem 4.2.

Theorem 6.2. *Let $u \in C^2(\Omega)$ be locally uniformly convex and affine maximal in a domain $\Omega \subset \mathbf{R}^n$. Then $u \in C^\infty(\Omega)$ and satisfies the affine maximal surface equation (1.4) in Ω.*

In particular we see from Theorem 6.2 that we need only assume $u \in C^2(\Omega)$ in Theorem 4.2 and moreover that Corollary 4.3 and Theorem 5.2 extend to C^2 graphs.

To complete this section, we extend the affine area functional to non-smooth convex functions and prove an upper semi-continuity result. First, we note from the equivalent representation (1.3),

$$\begin{aligned} A(u, \Omega) &= \int_{\mathcal{M}_\Omega} K^{1/(n+2)} \\ &\leq \left(\int_{\mathcal{M}_\Omega} K \right)^{1/(n+2)} \left(A_0(\mathcal{M}_\Omega) \right)^{(n+1)/(n+2)} \\ &\leq C(\Omega) \omega_n^{1/(n+2)} \end{aligned} \tag{6.11}$$

if $-1 \leq u \leq 0$ in Ω, where A_0 denotes the usual Euclidean area. By scaling, we then have the estimate

$$A(u, \Omega) \leq C \left(\operatorname{osc}_\Omega u \right)^{n/(n+2)}. \tag{6.12}$$

Next, if ω is a Borel subset of Ω with $\operatorname{dist}(\omega, \partial\Omega) \geq \delta$, we have

$$\begin{aligned} \int_\omega \left(\det D^2 u \right)^{1/(n+2)} &\leq |\omega|^{(n+1)/(n+2)} \left(\int_\omega \det D^2 u \right)^{1/(n+2)} \\ &\leq |\omega|^{(n+1)/(n+2)} \omega_n^{1/(n+2)} \left(\frac{\operatorname{osc} u}{\delta} \right)^{n/(n+2)} \end{aligned} \tag{6.13}$$

so that the functions $(\det D^2 u)^{1/(n+2)}$ are locally equi-integrable whenever the functions u are bounded. Now let u be an arbitrary convex function, and for $h > 0$, u_h be its mollification, namely,

$$u_h(x) = h^{-n} \int_\Omega \rho\Big(\frac{x-y}{h}\Big) u(y)dy \tag{6.14}$$

for $h \le \mathrm{dist}(x, \partial\Omega)$, where $\rho \ge 0, \in C_0^\infty(\mathbf{R})$, $\mathrm{supp}\,\rho \subset B_1(0)$ and $\int \rho = 1$. Since the second derivatives of u are signed measures, we can write

$$D^2 u = \mu^{(a)} + \mu^{(s)}, \tag{6.15}$$

where $\mu^{(a)}$ is absolutely continuous with respect to Lebesgue measure and $\mu^{(s)}$ is supported on a set $\mathcal{N}$ of Lebesgue measure zero. We let

$$\partial^2 u = [\partial_{ij} u] \ge 0, \ \in L^1_{loc}(\Omega) \tag{6.16}$$

denote the density of $\mu^{(a)}$ with respect to Lebesgue measure. It then follows that $D^2 u_h \to \partial^2 u$ almost everywhere in Ω [39]. However, since the functions $(\det D^2 u_h)^{1/(n+2)}$ are equi-integrable on subsets $\Omega' \subset\subset \Omega$ and hence relatively compact in $L^1_{loc}(\Omega)$, we then obtain

$$A(u_h, \Omega') \to \int_{\Omega'} (\det \partial^2 u)^{1/(n+2)} \tag{6.17}$$

for any subdomain $\Omega' \subset\subset \Omega$. The convergence result (6.17) is clearly also true for any sequence $\{u_m\} \subset C^2(\Omega)$ of convex functions, converging locally uniformly to u with $D^2 u_m \to \partial^2 u$ a.e.. Accordingly, we define the affine area of an arbitrary convex graph on a domain Ω, by

$$A(u, \Omega) = \int_\Omega \big(\det \partial^2 u\big)^{1/(n+2)}, \tag{6.18}$$

where $\partial^2 u$ denotes the density of the regular part of the Hessian matrix $D^2 u$. From our argument above, we have the approximation result

Lemma 6.3.

$$A(u, \Omega) = \lim_{h \to 0} A(u_h, \Omega_h), \tag{6.19}$$

where $\Omega_h = \{x \in \Omega \mid \mathit{dist}(x, \partial\Omega) > h\}$.

From the representation (6.18), we can then extend the formula (6.3) to arbitrary functions u, η such that $u + t\eta$ is convex for sufficiently small $t \ge 0$, namely

$$\frac{d}{dt} A(u + t\eta)\Big|_{t=0} = \frac{1}{n+2} \int_\Omega w U^{ij} \partial_{ij} \eta \tag{6.20}$$

where now $w = (\det \partial^2 u)^{-(n+1)/(n+2)}$, $[U^{ij}]$ denotes the cofactor matrix of $\partial^2 u$ and $[\partial_{ij}\eta]$ the density of the regular part of the signed measure $D^2\eta$ (which exists since η is a difference of convex functions). It follows that a convex function u is affine maximal if and only if

$$\int_\Omega wU^{ij}\partial_{ij}\eta \le 0 \tag{6.21}$$

for all such η with compact support in Ω. From Lemma 6.1, we then infer an extension of Theorem 6.2, namely that if $\partial^2 u = D^2 u_0$ for some locally uniformly convex function $u_0 \in C^2(\Omega)$, then $u_0 \in C^\infty(\Omega)$ and satisfies the affine maximal surface equation in Ω.

Finally we prove the upper semi-continuity of the extended affine area functional (6.18) with respect to uniform convergence.

Lemma 6.4. *Let $\{u_m\}$ be a sequence of convex functions in Ω, converging locally uniformly to u. Then*

$$\limsup_{m\to\infty} A(u_m, \Omega) \le A(u, \Omega). \tag{6.22}$$

Proof. By virtue of Lemma 6.3, it suffices to prove (6.22) for $u_m \in C^2(\Omega)$. Since $u_m \to u$ locally uniformly, $\det D^2 u_m \to \det D^2 u$ weakly as measures, that is, for any closed subset $F \subset \Omega$,

$$\limsup_{m\to\infty} \int_F \det D^2 u_m \le \int_F \det D^2 u. \tag{6.23}$$

For given $\varepsilon, \varepsilon' > 0$, let

$$\omega_k = \left\{x \in \Omega \;\middle|\; (k-1)\varepsilon \le \det D^2 u < k\varepsilon\right\},$$

$k = 1, 2, \cdots$, and $F_k \subset \omega_k$ be a closed set such that $|\omega_k - F_k| < \varepsilon'$. For each F_k, we have

$$\begin{aligned}
\limsup_{m\to\infty} \frac{1}{|F_k|}\int_{F_k} \left(\det D^2 u_m\right)^{1/(n+2)} &\le \limsup_{m\to\infty} \left(\frac{1}{|F_k|}\int_{F_k} \det D^2 u_m\right)^{1/(n+2)} \\
&\le \left(\frac{1}{|F_k|}\int_{F_k} \det D^2 u\right)^{1/(n+2)} \\
&\le (k\varepsilon)^{1/(n+2)}
\end{aligned}$$

so that

$$\begin{aligned}
\limsup_{m\to\infty} A(u_m, F_k) &\le (k\varepsilon)^{1/(n+2)}|F_k| \\
&\le A(u, F_k) + \frac{1}{n+2}\varepsilon^{1/(n+2)}|F_k|
\end{aligned} \tag{6.24}$$

since $\det D^2 u \geq (k-1)\varepsilon$ in F_k. Consequently

$$\limsup_{m\to\infty} A(u_m, \cup F_k) \leq A(u, \Omega) + \frac{1}{n+2}\varepsilon^{1/(n+2)}|\Omega|. \tag{6.25}$$

Using the equi-integrability (6.13) and sending $\varepsilon, \varepsilon' \to 0$, we conclude (6.22). □

Note that by considering polygonal approximations, we can have strict inequality in (6.22). Furthermore by combining Lemmas 6.3 and 6.4, we deduce a further representation for $A(u, \Omega)$, namely

$$A(u, \Omega) = \sup_{\{u_m\}\subset \mathcal{S}_u} \limsup_{m\to\infty} A(u_m, \Omega), \tag{6.26}$$

where $\mathcal{S}_u$ denotes the set of sequences of convex functions $\{u_m\} \subset C^2(\Omega)$, converging locally uniformly in Ω to the convex function u. We remark that we could have equivalently defined the extended affine area through the regular part of the Monge-Ampère measure. For other definitions see [27].

7. Example

In this section, we provide an example of affine maximal, convex graphs which does not satisfy the Bernstein property, and which violates the uniform strict convexity in high dimensions. Specifically we take $n = 10$ and define

$$u(x) = \sqrt{|x'|^9 + x_{10}^2}, \tag{7.1}$$

where $x' = (x_1, \cdots, x_9)$. It is readily shown that $u \in W^{2,1}_{loc}(\mathbf{R}^{10})$ so that $D^2u = \partial^2 u$ and we need to verify (6.21) to show that u is affine maximal. For $x \neq 0$, we consider the transformation

$$\begin{cases} y' = x', \\ y_{10} = x_{10} + u \\ v = u - x_{10} \end{cases} \tag{7.2}$$

so that the function v is given by

$$v(y) = \frac{|y'|^9}{y_{10}} \tag{7.3}$$

for $y_{10} > 0$. To show that v satisfies the affine maximal surface equation, we consider, more generally, functions of the form,

$$u = \frac{r^{2\alpha}}{t}, \tag{7.4}$$

where $\alpha \geq 1$, $r = |y'|$, $t = |y_n|$, $y' = (y_1, \cdots, y_{n-1})$. Then

$$
\begin{aligned}
u_r &= \frac{2\alpha r^{2\alpha-1}}{t}, \\
u_t &= -\frac{r^{2\alpha}}{t^2}, \\
u_{rr} &= 2\alpha(2\alpha-1)\frac{r^{2\alpha-2}}{t}, \\
u_{rt} &= -\frac{2\alpha r^{2\alpha-1}}{t^2}, \\
u_{tt} &= \frac{2r^{2\alpha}}{t^3},
\end{aligned} \tag{7.5}
$$

Denote

$$
\begin{aligned}
\Delta &= u_{rr}u_{tt} - u_{rt}^2 = 4\alpha(\alpha-1)\frac{r^{4\alpha-2}}{t^4}, \\
\mathcal{D} &= \det D^2 u = \left(\frac{u_r}{r}\right)^{n-2}\Delta = C\frac{r^{2n(\alpha-1)+2}}{t^{n+2}}, \\
w &= \mathcal{D}^{\frac{1}{n+2}-1} = C'\frac{t^{n+1}}{r^{\theta}},
\end{aligned} \tag{7.6}
$$

where

$$
\begin{aligned}
C &= 2^n\alpha^{n-1}(\alpha-1), \quad C' = C^{-(n+1)/(n+2)}, \\
\theta &= \frac{2(n+1)}{n+2}(n\alpha - n + 1).
\end{aligned}
$$

Also, denote

$$
\begin{aligned}
\widetilde{\Delta} &= u_{tt}w_{rr} + u_{rr}w_{tt} - 2u_{rt}w_{rt}, \\
&= \frac{C't^{n-2}}{r^{\theta-2\alpha+2}}\Big(2\theta(\theta+1) + n(n+1)2\alpha(2\alpha-1) - 4\alpha(n+1)\theta\Big).
\end{aligned} \tag{7.7}
$$

Then we have

$$
L[u] := u^{ij}w_{ij} = (n-2)\frac{r}{u_r}\frac{w_r}{r} + \frac{1}{\Delta}\widetilde{\Delta} = \frac{t^{n+2}}{r^{\theta+2\alpha}}K, \tag{7.8}
$$

where

$$
\begin{aligned}
K = C'\bigg[&-\frac{n-2}{2\alpha}\theta \\
&+ \frac{1}{2\alpha(\alpha-1)}\Big(\theta(\theta+1) + n(n+1)\alpha(2\alpha-1) - 2(n+1)\alpha\theta\Big)\bigg].
\end{aligned} \tag{7.9}
$$

For u to be affine maximal, we need $K = 0$, i.e.,

(7.10)
$$\theta(\theta+1) + n(n+1)\alpha(2\alpha-1) - 2(n+1)\alpha\theta - (n-2)(\alpha-1)\theta = 0.$$

Substituting for θ, we obtain the equivalent quadratic equation for α,

$$8\alpha^2 - (n^2 - 4n + 12)\alpha + 2(n-1)^2 = 0, \tag{7.11}$$

which is solvable for $n \geq 10$. In particular for $n = 10$, $\alpha = \frac{9}{2}$ and we conclude that the function (7.1) satisfies (1.4) for $x \neq 0$. To verify (6.21) in domains Ω containing the origin, we first observe, by virtue of Lemma 6.3, we may assume η is smooth away from the origin. We next estimate the integrand on the left hand side of (6.21) by

$$\begin{aligned} wU^{ij}\partial_{ij}\eta &\leq (\det D^2 u)^{1/(n+2)} \sum u^{ii} \sum |\partial_{ii}\eta| \\ &\leq \frac{C}{r^3} \sum |\partial_{ii}\eta| \end{aligned} \tag{7.12}$$

where $r = |x'|$, $t = x_{10}$ and since η is a difference of two convex functions, $\sum |\partial_{ii}\eta|$ can be regarded as a Borel measure, $\mu = \mu_\eta$, satisfying

$$|\mu(B_R)| \leq CR^{n-1} \tag{7.13}$$

for any ball B_R, (where C depends on η). Therefore, the integrability of the function, $wU^{ij}\partial_{ij}\eta$, follows from the following lemma whose proof is elementary and is omitted here.

Lemma 7.1. *Let μ be a Borel measure on $\mathbf{R}^n$ satisfying (7.13) for any ball B_r. Then, for any $\varepsilon > 0$,*

$$\left| \int_{B_1} \frac{d\mu}{|x|^{n-2-\varepsilon}} \right| < \infty, \tag{7.14}$$

where $x' = (x_1, \cdots, x_{n-1})$.

For sufficiently small $\delta > 0$, we set

$$\omega_\delta = \left\{ x \in \Omega \;\middle|\; |x'| < \delta, |x_{10}| < \delta \right\}.$$

Integrating by parts, we then have

$$\begin{aligned} \int_\Omega wU^{ij}\partial_{ij}\eta &= \int_{\omega_\delta} wU^{ij}\partial_{ij}\eta + \int_{\Omega\setminus\omega_\delta} wU^{ij}\eta_{ij} \\ &= \int_{\omega_\delta} wU^{ij}\partial_{ij}\eta + \int_{\partial\omega_\delta} wU^{ij}\eta_i\gamma_j - \int_{\partial\omega_\delta} \eta(wU^{ij})_i\gamma_j \\ &\to 0 \quad \text{as} \quad \delta \to 0 \end{aligned} \tag{7.15}$$

since u satisfies equation (1.4) in $\mathbf{R}^{10}\setminus\{o\}$. Consequently u is affine maximal in $\mathbf{R}^{10}$.

If $n > 10$, it is easy to verify that the function u, given by

$$u(x) = \sqrt{|x'|^9 + |x_{10}|^2} + |\widetilde{x}|^2,$$

is affine maximal, where $\widetilde{x} = (x_{11}, \cdots, x_n)$.

References

1. F.J. Almgren, Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem, Ann. Math. **74**(1966), 277–292
2. I.J. Bakelman, Convex analysis and nonlinear geometric elliptic equations, Springer-Verlag, Berlin, 1994
3. S.N. Bernstein, Sur un theoreme de geometrie et ses applications aux equations aux derivees partielles du type elliptique, Comm. de la Soc. Math de Kharkov (2eme ser.) **15**(1915–17), 38–45. See also: Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differential gleichungen vom elliptischen Typus, Math. Zeit. **26**(1927), 551–558
4. C. Blanc, F. Fiala, Le type d'une surface et sa courbure totale, Comm. Math. Helv., **14**(1941–42), 230–233
5. W. Blaschke, Vorlesungen über Differentialgeometrie, Berlin, 1923
6. E. Bombieri, E. De Giorgi, E. Giusti, Minimal cones and the Bernstein problem, Invent. math. **7**(1969), 243–268
7. L.A. Caffarelli, Interior $W^{2,p}$ estimates for solutions of Monge-Ampère equations, Ann. Math. **131**(1990), 135–150
8. L.A. Caffarelli, A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity, Ann. Math. **131**(1990), 129–134
9. L.A. Caffarelli, C.E. Gutiérrez, Properties of the solutions of the linearized Monge-Ampère equations, Amer. J. Math. **119**(1997), 423–465
10. E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, Mich. Math. J. **5**(1958), 1005–126
11. E. Calabi, Minimal immersions of surfaces in euclidean spheres, J. Diff. Geo. **1**(1967), 111–125
12. E. Calabi, Complete affine hypersurfaces I, Symposia Mathematica **10**(1972), 19–38
13. E. Calabi, Hypersurfaces with maximal affinely invariant area, Amer. J. Math. **104** (1982), 91–126
14. E. Calabi, Convex affine maximal surfaces, Results in Math. **13**(1988), 199–223
15. S.Y. Cheng, S.T. Yau, Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces, Ann. Math. **104**(1976), 407–419
16. S.Y. Cheng, S.T. Yau, Complete affine hypersurfaces I, The completeness of affine metrics, Comm. Pure Appl. Math. **39**(1986), 839–866
17. S.S. Chern, Brief survey of minimal submanifolds, Tagungsbericht, Oberwolfach, 1969
18. S.S. Chern, Affine minimal hypersurfaces. In Minimal submanifolds and geodesics, Proc. Japan-United States Sem., Tokyo, 1977, 17–30; see also Selected papers of S.S. Chern, Volume III, Springer, 1989, 425–438
19. K.S. Chou, X.J. Wang, Entire solutions to Monge-Ampère equations, Comm. Pure Appl. Math. **49**(1996) 529–539
20. E. De Giorgi, Una estensione del teoreme di Bernstein, Ann. Sci. Norm. Sup., Pisa **19**(1965), 79–85
21. Miguel De Guzman, Differentiation of integrals in $\mathbf{R}^n$, Lecture Notes in Math., **481**, Springer-Verlag, 1976
22. K. Ecker, G. Huisken, A Bernstein result for minimal graphs of controlled growth. J. Diff. Geo. **31**(1991), 397–400
23. D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, New York, 1983
24. J. van Heijenoort, On locally convex manifolds, Comm. Pure Appl. Math. **5**(1952), 223–242
25. W.Y. Hsiang, Minimal cones and the spherical Bernstein problem I, Ann. Math., **118**(1983), 61-73; II, Invent. math. **74**(1983), 351–369
26. K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, Math. Ann. **127**(1954), 130–134
27. K. Leichtweiss, Affine geometry of convex bodies, Johann Ambrosius Barth Verlag, Heidelberg, 1998

28. A.M. Li, Affine completeness and Euclidean completeness, Lecture Notes in Math. **1481**(1991), 116–126
29. A. Martinez, F. Milan, On the affine Bernstein problem, Geom. Dedicata **37**(1991), 295–302
30. C.B. Morrey, Multiple integrals in the calculus of variations, Springer, Berlin, 1966
31. K. Nomizu, T. Sasaki, Affine differential geometry, Cambridge University Press, 1994
32. A.V. Pogorelov, On the improper affine hyperspheres, Geom. Dedicata **1**(1972), 33–46
33. A.V. Pogorelov, The Minkowski multidimensional problem, J. Wiley, New York, 1978
34. R.T. Rockafellar, Convex analysis, Princeton Univ. Press, Princeton, 1970
35. R. Sacksteder, On hypersurfaces with no negative sectional curvatures, Amer. J. Math. **82**(1960), 609–630
36. J. Simons, Minimal varieties in Riemannian manifolds, Ann. Math. **88**(1968), 66–108
37. P. Tomter, The spherical Bernstein problem in even dimensions and related problems, Acta Math. **158**(1987), 189–212
38. S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. **28**(1975), 201–228
39. W.P. Ziemer, Weakly differentiable functions, Springer-Verlag, New York, 1989