Inventiones mathematicae (2024) 235:1019–1062
https://doi.org/10.1007/s00222-024-01235-7

SRB measures for C^{∞} surface diffeomorphisms

David Burguet[1]

Received: 5 November 2021 / Accepted: 16 January 2024 / Published online: 5 February 2024

Abstract
A C^{∞} smooth surface diffeomorphism admits an SRB measure if and only if the set $\{x,\ \limsup_n \frac{1}{n} \log \|d_x f^n\| > 0\}$ has positive Lebesgue measure. Moreover the basins of the ergodic SRB measures are covering this set Lebesgue almost everywhere. We also obtain similar results for C^r surface diffeomorphisms with $+\infty > r > 1$.

Mathematics Subject Classification 37C40 · 37D25

Contents

✉ D. Burguet
david.burguet@upmc.fr

[1] CNRS, Sorbonne Université, LPSM, 75005 Paris, France

1 Introduction

One fundamental problem in dynamics consists in understanding the statistical behaviour of the system. Given a topological system (X, f) we are more precisely interested in the asymptotic distribution of the empirical measures $\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}\right)_n$ for typical points x with respect to a reference measure. In the setting of differentiable dynamical systems the natural reference measure to consider is the Lebesgue measure on the manifold.

The basin of a f-invariant measure μ is the set $\mathcal{B}(\mu)$ of points whose empirical measures are converging to μ in the weak-$*$ topology. By Birkhoff's ergodic theorem the basin of an ergodic measure μ has full μ-measure. An invariant measure is said to be physical when its basin has positive Lebesgue measure. We may wonder when such measures exist and then study their basins.

In the works of Y. Sinai, D. Ruelle and R. Bowen [10, 38, 42] these questions have been successfully solved for uniformly hyperbolic systems. An SRB measure of a C^{1+} system is an invariant probability measure with at least one positive Lyapunov exponent almost everywhere, which has absolutely continuous conditional measures on unstable manifolds [45]. Physical measures may neither be SRB measures nor sinks (as in the famous figure-eight attractor), however hyperbolic ergodic SRB measures are physical measures [30]. For uniformly hyperbolic systems, there is a finite number of such measures and their basins cover a full Lebesgue subset of the manifold. Beyond the uniformly hyperbolic case such a picture is known for large classes of partially hyperbolic systems [1, 2, 9, 35]. Corresponding results have been established for unimodal maps with negative Schwartzian derivative [25]. SRB measures have been also deeply investigated for parameter families such as the quadratic family and Hénon maps [4–6, 24]. In his celebrated ICM's talk, M. Viana conjectured that a C^{1+} diffeomorphism admits an SRB measure, whenever the set of points with nonzero Lyapunov exponents has full Lebesgue measure. In recent works some weaker versions of the conjecture (with some additional assumptions of recurrence and Lyapunov regularity) have been proved [7, 18, 19]. Finally we mention that J. Buzzi, S. Crovisier, O. Sarig have also recently shown the existence of an SRB measure for C^∞ surface diffeomorphims when the set of points with a positive Lyapunov exponent has positive Lebesgue measure [16] (Corollary 2).

In this paper we define a general entropic approach to build SRB measures. We strongly believe that this approach may be used to recover the existence of SRB measures for weakly mostly expanding partially hyperbolic systems [1] and to give another proof of Ben Ovadia's criterion for C^{1+} diffeomorphisms in any dimension [7].

We state now the main results of our paper. Let $(M, \|\cdot\|)$ be a compact Riemannian surface and let Leb be a volume form on M, called Lebesgue measure. We consider a C^∞ surface diffeomorphism $f : M \circlearrowleft$. The maximal Lyapunov exponent at $x \in M$ is given by $\chi(x) = \limsup_n \frac{1}{n}\log\|d_x f^n\|$. When μ is a f-invariant probability measure, we let $\chi(\mu) = \int \chi(x)\,d\mu(x)$. For two Borel subsets A and B of M we write $A \overset{o}{\subset} B$ (resp. $A \overset{o}{=} B$) when we have $\mathrm{Leb}(A \setminus B) = 0$ (resp. $\mathrm{Leb}(A\Delta B) = 0$). For $c \in \mathbb{R}$ and $\Gamma \subset \mathbb{R}$ we also let $\{\chi > c\} := \{x \in M,\ \chi(x) > c\}$, $\{\chi = c\} := \{x \in M,\ \chi(x) = c\}$ and $\{\chi \in \Gamma\} := \{x \in M,\ \chi(x) \in \Gamma\}$.

Theorem 1 *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$, such that we have with $\Lambda = \{\chi(\mu_i),\ i \in I\} \subset \mathbb{R}_{>0}$:*

- $\{\chi > 0\} \overset{o}{=} \{\chi \in \Lambda\}$,
- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

Corollary 1 *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism. Then*

$$\{\chi > 0\} \overset{o}{\subset} \bigcup_{\mu \text{ SRB ergodic}} \mathcal{B}(\mu).$$

Corollary 2 (Buzzi-Crovisier-Sarig [16]) *Let $f : M \circlearrowleft$ be a C^∞ surface diffeomorphism.*

If $\mathrm{Leb}(\chi > 0) > 0$, *then there exists an SRB measure.*

In fact we establish a C^r, $1 < r < +\infty$, stronger version, which implies straightforwardly Theorem 1:

Main Theorem *Let $f : M \circlearrowleft$ be a C^r, $\mathbb{R} \ni r > 1$, surface diffeomorphism. Let $R(f) := \lim_n \frac{1}{n}\log^+ \sup_{x\in M}\|d_x f^n\|$. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$ with $\Lambda := \{\chi(\mu_i),\ i \in I\} \subset]\frac{R(f)}{r}, +\infty[$, such that we have*:

- $\left\{\chi > \frac{R(f)}{r}\right\} \overset{o}{=} \{\chi \in \Lambda\}$,
- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

In others terms, Lebesgue almost every point x with $\chi(x) > \frac{R(f)}{r}$ lies in the basin of an ergodic SRB measure μ with $\chi(x) = \chi(\mu)$.

When f is a C^{1+} topologically transitive surface diffeomorphism, there is at most one SRB measure, i.e. $\sharp I \leq 1$ [23]. If moreover the system is topologically mixing, then the SRB measure when it exists is Bernoulli [30]. By the spectral decomposition of C^r surface diffeomorphisms for $1 < r \leq +\infty$ [14] there are at most finitely many ergodic SRB measures with entropy and thus maximal exponent larger than a given constant $b > \frac{R(f)}{r}$. Therefore, in the Main Theorem, the set $\Lambda = \{\chi(\mu_i),\ i \in I\}$ is either finite or a sequence decreasing to $\frac{R(f)}{r}$. When r is finite, there may also exist ergodic SRB measures μ with $\chi(\mu) \leq \frac{R(f)}{r}$.

We prove in a forthcoming paper [11] that the above statement is sharp by building for any finite $r > 1$ a C^r surface diffeomorphism $f : M \circlearrowleft$ with a periodic saddle hyperbolic point p such that $\chi(x) = \frac{\chi(\delta_p)}{r} > 0$ for all $x \in U$ for some set $U \subset \mathcal{B}(\mu_p)$ with $\mathrm{Leb}(U) > 0$, where μ_p denotes the periodic measure associated to p (see [13] for such an example of interval maps).

In higher dimensions we let $\Sigma^k\chi(x) := \limsup_n \frac{1}{n}\|\Lambda^k d_x f^n\|$ where $\Lambda^k df$ denotes the action induced by f on the kth exterior power of TM for $k = 1, \ldots, d$ with d being the dimension of M. By convention we also let $\Sigma^0\chi = 0$. For any C^1 diffeomorphism (M, f) we have $\mathrm{Leb}(\Sigma^d\chi > 0) = 0$ (see [3]). The product of a figure-eight attractor with a surface Anosov diffeomorphism does not admit any SRB measure whereas χ is positive on a set of positive Lebesgue measure. However we conjecture:

Conjecture *Let $f : M \circlearrowleft$ be a C^∞ diffeomorphism on a compact manifold (of any dimension).*

If $\mathrm{Leb}\left(\Sigma^k\chi > \Sigma^{k-1}\chi \geq 0\right) > 0$, *then there exists an ergodic measure with at least k positive Lyapunov exponents, such that its entropy is larger than or equal to the sum of its k smallest positive Lyapunov exponents.*

In the present two-dimensional case the semi-algebraic tools used to bound the distortion and the local volume growth of C^∞ curves are elementary. This is a challenging problem to adapt this technology in higher dimensions.

When the empirical measures from $x \in M$ are not converging, the point x is said to have historic behaviour [39]. A set U is contracting when the diameter of $f^n U$ goes to zero when $n \in \mathbb{N}$ goes to infinity. In a contracting set the empirical measures of all points have the same limit set, however they may not converge. P. Berger and S. Biebler have shown that C^∞ densely inside the Newhouse domains [8] there are contracting domains with historic behaviour. In intermediate smoothness, such domains have been previously built in [27]. As a consequence of the Main Theorem, Lebesgue almost every point x with historic behaviour satisfies $\chi(x) \leq 0$ for C^∞ surface diffeomorphisms. We also show the following statement.

Theorem 2 *Let f be a C^∞ diffeomorphism on a compact manifold (of any dimension). Then Lebesgue a.e. point x in a contracting set satisfies $\chi(x) \leq 0$.*

We explain now in few lines the main ideas to build an SRB measure under the assumptions of the Main Theorem. The geometric approach for uniformly hyperbolic systems (see e.g. [17]) consists in considering a weak limit of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \mathrm{Leb}_{D_u}\right)_n$, where D_u is a local unstable disc and Leb_{D_u} denotes the normalized Lebesgue measure on D_u induced by its inherited Riemannian structure as a submanifold of M. Here we take a smooth C^r embedded curve D such that

$$\chi(x, v_x) := \limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > b > \frac{R(f)}{r}$$

for (x, v_x) in the unit tangent space T^1D of D with x in a subset B of D of positive Leb_D-measure. For x in B we define a subset $E(x)$ of positive integers, called the *geometric set*, such that the following properties hold for any $n \in E(x)$:

- the geometry of f^nD around f^nx is *bounded* meaning that for some uniform $\epsilon > 0$, the connected component $D_n^\epsilon(x)$ of f^nD intersected with the ball at f^nx of radius $\epsilon > 0$ is a curve with bounded s-derivative for $s \le r$,
- the distortion of df^{-n} on the tangent space of $D_n^\epsilon(x)$ is controlled,
- for some $\tau > 0$ we have $\frac{\|d_x f^l(v_x)\|}{\|d_x f^k(v_x)\|} \ge e^{(l-k)\tau}$ for any $l > k \in E(x)$.

We show that $E(x)$ has positive upper asymptotic density for x in a subset A of B of positive Leb_D-measure. Let $F : \mathbb{P}TM \circlearrowleft$ be the map induced by f on the projective tangent bundle $\mathbb{P}TM$. We build an SRB measure by considering a weak limit μ of a sequence of the form $\left(\frac{1}{\sharp F_n}\sum_{k\in F_n} F_*^k\mu_n\right)_n$ such that:

- $(F_n)_n$ is a Følner sequence, so that the weak limit μ will be invariant by F,
- for all n, the measure μ_n is the probability measure induced by Leb_D on $A_n \subset A$, the Leb_D-measure of A_n being not exponentially small,
- the sets $(F_n)_n$ are in some sense *filled with* the geometric set $E(x)$ for $x \in A_n$. Then the measure μ on $\mathbb{P}TM$ will be supported on the unstable Oseledec's bundle.

Finally we check with some Følner Gibbs property that the limit empirical measure μ projects to an SRB measure on M by using the Ledrappier-Young entropic characterization.

The paper is organized as follows. In Sect. 2 we recall for general sequences of integers the notion of asymptotical density and we build for any sequence E with positive upper density a Følner set F filled with E. Then we use a Borel-Cantelli argument to define our sets $(A_n)_n$ and the Følner sequence $(F_n)_n$. In Sect. 3, we study the maximal Lyapunov exponent and the entropy of the generalized empirical measure μ assuming some Gibbs property. We introduce the geometric set in Sect. 4 by using the Reparametrization Lemma of [12]. We build then SRB measures in Sect. 5 by using the abstract formalism of Sect. 2 and 3. Then we prove the covering property of the basins in Sect. 6 by the standard argument of absolute continuity of Pesin stable foliation. The last section is devoted to the proof of Theorem 2.

Comment: In a first version of this work, by following [12] (incorrectly) the author claimed that, at b-hyperbolic times n of the sequence $\left(\|d_x f^k(v_x)\|\right)_k$ for some $b > 0$, the geometry of f^nD at f^nx was bounded. J. Buzzi, S. Crovisier and O. Sarig gave then in [16] another proof of Corollary 2 by using their analysis of the entropic continuity of Lyapunov exponents from [15]. But our claim on the geometry at hyperbolic times is wrong in general and we manage to show it only when $\chi(x) > \frac{R(f)}{2}$. Here we correct our proof based on the Reparametrization Lemma of [12] by showing directly that the set of times with bounded geometry has positive upper asymptotic density on a set of positive Leb_D-measure.

2 Some asymptotic properties of integers

2.1 Asymptotic density

We first introduce some notations. The set of positive integers is denoted by $\mathbb{N}^*$. In the following we let $\mathcal{P}_{\mathbb{N}}$ and $\mathcal{P}_n$ be respectively the power sets of $\mathbb{N}$ and $\{0, 1, 2, \dots, n-1\}$, $n \in \mathbb{N}$. For $a \leq b \in \mathbb{N}$ we write $[\![a, b]\!]$ (resp. $[\![a, b[\![$, $]\!]a, b]\!]$) the interval of integers k with $a \leq k \leq b$ (resp $a \leq k < b$, $a < k \leq b$). The **connected components** of $E \in \mathcal{P}_{\mathbb{N}}$ are the maximal intervals of integers contained in E. An interval of integers $[\![a, b[\![$ is said to be E-**irreducible** when we have $a, b \in E$ and $[\![a, b[\![\cap E = \{a\}$. The **boundary** ∂E of E is the symmetric difference of E and $E+1$ with $E+1 := \{k+1,\ k \in E\}$. In particular $\partial\, [\![a, b[\![= \{a, b\}$. Observe that ∂E completely determines E. For $M \in \mathbb{N}^*$, we denote by E_M the union of the intervals $[\![a, b[\![$ with $a, b \in E$ and $0 < b - a \leq M$. We let $\mathfrak{N}$ be the set of increasing sequences of natural integers, which may be identified with the subset of $\mathcal{P}_{\mathbb{N}}$ given by infinite subsets of $\mathbb{N}$. For $\mathfrak{n} \in \mathfrak{N}$ we define the **generalized power set of** $\mathfrak{n}$ as $\mathcal{Q}_{\mathfrak{n}} := \prod_{n \in \mathfrak{n}} \mathcal{P}_n$.

We recall now the classical notion of upper and lower asymptotic densities. For $n \in \mathbb{N}^*$ and $F \in \mathcal{P}_{\mathbb{N}}$ we let $d_n(F)$ be the frequency of F in $[\![0, n[\![$:

$$d_n(F_n) = \frac{\sharp F \cap [\![0, n[\![}{n}.$$

The **upper and lower asymptotic densities** $\overline{d}(E)$ and $\underline{d}(E)$ of $E \in \mathcal{P}_{\mathbb{N}}$ are respectively defined by

$$\overline{d}(E) := \limsup_{n \in \mathbb{N}} d_n(E) \text{ and}$$

$$\underline{d}(E) := \liminf_{n \in \mathbb{N}} d_n(E).$$

We just write $d(E)$ for the limit, when the frequencies $d_n(E)$ are converging. For any $\mathfrak{n} \in \mathfrak{N}$ we let similarly $\overline{d}^{\mathfrak{n}}(E) := \limsup_{n \in \mathfrak{n}} d_n(E)$ and $\underline{d}^{\mathfrak{n}}(E) := \liminf_{n \in \mathfrak{n}} d_n(E)$. The concept of upper and lower asymptotic densities of $E \in \mathcal{P}_{\mathbb{N}}$ may be extended to generalized power sets as follows. For $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ we let

$$\overline{d}^{\mathfrak{n}}(\mathcal{F}) := \limsup_{n \in \mathfrak{n}} d_n(F_n) \text{ and}$$

$$\underline{d}^{\mathfrak{n}}(\mathcal{F}) := \liminf_{n \in \mathfrak{n}} d_n(F_n).$$

Again we just write $d^{\mathfrak{n}}(E)$ and $d^{\mathfrak{n}}(\mathcal{F})$ when the corresponding frequencies are converging.

2.2 Følner sequence and density along subsequences

We say that $E \in \mathcal{P}_{\mathbb{N}}$ is **Følner along** $\mathfrak{n} \in \mathfrak{N}$ when its boundary ∂E has zero upper asymptotic density with respect to $\mathfrak{n}$, i.e. $d^{\mathfrak{n}}(\partial E) = 0$. More generally $\mathcal{F} = (F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ with $\mathfrak{n} \in \mathfrak{N}$ is Følner when we have $d^{\mathfrak{n}}(\partial \mathcal{F}) = 0$ with $\partial \mathcal{F} = (\partial F_n)_{n \in \mathfrak{n}}$. In general this property seems to be weaker than the usual Følner property, which requires $\lim_{n\in\mathfrak{n}} \frac{\sharp \partial F_n}{\sharp F_n} = 0$. But in the following we will work with sequences $\mathcal{F}$ with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. In this case our definition coincides with the standard one.

Let $E, F \in \mathcal{P}_{\mathbb{N}}$ and $\mathfrak{n} \in \mathfrak{N}$. We say that F is **$\mathfrak{n}$-filled with** E or E is **dense in** F **along** $\mathfrak{n}$ when we have

$$d^{\mathfrak{n}}(F \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

Observe that $\left(\overline{d}(E_M)\right)_M$ is converging nondecreasingly to some $a \geq \overline{d}(E)$ when M goes to infinity. The limit a is in general strictly less than 1. For example if $E := \bigcup_n [\![2^{2n}, 2^{2n+1}]\!]$ one easily computes $\overline{d}(E_M) = \overline{d}(E) = 2/3$ for all M. In this case, the set E is moreover a Følner set.

Also $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ is said to be filled with E when we have with $\mathcal{F} \setminus E_M := (F_n \setminus E_M)_{n\in\mathfrak{n}}$:

$$d^{\mathfrak{n}}(\mathcal{F} \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

2.3 Følner set F filled with a given E with $\overline{d}(E) > 0$

Given a set E with positive upper asymptotic density we build a Følner set F filled with E by using a diagonal argument. More precisely we will build F by filling the holes in E of larger and larger size when going to infinity.

Lemma 1 *For any $E \in \mathcal{P}_{\mathbb{N}}$ with $\overline{d}(E) > 0$ there is a subsequence $\mathfrak{m} \in \mathfrak{N}$ and $F \in \mathcal{P}_{\mathbb{N}}$ with $\partial F \subset E$ such that*

- $d^{\mathfrak{m}}(F) \geq d^{\mathfrak{m}}(E \cap F) = \overline{d}(E)$;
- *F is Følner along $\mathfrak{m}$;*
- *E is dense in F along $\mathfrak{m}$.*

Proof By using a Cantor diagonal argument there is a subsequence $\mathfrak{N} \ni \mathfrak{p} \subset E$ with $d^{\mathfrak{p}}(E) = \overline{d}(E)$ such that the limits $\Delta_k := d^{\mathfrak{p}}(E_k) = \lim_{p\in\mathfrak{p}} d_p(E_k)$ exist for all $k \in \mathbb{N}^*$. The sequence $(\Delta_k)_k$ is nondecreasing and bounded from above by 1. We let $\Delta_\infty = \lim_{k\to+\infty} \Delta_k$. For $k \in \mathbb{N}^*$, we take $\mathfrak{m}_k \in \mathfrak{p}$, such that

$$\forall \mathfrak{m}_k \leq p \in \mathfrak{p}, \ \left|d_p(E_k) - \Delta_k\right| < \frac{1}{2^k}. \tag{2.1}$$

One can ensure that the sequence $(\mathfrak{m}_k)_{k\in\mathbb{N}^*}$ is increasing. We put

$$\mathfrak{m} = (\mathfrak{m}_k)_k \text{ and } F = \bigcup_k [\![\mathfrak{m}_k, \mathfrak{m}_{k+1}[\![\cap E_k.$$

Clearly we have $\partial F \subset \bigcup_k (\partial [\![\mathfrak{m}_k, \mathfrak{m}_{k+1}[\![) \cup (\partial E_k) \subset E$.

Any two integers $l < l'$ lying both in $E \setminus E_k$ satisfy $l' - l > k$, therefore $\overline{d}(E \setminus E_k) \leq 1/k$ for all $k \in \mathbb{N}^*$. Since F contains $[\![\mathfrak{m}_k, +\infty[\![\cap E_k$ for all k, we have then:

$$\begin{aligned}\forall k \in \mathbb{N}^*, \underline{d}^{\mathfrak{m}}(E \cap F) &\geq \underline{d}^{\mathfrak{m}}(E \cap E_k),\\ &\geq d^{\mathfrak{p}}(E) - \overline{d}(E \setminus E_k),\\ &\geq \overline{d}(E) - 1/k.\end{aligned}$$

By letting k go to infinity we get $\underline{d}^{\mathfrak{m}}(E\cap F)\geq \overline{d}(E)\geq \overline{d}^{\mathfrak{m}}(E\cap F)$, thus $d^{\mathfrak{m}}(E\cap F)=\overline{d}(E)$.

Let us prove now the Følner property of the set F. For $\mathfrak{m}_k<K\in\partial F\subset E$ either K or $K-1$ does not belong to F. But $K-1\geq \mathfrak{m}_k$ and $F\supset [\![\mathfrak{m}_k,+\infty[\![\cap E_k$. Consequently

- either K does not belong to E_k, then $[\![K,K+k[\![\subset\mathbb{N}\setminus E\subset\mathbb{N}\setminus\partial F$,
- or $K-1$ does not belong to E_k, then $[\![K-k,K[\![\subset\mathbb{N}\setminus E\subset\mathbb{N}\setminus\partial F$.

Therefore $\overline{d}^{\mathfrak{m}}(\partial F)\leq\frac{2}{k}$. As it holds for all k, the set F is Følner along $\mathfrak{m}$.

Finally we have $[\![0,\mathfrak{m}_k[\![\cap F\subset[\![0,\mathfrak{m}_k[\![\cap E_k$, therefore for any $M<k$ we get:

$$[\![0,\mathfrak{m}_k[\![\cap(F\setminus E_M)\subset[\![0,\mathfrak{m}_k[\![\cap(E_k\setminus E_M),$$

$$d_{\mathfrak{m}_k}(F\setminus E_M)\leq d_{\mathfrak{m}_k}(E_k)-d_{\mathfrak{m}_k}(E_M),\ \text{as } E_M\subset E_k,$$

$$d_{\mathfrak{m}_k}(F\setminus E_M)\leq\Delta_k-\Delta_M+\frac{1}{2^k}+\frac{1}{2^M},\ \text{by (2.1)}.$$

By taking the limits when k goes to infinity we get

$$\overline{d}^{\mathfrak{m}}(F\setminus E_M)\leq\Delta_\infty-\Delta_M+\frac{1}{2^M},$$

therefore

$$\overline{d}^{\mathfrak{m}}(F\setminus E_M)\xrightarrow{M\to+\infty}0.$$

By using once more a Cantor diagonal argument, we may finally assume the limits $d^{\mathfrak{m}}(F)$ and $d^{\mathfrak{m}}(F\setminus E_M)$, $M\in\mathbb{N}^*$, exist. □

2.4 Borel-Cantelli argument

Let $(X,\mathcal{A},\lambda)$ be a measure space with λ being a finite measure. A map $E:X\to\mathcal{P}_{\mathbb{N}}$ is said to be measurable, when for all $n\in\mathbb{N}$ the set $\{x,\ n\in E(x)\}$ belongs to $\mathcal{A}$ (equivalently writing E as an increasing sequence $\mathfrak{n}=(\mathfrak{n}_i)_{i\in\mathbb{N}}$ the integers valued functions $\mathfrak{n}_i$ are measurable). For such measurable maps E and $\mathfrak{n}$, the upper asymptotic density $\overline{d}^{\mathfrak{n}}(E)$ defines a measurable function.

Lemma 2 *Assume E is a measurable sequence of integers such that $\overline{d}(E(x))>\beta>0$ for x in a measurable set A of a positive λ-measure. Then there exist $\mathfrak{n}\in\mathfrak{N}$, measurable subsets $(A_n)_{n\in\mathfrak{n}}$ of X and $\mathcal{F}=(F_n)_{n\in\mathfrak{n}}\in\mathcal{Q}_{\mathfrak{n}}$ with $\partial F_n\subset E(x)$ for all $x\in A_n$, $n\in\mathfrak{n}$ such that*:

- $\lambda(A_n)\geq\frac{e^{-n\delta_n}}{n^2}$ *for all $n\in\mathfrak{n}$ with* $\delta_n\xrightarrow{\mathfrak{n}\ni n\to+\infty}0$;
- *$\mathcal{F}$ is a Følner sequence*;
- *E is dense in $\mathcal{F}$ uniformly on A_n, i.e.*

$$\limsup_{n\in\mathfrak{n}}\ \sup_{x\in A_n} d_n\left(F_n\setminus E_M(x)\right)\xrightarrow{M\to+\infty}0.$$

- $$\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \liminf_{n\in\mathfrak{n}} \inf_{x\in A_n} d_n\left(E(x)\cap F_n\right) \geq \beta.$$

Proof The sequences $\mathfrak{m}=(\mathfrak{m}_k)_{k\in\mathbb{N}}$ and F built in the previous lemma define measurable sequences on A. By taking a smaller subset A we may assume that $\mathfrak{m}_k(x)$ is bounded on A for all k and that the following sequences of functions are converging uniformly in $x\in A$ by using Egorov's theorem:

(1) $d_{\mathfrak{m}_k(x)}(\partial F(x)) \xrightarrow{k} 0$,
(2) $d^{\mathfrak{m}(x)}(F(x)\setminus E_M(x)) \xrightarrow{M\to+\infty} 0$,
(3) $d_{\mathfrak{m}_k(x)}(F(x)\setminus E_M(x)) \xrightarrow{k\to+\infty} d^{\mathfrak{m}}\left(F(x)\setminus E_M(x)\right)$ for any $M\in\mathbb{N}^*$,
(4) $d_{\mathfrak{m}_k(x)}(E(x)\cap F(x)) \xrightarrow{k} d^{\mathfrak{m}(x)}(E(x)\cap F(x)) \geq \beta$.

For a uniformly converging sequence of real bounded functions $f=\lim_n f_n$ we have $\sup f = \lim_n \sup f_n$ and $\inf f = \lim_n \inf f_n$. Applying this fact to the above third item we get:

$$\sup_{x\in A} d_{\mathfrak{m}_k(x)}(F(x)\setminus E_M(x)) \xrightarrow{k\to+\infty} \sup_{x\in A} d^{\mathfrak{m}(x)}\left(F(x)\setminus E_M(x)\right).$$

Together with the second item, we get:

$$\lim_k \sup_{x\in A} d_{\mathfrak{m}_k(x)}(F(x)\setminus E_M(x)) \xrightarrow{M} 0.$$

For the last item we obtain by taking the infimum:

$$\lim_k \inf_{x\in A} d_{\mathfrak{m}_k(x)}(E(x)\cap F(x)) = \inf_{x\in A} d^{\mathfrak{m}(x)}(E(x)\cap F(x)) \geq \beta.$$

By Borel-Cantelli Lemma, the subset $A_n := \{x\in A,\ n\in\mathfrak{m}(x)\}$ has λ-measure larger than $1/n^2$ for infinitely many $n\in\mathbb{N}$. We let $\mathfrak{n}$ be this infinite subset of integers. We observe firstly that:

$$\alpha_n := \sup_{x\in A_n} d_n(\partial F(x)) \xrightarrow{n\in\mathfrak{n}} 0. \tag{2.2}$$

For any n we let k_n be the largest integer k satisfying $\sup_{x\in A}\mathfrak{m}_k(x) < n$. As the functions $(\mathfrak{m}_k)_k$ are bounded on A, we get $k_n \xrightarrow{n\to+\infty} +\infty$. Then, by item (1), for any given $\epsilon>0$ we may find $k'_\epsilon\in\mathbb{N}$ such that $\sup_{x\in A} d_{\mathfrak{m}_k(x)}(\partial F(x)) < \epsilon$ for any $k\geq k'_\epsilon$. Therefore for any n so large that $k_n > k'_\epsilon$ and for any $x\in A_n\subset A$ we have $n=\mathfrak{m}_l(x)$ for some $l=l(x)>k_n$, therefore $d_n(\partial F(x)) < \epsilon$. This shows (2.2). By a similar argument, we can show that:

$$\limsup_{n\in\mathfrak{n}} \sup_{x\in A_n} d_n(F(x)\setminus E_M(x)) \leq \lim_k \sup_{x\in A} d_{\mathfrak{m}_k(x)}(F(x)\setminus E_M(x)) \xrightarrow{M} 0$$

and

$$\liminf_{n\in\mathfrak{n}} \inf_{x\in A_n} d_n\left(E(x)\cap F(x)\right) \geq \lim_k \inf_{x\in A} d_{\mathfrak{m}_k(x)}(E(x)\cap F(x)) \geq \beta.$$

As $\partial F(x) \cap [\![0, n[\![$ determines $F(x) \cap [\![0, n[\![$, there are at most $\sum_{k=1}^{[n\alpha_n]} \binom{n}{k}$ choices for $F(x) \cap [\![0, n[\![$, $x \in A_n$, and thus it may be fixed by dividing the measure of A_n by $\sum_{k=1}^{[n\alpha_n]} \binom{n}{k} = e^{n\delta_n}$ for some $\delta_n \xrightarrow{n} 0$. We let F_n be the common value of $F(x) \cap [\![0, n[\![$ for x in this new set A_n for any $n \in \mathfrak{n}$. Then the conclusions of the lemma hold with $(A_n)_{n \in \mathfrak{n}}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{n}}$. □

3 Empirical measures associated to Følner sequences

Let (X, T) be a topological system, i.e. X is a compact metrizable space and $T : X \circlearrowleft$ is continuous. We denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with the weak-$*$ topology and by $\mathcal{M}(X, T)$ the compact subset of invariant measures. We will write δ_x for the Dirac measure at $x \in X$. We let T_* be the induced (continuous) action on $\mathcal{M}(X)$, where for $\mu \in \mathcal{M}(X)$ the measure $T_*\mu$ is defined as $T_*\mu(A) = \mu(T^{-1}A)$ for any Borel set A. For $\mu \in \mathcal{M}(X)$ and a finite subset F of $\mathbb{N}$, we let μ^F be the empirical measure $\mu^F := \frac{1}{\sharp F} \sum_{k \in F} T_*^k \mu$.

3.1 Invariant measures

The following lemma is standard, but we give a proof for the sake of completeness. We fix $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{N}} \in \mathcal{Q}_{\mathfrak{n}}$.

Lemma 3 *Assume $\mathcal{F}$ is a Følner sequence and $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. Let $(\mu_n)_{n \in \mathfrak{n}}$ be a family in $\mathcal{M}(X)$ indexed by $\mathfrak{n}$. Then any limit of $\left(\mu_n^{F_n}\right)_{n \in \mathfrak{n}}$ is a T-invariant Borel probability measure.*

Proof Let $\mathfrak{n}'$ be a subsequence of $\mathfrak{n}$ such that $\left(\mu_n^{F_n}\right)_{n \in \mathfrak{n}'}$ is converging to some μ'. It is enough to check that $\left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right|$ goes to zero when $\mathfrak{n}' \ni n \to +\infty$ for any $\phi : X \to \mathbb{R}$ continuous.

This follows from the following inequalities:

$$\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n} = \frac{1}{\sharp F_n} \int \left(\sum_{\substack{k \in F_n \\ k-1 \notin F_n}} \phi \circ T^k - \sum_{\substack{k \notin F_n \\ k-1 \in F_n}} \phi \circ T^k \right) d\mu_n,$$

$$\left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right| \leq \sup_{x \in X} |\phi(x)| \frac{\sharp \partial F_n}{\sharp F_n},$$

$$\begin{aligned} \limsup_{n \in \mathfrak{n}} \left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right| &\leq \sup_{x \in X} |\phi(x)| \limsup_{n \in \mathfrak{n}} \frac{\sharp \partial F_n}{\sharp F_n}, \\ &\leq \sup_{x \in X} |\phi(x)| \frac{d^{\mathfrak{n}}(\partial \mathcal{F})}{\underline{d}^{\mathfrak{n}}(\mathcal{F})} = 0. \end{aligned}$$

□

3.2 Positive exponent of empirical measures for superadditive cocycles

We fix a general continuous superadditive cocycle $\Phi = (\phi_n)_{n\in\mathbb{N}}$ with respect to (X, T), i.e. $\phi_0 = 0$, $\phi_n : X \to \mathbb{R}$ is a continuous function for all n and $\phi_{n+m} \geq \phi_n + \phi_m \circ T^n$ for all m, n. By the subadditive ergodic theorem [26], the limit $\phi_*(x) = \lim_n \frac{\phi_n(x)}{n}$ exists for x in a set of full measure with respect to any invariant measure μ. In the proof of the main theorem we will only consider additive cocycles, but we think it could be interesting to consider general superadditive cocycles in other contexts.

Let $E : Y \to \mathcal{P}_\mathbb{N}$ be a measurable sequence of integers defined on a Borel subset Y of X. The set valued map E is said to be$a$**-large** with respect to Φ for some $a \in \mathbb{R}$ when for any $x \in Y$ we have $\phi_{q-p}(T^p x) \geq (q-p)a$ for all integers $q > p$ in $E(x)$. For a finite subset of integers J we let $\phi_J(x)$ be the sum of $\phi_{q-p}(T^p x)$ where $[\![p, q[\![$ runs over all connected components of J. By superadditivity of Φ we always have $\phi_J(x) \geq \phi_1(x)\sharp J$.

Lemma 4 *Let $(A_n)_{n\in\mathfrak{n}}$ and $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ as in Lemma 2 and let $(\mu_n)_{n\in\mathfrak{n}}$ be a family in $\mathcal{M}(X)$ indexed by $\mathfrak{n}$ with $\mu_n(A_n) = 1$ for all $n \in \mathfrak{n}$. Assume E is a-large with $a \in \mathbb{R}$. Then for any weak-$*$ limit μ of $\mu_n^{F_n} = \frac{1}{\sharp F_n}\sum_{k\in F_n} T_*^k\mu_n$, $n \in \mathfrak{n}$, we have*

$$\phi_*(x) \geq a \text{ for } \mu \text{ a.e. } x.$$

Proof Without loss of generality we can take $a = 0$ by considering the cocycle $(\phi_n - na)_n$, which is again superadditive. By taking a subsequence we can also assume that $\left(\mu_n^{F_n}\right)_{n\in\mathfrak{n}}$ is converging to μ. Fix $\alpha < 0$ and $M < N \in \mathbb{N}^*$. For $x \in A_n$, $n \in \mathfrak{n}$, we let

$$F_n^\alpha(x) := \left\{k \in F_n,\ \frac{\phi_N(T^k x)}{N} \leq \alpha\right\}.$$

For $l \in \mathbb{N}$ and $k \in [\![0, N[\![$ with $k + lN \in F_n$, the interval of integers $J_{k,l} = [\![k+lN, k+(l+1)N[\![$ may be written as

$$J_{k,l} = I_{k,l}^1 \coprod I_{k,l}^2 \coprod I_{k,l}^3 \coprod I_{k,l}^4$$

where

- $I_{k,l}^1$ is the union of disjoint $E(x)$-irreducible intervals of length less than M contained in $J_{k,l}$. As $E(x)$ is 0-large, we have $\phi_{I_{k,l}^1}(x) \geq 0$ by superadditivity of Φ,
- $I_{k,l}^2$ consists of a subinterval of an $E(x)$-irreducible interval of length less than M, which does not lie entirely in $J_{k,l}$ but may contain the right extreme of $J_{k,l}$. In particular $\sharp I_{k,l}^2 \leq M$, therefore $\phi_{I_{k,l}^2}(x) \geq -M\sup_{y\in X}|\phi_1(y)|$,
- $I_{k,l}^3 \subset F_n \setminus E_M(x)$,
- $I_{k,l}^4 \subset \mathbb{N} \setminus F_n$.

Note that the subsets $(I_{k,l}^j)_j$ depend on x, but we do not make this dependence explicit to simplify the notations. If $I_{k,l}^4$ is non empty then $J_{k,l}$ contains an element of ∂F_n, therefore for a fixed k we get from $\sharp\left(\bigcup_{l,\ I^4_{k,l}\neq\emptyset} J_{k,l}\right)\leq N\sharp\partial F_n$:

$$\sum_{\substack{l,\ k+lN\in F_n^\alpha\\ I^4_{k,l}\neq\emptyset}} \phi_{J_{k,l}}(x)\geq -N\sharp\partial F_n \sup_{y\in X}|\phi_1(y)|. \tag{3.1}$$

Then, if $I^4_{k,l}=\emptyset$ we have by superadditivity of Φ:

$$\begin{aligned}\phi_{J_{k,l}}(x)&\geq \phi_{I^1_{k,l}}(x)+\sum_{j=2,3}\phi_{I^j_{k,l}}(x),\\ &\geq -(\sharp I^3_{k,l}+M)\sup_{y\in X}|\phi_1(y)|.\end{aligned}$$

As F_n is a subset of $[\![0,n[\![$, the cardinality of $\{l,\ k+lN\in F_n\}$ is less than or equal to $\lceil\frac{n}{N}\rceil$. Therefore by summing this last inequality over l we obtain

$$\sum_{\substack{l,\ k+lN\in F_n^\alpha\\ I^4_{k,l}=\emptyset}} \phi_{J_{k,l}}(x)\geq -\left(\sharp(F_n\setminus E_M(x))+M\lceil\frac{n}{N}\rceil\right)\sup_{y\in X}|\phi_1(y)|. \tag{3.2}$$

By combining (3.1) and (3.2) we have

$$\sum_{l,\ k+lN\in F_n^\alpha} \phi_{J_{k,l}}(x)\geq -\left(N\sharp\partial F_n+\sharp(F_n\setminus E_M(x))+M\lceil\frac{n}{N}\rceil\right)\sup_{y\in X}|\phi_1(y)|.$$

After summing over $k\in[\![0,N[\![$ and dividing by $N\sharp F_n$ we obtain

$$\begin{aligned}\int_{\{\frac{\phi_N}{N}\leq\alpha\}}\frac{\phi_N}{N}\,d\delta_x^{F_n}&=\frac{1}{N\sharp F_n}\sum_{k,l,\ k+lN\in F_n^\alpha}\phi_{J_{k,l}}(x),\\ &\geq -\frac{1}{\sharp F_n}\left(N\sharp\partial F_n+\sharp(F_n\setminus E_M(x))+M\lceil\frac{n}{N}\rceil\right)\sup_{y\in X}|\phi_1(y)|.\end{aligned}$$

We integrate then with respect to μ_n (recall that A_n has full μ_n-measure):

$$\begin{aligned}&\int_{\{\frac{\phi_N}{N}\leq\alpha\}}\frac{\phi_N}{N}\,d\mu_n^{F_n}\\ &\quad\geq -\frac{1}{\sharp F_n}\left(N\sharp\partial F_n+\sup_{x\in A_n}\sharp(F_n\setminus E_M(x))+M\lceil\frac{n}{N}\rceil\right)\sup_{y\in X}|\phi_1(y)|,\\ &\quad\geq -\frac{1}{d_n(F_n)}\left(Nd_n(\partial F_n)+\sup_{x\in A_n}d_n(F_n\setminus E_M(x))+\frac{M}{n}\lceil\frac{n}{N}\rceil\right)\sup_{y\in X}|\phi_1(y)|,\end{aligned}$$

Since the sequence $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ is Følner and $\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \beta$, we get by taking the limsup when $n \in \mathfrak{n}$ goes to infinity:

$$\limsup_{n\in\mathfrak{n}} \int_{\{\frac{\phi_N}{N}\leq\alpha\}} \frac{\phi_N}{N}\, d\mu_n^{F_n} \geq -\frac{1}{\beta} \limsup_{n\in\mathfrak{n}} \left(\sup_{x\in A_n} d_n(F_n \setminus E_M(x)) + \frac{M}{N} \right) \sup_{y\in X} |\phi_1(y)|.$$

Then, the set $\{\frac{\phi_N}{N} \leq \alpha\}$ being closed, we get:

$$\int_{\{\frac{\phi_N}{N}\leq\alpha\}} \frac{\phi_N}{N}\, d\mu \geq \limsup_{n\in\mathfrak{n}} \int_{\{\frac{\phi_N}{N}\leq\alpha\}} \frac{\phi_N}{N}\, d\mu_n^{F_n}.$$

On the other hand we have:

$$\alpha\mu\left(\left\{\frac{\phi_N}{N} \leq \alpha\right\}\right) \geq \int_{\{\frac{\phi_N}{N}\leq\alpha\}} \frac{\phi_N}{N}\, d\mu,$$

therefore we get for all M:

$$\lim_{N\to+\infty} \mu\left(\left\{\frac{\phi_N}{N} \leq \alpha\right\}\right) \leq \frac{1}{-\alpha\beta} \limsup_{n\in\mathfrak{n}} \sup_{x\in A_n} d_n(F_n \setminus E_M(x)) \sup_{y\in X} |\phi_1(y)|.$$

By taking the limit in M we finally have $\lim_{N\to+\infty} \mu\left(\left\{\frac{\phi_N}{N} \leq \alpha\right\}\right) = 0$ by the third item of Lemma 2, therefore $\phi_* \geq \alpha$ almost everywhere. As it holds for any $\alpha < 0$, we conclude that $\phi_* = \lim_n \frac{\phi_n}{n} \geq 0$ almost everywhere. □

3.3 Entropy of empirical measures

Following Misiurewicz's proof of the variational principle, we estimate the entropy of empirical measures from below. For a finite partition P of X and a finite subset F of $\mathbb{N}$, we let P^F be the iterated partition $P^F = \bigvee_{k\in F} f^{-k}P$. When $F = [\![0, n[\![$, $n \in \mathbb{N}$, we just let $P^F = P^n$. We denote by $P(x)$ the element of P containing $x \in X$.

For a Borel probability measure μ on X, the static entropy $H_\mu(P)$ of μ with respect to a (finite measurable) partition P is defined as follows:

$$\begin{aligned} H_\mu(P) &= -\sum_{A\in P} \mu(A) \log \mu(A), \\ &= -\int \log \mu\,(P(x))\, d\mu(x). \end{aligned}$$

When μ is T-invariant, we recall that the measure theoretical entropy of μ with respect to P is then

$$h_\mu(P) = \lim_n \frac{1}{n} H_\mu(P^n)$$

and the entropy $h(\mu)$ of μ is

$$h(\mu) = \sup_P h_\mu(P).$$

We will use the two following standard properties of the static entropy [20]:

- for a fixed partition P, the map $\mu \mapsto H_\mu(P)$ is concave on $\mathcal{M}(X)$,
- for two partitions P and Q, the joined partition $P \vee Q$ satisfies

$$H_\mu(P \vee Q) \leq H_\mu(P) + H_\mu(Q). \tag{3.3}$$

Lemma 5 *Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ be a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. For any measurable finite partition P and $m \in \mathbb{N}^*$, there exist a sequence $(\epsilon_n)_{n\in\mathfrak{n}}$ converging to 0 such that*

$$\forall n \in \mathfrak{n},\ \frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - \epsilon_n.$$

Proof When F_n is an interval of integers, we have [32]:

$$\frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - \frac{3m \log \sharp P}{\sharp F_n}. \tag{3.4}$$

Consider a general set $F_n \in \mathcal{P}_n$. We decompose F_n into connected components $F_n = \coprod_{k=1,\ldots,K} F_n^k$. Observe $K \leq \sharp\partial F_n$. Then we get:

$$\begin{aligned}
\frac{1}{m} H_{\mu_n^{F_n}}(P^m) &\geq \sum_{k=1}^{K} \frac{\sharp F_n^k}{m \sharp F_n} H_{\mu_n^{F_n^k}}(P^m), \text{ by concavity of } \mu \mapsto H_\mu(P^m),\\
&\geq \frac{1}{\sharp F_n} \sum_{k=1}^{K} H_{\mu_n}(P^{F_n^k}) - \frac{3mK \log \sharp P}{\sharp F_n},\ \text{by applying (3.4) to each } F_n^k,\\
&\geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - 3m \log \sharp P \frac{\sharp \partial F_n}{\sharp F_n},\ \text{according to (3.3).}
\end{aligned}$$

This concludes the proof with $\epsilon_n = 3m \frac{\sharp\partial F_n}{\sharp F_n} \log \sharp P$, because $\mathcal{F}$ is a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. □

With the notations of Lemma 2 we let μ_n be the probability measure induced by λ on A_n, i.e. $\mu_n = \frac{\lambda(A_n \cap \cdot)}{\lambda(A_n)}$. In the following we consider an additive cocycle $\Psi = (\psi_n)_n$ associated to a continuous function $\psi : X \to \mathbb{R}$, i.e. $\psi_0 = 0$ and $\psi_n = \sum_{k=0}^{n-1} \psi \circ T^k$ for a positive integer n. Then for any finite subset of integers J we have $\psi_J = \sum_{k\in J} \psi \circ T^k$. The measure λ is said to satisfy the *Følner Gibbs property* with respect to the additive cocycle $\Psi = (\psi_n)_n$ and the Følner sequence $\mathcal{F} = (F_n)_n$ when:

$$\begin{aligned}
&\text{There exists } \epsilon > 0 \text{ such that}\\
&\text{we have for any partition } P \text{ with diameter less than } \epsilon: \qquad \text{(H)}\\
&\exists N\ \forall x \in A_n \text{ with } N < n \in \mathfrak{n},\quad \frac{1}{\lambda\left(P^{F_n}(x) \cap A_n\right)} \geq e^{\psi_{F_n}(x)}.
\end{aligned}$$

Proposition 3 *Under the above hypothesis (H), any weak-$*$ limit μ of $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ satisfies*

$$h(\mu) \geq \psi(\mu).$$

Proof Assume again without loss of generality that $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ is converging to μ. Take a partition P with $\mu(\partial P) = 0$ and with diameter less than ϵ. Then for $n \gg N \gg m$ we obtain:

$$\frac{1}{m}H_\mu(P^m) \geq \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}), \text{ by Lemma 5,}$$
$$\geq \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n} \int \left(-\log\lambda\left(P^{F_n}(x)\cap A_n\right) + \log\lambda(A_n)\right) d\mu_n(x).$$

Note that by the first item of Lemma 2 we have $\log\lambda(A_n) \geq -n\delta_n - 2\log n$, then

$$\frac{\log\lambda(A_n)}{\sharp F_n} \geq \frac{-n\delta_n - 2\log n}{d_n(F_n)n} \xrightarrow{\mathfrak{n}\ni n\to+\infty} 0 \text{ because } \delta_n \to 0 \text{ and } \underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0, \text{ thus}$$

$$\frac{1}{m}H_\mu(P^m) \geq \limsup_{n\in\mathfrak{n}} -\frac{1}{\sharp F_n}\int \log\lambda\left(P^{F_n}(x)\cap A_n\right) d\mu_n(x).$$

It follows from our definitions of ψ_{F_n} and $\delta_x^{F_n}$ that:

$$\forall x,\ \frac{\psi_{F_n}(x)}{\sharp F_n} = \int \psi\, d\delta_x^{F_n}.$$

By Hypothesis (H) we get therefore:

$$\frac{1}{m}H_\mu(P^m) \geq \limsup_{n\in\mathfrak{n}} \int \frac{\psi_{F_n}(x)}{\sharp F_n}\, d\mu_n(x),$$
$$\geq \limsup_{n\in\mathfrak{n}} \int \psi\, d\mu_n^{F_n},$$
$$\geq \psi(\mu).$$

Letting m go to infinity, we conclude that $h(\mu) \geq h_\mu(P) \geq \psi(\mu)$. □

4 Geometric times

Let $(M, \|\cdot\|)$ be a C^r, $r > 1$, smooth compact Riemannian manifold, not necessarily a surface for the moment. We denote by d the distance induced by the Riemannian structure on M. We also consider a distance $\hat{\mathrm{d}}$ on the projective tangent bundle $\mathbb{P}TM$ (compatible with the standard topology on $\mathbb{P}TM$), such that $\hat{\mathrm{d}}(\hat{x}, \hat{y}) \geq \mathrm{d}(\pi\hat{x}, \pi\hat{y})$ for all $\hat{x}, \hat{y} \in \mathbb{P}TM$ with $\pi : \mathbb{P}TM \to M$ being the natural projection. For a C^r map $f : M \to M$ or a C^r curve $\sigma : [0, 1] \to M$ we may define the norm $\|d^s f\|_\infty$ and $\|d^s\sigma\|_\infty$ for $1 \leq s \leq r$ as the supremum norm of the s-derivative of the induced maps through the charts of a given atlas or through the exponential map exp. In the following, to simplify the presentation we lead the computations as M was an Euclidean space or a flat torus. For a C^1 embedded curve $\sigma : I \to M$, I being a compact interval of $\mathbb{R}$, we let $\sigma_* = \sigma(I)$. The length of σ_* for the induced Riemannian metric is denoted by $\ell(\sigma_*)$. For $x \in \sigma_*$ we also let $v_x \in \mathbb{P}TM$ be the line tangent to σ_* at x and we write $\hat{x} = (x, v_x)$.

We denote by F the projective action $F : \mathbb{P}TM \circlearrowleft$ induced by f, i.e. $F(x, v) = \left(f(x), \frac{d_x f(v)}{\|d_x f(v)\|}\right)$, and we consider the additive derivative cocycle $\Phi = (\phi_k)_k$ for F on $\mathbb{P}TM$ given by $\phi(x, v) = \phi_1(x, v) = \log \|d_x f(v)\|$, where we have identified the line v of $T_x M$ with one of its unit generating vectors.

4.1 Bounded curve

Following [12] a C^r smooth curve $\gamma : [-1, 1] \to M$ is said to be **bounded** when

$$\max_{s=2,\dots,r} \|d^s \gamma\|_\infty \leq \frac{1}{6} \|d\gamma\|_\infty.$$

We first recall some basic properties of bounded curves (see Lemma 7 in [12]). A bounded curve has bounded distortion meaning that

$$\forall t, s \in [-1, 1], \ \frac{\|d\gamma(t)\|}{\|d\gamma(s)\|} \leq 3/2. \tag{4.1}$$

Indeed, we have for all $t, s \in [-1, 1]$,

$$\begin{aligned} \|d\gamma(t) - d\gamma(s)\| &\leq 2\|d^2\gamma\|_\infty, \\ &\leq \frac{1}{3}\|d\gamma(t)\|, \\ \text{therefore } \frac{2}{3}\|d\gamma(t)\| &\leq \|d\gamma(s)\|. \end{aligned}$$

The projective component of γ oscillates also slowly. If we identify M with $\mathbb{R}^2$,[1] we have

$$\begin{aligned} \|d\gamma(t)\| \cdot \sin \measuredangle(d\gamma(t), d\gamma(s)) \leq \|d\gamma(t) - d\gamma(s)\| \leq \frac{1}{3}\|d\gamma(t)\|, \\ \measuredangle(d\gamma(t), d\gamma(s)) \leq \pi/6. \end{aligned} \tag{4.2}$$

When moreover $\|d\gamma\|_\infty \leq \epsilon$ we say that γ is **strongly ϵ-bounded**. In particular such a map satisfies $\|\gamma\|_r := \max_{1 \leq s \leq r} \|d^s\gamma\|_\infty \leq \epsilon$, which is the standard C^r upper bound required for the reparametrizations in the usual Yomdin's theory. But this last condition does not allow to control the distortion along the curve in general.

[1] This will be always possible as we will only consider curves with diameter less than the radius of injectivity.

If γ is bounded then so is $\gamma_{a,b} = \gamma(a \cdot +b) : [-1, 1] \to M$ for any a, b with $|a| \leq \frac{2}{3}$ and $|a| + |b| \leq 1$:

$$\begin{aligned}\forall s \geq 2,\ \|d^s \gamma_{a,b}\|_\infty &\leq \frac{1}{6}|a|^s \|d\gamma\|_\infty,\\ &\leq \frac{1}{6}|a|^s \frac{3}{2}\|d\gamma(0)\|,\\ &\leq \frac{1}{6}|a|^{s-1}\|d\gamma(0)\|,\\ &\leq \frac{1}{6}\|d\gamma_{a,b}\|_\infty.\end{aligned}$$

As $\|d\gamma_{a,b}\|_\infty \leq |a|\|d\gamma\|_\infty$, if γ is moreover strongly ϵ-bounded, then $\gamma_{a,b}$ is strongly $|a|\epsilon$-bounded.

Lemma 6 *Let $\gamma : [-1, 1] \to M$ be a C^r bounded curve with $\|d\gamma\|_\infty \geq \epsilon$. Then there is a family of affine maps $\iota_j : [-1, 1] \circlearrowleft$, $j \in L := \underline{L} \cup \overline{L}$ such that:*

- *each $\gamma \circ \iota_j$ is strongly ϵ-bounded and $\|d(\gamma \circ \iota_j)(0)\| \geq \frac{\epsilon}{6}$,*
- *$[-1, 1]$ is the union of $\bigcup_{j\in\underline{L}} \iota_j([-1, 1])$ and $\bigcup_{j\in\overline{L}} \iota_j([-\frac{1}{3}, \frac{1}{3}])$,*
- *$\sharp\underline{L} \leq 2$ and $\sharp\overline{L} \leq 6\left(\frac{\|d\gamma\|_\infty}{\epsilon} + 1\right)$,*
- *for any $x \in \gamma_*$, we have $\sharp\{j \in L,\ (\gamma \circ \iota_j)_* \cap B(x, \epsilon) \neq \emptyset\} \leq 100$.*

Sketch of proof For the first three items it is enough to consider affine reparametrizations of $[-1, 1]$ with rate $\frac{2\epsilon}{3\|d\gamma\|_\infty}$. As the bounded map γ stays in a cone of opening angle $\pi/6$, its intersection with $B(x, \epsilon)$ is a curve of length less than 2ϵ. The last item follows then easily. □

Fix a C^r smooth diffeomorphism $f : M \circlearrowleft$. A curve $\gamma : [-1, 1] \to M$ is said**n-bounded** (resp. strongly **(n, ϵ)-bounded**) when $f^k \circ \gamma$ is bounded (resp. strongly ϵ-bounded) for $k = 0, \ldots, n$. A strongly ϵ-bounded curve γ is contained in the dynamical ball $B_n(x, \epsilon) := \{y \in M,\ \forall k = 0, \ldots, n-1,\ \mathrm{d}(f^k x, f^k y) < \epsilon\}$ with $x = \gamma(0)$.

Fix a C^r curve $\sigma : I \to M$. For $x \in \sigma_*$, a positive integer n is called an **(α, ϵ)-geometric time** of x when there exists an affine map $\theta_n : [-1, 1] \to I$ such that $\gamma_n := \sigma \circ \theta_n$ is strongly (n, ϵ)-bounded, $\gamma_n(0) = x$ and $\|d(f^n \circ \gamma_n)(0)\| \geq \frac{3}{2}\alpha\epsilon$. One can easily check that the curvature of $f^n \circ \sigma$ at $f^n x$ is bounded from above by $\frac{1}{\alpha\epsilon}$, when n is a (α, ϵ)-geometric time of x. It follows from the discussion just before Lemma 6 that, if n is a (α, ϵ)-geometric time of x, then it is also a (α, ϵ')-geometric time for $\epsilon' < \frac{2\epsilon}{3}$.

We let $D_n(x)$ and $H_n(x)$ be the images of $f^n \circ \gamma_n$ and γ_n respectively with γ_n as above of maximal length. Observe that for all $y = \gamma_n(t)$, $t \in [-1, 1]$, we have for any $0 \leq l < n$:

$$e^{\phi_{n-l}(F^l \hat{y})} = \frac{\|d(f^n \circ \gamma_n)(t)\|}{\|d(f^l \circ \gamma_n)(t)\|}.$$

The bounded distortion property of bounded curves (4.1) then implies:

$$\forall y, z \in H_n(x)\,\forall 0 \le l < n, \quad \frac{e^{\phi_{n-l}(F^l\hat{y})}}{e^{\phi_{n-l}(F^l\hat{z})}} \le \frac{9}{4}. \tag{4.3}$$

4.2 Reparametrization lemma

We consider a C^r smooth diffeomorphism $g : M \circlearrowleft$ and a C^r smooth curve $\sigma : I \to M$ with $r > 1$. To simplify the exposition we deal with $r \in \mathbb{N}$. The general case follows from standard arguments, see e.g. [12]. We state a global reparametrization lemma to describe the dynamics on σ_*. We will apply this lemma to $g = f^p$ for large p with f being the C^r smooth system under study. We denote by G the map induced by g on $\mathbb{P}TM$.

We will encode the dynamics of g on σ_* with a tree, in a similar way the symbolic dynamic associated to monotone branches encodes the dynamic of a continuous piecewise monotone interval map. A weighted directed rooted tree $\mathcal{T}$ is a directed rooted tree (by making all its edges point away from the root) whose edges are labelled. Here the weights on the edges are pairs of integers. Moreover the nodes of our tree will be coloured, either in blue or in red. The level of a node is the number of edges along the unique path between it and the root node.

We let $\mathcal{T}_n$ (resp. $\underline{\mathcal{T}_n}$, $\overline{\mathcal{T}_n}$) be the set of nodes (resp. blue, red nodes) of level n. For all $k \le n-1$ and for all $\mathbf{i}^n \in \mathcal{T}_n$, we also let $\mathbf{i}^n_k$ be the node of level k leading to $\mathbf{i}^n$. For $\mathbf{i}^n \in \mathcal{T}_n$, we let $k(\mathbf{i}^n) = (k_1(\mathbf{i}^n), k'_1(\mathbf{i}^n), k_2(\mathbf{i}^n) \cdots, k'_n(\mathbf{i}^n))$ be the $2n$-tuple of integers given by the sequence of labels along the path from the root $\mathbf{i}^0$ to $\mathbf{i}^n$, where $\left(k_l(\mathbf{i}^n), k'_l(\mathbf{i}^n)\right)$ denotes the label of the edge joining $\mathbf{i}^n_{l-1}$ and $\mathbf{i}^n_l$.

For $x \in \sigma_*$, we recall that $\hat{x} = (x, v_x) \in \mathbb{P}TM$ denotes the line tangent to σ at x. Then we let $k(x) \ge k'(\hat{x})$ be the following integers:

$$k(x) := \left[\log \|d_x g\|\right],$$

$$k'(\hat{x}) := \left[\log \|d_x g(v_x)\|\right].$$

Moreover for all $n \in \mathbb{N}^*$ we define:

$$k^n(x) = (k(x), k'(\hat{x}), k(gx), \dots k'(G^{n-2}\hat{x}), k(g^{n-1}x), k'(G^{n-1}\hat{x})).$$

For a $2n$-tuple of integers $\mathbf{k}^n = (k_1, k'_1, \dots k_n, k'_n)$ we consider then

$$\mathcal{H}(\mathbf{k}^n) := \left\{x \in \sigma_*,\ k^n(x) = \mathbf{k}^n\right\}.$$

We restate the Reparametrization Lemma (RL for short) proved in [12] in a *global* version. Let $\exp_x$ be the exponential map at x and let R_{inj} be the radius of injectivity of $(M, \|\cdot\|)$. For $\frac{R_{inj}}{2} > \epsilon > 0$ we let $g^x_{2\epsilon} = g \circ \exp_x(2\epsilon\cdot) : \{w \in T_xM,\ \|w\| \le 1\} \to M$. Then $\|d^s g^x_{2\epsilon}\|_\infty \le (2\epsilon)^s \sup_{\substack{w \in T_xM,\\ \|w\| \le 2\epsilon}} \|d^s(g \circ \exp_x)(w)\|$. In particular there is $\epsilon_0 = \epsilon_0(g) < \frac{R_{inj}}{2}$ depending only on M and $\|d^k g\|_\infty, k = 1, \dots, r$, such that $\|d^s g^x_{2\epsilon}\|_\infty \le 3\epsilon\|d_x g\|$ for all $s = 1, \dots, r$, all $x \in M$ and all $\epsilon \le \epsilon_0(g)$.

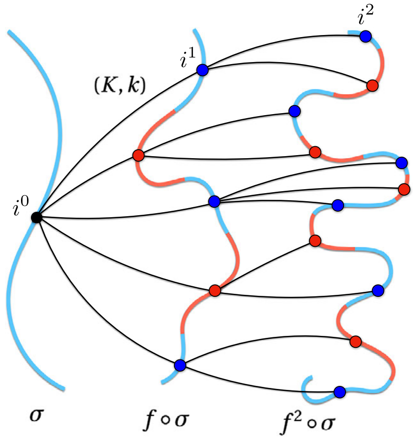

Fig. 1 Picture by J. Paik. Color figure online.

Reparametrization Lemma *Let $\epsilon_0(g)\geq\epsilon>0$ and let $\sigma:[-1,1]\to M$ be a strongly ϵ-bounded curve.*

Then there is $\mathcal{T}$, a bicoloured weighted directed rooted tree, and $(\theta_{\mathbf{i}^n})_{\mathbf{i}^n\in\mathcal{T}_n}$, $n\in\mathbb{N}$, families of affine reparametrizations of $[-1,1]$, such that for some universal constant C_r depending only on r:

(1) $\forall\mathbf{i}^n\in\mathcal{T}_n$, *the curve $\sigma\circ\theta_{\mathbf{i}^n}$ is strongly (n,ϵ)-bounded.*
(2) $\forall\mathbf{i}^n\in\mathcal{T}_n$, *the affine map $\theta_{\mathbf{i}^n}$ may be written as $\theta_{\mathbf{i}^n_{n-1}}\circ\varphi_{\mathbf{i}^n}$ with $\varphi_{\mathbf{i}^n}$ being an affine contraction with rate smaller than $1/100$. Moreover, when $\mathbf{i}^n_{n-1}$ belongs to $\overline{\mathcal{T}_{n-1}}$, we have also $\theta_{\mathbf{i}^n}([-1,1])\subset\theta_{\mathbf{i}^n_{n-1}}([-1/3,1/3])$.*
(3) $\forall\mathbf{i}^n\in\overline{\mathcal{T}_n}$, *we have $\|d\left(g^n\circ\sigma\circ\theta_{\mathbf{i}^n}\right)(0)\|\geq\epsilon/6$.*
(4) $\forall\mathbf{k}^n\in(\mathbb{Z}\times\mathbb{Z})^n$, *the set $\sigma^{-1}\mathcal{H}(\mathbf{k}^n)$ is contained in the union of*

$$\bigcup_{\substack{\mathbf{i}^n\in\overline{\mathcal{T}_n}\\ k(\mathbf{i}^n)=\mathbf{k}^n}}\theta_{\mathbf{i}^n}([-1/3,1/3])\ \textit{and}\ \bigcup_{\substack{\mathbf{i}^n\in\underline{\mathcal{T}_n}\\ k(\mathbf{i}^n)=\mathbf{k}^n}}\theta_{\mathbf{i}^n}([-1,1]).$$

Moreover any term of these unions have a non-empty intersection with $\sigma^{-1}\times\mathcal{H}(\mathbf{k}^n)$.
(5) $\forall\mathbf{i}^{n-1}\in\mathcal{T}_{n-1}$ *and $(k_n,k'_n)\in\mathbb{Z}\times\mathbb{Z}$ we have*

$$\sharp\left\{\mathbf{i}^n\in\overline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1}=\mathbf{i}^{n-1}\ \textit{and}\ (k_n(\mathbf{i}^n),k'_n(\mathbf{i}^n))=(k_n,k'_n)\right\}\leq C_r e^{\max\left(k'_n,\frac{k_n-k'_n}{r-1}\right)},$$

$$\sharp\left\{\mathbf{i}^n\in\underline{\mathcal{T}_n},\ \mathbf{i}^n_{n-1}=\mathbf{i}^{n-1}\ \textit{and}\ (k_n(\mathbf{i}^n),k'_n(\mathbf{i}^n))=(k_n,k'_n)\right\}\leq C_r e^{\frac{k_n-k'_n}{r-1}}.$$

The images of the curve σ together with the tree are represented in Figure 1.

Proof We argue by induction on n. For $n=0$ we let $\overline{\mathcal{T}_0}=\underline{\mathcal{T}_0}=\{\mathbf{i}^0\}$ and we just take $\theta_{\mathbf{i}^0}$ equal to the identity map on $[-1,1]$. Assume the tree and the associated reparametrizations have been built till the level n.

Fix $\mathbf{i}^n \in \mathcal{T}_n$ and let

$$\hat{\theta}_{\mathbf{i}^n} := \begin{cases} \theta_{\mathbf{i}^n}(\frac{1}{3}\cdot) & \text{if } \mathbf{i}^n \in \overline{\mathcal{T}_n}, \\ \theta_{\mathbf{i}^n} & \text{if } \mathbf{i}^n \in \underline{\mathcal{T}_n}. \end{cases}$$

We will define the children $\mathbf{i}^{n+1}$ of $\mathbf{i}^n$, i.e. the nodes $\mathbf{i}^{n+1} \in \mathcal{T}_{n+1}$ with $\mathbf{i}^{n+1}_n = \mathbf{i}^n$. The label on the edge joining $\mathbf{i}^n$ to $\mathbf{i}^{n+1}$ is a pair (k_{n+1}, k'_{n+1}) such that the $2(n+1)$-tuple $\mathbf{k}^{n+1} = (k_1(\mathbf{i}^n), \dots, k'_n(\mathbf{i}^n), k_{n+1}, k'_{n+1})$ satisfies $\mathcal{H}(\mathbf{k}^{n+1}) \cap \left(\sigma \circ \hat{\theta}_{\mathbf{i}^n}\right)_* \neq \emptyset$. We fix such a pair (k_{n+1}, k'_{n+1}) and the associated sequence $\mathbf{k}^{n+1}$. We let $\eta, \psi : [-1,1] \to M$ be the curves defined as:

$$\eta := \sigma \circ \hat{\theta}_{\mathbf{i}^n},$$
$$\psi := g^n \circ \eta.$$

We will make use of the two following well-known multivariate formulas for the derivatives of a product and a composition of C^r functions on $\mathbb{R}^d$. For positive integers m, p, q we let $M_{p,q}(\mathbb{R})$ be the set of real valued $p \times q$ matrices and we denote $A \cdot B \in M_{p,m}(\mathbb{R})$ the product of two matrices $A \in M_{p,q}(\mathbb{R})$ and $B \in M_{q,m}(\mathbb{R})$. We have with the standard multi-index notations:

- *General Leibniz rule:* Let $u : \mathbb{R}^d \to M_{p,q}(\mathbb{R})$ and $v : \mathbb{R}^d \to M_{q,m}(\mathbb{R})$ be C^r maps, then for any $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| := \sum_i \alpha_i \leq r$, we have

$$\partial^\alpha (u \cdot v) = \sum_{\beta : \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta u) \cdot (\partial^{\alpha - \beta} v). \tag{4.4}$$

- *Faà di Bruno's formula (see e.g. [22]):* Let $u : \mathbb{R}^d \to \mathbb{R}$ and $v = (v_1, \dots, v_d) : \mathbb{R}^e \to \mathbb{R}^d$ be C^r maps, then for any $\alpha \in \mathbb{N}^e$ with $|\alpha| \leq r$, we have

$$\partial^\alpha (u \circ v) = \sum_{\beta \in \mathbb{N}^d,\ |\beta| \leq |\alpha|} (\partial^\beta u) \circ v \times P_\beta \left(\left(\partial^\gamma v_i \right)_{\gamma, i} \right), \tag{4.5}$$

where $P_\beta\left((\partial^\gamma v_i)_{\gamma,i}\right)$ is a universal polynomial in $\partial^\gamma v_i$ for $i = 1, \dots, d$ and $\gamma \in \mathbb{N}^e$ with $|\gamma| \leq |\alpha|$.

First step: **Taylor polynomial approximation.** One computes for an affine map $\theta : [-1,1] \circlearrowleft$ with contraction rate b made precise later and with $y = \psi(t) \in g^n \mathcal{H}(\mathbf{k}^{n+1})$, $t \in \theta([-1,1])$:

$$\begin{aligned} \|d^r(g \circ \psi \circ \theta)\|_\infty &\leq b^r \left\| d^r \left(g^y_{2\epsilon} \circ \psi^y_{2\epsilon} \right) \right\|_\infty, \text{ with } \psi^y_{2\epsilon} := (2\epsilon)^{-1} \exp_y^{-1} \circ \psi, \\ &\leq b^r \left\| d^{r-1} \left(d_{\psi^y_{2\epsilon}} g^y_{2\epsilon} \circ d\psi^y_{2\epsilon} \right) \right\|_\infty. \end{aligned}$$

From Leibniz rule (4.4) we get for any $\alpha \in \mathbb{N}^2$ with $|\alpha| = r - 1$:

$$\left\| \partial^\alpha \left(d_{\psi^y_{2\epsilon}} g^y_{2\epsilon} \circ d\psi^y_{2\epsilon} \right) \right\|_\infty \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\partial^\beta (d_{\psi^y_{2\epsilon}} g^y_{2\epsilon})\|_\infty \|\partial^{\alpha-\beta}(d\psi^y_{2\epsilon})\|_\infty,$$

$$\leq \max_{s=0,\dots,r-1}\left\| d^s\left(d_{\psi^y_{2\epsilon}} g^y_{2\epsilon}\right)\right\|_\infty \|\psi^y_{2\epsilon}\|_r \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}.$$

By the multi-binomial formula we have $\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}=2^r$, so that we obtain finally

$$\|d^r(g\circ\psi\circ\theta)\|_\infty \leq b^r 2^r \max_{s=0,\dots,r-1}\left\| d^s\left(d_{\psi^y_{2\epsilon}} g^y_{2\epsilon}\right)\right\|_\infty \|\psi^y_{2\epsilon}\|_r.$$

By assumption on ϵ, we have $\|d^s g^y_{2\epsilon}\|_\infty \leq 3\epsilon\|d_y g\|$ for any $r\geq s\geq 1$. Moreover $\|\psi^y_{2\epsilon}\|_r \leq (2\epsilon)^{-1}\|d\psi\|_\infty \leq 1$ as ψ is strongly ϵ-bounded. Therefore by Faà di Bruno's formula (4.5), we get for some[2] constants $C_r>0$ depending only on r:

$$\max_{s=0,\dots,r-1}\|d^s\left(d_{\psi^y_{2\epsilon}} g^y_{2\epsilon}\right)\|_\infty \leq \epsilon C_r\|d_y g\|,$$

then,

$$\begin{aligned}
\|d^r(g\circ\psi\circ\theta)\|_\infty &\leq \epsilon C_r b^r\|d_y g\|\|\psi^y_{2\epsilon}\|_r,\\
&\leq C_r b^r\|d_y g\|\|d\psi\|_\infty,\\
&\leq (C_r b^{r-1}\|d_y g\|)\|d(\psi\circ\theta)\|_\infty,\\
&\leq (C_r b^{r-1}e^{k_{n+1}})\|d(\psi\circ\theta)\|_\infty,\ \text{because } y \text{ belongs to } g^n\mathcal{H}(\mathbf{k}^{n+1}),\\
&\leq e^{k'_{n+1}-4}\|d(\psi\circ\theta)\|_\infty,\ \text{by taking } b=\left(C_r e^{k_{n+1}-k'_{n+1}+4}\right)^{-\frac{1}{r-1}}.
\end{aligned}$$

The Taylor polynomial P at 0 of degree $r-1$ of $d(g\circ\psi\circ\theta)$ satisfies on $[-1,1]$:

$$\|P-d(g\circ\psi\circ\theta)\|_\infty \leq e^{k'_{n+1}-4}\|d(\psi\circ\theta)\|_\infty.$$

We may cover $[-1,1]$ by at most $b^{-1}+1$ such affine maps θ. This term $b^{-1}+1$ is the source of the factor $e^{\frac{k_n-k'_n}{r-1}}$ in the last item of RL.

Second step: **Bezout theorem.** Let $a_n := e^{k'_{n+1}}\|d(\psi\circ\theta)\|_\infty$. Note that for $s\in[-1,1]$ with $\eta\circ\theta(s)\in\mathcal{H}(\mathbf{k}^{n+1})$ we have

$$e^{k'\left(\widehat{\psi\circ\theta(s)}\right)}\|d(\psi\circ\theta)(s)\| \leq \|d(g\circ\psi\circ\theta)(s)\| \leq e^{1+k'\left(\widehat{\psi\circ\theta(s)}\right)}\|d(\psi\circ\theta)(s)\|,$$

$$e^{k'_{n+1}}\|d(\psi\circ\theta)(s)\| \leq \|d(g\circ\psi\circ\theta)(s)\| \leq e^{1+k'_{n+1}}\|d(\psi\circ\theta)(s)\|.$$

Then, as $\psi\circ\theta$ is bounded, we get by the bounded distortion property (4.1):

$$\frac{2}{3}e^{k'_{n+1}}\|d(\psi\circ\theta)\|_\infty \leq \|d(g\circ\psi\circ\theta)(s)\| \leq e^{1+k'_{n+1}}\|d(\psi\circ\theta)\|_\infty = a_n e.$$

In particular, $\|d(g\circ\psi\circ\theta)(s)\|\in[a_n e^{-1}, a_n e]$, therefore $\|P(s)\|\in[a_n e^{-3}, a_n e^3]$. Moreover if we have now $\|P(s)\|\in[a_n e^{-3}, a_n e^3]$ for some $s\in[-1,1]$ we get also $\|d(g\circ\psi\circ\theta)(s)\|\in[a_n e^{-4}, a_n e^4]$.

[2] Although these constants may differ at each step, they are all denoted by C_r.

By Bezout theorem the semi-algebraic set $\{s \in [-1,1],\ \|P(s)\| \in [e^{-3}a_n, e^3 a_n]\}$ is the disjoint union of closed intervals $(J_i)_{i\in I}$ with $\sharp I$ depending only on r (see e.g. Proposition 4.5 in [44]). Let θ_i be the composition of θ with an affine reparametrization from $[-1,1]$ onto J_i.

Third step: **Landau-Kolmogorov inequality.** By the Landau-Kolmogorov inequality on the interval (see Lemma 6 in [12]), we have for some constants $C_r \in \mathbb{N}$ and for all $1 \le s \le r$ with $|J_i|$ being the length of J_i:

$$\begin{aligned}\|d^s(g\circ\psi\circ\theta_i)\|_\infty &\le C_r\left(\|d^r(g\circ\psi\circ\theta_i)\|_\infty + \|d(g\circ\psi\circ\theta_i)\|_\infty\right),\\ &\le C_r\frac{|J_i|}{2}\left(\|d^r(g\circ\psi\circ\theta)\|_\infty + \sup_{t\in J_i}\|d(g\circ\psi\circ\theta)(t)\|\right),\\ &\le C_r a_n\frac{|J_i|}{2}.\end{aligned}$$

We cut again each J_i into $1000C_r$ intervals $\tilde{J}_i$ of the same length with $(\eta\circ\theta)(\tilde{J}_i)\cap\mathcal{H}(\mathbf{k}^{n+1})\neq\emptyset$. Let $\tilde{\theta}_i$ be the affine reparametrization from $[-1,1]$ onto $\theta(\tilde{J}_i)$. We check that $g\circ\psi\circ\tilde{\theta}_i$ is bounded:

$$\begin{aligned}\forall s=2,\dots,r,\ \|d^s(g\circ\psi\circ\tilde{\theta}_i)\|_\infty &\le (1000C_r)^{-2}\|d^s(g\circ\psi\circ\theta_i)\|_\infty,\\ &\le \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}a_n e^{-4},\\ &\le \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in J_i}\|d(g\circ\psi\circ\theta)(s)\|,\\ &\le \frac{1}{6}(1000C_r)^{-1}\frac{|J_i|}{2}\min_{s\in \tilde{J}_i}\|d(g\circ\psi\circ\theta)(s)\|,\\ &\le \frac{1}{6}\|d(g\circ\psi\circ\tilde{\theta}_i)\|_\infty.\end{aligned}$$

Last step: **Strongly ϵ-bounded curve.** Either $g\circ\psi\circ\tilde{\theta}_i$ is strongly ϵ-bounded and $\hat{\theta}_{\mathbf{i}^n}\circ\tilde{\theta}_i = \theta_{\mathbf{i}^{n+1}}$ for some $\mathbf{i}^{n+1}\in\underline{\mathcal{T}}_{n+1}$. Or we apply Lemma 6 to $g\circ\psi\circ\tilde{\theta}_i$: the new affine parametrizations $\hat{\theta}_{\mathbf{i}^n}\circ\tilde{\theta}_i\circ\iota_j$, $j\in\underline{L}$ (resp. $j\in\overline{L}$) then define $\theta_{\mathbf{i}^{n+1}}$ for a node $\mathbf{i}^{n+1}$ in $\underline{\mathcal{T}}_{n+1}$ (resp. $\overline{\mathcal{T}}_{n+1}$). Note finally that:

$$\begin{aligned}\sharp\overline{L} &\le 6\left(\frac{\|d(g\circ\psi\circ\tilde{\theta}_i)\|_\infty}{\epsilon}+1\right),\\ &\le 100\max(e^{k'_{n+1}}b,1),\ \text{ as } \psi \text{ is strongly } \epsilon\text{-bounded and } \|d\tilde{\theta}_i\|_\infty\le b,\\ &\le C_r\max\left(\frac{e^{k'_{n+1}}}{e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}},1\right),\end{aligned}$$

therefore

$$\sharp\left\{\mathbf{i}^{n+1}\in\overline{\mathcal{T}_{n+1}}\ \Big|\ \overset{\mathbf{i}^{n+1}_n=\mathbf{i}^n\text{ and}}{(k_{n+1}(\mathbf{i}^{n+1}),k'_{n+1}(\mathbf{i}^{n+1}))=(k_{n+1},k'_{n+1})}\right\}$$

$$\leq\sum_{\tilde{\theta}_i}C_r\max\left(\frac{e^{k'_{n+1}}}{e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}},1\right),$$

$$\leq\sharp\{\tilde{\theta}_i\}C_r\max\left(\frac{e^{k'_{n+1}}}{e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}},1\right),$$

$$\leq C_r e^{\max\left(k'_{n+1},\frac{k_{n+1}-k'_{n+1}}{r-1}\right)}.$$

and

$$\sharp\left\{\mathbf{i}^{n+1}\in\underline{\mathcal{T}_{n+1}}\ \Big|\ \overset{\mathbf{i}^{n+1}_n=\mathbf{i}^n\text{ and}}{(k_{n+1}(\mathbf{i}^{n+1}),k'_{n+1}(\mathbf{i}^{n+1}))=(k_{n+1},k'_{n+1})}\right\}\leq\sharp\{\tilde{\theta}_i\},$$

$$\leq C_r e^{\frac{k_{n+1}-k'_{n+1}}{r-1}}.\quad\square$$

As a corollary of the proof of RL we state a *local* reparametrization lemma, i.e. we only reparametrize the intersection of σ_* with some given dynamical ball (with respect to the projective action G induced by g). For $x\in\sigma_*$, $n\in\mathbb{N}$ and $\epsilon>0$ we let

$$B^G_\sigma(x,\epsilon,n):=\left\{y\in\sigma_*,\ \forall k=0,\ldots,n,\ \hat{\mathrm{d}}(G^k\hat{x},G^k\hat{y})<\epsilon\right\}.\tag{4.6}$$

For all $(x,v)\in\mathbb{P}TM$, we also let $w(x,v)=w_g(x,v):=\log\|d_xg\|-\log\|d_xg(v)\|$ and for all $n\in\mathbb{N}$ we let $w^n(x,v)=w^n_g(x,v):=\sum_{k=0}^{n-1}w(G^k(x,v))$ with the convention $w^0=0$. We consider $\epsilon>0$ as in the Reparametrization Lemma. We assume moreover that

$$[\hat{\mathrm{d}}((x,v),(y,w))<\epsilon]\Rightarrow[\big|\log\|d_xg(v)\|-\log\|d_yg(w)\|\big|<1\text{ and }\big|\log\|d_xg\|-\log\|d_yg\|\big|<1].\tag{4.7}$$

Corollary 3 *For any strongly ϵ-bounded curve $\sigma:[-1,1]\to M$ and for any $x\in\sigma_*$, we have for some constant C_r depending only on r:*

$$\forall n\in\mathbb{N},\ \ \sharp\left\{\mathbf{i}^n\in\mathcal{T}_n,\ (\sigma\circ\theta_{\mathbf{i}^n})_*\cap B^G_\sigma(x,\epsilon,n)\neq\emptyset\right\}\leq C_r^n e^{\frac{w^n(\hat{x})}{r-1}}.\tag{4.8}$$

Sketch of proof The Corollary follows from the Reparametrization Lemma together with the two following facts:

- for $y\in B^G_\sigma(x,\epsilon,n)$ we have $k^n(x)\simeq k^n(y)$ up to 1 on each coordinate,

- for any $\mathbf{i}^{n-1}$ there is at most $C_r e^{\frac{k(g^n x)-k'(G^n \hat{x})}{r-1}}$ nodes $\mathbf{i}^n \in \overline{\mathcal{T}}_n$ with $\mathbf{i}^n_{n-1} = \mathbf{i}^{n-1}$ and $\theta_{\mathbf{i}^n}([-1,1]) \cap \sigma^{-1} B^G_\sigma(x, \epsilon, n+1) \neq \emptyset$.

This last point is a consequence of the last item of Lemma 6 applied to the bounded map $g \circ \psi \circ \tilde{\theta}_i$ introduced in the third step of the proof of the reparametrization lemma. □

4.3 The geometric set E

We apply the Reparametrization Lemma to $g = f^p$ for some positive integer p. For $x \in \sigma_*$ we define the set $E_p(x) \subset p\mathbb{N}$ of integers mp such that there is $\mathbf{i}^m \in \overline{\mathcal{T}_m}$ with $k(\mathbf{i}^m) = k^m(x)$ and $x \in \sigma \circ \theta_{\mathbf{i}^m}([-1/3, 1/3])$.

Lemma 7 *There are $\alpha_p > 0$ and $\epsilon_p > 0$ depending only on r, f and p such that any $n \in E_p(x)$ is a (α_p, ϵ_p)-geometric time of x (with respect to f).*

Proof Write $x = \sigma \circ \theta_{\mathbf{i}^m}(b)$ with $b \in [-1/3, 1/3]$ and let $n = mp$. Then for $\epsilon = \epsilon_0(f^p)$, the curve $\gamma_n = \sigma \circ \theta_{\mathbf{i}^m}(b + \frac{2}{3}\cdot)$ is strongly (m, ϵ)-bounded with respect to f^p according to the discussion before Lemma 6. By item (3) of RL we have also $\|d(f^n \circ \gamma_n)(0)\| = \frac{2}{3}\|d(f^n \circ \sigma \circ \theta_{\mathbf{i}^m})(b)\| \geq \frac{4}{9}\|d(f^n \circ \sigma \circ \theta_{\mathbf{i}^m})(0)\| \geq \frac{4}{9}\frac{\epsilon}{6} = \frac{2}{27}\epsilon$. Consequently m is a $(\frac{4}{81}, \epsilon)$-geometric time of x with respect to f^p. For $0 < a \leq 1$ we let $\gamma_n^a = \gamma_n(a\cdot)$. Let $0 \leq n' \leq n$ and let $m' \in \mathbb{N}$ with $m'p \leq n' < (m'+1)p$. By arguing as in the first step of RL, we have for some constant $C > 1$ depending only on f, p and r:

$$\begin{aligned}\forall s = 2, \ldots, r,\ \|d^s(f^{n'} \circ \gamma_n^a)\|_\infty &\leq a^s \|d^s(f^{n'-m'p} \circ (f^{m'p} \circ \gamma_n))\|_\infty, \\ &\leq C a^s \|d(f^{m'p} \circ \gamma_n)\|_\infty,\end{aligned}$$

then as $f^{m'p} \circ \gamma_n$ is bounded:

$$\begin{aligned}\forall s = 2, \ldots, r,\ \|d^s(f^{n'} \circ \gamma_n^a)\|_\infty &\leq \frac{3}{2} C a^s \|d(f^{m'p} \circ \gamma_n)(0)\|, \\ &\leq \frac{3}{2} C a^s \max(1, \|df^{-1}\|)^p \|d(f^{n'} \circ \gamma_n)(0)\|, \\ &\leq \frac{3}{2} C a^{s-1} \max(1, \|df^{-1}\|)^p \|d(f^{n'} \circ \gamma_n^a)\|_\infty.\end{aligned}$$

We fix $a = \frac{1}{9C\max(1, \|df^{-1}\|)^p}$ so that γ_n^a is n-bounded with respect to f. As γ_n is strongly (m, ϵ)-bounded with respect to f^p, the curve γ_n^a is strongly (n, ϵ_p)-bounded with respect to f with $\epsilon_p = a\epsilon\|df\|_\infty^p$. Finally $\|d(f^n \circ \gamma_n^a)(0)\| = a\|d(f^n \circ \gamma_n)(0)\| \geq \frac{2}{27}a\epsilon = \frac{3}{2}\alpha_p\epsilon_p$ with $\alpha_p = \frac{4}{81\|df\|_\infty^p}$, therefore n is a (α_p, ϵ_p)-geometric time of x with respect to f. □

Observe now that if $k < m$ we have with $x = \sigma \circ \theta_{\mathbf{i}^m}(t)$ and $\theta_{\mathbf{i}^m}(t) = \theta_{\mathbf{i}^m_k}(s)$:

$$\begin{aligned} & e^{\phi_{mp-kp}(F^{kp}\hat{x})} \\ & \quad = \frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\|}{\left\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\right\|}, \\ & \quad \geq \frac{2}{3}\frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\|}{\left\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(t)\right\|}, \text{ since } \sigma \circ \theta_{\mathbf{i}^m} \text{ is } m\text{-bounded}, \\ & \quad \geq \frac{2}{3}\frac{\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\|}{\left\|d\left(f^{kp} \circ \sigma \circ \theta_{\mathbf{i}^m_k}\right)(s)\right\|} 100^{m-k}, \text{ by item (2) of RL}, \\ & \quad \geq \frac{2}{3\epsilon}\left\|d\left(f^{mp} \circ \sigma \circ \theta_{\mathbf{i}^m}\right)(0)\right\| 100^{m-k}, \text{ as } \sigma \circ \theta_{\mathbf{i}^n_k} \text{ is strongly } (k,\epsilon)\text{-bounded}, \\ & \quad \geq \frac{1}{9} 100^{m-k} \geq 10^{m-k}, \text{ by item (3) of RL}. \end{aligned}$$

Therefore E_p is τ_p-large with $\tau_p = \frac{\log 10}{p}$.

The next proposition is the key statement, which will ensure positive density of geometric times on a set of a positive Lebesgue measure of a curve with exponential growth length (see the beginning of Sect. 5.3). In the following, Leb_{σ_*} denotes the Lebesgue measure on σ_* induced by its inherited Riemannian structure as a submanifold of M. This is a finite measure with $\mathrm{Leb}_{\sigma_*}(M) = \ell(\sigma_*)$.

Proposition 4 *Let $f : M \circlearrowleft$ be a C^r diffeomorphism and $b > \frac{R(f)}{r}$. For p large enough there exists $\beta_p > 0$ such that*

$$\limsup_n \frac{1}{n} \log Leb_{\sigma_*}\left(\left\{x,\ d_n(E_p(x)) < \beta_p \text{ and } \|d_x f^n(v_x)\| \geq e^{nb}\right\}\right) < 0.$$

Proof Let

$$\mathcal{E}_n := \left\{x \in A,\ d_n(E_p(x)) < \beta_p \text{ and } \|d_x f^n(v_x)\| \geq e^{nb}\right\}.$$

It is enough to consider $n = mp \in p\mathbb{N}$. We apply the Reparametrization Lemma to $g = f^p$ with $\epsilon > 0$ being the scale. Let $\mathcal{T}$ be the corresponding tree and $(\theta_{\mathbf{i}^m})_{\mathbf{i}^m \in \mathcal{T}_m}$, $m \in \mathbb{N}$, its associated affine reparametrizations. Let $A_f := \log \|df\|_\infty + \log \|df^{-1}\|_\infty + 1$. We will show the following three claims later on:

(i) for a node $\mathbf{i}^m \in \mathcal{T}_m$ with $(\sigma \circ \theta_{\mathbf{i}^m})_* \cap \mathcal{E}_n \neq \emptyset$ the length of $(\sigma \circ \theta_{\mathbf{i}^m})_*$ is less than $3\epsilon e^{-nb}$,

(ii) the number of sequences $\mathbf{k}^m$ with $\mathcal{H}(\mathbf{k}^m) \cap \mathcal{E}_n \neq \emptyset$ is bounded from above by $(2pA_f + 1)^{2m}$,

(iii) for a fixed sequence $\mathbf{k}^m$ the number of nodes $\mathbf{i}^m \in \mathcal{T}_m$ with $k(\mathbf{i}^m) = \mathbf{k}^m$ and $(\sigma \circ \theta_{\mathbf{i}^m})_* \cap \mathcal{H}(\mathbf{k}^m) \cap \mathcal{E}_n \neq \emptyset$ is bounded from above by $2^m C_r^m e^m \|df^p\|_\infty^{\frac{m}{r}} \|df\|_\infty^{\beta_p p^2 m}$ for some constant C_r depending only on r.

Assume these three items already shown and let us conclude the proof of Proposition 4:

$$\begin{aligned}\mathrm{Leb}_{\sigma_*}(\mathcal{E}_n) &\le \sum_{\mathbf{k}^m,\ \mathcal{H}(\mathbf{k}^m)\cap\mathcal{E}_n\neq\emptyset} \mathrm{Leb}_{\sigma_*}(\mathcal{H}(\mathbf{k}^m)\cap\mathcal{E}_n),\\ &\le \sum_{\mathbf{k}^m,\ \mathcal{H}(\mathbf{k}^m)\cap\mathcal{E}_n\neq\emptyset}\ \sum_{\substack{\mathbf{i}^m\in\mathcal{T}_m, k(\mathbf{i}^m)=\mathbf{k}^m,\\ (\sigma\circ\theta_{\mathbf{i}^m})_*\cap\mathcal{H}(\mathbf{k}^m)\cap\mathcal{E}_n\neq\emptyset}} \ell((\sigma\circ\theta_{\mathbf{i}^m})_*),\ \text{by (4) of RL},\\ &\le \left[(2pA_f+1)^{2m}\right]\times\left[2^m C_r^m e^m \|df^p\|_\infty^{\frac{m}{r}}\|df\|_\infty^{\beta_p p^2 m}\right]\\ &\quad\times\left[3\epsilon e^{-nb}\right],\ \text{by using (i), (ii) and (iii).}\end{aligned}$$

Finally we obtain:

$$\begin{aligned}\limsup_{n\in p\mathbb{N}}\frac{1}{n}\log\mathrm{Leb}_{\sigma_*}(\mathcal{E}_n) &\le \frac{2}{p}\log(2pA_f+1)\\ &\quad+\frac{\log(2C_r e)}{p}+\frac{\log\|df^p\|_\infty}{pr}+p\beta_pA_f-b.\end{aligned}$$

As p goes to infinity the right member is bounded from above by $\frac{R(f)}{r}-b+A_f\limsup_p(p\beta_p)$. As b is larger than $\frac{R(f)}{r}$ one can choose firstly $p\in\mathbb{N}^*$ large then $\beta_p>0$ small in such a way this right member is negative.

We show now the three items (i), (ii), (iii):

(i) Let $\mathbf{i}^m\in\mathcal{T}_m$ with $(\sigma\circ\theta_{\mathbf{i}^m})_*\cap\mathcal{E}_n\neq\emptyset$. For $x=\sigma\circ\theta_{\mathbf{i}^m}(t)\in\mathcal{E}_n$ we have $\|d_xf^n(v_x)\|\ge e^{nb}$. Then by the distortion property (4.1) of the bounded maps $f^n\circ\sigma\circ\theta_{\mathbf{i}^m}$ and $\sigma\circ\theta_{\mathbf{i}^m}$ we get

$$\begin{aligned}\ell((\sigma\circ\theta_{\mathbf{i}^m})_*) &\le 2\|d(\sigma\circ\theta_{\mathbf{i}^m})\|_\infty,\\ &\le 3\|d(\sigma\circ\theta_{\mathbf{i}^m})(t)\|,\ \text{as } \sigma\circ\theta_{\mathbf{i}^m} \text{ is bounded},\\ &\le 3\frac{\|d(f^n\circ\sigma\circ\theta_{\mathbf{i}^m})(t)\|}{\|d_xf^n(v_x)\|},\\ &\le 3\epsilon e^{-nb}, \text{as } f^n\circ\sigma\circ\theta_{\mathbf{i}^m} \text{ is strongly } \epsilon\text{-bounded.}\end{aligned}$$

(ii) As the functions k and k' associated to g takes values in $[-pA_f, pA_f]$ the number of sequences $\mathbf{k}^m$ with $\mathbf{k}^m=k^m(x)$ for some $x\in\sigma_*$ is bounded from above by $(2pA_f+1)^{2m}$.

(iii) For a fixed sequence $\mathbf{k}^m$ we estimate now the number of nodes $\mathbf{i}^m\in\mathcal{T}_m$ whose path to the root is labelled with $\mathbf{k}^m$ and such that $(\sigma\circ\theta_{\mathbf{i}^m})_*$ has a non empty intersection with $\mathcal{E}_n\cap\mathcal{H}(\mathbf{k}^m)$. When x belongs to $(\sigma\circ\theta_{\mathbf{i}^m})_*$ for some $\mathbf{i}^m\in\mathcal{T}_m$ and satisfies $d_n(E_p(x))<\beta_p$, then we have $\sharp\left\{0<k<m,\ \mathbf{i}_k^m\in\overline{\mathcal{T}_k}\right\}\le n\beta_p$. But, by the estimates on the valence of $\mathcal{T}$ given in the last item of RL, the number of m-paths from the root labelled with $\mathbf{k}^m$ and with at most $n\beta_p$ red nodes are

less than $2^m C_r^m e^{\sum_i \frac{k_i-k_i'}{r-1}} \|df\|_\infty^{\beta_p p^2 m}$ for some constant C_r depending only on r (the factor 2^m is a rough upper bound for the number of ways to distribute the colours blue and red along the path). Then if $x \in \mathcal{H}(\mathbf{k}^m)$ satisfies $\|d_x f^n(v_x)\| \geq e^{nb}$, we have $e^{\sum_i \frac{k_i-k_i'}{r-1}} \leq e^m e^{m\frac{\log\|df^p\|_\infty - bp}{r-1}}$. But, as b is larger than $\frac{\log\|df^p\|_\infty}{pr}$ for large p, we get for such values of p: $\frac{\log\|df^p\|_\infty - bp}{r-1} \leq \frac{1-\frac{1}{r}}{r-1} \cdot \log\|df^p\|_\infty = \frac{\log\|df^p\|_\infty}{r}$. □

From now we fix p and the associated quantities satisfying the conclusion of Proposition 4 and we will simply write $E, \tau, \alpha, \epsilon, \beta$ for $E_p, \tau_p, \alpha_p, \epsilon_p, \beta_p$. The set $E(x)$ is called the **geometric set** of x.

4.4 Cover of F-dynamical balls by bounded curves

As a consequence of Corollary 3, we give now an estimate of the number of strongly (n, ϵ')-bounded curves reparametrizing the intersection of a given strongly ϵ'-bounded curve with a F-dynamical ball of length n and radius ϵ'. This estimate will be used in the proof of the Følner Gibbs property (Proposition 6).

For any $q \in \mathbb{N}^*$ we let $\omega_q : \mathbb{P}TM \to \mathbb{R}$ be the map defined for all $(x, v) \in \mathbb{P}TM$ by

$$\omega_q(x, v) := \frac{1}{q}\left(\log\|d_x f^q\| - \log\|d_x f^q(v)\|\right).$$

Note that $\omega_q = \frac{w_{f^q}}{q}$. We also write $(\omega_q^n)_n$ for the additive associated F-cocycle, i.e.

$$\omega_q^n(x, v) = \sum_{0\leq k<n} \omega_q(F^k(x, v)).$$

Recall that the dynamical ball $B_\sigma^F(x, \epsilon', n)$ has been defined in (4.6).

Lemma 8 *For any $q \in \mathbb{N}^*$, there exist $\epsilon_q' > 0$ and $B_q > 0$ such that for any strongly ϵ_q'-bounded curve $\sigma : [-1, 1] \to M$, for any $x \in \sigma_*$ and for any $n \in \mathbb{N}^*$ there exists a family $(\theta_i)_{i\in I_n}$ of affine maps of $[-1, 1]$ such that:*

- $B_\sigma^F(x, \epsilon_q', n) \subset \bigcup_{i\in I_n}(\sigma\circ\theta_i)_*$,
- *$\sigma\circ\theta_i$ is strongly (n, ϵ_q')-bounded (with respect to f) for any $i \in I_n$,*
- $\sharp I_n \leq B_q C_r^{\frac{n}{q}} e^{\frac{\omega_q^n(\hat{x})}{r-1}}$, *with C_r being a universal constant depending only on r.*

Proof Fix q. Let $\epsilon_q' = \epsilon/2$ with ϵ as in Corollary 3 for $g = f^q$. There is a finite family Θ of affine maps of $[-1, 1]$ with $\bigcup_{\theta\in\Theta}\theta_* = [-1, 1]$ such that for any strongly ϵ_q'-bounded map $\gamma : [-1, 1] \to M$ and for any $\theta \in \Theta$, the map $\gamma\circ\theta$ is strongly (q, ϵ_q')-bounded.

Fix now a strongly ϵ_q'-bounded curve $\sigma : [-1, 1] \to M$ and let $x \in \sigma_*$. We consider only the map $\theta \in \Theta$ such that $B_\sigma^F(x, \epsilon_q', n) \cap (\sigma\circ\theta)_* \neq \emptyset$. For such a map θ we let

$x_\theta \in B^F_\sigma(x, \epsilon'_q, n) \cap (\sigma \circ \theta)_*$. By the choice of ϵ'_q and the inequalities (4.7), we have $|w^{m_k}_{f^q}(F^k \hat{x}_\theta) - w^{m_k}_{f^q}(F^k \hat{x})| \leq 2$.

Take any $0 \leq k < q$ and let $m_k = \left[\frac{n-k}{q}\right]$. By applying Corollary 3 to "$g = f^q$", "$\sigma = f^k \circ \sigma \circ \theta$", "$x = f^k(x_\theta)$" and "$n = [\frac{n-k}{q}]$", we get a family $\Psi_{\theta,k}$ of affine maps of $[-1, 1]$ with $\sharp \Psi_{\theta,k} \leq C_r^{m_k} e^{\frac{w^{m_k}_{f^q}(F^k \hat{x}_\theta)}{r-1}} \leq C_r^{m_k} e^{\frac{w^{m_k}_{f^q}(F^k \hat{x})+2}{r-1}}$ such that

$$\bigcup_{\psi \in \Psi_{\theta,k}} (\sigma \circ \theta \circ \psi)_* \supset f^{-k} B^{F^q}_{f^k \circ \sigma \circ \theta}\left(f^k(x_\theta), \epsilon, \left[\frac{n-k}{q}\right]\right) \supset B^F_{\sigma \circ \theta}(x, \epsilon'_q, n)$$

such that $f^{mq+k} \circ \sigma \circ \theta \circ \psi$ is strongly ϵ-bounded for $\psi \in \Psi_{\theta,k}$ and integers m with $0 \leq mq + k \leq n$. Then $\Theta_k = \{\theta \circ \psi \circ \theta', \ \psi \in \Psi_{\theta,k}$ and $(\theta, \theta') \in \Theta^2\}$ satisfies the two first items of the conclusion. Moreover we have:

$$\sharp \Theta_k \leq C_r^{m_k} \sharp \Theta^2 e^{\frac{w^{m_k}_{f^q}(F^k \hat{x})+2}{r-1}}.$$

But for some constant A_q depending only on q we have

$$\min_{0 \leq k < q} e^{w^{m_k}_{f^q}(F^k \hat{x})} \leq \left(\prod_{0 \leq k < q} e^{w^{m_k}_{f^q}(F^k \hat{x})}\right)^{1/q} \leq A_q e^{\omega^n_q(\hat{x})}.$$

Take $0 \leq k = l < q$ achieving the minimum in $\min_{0 \leq k < q} e^{w^{m_k}_{f^q}(F^k \hat{x})}$. As $\sharp \Theta$ depends only on q, we get for some $B_q > 0$:

$$\sharp \Theta_l \leq B_q C_r^{\frac{n}{q}} e^{\frac{\omega^n_q(\hat{x})}{r-1}}.$$

This concludes the proof of the lemma by taking $(\theta_i)_{i \in I_n} := \Theta_l$. □

5 Existence of SRB measures

5.1 Entropy formula

By Ruelle's inequality [40], for any C^1 system, the entropy of an invariant measure is less than or equal to the integral of the sum of its positive Lyapunov exponents. For C^{1+} systems, the following entropy characterization of SRB measures was obtained by Ledrappier and Young:

Theorem 5 [28] *An invariant measure of a C^{1+} diffeomorphism on a compact manifold is an SRB measure if and only if it has a positive Lyapunov exponent almost everywhere and the entropy is equal to the integral of the sum of its positive Lyapunov exponents.*

As the entropy is harmonic (i.e. preserves the ergodic decomposition), the ergodic components of an SRB measures are also SRB measures.

5.2 Lyapunov exponents

We consider in this subsection a C^1 diffeomorphism $f : M \circlearrowleft$. Let $\|\|$ be a Riemannian structure on M. The (forward upper) Lyapunov exponent of (x, v) for $x \in M$ and $v \in T_x M$ is defined as follows (see [34] for an introduction to Lyapunov exponents):

$$\chi(x, v) := \limsup_{n \to +\infty} \frac{1}{n} \log \|d_x f^n(v)\|.$$

The function $\chi(x, \cdot)$ admits only finitely many values $\chi_1(x) > \cdots > \chi_{p(x)}(x)$ on $T_x M \setminus \{0\}$ and generates a flag $0 \subsetneq V_{p(x)}(x) \subsetneq \cdots \subsetneq V_1 = T_x M$ with $V_i(x) = \{v \in T_x M,\ \chi(x, v) \leq \chi_i(x)\}$. In particular, $\chi(x, v) = \chi_i(x)$ for $v \in V_i(x) \setminus V_{i+1}(x)$. The function p as well the functions χ_i and the vector spaces $V_i(x)$, $i = 1, \ldots, p(x)$ are invariant and depend Borel measurably on x. One can show the maximal Lyapunov exponent χ introduced in the introduction coincides with χ_1 (see the Appendix).

A point x is said to be **regular** when there exists a decomposition

$$T_x M = \bigoplus_{i=1}^{p(x)} H_i(x)$$

such that

$$\forall v \in H_i(x) \setminus \{0\}, \quad \lim_{n \to \pm \infty} \frac{1}{n} \log \|d_x f^n(v)\| = \chi_i(x)$$

with uniform convergence in $\{v \in H_i(x),\ \|v\| = 1\}$ and

$$\lim_{n \to \pm \infty} \frac{1}{|n|} \log \mathrm{Jac}\left(d_x f^n\right) = \sum_i \dim(H_i(x)) \chi_i(x).$$

In particular we have $V_i(x) = \bigoplus_{j=i}^{p(x)} H_j(x)$ for all i. The set $\mathcal{R}$ of regular points is an invariant measurable set of full measure for any invariant measure [29]. The invariant subbundles H_i are called the Oseledec's bundles. We also let $\mathcal{R}^* := \{x \in \mathcal{R},\ \forall i\ \ \chi_i(x) \neq 0\}$. For $x \in \mathcal{R}^*$ we put $E_u(x) = \bigoplus_{i,\ \chi_i(x) > 0} H_i(x)$ and $E_s(x) = \bigoplus_{i,\ \chi_i(x) < 0} H_i(x)$.

In the following we only consider surface diffeomorphisms. Therefore we always have $p(x) \leq 2$ and when $p(x)$ is equal to 1, we let $\chi_2(x) = \chi_1(x)$. When ν is f-invariant we let $\chi_i(\nu) = \int \chi_i \, d\nu$.

5.3 Building SRB measures

Assume f is a C^r, $r > 1$, surface diffeomorphism and $\limsup_n \frac{1}{n} \log \|d_x f^n\| > b > \frac{R(f)}{r}$ on a set of positive Lebesgue measure as in the Main Theorem. Take p (depending only on $b - \frac{R(f)}{r} > 0$) as in Proposition 4, then $\epsilon = \epsilon_0(f^p)$ as in RL. From now $\beta = \beta_p$ is also fixed. By using Fubini's theorem as in [13] there is a C^r smooth embedded curve $\sigma : I \to M$, which can be assumed to be strongly ϵ-bounded, and a subset A of σ_* with $\mathrm{Leb}_{\sigma_*}(A) > 0$, such that we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > b$

for all $x \in A$ (recall v_x is the line tangent to σ_* at x, which is identified with an associated unit vector). We can also assume that the countable set of periodic sources has an empty intersection with σ_*.

It follows from Proposition 4 that

$$\sum_n \mathrm{Leb}_{\sigma_*}\left(\left\{x \in A,\ d_n(E(x)) < \beta \text{ and } \|d_x f^n(v_x)\| \geq e^{nb}\right\}\right) < +\infty.$$

Therefore, by taking a smaller subset A (still with positive Leb_{σ_*}-measure), we may assume that there is $N > 0$ such that for any $n > N$ it holds that

$$\forall x \in A,\ \left[\|d_x f^n(v_x)\| \geq e^{nb}\right] \Rightarrow [d_n(E(x)) \geq \beta].$$

As we have $\limsup_n \frac{1}{n}\log\|d_x f^n(v_x)\| > b$ for all $x \in A$, the set of geometric times has positive upper density in A:

$$\forall x \in A,\ \overline{d}(E(x)) \geq \beta.$$

We prove now the existence of an SRB measure. This is a first step in the proof of the Main Theorem. For any $q \in \mathbb{N}^*$ we let:

$$\psi_q = \phi - \frac{\omega_q}{r-1}.$$

We will apply the results of the first sections to the projective action $F : \mathbb{P}TM \circlearrowleft$ induced by f, where we consider:

- the additive derivative cocycle $\Phi = (\phi_k)_k$ given by $\phi_k(x, v) = \log\|d_x f^k(v)\|$,
- the measure $\lambda = \lambda_\sigma$ on $\mathbb{P}TM$ given by $\mathfrak{s}^* \mathrm{Leb}_{\sigma_*}$ with $\mathfrak{s} : x \in \sigma_* \mapsto (x, v_x)$,
- the geometric set E, which is τ-large with respect to Φ,
- the additive cocycles Ψ_q associated to $\psi_q - \delta_q$ for any $q \in \mathbb{N}^*$.

The topological extension $\pi : (\mathbb{P}TM, F) \to (M, f)$ is principal[3] by a straightforward application of Ledrappier-Walters variational principle [31] and Lemma 3.3 in [41]. In fact this holds in any dimension and more generally for any finite dimensional vector bundle morphism instead of $df : TM \circlearrowleft$.

Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ and $(A_n)_{n\in\mathfrak{n}}$ be the sequences associated to E given by Lemma 2. As in Proposition 3 we denote by μ_n the probability measure induced by λ on A_n, i.e. $\mu_n = \frac{\lambda(A_n\cap\cdot)}{\lambda(A_n)}$. Rigorously E should be defined on the projective tangent bundle, but as π is one-to-one on $\mathbb{P}T\sigma_*$ there is no confusion. In the same way we see the sets A_n, $n \in \mathbb{N}$, as subsets of $A \subset \sigma_*$.

Any weak-$*$ limit μ of $\mu_n^{F_n} := \frac{1}{\sharp F_n}\sum_{k\in F_n} F_*^k \mu_n$ is invariant under F and thus supported by Oseledec's bundles. Let $\nu = \pi\mu$. By Lemma 4, μ is supported by the unstable bundle E_u and $\phi_*(\hat{x}) > \tau$ for μ a.e. $\hat{x} \in \mathbb{P}TM$. Note also that $\phi_*(\hat{x}) \in \{\chi_1(\pi\hat{x}), \chi_2(\pi\hat{x})\}$ for μ-almost every $\hat{x}$. We claim that $\phi_*(\hat{x}) = \chi_1(\pi\hat{x})$. If not ν would have an ergodic component with two positive exponents. It is well known such a measure is necessarily a periodic measure associated to a periodic source S (see e.g.

[3] i.e. $h_f(\pi\mu) = h_F(\mu)$ for all F-invariant measure μ.

Proposition 4.4 in [36]). But there is an open neighborhood U of the orbit of S with $f^{-1}U \subset U$ and $\sigma_* \cap U = \emptyset$. In particular we have $\pi\mu_n^{F_n}(U) = 0$ for all n because $\pi\mu_n^{F_n}\left(\bigcup_{N\in\mathbb{N}} f^N\sigma_*\right) = 1$ and $f^N\sigma_* \cap U = f^N(\sigma_* \cap f^{-N}U) \subset f^N(\sigma_* \cap U) = \emptyset$. By taking the limit in n we would obtain $\nu(U) = \nu(S) = 0$. Therefore $\phi_*(\hat{x}) = \chi_1(\pi\hat{x}) > \tau$ for μ-almost every $\hat{x}$ and $\chi_1(x) > \tau > 0 \geq \chi_2(x)$ for ν-almost every x.

We conclude the construction of an SRB measure by assuming the following proposition, whose proof is given in the next section.

Proposition 6 *There exists an infinite sequence of positive real numbers* $(\delta_q)_q$ *with* $\delta_q \xrightarrow{q\to\infty} 0$ *such that the property* (H) *with respect to* $\mathcal{F}$ *holds with respect to the additive cocycle on* $\mathbb{P}TM$ *associated to the observable* $\psi_q - \delta_q$ *for any* $q \in \mathbb{N}^*$.

Then by Proposition 3 and Proposition 6 we obtain:

$$\begin{aligned} h(\nu) = h(\mu) &\geq \int \psi_q \, d\mu - \delta_q, \\ &\geq \int \phi \, d\mu - \frac{1}{r-1}\int \omega_q \, d\mu - \delta_q, \\ &\geq \chi_1(\nu) - \frac{1}{r-1}\left(\frac{1}{q}\int \log \|d_x f^q\| \, d\nu(x) - \chi_1(\nu)\right) - \delta_q. \end{aligned}$$

By a standard application of the subadditive ergodic theorem, we have

$$\frac{1}{q}\int \log \|d_x f^q\| \, d\nu(x) \xrightarrow{q\to+\infty} \int \chi_1(x) \, d\nu(x) = \chi_1(\nu).$$

Therefore $h(\nu) \geq \chi_1(\nu)$, since $\delta_q \xrightarrow{q\to\infty} 0$. Then Ruelle's inequality implies $h(\nu) = \chi_1(\nu)$. According to Ledrappier-Young characterization (Theorem 5), the measure ν is an SRB measure of f. Note also that any ergodic component ξ of ν is also an SRB measure, therefore $h(\xi) = \chi_1(\xi) > \tau$. But by Ruelle inequality applied to f^{-1}, we get also $h(\xi) \leq -\chi_2(\xi)$. In particular we have $\chi_1(x) > \tau > 0 > -\tau > \chi_2(x)$ for ν-almost every x.

5.4 Proof of the Følner Gibbs property (H)

In this subsection we prove Proposition 6. We will show that for any $\delta > 0$ there is q arbitrarily large and $\epsilon'_q > 0$ such that we have for any partition P of $\mathbb{P}TM$ with diameter less than ϵ'_q:

$$\exists n_* \, \forall x \in A_n \subset \sigma_* \text{ with } n_* < n \in \mathfrak{n}, \quad \frac{1}{\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A_n\right)} \geq e^{-\delta\sharp F_n} e^{\psi_q^{F_n}(\hat{x})}, \tag{5.1}$$

where we denote $\psi_q^{F_n}(\hat{x}) := \sum_{k\in F_n} \psi_q(F^k\hat{x})$ to simplify the notations.

For $G \subset \mathbb{N}$ we let A^G be the set of points $x \in A$ with $G \subset E(x)$. When $G = \{k\}$ or $\{k, l\}$ with $k, l \in \mathbb{N}$, we just let $A^G = A^k$ or $A^{k,l}$. We recall that $\partial F_n \subset E(x)$ for all $x \in A_n$, in others terms $A_n \subset A^{\partial F_n}$. We will show (5.1) for $A^{\partial F_n}$ in place of A_n.

Fix the error term $\delta > 0$. Let q be so large that $C_r^{1/q} < e^{\delta/3}$ and let $\epsilon'_q > 0$ as in Lemma 8 (with C_r being the universal constant in the same lemma). Without loss of generality we may assume $\epsilon'_q < \frac{\epsilon}{81}$. Recall that ϵ corresponds to the fixed scale in the definition of the geometric set E. We can also ensure that

$$\forall \hat{x}, \hat{y} \in \mathbb{P}TM \text{ with } \hat{\mathrm{d}}(\hat{x}, \hat{y}) < \epsilon'_q, \quad |\phi(\hat{x}) - \phi(\hat{y})| < \delta/3. \tag{5.2}$$

Let us remember some notations and definitions introduced just before Sect. 4.2. For $x \in \sigma_*$ the curve $D_n(x)$ denotes the image of $f^n \circ \gamma_n$ where $\gamma_n = \sigma \circ \theta_n$ is the subcurve of maximal length satisfying the following three items:

- $\theta_n : [-1, 1] \circlearrowleft$ is affine,
- γ_n is strongly (n, ϵ)-bounded,
- $\gamma_n(0) = x$.

The integer n is a (α, ϵ)-geometric time of x, when $\|d(f^n \circ \gamma_n)(0)\| \geq \frac{3}{2}\alpha\epsilon$. We define the semi-length of $D_n(x)$ as the minimum of the lengths of $f^n \circ \gamma_n([0, 1])$ and $f^n \circ \gamma_n([-1, 0])$. The semi-length of $D_n(x)$ is larger than $\alpha\epsilon$ at a (α, ϵ)-geometric time n.

In the next three lemmas we consider a strongly ϵ-bounded curve σ.

Lemma 9 *For any subset N of M, any $k \in \mathbb{N}$ and any ball B_k of radius less than ϵ'_q, there exists a finite family $(y_j)_{j \in J}$ of $A^k \cap f^{-k} B_k \cap N$ such that*:

- $B_k \cap f^k(A^k \cap N) \subset \bigcup_{j \in J} D_k(y_j)$,
- $D_k(y_j)$, $j \in J$, *are pairwise disjoint.*

Proof For $y, y' \in A^k \cap f^{-k} B_k \cap N$ we let $y \sim y'$ when $D_k(y) \cap D_k(y') \neq \emptyset$. We claim that $[y \sim y'] \Rightarrow [D_k(y) \cap B_k = D_k(y') \cap B_k]$. In particular $\sim$ defines an equivalence relation on $A^k \cap f^{-k} B_k \cap N$ (with finite quotient set). Then if $(y_i)_i$ is a family of representatives, the curves $D_k(y_j)$, $j \in J$, are pairwise disjoint and $B_k \cap f^k(A^k \cap N) \subset \bigcup_{j \in J} D_k(y_j)$. It remains to show our claim. For $y, y' \in A^k \cap f^{-k} B_k \cap N$ with $D_k(y) \cap D_k(y') \neq \emptyset$, the curves $D_k(y)$, $D_k(y')$ and $D_k(y) \cup D_k(y')$ lie in a cone with opening angle $\pi/6$ by (4.2) and their length are larger than $\frac{4}{81}\epsilon > 4\epsilon'_q$. By elementary Euclidean geometric arguments, the intersection of one of these curves with $2B_k$ is a curve crossing $2B_k$, i.e. its two endpoints lies in the boundary of $2B_k$ (see Figure 2). Two such subcurves of $B_k \cap (f^k \circ \sigma)_*$ if not disjoint are necessarily equal. Therefore $\big(D_k(y) \cup D_k(y')\big) \cap 2B_k = D_k(y) \cap 2B_k = D_k(y') \cap 2B_k$. □

As the distortion is bounded on $D_k(y_j)$, $j \in J$, by (4.3), we get:

$$\sum_{j \in J} \frac{4}{9} e^{-\phi_k(\hat{y}_j)} \ell\left(D_k(y_j)\right) \leq \sum_{j \in J} \ell\left(f^{-k} D_k(y_j)\right).$$

The curves $(D_k(y_i))_i$, being pairwise disjoint, we have:

$$\sum_{j \in J} \ell\left(f^{-k} D_k(y_j)\right) \leq \ell(\sigma_*) \leq 2\epsilon,$$

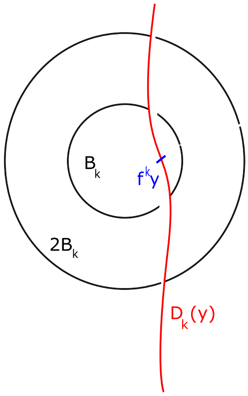

Fig. 2 The curve $D_k(y)$ crossing the ball $2B_k$

therefore

$$\sum_{j\in J}\frac{4}{9}e^{-\phi_k(\hat{y}_j)}\ell\left(D_k(y_j)\right)\leq 2\epsilon.$$

The semi-length of $D_k(y_j)$ is larger than $\alpha\epsilon$ because y_j belongs to A^k, so that we obtain finally:

$$\sum_{j\in J}e^{-\phi_k(\hat{y}_j)}\leq\frac{9}{4\alpha}. \tag{5.3}$$

Below we consider the dynamical ball $B_\sigma^F(x,\epsilon_q',k)$ defined in (4.6).

Lemma 10 *For any subset N of M and any dynamical ball $B_{[\![0,k]\!]}:=B_\sigma^F(x,\epsilon_q',k)$, there exists a finite family $(z_i)_{i\in I}$ of $A^k\cap B_{[\![0,k]\!]}\cap N$ such that*

- $f^k\left(A^k\cap B_{[\![0,k]\!]}\cap N\right)\subset\bigcup_{i\in I}D_k(z_i)$,
- $D_k(z_i)$, $i\in I$, *are pairwise disjoint,*
- $\sharp I\leq B_q e^{\delta k/3}e^{\frac{\omega_q^k(\hat{x})}{r-1}}$ *for some constant B_q depending only on q.*

Proof As in the previous lemma we consider the subcurves $D_k(z)$ for $z\in A^k\cap B_{[\![0,k]\!]}\cap N$. By Lemma 8 we can reparametrize $B_{[\![0,k]\!]}$ by a family of strongly (k,ϵ_q')-bounded curves with cardinality less than $B_qC_r^{\frac{k}{q}}e^{\frac{\omega_q^k(\hat{x})}{r-1}}$. Each of these curves is contained in some $f^{-k}D_k(z)$ with $z\in A^k\cap B_{[\![0,k]\!]}$. Arguing as in the proof of Lemma 9, such curves can be chosen pairwise disjoint. □

Lemma 11 *For any dynamical ball $B_{[\![k,l]\!]}:=f^{-k}B^F_{f^k\circ\sigma}(f^kx,\epsilon_q',l-k)$, there exists a finite family $(y_i)_{i\in I}$ of $A^{k,l}\cap B_{[\![k,l]\!]}$ and a partition $I=\coprod_{j\in J}I_j$ of I with $j\in I_j$ for all $j\in J\subset I$ such that*

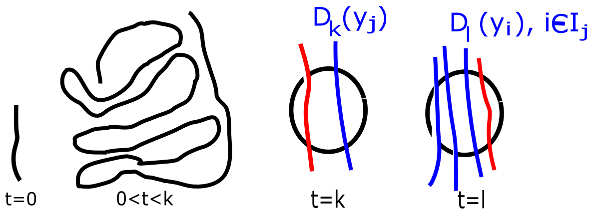

Fig. 3 For $0 \leq t < k$ the image of $f^t \circ \sigma$ in black may be large and the disks $D_t(y_i)$ are scattered through the surface. For $t = k$, the sets $D_k(y_j)$ for $j \in J$ are covering $(f^k \circ \sigma)_* \cap B_k$. For $t = l$, we drew in blue the sets $D_l(y_i) \subset f^{l-k} D_k(y_j)$ for $i \in I_j$ (Color figure online)

- $f^l(A^{k,l} \cap B_{[\![k,l]\!]}) \subset \bigcup_{i \in I} D_l(y_i)$,
- $D_l(y_i)$, $i \in I$, *are pairwise disjoint,*
- $\forall j \in J\ \forall i, i' \in I_j$, $D_k(y_i) \cap B(f^k x, \epsilon'_q) = D_k(y_{i'}) \cap B(f^k x, \epsilon'_q)$,
- $\forall j \in J$, $\sharp I_j \leq B_q e^{\delta(l-k)/3} e^{\frac{\omega_q^{l-k}(F^k \hat{x})}{r-1}}$ *for some constant* B_q *depending only on* q.

Proof We first apply Lemma 9 to σ and $N = A^{k,l} \cap B_{[\![k,l]\!]}$ to get the collection of strongly ϵ-bounded curves $\left(D_k(y_j)\right)_{j \in J}$. For $j \in J$ we let σ_j^k be the strongly ϵ-bounded curve σ given by $D_k(y_j)$. Then we apply Lemma 10 to each σ_j^k for $j \in J$ and $N = f^k(B_{[\![k,l]\!]} \cap A_k)$ to get a family $(z_i)_{i \in I_j}$ of $D_k(y_j) \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k)$ satisfying:

- $f^{l-k}\left(D_k(y_j) \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k)\right) \subset \bigcup_{j \in J} D_k(z_i)$,
- $D_{l-k}(z_i)$, $i \in I_j$, are pairwise disjoint,
- $\sharp I_j \leq B_q e^{\delta(l-k)/3} e^{\frac{\omega_q^{l-k}(F^k \hat{x})}{r-1}}$.

For all $j \in J$ we can take $j \in I_j$ and $z_j = f^k(y_j)$. We conclude the proof by letting $y_i = f^{-k} z_i \in A^{k,l} \cap B_{[\![k,l]\!]}$ for all $i \in I := \coprod_{j \in J} I_j$. See figure 3. □

We prove now (H). Recall that $\lambda = \lambda_\sigma$ is the push-forward on $\mathbb{P}TM$ of the Lebesgue measure on σ_*. As $\sharp \partial F_n = o(n)$ and $\underline{d}_n(\mathcal{F}) > 0$ by Lemma 2, it is enough to show there is a constant C such that for any strongly ϵ-bounded curve σ we have for $x \in A^{\partial F_n}$:

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \leq C^{\sharp \partial F_n} e^{2\delta \sharp F_n/3} e^{-\psi_q^{F_n}(\hat{x})}. \tag{5.4}$$

To prove (5.4) we argue by induction on the number of connected components of F_n. Let $[\![k,l[\![$, $0 \leq k \leq l$, be the first connected component of F_n and write $G_{n-l} = \mathbb{N}^* \cap (F_n - l)$. As P has diameter less than ϵ'_q, the set $\mathbb{P}T\sigma_* \cap P^{F_n}(\hat{x})$ is contained in the intersection of $\pi^{-1} B_{[\![k,l]\!]} = \pi^{-1}\left(f^{-k} B^F_{f^k \circ \sigma}(f^k x, \epsilon'_q, l-k)\right)$ and $F^{-l} P^{G_{n-l}}(F^l \hat{x})$. Then with the notations of Lemma 11 we get:

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right)$$

$$\leq \lambda_\sigma \left(\coprod_{i\in I} F^{-l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i) \right) \right),$$

$$\leq \lambda_\sigma \left(\coprod_{j\in J} F^{-k} \left(\coprod_{i\in I_j} F^{-(l-k)} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i) \right) \right) \right)$$

We recall that σ_j^k denotes the strongly ϵ-bounded curve σ given by $D_k(y_j)$ for $j \in J$. By the bounded distortion property (4.3) we get:

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n} \right)$$
$$\leq 3 \sum_{j\in J} e^{-\phi_k(\hat{y}_j)} \lambda_{\sigma_j^k} \left(\coprod_{i\in I_j} F^{-(l-k)} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \cap D_l(y_i) \right) \right).$$

By using again the bounded distortion property (now between the times k and l) we get with σ_i^l being the curve associated to $D_l(y_i)$:

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n} \right)$$
$$\leq 9 \sum_{j\in J} e^{-\phi_k(\hat{y}_j)} \sum_{i\in I_j} e^{-\phi_{l-k}(F^k \hat{y}_i)} \lambda_{\sigma_i^l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \right).$$

We may assume that any $\hat{y}_i$, $i \in I$, lies in $P^{F_n}(\hat{x})$. In particular we have $|\phi_{l-k}(F^k \hat{y}_i) - \phi_{l-k}(F^k \hat{x})| < (l-k)\delta/3$ by (5.2). Then

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n} \right) \leq 9 \left(\sum_{j\in J} e^{-\phi_k(\hat{y}_j)} \right) e^{\delta(l-k)/3} e^{-\phi_{l-k}(F^k \hat{x})} \sup_j \sharp I_j$$
$$\times \sup_{i\in I} \lambda_{\sigma_i^l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \right).$$

By (5.3) and the last item of Lemma 11 we obtain:

$$\lambda_\sigma \left(P^{F_n}(\hat{x}) \cap \pi^{-1} \partial A^{F_n} \right)$$
$$\leq \frac{81}{4\alpha} B_q e^{2\delta(l-k)/3} e^{-\phi_{l-k}(F^k \hat{x}) + \frac{\omega_q^{l-k}(F^k \hat{x})}{r-1}} \sup_{i\in I} \lambda_{\sigma_i^l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \right),$$
$$\leq \frac{81}{4\alpha} B_q e^{2\delta(l-k)/3} e^{-\psi_q^{[\![k,l]\!]}(\hat{x})} \sup_{i\in I} \lambda_{\sigma_i^l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \right).$$

By induction hypothesis (5.4) applied to G_{n-l} for each σ_i^l, we have for all $i \in I$:

$$\lambda_{\sigma_i^l} \left(\pi^{-1} A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l \hat{x}) \right) \leq C^{\sharp \partial G_{n-l}} e^{2\delta \sharp G_{n-l}/3} e^{-\psi_q^{G_{n-l}}(F^l \hat{x})}.$$

Note that $\sharp\partial F_n = \sharp\partial G_{n-l} + 2$ and $\sharp F_n = (l-k+1) + \sharp G_{n-l}$. We conclude by taking $C = \sqrt{\frac{81 B_q}{4\alpha}}$ that:

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1} A^{\partial F_n}\right) \leq \frac{81}{4\alpha} B_q e^{2\delta\sharp F_n/3} C^{\sharp\partial G_{n-l}} e^{-\psi_q^{F_n}(\hat{x})},$$
$$\leq C^{\sharp\partial F_n} e^{2\delta\sharp F_n/3} e^{-\psi_q^{F_n}(\hat{x})}.$$

This completes the proof of (5.1).

6 End of the proof of the Main Theorem

6.1 The covering property of the basins

For $x \in M$ the stable/unstable manifold $W^{s/u}(x)$ at x are defined as follows:

$$W^s(x) := \{y \in M,\ \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0\},$$

$$W^u(x) := \{y \in M,\ \limsup_{n\to+\infty} \frac{1}{n} \log d(f^{-n} x, f^{-n} y) < 0\}.$$

For a subset Γ of M we let $W^s(\Gamma) = \bigcup_{x\in\Gamma} W^s(x)$. Let $\mathcal{R}^*$ be the set of regular points and $E_{s/u}$ be the subbundles of TM as defined in Sect. 5.2. According to Pesin's theory, there are a nondecreasing sequence of compact, a priori non-invariant, sets $(K_n)_n$ (called the Pesin blocks) with $\mathcal{R}^* = \bigcup_n K_n$ and two families of embedded C^∞ discs $(W^s_{loc}(x))_{x\in\mathcal{R}^*}$ and $(W^u_{loc}(x))_{x\in\mathcal{R}^*}$ (called the local stable and unstable manifolds) such that:

- $W^{s/u}_{loc}(x)$ are tangent to $E_{s/u}$ at x,
- the splitting $E_u(x) \oplus E_s(x)$ and the discs $W^{s/u}_{loc}(x)$ are continuous on $x \in K_n$ for each n.

For $\gamma > 0$ and $x \in \mathcal{R}^*$ we let $W^{s/u}_\gamma(x)$ be the connected component of $B(x,\gamma) \cap W^{s/u}_{loc}(x)$ containing x.

Proposition 7 *The set* $\left\{\chi > \frac{R(f)}{r}\right\}$ *is covered by the basins of ergodic SRB measures* μ_i, $i \in I$, *up to a set of zero Lebesgue measure.*

In fact we prove a stronger statement by showing that $\left\{\chi > \frac{R(f)}{r}\right\}$ is contained Lebesgue a.e. in $W^s(\Gamma)$ where Γ is any f-invariant subset of $\bigcup_{i\in I} \mathcal{B}(\mu_i)_{i\in I}$ with $\mu_i(\Gamma) = 1$ for all $i \in I$.

So far we only have used the characterization of SRB measure in terms of entropy (Theorem 5). In the proof of Proposition 7 we will use the absolute continuity property of SRB measures. Let μ be a Borel measure on M. We recall a measurable partition ξ in the sense of Rokhlin [37] is said to be μ-subordinate to W^u

when $\xi(x) \subset W^u(x)$ and $\xi(x)$ contains an open neighborhood of x in the topology of $W^u(x)$ for μ-almost every x. The measure μ is said to have **absolutely continuous conditional measures on unstable manifolds** if for every measurable partition ξ μ-subordinate to W^u, the conditional measures μ_x^ξ of μ with respect to ξ satisfy $\mu_x^\xi \ll Leb_{W^u(x)}$ for μ-almost every x.

Proof We argue by contradiction. Take Γ as above. Assume there is a Borel set B with positive Lebesgue measure contained in the complement of $W^s(\Gamma)$ such that we have $\chi(x) > b > \frac{R(f)}{r}$ for all $x \in B$. Then we follow the approach of Sect. 5.3. We consider a C^r smooth disc σ with $\chi(x, v_x) > b$ for $x \in B' \subset B$, $\mathrm{Leb}_{\sigma_*}(B') > 0$. One can then define the geometric set E on a subset B'' of B with $\mathrm{Leb}_{\sigma_*}(B'') > 0$. We also let τ, β, α and ϵ be the parameters associated to E. Recall that:

- E is τ-large with respect to the derivative cocycle Φ,
- $\overline{d}(E(x)) \geq \beta > 0$ for $x \in B''$,
- $D_k(y) = f^k(H_k(y))$ has semi-length larger than $\alpha\epsilon$ when $k \in E(y)$, $y \in B''$.

Let B''' be the subset of B'' given by density points of B'' with respect to Leb_{σ_*}. In particular, we have

$$\forall x \in B''', \quad \frac{\mathrm{Leb}_{\sigma_*}\left(H_k(x) \cap B''\right)}{\mathrm{Leb}_{\sigma_*}(H_k(x))} \xrightarrow{k \to +\infty} 1.$$

We choose a subset A of B''' with $\mathrm{Leb}_{\sigma_*}(A) > 0$ such that the above convergence is uniform in $x \in A$. Then from this set A and the geometric set E on A we may build $\mathfrak{n}$, $(F_n)_{n \in \mathfrak{n}}$ and $(\mu_n^{F_n})_{n \in \mathfrak{n}}$ as in Sects. 2 and 3. As proved in Sect. 5.3 any limit measure μ of $\mu_n^{F_n}$ is supported on the unstable bundle and projects to an SRB measure ν with $\chi_1(x) \geq \tau > 0 > -\tau \geq \chi_2(x)$ for ν a.e. x. The measure ν is a barycenter of ergodic SRB measures with such exponents, in particular $\nu(\Gamma) = 1$. Take $P = K_N$ a Pesin block with $\nu(P) > 1 - \frac{\beta}{2}$. We let θ and l be respectively the minimal angle between E_u and E_s and the minimal length of the local stable and unstable manifolds on P.

Let ξ be a measurable partition ν-subordinate to W^u with diameter less then $\gamma > 0$. We have $\nu(P) = \nu(\Gamma \cap P) = \int \nu_x^\xi(\Gamma \cap P)\, d\nu(x)$ and $\nu_x^\xi \ll Leb_{W_\gamma^u(x)}$ for ν a.e. x. Then

$$\begin{aligned} \nu\left(x,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Gamma \cap P) = 0\right) &\leq \nu\left(x,\ \nu_x^\xi(\Gamma \cap P) = 0\right), \\ &\leq 1 - \int \nu_x^\xi(\Gamma \cap P)\, d\nu(x), \\ &\leq \quad 1 - \nu(P) < \frac{\beta}{2}. \end{aligned}$$

Therefore we get for some $c > 0$:

$$\nu\left(x,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Gamma \cap P) > c\right) > 1 - \frac{\beta}{2}.$$

We let $G = \{x \in \Gamma \cap P,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Gamma \cap P) > c\}$. Observe that:

$$\nu(G) \geq \nu\left(x,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Gamma \cap P) > c\right) - \nu(M \setminus P) > 1 - \beta.$$

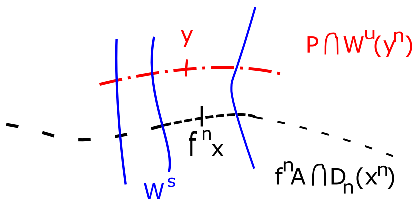

Fig. 4 Holonomy of the local stable foliation between the transversals $D_n(x^n)$ and $W^u_\gamma(y^n)$ (Color figure online)

For $x \in \sigma_*$ and $y \in P$ we use the following notations:

$$\hat{x}_\sigma = (x, v_x) \in \mathbb{P}T\sigma_* \quad \hat{y}_u = (y, v^u_y) \in \mathbb{P}TM,$$

where v^u_y is the element of $\mathbb{P}TM$ representing the line $E_u(y)$. Let $\hat{G}^\gamma_u$ be the open $\gamma/8$-neighborhood of $\hat{G}_u := \{\hat{y}_u,\ y \in G\}$ in $\mathbb{P}TM$. Recall $E(x)$ denotes the set of geometric times of x. We let for $n \in \mathfrak{n}$:

$$\zeta_n := \int \frac{1}{\sharp F_n} \sum_{k \in E(x) \cap F_n} \delta_{F^k \hat{x}_\sigma}\, d\mu_n(\hat{x}_\sigma).$$

Observe that $\zeta_n(\mathbb{P}TM) \geq \inf_{x \in A_n} d_n(E(x) \cap F_n)$. By the last item in Lemma 2, we have $\liminf_{n \in \mathfrak{n}} \inf_{x \in A_n} d_n(E(x) \cap F_n) \geq \beta$. Therefore there is a weak-$*$ limit $\zeta = \lim_k \zeta_{p_k}$ with $\zeta \leq \mu$ and $\zeta(\mathbb{P}TM) \geq \beta$. From $\mu(\hat{G}^\gamma_u) \geq \mu(\hat{G}_u) = \nu(G) > 1 - \beta$ we get $0 < \zeta(\hat{G}^\gamma_u) \leq \lim_k \zeta_{p_k}(\hat{G}^\gamma_u)$. Note also $\hat{A}_\sigma := \{\hat{y}_\sigma,\ y \in A\}$ has full μ_n-measure for all n. In particular, for infinitely many $n \in \mathbb{N}$ there is $(x^n, v_{x^n}) = \hat{x}^n_\sigma \in \hat{A}_\sigma$ with $F^n \hat{x}^n_\sigma \in \hat{G}^\gamma_u$ and $n \in E(x^n)$. Let $\hat{y}^n_u = (y^n, v^u_{y^n}) \in \hat{G}_u$ which is $\gamma/8$-close to $F^n \hat{x}^n_\sigma$. Then for $\gamma \ll \delta \ll \min(\theta, l, \alpha\epsilon)$ independent of n, the curve $D^\delta_n(x^n) := D_n(x^n) \cap B(f^n x^n, \delta)$ is transverse to $W^s(P \cap \Gamma \cap W^u_\gamma(y^n))$ and may be written as $\exp_{y^n}\left(\Gamma_\psi\right)$ where Γ_ψ is the graph of a C^r smooth function $\psi : E \subset E_u(y^n) \to E_s(y^n)$ with $\|d\psi\| < L$ for a universal constant L.

By Theorem 8.6.1 in [34] the associated holonomy map $h : W^u_\gamma(y^n) \to D^\delta_n(x^n)$, represented in Figure 4, is absolutely continuous and its Jacobian is bounded from below by a positive constant depending only on the Pesin block $P = K_N$ (not on x^n and y^n). Since we have $\mathrm{Leb}_{W^u_\gamma(x)}(\Gamma \cap P) > c$, we get for some constant c' independent of n:

$$\mathrm{Leb}_{D_n(x^n)}\left(W^s(\Gamma \cap P)\right) \geq c'. \tag{6.1}$$

The distortion of df^n on $H_n(x^n)$ being bounded by 3, we get (recall $f^n H_n(x^n) = D_n(x^n)$):

$$\frac{\mathrm{Leb}_{D_n(x^n)}\,(D_n(x^n) \setminus f^n B)}{\mathrm{Leb}_{D_n(x^n)}\,(D_n(x^n))} \leq 9 \frac{\mathrm{Leb}_{H_n(x^n)}\,(H_n(x^n) \setminus B)}{\mathrm{Leb}_{H_n(x^n)}\,(H_n(x^n))} \xrightarrow{n \to \infty} 0.$$

As $D_n(x^n)$ is the image of a strongly ϵ-bounded curve, its length is bounded from above by 2ϵ, so that we get:

$$(2\epsilon)^{-1}\mathrm{Leb}_{D_n(x^n)}\left(D_n(x^n)\setminus f^nB\right)\leq \frac{\mathrm{Leb}_{D_n(x^n)}(D_n(x^n)\setminus f^nB)}{\mathrm{Leb}_{D_n(x^n)}(D_n(x^n))}\xrightarrow{n\to\infty}0. \quad (6.2)$$

It follows from (6.1) and (6.2) that for n large enough, there exists $x\in f^nB\cap W^s(\Gamma\cap P)$, in particular $B\cap f^{-n}W^s(\Gamma)=B\cap W^s(\Gamma)\neq\emptyset$. This contradicts the definition of B. □

6.2 The maximal exponent

In Proposition 7 we proved that Lebesgue almost every point x with $\chi(x)>\frac{R(f)}{r}$ lies in the basin of an ergodic SRB measure μ. To complete the proof of Theorem 1 it remains to show $\chi(x)=\chi(\mu)$ for a.e. such points x.

For uniformly hyperbolic systems, we have

$$\Sigma\chi(x):=\max_k\Sigma^k\chi(x)=\lim_n\frac{1}{n}\log\mathrm{Jac}(d_xf^n_{E_u})=\lim_n\int\log\mathrm{Jac}(d_yf_{E_u})\,d\delta^n_x.$$

As the geometric potential $y\mapsto\log\mathrm{Jac}(d_yf_{E_u})$ is continuous in this case, any point in the basin of an SRB measure μ satisfies $\Sigma\chi(x)=\int\Sigma\chi(y)\,d\mu(y)$. As the geometric potential is not continuous in our context, the proof of this last point is not straightforward.

As mentioned after Proposition 7, we proved in Sect. 6.1 that $\left\{\chi>\frac{R(f)}{r}\right\}$ is contained Lebesgue a.e. in $W^s(\Gamma)$ where Γ is any f-invariant subset of $\bigcup_{i\in I}\mathcal{B}(\mu_i)_{i\in I}$ with $\mu_i(\Gamma)=1$ for all i. For such a set Γ we have $W^s(\Gamma)\subset\bigcup_{i\in I}\mathcal{B}(\mu_i)_{i\in I}$ and therefore it is enough to find such a set Γ satisfying $\chi(x)=\chi(\mu_i)$ for $x\in W^s(\Gamma)\cap\mathcal{B}(\mu_i)$, $i\in I$.

Let $\mathcal{R}^{+*}$ denote the invariant subset of Lyapunov regular points x of (M,f) with $\chi_1(x)>0>\chi_2(x)$. Such a point admits so called regular neighborhoods (or ϵ-Pesin charts):

Lemma 12 [33] *For a fixed $\epsilon>0$ there exists a measurable function $q=q_\epsilon:\mathcal{R}^{+*}\to(0,1]$ with $e^{-\epsilon}<q(fx)/q(x)<e^\epsilon$ and a collection of embeddings $\Psi_x:B(0,q(x))\subset T_xM=E_u(x)\oplus E_s(x)\sim\mathbb{R}^2\to M$ with $\Psi_x(0)=x$ such that $f_x=\Psi_{fx}^{-1}\circ f\circ\Psi_x$ satisfies the following properties*:

- $$d_0f_x=\begin{pmatrix}a^1_\epsilon(x) & 0\\ 0 & a^2_\epsilon(x),\end{pmatrix}$$
  with $e^{-\epsilon}e^{\chi_i(x)}<a^i_\epsilon(x)<e^\epsilon e^{\chi_i(x)}$ for $i=1,2$,
- *the C^1 distance between f_x and d_0f_x is less than ϵ,*
- *there exists a constant K and a measurable function $A=A_\epsilon:\mathcal{R}^{+*}\to\mathbb{R}$ such that for all $y,z\in B(0,q(x))$*

$$Kd(\Psi_x(y),\Psi_x(z))\leq\|y-z\|\leq A(x)d(\Psi_x(y),\Psi_x(z)),$$

with $e^{-\epsilon} < A(fx)/A(x) < e^{\epsilon}$.

For any $i \in I$ we let

$$E_i := \{x,\ \chi(x) = \chi(\mu_i)\}.$$

The set E_i has full μ_i-measure by the subadditive ergodic theorem. Put $\Gamma_i = \mathcal{B}(\mu_i) \cap E_i \cap \mathcal{R}^{+*}$ and $\Gamma = \bigcup_i \Gamma_i$. Clearly Γ is f-invariant. We finally check that $\chi(x) = \chi(\mu_i)$ for $x \in W^s(\Gamma_i)$.

Lemma 13 *If* $y \in W^s(x)$ *with* $x \in \mathcal{R}^{+*}$, *then* $\chi(y) = \chi(x)$.

Proof Fix $x \in \mathcal{R}^{+*}$ and $\delta > 0$. We apply Lemma 12 with $\epsilon \ll \chi_1(x)$. For $\alpha > 0$ we let $\mathcal{C}_\alpha$ be the cone $\mathcal{C}_\alpha = \{(u, v) \in \mathbb{R}^2,\ \alpha\|u\| \geq \|v\|\}$. We may choose $\alpha > 0$ and $\epsilon > 0$ so small that for all $k \in \mathbb{N}$ we have $d_z f_{f^k x}(\mathcal{C}_\alpha) \subset \mathcal{C}_\alpha$ and $\|d_z f_{f^k x}(v)\| \geq e^{\chi_1(x)-\delta}$ for all $v \in \mathcal{C}_\alpha$ and all $z \in B(0, q_\epsilon(f^k x))$.

Let $y \in W^s(x)$. There is $C > 0$ and λ such that $d(f^n x, f^n y) < C\lambda^n$ holds for all $n \in \mathbb{N}$. We can choose $\epsilon \ll \lambda$. In particular there is $N > 0$ such that $f^n y$ belongs to $\Psi_{f^n x} B(0, q(f^n x))$ for $n \geq N$ since we have $A(f^n x) < e^{\epsilon n} A(x)$ and $q(f^n x) > e^{\epsilon n} q(x)$. Let $z \in B(0, q(f^N x))$ with $\Psi_{f^N x}(z) = y$. Then for all $v \in \mathcal{C}_\alpha$ and for all $n \geq N$ we have $\|d_z \left(\Psi^{-1}_{f^{n-N} x} \circ f^{n-N} \circ \Psi_{f^N x}\right)(v)\| \geq e^{(n-N)(\chi_1(x)-\delta)}$. As the conorm of $d_{f^{n-N} y} \psi_{f^n x}$ is bounded from above by $A(f^n x)^{-1}$ for all n we get

$$\begin{aligned}
\chi(y) &= \limsup_n \frac{1}{n} \log \|d_y f^{n-N}\|, \\
&= \limsup_n \frac{1}{n} \log \|d_z \left(f^{n-N} \circ \Psi_{f^N x}\right)\|, \\
&\geq \limsup_{n\to+\infty} \frac{1}{n} \log \left(A(f^n x)^{-1} \left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right), \\
&\geq \chi_1(x) - \delta - \epsilon.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\| &\leq \prod_{k=N}^{n-1} \sup_{t \in B(0, q(f^k x))} \|d_t f_{f^k x}\|, \\
&\leq \left(e^{\chi_1(x)+\epsilon} + \epsilon\right)^{n-N}, \\
&\leq e^{(n-N)(\chi_1(x)+2\epsilon)}.
\end{aligned}$$

Then it follows from $\|d_{f^{n-N} y} \psi_{f^n x}\| \leq K$:

$$\chi(y) \leq \limsup_{n\to+\infty} \frac{1}{n} \log \left(\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right),$$

$$\leq \chi_1(x) + 2\epsilon.$$

As it holds for arbitrarily small ϵ and δ we get $\chi(y) = \chi_1(x) = \chi(x)$. $\square$

We conclude with $\Lambda = \{\chi(\mu_i),\ i \in I\}$ that for Lebesgue a.e. point x, we have $\chi(x) \in \left]-\infty, \frac{R(f)}{r}\right] \cup \Lambda$ and that $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i \in I,\, \chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ for all $\lambda \in \Lambda$. The proof of the Main Theorem is now complete. It follows also from Lemma 13, that the converse statement of Corollary 2 holds: if (f, M) admits an SRB measure then $\mathrm{Leb}(\chi > 0) > 0$.

7 Nonpositive exponent in contracting sets

In this last section we show Theorem 2. For a dynamical system (M, f) a subset U of M is said to be **almost contracting** when for all $\epsilon > 0$ the set $E_\epsilon = \{k \in \mathbb{N},\ \mathrm{diam}(f^k U) > \epsilon\}$ satisfies $\overline{d}(E_\epsilon) = 0$. In [21] the authors build subsets with historic behaviour and positive Lebesgue measure which are almost contracting but not contracting. We will show Theorem 2 for almost contracting sets.

We borrow the next lemma from [12] (Lemma 4 therein), which may be stated with the notations of Sect. 5 as follows:

Lemma 14 *Let $f : M \circlearrowleft$ be a C^∞ diffeomorphism and let U be a subset of M with* $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ *for some $a > 0$. Then for all $\gamma > 0$ there is a C^∞ smooth embedded curve $\sigma : [0, 1] \to M$ and $I \subset \mathbb{N}$ with $\sharp I = \infty$ such that*

$$\forall n \in I,\ \mathrm{Leb}_{\sigma_*}\left(\{x \in U \cap \sigma_*,\ \|d_x f^n(v_x)\| > e^{na}\}\right) > e^{-n\gamma}.$$

We are now in a position to prove Theorem 2 for almost contracting sets.

Proof of Theorem 2 We argue by contradiction by assuming $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ for some $a > 0$ with U being a almost contracting set. By Yomdin's Theorem on one-dimensional local volume growth for C^∞ dynamical systems [43] there is $\epsilon > 0$ so small that

$$v^*(f, \epsilon) := \sup_\sigma \limsup_{n\to\infty} \frac{1}{n} \sup_{x \in M} \log \mathrm{Leb}_{(f^n\circ\sigma)_*}\left(f^n B_n(x, \epsilon)\right) < a/2, \tag{7.1}$$

where the supremum holds over all C^∞ smooth embedded curves $\sigma : [0, 1] \to M$. As U is almost contracting, there are subsets $(C_n)_{n\in\mathbb{N}}$ of M with $\lim_n \frac{\log \sharp C_n}{n} = 0$ satisfying for all n:

$$U \subset \bigcup_{x \in C_n} B_n(x, \epsilon). \tag{7.2}$$

Fix an error term $\gamma \in]0, \frac{a}{2}[$. Then by Lemma 14 there is a C^∞-smooth curve $\sigma_* \subset U$ and an infinite subset I of $\mathbb{N}$ such that for all $n \in I$:

$$\sum_{x \in C_n} \mathrm{Leb}_{(f^n\circ\sigma)_*}\left(f^n B_n(x, \epsilon)\right) \geq \mathrm{Leb}_{(f^n\circ\sigma)_*}(f^n(U \cap \sigma_*)),$$

$$\begin{aligned} &\geq e^{na}\mathrm{Leb}_{\sigma_*}\left(\{x \in U \cap \sigma_*,\ \|d_x f^n(v_x)\| > e^{na}\}\right), \\ &\geq e^{n(a-\gamma)} \text{ by (7.1)}, \\ \sharp C_n \sup_{x\in M} \mathrm{Leb}_{(f^n\circ\sigma)_*}\left(f^n B_n(x,\epsilon)\right) &\geq e^{n(a-\gamma)} \text{ by (7.2)}. \end{aligned}$$

Therefore we get the contradiction $v^*(f,\epsilon) > a-\gamma > a/2$. □

Appendix

Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a sequence in $M_d(\mathbb{R}^d)$. For any $n \in \mathbb{N}$ we let $A^n = A_{n-1}\dots A_1A_0$. We define the Lyapunov exponent $\chi(\mathcal{A})$ of $\mathcal{A}$ with respect to $v \in \mathbb{R}^d \setminus \{0\}$ as

$$\chi(\mathcal{A}, v) := \limsup_n \frac{1}{n}\log\|A^n(v)\|,$$

Lemma 15

$$\sup_{v\in\mathbb{R}^d\setminus\{0\}} \chi(\mathcal{A}, v) = \limsup_n \frac{1}{n}\log \left|\left|\left| A^n \right|\right|\right|.$$

Proof The inequality $\leq$ is obvious. Let us show the other inequality. Let $v_n \in \mathbb{R}^d$ with $\|v_n\| = 1$ and $\|A^n(v_n)\| = |||A^n|||$. Then take $v = \lim_k v_{n_k}$ with $\lim_k \frac{1}{n_k}\log|||A^{n_k}||| = \limsup_n \frac{1}{n}\log|||A^n|||$. We get

$$\begin{aligned} \|A^{n_k}(v)\| &\geq \|A^{n_k}(v_k)\| - \|A^{n_k}(v-v_k)\|, \\ &\geq \left|\left|\left|A^{n_k}\right|\right|\right|(1-\|v-v_k\|), \\ \limsup_k \frac{1}{n_k}\log\|A^{n_k}(v)\| &\geq \limsup_n \frac{1}{n}\log\left|\left|\left|A^n\right|\right|\right|. \end{aligned}$$

□

Acknowledgement The author would like to thank the anonymous referees, whose very thorough reports led to many improvements and clarifications.

References

1. Alves, J.F., Dias, C.L., Luzzatto, S., Pinheiro, V.: SRB measures for partially hyperbolic systems whose central direction is weakly expanding. J. Eur. Math. Soc. **19**(10), 2911–2946 (2017)
2. Alves, J.F., Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly expanding. Invent. Math. **140**(2), 351–398 (2000)
3. Araujo, A.: A existencia de atratores hiperbolicos para difeormorfismos de superficies (Portuguese) Preprint IMPA Serie F, No 23/88 (1988)
4. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. Ann. Math. **133**, 73–169 (1991)
5. Benedicks, M., Young, L.-S.: Sinai-Bowen-Ruelle measures for certain Henon maps. Invent. Math. **112**(3), 541–576 (1993)
6. Benedicks, M., Viana, M.: Solution of the basin problem for Hénon-like attractors. Invent. Math. **143**(2), 375–434 (2001)

7. Ovadia, S.B.: Hyperbolic SRB measures and the leaf condition. Commun. Math. Phys. **387**, 1353–1404 (2021)
8. Berger, P., Biebler, S.: Emergence of wandering stable components. arXiv:2001.08649
9. Bonatti, C., Viana, M.: SRB measures for partially hyperbolic systems whose central direction is mostly contracting. Isr. J. Math. **115**, 157–193 (2000)
10. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lect. Notes in Math., vol. 470. Springer, Berlin (1975)
11. Burguet, D.: C^r Counter-examples to Viana conjecture for any finite r. In preparation
12. Burguet, D.: Symbolic extensions in intermediate smoothness on surfaces. Ann. Sci. Éc. Norm. Supér. (4) **45**(2), 337–362 (2012)
13. Burguet, D.: Entropy of physical measures for C^∞ dynamical systems. Commun. Math. Phys. **375**, 1201–1222 (2020)
14. Buzzi, J., Crovisier, S., Sarig, O.: Measures of maximal entropy for surface diffeomorphisms. Ann. Math. (2) **195**, 421–508 (2022)
15. Buzzi, J., Crovisier, S., Sarig, O.: Continuity properties of Lyapunov exponents for surface diffeomorphisms. Invent. Math. **230**, 767–849 (2022)
16. Buzzi, J., Crovisier, S., Sarig, O.: On the existence of SRB measures for C^∞ surface diffeomorphisms. To appear in Int. Math. Res. Not.
17. Climenhaga, V., Luzzatto, S., Pesin, Y.: The geometric approach for constructing Sinai–Ruelle–Bowen measures. J. Stat. Phys. **166**, 467–493 (2017)
18. Climenhaga, V., Luzzatto, S., Pesin, Y.: SRB measures and Young towers for surface diffeomorphism. Ann. Henri Poincaré **23**, 973–1059 (2022)
19. Climenhaga, V., Dolgopyat, D., Pesin, Y.: Non-stationary non-uniform hyperbolicity: SRB measures for dissipative maps. Commun. Math. Phys. **346**, 553–602 (2016)
20. Downarowicz, T.: Entropy in Dynamical Systems, vol. 18. Cambridge University Press, Cambridge (2011)
21. Guarino, P., Guihéneuf, P.-A., Santiago, B.: Dirac physical measures on saddle-type fixed points. J. Dyn. Differ. Equ. to appear
22. Hernández Encinas, L., Muñoz Masqué, J.: A short proof of the generalized Faà di Bruno's formula. Appl. Math. Lett. **16**, 975–979 (2003)
23. Rodriguez Hertz, F., Rodriguez Hertz, M.A., Tahzibi, A., Ures, R.: Uniqueness of SRB measures for transitive diffeomorphisms on surfaces. Commun. Math. Phys. **306**(1), 35–49 (2011)
24. Jakobson, M.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. **81**, 39–88 (1981)
25. Keller, G.: Exponents, attractors and Hopf decompositions for interval maps. Ergod. Theory Dyn. Syst. **10**(4), 717–744 (1990)
26. Kingman, J.F.C.: Subadditive ergodic theory. Ann. Probab. **1**, 883–909 (1973)
27. Kiriki, S., Soma, T.: Takens' last problem and existence of non-trivial wandering domains. Adv. Math. **306**, 524–588 (2017)
28. Ledrappier, F., Young, L.-S.: The metric entropy of diffeomorphisms, I. Characterization of measures satisfying Pesin's entropy formula. Ann. Math. **122**, 509–539 (1985)
29. Oseledec, V.I.: A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. Trans. Mosc. Math. Soc. **19**, 197–231 (1968)
30. Ledrappier, F.: Proprietes ergodiques des mesures de Sinai. Publ. Math. IHES **59**, 163–188 (1984)
31. Ledrappier, F., Walters, P.: A relativised variational principle for continuous transformations. J. Lond. Math. Soc. (2) **16**(3), 568–576 (1977)
32. Misiurewicz, M.: A short proof of the variational principle for $\mathbb{Z}^d$ actions. Astérisque **40**, 147–158 (1976)
33. Pesin, Ja.B.: Families of invariant manifolds that correspond to nonzero characteristic exponents. Izv. Akad. Nauk SSSR, Ser. Mat. **40**, 1332–1379 (1976)
34. Pesin, Y., Barreira, L.: Lyapunov Exponents and Smooth Ergodic Theory. University Lecture Series, vol. 23. Am. Math. Soc., Providence (2001)
35. Pesin, Ya.B., Sinai, Y.G.: Gibbs measures for partially hyperbolic attractors. Ergod. Theory Dyn. Syst. **2**, 417–438 (1982)
36. Pollicott, M.: Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds. London Mathematical Society Lecture Note Series, vol. 180. Cambridge University Press, Cambridge (1993)
37. Rokhlin, V.A.: Lectures on the entropy theory of measure-preserving transformations. Russ. Math. Surv. **22**(5), 3–56 (1967)

38. Ruelle, D.: A measure associated with axiom a attractors. Am. J. Math. **98**, 619–654 (1976)
39. Ruelle, D.: Historical behaviour in smooth dynamical systems. In: Global Analysis of Dynamical Systems, pp. 63–66 (2001)
40. Ruelle, D.: An inequality for the entropy of differentiable maps. Bol. Soc. Bras. Mat. **9**(1), 83–87 (1978)
41. Sacksteder, R., Shub, M.: Entropy on sphere bundles. Adv. Math. **28**(2), 174–177 (1978)
42. Sinai, Ya.: Gibbs measure in ergodic theory. Russ. Math. Surv. **27**, 21–69 (1972)
43. Yomdin, Y.: Volume growth and entropy. Isr. J. Math. **57**(3), 285–300 (1987)
44. Yomdin, Y., Comte, G.: Tame Geometry with Application in Smooth Analysis. Lecture Notes in Mathematics, vol. 1834 (2004)
45. Young, L.-S.: What are SRB measures, and which dynamical systems have them? J. Stat. Phys. **108**, 733–754 (2002)