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The universal p-adic Gross–Zagier formula

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Abstract Let G be the group $(\mathrm{GL}_2 \times \mathrm{GU}(1))/\mathrm{GL}_1$ over a totally real field F, and let $\mathscr{X}$ be a Hida family for G. Revisiting a construction of Howard and Fouquet, we construct an explicit section $\mathscr{P}$ of a sheaf of Selmer groups over $\mathscr{X}$. We show, answering a question of Howard, that $\mathscr{P}$ is a universal Heegner class, in the sense that it interpolates geometrically defined Heegner classes at all the relevant classical points of $\mathscr{X}$. We also propose a 'Bertolini–Darmon' conjecture for the leading term of $\mathscr{P}$ at classical points. We then prove that the p-adic height of $\mathscr{P}$ is given by the cyclotomic derivative of a p-adic L-function. This formula over $\mathscr{X}$ (which is an identity of functionals on some universal ordinary automorphic representations) specialises at classical points to all the Gross–Zagier formulas for G that may be expected from representation-theoretic considerations. Combined with a result of Fouquet, the formula implies the p-adic analogue of the Beilinson–Bloch–Kato conjecture in analytic rank one, for the selfdual motives attached to Hilbert modular forms and their twists by CM Hecke characters. It also implies one half of the first example of a non-abelian Iwasawa main conjecture for deriva-

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tives, in $2[F : \mathbf{Q}]$ variables. Other applications include two different generic non-vanishing results for Heegner classes and p-adic heights.

Contents

1 Introduction and statements of the main results

A beautiful construction of Heegner and Birch, based on the modularity of elliptic curves and the theory of complex multiplication, attaches to an elliptic curve $A/\mathbf{Q}$ and an imaginary quadratic field E a point $P \in A(E)$. The work of Gross–Zagier [45] related the height of P to the derivative of the L-function $L'(A_E, 1)$, with striking applications to the Birch and Swinnerton-Dyer conjecture. An analogous result in p-adic coefficients was proved by Perrin-Riou [84] soon thereafter, if A has good ordinary reduction at the prime p.

The decade following those works saw a pair of similar results, by Nekovář [78] and Zhang [109], relating Heegner cycles on Kuga–Sato varieties to (p-adic) L-functions of higher-weight modular forms. We may single out two major innovations in the approach to Heegner points and Gross–Zagier formulas since then,[1] both answering the question of what 'other' Heegner points there are and how they fit together.

The first one starts from the observation by Mazur [73] and Perrin-Riou [85] that Heegner points should vary p-adically in anticyclotomic families, in the same way that the L-function of the elliptic curve A_E does; this observation inspired Howard [57] to prove a generalisation to such families of Perrin-Riou's formula. Howard later significantly expanded the scope of Mazur and Perrin-Riou's idea by proving that the Kummer classes of Heegner points also vary in Hida families of modular forms [58]; the question of the relation of the resulting 'big' classes to Heegner cycles was left open.

The second innovation was the observation by Gross [46] that Heegner points can be viewed as elements of spaces of H'-invariant linear functionals on an automorphic representation of $(\mathrm{G} \times \mathrm{H})'$ (these reductive groups will be defined below),[2] so that the tools of representation theory may be brought in to conceive and prove more general formulas: a programme whose main

[1] Two other recent ideas that our work does not touch upon are nevertheless too important to be ignored: the conjecture of Darmon and Guitart–Madseu–Şengün that there should exist Heegner points attached to any quadratic extension of number fields (see [27,48]), and the formulas for the p-adic *logarithms* of Heegner points of [8,68].

[2] N.B.: the notation G used in the informal abstract differs from the notation of the paper.

achievement, in complex coefficients, is the work of Yuan–Zhang–Zhang [107] on Heegner points on Shimura curves.

In this work, we combine those two approaches. We construct Heegner classes for the Galois representation over a Hida family for $(\mathrm{G} \times \mathrm{H})'$, show that they specialise to (cohomological) Heegner cycles at all classical points, and prove a formula for their p-adic heights that is universal in the sense that it specialises to all the p-adic formulas suggested by the framework of Gross. (The analogous complex Gross–Zagier formulas are not currently known[3] for motives of higher weight.) We obtain various applications to the arithmetic of motives attached to Hilbert modular forms.

In the rest of this first section we state our main theorems, and complete the discussion of their history.

We begin in Sect. 1.1 by presenting the results concerning the p-adic Beĭlinson–Bloch–Kato conjecture (Theorem A); they are applications of the general p-adic Gross–Zagier formula for a fixed representation, stated as Theorem B in Sect. 1.2.

In Sect. 1.3 we outline the construction and properties of the universal family of Heegner classes (Theorem C), referring to the "Bertolini–Darmon" conjecture of Sect. 7.3 for a further study of its classical specialisations. In Sect. 1.4 we state the universal formula of the title (Theorem D); a complementary 'Waldspurger' analogue will be proved in Sect. 7.2 (Theorem H).

Finally, in Sect. 1.5 we discuss some further applications: the first non-abelian example of an Iwasawa main conjecture for derivatives of p-adic L-functions (Theorem E); and two results on the generic non-vanishing of p-adic heights and Heegner cycles: one for CM motives (Theorem F), the other for Hida families containing a rank-0 elliptic curve with split multiplicative reduction (Theorem G). A further application, to a criterion for certain Bloch–Kato Selmer groups to be of rank *zero*, will appear separately.

1.1 The p-adic Beĭlinson–Bloch–Kato conjecture in analytic rank 1

The primary motivation for our work comes from the generalisations of the Birch and Swinnerton-Dyer (BSD) conjecture and its p-adic analogue, as proposed by Beĭlinson, Bloch–Kato, and Perrin-Riou [3,11,86]. Recall that if $A/\mathbf{Q}$ is an elliptic curve, (BSD) is equivalent to the following statements. Denote by r_{an} and $L^*(A, 1)$ the order of vanishing and leading term of $L(A, s)$ at $s = 1$. Then $L^*(A, 1) > 0$ and for every prime p,

(a) the Selmer group $\mathrm{Sel}(V_pA) := (\varprojlim_n \mathrm{Sel}_{p^n}(A)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ has dimension equal to r_{an};
(b) the divisible part of $\text{Ш}(A)[p^\infty]$ vanishes;

[3] See however the very recent [87]. (Note added during revision.)

(c) the p-adic valuations of $L^*(A,1)/\Omega_A R_A$ and $|\text{Ш}(A)[p^\infty]| \prod_{v\nmid\infty} c_v(A)$ are equal.

1.1.1 Selmer groups according to Bloch–Kato and Nekovář

If E is a number field and V is a geometric p-adic representation of its Galois group G_E, Bloch and Kato [11] have proposed an analogue

$$H^1_f(E,V)$$

of the Selmer group of A; it is an L-vector-subspace (where L is the field of scalars for V) of the first Galois cohomology group of V, consisting of those classes satisfying certain local conditions. According to the resulting variant of the conjecture of Beĭlinson [3], the dimension $\dim_L H^1_f(E,V)$ should equal the order of vanishing of the L-function $L(V^*(1),s)$ at $s=0$.[4]

Another definition of Selmer groups was proposed by Greenberg when V satisfies an ordinariness condition at the places above a prime p; specialised to the cases of interest to us, it recovers the Bloch–Kato Selmer groups. Nekovář observed that a variation of Greenberg's definition works well in p-adic families, and developed this observation into the theory of Selmer complexes [81], that provides the foundation for the present work (Sect. 5). For nice p-adic families of G_E-representations, the theory allows to define groups

$$\widetilde{H}^i_f(E,V)$$

for all i.

1.1.2 The p-adic Beĭlinson–Bloch–Kato conjecture for Hilbert modular forms

Our main arithmetic results concern the p-adic analogue of the Beĭlinson–Bloch–Kato conjecture for the Galois representations attached to Hilbert modular forms and their twists by Hecke characters of CM fields.

Fix throughout the rest of this paper a rational prime p. Let F be a totally real field, let E be a CM quadratic extension of F, and let

$$\mathrm{G}_0 := \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2, \quad \mathrm{H} := \mathrm{Res}_{E/\mathbf{Q}}\mathrm{G}_m.$$

[4] Provided V contains no copies of the trivial representation. Of course in general the meromorphic continuation of $L(V,s)$ is itself conjectural. Note that when V is self-dual, or E is a CM field and V is conjugate-self-dual, we have $L(V,s)=L(V^*(1),s)$.

Let L be a finite extension of $\mathbf{Q}_p$ splitting F. A pair of cohomological weights for G_0 and H is a pair of tuples $\underline{w} := (w; (w_\sigma)_{\sigma\colon F\hookrightarrow L}), \underline{l} = (l; (l_\sigma)_{\sigma\colon F\hookrightarrow L})$, each consisting of $[F : \mathbf{Q}] + 1$ integers of the same parity, such that $w_\sigma \geq 2$ for all $\sigma\colon F \hookrightarrow L$. In this paper we will only consider cohomological weights and therefore omit the adjective 'cohomological'. By a "Hilbert modular form over L of weight $\underline{w}$" (respectively a "Hecke character of E over L of weight $\underline{l}$") we mean a cuspidal automorphic representation of $\mathrm{G}_0(\mathbf{A})$ (respectively $\mathrm{H}(\mathbf{A})$) over L of weight $\underline{w}$ (respectively weight $\underline{l}$) as defined in Definition 2.4.1 below.

If π_0 is a Hilbert modular form and χ a Hecke character over L, we denote by $\Pi_0 = \pi_0 \otimes \chi$ the associated representation of $\mathrm{G}_0 \times \mathrm{H}$. We denote by V_{π_0} and V_χ the corresponding 2- (respectively 1-) dimensional representations of G_F (respectively G_E), normalised so that $L(V_{\pi_0}, s) = L(s + 1/2, \pi_0)$, and we let

$$V := V_{\Pi_0} := V_{\pi_0|G_E} \otimes V_\chi.$$

Let ω_{π_0} be the central character of π_0 and let $\omega_\chi := \chi|_{F^\times_{\mathbf{A}^\infty}}$. If $\omega_{\pi_0}\omega_\chi = 1$, then V is conjugate-self-dual and pure of weight -1, and the epsilon factor $\varepsilon(V) \in \{\pm 1\}$.

Let $\Gamma_F := F^\times_{\mathbf{A}^\infty}/F^\times \hat{\mathscr{O}}_F^{p,\times}$ (identified with the Galois group of the maximal abelian extension of F unramified outside p by class field theory), and let

$$\mathscr{E}_{\mathrm{Z}/L} := \mathrm{Spec}\,(\mathbf{Z}_p[\![\Gamma_F]\!]_L).$$

(We will also simply write $\mathscr{E}_{\mathrm{Z}}$ for $\mathscr{E}_{\mathrm{Z}/\mathbf{Q}_\mathrm{p}}$.) Suppose that π_0 is *ordinary* in the sense of Definition 2.4.3; equivalently, for all $v|p$ the associated G_{F_v}-representation $V_{\pi_0,v}$ reduces nontrivially as

$$0 \to V^+_{\pi_0,v} \to V_{\pi_{0,v}} \to V^-_{\pi_0,v} \to 0,$$

and G_{F_v} acts on $V^+_{\pi_0,v}$ by the product of the cyclotomic character χ_{cyc} and a character α_v° valued in p-adic units. We may attach to V a meromorphic p-adic L-function

$$\mathscr{L}_p(V_{(\pi_0,\chi)}, s) \in \mathscr{K}(\mathscr{E}_{\mathrm{Z}/L})$$

where the variable $s \in \mathscr{E}_{\mathrm{Z}/L}$ may be thought of as a p-adic character of Γ_F; we use the synonym $\chi_{F,s}$ when we want to emphasise such nature of s, and we denote by "$s = 0$" the trivial character $\chi_{F,0} = \mathbf{1}$.[5] More precisely, working in

[5] Other authors consider p-adic L-functions of a variable $s' \in \mathbf{Z}_p$. In our language this corresponds to restricting $\mathscr{L}_p(V, s)$ along the embedding $\mathbf{Z}_p = \mathrm{Spec}\,\mathbf{Z}_p[\![\mathbf{Z}_p]\!]_{\mathbf{Q}_p}(\mathbf{Q}_p) \to \mathscr{E}_{\mathrm{Z}/L}(\mathbf{Q}_p)$, $s' \mapsto \chi^{s'}_{\mathrm{cyc},F}$ where $\chi_{\mathrm{cyc},F} =$(1.8.1) is the cyclotomic character of F.

terms of the multivariable function $\mathscr{L}_p(\mathscr{V}^\sharp)$ of Theorem 1.4.1 below, we may define $\mathscr{L}_p(V_{(\pi,\chi)})$ as the restriction

$$\mathscr{L}_p(V_{(\pi,\chi)}, s) := \mathscr{L}_p(\mathscr{V}^\sharp)(z_s) \tag{1.1.1}$$

where z_s corresponds to the family of representations $V_{\pi|G_E} \otimes \chi\chi_{F,s|G_E}$.

If $\varepsilon(V) = -1$, then $\mathscr{L}_p(V_{(\pi_0,\chi)}, 0) = 0$ and we denote by $\mathscr{L}'_p(V_{(\pi_0,\chi)}, 0) = \mathrm{d}\mathscr{L}_p((V_{(\pi_0,\chi)})(0) \in T_0\mathscr{E}_{Z/L} = \Gamma_F\hat{\otimes}L$ its first derivative.

Theorem A *Let π_0 be a Hilbert modular form over L of weight $\underline{w}$, and let χ be a Hecke character of E over L of weight $\underline{l}$. Let $V := V_{\pi_0|G_E} \otimes V_\chi$. Suppose that:*

- (wt) $|l_\sigma| < w_\sigma$ *for all* $\sigma : F \hookrightarrow L$*;*
- (sd) $\omega_{\pi_0}\omega_\chi = 1$ *(which implies* $w + l = 0$*);*
- (ε) $\varepsilon(V) = -1$*;*
- (ord) π_0 *is ordinary;*
- (n-exc) V *is* not exceptional*: for no place* $w|v|p$ *of* E *is* $V_w^- := V^-_{\pi_0,v|G_{E_w}} \otimes \chi_w$ *the trivial representation.*

1. *We have*

$$\mathscr{L}'_p(V_{(\pi,\chi)}, 0) \neq 0 \Longrightarrow \dim_L \widetilde{H}^1_f(E, V) \geq 1,$$

and we can exhibit an explicit nonzero element of $\widetilde{H}^1_f(E, V) = H^1_f(E, V)$*, whose p-adic height (cf. Proposition 5.3.3) is also non-zero.*

2. *Let $T \subset V$ be a stable lattice. If $\mathscr{L}'_p(V_{(\pi,\chi)}, 0) \neq 0$ and moreover the conditions of* [41, Theorem B.(i)] *are satisfied, then:*
   (a) *we have*

$$\dim_L \widetilde{H}^1_f(E, V) = 1;$$

   (b) *let $R_T \in \mathscr{O}_L\hat{\otimes}_{\mathbf{Z}_p}\Gamma_F$ be the regulator of the height pairing* (1.2.12) *on $\widetilde{H}^1_f(E, T) \times \widetilde{H}^1_f(E, T^*(1))$. Then*

$$\mathscr{L}'_p(V_{(\pi,\chi)}, 0) \succeq_{\mathbf{Z}_p} R_T \cdot |\widetilde{H}^2_f(E, T)_{\mathrm{tors}}|$$

   in $L\hat{\otimes}\Gamma_F$.

In the last formula we have used the following suggestive notation.

Notation For a domain A with fraction field K and two A-submodules m_1, m_2 of a K-vector space M we write $m_1 \succeq_A m_2$ if $m_1 \subseteq m_2$; the notation is extended to the case where some m_i is an element of M, in which case we interpret it as Am_i.

Part 1 will be an immediate consequence of Theorem B, the Jacquet–Langlands correspondence, and the observation following (1.2.5) below. For a list of previous results in the direction of part 1 we refer to the discussion following Theorem B. Let us note, for now, that an analogue of this result in complex coefficients is not known.

Part 2 follows from invoking the results of Fouquet in [41], that generalise the bounds on Tate–Shafarevich groups of elliptic curves obtained by Kolyvagin using the methods of Euler systems.

Remark 1.1.1 Condition (n-exc) guarantees that $\mathscr{L}_p(V_{(\pi,\chi)}, s)$ has no exceptional zeros at $s = 0$, and it is equivalent to the identity $\widetilde{H}^1_f(E, V) = H^1_f(E, V)$. We will also equivalently say that Π is not exceptional. For a characterisation of this condition, see Lemma 6.4.6.

Remark 1.1.2 In the simplest case where $F = \mathbf{Q}$, π_0 is a modular form with rational Fourier coefficients of weight $w_\sigma = 2$, and $\chi = \mathbf{1}$, the representation $V_{\pi_0} = V_{\mathfrak{p}}A$ is the rational p-adic Tate module of an elliptic curve $A/\mathbf{Q}$. In this case $H^1_f(E, V) = \mathrm{Sel}(V_p A_E)$, and letting $T = T_p A_E$, the group $\widetilde{H}^2_f(E, T)_{\mathrm{tors}}$ equals [12, (1.36)] the quotient of $Ш(A_E)[p^\infty]$ by its divisible submodule $Ш(A_E)_{p\text{-div}}$.

The group $Ш(A_E)_{p\text{-div}}$, conjecturally 0, measures the failure of $\mathrm{Sel}(V_p A_E)$ to be generated by the classes of points in $A(E)$. We do not address in this paper the analogous conjecture from [11] that $H^1_f(E, V)$ should be generated by the classes of algebraic cycles. Nevertheless our construction of a generator is sufficiently geometric to provide a good starting point to establish this conjecture, cf. Remark 6.2.2.

1.1.3 A variant for selfdual Hilbert modular forms

Suppose that π_0 is an ordinary Hilbert modular form, $\omega_{\pi_0} = 1$ (so that $w = 0$), and $\varepsilon(V_{\pi_0}) = -1$. Assume that either $[F : \mathbf{Q}]$ is odd or there is a place $v \nmid p\infty$ of F such that $\pi_{0,v}$ is not a principal series. Suppose that for no $v|p$ is $\pi_{0,v}$ the Steinberg representation. Let $L_p(V_{\pi_0}, s)$ be the p-adic L-function of V_{π_0} constructed in [30]. If $L'_p(V_{\pi_0}, 0) \neq 0$, then the conclusions (1) and (2a) of the previous theorem hold with (E, V) replaced by (F, V_{π_0}). (This is proved by a standard argument based on the choice of a suitable auxiliary E to reduce to the previous theorem.) A similar remark (at least for part (1)) applies when π_0 has CM by E, cf. the proof of Theorem F in Sect. 7.1.4

1.1.4 Addendum to the historical overview: higher-rank cases

The general overview sketched in our opening page ignored a third important theme: Gross's framework has been generalised in [42] to study special cycles attached to other pairs of groups (G, H). Several works have explored the consequences towards the Beĭlinson–Bloch–Kato conjecture of the possible non-vanishing of those cycles, most notably [69]. On the other hand, non-vanishing *criteria* in terms of L-functions have been obtained in a considerably more limited set of cases, mostly related to triple-product L-functions [9,10,28,106,108].[6] The relation with cyclotomic p-adic L-functions has not been studied beyond Heegner cycles.

1.2 The p-adic Gross–Zagier formula for arbitrary weight

Theorem A, like analogous previous results [31,32,35,78,84,96], is an application of an explicit formula for the p-adic heights of a certain Selmer class (here rather a collection of classes). When the weights are *trivial*, that is $\underline{w} = (0; (2, \ldots, 2))$ and $\underline{l} = (0; (0, \ldots, 0))$, this is the class of a *Heegner* 0-cycle coming from CM points on quaternionic Shimura curves; this is the case studied in [32,35], and earlier in complex coefficients by Yuan–Zhang–Zhang [107]. In general, it is the class of a 0-cycle supported at CM points, with coefficients in a local system corresponding to the weight of the representation. The specific choice of the (tower of) Shimura curves is dictated by the local root numbers of V, see the discussion preceding Definition 1.2.1.

1.2.1 Algebraic groups and Shimura varieties

Let $\mathbf{B}$ be a quaternion algebra over $F_{\mathbf{A}}$ (where $\mathbf{A}$ denotes the adèles of $\mathbf{Q}$) with ramification set $\Sigma \sqcup \{v|\infty\}$ satisfying $|\Sigma| \equiv [F : \mathbf{Q}] - 1 \pmod 2$. Then $\mathrm{G}(\mathbf{A}) := \mathbf{B}^\times$ is *not* the points of an algebraic group 'G' over $\mathbf{Q}$, but we will still find convenient to use this suggestive notation and refer to G as an *incoherent* algebraic group over $\mathbf{Q}$ (see Sect. 2.1.1 for a more formal treatment). Let $\mathrm{H} = \mathrm{Res}_{E/\mathbf{Q}}\mathbf{G}_m$ as above, and let $\mathrm{Z} := \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$, that admits natural central embeddings in G and H.

The list of (coherent or incoherent) groups of interest in this paper, often denoted collectively by G_*, is

$$\mathrm{G}, \quad \mathrm{H}, \quad \mathrm{G} \times \mathrm{H}, \quad (\mathrm{G} \times \mathrm{H})' := (\mathrm{G} \times \mathrm{H})/\mathrm{Z}, \quad \mathrm{H}' := \mathrm{H}/\mathrm{Z}, \tag{1.2.1}$$

[6] In a related context, see also the very recent breakthrough of Li–Liu [66]. (Note added during revision.)

where Z is embedded diagonally in the product group.[7] We suppose that for every $v \in \Sigma$, E_v/F_v is nonsplit. Then there is unique $\mathbf{B}^\times$-conjugacy class of $F_\mathbf{A}$-embeddings $E_\mathbf{A} \hookrightarrow \mathbf{B}$, of which we fix one. It induces an embedding $\mathrm{e}\colon \mathrm{H} \hookrightarrow \mathrm{G}$.

To the above groups and suitable Shimura data (Sect. 2.3.1), we associate corresponding towers of compactified Shimura varieties X_*, respectively denoted

$$X_{/F}, \quad Y_{/E}, \quad X \times_F Y_{/E}, \quad Z_{/E}, \quad Y'_{/E}. \tag{1.2.2}$$

They are curves except for Y, Y' that have dimension 0. The embedding e induces a diagonal embedding $\mathrm{H}' \hookrightarrow (\mathrm{G} \times \mathrm{H})'$, hence a morphism of Shimura varieties

$$\mathrm{e}'\colon Y' \to Z.$$

1.2.2 p-Adic automorphic representations

It is more natural to parametrise "cohomological automorphic representations over a p-adic field L" of a group G_* by irreducible algebraic representations W of G_*.[8]

Let $G_{*,\infty}$ be $\mathrm{G}_*(\mathbf{Q}_p)$ with the Zariski topology (and for later purposes let $G_{*,p} := \mathrm{G}_*(\mathbf{Q}_p)$ with the p-adic topology, $G_* := G_{*,p} \times G_{*,\infty}$). We *redefine* throughout this work

$$\mathrm{G}_*(\mathbf{A}) := \mathrm{G}_*(\mathbf{A}^\infty) \times G_{*,\infty}.$$

Let W be an (algebraic) representation of $G_{*,\infty}$ over L, and let $\mathscr{W}$ be the corresponding étale local system on the tower X_*. Then we define a (cuspidal, cohomological) *automorphic representation of* $\mathrm{G}_*(\mathbf{A})$ *over* L *of weight* W to be a representation

$$\Pi = \Pi^\infty \otimes W$$

7 In fact, the (incoherent) group that truly underlies our constructions is $(\mathrm{G} \times_\mathrm{Z} \mathrm{H})' = \{(g,h) \mid \nu_\mathrm{G}(g) = \nu_\mathrm{H}(h)\}$ (where $\nu_?\colon ? \to \mathrm{Z}$ arises from the reduced norm map of $\mathbf{B}$ (for $? = \mathrm{G}$) or from the norm of E/F (for $? = \mathrm{H}$). That is, the universal Heegner class and the other associated objects described below descend to the ordinary eigenvariety for $(\mathrm{G} \times_\mathrm{Z} \mathrm{H})'$ (a quotient of the one for $(\mathrm{G} \times \mathrm{H})'$). Nevertheless, for the sake of simplicity we will content ourselves with working with $(\mathrm{G} \times \mathrm{H})'$.

8 See Definition 2.4.1: the W/L of interest to us are in bijection with (finite) G_L-orbits of cohomological 'numerical' weights as defined above. From now on all numerical or representation-theoretic weights will be tacitly understood to be cohomological.

of $\mathrm{G}_*(\mathbf{A})$ occurring in $H^\bullet(X_{*,\overline{E}}, \mathscr{W}^\vee)\otimes W$.[9] (Here and in the rest of the paper, groups and Hecke algebras act on Shimura varieties and their homology on the *right*, on cohomology and on automorphic forms on the *left*. Left and right algebraic representations W are identified via $w.g := g^{-1}.w$.)

1.2.3 Automorphic and Galois representations

Let $\Pi = \pi \otimes \chi$ be a cuspidal automorphic representation of $(\mathrm{G}\times\mathrm{H})'(\mathbf{A})$ over L of weight $W = W_\mathrm{G}\otimes W_\mathrm{H}$. Let $V = V_\Pi = V_{\pi|G_E}\otimes V_\chi$ be the associated G_E-representation.

For a smooth proper variety Z' of dimension d over a characteristic-zero field F' and a p-adic local system $\mathscr{W}'$, define

$$H_i(Z', \mathscr{W}') := H^{2d-i}_{\text{ét}}(Z', \mathscr{W}'(d))$$

for all $0 \le i \le 2d$. For each level $K \subset (\mathrm{G}\times\mathrm{H})'(\mathbf{A}^\infty)$, let $\overline{Z}_K := Z_K \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \overline{E}$. We use the notation

$$\mathrm{H}_i(\overline{Z}_K, \mathscr{W}) := H_i(\overline{Z}_K, \mathscr{W}) \otimes W^\vee$$

and similarly for the other Shimura varieties over $F, E, \overline{E}$ under consideration. Thanks to work of Carayol we can construct an injection (an isomorphism unless V is decomposable) of $(\mathrm{G}\times\mathrm{H})'(\mathbf{A})$-representations

$$\Pi \hookrightarrow \varinjlim_K \mathrm{Hom}_{L[G_E]}(\mathrm{H}_1(\overline{Z}_K, \mathscr{W}), V_\Pi). \qquad (1.2.3)$$

1.2.4 Heegner cycles

Suppose that W satisfies (wt), then $W^{H'_\infty} \cong W_{H'_\infty}$ is 1-dimensional, and e' induces a canonical system of maps

$$\mathrm{H}_0(Y'_{V'}, L) \to \mathrm{H}_0(Z_K, \mathscr{W})$$

for all $V' \subset \mathrm{H}'(\mathbf{A}^\infty)\cap K$. The image $\Delta^\circ_{W,f_\infty} \in \mathrm{H}_0(Z_K, \mathscr{W})$ of the normalised fundamental class

$$[Y'_{V'}] = |Y'(\overline{E})|^{-1}\cdot \sum_{y\in Y'(\overline{E})} [y] \in \varprojlim_{V'} \mathrm{H}_0(Y'_V, L)$$

is well-defined and (after a modification if W is trivial) belongs to the kernel $\mathrm{H}_0(Z_K, \mathscr{W})_0$ of $\mathrm{H}_0(Z_K, \mathscr{W}) \to \mathrm{H}_0(\overline{Z}_K, \mathscr{W})$. The images of $\Delta^\circ_{W,f_\infty}$ under the

9 This approach is inspired by the work of Emerton [37].

Abel–Jacobi maps $\mathrm{AJ}\colon \mathrm{H}_0(Z_K, \mathscr{W})_0 \to H^1(E, \mathrm{H}_1(\overline{Z}_K, \mathscr{W}))$ are compatible under pushforward along the tower Z_K and invariant under the $\mathrm{H}'(\mathbf{A})$-action, hence they define an element

$$P_W := \lim \mathrm{AJ}(\Delta^{\circ}_{W,-}) \in \varprojlim_K H^1(E, \mathrm{H}_1(\overline{Z}_K, \mathscr{W})^{H'(\mathbf{A})})$$

Via (1.2.3), P_W defines an $\mathrm{H}'(\mathbf{A})$-invariant functional

$$P_\Pi\colon \Pi_{H'_\infty} \to H^1(G_E, V_\Pi), \tag{1.2.4}$$

whose image should lie in $H^1_f(E, V_\Pi) \subset H^1(E, V_\Pi)$ (see Remark 6.2.2 for a stronger conjecture). We show in Proposition 6.4.7 that this is the case if $\mathbf{B}_p$ is split and Π is *ordinary* and not exceptional, which we define to mean that $\mathbf{B}_p$ is split and the Jacquet–Langlands transfer Π_0 of Π to $\mathrm{G}_0 \times \mathrm{H}$ (which is thus the 'identity' at p) satisfies those properties.

Our formula will give a criterion for the nonvanishing of P_Π.

1.2.5 Multiplicity one

Representation theory provides a necessary condition. The space

$$(\Pi)^{*,\mathrm{H}'(\mathbf{A})} = \mathrm{Hom}_{\,\mathrm{H}'(\mathbf{A})}(\Pi, L)$$

is known, by a theorem of Waldspurger, Tunnell, and Saito [90,102], to be nonzero if and only if the following condition is satisfied for all v:

(ε_v) Define $\varepsilon(\mathbf{B}_v) := +1$ (respectively -1) if $\mathbf{B}_v$ is split (respectively nonsplit). Let $\varepsilon(V_v) := \prod_{w|v} \varepsilon(V_w)$, $\chi_v(-1) := \prod_{w|v} \chi_w(-1)$; then

$$\varepsilon_v^{\mathrm{G}}(V) := \varepsilon(V_v)\chi_v(-1)\eta_v(-1)\varepsilon(\mathbf{B}_v) = +1. \tag{1.2.5}$$

If this is the case, $(\Pi)^{*,\mathrm{H}'(\mathbf{A})}$ is 1-dimensional and moreover the global root number $\varepsilon(V) = -1$. Conversely, if V is as in Theorem A and in particular satisfies $\varepsilon(V) = -1$, there exists a unique incoherent totally definite quaternion algebra $\mathbf{B}$ verifying (ε_v).

The conditions (ε_v) for a finite v generalise the classical "Heegner condition". For $v|p$, if π is ordinary the condition (ε_v) is satisfied unless v is nonsplit in E and π is exceptional at v (Lemma 6.4.6). The condition (ε_∞) is equivalent to (wt).

Definition 1.2.1 We say that Π is *locally distinguished by* H', or simply *locally distinguished*, if it satisfies conditions (ε_v) for all v.

1.2.6 Local toric periods

Assume that Π is locally distinguished, and let $\Pi^\vee$ denote the contragredient representation of Π. Then we know an explicit a generator of

$$\Pi^{*,\mathrm{H}'(\mathbf{A})} \otimes (\Pi^\vee)^{*,\mathrm{H}'(\mathbf{A})} \tag{1.2.6}$$

as a product of local pairings, which we now define. The pair $P_\Pi \otimes P_{\Pi^\vee}$ will be measured against this generator.

For v a finite place of F, let Π_v be the local component of Π, a representation of $(\mathbf{B}_v^\times \times E_v^\times)/F_v^\times \supset H_v' := E_v^\times/F_v^\times$; let dt_v be a Haar measure on H_v'. For $v = \infty$, let $\Pi_\infty = W$ and let dt_∞ be a formal symbol synonymous with a constant $\mathrm{vol}(H'_\infty, dt_\infty) \in L$. In all cases, let $\Pi_v^{*,H_v'} := \mathrm{Hom}_{H_v'}(\Pi_v, L)$ and let $(\ ,\ )_v$ be an invariant pairing on $\Pi_v \otimes \Pi_v^\vee$.

Let V_v (respectively $V_{\pi,v}$) be the restriction to $G_{E_v} := \prod_{w|v} G_{E,w}$ (respectively G_{F_v}) of the Galois representation associated with Π (respectively π) if v is finite, and the Hodge structure associated with W (reps. W_{G}) if $v = \infty$. Let us also introduce the convenient notation

$$\text{“}V_{(\pi,\chi),v} := (V_{\pi,v} \otimes \mathrm{Ind}_{F_v}^{E_v} \chi_v) \ominus \mathrm{ad}(V_\pi)(1)\text{”}$$

(to be thought of as referring to a ‘virtual motive’).

Let $\eta\colon F_{\mathbf{A}}^\times/F^\times \to \{\pm 1\}$ be the character associated with E/F, and let

$$\mathscr{L}(V_{(\pi,\chi),v}, 0) := \frac{\zeta_{F,v}(2) L(V_v, 0)}{L(1, \eta_v) L(\mathrm{ad}(V_{\pi,v}), 1)} \cdot \begin{cases} 1 & \text{if } v \text{ is finite} \\ \pi^{-[F:\mathbf{Q}]} & \text{if } v = \infty \end{cases} \in L. \tag{1.2.7}$$

Then

$$Q_{v,(,)_v,dt_v}(f_{1,v}, f_{2,v}) := \mathscr{L}(V_{(\pi,\chi),v}, 0)^{-1} \int_{H_v'} (\Pi_v(t) f_{1,v}, f_{2,v})_v \, dt_v$$

is an explicit generator of $\Pi_v^{*,H_v'} \otimes_L (\Pi_v^\vee)^{*,H_v'}$. Here for $v \nmid \infty$ the integral is absolutely convergent (after making any choice of $L \hookrightarrow \mathbf{C}$), and for $v = \infty$ we understand

$$\int_{H'_\infty} \Pi_\infty(t)dt_\infty := \mathrm{vol}(H'_\infty, dt_\infty)\cdot \mathrm{p}_{H'_\infty}\colon W \to W_{H'_\infty} = W^{H'_\infty},$$

where $\mathrm{p}_{H'_\infty}$ is the natural projection.

Given $f_{3,v}, f_{4,v} \in \Pi_v \otimes \Pi_v^\vee$ such that $(f_{3,v}, f_{4,v})_v \neq 0$, the quantity

$$Q_{v,dt_v}\left(\frac{f_{1,v}\otimes f_{2,v}}{f_{3,v}\otimes f_{4,v}}\right) := \frac{Q_{v,(,)_v,dt_v}(f_{1,v}, f_{2,v})}{(f_{3,v}, f_{4,v})_v} \tag{1.2.8}$$

is independent of the choice of $(\ ,\ )_v$; it equals $\mathrm{vol}(\mathscr{O}_{E,v}^\times/\mathscr{O}_{F,v}^\times, dt_v)$ if all the data are unramified.

Fix a choice of measures dt_v such that for $dt = \prod_v dt_v$,

$$\mathrm{vol}(\mathrm{H}'(\mathbf{Q})\backslash \mathrm{H}'(\mathbf{A}), dt) := \mathrm{vol}(\mathrm{H}'(\mathbf{Q})\backslash \mathrm{H}'(\mathbf{A}^\infty), \textstyle\prod_{v\nmid\infty} dt_v)\cdot \mathrm{vol}(H'_\infty, dt_\infty) = 1. \tag{1.2.9}$$

Then we define for $f_1 \in \Pi_{H'}$, $f_2 \in \Pi_{H'}^\vee$, $f_3 \in \Pi$, $f_4 \in \Pi^\vee$ such that $\prod_v (f_{3,v}, f_{4,v})_v \neq 0$:

$$Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right) := \prod_v Q_{v,dt_v}\left(\frac{f_{1,v}\otimes f_{2,v}}{f_{3,v}\otimes f_{4,v}}\right).$$

1.2.7 Global pairings and p-adic heights

Let $V^\iota := V_{\Pi^\vee}$. Fix a Galois-equivariant pairing

$$V \otimes V^\iota \to L(1). \tag{1.2.10}$$

Poincaré duality provides a canonical Galois- and Hecke- equivariant pairing $\mathrm{H}_1(\overline{Z}_K, \mathscr{W}) \otimes \mathrm{H}_1(\overline{Z}_K, \mathscr{W}^\vee) \to L(1)$. Via (1.2.3) and (1.2.10), it induces dual pairings $(\ ,\ )_\Pi^K\colon \Pi^K \otimes \Pi^{\vee,K} \to L$ for all K. Letting L_K be the Hodge bundle on Z_K, the following pairing ((4.1.7) in the text) is well defined:

$$(\ ,\ )_\Pi := \lim_K\, (\dim W \cdot \deg(L_K))^{-1} \cdot (\ ,\ )_{\Pi^K}\colon \Pi \otimes \Pi^\vee \to L.$$

On the other hand, if π is ordinary the restriction V_w of V to G_{E_w}, $w|p$, is reducible

$$0 \to V_w^+ \to V_w \to V_w^- \to 0, \tag{1.2.11}$$

and there is an analogous reduction for V^ι such that V_w^+ and $V_w^{\iota,+}$ are exact orthogonal of each other under (1.2.10). These data allow to define a height pairing

$$h_V\colon \widetilde{H}^1_f(E,V)\otimes \widetilde{H}^1_f(E,V^\iota)\to L\hat{\otimes}\Gamma_F \tag{1.2.12}$$

on Nekovář's Selmer groups as in Proposition 5.3.3. When W is trivial, the representation $V = V_{\mathfrak{p}}A_E \otimes \chi$ is a factor of the Tate module of an abelian variety, and (under (n-exc)) the pairing h_V coincides with all other p-adic height pairings on abelian varieties defined in the literature: see [32] for a review.

1.2.8 The formula

We can now state the p-adic Gross–Zagier formula for V.

Theorem B *Let $\Pi = \pi\otimes\chi$ be an ordinary, locally distinguished, non-exceptional automorphic representation of* $(\mathrm{G}\times\mathrm{H})'(\mathbf{A})$ *over L. Let $V = V_\Pi$.*

The image of P_Π lies in $H^1_f(E,V)$, and for all $f_1\in\Pi_{H'_\infty}$, $f_2\in\Pi^\vee_{H'_\infty}$, $f_3\in\Pi$, $f_4\in\Pi^\vee$ such that $(f_3,f_4)_\Pi\neq 0$, we have

$$\frac{h_V(P_\Pi(f_1),P_{\Pi^\vee}(f_2))}{(f_3,f_4)_\Pi} = e_{p\infty}(V_{(\pi,\chi)})^{-1}\cdot \mathscr{L}'_p(V_{(\pi,\chi)},0)\cdot Q\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right),$$

where $e_{p\infty}(V_{(\pi,\chi)})\in L^\times$ is the p-interpolation factor for $\mathscr{L}_p(V_{(\pi,\chi)},s)$ defined in (1.4.6) *below.*

When $\mathrm{G} = \mathrm{GL}_{2/\mathbf{Q}}$, V is crystalline at p, p splits in E, χ is unramified and the f_i are newforms, a version of this result was proved by Perrin-Riou [84] when W is trivial, and by Nekovář [78] and Shnidman [96] when W has even weights. The general case with trivial W was proved in [32,35].

Remark 1.2.2 Establishing Gross–Zagier formulas in this generality has proven useful for arithmetic applications, such as those in [14,15,100] and Theorem F below.

Explicit versions of the formula can be obtained by evaluating the functional Q at well-chosen f_i. This is a local problem, solved in [18].

Remark 1.2.3 For a variant of Theorem B that is valid in the exceptional case as well, see Theorem B$^{\mathrm{ord}}$. That variant is often trivially $0=0$ in the exceptional case, but not always, and indeed Remark 7.3.4 sketches a new proof of the Greenberg–Stevens theorem [43] based on it. For a further discussion going beyond any trivial or non-trivial vanishing, see Remark 1.3.3 and Sect. 7.3.

1.3 The universal Heegner classes

We explain the interpolation of the Heegner cycles P_Π as Π varies over a Hida family for $(\mathrm{G}\times\mathrm{H})'$.

Suppose from now on that $\mathbf{B}_p$ is split and fix an isomorphism $\mathrm{G}_{\mathbf{Q}_p} \cong \mathrm{Res}_{F_p/\mathbf{Q}_p}\mathrm{GL}_2$, giving a model of G (hence $(\mathrm{G}\times\mathrm{H})'$) over $\mathbf{Z}_{(p)}$. We let $N_{\mathrm{G},0} := \begin{pmatrix} 1 & \mathscr{O}_{F,p} \\ & 1 \end{pmatrix} \subset \mathrm{G}(\mathbf{Q}_p)$ and N_0 be the image of $N_{\mathrm{G},0}$ in $(G \times H)'_p$. Finally we denote by U_p the usual operator in the Iwahori–Hecke algebra of $(G \times H)'_p$, and by $\mathrm{U}_{p\infty}$ its product with $(\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), 1) \in (G \times H)'_\infty$.

For a localisation M of a finite $\mathbf{Z}_p$-module M° on which the operator $\mathrm{U}_{p\infty}$ acts (on the left or the right), we denote by $\mathrm{M}^{\mathrm{ord}}$ the image of M under Hida's ordinary projector

$$e^{\mathrm{ord}} = \lim \mathrm{U}_{p\infty}^{n!}.$$

1.3.1 Hida families for $(\mathrm{G} \times \mathrm{H})'$

Pick an arbitrary $(\mathrm{G} \times \mathrm{H})'(\mathbf{Z}_p)$-stable lattice $W^\circ \subset W$, yieding a sub-local system $\mathscr{W}^\circ \subset \mathscr{W}$. Then we define, for any $K = K^p K_p$ with $K_p \supset N_0$,

$$M^\circ_{W,K} := (H^1(\overline{Z}_K, \mathscr{W}^\circ)\otimes(W^{\circ,\vee})_{N_0})^{\mathrm{ord}}, \quad M_{W,K} := M^\circ_{W,K}\otimes_{\mathscr{O}_L} L. \quad (1.3.1)$$

Let $K^p \subset (\mathrm{G} \times \mathrm{H})'(\mathbf{A}^{p\infty})$ be an open compact subgroup. Consider the ordinary completed homology of $\overline{Z}_{K^p}$

$$M_{K^p} := (\varprojlim_{K_p \supset N_0} M^\circ_{K^p K_p}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,$$

where $M^\circ_K =$(1.3.1) with W the trivial representation, and the limit is over K such that $K_p \supset N_0$ ("level $\Gamma^1_1(p^\infty)$"). By the work of Hida, M_{K^p} is a finite flat module over a certain weight algebra $\Lambda = \Lambda_{K^p} \simeq \mathbf{Q}_p[\Delta] \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![T_1, \ldots, T_{2[F:\mathbf{Q}]+1+\delta_{F,p}}]\!]$ where Δ is a finite group and $\delta_{F,p}$ is the Leopoldt defect of F.

Let $\mathbf{T}^{\mathrm{sph,ord}}_{K^p,\mathbf{Q}_p} \subset \mathrm{End}_\Lambda(M_{K^p})$ be the image of the algebra generated by the spherical Hecke operators and the operators U_v, $v|p$. The 'ordinary eigenvariety'

$$\mathscr{E}^{\mathrm{ord}} = \mathscr{E}^{\mathrm{ord}}_{K^p} := \mathrm{Spec}\, \mathbf{T}^{\mathrm{sph,ord}}_{K^p,\mathbf{Q}_p}$$

contains a dense subset $\mathscr{E}^{\mathrm{ord,cl}}$ (more precisely a reduced 0-dimensional ind-subscheme) of regular points, in bijection with the set of $G_{\mathbf{Q}_p}$-orbits of those ordinary automorphic representations Π of $(\mathrm{G}\times\mathrm{H})'$ over $\mathbf{Q}_p$ such that $\Pi^{K^p} \neq 0$.

Let us fix an irreducible component

$$\mathscr{X} \subset \mathscr{E}^{\mathrm{ord}}_{K^p}$$

that is a *Hida family* for $(\mathrm{G}\times\mathrm{H})'$. We let $\mathscr{X}^{\mathrm{cl}} := \mathscr{X}\cap\mathscr{E}^{\mathrm{ord,cl}}$.

Definition 1.3.1 A Hida family $\mathscr{X}$ for $(\mathrm{G}\times\mathrm{H})'$ is said to be *locally distinguished* (by H') if it satisfies the conditions

$(\varepsilon_v)'$ for every (equivalently,[10] one) classical point $z\in\mathscr{X}$ (of weight satisfying (wt)), the Galois representation $\mathscr{V}_z$ attached to the representation Π_z satisfies (ε_v)

for all $v\nmid p\infty$.

1.3.2 Sheaves on $\mathscr{X}$

The Hida family $\mathscr{X}$ comes with a coherent sheaf $\mathscr{M}_{K^p}$ corresponding to M_{K^p}; moreover in fact for each $K^{p\prime}\subset K^p$ the module $M_{K^{p\prime}}$ gives rise to a coherent $\mathscr{O}_{\mathscr{X}}$-module

$$\mathscr{M}_{K^{p\prime}}$$

with $\mathscr{O}_{\mathscr{X}}$-linear Hecke- and Galois actions. Fix an arbitrary $K^{p\prime}\subset K^p$, 'sufficiently large' at the places in Σ.[11] Let S be a finite set of primes, not containing those above p, such that all data G, H, $K^{p\prime}$ are unramified outside Sp. Let $G_{E,Sp}$ be the Galois group of the maximal extension of E unramified outside Sp. We prove in the text that the following statements are true up to replacing $\mathscr{X}$ by an open subset containing $\mathscr{X}^{\mathrm{cl}}$:

- there exists a locally free sheaf $\mathscr{V}$ of rank 2 with a $G_{E,Sp}$-action, such that for all $z\in\mathscr{X}^{\mathrm{cl}}$, the representation $\mathscr{V}_{|z}$ is associated with Π_z via the Langlands correspondence;
- for each $w|p$ there is an exact sequence of $\mathscr{O}_{\mathscr{X}}[G_{E,w}]$-modules

$$0\to\mathscr{V}_w^+\to\mathscr{V}_w\to\mathscr{V}_w^-\to 0, \tag{1.3.2}$$

  where the $\mathscr{V}_w^\pm$ are line bundles over $\mathscr{X}$, specialising to (1.2.11) at all $z\in\mathscr{X}^{\mathrm{cl}}$;
- assume from now on that $\mathscr{X}$ is locally distinguished. There is a locally free $\mathscr{O}_{\mathscr{X}}$-module

$$\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$$

  interpolating the spaces of $(E_\Sigma^\times/F_\Sigma^\times)$-coinvariants, $K^{p\prime}$-invariants of Π_z^{ord} for $z\in\mathscr{X}^{\mathrm{cl}}$;

10 By [34, Corollary 5.3.3].

11 In the sense that for each $z\in\mathscr{X}^{\mathrm{cl}}$, $v\in\Sigma$, the finite-dimensional constituent $\Pi_{z,v}$ of Π_z is fixed by K_v.

– we have a map of Hecke modules over $\mathscr{O}_{\mathscr{X}}$

$$\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \to \mathrm{Hom}_{\mathscr{O}_{\mathscr{X}}[G_{E,Sp}]}(\mathscr{M}_{K^{p\prime}}^{H'_\Sigma}, \mathscr{V}) \tag{1.3.3}$$

whose specialisations over $\mathscr{X}^{\mathrm{cl}}$ are deduced by (1.2.3).

1.3.3 The universal Heegner class

We construct in the appendix (Proposition A.2.4) an operator $\gamma_{H'}^{\mathrm{ord}}$, that is the key to the interpolation of Heegner cycles. It is a limit of of Hecke operators at $p\infty$, intertwining toric and ordinary parts:

$$\mathrm{H}_1(\overline{Z}_{K^p}, \mathscr{W})^{H'} \xrightarrow{\cdot\gamma_{H'}^{\mathrm{ord}}} \mathrm{H}_1(\overline{Z}_{K^p}, \mathscr{W})^{\mathrm{ord}} = M_{W,K^p}^{\mathrm{ord}}$$

$$\Pi_{H'}^{K^p} \xleftarrow{\gamma_{H'}^{\mathrm{ord}}\cdot} \Pi^{K^p,\mathrm{ord}}.$$

Consider the class

$$P_{W,K^{p\prime}}^{\mathrm{ord}} := P_{W,K^{p\prime}}\,\gamma_{H'}^{\mathrm{ord}} \in H^1(G_{E,Sp}, M_{W,K^{p\prime}}).$$

It is invariant under $\mathrm{H}'(\mathbf{A}^{p\infty})$, hence:

– as $K^{p\prime}$ varies, it defines an $\mathrm{H}'(\mathbf{A}^{p\infty})$-invariant functional

$$P_{\Pi}^{\mathrm{ord}} = P_W \circ \gamma_{H'}^{\mathrm{ord}} \colon \Pi^{\mathrm{ord}} \to H^1(E, V_\Pi) \tag{1.3.4}$$

and in fact, as we shall prove, valued in $H^1_f(E, V_\Pi)$.

– restricting (without loss of generality as we will see in a moment) to the case where W is trivial, its localisation over $\mathscr{X}$ defines a global section $\mathscr{P}_{K^{p\prime}}$ of $H^1(G_{E,Sp}, \mathscr{M}_{K^{p\prime}}^{H'_\Sigma})$.

Using Nekovář's theory of Selmer complexes we show that the universal class $\mathscr{P}_{K^{p\prime}}$ is a section of a sheaf of Selmer groups $\widetilde{H}^1_f(E, \mathscr{M}_{K^{p\prime}}^{H'_\Sigma})$, where the subscript f signifies a local condition at p coming from (1.3.2), and for Selmer groups we use E in place of $G_{E,Sp}$ for short. Then by (1.3.3) the class $\mathscr{P}_{K^{p\prime}}$ defines a map of $\mathscr{O}_{\mathscr{X}}$-modules

$$\mathscr{P}_{K^{p\prime}} \colon \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \to \widetilde{H}^1_f(E, \mathscr{V}).$$

When $\mathrm{G} = \mathrm{GL}_{2/\mathbf{Q}}$, the value of $\mathscr{P}_{K^{p\prime}}$ on a family of newforms is the class originally defined by Howard in [58]. (The statement that the fibre of $\mathscr{P}_{K^{p\prime}}$ at *all* classical points lands in the Selmer group is in new even in the context of

[58].) There, Howard asked whether his class interpolates Heegner cycles at all classical points of $\mathscr{X}$. The first part of the following theorem summarises the results described above. The second part, whose proof is simple and direct, provides an affirmative answer to the generalisation of Howard's question.[12]

Theorem C *Let $\mathscr{X}$ be a locally distinguished Hida family for $(\mathrm{G}\times\mathrm{H})'$. There exist an open subset $\mathscr{X}'\subset\mathscr{X}$ containing $\mathscr{X}^{\mathrm{cl}}$ and a map*

$$\mathscr{P}_{K^{p\prime}}\colon \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}\to H^1_f(G_{E,Sp},\mathscr{V})$$

of sheaves over $\mathscr{X}'$, satisfying the following properties:

1. *$\mathscr{P}_{K^{p\prime}}$ is invariant under the action of the away-from-$p\Sigma$-Hecke algebra of H';*
2. *for all $z\in\mathscr{X}^{\mathrm{cl}}$ corresponding to a representation Π_z satisfying (wt), denote by $P^{\mathrm{ord}}_{\Pi_z,K^{p\prime}}$ the restriction of (1.3.4) to $(\Pi_z)_{H'_\Sigma}^{K'_p,\mathrm{ord}}$; then*

$$\mathscr{P}_{K^{p\prime}|z}=P^{\mathrm{ord}}_{\Pi_z,K^{p\prime}}$$

under the natural map $H^1(G_{E,Sp},\mathscr{V})_{|z}\to H^1(G_{E,Sp},V_{\Pi_z})$.

An answer to Howard's question in its original context was earlier given by Castella [22,23] by an indirect method, under the assumption that p splits in E.

Remark 1.3.2 It follows from the results of [26] that, under mild conditions, the class $\mathscr{P}$ is non-torsion over $\mathscr{X}$, cf. the discussion after [41, Theorem B].

Remark 1.3.3 Theorem C is far from being the last word on $\mathscr{P}$: first, the class $\mathscr{P}$ may vanish at some classical points; second, we can consider its specialisation in Nekovář's Selmer group $\widetilde{H}^1_f(E,V_{\Pi_z})$, which equals $H^1_f(E,V_{\Pi_z})$ when z is not exceptional but is larger otherwise. In Sect. 7.3, we address both problems by proposing a conjecture for the order of vanishing and leading term of $\mathscr{P}$ at any classical point, generalising conjectures by Bertolini–Darmon. The same Conjecture Pf will also give a prediction for the leading terms of universal toric periods on distinguished Hida families for *coherent* quaternionic groups, discussed in Sect. 7.2, and in that case we will describe some new evidence in higher rank coming from the 'plectic' world via [39].

[12] The question in [58] was phrased in terms of the Abel–Jacobi classes of Heegner cycles in a suitable Chow group, defined in that case in [78]; these classes are identical to the $P_\Pi(f)$ from (1.2.4): see [78, § I.2].

1.4 The universal formula

We first recall the p-adic L-function constructed in [36], then state our formula for the p-adic height of $\mathscr{P}_{K^{p\prime}}$.

At times we refer to the main body of the paper for the precise definition of some of the objects.

1.4.1 Dualities over Hida families

The space $\mathscr{E}^{\mathrm{ord}}$ is endowed with an involution ι corresponding to $\Pi_z \mapsto \Pi_z^\vee$. Fix a locally distinguished Hida family $\mathscr{X}$; then the constructions of Sect. 1.3 can be performed over $\mathscr{X}$. Denoting by $(-)^\iota$ the pullback under ι of an object over $\mathscr{X}$, we have dualities

$$\mathscr{V} \otimes \mathscr{V}^\iota \to \mathscr{O}_{\mathscr{X}}(1) \tag{1.4.1}$$

interpolating (1.2.10). These data, together with their deformation to a Hida family $\mathscr{X}^\sharp$ for $\mathrm{G} \times \mathrm{H}$, allow to define a height pairing as in Proposition 5.3.4,

$$h_{\mathscr{V}/\mathscr{V}^\sharp}\colon \widetilde{H}^1_f(E, \mathscr{V}) \otimes_{\mathscr{O}_{\mathscr{X}}} \widetilde{H}^1_f(E, \mathscr{V}^\iota) \to \mathscr{N}^*_{\mathscr{X}/\mathscr{X}^\sharp} \cong \mathscr{O}_{\mathscr{X}} \hat{\otimes} \Gamma_F. \tag{1.4.2}$$

As usual after possibly restricting to an open subset containing $\mathscr{X}^{\mathrm{cl}}$, we construct:

- pairings

$$((\ ,\ ))\colon \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \otimes_{\mathscr{O}_{\mathscr{X}}} (\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})^\iota \to \mathscr{O}_{\mathscr{X}}$$

  interpolating the $p\infty$-modification $(\ ,\ )^{\mathrm{ord}}_\Pi :=$(4.1.8) of $(,)_\Pi$;
- $\mathscr{O}_{\mathscr{X}}^\times$-module maps

$$\mathscr{Q}\colon (\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \otimes_{\mathscr{O}_{\mathscr{X}}^\times} \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord},\iota}) \otimes_{\mathscr{O}_{\mathscr{X}}^\times} (\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \otimes_{\mathscr{O}_{\mathscr{X}}^\times} \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord},\iota})^{\times,-1} \to \mathscr{K}_{\mathscr{X}}$$

  interpolating the $p\infty$-modification $Q^{\mathrm{ord}} =$ (4.3.3) of Q. Here, $\mathscr{K}_{\mathscr{X}}$ is the sheaf of fractions of $\mathscr{O}_{\mathscr{X}}$ and the superscript '$\times, -1$' denotes the subgroup of those $f_3 \otimes f_4$ satisfying $((f_3, f_4)) \neq 0$ and suggests the 'denominator' invariance of the pairing in the last two variables.

1.4.2 The p-adic L-function

Let $\mathscr{E}_0^{\sharp,\mathrm{ord}} := \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_0\times\mathrm{H}}$ be the ordinary eigenvariety for $\mathrm{G}_0 \times \mathrm{H}$ (see [50,51]); for appropriate choices of tame levels, there is a map $\iota_{\mathrm{JL}}\colon \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}\times\mathrm{H}} \to \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_0\times\mathrm{H}}$, which is a closed immersion onto a union of irreducible components. Let $\mathscr{X}_0^\sharp := \iota_{\mathrm{JL}}(\mathscr{X}^\sharp) \subset \mathscr{E}^{\sharp,\mathrm{ord}}$. We recall the p-adic L-function on $\mathscr{X}^\sharp$ constructed in [36].

Let $\mathscr{X}_0^{\sharp,\mathrm{cl}} = \iota_{\mathrm{JL}}(\mathscr{X}^{\sharp,\mathrm{cl}}) \subset \mathscr{X}_0^\sharp$ be the ind-scheme of classical points. If $(x, y) \in \mathscr{X}_0^{\sharp,\mathrm{cl}}(\mathbf{C})$ is a geometric point corresponding to a closed point $(x_0, y_0) \in \mathscr{X}_0^{\sharp,\mathrm{cl}}$ together with an embedding $\iota\colon \mathbf{Q}_p(x_0, y_0) \hookrightarrow \mathbf{C}$, we denote $\pi_x = \pi_{x_0}^\iota, \chi_y = \chi_{y_0}^\iota$, which are complex automorphic representations of $\mathrm{G}_0(\mathbf{A})$ and $\mathrm{H}(\mathbf{A})$ respectively. We then denote $V^\sharp_{(x_0,y_0)} := V_{(\pi_{x_0},\chi_{y_0})}$ and let

$$\mathscr{L}(V^\sharp_{(x,y)}, 0) = \prod_v \iota\mathscr{L}(V^\sharp_{(x,y),v}, 0)$$

be the product (defined by analytic continuation) of *all* of the factors (1.2.7).

Recall that if W is a complex Weil–Deligne representation of the Weil group of a local field F_v and $\psi_v\colon F_v \to \mathbf{C}^\times$ is a nontrivial character, the inverse Deligne–Langlands γ-factor is[13]

$$\gamma(W, \psi_v)^{-1} = L(W)/\varepsilon(W, \psi_v)L(W^*(1)), \tag{1.4.3}$$

and $\psi_{E,w} = \psi_v \circ \mathrm{Tr}_{E_w/F_v}$.

If $\pi = \pi_{x_0}$, $\chi = \chi_{y_0}$ are as just above (with weights $\underline{w} = \underline{w}_{x_0}$, $\underline{l} = \underline{l}_{y_0}$), let $\mathrm{ad}(V_{\pi,v})(1)^{++} := \mathrm{Hom}\,(V^-_{\pi,v}, V^+_{\pi,v})$. Let $\psi = \prod_v \psi_v\colon F\backslash\mathbf{A}_F \to \mathbf{C}^\times$ be the standard additive character such that $\psi_\infty(\cdot) = e^{2\pi i \mathrm{Tr}_{F_\infty/\mathbf{R}}(\cdot)}$; let $\psi_E = \prod_w \psi_{E,w} = \psi \circ \mathrm{Tr}_{\mathbf{A}_E/\mathbf{A}_F}$. For a place $v|p$ of F, let d_v be a generator of the different ideal of F_v, and define

$$\begin{aligned} e_v(V_{(\pi_x,\chi_y)}) = |d_v|^{-1/2} \frac{\prod_{w|v} \gamma(\iota\mathrm{WD}(V^+_{\pi,v|G_{E,w}} \otimes V_{\chi,w}), \psi_{E,w})^{-1}}{\gamma(\iota\mathrm{WD}(\mathrm{ad}(V_{\pi,v})(1))^{++}, \psi_v)^{-1}} \\ \cdot \mathscr{L}(V_{(\pi^\iota,\chi^\iota),v})^{-1}, \end{aligned} \tag{1.4.4}$$

where $\iota\mathrm{WD}$ is the functor from potentially semistable Galois representations to complex Weil–Deligne representations of [38]. Finally, we define

$$\begin{aligned} e_\infty(V_{(\pi^\iota,\chi^\iota)}) &:= i^{(w+l)[F:\mathbf{Q}]}, \\ e_{p\infty}(V_{(\pi^\iota,\chi^\iota)}) &:= e_\infty(V_{(\pi^\iota,\chi^\iota)}) \cdot \prod_{v|p} e_v(V_{(\pi^\iota,\chi^\iota)}). \end{aligned} \tag{1.4.5}$$

[13] The normalisations of L- and ε-factors are as in [98].

At least if $w+l=0$, these belong to $\iota\mathbf{Q}_p(x_0,y_0)$, and we may define

$$e_{p\infty}(V_{(\pi,\chi)}) := \iota^{-1} e_{p\infty}(V_{(\pi^\iota,\chi^\iota)}). \tag{1.4.6}$$

The following is the main theorem of [36].

Theorem 1.4.1 *There exists a meromorphic function*

$$\mathscr{L}_p(\mathscr{V}^\sharp) \in \mathscr{K}(\mathscr{X}_0^\sharp)$$

whose polar locus $\mathscr{D}$ does not intersect $\mathscr{X}_0^{\rm cl}$, uniquely characterised by the following property.

For each $z=(x,y)\in\mathscr{X}_0^{\sharp,\rm cl}(\mathbf{C})-\mathscr{D}(\mathbf{C})$ corresponding to an automorphic representation $\pi_x\otimes\chi_y$ of $\mathrm{G}_0(\mathbf{A})\times\mathrm{H}(\mathbf{A})$ of weight $(\underline{w}_x,\underline{l}_y)$ satisfying the conditions

$$|l_{y,\sigma}| < w_{x,\sigma}, \quad |w_x+l_y| \le w_{x,\sigma}-|l_{y,\sigma}|-2 \ \ \textit{for all } \sigma\colon F\hookrightarrow\mathbf{C},$$

we have

$$\mathscr{L}_p(\mathscr{V})(x,y) = e_{p\infty}(V_{(\pi_x,\chi_y)})\cdot\mathscr{L}(V_{(\pi_x,\chi_y)},0). \tag{1.4.7}$$

1.4.3 Main theorem

Under the condition of local distinction of $\mathscr{X}$, the function $\mathscr{L}_p(\mathscr{V}^\sharp)$ vanishes identically on $\mathscr{X}_0$. Let $\mathscr{N}^*_{\mathscr{X}_0/\mathscr{X}_0^\sharp} = \mathscr{I}_{\mathscr{X}_0}/\mathscr{I}^2_{\mathscr{X}_0}\otimes_{\mathscr{O}_{\mathscr{X}_0^\sharp}}\mathscr{O}_{\mathscr{X}_0}\cong\mathscr{O}_{\mathscr{X}_0}\hat{\otimes}\Gamma_F$ be the conormal sheaf and let

$$\mathrm{d}^\sharp\mathscr{L}_p(\mathscr{V}) := \mathrm{d}_{\mathscr{X}_0/\mathscr{X}_0^\sharp}\mathscr{L}_p(\mathscr{V}^\sharp) \quad \in\mathscr{K}(\mathscr{X}_0)\hat{\otimes}\Gamma_F = \mathscr{K}(\mathscr{X})\hat{\otimes}\Gamma_F$$

be the image of $\mathscr{L}_p(\mathscr{V}^\sharp)$.

Theorem D *Let $\mathscr{X}$ be a locally distinguished Hida family for* $(\mathrm{G}\times\mathrm{H})'$. *Abbreviate $\Pi^{(\iota)} := \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord},(\iota)}$, $\mathscr{O} := \mathscr{O}_{\mathscr{X}}$, $\mathscr{K} := \mathscr{K}_{\mathscr{X}}$.*

Then there is an open subset $\mathscr{X}'\subset\mathscr{X}$ containing $\mathscr{X}^{\rm cl}$ such that all of the above constructions can be made over $\mathscr{X}'$, and

$$\frac{h_{\mathscr{V}/\mathscr{V}^\sharp}(\mathscr{P}(f_1),\mathscr{P}^\iota(f_2))}{((f_3,f_4))} = \mathrm{d}^\sharp\mathscr{L}_p(\mathscr{V}^\sharp)\cdot\mathscr{Q}\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right),$$

an equality of $\mathscr{K}\hat{\otimes}_{\mathbf{Z}_p}\Gamma_F$-valued $\mathscr{O}$-linear functionals on $(\Pi\otimes_{\mathscr{O}}\Pi^\iota)\otimes_{\mathscr{O}^\times}(\Pi\otimes_{\mathscr{O}}\Pi^\iota)^{\times,-1}$.

The formula of the theorem in fact also holds at exceptional points $z \in \mathscr{X}^{\mathrm{cl}}$, see Theorem B $^{\mathrm{ord}}$.

1.5 Applications

We turn to some arithmetic applications of the main theorems (in addition to Theorem A).

1.5.1 On the Iwasawa Main Conjecture for derivatives

We use the notation introduced after Theorem A.

Theorem E *Let $\mathscr{X}$ be a locally distinguished Hida family for* $(\mathrm{G} \times \mathrm{H})'$, *satisfying the further conditions of* [41, Theorem B.(iii)]. *Let $\mathscr{X}' \subset \mathscr{X}$ be the open subset of Theorem* D*; up to shrinking $\mathscr{X}'$ we may assume it is a regular scheme. Let $\mathscr{R} \subset \mathscr{O}_{\mathscr{X}'}\hat{\otimes}\Gamma_F$ be the regulator of the height pairing* (1.4.2) *over $\mathscr{X}'$. Then*

$$\mathrm{d}^\sharp \mathscr{L}_p(\mathscr{V}) \succeq_{\mathscr{O}_{\mathscr{X}'}} \mathscr{R} \cdot \mathrm{char}_{\mathscr{O}_{\mathscr{X}'}}(\widetilde{H}^2_f(E, \mathscr{V})_{\mathscr{O}_{\mathscr{X}'}\text{-}tors}).$$

The proof, based on Theorem D and [41, Theorem B.(iii)], is virtually identical to that of [32, Theorem D], based on Theorem C.4 *ibid.* and [41, Theorem B.(ii)].

1.5.2 Generic non-vanishing of p-adic heights for self-dual CM motives

It is conjectured that cyclotomic p-adic height pairings are non-vanishing (and even non-degenerate). Results in this direction have been quite rare. The next theorem generalises a variant of the main theorem of [14], to which we refer for a discussion of the background.

Consider the set of locally algebraic Hecke characters

$$\chi\colon E^\times \backslash E^\times_{\mathbf{A}} \to \mathbf{Q}_p(\chi)^\times.$$

satisfying the special self-duality condition

$$\chi_{|F^\times_{\mathbf{A}}} = \eta \cdot \chi_{\mathrm{cyc},F}. \tag{1.5.1}$$

This is precisely the set of classical points of the closed subspace

$$\mathscr{E}^{\mathrm{ord,sd}}_{\mathrm{H}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{H}} := \bigcup_{V^p \subset \mathrm{H}(\mathbf{A}^p)} \mathscr{E}^{\mathrm{ord}}_{\mathrm{H},V^p}$$

cut out by the condition (1.5.1) on continuous characters. The space $\mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}}$ is a torsor for $\mathscr{E}_{\mathrm{H}'}^{\mathrm{ord}}$; in particular it is smooth of dimension $[F:\mathbf{Q}]$. Let $\mathscr{Y}\subset\mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}}$ be an irreducible component; then there is a sign $\epsilon\in\{\pm 1\}$ such that for all $x\in\mathscr{Y}^{\mathrm{cl}}$, $\varepsilon(1,\chi)=\epsilon$; we then say that $\mathscr{Y}^{\mathrm{cl}}$ has type ϵ.

Denote by $h_E^-=h_E/h_F$ the relative class number of E/F and by D_F the absolute discriminant of F.

Theorem F *Let $\mathscr{Y}\subset\mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}}$ be an irreducible component of type -1. Suppose that all primes $v|p$ of F split in E, the extension E/F is ramified, and $p\nmid 2D_Fh_E^-$.*

Then, there exists a non-empty open subset $\mathscr{Y}'\subset\mathscr{Y}$ such that for all $y\in\mathscr{Y}^{\mathrm{cl}}\cap\mathscr{Y}'$, the Selmer group $H^1_f(E,\chi_y)$ is nonzero and the p-adic height pairing

$$h\colon H^1_f(E,\chi_y)\otimes H^1_f(E,\chi_y^{-1}(1))\to\mathbf{Q}_p(y)\hat{\otimes}\Gamma_F$$

is non-vanishing.

1.5.3 Non-vanishing of universal Heegner classes along some classical Hida families

Part 3 of the following theorem is also a contribution to the non-vanishing conjecture for p-adic heights. Parts 1 and 2 provide, to the best of the author's knowledge, the first piece of theoretical evidence towards conjectures of Greenberg [44] and Howard [58].

Theorem G *Let $\mathscr{X}_0$ be a Hida family for $\mathrm{PGL}_{2/\mathbf{Q}}$, and let $\mathscr{X}_0^\sharp$ be the Hida family for $\mathrm{GL}_{2/\mathbf{Q}}$ containing $\mathscr{X}_0$. Denote by $\mathscr{V}_0$, $\mathscr{V}_0^\sharp$ the associated rank-2 representations of $G_{\mathbf{Q}}$.*

Suppose that $\mathscr{X}_0$ contains a point corresponding to an elliptic curve A with split multiplicative reduction at p, satisfying $L(A,1)\neq 0$. Then:

1. *a universal Heegner class $\mathscr{P}_0$ is nonvanishing along $\mathscr{X}_0$;*
2. *the Selmer group $\widetilde{H}^1_f(\mathbf{Q},\mathscr{V}_0)$ has generic rank 1, generated by $\mathscr{P}_0$;*
3. *the p-adic height pairing $h_{\mathscr{V}_0/\mathscr{V}_0^\sharp}$ is non-vanishing.*

1.6 Outline of the proofs

The basic strategy to prove the main results is very simple. When W is trivial, Theorem B was proved in [32,35] under some technical assumptions. As the set of points of trivial weight in $\mathscr{X}^{\mathrm{cl}}$ satisfying those assumptions is still dense in $\mathscr{X}$, this suffices to deduce Theorem D once its terms are defined; by a multiplicity-one argument and an explicit local computation, this in turn

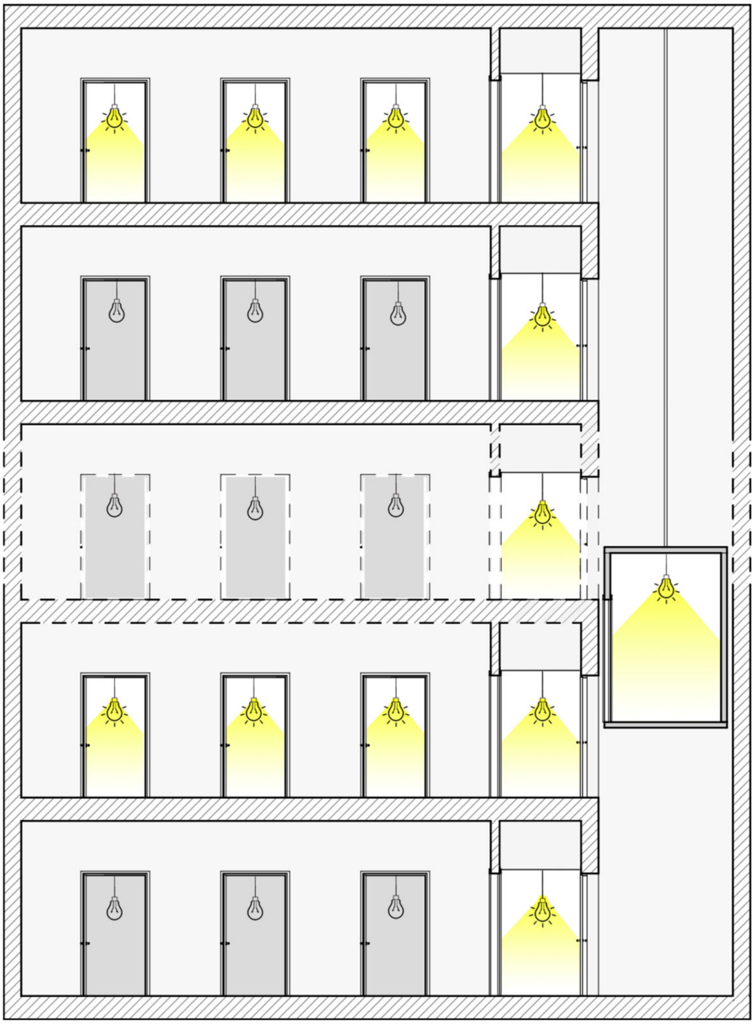

Fig. 1 An illustration of the proof of Theorems B and D. Each of the (infinitely many) floors corresponds to a representation Π as in Theorem B, each apartment to a quadruple $\mathbf{f} = (f_1, f_2, f_3, f_4)$, and the building to a Hida family. A light being on indicates that the corresponding Gross–Zagier formula is proven. On 'most' floors corresponding to a Π of trivial weight, all lights are on by [32,35]. In this paper, we construct the lift corresponding to the formula of Theorem D, with doors (interpolation statements) onto special apartments in each floor (the formulas of Theorem $\mathrm{B}^{\mathrm{ord}}$ in Sect. 7.1.1, equivalent to Theorem B for certain special quadruples $\mathbf{f}$). As soon as the lights in a dense set of floors in the building are on, the light in the lift is on; this allows to turn on the light in all the special apartments. Finally, the multiplicity-one principle allows to propagate the electricity among different apartments on the same floor

implies Theorem B for all W. Much of this work is therefore an exercise in p-adic interpolation to construct the objects of Sects. 1.3–1.4; the table of contents, and the internal references given so far, should suffice to guide the reader through the paper.

The proof of Theorem C is completed in Sect. 6.4.3 and the proof of Theorems B, D is completed in Sect. 7.1, where we also prove Theorem F. Theorem G is proved at the end of Sect. 7.3 using a known case of the conjecture made there. Constructions and calculations of a local nature are gathered in “Appendix A”.

We highlight some of the key tools we use (many have already been mentioned):

- Nekovář’s theory of Selmer complexes and p-adic heights ([81], see also [103, Appendix C]), applied to Hida theory;
- the local Langlands correspondence in families as described in [34], that is necessary for the interpolation of the terms Q_v;
- Emerton’s point of view [37] on p-adic cohomological automorphic representations as having a component at ‘infinity’ that is an algebraic representation of the relevant group; in our context, this further allows to properly consider ‘incoherent’ reductive groups;
- the multiplicity-one result for $\mathrm{H}'(\mathbf{A})$-invariant functionals;
- the definition and study of semi-local operators at $p\infty$, as the key to transitioning between ordinary and anti-ordinary or toric parts of a module;
- the explicit evaluation of certain local toric periods in terms of gamma factors.

We view the framework introduced in the appendix as the main technical novelty contributed by the present work, and we hope that the underlying approach will prove useful in many other contexts.[14]

Further directions

We have not paid attention to the integral aspects; doing so may also remove the need to restrict to open subsets of $\mathscr{X}$ at various points, e.g. by restricting to newforms or using the local Langlands correspondence in integral families of Emerton, Helm, and Moss (see references in [34]). (However, this would require imposing some residual irreducibility assumptions for the representation $\mathscr{V}_v$.) This may lead to non-vanishing results for higher-weight Heegner cycles, automorphic toric periods, and L-values: an example we have in mind is the anticyclotomic non-vanishing result of [25], based on a construction not unlike that of Theorem H.

In a different direction, all of the constructions of this paper could be generalised, with work, to the context of eigenvarieties; the Gross–Zagier formulas should also extend to that context.

[14] Cf. the work [67] discussed in Sect. 1.7 below.

1.7 Related contemporary work

After a first version of this paper was made publicly available, the following partly related works have appeared.

- In [63], the authors construct universal Heegner classes for Coleman families of elliptic modular forms (with classical restrictions); then they prove that these classes interpolate the images of Heegner cycles, by a method not dissimilar to that of the present work. Similar results are also independently proved in [83] in the ordinary case, and (by a different method) in [16] in the case where p splits in the field of complex multiplications.
- In [17], the authors use [16,63] and a strategy similar to the one of the present paper to prove the p-adic Gross–Zagier formula for critical-slope refinements of elliptic modular forms, conditionally on work in preparation of Kobayashi on such formula for small-slope refinements. Their idea is to deduce, from the latter, a p-adic Gross–Zagier formula in a Coleman family, within which the objects considered by Kobayashi form a dense subset; then specialise the formula to other classical points.
- In [67], Loeffler gives a method to construct p-adic families of cohomology classes attached to inclusions of reductive groups $\mathrm{H}_1 \subset \mathrm{H}_2$ such that $\mathrm{H}_2/\mathrm{H}_1$ is a spherical variety. His local-at-p construction vastly generalises the one of Proposition A.2.4. A difference is that in [67], the weight variation is not addressed (accordingly, that construction does not use the 'infinite' place).

1.8 Notation

Throughout the paper we use the following notation unless otherwise noted.

- $\mathbf{A}$ is the ring of adèles of $\mathbf{Q}$;
- the fields F and E are as in the introduction, $\eta = \eta_{E/F} \colon F_{\mathbf{A}}^{\times}/F^{\times} \to \{\pm 1\}$ is the associated quadratic character, and we denote by $\overline{E}$ a fixed algebraic closure of E;
- we denote by G_E the absolute Galois group of a field E; if E is a number field and S is a finite set of places, we denote by $G_{E,S}$ the Galois group of the maximal extension of E unramified outside $S\infty$;
- for a place w of a number field E, we denote by ϖ_w a fixed uniformiser at w, and by q_w the cardinality of the residue field;
- the class field theory isomorphism is normalised by sending uniformisers to geometric Frobenii; for E a number field (respectively a local field), we will then identify characters of G_E with characters of $E_{\mathbf{A}}^{\times}/E^{\times}$ (respectively $E^{\times}$) without further comment;
- let $\mu \subset \mathbf{Q}_p^{\times}$ be the subgroup of roots of unity, and let $\langle\cdot\rangle_p \colon \mathbf{Q}_p^{\times} \to 1 + 2p\mathbf{Z}_p \subset \mathbf{Q}_p^{\times}$ be the unique continuous character such that $x_p\langle x\rangle_p^{-1}$ has

values in μ. The p-adic cyclotomic character of $\mathbf{Q}$ is

$$\chi_{\mathrm{cyc},\mathbf{Q}}(x) := |x|_{\mathbf{A}^\infty}\langle x_p\rangle_p,$$

a character on $\mathbf{A}^{\infty,\times}/\mathbf{Q}^\times$. If E is a number field, the p-adic cyclotomic character of E is the character

$$\chi_{\mathrm{cyc},E} = \chi_{\mathrm{cyc},\mathbf{Q}} \circ N_{E/\mathbf{Q}}\colon E^\times_{\mathbf{A}^\infty}/E^\times \to \mathbf{Q}_p^\times. \tag{1.8.1}$$

2 Automorphic and Galois representations

In this section we define the basic set up regarding ordinary automorphic representations for our groups, and the associated Galois representations.

2.1 Groups

We introduce our notation on groups and related objects.

2.1.1 Incoherent reductive groups

Let F be a global field. For the purposes of this discussion, a 'coherent' reductive group over F is just a reductive algebraic group in the usual sense. The following notion is probably appropriate only in the context of orthogonal or unitary groups, cf. [47]; we do not explicitly restrict to that case just for the sake of brevity.

An *F-incoherent reductive group* G *over* F is a collection of reductive groups G_v/F_v, for v a place of F, such that for each place w of F there is a coherent reductive group $\mathrm{G}(w)/F$ that is w-nearby to G in the following sense: for each place $v \neq w$, $\mathrm{G}(w) \times_F F_v \cong \mathrm{G}_v$, and the groups $\mathrm{G}(w) \times_F F_w$ and G_w are non-isomorphic inner forms of each other.

Let F/F_0 be a finite extension of global fields. An *F-incoherent reductive group* G *over* F_0 is a collection of reductive groups $\mathrm{G}_{v_0}/F_{0,v_0}$, indexed by the places v_0 of F_0, satisfying the following. For each v_0, we may write $\mathrm{G}_{v_0} = \mathrm{Res}_{F_{v_0}/F_{0,v_0}}\mathrm{G}_{F,v_0} := \prod_{v|v_0} \mathrm{Res}_{F_v/F_{0,v_0}} G_{F,v}$ for a collection of reductive groups $G_{F,v}/F_v$ that forms an F-incoherent algebraic group G_F over F. In this situation, we write $\mathrm{G} = \mathrm{Res}_{F/F_0}\mathrm{G}_F$. We write just 'incoherent' when F is unimportant or understood from context. We also write $\mathrm{G}(F_{v_0}) := \mathrm{G}_{v_0}(F_{v_0})$ for short.

By definition, for all but finitely many v_0, the group G_{v_0} is unramified. In particular, if S is a finite set of places of F_0, it makes sense to consider the restricted tensor product $\mathrm{G}(\mathbf{A}^S) := \prod'_{v_0 \notin S} \mathrm{G}(F_{v_0})$.

It will be convenient to consider a p-adic variant in the case where $F = \mathbf{Q}$ and $\mathrm{G}_{F,\infty}$ is anisotropic modulo its centre (so that all its admissible representations are finite-dimensional). In this case we *redefine*, for any finite set of finite places S,

$$\mathrm{G}(\mathbf{A}^S) := \mathrm{G}(\mathbf{A}^{S\infty}) \times G_\infty,$$

where $\mathrm{G}_\infty := \mathrm{G}_p(\mathbf{Q}_p)$ with the Zariski topology.

The main example of interest to us is the following: F is our totally real number field, $F_0 = \mathbf{Q}$, and $\mathrm{G}_{F_v} = \mathbf{B}_v^\times$. The conditions are satisfied since, for each place w, there is a quaternion algebra $B(w)$ over F such that $\mathbf{B}_v \cong B(w)_v$ if and only if $w \neq v$. Other examples are obtained as follows: if G is an incoherent group and H is a coherent group, the product $\mathrm{G} \times \mathrm{H}$ (whose precise definition is left to the reader) is an incoherent group.

2.1.2 Hecke algebras

Let G be a coherent or incoherent reductive group over $\mathbf{Q}$, A a ring.

If S is a finite set of primes of $\mathbf{Q}$ different from p, let

$$\mathscr{H}_{\mathrm{G},A} := C_{\mathrm{c}}^\infty(\mathrm{G}(\mathbf{A}^{p\infty}), A), \quad \mathscr{H}_{\mathrm{G}}^S := C_{\mathrm{c}}^\infty(\mathrm{G}(\mathbf{A}^{Sp\infty}), A)$$

be the Hecke algebras. If $U \subset \mathrm{G}(\mathbf{A}^\infty)$ is a compact open subgroup we let $\mathscr{H}_{\mathrm{G},U,A}$ and $\mathscr{H}_{G,U,A}^S$ be the respective subalgebras of functions that are bi-U-invariant. If S is *U-spherical* in the sense that U_v is maximal for all $v \notin S$, we say that $\mathscr{H}_{\mathrm{G},U}^S$ is a *spherical* Hecke algebra.

If M is an A-module with a smooth A-module action by $\mathscr{H} = \mathscr{H}_{\mathrm{G}}$, $\mathscr{H}_{\mathrm{G}}^S$, $\mathscr{H}_{\mathrm{G},U}$, or $\mathscr{H}_{\mathrm{G},U}^S$, we let $\mathscr{H}(M) \subset \mathrm{End}_A(M)$ by the image of $\mathscr{H}$. We define the *spherical Hecke algebra* acting on M to be

$$\mathscr{H}_{\mathrm{G}}^{\mathrm{sph}} := \varprojlim_{S,U} \mathscr{H}_{\mathrm{G},U}^S(M) \subset \mathrm{End}_A(M)$$

if the limit, taken over pairs (S, U) such that S is U-spherical, stabilises. It is equipped with an involution ι coming from the involution on $\mathrm{G}_*(\mathbf{A})$.

2.1.3 Subgroups of G_*

We restrict, for the rest of this subsection, to the groups in (1.2.1), denoted collectively by G_*. Assuming that $\mathbf{B}_p$ is split, we fix an identification $G := \mathrm{G}(\mathbf{Q}_p) \cong \mathrm{GL}_2(F_p)$ for the rest of the paper, by which we obtain $\mathbf{Z}_p$-models $\mathrm{G}_{*/\mathbf{Z}_p}$ for all of the groups $\mathrm{G}_{*/\mathbf{Q}_p}$.

Let $N_{\mathrm{G}} \subset \mathrm{G}(\mathbf{Q}_p) \cong \mathrm{GL}_2(F_p)$ be the subgroup of unipotent matrices. Let $N_{\mathrm{H}} = \{1\} \subset \mathrm{H}(\mathbf{Q}_p)$, and for $? = \emptyset$ (respectively $? = {}'$), let $N_{(\mathrm{G}\times\mathrm{H})^?} := N_{\mathrm{G}} \times \{1\}$ (respectively its image in $(\mathrm{G} \times \mathrm{H})'(\mathbf{Q}_p)$). Finally, let $N_{\mathrm{G}_*,0} := N_{\mathrm{G}_*} \cap \mathrm{G}_{*/\mathbf{Z}_p}(\mathbf{Z}_p)$.

Let $T_{\mathrm{G}_*} \subset \mathrm{G}_*(\mathbf{Q}_p)$ be the maximal torus consisting of diagonal matrices when $\mathrm{G}_* = \mathrm{G}$ and compatible with this choice when G_* is any other group. Let $T_{\mathrm{G},0} := T_{\mathrm{G}_*} \cap \mathrm{G}_*(\mathbf{Z}_p)$ the integral subgroup. Let $T^+_{\mathrm{G}_*} \subset T_{\mathrm{G}_*}$ be the normaliser of $N_{\mathrm{G}_*,0}$ in T_{G_*}, so that $T^+_{\mathrm{G}} := \prod_{v|p} T^+_{\mathrm{G},v}$ with

$$T^+_{\mathrm{G},v} := \{\left(\begin{smallmatrix} t_1 & \\ & t_2 \end{smallmatrix}\right) \, : \, v(t_1) \geq v(t_2)\}.$$

2.1.4 *Involutions*

We denote by ι the involutions on $\mathscr{H}^{\mathrm{sph}}_{\mathrm{G}}$ induced by $g \mapsto g^{\mathrm{T},-1}$, and on H induced by $h \mapsto h^{-1}$.

We also denote by ι the involution of T_{G_*} deduced by the involutions

$$t \mapsto t^{\iota} := t\nu(t)^{-1}, \tag{2.1.1}$$

where ν denotes the reduced norm if $\mathrm{G}_* = \mathrm{G}$, the norm $N_{E/F}$ if $\mathrm{G} = \mathrm{H}$. It preserves the sub-semigroups $T^+_{\mathrm{G}_*}$.

2.1.5 *Congruence subgroups*

Let $G = \mathrm{GL}_2(F_p)$, $H = E_p^{\times}$, $H' = E_p^{\times}/F_p^{\times}$, $(G \times H)' := (G \times H)/F_p^{\times}$ where $F_p^{\times}$ is identified with the centre of $G \times H$.

For $r \in \mathbf{N}$, define the compact subgroups $U(\varpi_v^r) \subset U_1^1(\varpi_v^r) \subset \mathrm{GL}_2(F_v)$ by

$$U_1^1(\varpi_v^r) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_{F,v}) : a - 1 \equiv d - 1 \equiv c \equiv 0 \pmod{\varpi_v^r}\},$$
$$U(\varpi_v^r) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_{F,v}) : a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod{\varpi_v^r}\}.$$

For each place $v|p$ of F, we fix $\epsilon_v \in F_v^{\times}$ such that $E_v = F_v(\sqrt{\epsilon_v})$; for technical reasons it will be convenient to *assume that* $v(\epsilon_v) \geq 1$.

For $\underline{r} = (r_v) \in \mathbf{N}^{\{v|p\}}$, we define the compact open subgroups $V_{F,v,r_v} := 1 + \varpi_v^{r_v}\mathscr{O}_{F,v} \subset F_v^{\times}$ and

$$V_{p,\underline{r}} = \prod_{v|p} V_{v,r_v} \subset H = \prod_{v|p} E_v^{\times}, \quad U_{p,\underline{r}} = \prod_{v|p} U_{v,r_v} \subset G = \prod_{v|p} \mathrm{GL}_2(F_v)$$

as follows:

$$V_{v,r_v} := \begin{cases} V_{F,v,r_v}(1+\varpi^{r_v}\mathscr{O}_{E,w}) & \text{if } v \text{ splits in } E, \\ V_{F,v,r_v}(1+\sqrt{\epsilon_v}\varpi^{r_v}\mathscr{O}_{E,w}) & \text{if } v \text{ is nonsplit in } E, \end{cases}$$
$$U_{v,r_v} := U_1^1(\varpi_v^{r_v}).$$

We also define $V'_{p,\underline{r}} := V_{p,\underline{r}}F_p^\times/F_p^\times \subset H'$, and

$$K_{p,\underline{r}},\, K_p(p^{\underline{r}}) \subset (G\times H)'$$

to be the images of $U_{p,\underline{r}} \times V_{p,\underline{r}}$, $U_p(p^{\underline{r}}) \times V_{p,\underline{r}}$ respectively.

We also denote

$$T_{\mathrm{G}_*,r} := T_{\mathrm{G}_*} \cap U_{*,p}(p^r).$$

If $p\mathscr{O}_{F,p} = \prod_v \varpi_v^{e_v}\mathscr{O}_{F,v}$, we associate to an integer r the tuple $\underline{r} := (e_v r)_{v|p}$. Denoting by U_* any of the symbols U, V, K, we then let $U_{*,p,r} := U_{*,p,\underline{r}}$, $U_{*,p}(p^r) := U_{*,p}(p^{\underline{r}})$.

2.2 Algebraic representations

We set up some notation for algebraic representations of a (coherent or incoherent) reductive group G over $\mathbf{Q}$, then discuss in some more detail the situation for the groups of interest to us. Let L be an extension of $\mathbf{Q}_p$, W a finite-dimensional irreducible algebraic (left) representation of G over L. Throughout the paper, we tacitly identify left and right algebraic representations of G via $g.w = w.g^{-1}$.

2.2.1 Highest-weight character

We suppose that $\mathrm{G} = \mathrm{G}_*$ is one of the groups of Sect. 2.1. Let $T_{\mathrm{G}_*} \subset G_*$ be the fixed torus and let $N_{\mathrm{G}_*} \subset G_*$ be the fixed unipotent subgroup. If W is an irreducible left (respectively right) representation of G_*, we denote by σ_W the character by which T_{G_*} acts on the line of highest-weight vectors $W^{N_{\mathrm{G}_*}}$ (respectively highest-weight covectors $W_{N_{\mathrm{G}_*}}$).

The highest-weight character of W is related to that of its dual by

$$\sigma_{W^\vee}(t) = \sigma_W(t^\iota), \tag{2.2.1}$$

where ι is the involution (2.1.1).

2.2.2 Quaternionic special case

Suppose that $\mathrm{G}(\mathbf{A}^\infty)$ is the group of units of a quaternion algebra $\mathbf{B}^\infty$ over $\mathbf{A}^\infty$. Let L be an extension of $\mathbf{Q}_p$ splitting F and B_p. A *(cohomological) weight* for G over L is a list $\underline{w} = (w; (w_\sigma)_{\tau\colon F\hookrightarrow L})$ of $[F : \mathbf{Q}] + 1$ integers of the same parity such that $w_\sigma \geq 2$ for all $\sigma\colon F \hookrightarrow L$. Denote by $\mathrm{Std}_\sigma \cong (L^{\oplus 2})^*$ (respectively, $\mathrm{Nm}_\sigma \cong L$) the standard (respectively, reduced norm) representation of $\mathrm{G}(\mathbf{Q}_p) = B_p^\times$ factoring through $(B_p \otimes_{F_p,\sigma} L)^\times \cong \mathrm{GL}_2(F_p \otimes_\sigma L)$. We associate to the weight $\underline{w}$ the algebraic representation

$$W_{\mathrm{G},\underline{w}} := \bigotimes_{\sigma \in \mathrm{Hom}\,(F,L)} \mathrm{Sym}^{w_\sigma - 2}\mathrm{Std}_\sigma \otimes \mathrm{Nm}_\sigma^{(w-w_\sigma+2)/2} \tag{2.2.2}$$

of $\mathrm{G}_{/\mathbf{Q}_p}$, whose dual is $W_{\mathrm{G},\underline{w}^\vee}$ with $\underline{w}^\vee = (-w; (w_\sigma))$.

Suppose for a moment that $L/\mathbf{Q}_p$ is Galois, then $\mathrm{Gal}(L/\mathbf{Q}_p)$ acts on the set of all weights $\underline{w}$ and, letting $L(\underline{w}) \subset L$ be the fixed field of the stabiliser of $\underline{w}$, the representation $W_{\mathrm{G},\underline{w}}$ descends to a representation over $L(\underline{w})$. It is then convenient to use the following terminology: if W is an algebraic representation of G over L and $\underline{w}$ is a cohomological weight over a finite extension L'/L, we say that W is of weight $\underline{w}$ (with respect to $L \hookrightarrow L'$) if $W \otimes_L L' \cong W_{\mathrm{G},\underline{w}}$.

Explicitly, $W_{\mathrm{G},\underline{w}}$ may be described as the space of tuples $p = (p_\sigma)_{\sigma\colon F\hookrightarrow L}$ such that $p_\sigma \in L[x_\sigma, y_\sigma]$ is a homogeneous polynomial of degree $w_\sigma - 2$, with action on each σ-component by

$$g.p_\sigma(x, y) = \det(\sigma g)^{\frac{w-w_\sigma+2}{2}} \cdot p_\sigma((x, y)\sigma g). \tag{2.2.3}$$

The representation $W_{\mathrm{G},\underline{w}}$ admits a natural $\mathscr{O}_L$-lattice, stable under the action of a maximal order in $\mathrm{G}(\mathbf{Q}_p)$,

$$W^{*,\circ}_{\mathrm{G},\underline{w}} \subset W^*_{\mathrm{G},\underline{w}} \tag{2.2.4}$$

consisting of tuples of polynomials with coefficients in $\mathscr{O}_L$.

If $W = W_{\mathrm{G},\underline{w}}$, we have $\sigma_W := \otimes_v \sigma_{W,v}\colon T_v \to L^\times$ with

$$\sigma_{W,v}\colon \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \prod_{\sigma\colon F_v \hookrightarrow L} \sigma(t_1)^{\frac{w+w_\sigma-2}{2}} \sigma(t_2)^{\frac{w-w_\sigma+2}{2}}. \tag{2.2.5}$$

By abuse of notation we still denote by $\sigma_W = \otimes \sigma_{W,v}$ the algebraic character of $F_p^\times$ defined by

$$\sigma_{W,v}(x) := \sigma_{\underline{w},v}\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\right).$$

2.2.3 Toric special case

Let L be a finite extension of $\mathbf{Q}_p$ splitting E. A *cohomological weight* for H is a list $\underline{l} := (l, (l_\sigma)_{\sigma\colon F\hookrightarrow L})$ of $[F : \mathbf{Q}] + 1$ integers of the same parity. For each $\sigma\colon F \hookrightarrow L$, fix an arbitrary extension σ' of σ to E (this choice will only intervene in the numerical labelling of representations of H). We let

$$W_{\mathrm{H},\underline{l}} := \bigotimes_{\sigma\in\mathrm{Hom}\,(F,L)} \sigma'^{l_\sigma} \otimes \sigma \circ \mathrm{Nm}_{E_p/F_p}^{\frac{l-l_\sigma}{2}}, \tag{2.2.6}$$

as a 1-dimensional vector space over L with action by $\mathrm{H}(\mathbf{Q}_p) = E_p^\times$. After choosing an identification of this space with L, it admits a lattice $W^\circ_{\mathrm{H},\underline{l}}$, stable under the action of $\mathscr{O}_{E,p}^\times$. If W is an algebraic representation of H over L and $\underline{l}$ is a cohomological weight over a finite extension L'/L, we say that W is of weight $\underline{l}$ (with respect to $L \hookrightarrow L'$) if $W \otimes_L L' \cong W_{\mathrm{H},\underline{l}}$.

2.3 Shimura varieties and local systems

We again write G_* to denote any of the groups (1.2.1).

2.3.1 Shimura varieties

For τ an infinite place of F, let $\mathrm{G}_\tau = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{G}_F(\tau)$ be the τ-nearby group as in Sect. 2.1.1. Consider the Shimura datum $(\mathrm{G}_\tau, \{h_{\mathrm{G},\tau}\})$, where $h_{\mathrm{G},\tau}\colon \mathrm{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m \to \mathrm{G}_\mathbf{R}$ the Hodge cocharacter of [19, §0.1]. Let $h_\mathrm{H}\colon \mathrm{S} \to \mathrm{H}_\mathbf{R}$ be the unique cocharacter such that $\mathrm{e}_\mathbf{R} \circ h_\mathrm{H} = h_\mathrm{G}$. By products and projections we deduce Hodge cocharacters $h_{\mathrm{G}_*,\tau}$, hence Shimura data $(\mathrm{G}_{*,\tau}, h_{\mathrm{G}_{*,\tau}})$, for any of the groups (1.2.1); from $h_{\mathrm{H},\tau}$ we also obtain an extension of τ to an embedding $\tau\colon E \hookrightarrow \mathbf{C}$. Then we obtain towers of Shimura varieties $X_{*,\tau}/\tau E_*$, where the reflex field $E_* := E$ unless $\mathrm{G}_* = \mathrm{G}$, in which case $E_* = F$. These data descend to E_*: there are towers

$$X_*/E_*$$

such that $X_* \times_{\mathrm{Spec}\,E_*} \mathrm{Spec}\,\tau E_* = X_{*,\tau}$, see [107, § 3.1]. Throughout this paper, we will also use the notation $\overline{X}_* := X \times_{\mathrm{Spec}\,E_*} \mathrm{Spec}\,\overline{E}_*$.

We will use also the specific names (1.2.2) for those varieties; an explicit description of some of them is as follows:

$$X_{U,\tau}(\mathbf{C}) \cong B(\tau)^\times\backslash\mathfrak{h}^\pm \times \mathbf{B}^{\infty\times}/U \cup \{\text{cusps}\}, \quad Y_V(E^{\mathrm{ab}}) \cong E^\times\backslash E^\times_{\mathbf{A}^\infty}/V,$$
$$Z_K \cong X_U \times Y_V/\Delta_{U,V},$$

$$\Delta_{U,V} := F^{\times}_{\mathbf{A}^\infty}/F^{\times} \cdot ((U \cap F^{\times}_{\mathbf{A}^\infty}) \cap (V \cap F^{\times}_{\mathbf{A}^\infty})) \tag{2.3.1}$$

if $K \subset (\mathrm{G} \times \mathrm{H})'(\mathbf{A}^\infty)$ is the image of $U \times V$.

2.3.2 *Automorphic local systems*

Let W be an irreducible cohomological right algebraic representation of G_* over L, let $U_* \subset \mathrm{G}_*(\mathbf{A}^\infty)$ be a sufficiently small (in the sense of Lemma 2.3.1 below) compact open subgroup, let $W^\circ \subset W$ be a $U_{*,p}$-stable $\mathscr{O}_L$-lattice, and let $U_{*,p,n} \subset U_{*,p}$ be a subgroup acting trivially on $W^\circ/p^n W^\circ$.

Lemma 2.3.1 *If U_*^p is sufficiently small (a condition independent of n), then:*

1. *The quotient $\overline{\mathrm{G}}_n := U_{*,p}/U_{*,p,n}(\mathrm{Z}_{\mathrm{G}_*}(\mathbf{Q}) \cap U_*)_p$ acts freely on $X_{U_*^p U_{*,p,n}}$, hence $X_{*,U_*^p U_{*,p,n}} \to X_{*,U_*}$ is an étale cover with Galois group $\overline{\mathrm{G}}_{*,n}$.*
2. *The group $\mathrm{Z}_{\mathrm{G}_*}(\mathbf{Q}) \cap U_*$ acts trivially on W°.*

Proof The first assertion is [19, Lemme 1.4.1.1] when $\mathrm{G}_* = \mathrm{G}$ (other cases are similar or easier). For the second assertion, we may reduce to the case $\mathrm{G}_* = \mathrm{G}$ or $\mathrm{G}_* = \mathrm{H}$, with centre $\mathrm{Z}_{\mathrm{G}_*} = \mathrm{Res}_{E_*/\mathbf{Q}}\mathrm{G}_m$. For any U_*, the group $\mathrm{Z}_{\mathrm{G}_*}(\mathbf{Q}) \cap U_*$ has has finite index in $\mathscr{O}_{E_*}^{\times}$, therefore for sufficiently small U_*^p it is contained in the finite-index subgroup $\mathscr{O}_F^{\times,1} := \{z \in \mathscr{O}_F^{\times} : N_{F/\mathbf{Q}}(z) = 1\} \subset \mathscr{O}_{E_*}^{\times}$. But since W is of cohomological weight, the group $\mathscr{O}_F^{\times,1}$ acts trivially. □

Assume first that X_* is compact. Then, by the lemma,

$$(X_{*,U_*^p U_{*,p,n}} \times W^\circ/p^n W^\circ)/\overline{\mathrm{G}}_{*,n} \tag{2.3.2}$$

defines a locally constant étale $\mathscr{O}_L/p^n\mathscr{O}_L$-module $\mathscr{W}^n$ over X_{*,U_*}. We let

$$\mathscr{W}^\circ := (\varprojlim_n \mathscr{W}^n),$$

an $\mathscr{O}_L$-local system on X_{*,U_*}, and consider

$$\mathscr{W} := \mathscr{W}^\circ \otimes_{\mathscr{O}_L} L.$$

The L-local system $\mathscr{W}$ is compatible with pullback in the tower $\{X_{*,U_*}\}$ and, up to isomorphism, independent of the choice of lattice W°. When X_* is the compactification of a noncompact Shimura variety X'_* (essentially only when $\mathrm{G} = \mathrm{GL}_{2/\mathbf{Q}}$), we perform the above construction on X'_* and then push the resulting sheaf forward to X_*.

2.4 Ordinary automorphic representations

Keep the assumption that G_* is one of the groups in (1.2.1).

2.4.1 p-adic automorphic representations

Let L be an extension of $\mathbf{Q}_p$, W a finite-dimensional irreducible algebraic left representation of $G_{*,\infty} = \mathrm{G}_*(\mathbf{Q}_p)$ over L.

Definition 2.4.1 A (regular algebraic cuspidal) *automorphic representation of* $\mathrm{G}_*(\mathbf{A})$ *over* L of weight W is an irreducible admissible locally algebraic representation π of

$$\mathrm{G}_*(\mathbf{A}) := \mathrm{G}(\mathbf{A}^\infty) \times G_{*,\infty}$$

that can be factored as

$$\pi = \pi^\infty \otimes W$$

such that $\mathrm{G}_*(\mathbf{A}^\infty)$ acts smoothly on π^∞, $\mathrm{G}_{*,\infty}$ acts algebraically, and π occurs as a subrepresentation of

$$\mathrm{H}^\bullet(\overline{X}_*, \mathscr{W}^\vee) = \varinjlim_{U \subset \mathrm{G}_*(\mathbf{A}^\infty)} H^\bullet(\overline{X}_{*,U}, \mathscr{W}^\vee) \otimes W,$$

where X_* is the compactified Shimura variety attached to G_*, and $\mathscr{W}^\vee$ is the local system on X attached to $W^\vee$.

In the quaternionic or toric case, we say that π is of weight $\underline{w}$ (a cohomological weight for G over some finite extension L'/L) if W is of weight $\underline{w}$ as defined after (2.2.2) (respectively (2.2.6)).[15]

We will use subscripts p, respectively ∞, respectively $p\infty$, to denote an element of $\mathrm{G}(\mathbf{A})$ in the copy of $\mathrm{G}(\mathbf{Q}_p)$ contained in $\mathrm{G}(\mathbf{A}^\infty)$, respectively in the 'algebraic copy' G_∞, respectively the diagonal copy in the product of the previous two.

Remark 2.4.2 The previous definition follows the work of Emerton [37]. It slightly departs from it in that in [37], one restricts to considering the action of the product of $\mathrm{G}(\mathbf{A}^{p\infty})$ and the diagonal copy of $\mathrm{G}(\mathbf{Q}_p)$. While this is indeed the part that acts integrally, we do have use for the non-integral action of each individual copy (cf. Sect. A.2). The corresponding local notions are introduced in Definition A.1.1.

[15] These notions depend of course on $L \hookrightarrow L'$; nevertheless they will only be used to impose conditions on the weights that are invariant under the Galois group of L.

2.4.2 Quaternionic special case and ordinarity

Suppose that $\mathrm{G}_* = \mathrm{G}$ and $\mathbf{B}_p$ is split, or that $\mathrm{G}_* = \mathrm{G}_0 = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_2$ for a totally real field F. An automorphic representation π over L of weight $W_{\mathrm{G}_*,\underline{w}}$ is also said to be of weight $\underline{w}$.

Definition 2.4.3 We say that an automorphic representation π of $\mathrm{G}_*(\mathbf{A})$ over L of classical weight $W = W_{\mathrm{G}_*,\underline{w}}$ is *ordinary* at v with unit character α_v° if there exists a smooth character α_v of T_v such that π_v is the unique irreducible subrepresentation of $\mathrm{Ind}(\alpha_v \cdot (|\ |_v, |\ |_v^{-1}))$ and the locally algebraic character

$$\alpha_v^\circ := \alpha_v \sigma_{W,v} \colon T_v \to L^\times \tag{2.4.1}$$

takes values in $\mathscr{O}_L^\times$.[16]

(It follows from the parity conditions on the weights that the indicated subrepresentation is always infinite-dimensional; moreover if π_v is ordinary then the character α_v of T_v is uniquely determined by π_v.) We say that π is *ordinary* if it is ordinary at all $v|p$.

Let $v|p$ be a prime of F and ϖ_v a uniformiser. For $t \in T_v^+$ or $x \in F_v^\times$ with $v(x) \geq 0$, define the double coset operators

$$\begin{aligned} \mathrm{U}_t &:= [U_1^1(\varpi_v^r)\, t\, U_1^1(\varpi_v^r)_v], \\ \mathrm{U}_x &:= \mathrm{U}_{\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)}, \\ \mathrm{U}_v &:= \mathrm{U}_{\varpi_v}, \end{aligned} \tag{2.4.2}$$

which act on the N_0-fixed vectors of any locally algebraic representation of $\mathrm{GL}_2(F_v)$ (see also Sect. A.1 for further details). Then π is ordinary at v with unit character α_v° if and only if, for sufficiently large r, the space of $U_1^1(\varpi_v^r)$-fixed vectors in the locally algebraic representation

$$\pi_v \otimes_L W$$

of $\mathrm{GL}_2(F_v)$ contains a (necessarily unique) line of eigenvectors for the diagonal action of the operators U_x, $x \in F_v^\times$, with eigenvalue $\alpha^\circ(x)$. Specifically, if $w_v \in \pi_v$ is a U_x-eigenvector of eigenvalue $\alpha_v(\varpi_v)$, then such line is

$$\pi_v^{\mathrm{ord}} := L w_v \otimes W^{N_v},$$

where W^{N_v} is the line of highest-weight vectors of W.

[16] This notion agrees with the notion of π being *nearly ordinary* as defined in the work of Hida (e.g. [50]).

If π is an automorphic representation that is ordinary at all $v|p$, extend α_v° to a character of T_v^+ by the formula (2.4.1), and let $\alpha^\circ = \prod_{v|p} \alpha_v^\circ \colon T^+ = \prod T_v^+ \to L^\times$; then we define

$$\pi^{\rm ord} := \pi^{p\infty} \otimes \otimes_{v|p} \pi_v^{\rm ord} \tag{2.4.3}$$

as a smooth representation of $\mathrm{G}(\mathbf{A}^{p\infty})$ and a locally algebraic representation of T^+ on which T^+ acts by $\mathrm{U}_t \mapsto \alpha_v^\circ(t)$.

2.4.3 Toric special case

Suppose now that E is a CM field and that $\mathrm{G}_* = \mathrm{H} := \mathrm{Res}_{E/\mathbf{Q}}\mathrm{G}_m$. Then a p-adic automorphic representation of H of weight $W_{\mathrm{H},\underline{l}}$ is simply the space of scalar multiples of a locally algebraic character $\chi \colon E^\times \backslash E^\times_{\mathbf{A}^\infty} \to L^\times$ whose restriction to a sufficiently small open subgroup of $E_p^\times$ coincides with the character of $W_{\mathrm{H},\underline{l}}$.

2.4.4 Convention

We use the convention that all automorphic representations of $\mathrm{H}(\mathbf{A}^\infty)$ are ordinary, and that a representation $\pi \otimes \chi$ of $\mathrm{G} \times \mathrm{H}$ of cohomological weight is cuspidal and ordinary if π and χ are.

2.5 Galois representations

Let G be as in Sect. 2.4.2.

2.5.1 Galois representations attached to automorphic representations of $\mathrm{G}(\mathbf{A})$

The following notation is used throughout the paper: if V is a representation of G_F and v is a prime of F, we denote by V_v the restriction of V to a decomposition group at v.

Theorem 2.5.1 (Ohta, Carayol, Saito). *Let L be a finite extension of $\mathbf{Q}_p$, let W be an irreducible algebraic representation of* G *over L, and let π be an automorphic representation of* $\mathrm{G}(\mathbf{A}^\infty)$ *of weight W over L. Let S be a finite set of non-archimedean places of F containing all the places at which π is ramified and the places above p. There exists a* 2*-dimensional L-vector space V_π and an absolutely irreducible Galois representation*

$$\rho = \rho_\pi \colon G_{F,S} \to \mathrm{Aut}(V_\pi)$$

uniquely determined by the property that for every finite place $v \notin S$ of F,

$$\mathrm{Tr}(\rho(\mathrm{Fr}_v)) = q_v^{-1}\lambda_{\pi_v}(T_v), \tag{2.5.1}$$

where Fr_v *is a geometric Frobenius,* $T_v \in \mathscr{H}^{\mathrm{sph}}_{\mathrm{GL}_2(F_v)}$ *is the element corresponding to the double class* $K_0(1)\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix}K_0(1)$, *and* $\lambda_{\pi_v}\colon \mathscr{H}^{\mathrm{sph}}_{\mathrm{GL}_2(F_v)} \to L$ *is the character giving the action on* $\pi_v^{K_0(1)}$.

For a prime v of F, let ρ_v be the restriction of ρ to a decomposition group at v.

1. *The representation ρ_v is unramified for almost all v and potentially semistable for $v|p$. For every finite place v, the Weil–Deligne representation r_v attached to ρ_v is associated with π_v via the local Langlands correspondence normalised "à la Hecke"* [29, § 3.2]*:*

$$L(s, r_v) = L(s + 1/2, \pi_v).$$

2. *For every finite place v, r_v satisfies the weight-monodromy conjecture: its monodromy filtration is pure of weight $w - 1$. The monodromy filtration is trivial if and only if π_v is not a special representation.*
3. *For any archimedean place v, the representation ρ_v is odd, that is if $c_v \in G_{F_v}$ is the complex conjugation,* $\det \rho_v(c_v) = -1$.
4. *If $W = W_{\mathrm{G},\underline{w}}$ with $\underline{w} = (w; (w_\sigma)_{\sigma\colon F\hookrightarrow \overline{L}})$, then for each $v|p$ and $\sigma\colon F_v \hookrightarrow \overline{L}$,*
   - *the σ-Hodge–Tate weights*[17] *of $\rho_v \otimes_L \overline{L}$ are*

$$-1 - \frac{w + w_\sigma - 2}{2}, \quad -\frac{w - w_\sigma + 2}{2}.$$

   - *if π is ordinary at v in the sense of Definition 2.4.3, then there is a unique exact sequence in the category of $G_{F,v}$-representations*

$$0 \to V^+_{\pi,v} \to V_{\pi,v} \to V^-_{\pi,v} \to 0, \tag{2.5.2}$$

   such that $V^{\pm}_{\pi,v}$ is 1-dimensional.

[17] If V is a Hodge–Tate representation of G_{F_v} over $\overline{L}$ and $\sigma\colon F_v \hookrightarrow \overline{L}$, the σ-Hodge–Tate weights of V are the degrees in which the graded module

$$(\oplus_n \mathbf{C}_v(n) \otimes_{\overline{F}_v,\sigma} V)^{G_{F_v}}$$

is nonzero; here $\mathbf{C}_v$ is a completion of $\overline{F}_v$ and, in the tensor product, σ is extended to an isomorphism $\overline{F}_v \to \overline{L}$. In particular our convention is that the Hodge–Tate weight of the cyclotomic character of $\mathbf{Q}_p$ is -1.

The Galois group $G_{F,v}$ acts on $V^{+}_{\pi,v}$ by the character

$$\alpha^{\circ}_{\pi,v}\chi_{\mathrm{cyc},v} \colon F_v^{\times} \to L^{\times},$$

where $\alpha^{\circ}_{\pi,v}$ is (2.4.1).

Proof The construction and statements 1 and 2 for $v|p$ are the main results of Carayol in [19]. Statements 1 and 2 for $v|p$ were proved by Saito [91, Theorems 2.2, 2.4]. For the last two statements, we refer to [101, Proposition 6.7] and references therein; note that in comparison with the notation of [101], our ρ equals their $\rho_f(1)$, and our $(w;\underline{w})$ is their $(w-2,\underline{k})$. □

2.5.2 Realisation in the homology of Shimura varieties

Let G_* be again one of the groups of (1.2.1). We introduce a new piece of notation. Let

$$G_{F,E} := G_F \times G_E,$$

and similarly for a finite set of places S, $G_{F,E,S} := G_{F,S}\times G_{E,S}$. If $\mathrm{G}_* = \mathrm{G}\times\mathrm{H}$ or $(\mathrm{G}\times\mathrm{H})'$, we *redefine*

$$G_{E_*} := G_{F,E}.$$

(This is an abuse of notation, as we have not redefined E_*.) This product of Galois groups acts on the homology $\overline{X}_*$: this is clear by the Künneth formula in the case of $\mathrm{G}\times\mathrm{H}$, and follows from that case and the Galois-invariance of the quotient map for $(\mathrm{G}\times\mathrm{H})'$.

The following is the main result of [20] in the special case $\mathrm{G}_* = \mathrm{G}$; the general case may be deduced from the special case together with the case $\mathrm{G}_* = \mathrm{H}$ (that is class field theory).

Proposition 2.5.2 (Carayol). *Let $U_* \subset \mathrm{G}_*(\mathbf{A}^{\infty})$ be a compact open subgroup, W be an irreducible right algebraic representation of G_* over L, $\mathscr{W}$ the local system on X_{*,U_*} associated with W. Let L' be a sufficiently large finite Galois extension of L.*

Then there is an isomorphism of $\mathscr{H}_{\mathrm{G}_*,U_*,L}[G_{E_*,S}]$-modules

$$\mathrm{H}_d(\overline{X}_{*,U_*},\mathscr{W})\otimes_L L' \cong \bigoplus_{\pi} \pi^{\vee,U_*}\otimes V_{\pi}, \tag{2.5.3}$$

equivariant for the action of $\mathrm{Gal}(L'/L)$, where π runs through all equivalence classes of automorphic representations of $\mathrm{G}_*(\mathbf{A})$ of weight W over L'.

3 Sheaves on Hida families

We construct the universal Hecke- and Galois- modules over Hida families for $(\mathrm{G}\times\mathrm{H})'$ and prove a local-global compatibility result. We claim no originality for the results of Sects. 3.1–3.2.1.

3.1 Hida theory

We let G_* denote any of the groups G, H, $(\mathrm{G}\times\mathrm{H})$, $(\mathrm{G}\times\mathrm{H})'$, and let $r\in\mathbf{N}^{\{v|p\}}$. We will use the notation from Sect. 2.1. For $U_*^p\subset\mathrm{G}(\mathbf{A}^{p\infty})$ we let $X_{*,U_*^p,r}:=X_{*,U_*^pU_{*,r}}$ be the corresponding Shimura variety.

When M is a $\mathbf{Z}_p$-module with action by $T^+_{\mathrm{G}_*}$, arising as limit of ordinary parts of p-adic coadmissible $\mathrm{G}_*(\mathbf{Q}_p)\times G_{*,\infty}$-modules (see Definition A.1.2 and Sect. A.1.3), we denote this action by

$$t\mapsto \mathrm{U}_t$$

and adopt the notation of (2.4.2).

3.1.1 Weight spaces

Let $U_*^p\subset\mathrm{G}_*(\mathbf{A}^{p\infty})$ be a compact open subgroup, and define $Z_{\mathrm{G}_*,U_*^p}\subset\mathrm{Z}_{\mathrm{G}}(\mathbf{Q})$ by

$$\begin{aligned}
Z_{\mathrm{G},U^p}&:=\mathrm{Z}_{\mathrm{G}}(\mathbf{Q})\cap U^pT_{\mathrm{G},0},\\
Z_{\mathrm{H},V^p}&:=\mathrm{H}(\mathbf{Q})\cap V^pT_{\mathrm{H},0}=\mathscr{O}_E^\times\cap V^p,\\
Z_{(\mathrm{G}\times\mathrm{H})',K^p}&:=\text{the image in } T_{(\mathrm{G}\times\mathrm{H})'} \text{ of}\\
Z_{\mathrm{G}\times H,U^p\times V^p}&:=Z_{\mathrm{G},U^p}\times Z_{\mathrm{H},V^p}\quad\text{if } K^p \text{ is the image of } U^p\times V^p.
\end{aligned}$$

In all cases, let $\overline{T}_{\mathrm{G}_*,U_*^p,0}:=T_{\mathrm{G},0}/\overline{Z_{\mathrm{G}_*,U_*}}$, where $\overline{\square}$ denotes the closure for the p-adic topology, and let $\overline{T}_{\mathrm{G}_*,U_*^p,r}\subset\overline{T}_{\mathrm{G},U_*^p,0}$ be the image of $T_{\mathrm{G}_*,U_*^p,r}$. Let

$$\Lambda^\circ_{\mathrm{G}_*,U_*^p}:=\mathbf{Z}_p[\![\overline{T}_{\mathrm{G}_*,0}]\!],$$

and for an irreducible algebraic representation W of G_* consider the ideals

$$I_{\mathrm{G}_*,U_*^p,W,r,L}:=([t]-\sigma_W^{-1}(t))_{t\in\overline{T}_{\mathrm{G}_*,U_*^p,r}})\subset\Lambda^\circ_{\mathrm{G}_*,U_*^p}\otimes\mathscr{O}_L. \tag{3.1.1}$$

For each fixed W and varying r, the ideals $I_{\mathrm{G}_*,U_*^p,W,r}$ form a fundamental system of neighbourhoods of zero in $\Lambda^\circ_{\mathrm{G}_*,U_*^p}\otimes\mathscr{O}_L$, so that

$$\Lambda_{\mathrm{G}_*,U_*^p}:=\Lambda^\circ_{\mathrm{G}_*,U_*^p}\otimes_{\mathscr{O}_L}L=(\varprojlim_r\Lambda_{\mathrm{G}_*,U_*^p,W,r})\otimes_{\mathscr{O}_L}L \tag{3.1.2}$$

with

$$\Lambda^\circ_{\mathrm{G}_*,U_*^p,W,r}:=\Lambda^\circ_{\mathrm{G}_*,U_*^p}/I_{\mathrm{G}_*,U_*^p,W,r}\cong\mathscr{O}_L[\overline{T}_{\mathrm{G}_*,U_*^p,0}/\overline{T}_{\mathrm{G}_*,U_*^p,r}], \tag{3.1.3}$$

where the isomorphism is given by $[t]\mapsto\sigma_W^{-1}(t)[\bar{t}]$. When $W=\mathbf{Q}_p$, we omit W from the notation. We also omit the subscript U_*^p when it is unimportant or understood from context.

Writing $\overline{T}_{\mathrm{G}_*,U_*^p,0}\cong\Delta\times\mathbf{Z}_p^{\mathrm{d}(\mathrm{G}_*)}$ for a finite torsion group Δ, we have an isomorphism $\Lambda_{\mathrm{G}_*,U_*^p}\cong\mathbf{Z}_p[\Delta]\otimes\mathbf{Z}_p[\![X_1,\dots X_{\mathrm{d}(\mathrm{G}_*)}]\!]$ for an integer $\mathrm{d}(\mathrm{G}_*)$ given by[18]

$$\mathrm{d}(\mathrm{G})=\mathrm{d}(\mathrm{H})=[F:\mathbf{Q}]+1+\delta,\quad\mathrm{d}((\mathrm{G}\times\mathrm{H})')=2[F:\mathbf{Q}]+1+\delta,$$

where $\delta=\delta_{F,p}$ is the Leopoldt defect of F at p; see [41, § 2.2.3.3] for $\mathrm{d}(\mathrm{G})$.

Definition 3.1.1 The *weight space* is

$$\mathfrak{W}_{\mathrm{G}_*,U_*^p}:=\mathrm{Spec}\,\Lambda_{\mathrm{G}_*,U_*^p}\otimes\mathbf{Q}_p.$$

Let W be an irreducible cohomological algebraic representation of G_*. The zero-dimensional subscheme of *classical points of weight* W and level r is

$$\mathfrak{W}^{\mathrm{cl},W}_{\mathrm{G}_*,U_*^p,r}:=\mathrm{Spec}\,\Lambda_{\mathrm{G}_*,U_*^p,r,W}.$$

The ind-subschemes of all classical points of weight W and of of all classical points are respectively

$$\mathfrak{W}^{\mathrm{cl},W}_{\mathrm{G}_*,U_*^p}:=\bigcup_{r\geq0}\mathfrak{W}^{\mathrm{cl},W}_{\mathrm{G}_*,U_*^p,r},\quad\mathfrak{W}^{\mathrm{cl}}_{\mathrm{G}_*,U_*^p}:=\bigcup_W\mathfrak{W}^{\mathrm{cl},W}_{\mathrm{G}_*,U_*^p},$$

where as usual the union runs through the algebraic representations of cohomological weight.

[18] We would have $\mathrm{d}((\mathrm{G}\times_{\mathrm{Z}}\mathrm{H})'=2[F:\mathbf{Q}]$ for the group of the footnote after (1.2.1), which explains the way the number of variables of Theorem E is counted in the abstract.

3.1.2 *Ordinary completed homology*

Let W be an irreducible right algebraic representation of $\mathrm{G}_*(\mathbf{Q}_p)$ over L, and fix a $\mathrm{G}(\mathbf{Z}_p)$-stable $\mathscr{O}_L$-lattice $W^\circ \subset W$. Let $\mathscr{W}$ be the local system attached to W, and for $U_*^p \subset \mathrm{G}_*(\mathbf{A}^{p\infty})$, $r \geq 0$ consider the *ordinary parts*

$$\mathrm{H}_d^{\text{ét}}(\overline{X}_{*,U_*^pU_{*,r}}, \mathscr{W}^\circ)^{\text{ord}} := (H_d^{\text{ét}}(\overline{X}_{*,U_*^pU_{*,r}}, \mathscr{W}^\circ) \otimes_{\mathscr{O}_L[T_{\mathrm{G}_*}^+]} (W^{\circ,\vee})_{N_0})^{\text{ord}},$$
$$\mathrm{H}_d^{\text{ét}}(\overline{X}_{*,U_*^pU_{*,r}}, \mathscr{W})^{\text{ord}} = \mathrm{H}_d^{\text{ét}}(\overline{X}_{*,U_*^pU_{*,r}}, \mathscr{W}^\circ)^{\text{ord}} \otimes_{\mathscr{O}_L} L$$

with respect to the action of $T_{\mathrm{G}_*}^+$ by $\mathrm{U}_t \otimes t$, as defined in Sect. A.1.3. The *ordinary completed homology* of X_{G_*,U_*^p} is

$$M_{\mathrm{G}_*,U_*^p,W}^\circ := \varprojlim_r \mathrm{H}_d^{\text{ét}}(\overline{X}_{*,U_*^pU_{*,p,r}}, \mathscr{W}^\circ)^{\text{ord}},$$

an $\mathscr{O}_L$-module. It depends on the choice of lattice $W^\circ \subset W$, whereas the L-vector space

$$M_{\mathrm{G}_*,U_*^p,W} := M_{\mathrm{G}_*,U_*^p,W}^\circ \otimes_{\mathscr{O}_L} L$$

does not. When $\mathscr{W} = \mathbf{Q}_p$ is the trivial local system, we omit it from the notation, thus

$$M_{\mathrm{G}_*,U_*^p} = M_{\mathrm{G}_*,U_*^p,\mathbf{Q}_p}.$$

3.1.3 *Independence of weight and Control Theorem*

For a $\mathbf{Z}_p$-algebra A, let $\mathscr{H}_{\mathrm{G}_*,U_*^p,p,A}^{\text{ord}} := A[T_{\mathrm{G}_*}] \otimes_{\mathbf{Z}_p[T_{\mathrm{G}_*,0}^+]} \Lambda_{\mathrm{G}_*,U_*^p}^\circ$. For $? = S, \emptyset, \text{sph}$, consider the $\Lambda_{\mathrm{G}_*,U_*^p,A}$-algebra

$$\mathscr{H}_{\mathrm{G}_*,U_*^p,A}^{?,\text{ord}} := \mathscr{H}_{\mathrm{G}_*,U_*^p,A}^{?} \otimes_A \mathscr{H}_{\mathrm{G}_*,U_*^p,p,A}^{\text{ord}}. \tag{3.1.4}$$

For every irreducible algebraic representation W over L and $\mathscr{O}_L$-algebra A, the space $M_{\mathrm{G}_*,U_*^p,W}^\circ \otimes A$ is a module over $\mathscr{H}_{\mathrm{G}_*,U_*^p,A}^{\text{ord}}$, where $[t] \in A[T_{\mathrm{G}_*}^+]$ acts by the double coset operator U_t.

The base ring A will be omitted from the notation when it can be understood from the context.

Let $U_{*,r} = U_*^p U_{*,r,p}$ be as in Sect. 2.1 and let $X_{*,r} := X_{*,U_{*,r}}$.

Proposition 3.1.2 *Let W be an irreducible right algebraic representation of $\mathrm{G}_{*/\mathbf{Q}_p}$ over L, $\mathscr{W}$ the corresponding local system. Then:*

1. *If* $\mathrm{G}_* = \mathrm{G}, \mathrm{H}$*, then* $M^\circ_{\mathrm{G}_*,U^p_*,W}$ *is a projective* $\Lambda^\circ_{\mathrm{G}_*,U^p_*} \otimes \mathscr{O}_L$*-module of finite type. For all of the groups* G_**, the* $\Lambda^\circ_{(\mathrm{G}\times\mathrm{H})'} \otimes \mathscr{O}_L$*-module* $M^\circ_{(\mathrm{G}\times\mathrm{H})',K^p,W}$ *is of finite type, and* $M_{(\mathrm{G}\times\mathrm{H})',K^p,W}$ *is a projective* $\Lambda_{(\mathrm{G}\times\mathrm{H})'} \otimes L$*-module of finite type.*
2. *We have natural* $\mathscr{H}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*}$*-equivariant isomorphisms*

$$j_W \colon M_{\mathrm{G}_*,U^p_*} \otimes \mathscr{O}_L \cong M_{\mathrm{G}_*,U^p_*,W}. \tag{3.1.5}$$

3. *Consider*

$$M_{\mathrm{G}_*,U^p_*,W,r} := M_{\mathrm{G}_*,U^p_*} \otimes_{\Lambda_{\mathrm{G}_*,U^p_*}} \Lambda_{\mathrm{G}_*,U^p_*,W,r}. \tag{3.1.6}$$

There is a natural $\mathscr{H}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*}$*-equivariant isomorphism*

$$M_{\mathrm{G}_*,U_*,W,r} \cong \mathrm{H}_d(\overline{X}_{*,r}, \mathscr{W})^{\mathrm{ord}}.$$

Proof We first treat part 1 when $W = \mathbf{Q}_p$. Then we will deal with part 2, which implies that part 1 holds for any W.

If $\mathrm{G}_* = \mathrm{G}$, the result is proved in [55, Thoerem 1.2, cf. also Remark 1.1]. If $\mathrm{G}_* = \mathrm{H}$, $U^p_* = V^p$, then the module under consideration is isomorphic to $\mathbf{Z}_p[\![E^\times\backslash E^\times_{\mathbf{A}^\infty}/V^p]\!]$, which is finite free over $\Lambda^\circ_{\mathrm{H},V^p} = \mathbf{Z}_p[\![\overline{\mathscr{O}_E^\times \cap V^p}\backslash\mathscr{O}^\times_{E,p}]\!]$ as $\overline{\mathscr{O}_E^\times \cap V^p}\backslash\mathscr{O}^\times_{E,p} \subset E^\times\backslash E^\times_{\mathbf{A}^\infty}/V^p$ is a subgroup of finite index.

If $\mathrm{G}_* = \mathrm{G}\times\mathrm{H}$, by the Künneth formula we have $M^\circ_{\mathrm{G}\times\mathrm{H},U^p\times V^p} = M^\circ_{\mathrm{G},U^p}\hat{\otimes}M^\circ_{\mathrm{H},V^p}$, which by the previous results is a finite type projective module over $\Lambda^\circ_{\mathrm{G}\times\mathrm{H}} = \Lambda^\circ_{\mathrm{G}}\hat{\otimes}\Lambda^\circ_{\mathrm{H}}$. Finally, if $\mathrm{G}_* = (\mathrm{G}\times\mathrm{H})'$ and K^p is the image of $U^p \times V^p$, by the description of Z_K in (2.3.1) we have

$$M^\circ_{(\mathrm{G}\times\mathrm{H})',K^p} = (M^\circ_{\mathrm{G}\times\mathrm{H},U^p\times V^p})/(F^\times_{\mathbf{A}^\infty}/F^\times \cdot ((U^p \cap F^\times_{\mathbf{A}^{p\infty}}) \cap (V^p \cap F^\times_{\mathbf{A}^{p\infty}}))) \tag{3.1.7}$$

As $M^\circ_{\mathrm{G}\times\mathrm{H},U^p\times V^p}$ is a projective $\Lambda^\circ_{\mathrm{G}\times\mathrm{H},U^p\times V^p}$-module of finite type, the quotient $M^\circ_{\mathrm{G},\times\mathrm{H},U^p\times V^p}/F, p^\times = M_{\mathrm{G},\times\mathrm{H},U^pN_{\mathrm{G},0}\times V^p} \otimes_{\Lambda^\circ_{\mathrm{G}\times\mathrm{H}}} \Lambda^\circ_{(\mathrm{G}\times\mathrm{H})'}$ is a projective $\Lambda^\circ_{(\mathrm{G}\times\mathrm{H})',K^p}$-module of finite type, and $M^\circ_{(\mathrm{G}\times\mathrm{H})',K^p}$ is its quotient by the free action of the finite group $F^\times_{\mathbf{A}^\infty}/F^\times \cdot F^\times_p((U^p \cap F^\times_{\mathbf{A}^{p\infty}}) \cap (V^p \cap F^\times_{\mathbf{A}^{p\infty}})))$. After inverting p, the quotient map admits a section, hence $M_{(\mathrm{G}\times\mathrm{H})',K^p}$ is projective over $\Lambda_{(\mathrm{G}\times\mathrm{H})'}$.

We now turn to part 2. As above it suffices to prove the result when $\mathrm{G}_* = \mathrm{G}, \mathrm{H}$. Let $\mathrm{G}_* = \mathrm{G}$, and suppose that $W = W^*_{\mathrm{G},\underline{w}}$. Let $W^\circ \subset W$ be the lattice of (2.2.4), $r \geq 1$. We have a Λ_r-linear map

$$j_{W,r} \colon H_1(\overline{X}_r, \mathbf{Z}/p^r\mathbf{Z})^{\mathrm{ord}} \otimes_{\Lambda_r} W^{\circ,N_0}/p^r \to H_1(\overline{X}_r, \mathscr{W}^\circ/p^r\mathscr{W}^\circ) \tag{3.1.8}$$

induced by cap product[19] via the isomorphism of Λ_r-modules $H^0(\overline{X}_r, \mathscr{W}^\circ/p^r\mathscr{W}^\circ) \cong W^{\circ,N_0}/p^r$.

The maps (3.1.8) are compatible with variation in r, and taking limits we obtain the map (3.1.5), which Hida [49, § 8], [55, Theorem 2.4] proved to be an isomorphism; the asserted equivariance properties are clear from the construction.

When $\mathrm{G}_* = \mathrm{H}$ the construction is similar but easier, as each W is 1-dimensional and each of the analogous maps $j_{W,r}$ is an isomorphism.

Finally, we address part 3. As above we may reduce to the case $W = \mathbf{Q}_p$ and $\mathrm{G}_* = \mathrm{H}$ or $\mathrm{G}_* = \mathrm{G}$. The former is clear, and the latter is, in view of part 2, equivalent to the statement

$$M_{\mathrm{G},U^p,\mathscr{W}} \otimes_{\Lambda_{\mathrm{G},U^p}} \Lambda_{\mathrm{G},U^p,r} \cong \mathrm{H}_d(\overline{X}_r, \mathscr{W})^\circ,$$

which is the control theorem of [55, Theorem 1.2 (3)]. □

3.1.4 Ordinary eigenvarieties

The space $M^\circ_{\mathrm{G}_*,U^p_*}$ has the structure of an $\mathscr{H}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*}$-module (in particular of $\Lambda^\circ_{\mathrm{G}_*,U^p_*}$-module), and for $? = \emptyset$, sph and A a $\mathbf{Z}_p$-algebra, we let

$$\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p_*,A}$$

be the image of $\mathscr{H}^{?,\mathrm{ord}}_{\mathrm{G}_*,U^p_*,A}$ in $\mathrm{End}_A(M^\circ_{\mathrm{G}_*,U^p_*} \otimes A)$, that is independent of the particular spherical Hecke algebra chosen when $? = $ sph. When $A = \mathbf{Z}_p$ we omit it from the notation.

We may now define

$$\mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*} := \mathrm{Spec}\,\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p_*,\mathbf{Q}_p}.$$

When $\mathrm{G}_* = \mathrm{H}$, we will omit the superscript 'ord'.

Let

$$\kappa_{\mathrm{G}_*} : \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*} \to \mathfrak{W}_{\mathrm{G}_*,U^p_*}.$$

Referring to Definition 3.1.1, the zero-dimensional (ind)-subscheme of classical points (respectively classical points of weight W, for an algebraic

[19] I am grateful to David Loeffler and Sarah Zerbes for explaining to me this point of view on the Control Theorem.

representation W of G_*, respectively classical points of weight W and level r) is

$$\mathscr{E}^{\mathrm{ord,cl},(W)}_{\mathrm{G}_*,U^p_*,(r)} := \mathfrak{W}^{\mathrm{cl},(W)}_{\mathrm{G}_*,U^p_*,(r)} \times_{\mathfrak{W}_{\mathrm{G}_*,U^p_*},\kappa} \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*}.$$

We denote by

$$\mathscr{M}_{\mathrm{G}_*,U^p_*}$$

the sheaf on $\mathscr{E}^{\mathrm{ord}}_{\mathrm{G}_*,U^p_*}$ corresponding to M_{G_*,U^p_*}.

Notation When $\mathrm{G}_* = (\mathrm{G}\times\mathrm{H})'$, we omit the subscripts, thus e.g. for $K^p \subset (\mathrm{G}\times\mathrm{H})'(\mathbf{A}^{p\infty})$ we write

$$\mathscr{E}^{\mathrm{ord}}_{K^p} := \mathscr{E}^{\mathrm{ord}}_{(\mathrm{G}\times\mathrm{H})',K^p}.$$

By (3.1.7), $\mathbf{T}^{\mathrm{sph,ord}}_{K^p}$ is a quotient of $\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}\times\mathrm{H},U^p\times V^p}$ and correspondingly we have a closed immersion

$$\mathscr{E}^{\mathrm{ord}}_{K^p} \hookrightarrow \mathscr{E}^{\mathrm{ord}}_{\mathrm{G}\times\mathrm{H},U^p\times V^p}. \tag{3.1.9}$$

Proposition 3.1.3 *The ring $\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p_*}$ is finite flat over $\Lambda^\circ_{\mathrm{G}_*,U^p_*}$, hence semi-local. The maximal ideals of $\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p}$ are in bijection with $G_{\mathbf{F}_p}$-orbits of characters $\overline{\lambda}\colon \mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p_*} \to \overline{\mathbf{F}}_p$.*

Proof The first statement is easy for the group H and it is proved in [55] for the group G. Together they imply the statement for $\mathrm{G}\times\mathrm{H}$ and hence $(\mathrm{G}\times\mathrm{H})'$. As $\mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G}_*,U^p}$ is topologically finitely generated over $\mathbf{Z}_p$, the residue fields of its maximal ideals are finite extensions of $\mathbf{F}_p$; this implies the second statement. □

Lemma 3.1.4 *Let W be an irreducible algebraic representation of* G_*. *The set $\mathscr{E}^{\mathrm{ord,cl},W}_{\mathrm{G}*,U_*}$ of classical points of weight W is Zariski-dense in $\mathscr{E}^{\mathrm{ord}}_{G_*,U^p_*}$.*

Proof By the previous proposition, the map κ_{G_*} is finite hence closed. Then the Zariski-density of $\mathscr{E}^{\mathrm{ord,cl},W}_{\mathrm{G}*,U_*} = \kappa^{-1}_{\mathrm{G}_*}(\mathfrak{W}^W_{\mathrm{G}_*,U^p_*})$ reduces to the Zariski-density of $\mathfrak{W}^W_{\mathrm{G}_*,U^p_*} \subset \mathfrak{W}_{\mathrm{G}_*,U^p_*}$, which follows from (3.1.2); cf. aso [97, Lemma 3.8]. □

3.1.5 Abelian case

The structure of the eigenvariety for the abelian groups $\mathrm{H} := \mathrm{Res}_{E/\mathbf{Q}}\mathrm{G}_m$ and $\mathrm{Z} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{G}_m$ is very simple, and we make it explicit for the group H: we have

$$M_{\mathrm{H},V^p} := \hat{H}_0(\overline{Y}_{V^p}) = \mathbf{Z}_p[\![Y_{V^p}(\overline{E})]\!] \otimes \mathbf{Q}_p,$$

the set $Y_{V^p}(\overline{E})$ is a principal homogeneous space for $\Gamma_{E,V^p} := \mathrm{H}(\mathbf{Q})\backslash\mathrm{H}(\mathbf{A}^\infty)/V^p = E^\times\backslash E^\times_{\mathbf{A}^\infty}/V^p$, and

$$\mathscr{E}_{\mathrm{H},V^p} = \mathrm{Spec}\, \mathbf{Z}_p[\![\Gamma_{E,V^p}]\!]_{\mathbf{Q}_p}.$$

(We omit the superscript 'ord' which is meaningless here.) The classical points $\mathscr{E}^{\mathrm{cl}}_{\mathrm{H},V^p} \subset \mathscr{E}_{\mathrm{H},V^p}(\overline{\mathbf{Q}}_p)$ parametrise locally algebraic characters of Γ_{E,V^p}. Finally, the sheaf $\mathscr{M}_{\mathrm{H},V^p}$ is a trivial line bundle, with actions by G_E given by the universal character

$$\chi_{\mathrm{univ}} \colon G_E \to \Gamma_{E,V^p} \to \mathbf{Z}_p[\![\Gamma_E]\!]^\times, \tag{3.1.10}$$

and by $\mathrm{H}(\mathbf{A}^\infty)$ given by the inverse $\chi^{-1}_{\mathrm{H},\mathrm{univ}}$ of the corresponding automorphic character. We may formally write

$$\mathscr{M}_{\mathrm{H},V^p} = \chi^{-1}_{\mathrm{H},\mathrm{univ}} \otimes \chi_{\mathrm{univ}} \tag{3.1.11}$$

as a tensor product of two trivial sheaves, the first one endowed with the $\mathrm{H}(\mathbf{A})$-action only, and the second one with the Galois action by χ_{univ} only.

3.1.6 Fibres of the sheaves $\mathscr{M}$

Let

$$(\lambda^p, \lambda_p) \colon \mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G},U^p} \to \mathscr{O}(\mathscr{E}_{\mathrm{G},U^p})$$

be the tautological character, and define

$$\begin{aligned} \alpha^\circ \colon F_p^\times &\to \mathscr{O}(\mathscr{E}_{\mathrm{G},U^p})^\times \\ x &\mapsto \lambda_p(\mathrm{U}_x). \end{aligned} \tag{3.1.12}$$

Proposition 3.1.5 *Let $x \in \mathscr{E}^{\mathrm{ord,cl},W}_{\mathrm{G},U^p}$ be a classical point of weight W and level r.*

Then:

1. *Let $U := U^pU_{p,r}$, and let $\mathscr{W}$ be the local system on X associated with W. We have an isomorphism of $\mathscr{H}^{\mathrm{ord}}_{\mathrm{G},U^p,\mathbf{Q}_p(x)}$-modules*

$$(\mathscr{M}_{\mathrm{G},U^p})_x \cong \mathrm{H}_1(X_{U,\overline{F}}, \mathscr{W}_{\mathrm{G},\underline{w}})^{\mathrm{ord}} \otimes_{\mathscr{H}^{\mathrm{sph}}_{\mathrm{G},U^p},\lambda_x} \mathbf{Q}_p(x).$$

2. *There exists a unique automorphic representation π_x of $\mathrm{G}(\mathbf{A})$ over $\mathbf{Q}_p(x)$ of spherical character λ_x^p, weight W, and unit character α_x°. It satisfies the property*

$$\pi_x^{\mathrm{ord},U^p} \cong \mathrm{Hom}_{\mathbf{Q}_p(x)[G_{F,S}]}((\mathscr{M}_{\mathrm{G},U^p})_x, \rho_x) \tag{3.1.13}$$

as left $\mathscr{H}^{\mathrm{ord}}_{\mathrm{G},U} \otimes \mathbf{Q}_p(x)$-modules.

Proof Part 1 follows from Proposition 3.1.2.3.

For part 2, fix an embedding $\mathbf{Q}_p(x) \hookrightarrow \overline{\mathbf{Q}}_p$. By strong multiplicity-one, a representation $\overline{\pi}$ over $\overline{\mathbf{Q}}_p$ with character λ_x^p is unique if it exists. By comparing part 1 with (2.5.3), we find that $\overline{\pi}$ exists and that for such $\overline{\pi}$ property (3.1.13) holds after base-change to $\overline{\mathbf{Q}}_p$. Let $V_{\overline{\pi}}$ be the Galois representation associated with π by Theorem 2.5.1, then by looking at Frobenius traces, we see that $V_{\overline{\pi}}$ has a model V_π over $\mathbf{Q}_p(x)$. It follows again from (2.5.3) that $\pi := \lim_{U'} \mathrm{Hom}\,(H_1(\overline{X}_U, \mathscr{W}), V_\pi)$ is a model of $\overline{\pi}$ that satisfies (3.1.13). □

In the rest of the paper, we will use without further comment the notation π_x for the representation of $\mathrm{G}(\mathbf{A})$ defined above, for $x \in \mathscr{E}^{\mathrm{ord,cl}}_{\mathrm{G},U^p}$.

Corollary 3.1.6 *Let $z \in \mathscr{E}^{\mathrm{ord,cl}}_{K^p}$ be a classical point, and write $z = (x, y)$ via (3.1.9) and $L := \mathbf{Q}_p(z)$. Let ω_x be the central character of π_x, let $\chi_{\mathrm{H},y}$ be the character of $\mathrm{H}(\mathbf{A}^\infty)$ obtained by specialising $\chi_{\mathrm{H,univ}}$, and let χ_y be the corresponding locally algebraic character of $G_{E,S}$. Write $L := \mathbf{Q}_p(z)$. Then $\omega_z := \chi_{y|F_{\mathbf{A}^\infty,\times}}\omega_x = \mathbf{1}$, and*

$$\mathscr{M}_{K^p,z} \cong ((\pi_x^{p,\vee,})^{U^p} \otimes_L \chi_{\mathrm{H},y}^{-1,p}) \otimes_L (V_{x|_{G_{E,S}}} \otimes_{L[G_{E,S}]} \chi_y)$$

as $\mathscr{H}^{K_S}_{S,L} \otimes_L L[G_{E,S}]$-modules. Here, $G_{E,S}$ acts trivially on the first two tensor factors, and the natural action of $\mathscr{H}^{U_S\times V_S}_{\mathrm{G}\times\mathrm{H}}$ on the first two factors is extended trivially to the whole tensor product, and it factors to an action of $\mathscr{H}^{K_S}_S$.

Proof Let $\lambda^p_{x,F}$ and $\lambda^p_{y,F}$ be the restrictions of the characters λ_x, λ_y to $\mathbf{Z}[F^\times_{\mathbf{A}^{Sp\infty,\times}}/K_F^{Sp}]$, and let λ_F be the restriction of $\lambda_{F,x}\lambda_{F,y}$ to $\Delta' = F^\times_{\mathbf{A}^{Sp\infty,\times}}/K_F^{Sp}$. As this groups acts trivially on M_{K^p} by (3.1.7), we have $\lambda_F = \mathbf{1}$. On the other hand λ_F equals the restriction of ω_z to Δ'. We deduce that

ω_z factors through $C = F^{\times}_{\mathbf{A}^\infty}/F^{\times}F_{\mathbf{A}^{Sp\infty}}K_{F,S}K_{F,p}$ for some open compact $K_{F,p} \subset F_p^{\times}$. By weak approximation, $C = \{1\}$, therefore $\omega_z = \mathbf{1}$.

By Proposition 3.1.5, (3.1.11), and (3.1.7), the asserted result holds provided we quotient the right-hand side by the action of $F^{\times}_{\mathbf{A}^\infty}$, however this group acts by ω_z, hence trivially. □

Proposition 3.1.7 *The natural map $\kappa \colon \mathscr{E}^{\mathrm{ord}}_{K^p} \to \mathfrak{W}_{K^p}$ is étale over a neighbourhood of the classical points in $\mathfrak{W}^{\mathrm{cl}}_{K^p}$. In particular, the space $\mathscr{E}^{\mathrm{ord}}_{K^p}$ is regular at all $z \in \mathscr{E}^{\mathrm{ord,cl}}_{K^p}$.*

Proof As κ is finite flat by Proposition 3.1.3, it suffices to check that the fibre of κ over any $x \in \mathfrak{W}^{\mathrm{cl}}_{K^p}(\overline{\mathbf{Q}}_p)$ is isomorphic to $\overline{\mathbf{Q}}_p^m$ for some m. By 3.1.2, 3.1.6 and (2.5.3), this fibre is the spectrum of the image A_x of $\mathscr{H}^{\mathrm{sph,ord}}_{K^p}$ in $\bigoplus_{z\in\kappa^{-1}(x)}(\Pi_z^{K^p,\mathrm{ord}})^{\oplus 2}$, where the Π_z form a list of distinct irreducible representations of $(\mathrm{G}\times\mathrm{H})'(\mathbf{A}^{p\infty})$ over $\overline{\mathbf{Q}}_p$. By strong multiplicity-one, we have $A_x \cong \oplus_z \overline{\mathbf{Q}}_p$. This proves étaleness. As $\mathfrak{W}_{K^p}$ is regular, we deuce that so is $\mathscr{E}^{\mathrm{ord}}_{K^p}$ in a neighbourhood of classical points. □

3.2 Galois representations in families

We recall the existence of a universal family of Galois representations over $\mathscr{X}$.

3.2.1 Representations associated with irreducible pseudocharacters

Recall that an n-dimensional pseudocharacterof G over a scheme $\mathscr{X}$ is a function $T \colon G \to \mathscr{O}(\mathscr{X})$ that 'looks like' the trace of an n-dimensional representation of G over $\mathscr{O}(\mathscr{X})$, see [88] for the precise definition. A pseudocharacter T is said to be (absolutely) irreducible at a point $x \in \mathscr{X}$ if, for any (equivalently, all) geometric point $\overline{x}$ of $\mathscr{X}$ with image x, the pullback $\overline{x}^*T$ is not the sum of two pseudocharacters of dimensions k, $n-k$ with $0 < k < n$. The *irreducibility locus* of T is the set of points of $\mathscr{X}$ at which T is irreducible; it is open [24, § 7.2.3].

We start by proving that, if T is irreducible, a representation with trace T is essentially unique when it exists.

Lemma 3.2.1 *Let $\mathscr{X}$ be an integral scheme and let $\mathscr{V}_1$, $\mathscr{V}_2$ be vector bundles of rank $n > 0$ over $\mathscr{X}$. Suppose that there is an isomorphism $F \colon \mathrm{End}_{\mathscr{O}_\mathscr{X}}(\mathscr{V}_1) \to \mathrm{End}_{\mathscr{O}_\mathscr{X}}(\mathscr{V}_2)$. Then there is an invertible $\mathscr{O}_\mathscr{X}$-module $\mathscr{L}$ and an isomorphism*

$$g \colon \mathscr{V}_1 \cong \mathscr{V}_2 \otimes \mathscr{L}$$

inducing F *in the sense that* $F(T) \otimes \mathrm{id}_{\mathscr{L}} = gTg^{-1}$ *for all sections* T *of* $\mathrm{End}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{V}_1)$.

Proof By [65, Ch. IV], any automorphism of an Azumaya algebra (such as $\mathrm{End}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{V}_i)$) is Zariski-locally inner. Therefore there exists an open cover $\{U_i\}$ of $\mathscr{X}$ and isomorphisms $g_i\colon \Gamma(U_i, \mathscr{V}_1) \to \Gamma(U_i, \mathscr{V}_2)$ such that

$$F(T) = g_i T g_i^{-1} \tag{3.2.1}$$

for all $T \in \mathrm{End}_{\mathscr{O}_{\mathscr{X}}(U_i)}(V_1)$. Let $U_{ij} := U_i \cap U_j$ and

$$c_{ij} := g_i^{-1} g_j, \tag{3.2.2}$$

an automorphism of $\mathscr{V}_1$ over U_{ij}. By (3.2.1), c_{ij} commutes with every $T \in \mathrm{End}_{\mathscr{O}_{\mathscr{X}}(U_{ij})}(\mathscr{V}_1)$, hence it is a scalar in $\mathscr{O}_{\mathscr{X}}(U_{ij})^\times$. One verifies easily that the c_{ij} form a cocycle in $H^1(\mathscr{X}, \mathscr{O}_{\mathscr{X}}^\times)$. Let $\mathscr{L}$ denote the associated invertible sheaf, which is trivialised by the cover $\{U_i\}$. Then we may view $g_i\colon \Gamma(U_i, \mathscr{V}_1) \to \Gamma(U_i, \mathscr{V}_2 \otimes \mathscr{L})$. By (3.2.2), the g_i glue to the desired isomorphism $g\colon \mathscr{V}_1 \cong \mathscr{V}_2 \otimes \mathscr{L}$. □

Lemma 3.2.2 *Let* $\mathscr{X}$ *be an integral scheme and* $\mathscr{T}\colon G_{F,S} \to \mathscr{O}(\mathscr{X})$ *an irreducible pseudocharacter of dimension* n. *Let* $\mathscr{V}_1$, $\mathscr{V}_2$ *be representations of* $G_{F,S}$ *with trace* $\mathscr{T}$. *Then there exist a line bundle* $\mathscr{L}$ *with trivial Galois action and a* $G_{F,S}$*-equivariant isomorphism*

$$\mathscr{V}_1 \cong \mathscr{V}_2 \otimes \mathscr{L}.$$

Proof Write $G = G_{F,S}$ and let $\mathscr{A} := \mathscr{O}_{\mathscr{X}}[G]/\mathrm{Ker}\,(\mathscr{T})$. By [88, Theorem 5.1], $\mathscr{A}$ is an Azumaya algebra of rank 4. By [92, Corollary 2.9], the two natural injective maps $\alpha_i\colon \mathscr{A} \to \mathrm{End}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{V}_i)$ are isomorphisms. Then we conclude by the previous lemma. □

3.2.2 Galois representations in ordinary families

We prove the analogue in Hida families of Theorem 2.5.1.

Lemma 3.2.3 *Let* $\overline{\lambda}\colon \mathbf{T}^{\mathrm{sph,ord}}_{\mathrm{G},U^p} \to \overline{\mathbf{F}}_p$ *be a character. Then there is a unique semisimple representation* $\overline{\rho}\colon G_{F,S} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *such that* $\mathrm{Tr}(\overline{\rho}(\mathrm{Fr}_v)) = q_v^{-1}\overline{\lambda}(T_v)$ *for all* $v \notin S$.

Proof The existence follows by lifting $\overline{\lambda}$ to the character λ_x associated with a classical point x (that is possible thanks to Lemma 3.1.4), then taking the semisimplification of the reduction modulo p of a lattice in the representation $\rho_x := \rho_{\pi_x}$ of Theorem 2.5.1; the uniqueness is a consequence of the Brauer–Nesbitt theorem. □

By Proposition 3.1.3 we may decompose $\mathbf{T}^{\rm sph,ord}_{\rm G,U^p}\cong\prod_{\mathfrak{m}}\mathbf{T}^{\rm sphord}_{{\rm G},U^p,\mathfrak{m}}$ and consequently the generic fibre of the associated schemes also decomposes as

$$\mathscr{E}^{\rm ord}_{{\rm G},U^p}\cong\coprod\mathscr{E}^{\rm ord}_{{\rm G},U^p,\mathfrak{m}}. \tag{3.2.3}$$

We will say that a connected subset $\mathscr{X}\subset\mathscr{E}^{\rm ord}_{{\rm G},U^p}$ has residual representation $\overline{\rho}\colon G_F\to{\rm GL}_2(\overline{\mathbf{F}}_p)$ if $\mathscr{X}$ is contained in some $\mathscr{E}^{\rm ord}_{{\rm G},U^p,\mathfrak{m}}$ such that the character $\lambda_{\mathfrak{m}}\otimes_{\mathbf{F}_p(\mathfrak{m})}\overline{\mathbf{F}}_p$ associated with $\mathfrak{m}$ is the character of $\overline{\rho}$.

Proposition 3.2.4 *Let $\mathscr{X}_{\rm G}$ be an irreducible component of $\mathscr{E}_{\rm G}$ (that is, a Hida family). Then there exist:*

- *an open subset $\mathscr{X}'_{\rm G}\subset\mathscr{X}_{\rm G}$ containing $\mathscr{X}^{\rm cl}_{\rm G}:=\mathscr{X}_{\rm G}\cap\mathscr{E}^{\rm cl}_{{\rm G},U^p}$;*
- *a locally free $\mathscr{O}_{\mathscr{X}'_{\rm G}}$-module $\mathscr{V}_{\rm G}$ of rank two over $\mathscr{X}'_{\rm G}$, such that*

$$\mathscr{V}_{{\rm G},x}\cong V_{\pi_x}$$

  for all $x\in\mathscr{X}^{\rm cl}_{\rm G}$;
- *a filtration*

$$0\to\mathscr{V}^+_{{\rm G},v}\to\mathscr{V}_{{\rm G},v}\to\mathscr{V}^-_{{\rm G},v}\to 0, \tag{3.2.4}$$

  where the $\mathscr{V}^{\pm}_{{\rm G},v}$ are locally free $\mathscr{O}_{\mathscr{X}'_{\rm G}}$-modules of rank 1, and G_{F_v} acts on $\mathscr{V}^+_{{\rm G},v}$ by the character associated, via local class field theory, with the character

$$\alpha^{\circ}_{|F_v^\times}\langle\ \rangle_{F_v} \tag{3.2.5}$$

  deduced from (3.1.12).

The representation $\mathscr{V}_{\rm G}$ is uniquely determined up to automorphisms and twisting by line bundles with trivial Galois action.

The result is due to Hida and Wiles ([41, § 3.2.3] and references therein), except for the existence of (3.2.4) when the residual Galois representation of $\mathscr{X}_{\rm G}$ is reducible.

Proof Let $\mathscr{T}: G_{F,S}\to\mathscr{O}(\mathscr{X}_{\rm G})$ be the pseudocharacter defined by $\mathscr{T}({\rm Fr}_v)=q_v^{-1}\lambda(T_v)$, where $\lambda\colon\mathbf{T}^{\rm sph}_{{\rm G},U^p}\to\mathscr{O}(\mathscr{X}_{\rm G})$ is the tautological character. Let $\mathscr{X}^{\rm irr}_{\rm G}\subset\mathscr{X}_{\rm G}$ be the (open) irreducibility locus. By Theorem 2.5.1, $\mathscr{X}^{\rm cl}_{\rm G}\subset\mathscr{X}^{\rm irr}_{\rm G}$. By Lemma 3.2.2, a representation $\mathscr{V}_{\rm G}$ is unique up to Galois-trivial twists if it exists. We show existence.

By [88, Theorem 5.1], $\mathscr{A} := \mathscr{O}_{\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}}[G_{F,S}]/\mathrm{Ker}\,(\mathscr{T})$ is an Azumaya algebra of rank 4 over $\mathscr{E}_{\mathrm{G}'}$ and the natural map

$$\rho\colon G_{F,S} \to \mathscr{A}^{\times}$$

satisfies $\mathrm{Tr} \circ \rho = \mathscr{T}$ (where Tr is the reduced trace of $\mathscr{A}$). Let $c \in G_{F,S}$ be a complex conjugation; we have an isomorphism $\mathscr{A} = \mathscr{A}(\rho(c) - 1) \oplus \mathscr{A}(\rho(c) + 1) =: \mathscr{V}_{+1} \oplus \mathscr{V}_{-1}$. Each of the c-eigen-summands $\mathscr{V}_{\pm 1}$ is a locally free $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}}$-module (since so is $\mathscr{A}$), whose rank is 2: indeed at any classical geometric point $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}(\mathbf{C}_p)$, the specialisation ρ_x is odd, hence we can pick an isomorphism $\mathscr{A}_x \cong M_2(\mathbf{C}_p)$ sending $\rho_x(c)$ to $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ from which it is immediate that $\mathscr{V}_{\pm 1,x}$ has rank 2; since classical points are dense, we conclude that $\mathscr{V}_{\pm 1}$ also has rank 2.

Let $\mathscr{V}_{\mathrm{G}}$ be either of $\mathscr{V}_{1,\pm}$. By [92, Corollary 2.9 (a)], the natural map

$$\mathscr{A} \to \mathrm{End}_{\,\mathscr{O}_{\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}}}(\mathscr{V}_{\mathrm{G}})$$

is an isomorphism; we view it as an identification to obtain a representation ρ' with trace $\mathscr{T}$. As an irreducible 2-dimensional Galois representation over a field is uniquely determined by its trace, the representation $\mathscr{V}_{\mathrm{G},x}$ is isomorphic to V_{π_x}.

We now show the existence of the filtration up to further restricting the base. Fix a place $v|p$ of F, and let $\det_v\colon G_{F_v} \to \mathscr{O}(\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}})^{\times}$ be the character giving the action on $\det \mathscr{V}_{\mathrm{G},v}$. Let $\mathscr{V}_{0,v}^{+}$ be the trivial sheaf $\mathscr{O}_{\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}}$ with G_{F_v}-action by the character (3.2.5), $\mathscr{V}_{0,v} := \mathscr{V}_{\mathrm{G},v}$, $\mathscr{V}_{0,v}^{-} := (\mathscr{V}_{0,v}^{+})^{-1}(\det_v)$. Finally, for $? = +, -.\emptyset$, let

$$\mathscr{W}_v^{?} := \mathrm{Hom}_{\,\mathscr{O}_{\mathscr{X}'_{\mathrm{G}}}}(\mathscr{V}_{0,v}^{+}, \mathscr{V}_{0,v}^{?}).$$

Then for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, by Theorem 2.5.1 we have exact sequences

$$0 \to \mathscr{W}_{v,x}^{+} = \mathbf{Q}_p(x) \to \mathscr{W}_x \to \mathscr{W}_{v,x}^{-}, \tag{3.2.6}$$

which we wish to extend to a neighbourhood of $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$. From a consideration of weights based on Theorem 2.5.1, we see that for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, $H^0(F_v, \mathscr{W}_{v,x}^{-}) = H^2(F_v, \mathscr{W}_{v,x}^{-}) = 0$. Then from (3.2.6) we deduce

$$H^2(F_v, \mathscr{W}_{v,x}) = 0 \tag{3.2.7}$$

for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, and from the Euler–Poincaré formula and (3.2.6) we deduce that

$$\dim_{\mathbf{Q}_p(x)} H^1(F_v, \mathscr{W}_{v,x}^-) = 1 + [F_v : \mathbf{Q}_p], \quad H^1(F_v, \mathscr{W}_{v,x}) = 1 + 2[F_v : \mathbf{Q}_p] \tag{3.2.8}$$

for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$.

By Proposition 5.2.3.3 below, (3.2.7) and (3.2.8) imply that the natural map

$$H^0(F_v, \mathscr{W}) \otimes_{\mathscr{O}_{\mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}}} \mathbf{Q}_p(x) \to H^0(F_v, \mathscr{W}_{v,x}) \cong \mathbf{Q}_p(x)$$

is an isomorphism for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$. Hence the sheaf $\mathscr{L} := H^0(F_v, \mathscr{W}_v)$ is locally free of rank one in a neighborhood $\mathscr{X}_{\mathrm{G}}' \subset \mathscr{X}_{\mathrm{G}}^{\mathrm{irr}}$ of $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$. Defining

$$\mathscr{V}_{\mathrm{G},v}^+ := \mathscr{L} \otimes_{\mathscr{O}_{\mathscr{X}_{\mathrm{G}}'}} \mathscr{V}_{0,v}^+,$$

the natural map $\mathscr{V}_{\mathrm{G},v}^+ \to \mathscr{V}_{\mathrm{G},v|\mathscr{X}_{\mathrm{G}}'}$ is injective, and its cokernel $\mathscr{V}_{\mathrm{G},v}^-$ has rank one at each $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$. Up to further restricting $\mathscr{X}_{\mathrm{G}}'$, $\mathscr{V}_{\mathrm{G},v}^-$ is also locally free of rank one. It follows immediately from the construction that the exact sequence

$$0 \to \mathscr{V}_{\mathrm{G},v}^+ \to \mathscr{V}_{\mathrm{G},v} \to \mathscr{V}_{\mathrm{G},v}^- \to 0$$

has the asserted properties. □

Proposition 3.2.5 *In the situation of Proposition 3.2.4, the natural injective map*

$$i \colon \mathscr{O}_{\mathscr{X}_{\mathrm{G}}'} \to \mathrm{End}_{\mathscr{O}_{\mathscr{X}_{\mathrm{G}}'}[G_{F,S}]}(\mathscr{V}_{\mathrm{G}})$$

is an isomorphism over an open subset $\mathscr{X}_{\mathrm{G}}'' \subset \mathscr{X}_{\mathrm{G}}$ containing $\mathscr{X}^{\mathrm{cl}}$.

Proof By Theorem 2.5.1, ρ_x is absolutely irreducible for all $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$.

We deduce that for each $x \in \mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$, the map i_x is an isomorphism. Then we may take for $\mathscr{X}_{\mathrm{G}}''$ the open complement of the support of $\mathrm{Coker}(i)$. □

3.3 Universal ordinary representation and local-global compatibility

The idealised description of what is achieved in this subsection would be to define a universal ordinary automorphic representation of $\mathrm{G}(\mathbf{A}^\infty)$ over an irreducible component $\mathscr{X}$ of $\mathscr{E}_{\mathrm{G}}^{\mathrm{ord}}$; then show that it decomposes as the product of the representations of the local groups $\mathbf{B}_v^\times$, for $v \nmid p$,[20] associated to $\mathscr{V}|_{G_{F,v}}$

[20] The action of $\mathbf{B}_p^\times$ has already been traded for an action of the torus, subsumed into the $\mathscr{E}_{\mathrm{G}}$-module structure.

by a local Langlands correspondence in families. The definition should be an elaboration of

$$\text{“}\Pi_{\mathrm{G}} := \lim_{U^{p\prime}} \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}}[G_{F,E}]}(\mathscr{M}_{U^{p\prime}}, \mathscr{V}_{\mathrm{G}})\text{”}. \tag{3.3.1}$$

For technical reasons, a few modifications are necessary:

- the local Langlands correspondence in families is not defined for the unit groups of division algebras;[21] therefore we “remove” the components at the ramification primes Σ of $\mathbf{B}$, in the following way: we consider a component of $\mathscr{E}$ rather than $\mathscr{E}_{\mathrm{G}}$, and we take H'_{Σ}-coinvariants in an analogue Π of (3.3.1). For sufficiently large levels, this isolates a local factor of Π that is generically free of rank one along locally distinguished Hida families;
- in the limit in (3.3.1), we fix an arbitrarily large finite set of primes Σ', disjoint from Σ and from S_p, and we let only the Σ'-component of $U^{p\prime}$ shrink, so as to get a representation of $\mathbf{B}^{\times}_{\Sigma'}$;
- we replace the abstractly constructed $\mathscr{V} = \mathscr{V}_{\mathrm{G}} \otimes \mathscr{V}_{\mathrm{H}}$ (where $\mathscr{V}_{\mathrm{H}} = \chi_{\mathrm{univ}}$) by a more geometric incarnation using the sheaf $\mathscr{M}$ in ‘new’ level (with respect to the chosen irreducible component).

We use the correspondence studied in [34], with the caveat that strictly speaking the normalisation chosen there differs by the one fixed here in Theorem 2.5.1.1 by a Tate twist. This is only a matter of book-keeping, and in order to avoid excessive notational burden, we do not signal such Tate twists when referring to the results of [34] in the rest of this paper.

3.3.1 Irreducible components

Let $\mathscr{X}_{\mathrm{G}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p}$ be an irreducible component. Fix a place v of F not in $\Sigma \cup S_p$. Recall that the v-level of a representation π_v of $\mathrm{GL}_2(F_v)$ is the smallest m such that $\pi_v^{U_1(\varpi_v^m)} \neq 0$, where $U_1(\varpi_v^m) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathscr{O}_{F,v}) \,:\, c \equiv d - 1 \equiv 0 \pmod{\varpi_v^m \mathscr{O}_{F,v}}\}$. Let $m_{x,v}$ be the v-level of $x \in \mathscr{X}^{\mathrm{cl}}$.

Lemma 3.3.1 *The function $x \mapsto m_{x,v}$ is constant on $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}}$.*

Proof By [20], $m_{x,v}$ equals the conductor of the G_{F_v}-representation $\mathscr{V}_x$; as all those Galois representations are pure, we may conclude by [89, Theorem 3.4]. □

We may then define the v-level m_v of $\mathscr{X}_{\mathrm{G}}$ to be the common value of the $m_{x,v}$ for $x \in \mathscr{X}^{\mathrm{cl}}$. By the following lemma, it is not restrictive to make the

[21] There is an essential reason for this, namely the possible presence of Schur indices in representations of those groups.

following assumption: *for all* $v \notin \Sigma \cup S_p$, *we have* $U_v = U_1(\varpi_v^{m_v(\mathscr{X}_{\mathrm{G}})})$. (We say that $\mathscr{X}_{\mathrm{G}}$ is a v-new component of $\mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p}$.)

Lemma 3.3.2 *Let* $'\mathscr{X}_{\mathrm{G}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^{p\prime}}$ *be an irreducible component, and suppose that* $U^{p\prime} = \prod_{v\nmid p} U'_v$. *Let* m_v *be the level of* $\mathscr{X}_{\mathrm{G}}$ *and let* $U^p = \prod_{v\nmid p} U_v$, *with* $U_v = U_1(\varpi_v^{m_v}) \supset U'_v$ *for all* $v \notin \Sigma \cup S_p$, *and* $U_v = U'_v$. *There exists a unique irreducible component* $\mathscr{X}_{\mathrm{G}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p}$ *whose image under the natural embedding* $\mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^{p\prime}}$ *is* $'\mathscr{X}_{\mathrm{G}}$.

Proof Let $x' \in {'\mathscr{X}}^{\mathrm{cl}}_{\mathrm{G}}$ be any classical point. By [19], its level (that is, the level of $\pi_{x,v}$) is m_v if and only if $\pi_{x,v}$ already occurs in the cohomology of $X_{\overline{F}}$ at v-level m_v, equivalently if and only if (the system of Hecke- and U_v-eigenvalues associated with) $\pi_{x,v}$ occurs in a quotient of $\mathscr{M}_{U^p}$; that is, if x' comes from a point x of $\mathscr{E}_{\mathrm{G},U^p}$. Let $\mathscr{X}_{\mathrm{G}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p}$ be the irreducible component containing x, which is unique by Proposition 3.1.7. As $\mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^{p\prime}}$ are equidimensional of the same dimension, the image of $\mathscr{X}_{\mathrm{G}}$ in $\mathscr{E}_{\mathrm{G},U^{p\prime}}$ is an irreducible component, necessarily $'\mathscr{X}_{\mathrm{G}}$. □

We now deal with the level at Σ.

Lemma 3.3.3 *Let* $v \in \Sigma$. *There exists a compact open* $U'_v \subset U_v$ *such that for every classical point* $x' \in \mathscr{X}_{\mathrm{G}}$, *we have* $\pi_{x',v}^{U'_v} = \pi_{x',v}$, *where* $\pi_{x,v}$ *is the local component at* v *of* $\pi_x \otimes \overline{\mathbf{Q}}_p$.

Proof Fix a classical point $x \in \mathscr{X}_{\mathrm{G}} \subset \mathscr{E}^{\mathrm{ord}}_{\mathrm{G},U^p}$, and let $U'_v \subset U_v$ be such that $\pi_{x,v}^{U'_v} = \pi_{x,v}$. (This will hold for sufficiently small U'_v as $\pi_{x,v}$ is finite-dimensional.) We show that U'_v satisfies the desired property at all classical $x' \in \mathscr{X}_{\mathrm{G}}$. Let $\mathfrak{X}_{v/\mathbf{Q}_p}$ be the Bernstein variety of $\mathrm{GL}_2(F_v)$, a scheme over $\mathbf{Q}_p$ (see [34], to which we refer for more background). By [34, Theorem 3.2.1], the representation $\mathscr{V}_{\mathrm{G}}$ of G_{F_v} gives a map $f\colon \mathscr{X}_{\mathrm{G}} \to \mathfrak{X}_{v/\mathbf{Q}_p}$, compatibly with the local Langlands correspondence in the sense that for all $x \in \mathscr{X}_{\mathrm{G}}$, $f(x)$ is the point corresponding to the supercuspidal support of the representation $\pi'_{x,v}$ of $\mathrm{GL}_2(F_v)$ over $\mathbf{Q}_p(x)$ attached to the representation $\mathscr{V}_{\mathrm{G},x}$. Note that for classical points x, $\pi_{x,v} = \mathrm{JL}_v(\pi'_{x,v} \otimes_{\mathbf{Q}_p(x)} \overline{\mathbf{Q}_p})$, where JL_v is the Jacquet–Langlands correspondence.

After base-change to $\overline{\mathbf{Q}}_p$, we may consider the finitely many maps $f_i\colon \mathscr{X}_i \to \mathfrak{X}_{v/\overline{\mathbf{Q}}_p}$, where the $\mathscr{X}_i$ are the connected components of $\mathscr{X}_{\mathrm{G},\overline{\mathbf{Q}}_p}$. The image of f_i is contained in a connected component $\mathfrak{X}_i$ of $\mathfrak{X}_{v/\overline{\mathbf{Q}}_p}$. These components are in bijection with inertial classes of supercuspidal supports for $\mathrm{GL}_2(F_v)$, and for the class $\sigma = \sigma_i$ of $\mathfrak{X}_i$ there are three possibilities:

– σ corresponds to the class of a supercuspidal representation σ_0 of $\mathrm{GL}_2(F_v)$ over $\overline{\mathbf{Q}}_p$. In this case, there is an unramified character $\omega\colon F_v^\times \to \mathscr{O}(\mathfrak{X}_i)^\times$

such that every $y' \in \mathfrak{X}_i$ corresponds to $\sigma_0 \otimes \omega_{y'}$. Hence for every classical $x' \in \mathscr{X}_i$, we have

$$\pi_{x',v} \cong \mathrm{JL}_v(\sigma_0 \otimes \omega_{f_i(x')}) = \mathrm{JL}_v(\sigma_0) \otimes \omega_{f_i(x')} \cong \pi_{x,v} \otimes \omega_{f_i(x')}.$$

As $\omega_{f(x')}$ is unramified, it follows that $\pi_{x',v}^{U_v} = \pi_{x',v}$.

– σ corresponds to the class of the supercuspidal support of $\mathrm{St} \otimes \omega_0$, where St is the Steinberg representation and $\omega_0 \colon F_v^\times \to \overline{\mathbf{Q}}_p^\times$ is a character. Then there exist a closed subset $\mathfrak{X}_i' \subset \mathfrak{X}_i$ and an unramified character $\omega \colon F_v^\times \to \mathscr{O}(\mathfrak{X}_i)^\times$ such that every $y' \in \mathfrak{X}_i'$ corresponds to the supercuspidal support of $\mathrm{St} \otimes \omega_0\omega_{y'}$, and such that every $y' \in \mathfrak{X}_i - \mathfrak{X}_i'$ corresponds to the support of an irreducible principal series representation. It follows that for every classical $x' \in \mathscr{X}_i$, the image $f_i(x') \in \mathfrak{X}_i'$ (since $\pi_{x,v}' \otimes \overline{\mathbf{Q}}_p$ is in the domain of JL_v), and that $\pi_{x',v} = \mathrm{JL}_v(\mathrm{St} \otimes \omega_0\omega_{f_i(x')}) = \omega_0\omega_{f_i(x')} \circ \mathrm{Nm}$. We conclude as above.
– no element of the inertial class σ is the supercuspidal support of a special or supercuspidal representation. This case is excluded as only those representations are in the image of the Jacquet–Langlands correspondence. □

3.3.2 Galois representation from geometry

Let $U^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$, $V^p \subset \mathrm{H}(\mathbf{A}^{p\infty})$ be compact open subgroups. We will consider various compact open subgroups $U^p \subset U_*^p \subset \mathrm{G}(\mathbf{A}^{p\infty})$, and will correspondingly denote by K_*^p be the image of $U_*^p \times V^p$ in $(\mathrm{G} \times \mathrm{H})'(\mathbf{A}^{p\infty})$. Let $\mathscr{X}$ be an irreducible component of $\mathscr{E}_{K^p}^{\mathrm{ord}} \subset \mathscr{E}_{\mathrm{G},U^p}^{\mathrm{ord}} \times \mathscr{E}_{\mathrm{H},V^p}$, and let $\mathscr{X}_\mathrm{G} \subset \mathscr{E}_{\mathrm{G},U^p}^{\mathrm{ord}}$ be the irreducible component such that $\mathscr{X} \subset \mathscr{X}_{\mathrm{G}\times\mathrm{H}} := \mathscr{X}_\mathrm{G} \times \mathscr{E}_{\mathrm{H},V^p}$.

Suppose from now on that $\mathscr{X}$ is locally distinguished by H' (Definition 1.3.1). Let $\mathscr{V}_\mathrm{G}$ be the $\mathscr{O}_{\mathscr{X}_\mathrm{G}''}[G_{F,S}]$-module constructed in Proposition 3.2.4 and Proposition 3.2.5, and let $\mathscr{V}_\mathrm{H}$ be the universal character χ_{univ} of $G_{E,S}$ from (3.1.10). Let

$$\mathscr{X}^{(0)} := \mathscr{X} \cap (\mathscr{X}_\mathrm{G}'' \times \mathscr{E}_{\mathrm{H},V^p})$$

an open subset, and consider the $G_{F,E,S}$-representation

$$\mathscr{V}' := (\mathscr{V}_\mathrm{G} \boxtimes \mathscr{V}_\mathrm{H})_{|\mathscr{X}^{(0)}}$$

We define another sheaf $\mathscr{V}$ with $G_{F,E,S}$-action, that will provide a more convenient and concrete substitute for $\mathscr{V}'$ on (an open subset of) $\mathscr{X}^{(0)}$.

Let

$$U_0^{p\prime} = U^{\Sigma p} \prod_{v\in\Sigma} U_v',$$

with U_v' as in Lemma 3.3.3. Let $K_0^{p\prime} = (U_0^{p\prime} \times V^p) F^\times_{\mathbf{A}^{p\infty}} / F^\times_{\mathbf{A}^{p\infty}}$, and let

$$\mathscr{V} := \mathscr{M}_{K_0^{p\prime}}^{H_\Sigma'},$$

viewed a sheaf over $\mathscr{X}$.

Lemma 3.3.4 *The sheaf $\mathscr{V}$ is a direct summand of $\mathscr{M}_{K_0^{p\prime}}$.*

Proof The group $H_\Sigma' = \prod_{v\in\Sigma} E_v^\times / F_v^\times$ acts on the locally free sheaf $\mathscr{M}_{K_0^{p\prime}}$ through a quotient by an open subgroup. Since H_Σ' is compact, such a quotient is finite; therefore the inclusion $\mathscr{V} \subset \mathscr{M}_{K_0^{p\prime}}$ splits. □

Proposition 3.3.5 *There is an open subset $\mathscr{X}^{(1)} \subset \mathscr{X}$ containing all classical points such that $\mathscr{V}$ is locally free of rank 2 along $\mathscr{X}^{(1)}$. For every $z = (x, y) \in \mathscr{X}^{\rm cl}$ we have*

$$\mathscr{V}_z \cong V_x \otimes \chi_y$$

as a $G_{F,E,S}$-representation.

Proof By Corollary 3.1.6, for $z = (x, y) \in \mathscr{X}^{\rm cl}$ we have

$$\mathscr{M}_{K_0^p,(x,y)} \cong (\pi_x^{\vee,p,U_0^p} \otimes \chi_y^{-1,p}) \otimes (V_x \otimes \chi_y)$$

(where the first pair of factors is a representations of G × H($\mathbf{A}^{p\infty}$) and the second one is a a representation of G_E). By Lemma 3.3.4, taking H_Σ'-invariants commutes with specialisation, and we find that

$$\mathscr{V}_z \cong (\pi_x^{\vee,\Sigma p,U_0^{\Sigma p}} \otimes \chi_y^{-1,\Sigma p}) \otimes (\pi_{x,\Sigma}^\vee \otimes \chi_{y,\Sigma}^{-1})^{H_\Sigma'} \otimes (V_x \otimes \chi_y). \quad (3.3.2)$$

The first factor is 1-dimensional by the theory of local newforms, and the second factor is 1-dimensional by assumption $(\varepsilon_v)'$.

Since the fibre-rank of $\mathscr{V}$ is 2 in the dense set $\mathscr{X}^{\rm cl}$, there is an open neighbourhood of this set over which $\mathscr{V}$ is locally free of rank 2. □

Corollary 3.3.6 *There exist: an open subset $\mathscr{X}^{(2)} \subset \mathscr{X}^{(0)} \cap \mathscr{X}^{(1)}$ containing $\mathscr{X}^{\rm cl}$ such that*

$$\mathrm{End}_{\mathscr{O}_{\mathscr{X}^{(2)}}[G_{F,E,S}]}(\mathscr{V}) = \mathscr{O}_{\mathscr{X}^{(2)}},$$

an invertible sheaf $\mathscr{L}$ over $\mathscr{X}^{(2)}$ with trivial Galois action, and a $G_{F,E,S}$-equivariant isomorphism of sheaves on $\mathscr{X}^{(2)}$

$$\mathscr{V} \cong \mathscr{L} \otimes \mathscr{V}'.$$

Proof By Proposition 3.3.5 and the construction of $\mathscr{V}'$, the representations $\mathscr{V}$, $\mathscr{V}'$ have a common trace $\mathscr{T}\colon G_{F,E,S} \to \mathscr{O}(\mathscr{X}^{(1)})$. Since this is an irreducible pseudocharacter, the assertions follow from Lemma 3.2.2 and (the argument of) Proposition 3.2.5. □

3.3.3 The universal ordinary representation

In what follows, all sheaves $\mathscr{M}_{K^p_*}$ will be considered as sheaves over $\mathscr{X}$ (or open subsets of $\mathscr{X}$). Note that, as the action of H'_Σ on $\mathscr{M}_{\mathrm{G}}$, $\mathscr{M}_{\mathrm{H}}$ commutes with the Galois action, the sheaves $\mathscr{M}_{K^p_*}$ retain an action of $G_{F,E,S}$.

We will use the following well-known fact.

Lemma 3.3.7 *Let R be a ring and let $T\colon M \to N$ be a map of free R-modules of the same rank. The set of those $x \in \operatorname{Spec} R$ such that $T \otimes R/\mathfrak{p}_x$ is an isomorphism is open in* $\operatorname{Spec} R$.

Proof The locus is the complement of $V(\det T)$. □

In what follows, similarly to Sect. 2.1.5, if '?' is any decoration, $U^p_?$ is a subgroup of $\mathrm{G}(\mathbf{A}^{p\infty})$, and $V^p \subset \mathrm{H}(\mathbf{A}^{p\infty})$ is a fixed subgroup, we denote by $K^p_? \subset (\mathrm{G}\times\mathrm{H})'(\mathbf{A}^{p\infty})$ the image of $U^p_? \times V^p$.

Proposition 3.3.8 *Fix a finite set of primes Σ' disjoint from $\Sigma \cup S_p$, such that U_v is maximal for all $v \notin \Sigma' \cup \Sigma \cup S_p$, and consider the set $\mathscr{U}$ of subgroups $U^{p\prime} = \prod_{v\nmid p} U'_v \subset U^p$ with U'_v as in Lemma 3.3.3 for all $v \in \Sigma$, and $U'_v = U_v$ for all $v \notin \Sigma' \cup \Sigma \cup S_p$. (In particular $U'_0 \in \mathscr{U}$.)*

1. *There exists a cofinal sequence $(U^{p\prime}_i)_{i\geq 0} \subset \mathscr{U}$, and open subsets $\mathscr{X}_i \subset \mathscr{X}^{(2)} \subset \mathscr{X}$ containing $\mathscr{X}^{\mathrm{cl}}$ such that $\mathscr{X}_j \subset \mathscr{X}_i$ for $i \leq j$, satisfying the following: there are integers r_i and $G_{F,E}$-equivariant maps*
$$T_i\colon \mathscr{V}^{\oplus r_i} = (\mathscr{M}^{H'_\Sigma}_{K^{p\prime}_0})^{\oplus r_i} \to \mathscr{M}^{H'_\Sigma}_{K^{p\prime}_i}$$
that are isomorphisms over $\mathscr{X}_i$.
2. *For each $U^{p\prime} \in \mathscr{U}$, there is an open subset $\mathscr{X}_{U^{p\prime}} \subset \mathscr{X}^{(2)}$ containing $\mathscr{X}^{\mathrm{cl}}$ such that the restriction to $\mathscr{X}_{U^{p\prime}}$ of*
$$\Pi^{K^{p\prime},\mathrm{ord}}_{H'_\Sigma} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}}[G_{F,E,S}]}(\mathscr{M}^{H'_\Sigma}_{K^{p\prime}}, \mathscr{V}) \tag{3.3.3}$$

is a locally free $\mathscr{O}_{\mathscr{X}_{U^{p\prime}}}$-module, and we have an isomorphism of locally free sheaves with Hecke- and Galois- actions

$$\mathscr{M}_{K^{p\prime}}^{H'_\Sigma} \cong (\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})^\vee \otimes \mathscr{V}.$$

Moreover $\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} \subset \Pi_{H'_\Sigma}^{K^{p\prime\prime},\mathrm{ord}}$ for $U^{p\prime\prime} \subset U^{p\prime}$ via the natural projections $\mathscr{M}_{K^{p\prime\prime}}^{H'_\Sigma} \to \mathscr{M}_{K^{p\prime}}^{H'_\Sigma}$.

3. *The $\mathscr{H}_{\mathrm{G}\times\mathrm{H},\Sigma'}^{K^{p\prime}}$-module $\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ is generated by $\Pi_{H'_\Sigma}^{K_0^{p\prime},\mathrm{ord}}$ over $\mathscr{X}_{U^{p\prime}}$.*
4. *For each $z = (x, y) \in \mathscr{X}^{\mathrm{cl}}$, we have*

$$(\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})_z \cong (\pi_x^{U^{p\prime},\mathrm{ord}})_{H'_\Sigma} \otimes \chi_y,$$

with the notation of (2.4.3).

Proof It suffices to prove part 1 for a sequence of subgroups $U_i^{p\prime} = \prod_{v\nmid p} U'_{i,v}$ that are $\mathbf{B}_S^\times$-*conjugate* to a cofinal sequence (if $U_i^{p\prime\prime} = g_i U_i^{p\prime} g_i^{-1}$ is cofinal and $(U_i^{p\prime}, T_i)$ satisfies the desired condition, then $(U_i^{p\prime\prime}, g_i^{-1} \circ T_i)$ also satisfies the desired condition). We thus take any sequence with $U'_{i,v} = U_1(\varpi^{m_{i,v}})$ for $v \notin \Sigma' \cup \Sigma \cup S_p$, with $m_{i,v} \geq m_v$ and such that $\min_{v\in\Sigma'} m_{i,v} \to \infty$.

Let $r_i = \prod_v (1 + m_{i,v} - m_v)$. By the local theory of oldforms of [21] and the isomorphisms (3.3.2) and

$$(\mathscr{M}_{K_i^{p\prime}})_z^{H'_\Sigma} \cong (\pi_x^{\vee,\Sigma p, U_i^{\Sigma p\prime}} \otimes \chi_y^{-1,\Sigma p}) \otimes (\pi_{x,\Sigma}^\vee \otimes \chi_{y,\Sigma}^{-1})^{H'_\Sigma} \otimes (V_x \otimes \chi_y), \tag{3.3.4}$$

there are Hecke operators

$$T_{v,j_v} \colon \mathscr{M}_{K_0^{p\prime}}^{H'_\Sigma} \to \mathscr{M}_{K_i^{p\prime}}^{H'_\Sigma}$$

such that the map $T_i := \prod_{v\in S} \oplus_{j_v} T_{i,v,j_v}$ is an isomorphism after specialisation at any z in the dense set $\mathscr{X}^{\mathrm{cl}}$. Hence T_i is an isomorphism in an open neighbourhood $\mathscr{X}_i$ of $\mathscr{X}^{\mathrm{cl}}$ (which we possibly shrink to make sure it is contained in $\mathscr{X}^{(2)}$). Together with Lemma 3.3.7, this concludes the proof of part 1. Part 2 is a consequence of part 1 and the absolute irreducibility of $\mathscr{V}$, in the special case $U^{p\prime} = U_i^{p\prime}$, with $\mathscr{X}_{U^{p\prime}} = \mathscr{X}_i$. The general case is deduced from the special case: if $U^{p\prime} \subset U_i^{p\prime}$, let $\mathscr{X}_{U^{p\prime}} := \mathscr{X}_i$ and take on both sides the locally free summands consisting of $U^{p\prime}$-invariants (for the first assertion) or coinvariants (for the second assertion). For part 3, we may again reduce to the special case $U^{p\prime} = U_i^{p\prime}$; then the space $\Pi^{U_i^{p\prime}}$ is generated by the images of the transposes of various "oldforms" degeneracy maps T_i from part 1, that are elements of the Hecke algebra $\mathscr{H}_{\mathrm{G},S}^{U^{p\prime}}$. Finally, part 4 follows from (3.3.4). □

Definition 3.3.9 (*Universal ordinary representation*). Let $\mathscr{U}$ be as in Proposition 3.3.8, and fix an arbitrary $U' \in \mathscr{U}$. Let

$$\mathscr{X}^{(3)} := \mathscr{X}_{U^{p\prime}}$$

be as in Proposition 3.3.8, and let

$$\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}^{(3)}}[G_{F,E,S}]}(\mathscr{M}_{K^{p\prime}}^{H'_\Sigma}, \mathscr{V})$$

as in (3.3.3). The *universal ordinary representation*

$$\Pi_{H'_\Sigma}^{K^{Sp},\mathrm{ord}} \subset {}'\Pi_{H'_\Sigma}^{K^{Sp},\mathrm{ord}} := \varinjlim_{U^{p\prime\prime}\in\mathscr{U}} \Pi^{K^{p\prime\prime},\mathrm{ord}},$$

is the $\mathscr{O}_{\mathscr{X}^{(3)}}[(\mathbf{B}_{\Sigma'}^\times \times E_{\Sigma'}^\times)/F_{\Sigma'}^\times]$-submodule generated by $\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$.

3.3.4 Local-global compatibility

The next theorem describes $\Pi_{H'_\Sigma}^{K^{Sp\prime},\mathrm{ord}}$, as a sheaf with an action by $\mathbf{B}_{\Sigma'}^\times \times E^{\Sigma\infty,\times}$, in terms of the local Langlands correspondence in families of [34], denoted by

$$\mathscr{V}_\mathrm{G} \mapsto \pi_{\mathrm{G},\Sigma'}(\mathscr{V}_\mathrm{G}).$$

This correspondence attaches, to any family $\mathscr{V}_\mathrm{G}$ of representations of $\prod_{v\in\Sigma'} G_{F_v}$ on a rank-2 locally free sheaf over a Noetherian scheme $\mathscr{Y}/\mathbf{Q}$, a family of representations of $\mathrm{GL}_2(F_{\Sigma'})$ on a torsion-free sheaf over $\mathscr{Y}$. The latter representation is *co-Whittaker* in the sense of [34, Definition 4.2.2]; in particular it admits a unique Whittaker model.

Theorem 3.3.10 (Local-global compatibility). *Let*

$$\pi_{\mathrm{G},\Sigma'}(\mathscr{V}_\mathrm{G})$$

be the representation of $\mathrm{GL}_2(F_{\Sigma'})$ *over* $\mathscr{X}^{(3)}$ *associated with* $\mathscr{V}_\mathrm{G}$ *by the local Langlands correspondence in families for* $\mathrm{GL}_2(F_{\Sigma'})$ *of* [34]*; let* $\chi_{\mathrm{H,univ},\Sigma'}$ *be the pullback to* $\mathscr{X}^{(3)}$ *of the sheaf* $\chi_{\mathrm{H,univ}}$ *of* (3.1.11)*, with the* $\mathrm{H}(\mathbf{A}^\infty)$*-action restricted to* $E_{\Sigma'}^\times$.

Then there exist an open subset $\mathscr{X}^{(4)} \subset \mathscr{X}^{(3)} \subset \mathscr{X}$ *containing* $\mathscr{X}^{\mathrm{cl}}$*, a line bundle* $\Pi_{H'_\Sigma}^{K^{Sp\prime},S,\mathrm{ord}}$ *over* $\mathscr{X}^{(4)}$*, and an isomorphism of* $\mathscr{O}_{\mathscr{X}^{(4)}}[\mathrm{GL}_2(F_{\Sigma'}) \times E_{\Sigma'}^\times]$*-modules*

$$\Pi_{H'_\Sigma}^{K^{Sp\prime},\mathrm{ord}} \cong (\pi_{\mathrm{G},\Sigma'}(\mathscr{V}_\mathrm{G}) \otimes_{\mathscr{O}_{\mathscr{X}^{(3)}}} \chi_{\mathrm{H,univ},\Sigma'}) \otimes \Pi_{H'_\Sigma}^{K^{Sp\prime},S,\mathrm{ord}}.$$

Proof For $* = \emptyset, '$, consider

$${}^*\pi'_{\mathrm{G},\Sigma'} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}^{(3)}}[E_S^\times]}(\chi_{\mathrm{H,univ},S}, {}^*\Pi_{H'_\Sigma}^{K^{Sp},\mathrm{ord}}),$$

a torsion-free sheaf over $\mathscr{X}^{(3)}$ with action by $\mathbf{B}_{\Sigma'}^\times = \mathrm{GL}_2(F_{\Sigma'})$. There is an obvious isomorphism

$${}^*\Pi_{H'_\Sigma}^{K^{Sp},\mathrm{ord}} \cong {}^*\pi'_{\mathrm{G},\Sigma'} \otimes \chi_{\mathrm{H,univ},\Sigma'}. \tag{3.3.5}$$

By Proposition 3.3.8.4 and the local freeness of each ${}'\Pi_{H'_\Sigma}^{K^{p\prime\prime}}$ near $\mathscr{X}^{\mathrm{cl}}$, the fibre of $({}'\pi'_{\mathrm{G},S})^{U'_S}$ at any $z = (x, y) \in \mathscr{X}^{\mathrm{cl}}$ equals $\pi_{\mathrm{G},S}(\mathscr{V}_{\mathrm{G},x})^{U''_S}$; by Proposition 3.3.8.3 the same is true if one replaces $(\pi'_{\mathrm{G},\Sigma'})^{U''_{\Sigma'}}$ by the submodule $(\pi'_{\mathrm{G},\Sigma'})^{U''_S}$. In conclusion, taking the limit over $U^{p\prime\prime} \in \mathscr{U}$ we find that the smooth, finitely generated, admissible $\mathscr{O}_{\mathscr{X}^{(3)}}[\mathrm{GL}_2(F_{\Sigma'})]$-module $\pi'_{\mathrm{G},\Sigma'}$ satisfies

$$\pi'_{\mathrm{G},\Sigma',(x,y)} \cong \pi_{\mathrm{G},\Sigma'}(\mathscr{V}_{\mathrm{G},x}).$$

for all $(x, y) \in \mathscr{X}^{\mathrm{cl}}$. Then by [34, Theorem 4.4.3], there exist an open subset $\mathscr{X}^{(4)} \subset \mathscr{X}^{(3)}$ containing $\mathscr{X}^{\mathrm{cl}}$ and a line bundle that we denote by $\Pi_{H'_\Sigma}^{K^{Sp},S,\mathrm{ord}}$, such that

$$\pi'_{\mathrm{G},\Sigma'} \cong \pi_{\mathrm{G},\Sigma'}(\mathscr{V}_\mathrm{G}) \otimes \Pi_{H'_\Sigma}^{K^{Sp},S,\mathrm{ord}}.$$

Substituting in (3.3.5) gives the desired result. □

4 Pairings

4.1 Global dualities

We construct Hecke- and/or Galois-equivariant duality pairings on the sheaves constructed in the previous section. The results of this somewhat technical subsection are summarised in Propositions 4.1.7, 4.1.8.

4.1.1 Pairings, symmetries and involutions

If $\epsilon \in \{\pm 1\}$, R is a ring or sheaf of rings, and M, J are R-modules, an R-bilinear pairing $(\ ,\ ): M \otimes M \to J$ is said to be ϵ-*symmetric* if it satisfies $(m, m') = \epsilon \cdot (m', m)$. If R is equipped with an involution ι, denote by $(\cdot)^\iota$ both the functor $\otimes_{R,\iota} R$ and the maps $m \mapsto m \otimes 1$; an (R, ι)-sesquilinear pairing $(\ ,\ ): M \otimes M^\iota \to J$ is said to be ϵ-*hermitian* if it satisfies $(m, n) = \epsilon \cdot \iota((n^\iota, m^\iota)^\iota)$. We will also use the prefix 'skew-' (respectively no prefix) if $\epsilon = -1$ (respectively $+1$).

4.1.2 Involutions

We denote by the same name ι the involutions on $\mathscr{H}^{\rm sph}_{\rm G_*}$, $\mathscr{H}^{\rm sph, ord}_{\rm G_*}$ $\Lambda_{{\rm G}_*, U_*^p}$, $\mathscr{E}^{\rm ord}_{\rm G_*}$ deduced from those of Sect. 2.1.4. If M is a module over any of the above rings (or sheaf of modules over any of the above spaces), we let $M^\iota = \iota^* M$.

Lemma 4.1.1 *Let W be an irreducible algebraic representation of* $\rm G_*$ *over L.*

1. *We have $\sigma_{W^\vee}(t) = \sigma_W(t^\iota)$ for all $t \in T_{\rm G_*}$.*
2. *If $\pi^{\rm ord}$ is the ordinary part of an automorphic representation of* ${\rm G}_*(\mathbf{A}^\infty)$ *over L of weight W, unramified of level U^S_* outside of a finite set of primes S, then there is an isomorphism of $\mathscr{H}^{\rm sph, ord}_{{\rm G}_*, U^S}$-modules $\pi^{\vee, U^S_*, \rm ord} \cong (\pi^{U^S_*, \rm ord})^\iota$.*
3. *There is an identification*

$$(\mathscr{E}^{{\rm cl}, W}_{G_*})^\iota = \mathscr{E}^{{\rm cl}, W^\vee}_{\rm G_*}$$

such that $\pi^{\rm ord}_{\iota(x)} = (\pi_x)^{\vee, \rm ord}$.

Proof All results can be reduced to the case ${\rm G}_* = {\rm H}$, which is trivial, or ${\rm G}_* = {\rm G}$, that we address. Part 1 follows from the explicit description of σ_W in (2.2.5) and $W^\vee_{{\rm G}, (w; (w_\sigma))} \cong W_{{\rm G}, (-w; (w_\sigma))}$ (see (A.4.2) below for an explicit duality).

For part 2, we use $\pi^\vee = \pi \otimes \omega^{-1}$ where ω is the central character of π, and verify the statement separately for the spherical Hecke algebra and for the operators ${\rm U}_t$. For the former, it is well known that the spherical Hecke algebra is generated by operators $T(z)$ and $T(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)) = T(\left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right))$ for $z, x \in F_S^\times$; denoting by $\lambda_{\pi^?}(\cdot)$ the eigenvalue of $T(\cdot)$ on $(\pi^?)^{U^S}$, we then have $\lambda_\pi(z^\iota) = \lambda_\pi(z^{-1}) = \omega(z)^{-1} = \lambda_{\pi^\vee}(z)$, and

$$\lambda_\pi((\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)^\iota) = \lambda_\pi(x^{-1}(\left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right)) = \omega(x)^{-1}\lambda_\pi(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)) = \lambda_{\pi^\vee}(\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right))$$

as desired.

For the operators U_t, we verify that if π is ordinary at v with unit character $\alpha_v^\circ = \alpha_v \sigma_W^{-1}$ (as a character of $T^+_{\mathrm{G},v}$), then $\pi^\vee$ is ordinary at v with unit character

$$\alpha_v^{\circ,\iota} \colon t \mapsto \alpha_v^\circ(t^\iota).$$

This follows from observing

$$\pi_v^\vee \cong \mathrm{Ind}(\alpha_v^\iota \cdot (|\ |_v, |\ |_v^{-1})),$$
$$\alpha_v^\circ(t^\iota) = \alpha_v^\circ(t)\alpha_v^\circ(\nu(t))^{-1} \in \mathscr{O}_F^\times.$$

Finally, part 3 follows from parts 1 and 2. □

4.1.3 Homological and cohomological dualities

We shall define various pairings $\langle\ ,\ \rangle_?$ in the (ordinary, completed) homology of Shimura varieties, starting from the Poincaré duality pairings. Then we will use them to construct corresponding pairings $(\ )_?$ on spaces of representations, as follows.

Construction 4.1.2 *Let A be a ring, G a group, and let M_1, M_2, V_1, V_2, $A(d)$ be $A[G]$-modules, projective and of finite type over A; denote*

$$V^D := \mathrm{Hom}\,(V, A(d)).$$

Let $f_i \in \pi_i := \mathrm{Hom}\,(M_i, V_i)$ be $A[G]$-maps, suppose we have fixed an identification $V_2^D \cong V_1$; let $\langle\ ,\ \rangle$ be a perfect pairing $M_1 \times M_2 \to A(d)$, inducing $u_{\langle,\rangle} \colon M_2^D \to M_1$. Let $f_2^D \colon V_2^D \cong V_1 \to M_2^D$ be the dual map, then we define a pairing on $\pi_1 \times \pi_2$ by

$$(f_1, f_2)_{\langle,\rangle} = f_1 \circ u_{\langle,\rangle}(f_2^D) \in \mathrm{End}_{A[G]}(V_1). \tag{4.1.1}$$

4.1.4 Homological dualities/1

Fix lattices W° and $W^{\vee,\circ}$ on any right algebraic representation of G_* over L, and denoted by $\langle\ ,\ \rangle^W \colon W \otimes W^\vee \to L$ the natural invariant pairing. This may not preserve the lattices but it does so up to a bounded denominator which we denote by $p^{-|W|}$.[22]

[22] With respect to the model in (A.4.2), we have $|W| = \mathrm{ord}_p\left(\binom{k-2}{(k-2+l)/2}\right)$ for the representation (A.2.2) of G.

We may then consider the Poincaré duality pairings

$$\langle , \rangle_{U_*,W} : H_d(\overline{X}_{*,U_*}, \mathscr{W}) \times H_d(\overline{X}_{*,U_*}, \mathscr{W}^\vee) \to H_0(\overline{X}_{*,U_*}, \mathscr{W} \otimes \mathscr{W}^\vee) \to L(d), \quad (4.1.2)$$

where the second map is induced by $\langle , \rangle^W$ and summation over the connected components of $\overline{X}_{*,U_*}$. These pairings are integral up to a bounded denominator $p^{-|W|}$ and satisfy

$$\langle xT, y\rangle_{U_*,W} = \langle x, yT^\iota\rangle_{U_*,W}$$

for any T in $\mathscr{H}^p_{\mathrm{G}_*,U_*}$, as well as the projection formula

$$\langle \mathrm{p}_{U'_*/U_*,*}(x), y\rangle_{W,U_*} = \langle x, \mathrm{p}^*_{U'_*/U_*}(y)\rangle_{W,U'_*} \quad (4.1.3)$$

for all pairs of levels $U'_* \subset U_*$; here $\mathrm{p}_{U'_*/U_*} : X_{U'_*} \to X_{U_*}$ is the projection.

4.1.5 Homological dualities/2

We start to promote and modify the Poincaré duality pairings. The following lemma is clear.

Lemma 4.1.3 *Let R be a ring, S a finite R-algebra, M a finite S-module.*

1. *Suppose that S is étale over R. Then there is a natural isomorphism*

$$\alpha : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, R)) \to \mathrm{Hom}_S(M, S)$$

where the first map is $\lambda \mapsto (m \mapsto (s \mapsto \lambda(sm))$. and the second one comes from the isomorphism $S \cong \mathrm{Hom}_R(S, R)$ induced by the relative trace map.

2. *Suppose that $S = R[T]$ for a finite abelian group T, then there is an isomorphism $\beta : \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_{R[T]}(M, R[T])$ given by $\lambda \mapsto (m \mapsto \sum_t \lambda(tm)[t^{-1}])$.*

If $S = R[T]$ is étale over R then we have $\alpha(\lambda) = |T|^{-1}\beta(\lambda)$.

If $S = R[T]$ for a finite abelian group T, one verifies that the isomorphism of the lemma is given by

$$\langle\ ,\ \rangle \mapsto \langle\langle\ ,\ \rangle\rangle, \quad \langle\langle x, y\rangle\rangle := \sum_{t\in T} \langle x, ty\rangle[t^{-1}].$$

We may apply case 2 of the lemma to

$$M = M_{\mathrm{H},V^p,r,W} \otimes M_{\mathrm{H},V^p,r,W^\vee}, \quad R = L,$$

$$S = \Lambda_{\mathrm{H},V^p,r} := \Lambda^{\circ}_{\mathrm{H},V^p,r} \otimes_{\mathscr{O}_L} L \cong L[\overline{T}_{\mathrm{H},0}/\overline{T}_{\mathrm{H},r}]$$

(with the isomorphism of (3.1.3)). We obtain, from the pairings $\langle\ ,\ \rangle_{V_r,W}$, pairings

$$\langle\langle\ ,\ \rangle\rangle_{V^p,W,r} \colon M_{\mathrm{H},V^p,W,r} \otimes_{\Lambda_{\mathrm{H},W,r}} M^{\iota}_{\mathrm{H},V^p,W^{\vee},r} \to \Lambda_{\mathrm{H},V^p,W,r} \otimes L,$$

and thanks to an easily verified compatibility, a well-defined pairing

$$\begin{aligned}\langle\langle\ ,\ \rangle\rangle_{V^p,W} \colon M_{\mathrm{H},V^p,W} \otimes_{\Lambda_H} M^{\iota}_{\mathrm{H},V^p,W^{\vee}} &\to \Lambda_{\mathrm{H}} \otimes L = \mathscr{O}_{\mathscr{E}_{\mathrm{H}}} \otimes L \\ x \otimes y &\mapsto \lim_r \langle\langle x_r, y_r\rangle\rangle_{V^p,W,r}.\end{aligned} \tag{4.1.4}$$

4.1.6 Automorphic inner products

Let

$$\mathrm{v}(U_*) := \mathrm{vol}(X_{*,U_*}(\mathbf{C})),$$

where 'vol' denotes the volume with respect to the metric deduced from the hyperbolic metric $dxdy/2\pi y^2$ (using the complex uniformisation (2.3.1)), when $\mathrm{G}_* = \mathrm{G}$, and the counting metric, when $\mathrm{G}_* = \mathrm{H}$. By [107, Lemma 3.1], $\mathrm{v}(U_*) \in \mathbf{Q}^{\times}$ and, when $d = \dim X_* = 1$, it equals the degree of the Hodge bundle L_{U_*} defined as in *loc.cit*. We have

$$\deg \mathrm{p}_{U'_*,U_*} = \mathrm{v}(U'_*)/\mathrm{v}(U_*) = \mathrm{Z}_{\mathrm{G}_*}(\mathbf{Q}) \cap U_* \backslash U_*/U'_*, \tag{4.1.5}$$

where the last equality can be easily seen e.g. from the complex uniformisation (2.3.1). We set for any $r \geq 1$

$$\mathrm{v}(U^p_*) := \frac{\mathrm{v}(U^p_* U_{*,0}(p^r)_p)}{p^{dr[F:\mathbf{Q}]}}, \tag{4.1.6}$$

where $U_{*,0}(p^r)_p \subset \mathrm{G}_*(\mathbf{Q}_p)$ is a maximal compact subgroup if $\mathrm{G}_* = \mathrm{H}, \mathrm{H}'$, it is the group of those matrices that are upper triangular modulo p^r if $\mathrm{G}_* = \mathrm{G}$, and it is deduced from those by product and quotient if $\mathrm{G}_* = \mathrm{G} \times \mathrm{H}$, $(\mathrm{G} \times \mathrm{H})'$. The right hand side of (4.1.6) is independent of $r \geq 1$.

Let π be an automorphic representation of $\mathrm{G}_*(\mathbf{A}^{\infty})$ of weight W^* over L, V_π the corresponding G_{E_*}-representation. Then we have an isomorphism $V_{\pi^{\vee}} \cong V^*_\pi(1)$, hence we may use Construction 4.1.2 with $A = L$, $G = G_{E_*}$, $M_1 = \mathrm{H}_d(\overline{X}_{*,U_*}, \mathscr{W})$, $M_2 = \mathrm{H}_d(\overline{X}_{*,U_*}, \mathscr{W}^{\vee})$, $V_1 = V_\pi$, $V_2 = V_{\pi^{\vee}}$ and the pairings (4.1.2). Using (2.5.3), we obtain

$$(\ ,\ )_{\pi,U_*} := (\ ,\ )_{\langle,\rangle_{U_*,W}} \colon \pi^{U_*} \times \pi^{\vee,U_*} \to L.$$

One verifies thanks to (4.1.3) and (4.1.5) that the pairing

$$(\, , \,)_\pi := \lim_{U_*} (\dim W \cdot \mathrm{v}(U_*))^{-1} \cdot (\, , \,)_{\pi, U_*} : \pi \times \pi^\vee \to L. \quad (4.1.7)$$

is well-defined.

When $\mathrm{G}_* = \mathrm{H}$, denoting $\pi = \chi_{\mathrm{H}}$, we may alternatively apply Construction 4.1.2 to A, M_1, M_2, V_1, V_2 as above and the image of the pairings $\langle\langle \, , \, \rangle\rangle_{V^p, rW}$ under the map $\Lambda_{\mathrm{H},r,W} \to L$ given by $[t] \mapsto \chi_{\mathrm{H}}(t)$, and denote the resulting pairings on $\chi_{\mathrm{H}} \times \chi_{\mathrm{H}}^{-1}$ by $(\, , \,)_{\langle\langle , \rangle\rangle_{\chi_{\mathrm{H}}, V^p, r}}$. As $|\overline{T}_{\mathrm{H},0}/\overline{T}_{\mathrm{H},r}| \cdot \mathrm{v}(V^p) = \mathrm{v}(V^p V_{p,r})$ by (4.1.5), we have

$$(\, , \,)_{\chi_{\mathrm{H}}} = \mathrm{v}(V^p)^{-1} (\, , \,)_{\langle\langle , \rangle\rangle_{\chi_{\mathrm{H}}, V^p, r}},$$

and in particular the right-hand side is independent of V^p.

Assume for the rest of this subsection that $\mathrm{G}_* = \mathrm{G}, \mathrm{G} \times \mathrm{H}, (\mathrm{G} \times \mathrm{H})'$. Then we need a twist in order to isolate the toric action and to obtain the ι-equivariance of the pairings under the action of the $\mathrm{U}_{p\infty}$-operators.

Let $\pi = \pi^\infty \otimes W$ be an ordinary representation of $\mathrm{G}_*(\mathbf{A})$. Using the transformation $w_{\mathrm{a}}^{\mathrm{ord}}$ defined in Proposition A.2.1, we define a pairing

$$(f_1, f_2)_\pi^{\mathrm{ord}} := \dim W \cdot (w_{\mathrm{a}}^{\mathrm{ord}} f_1, f_2)_\pi : \pi^{\mathrm{ord}} \times \pi^{\vee, \mathrm{ord}} \to L. \quad (4.1.8)$$

See Lemma A.2.2 for its nondegeneracy.

4.1.7 Homological dualities/3

Analogously to the previous paragraph, we define a twisted Poincaré pairing

$$\begin{aligned} \mathrm{H}_1(\overline{X}_{*,U_*^p,r}, \mathscr{W})^{\mathrm{ord}} \otimes \mathrm{H}_1(\overline{X}_{*,U_*^p,r}, \mathscr{W}^\vee)^{\mathrm{ord}} &\to L(1) \\ \langle x, y \rangle_{U_*^p, W, r}^{\mathrm{ord}} &:= \langle x, y w_{\mathrm{a}}^{\mathrm{ord}} \rangle_{U_*^p U_{*,p,r}, W}, \end{aligned} \quad (4.1.9)$$

of which we will especially consider the restriction to the ordinary parts of homology.

Lemma 4.1.4 *Let π be an ordinary representation of $\mathrm{G}_*(\mathbf{A})$, and identify*

$$\pi^{\mathrm{ord}} = \mathrm{Hom}_{L[G_{E_*}]}(\mathrm{H}_1(\overline{X}_{*,U_*^p,r}, \mathscr{W})^{\mathrm{ord}}, V_\pi)$$

for sufficiently large r similarly to Proposition 3.1.5. Then Construction 4.1.2 provides a pairing $(,)_{\langle,\rangle_{W,r}^{\mathrm{ord}}}$ on $\pi^{\mathrm{ord}} \times \pi^{\vee,\mathrm{ord}}$; it is related to (4.1.8) by

$$(,)_\pi^{\mathrm{ord}} = \mathrm{v}(U_*)^{-1} \cdot (,)_{\langle,\rangle_{U*,W,r}^{\mathrm{ord}}}. \quad (4.1.10)$$

Proof This follows by chasing the definitions. □

By applying case 2 of Lemma 4.1.3 as in (4.1.4), corrected by a factor $p^{r[F:\mathbf{Q}]}$,[23] we obtain from (4.1.9) pairings

$$\begin{aligned} \langle\langle , \rangle\rangle_{U_*^p,W,r} \colon \mathrm{H}_1(\overline{X}_{*,U_*^p,r}, \mathscr{W}) \otimes_{\mathbf{Z}_p} \mathrm{H}_1(\overline{X}_{*,U_*^p,r}, \mathscr{W}^\vee) &\to \Lambda_{\mathrm{G}_*,U_*^p,r}(1) \\ x \otimes y \mapsto p^{r[F:\mathbf{Q}]} \sum_{t \in \overline{T}_{\mathrm{G}_*,0}/\overline{T}_{\mathrm{G}_*,r}} \langle x, y \rangle^{\mathrm{ord}}_{U_*^p,W,r} \end{aligned} \tag{4.1.11}$$

Lemma 4.1.5 *The parings (4.1.11) satisfy $\langle\langle x_r T, y_r \rangle\rangle_{W,r} = \langle\langle x_r, y_r T^\iota \rangle\rangle_{U_*^p,W,r}$ for all $T \in \mathscr{H}^{\mathrm{ord}}_{\mathrm{G},r}$ and all $x_r \in H_d(\overline{X}_{*,U_*^p,r}, \mathscr{W})$, $y_r \in H_d(\overline{X}_{*,U_*^p,r}, \mathscr{W}^\vee)$.*

For $z \in M_{\mathrm{G}_*,\mathscr{W}}$, denote by z_r its image in $M_{\mathrm{G}_*,\mathscr{W},r} := H_d(\overline{X}_{*,U_*^p,r}, \mathscr{W})^{\mathrm{ord}}$. The pairing

$$\begin{aligned} \langle\langle\ ,\ \rangle\rangle_{\Lambda,U_*^p,W} \colon M_{\mathrm{G}_*,U_*^p,\mathscr{W}} \otimes_{\mathscr{H}^{\mathrm{ord}}_{\mathrm{G}}} M^\iota_{\mathrm{G}_*,U_*^p,\mathscr{W}^\vee} &\to \Lambda_{\mathrm{G}_*,U_*^p}(d) \otimes L \\ \langle\langle x, y^\iota \rangle\rangle_{\Lambda,U_*^p,W} &:= \lim_r \langle\langle x_r, y_r^\iota \rangle\rangle_{U_*^p,W,r} \end{aligned} \tag{4.1.12}$$

is well-defined.

The above construction is a minor variation on the one of [41, § 2.2.4], to which we refer for the proof of the lemma. As usual, when $W = \mathbf{Q}_p$ we shall omit it from the notation.

Lemma 4.1.6 *The diagram*

$$\begin{array}{ccc} M_{\mathrm{G}_*,U_*^p,\mathscr{W}} \otimes_{\mathscr{H}^\circ_{\mathrm{G}_*}} M^\iota_{\mathrm{G}_*,U_*^p,\mathscr{W}^\vee} & \xrightarrow{\langle\langle , \rangle\rangle_{\Lambda,U_*^p,W}} & \Lambda_{\mathrm{G}_*,U_*^p}(d) \otimes L \\ \uparrow{\scriptstyle j_W \otimes j_{W^\vee}} & & \uparrow{\scriptstyle \cong} \\ M_{\mathrm{G}_*,U_*^p} \otimes_{\mathscr{H}^\circ_{\mathrm{G}_*}} M^\iota_{\mathrm{G}_*,U_*^p} & \xrightarrow{\langle\langle , \rangle\rangle_{\Lambda,U_*^p}} & \Lambda_{\mathrm{G}_*,U_*^p}(d) \otimes L, \end{array}$$

where the left vertical map comes from Proposition 3.1.2.2 and the right vertical map is $[t] \mapsto \sigma_W^{-1}(t)[t]$, is commutative.

Proof For simplicity we write down the proof for the group $\mathrm{G}_* = \mathrm{G}$ and we drop the subscripts U^p. Poincaré duality and the pairings $\langle\ ,\ \rangle^W$ preserve integral structures up to $p^{-|W|}$. Then by construction it suffices to show the identity

$$\langle j_{W,r}(x), j_{W^\vee,r}(y) \rangle^{\mathrm{ord}}_W \equiv \langle x, y \rangle^{\mathrm{ord}} \pmod{p^{r-|W|}\mathscr{O}_L}$$

[23] This factor accounts for the '$K_0(p^r)$'-part of the level.

for all $r \geq 1$ and x, y in $H_d(\overline{X}_r, \mathbf{Z}_p)$.

By definition in (3.1.8), we have

$$j_{W,r}(x) = x \otimes \zeta \otimes \zeta^*,$$

where $\zeta_r \in W^{\circ,N_0}/p^r$ and $\zeta_r^* \in W^{\circ}_{N_0}/p^r$ are elements pairing to 1; we denote by $\zeta_r^{\vee}$, $\zeta_r^{\vee,*}$ the analogous elements for $j_{W^{\vee},r}$. Then we need to show that

$$\langle (x \otimes \zeta_r \otimes \zeta_r^*) w_{\rm a}^{\rm ord}, y \otimes \zeta_r^{\vee} \otimes \zeta_r^{\vee *} \rangle = \langle x, y \rangle.$$

By the definition of $w_{\rm a}^{\rm ord}$ in Proposition A.2.1, this reduces to the identity

$$\langle \zeta_r w_0, \zeta_r^{\vee} \rangle \cdot \langle \zeta_r^* w_0, \zeta_r^{\vee,*} \rangle = \langle \zeta_r, \zeta_r^* \rangle \cdot \langle \zeta_r^{\vee}, \zeta_r^{\vee,*} \rangle = 1,$$

which can be immediately verified using an explicit model for the pairing such as given in (A.4.2). □

4.1.8 Dualities over Hida families

Let $\mathscr{X}$ be an irreducible component of $\mathscr{E}^{\rm ord}_{K^p}$. By Proposition 3.1.7, the map $\mathscr{E}^{\rm ord}_{K^p} \to \operatorname{Spec} \Lambda_{\mathbf{Q}_p}$ is étale in a neighbourhood $\mathscr{X}'$ of $\mathscr{X}^{\rm cl}$, hence we may apply case 1 of Lemma 4.1.3 to deduce from (4.1.12) a pairing

$$\langle\langle\ ,\ \rangle\rangle_{K^p{\rm ord}} : \mathscr{M}_{K^p{\rm ord}} \otimes_{\mathscr{O}_{\mathscr{X}'}} \mathscr{M}^{\iota}_{K^{p\prime}} \to \mathscr{O}_{\mathscr{X}'}(1). \qquad (4.1.13)$$

We summarise the situation.

Proposition 4.1.7 (Duality). *Let $\mathscr{X}^{(5)} \supset \mathscr{X}^{\rm cl}$ be the intersection of the subset $\mathscr{X}^{(4)}$ of Theorem 3.3.10 with the locus where the map $\mathscr{X} \to \operatorname{Spec} \Lambda_{\mathbf{Q}_p}$ is étale. There exist*

- *a perfect, G_E-equivariant, skew-hermtian pairing*

$$\mathscr{M}^{H'_{\Sigma}}_{K^{p\prime}} \otimes_{\mathscr{O}_{\mathscr{X}^{(5)}}} \mathscr{M}^{H'_{\Sigma},\iota}_{K^{p\prime}} \to \mathscr{O}_{\mathscr{X}^{(5)}}(1). \qquad (4.1.14)$$

  induced from (4.1.13);
- *a perfect, G_E-equivariant, skew-hermtian pairing*

$$\mathscr{V} \otimes_{\mathscr{X}^{(5)}} \mathscr{V}^{\iota} \to \mathscr{O}_{\mathscr{X}^{(5)}}(1).. \qquad (4.1.15)$$

- *a perfect pairing*

$$((\ ,\ )) := {\rm v}(K^{p\prime})^{-1} \cdot (\ ,\ )_{\langle\langle,\rangle\rangle_{K^{p\prime}}} : \Pi^{K^{p\prime}}_{H'_{\Sigma}} \otimes_{\mathscr{O}_{\mathscr{X}^{(5)}}} (\Pi^{K^{p\prime}}_{H'_{\Sigma}})^{\iota} \to \mathscr{O}_{\mathscr{X}^{(5)}}, \qquad (4.1.16)$$

where $(\ ,\ )_{\langle\langle,\rangle\rangle_{K^{p\prime}}}$ *is deduced from* (4.1.14)*,* (4.1.15) *and the isomorphism of Proposition* 3.3.8.2 *via Construction* 4.1.2*.*

Proof Observe that the natural map

$$(\mathscr{M}_{K^{p\prime}}^{H'_\Sigma})^* \to ((\mathscr{M}_{K^{p\prime}})_{H'_\Sigma})^* = (\mathscr{M}_{K^{p\prime}}^*)^{H'_\Sigma}$$

(where * denotes $\mathscr{O}_{\mathscr{X}^{(5)}}$-dual) is an isomorphism; as (4.1.13) is equivariant for the action of the full Hecke algebra, this implies that its restriction (4.1.14) is perfect. It is skew-hermitian by Lemma 4.1.5 and the fact that the Poincaré pairing (4.1.2), when W is trivial, is skew-symmetric.

We find the pairing (4.1.15) by specialising (4.1.14) to $K^{p\prime} = K^p$, and the pairing (4.1.16) as described. □

4.1.9 Specialisations

We describe the specialisation of the pairing $((,))$ just constructed.

For each algebraic representation W of $(\mathrm{G} \times \mathrm{H})$, denote by

$$\mathscr{X}_r^{\mathrm{cl},W} := \mathscr{X} \cap \mathscr{E}_{K^p,r}^{\mathrm{cl},W},$$

the set of classical points of weight W (omitted from the notation when $W = \mathbf{Q}_p$) and level r. Denote by a subscript 'W, r' the pullbacks of sheaves or global sections from $\mathscr{X}^{(5)}$ to $\mathscr{X}_r^{\mathrm{cl},W}$ (which is a finite étale scheme over $\mathbf{Q}_p$). We let

$$V_{W,r} := \Gamma(\mathscr{X}_r^{\mathrm{cl},W}, \mathscr{V}_{W,r}), \quad \Pi_{H'_\Sigma,W,r}^{K^{p\prime},\mathrm{ord}} := \Gamma(X_r^{\mathrm{cl},W}, \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}),$$
$$M_{K^{p\prime},W,r}^{H'_\Sigma} := \Gamma(\mathscr{X}_r^{\mathrm{cl},W}, \mathscr{M}_{K^{p\prime}}^{H'_\Sigma}) = H_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W})^{\mathrm{ord},H'_\Sigma},$$

where the last equality is by Proposition 3.1.2.3. We denote

$$((\ ,\ ))_{W,r} := \mathrm{v}(K^{p\prime})^{-1} \cdot (\ ,\ )_{\langle\langle,\rangle\rangle_{K^{p\prime},W,r}} : \Pi_{H'_\Sigma,W,r}^{K^{p\prime},\mathrm{ord}} \times \Pi_{H'_\Sigma,W,r}^{K^{p\prime},\mathrm{ord},\iota} \to \mathscr{O}(\mathscr{X}_r^{\mathrm{cl},W}).$$

Proposition 4.1.8 *Let* f_1*, respectively* f_2 *be global*[24] *sections of* $\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}} = \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathscr{X}^{(5)}}[G_{F,E}]}(\mathscr{M}_{K^{p\prime}}^{H'_\Sigma}, \mathscr{V})$*, respectively* $(\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})^\iota$*. Let*

$$f_{1,W,r} : M_{K^{p\prime},W,r}^{H'_\Sigma} \to V_{W,r}, \quad f_{2,W,r} : (M_{K^{p\prime},W,r}^{H'_\Sigma})^\iota \to V_{W,r}^\iota$$

[24] The same statements hold with some extra notational burden if f_1, f_2 are only defined over an open subset of $\mathscr{X}^{(5)}$.

be $\mathscr{O}_{\mathscr{X}_r^{\mathrm{cl},W}}[G_{F,E}]$*-linear maps.*

Let $x \in \mathscr{X}_r^{\mathrm{cl},W}$, $\pi := \pi(x)$ *and let*

$$f_{1,x}\colon \mathrm{H}_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W}) \to V_\pi, \quad f_{2,x}\colon \mathrm{H}_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W}^\vee) \to V_{\pi^\vee}$$

be $\mathbf{Q}_p(x)[G_{F,E}]$*-linear maps.*

The following hold.

1. *Suppose that for* $i = 1, 2$*, the map* $f_{i,W,r}$ *arises as the specialisation of* f_i*. Then*

$$((f_1, f_2))|_{\mathscr{X}_r^{\mathrm{cl},W}} = ((f_{1,W,r}, f_{2,W,r}))_{W,r} \quad in\ \mathscr{O}(\mathscr{X}_r^{\mathrm{cl},W}).$$

2. *Suppose that for* $i = 1, 2$*, the map* $f_{i,x}$ *factors through the projection*

$$\begin{aligned}\mathrm{p}^?\colon \mathrm{H}_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W}^?) \to \mathrm{H}_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W}^?)^{\mathrm{ord}}_{H'_\Sigma} &\cong H_1(\overline{Z}_{K^{p\prime},r}, \mathscr{W})^{\mathrm{ord},H'_\Sigma} \\ &= M^{H'_\Sigma}_{K^{p\prime},W,r},\end{aligned}$$

where $? = \emptyset$ *if* $i = 1$, $? = \vee$ *if* $i = 2$*; and that* $f_{i,x}$ *coincides with the specialisation of* $f_{i,W,r}$ *at* x*. Then*

$$((f_1, f_2))_{W,r}(x) = (f_{1,x}, f_{2,x})^{\mathrm{ord}}_\pi = \dim W \cdot (w_{\mathrm{a}}^{\mathrm{ord}} f_1, f_2)_\pi \quad in\ \mathbf{Q}_p(x). \tag{4.1.17}$$

Proof We simplify the notation by omitting the superscripts H'_Σ and subscripts $K^{p\prime}$; moreover we ignore the normalisations $\mathrm{v}(K^{p\prime})^{-1}$ that are present in all of the pairings to be compared.

Part 1 follows from the definition (4.1.12) if $W = \mathbf{Q}_p$, and similarly we can also identify $((\ ,\ ))_{W,r}$ with the restriction to $\mathscr{X}_r^{\mathrm{cl},W}$ of the pairing on functions on M_W deduced from $\langle\langle\ ,\ \rangle\rangle_W$ via Construction 4.1.2. By Lemma 4.1.6 this implies that the desired statement holds for all W.

For Part 2, let $H_{W,r} := \mathscr{O}(\mathscr{X}_r^{\mathrm{cl},W})$. First notice that, by the construction of case 1 of Lemma 4.1.3, the diagram of $H_{W,r}$-modules

$$\begin{array}{ccc}\mathrm{Hom}_{H_{W,r}}(M^\iota_{W,r}, H_{W,r}(1))_{\overline{\mathbf{Q}}_p} & \xrightarrow{u_{\langle\langle,\rangle\rangle W,r}} & M_{W,r} \\ \downarrow & & \downarrow \\ \mathrm{Hom}\,(M^\iota_{W,r,\Lambda_{W,r}}, \Lambda_{W,r}(1)) & \xrightarrow{u_{\langle\langle,\rangle\rangle \Lambda,W,r}} & M_{W,r}\end{array}$$

is commutative.

On the other hand, let $x \in \mathscr{X}_{\rm cl}^{W,r}$ and let α_x be the associated character of T^+. By definition in (4.1.11), the pairing $\langle\langle , \rangle\rangle_{\Lambda,W}$ specialises, on $M_{W,r|x} \otimes M^\iota_{W,r|x}$, to

$$(x, y) \mapsto \sum_{t\in\overline{T}_0/\overline{T}_r} \langle x, ty\rangle^{\rm ord}_{U_*^p,W,r}[t^{-1}](x) = \sum_{t\in\overline{T}_0/\overline{T}_r} \alpha_x(t)\langle x, y\rangle^{\rm ord}_{U_*^p,W,r}\alpha_x^{-1}(t)$$
$$= p^{r[F:\mathbf{Q}]}|\overline{T}_0/\overline{T}_r| \cdot \langle x, y\rangle^{\rm ord}_{U_*^p,W,r}.$$

It follows that $u_{\langle\langle,\rangle\rangle_{W,r}}$ specialises at x to $p^{-r[F:\mathbf{Q}]}|\overline{T}_0/\overline{T}_r|^{-1}u_{\langle,\rangle'_{W,r}}$, hence that the specialisation of $((,))(x) = \mathrm{v}(K^{p\prime})^{-1}(,)_{\langle\langle,\rangle\rangle_{W,r}}(x)$ is

$$\frac{(,)_{\langle,\rangle^{\rm ord}_{W,r}}}{p^{r[F:\mathbf{Q}]}|\overline{T}_0/\overline{T}_r|\mathrm{v}(K^{p\prime})}$$
$$= \frac{p^{r[F:\mathbf{Q}]}\cdot \mathrm{v}(K^{p\prime}K_1^1(p^r)_p)(,)^{\rm ord}_\pi}{p^{r[F:\mathbf{Q}]}[\mathrm{v}(K^{p\prime}K_1^1(p^r)_p)/\mathrm{v}(K^{p\prime}K_0(p)^r_p)]\mathrm{v}(K^{p\prime}K_0(p^r)_p)}$$
$$= (,)^{\rm ord}_\pi,$$

where we have used $|\overline{T}_0/\overline{T}_r| = \mathrm{v}(K^{p\prime}K_0(p^r)_p)/\mathrm{v}(K^{p\prime}K_1^1(p)^r_p)$ (by (4.1.5)) and (4.1.10).

This establishes the first equality of (4.1.17); the second one is just a reminder of (4.1.8). □

4.2 Local toric pairings

Let F be a non-archimedean local field, E a quadratic étale algebra over F with associated character $\eta\colon F^\times \to \{\pm 1\}$, B a quaternion algebra over F, $G = B^\times$, $H = E^\times$, $H' = H/F^\times$, and suppose given an embedding $H \hookrightarrow G$

4.2.1 Definition of the pairing

Let π be a smooth irreducible representation of G over a finite extension L of $\mathbf{Q}_p$, with a central character $\omega\colon F^\times \to L^\times$. Let $\chi : E^\times \to L^\times$ a character such that $\chi|_{F^\times}\cdot\omega = \mathbf{1}$. We identify χ with a representation $L\chi$ of $E^\times$ on L, and when more precision is needed we denote by e_χ the basis element corresponding to the character χ in $L\chi$. Let $\Pi := \pi\otimes\chi$, a representation of $(G\times H)' = (G\times H)/F^\times$ over L. We assume that π is essentially unitarisable, that is that for any embedding $\iota\colon L \hookrightarrow \mathbf{C}$, a twist of $\iota\pi$ is isomorphic to the space of smooth vectors of a unitary representations. (This holds automatically if π arises as the local component of a cuspidal automorphic representation over L.) Let $\pi^\vee$ be the smooth dual, $\Pi^\vee := \pi^\vee \otimes \chi^{-1}$

Assume from now on that the modified local sign $\varepsilon(\Pi) =$(1.2.5) equals $+1$. Then, by the result of Tunnell and Saito mentioned in the introduction, the space

$$\Pi^{*,H'} := \mathrm{Hom}_{H'}(\Pi, L)$$

has dimension 1 over L. Moreover the choices of an invariant pairing $(\ ,\ )$ on $\Pi \otimes \Pi^\vee$ and a Haar measure dt on H' give a generator

$$Q = Q_{(\ ,\ ),dt} \in \Pi^{*,H'} \otimes_L (\Pi^\vee)^{*,H'}$$

defined by the absolutely convergent integral

$$Q_{(,)}(f_1, f_2) := \mathscr{L}(V_v, 0)^{-1} \cdot \iota^{-1} \int_{E^\times/F^\times} (\iota\Pi(t) f_1, \iota f_2)\, dt; \qquad (4.2.1)$$

for any $\iota\colon L \hookrightarrow \mathbf{C}$; here $\mathscr{L}(V_v, 0) =$(1.2.7).

Recall also from the introduction (1.2.8) that

$$Q_{dt}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right) := \frac{Q_{(,),dt}(f_1, f_2)}{(f_3, f_4)} \qquad (4.2.2)$$

is independent of $(\ ,\ )$ whenever it is defined.

We study the pairing, or some of its variations, in a few different contexts.

4.2.2 *Interpretation in the case $E = F \oplus F$*

In this case $G = \mathrm{GL}_2(F)$, and the integral (4.2.1) has an interpretation as product of zeta integrals. Let $\mathscr{K}(\pi)$ and $\mathscr{K}(\pi^\vee)$ be Kirillov models over L as in [32, § 2.3]. By [32, Lemma 2.3.2], the L-line of invariant pairings on $\mathscr{K}(\pi) \times \mathscr{K}(\pi^\vee)$ is generated by an element $(\ ,\ )$ such that, for each $\iota\colon L \hookrightarrow \mathbf{C}$, we have

$$\iota(f, f^\vee) = \frac{\zeta(2)}{L(1, \pi \times \pi^\vee)} \cdot \int_{F^\times} \iota f(y) \iota f^\vee(y) d^\times y, \qquad (4.2.3)$$

where the integral is absolutely convergent (as $\iota\pi$ is essentially unitarisable) and $d^\times y$ is any L-valued Haar measure. Identify $E^\times$ with the diagonal torus in $\mathrm{GL}_2(F)$ and write $\chi = (\chi_1, \chi_2)$ according to the decomposition $E = F \oplus F$; noting that $\chi_2 = \omega^{-1}\chi_1$ and $\pi = \pi^\vee \otimes \omega^{-1}$, we indentify $Q_{(,),dt}$ with

$$\iota Q_{(,),dt}(f \otimes e_\chi, f^\vee \otimes e_{\chi^{-1}}) \doteq L(1/2, \iota\pi_E \otimes \iota\chi)^{-1}$$

$$\begin{aligned}
&\int_{E^\times/F^\times} \iota\chi(t)(\iota\pi(t)f, \iota f^\vee)|t|^s\, dt|_{s=0} \\
&= L(1/2, \iota\pi\otimes\iota\chi_1)^{-1}\int_{F^\times} \iota f^\vee(t)\iota\chi_1(t)|t|^s d^\times t|_{s=0} \\
&\quad \cdot L(1/2, \iota\pi\otimes\iota\chi_1)^{-1}\int_{F^\times} \iota f^\vee(y)\iota\chi_1(y)|y|^s d^\times y|_{s=0} \\
&= (L(1/2, \iota\pi\otimes\iota\chi_1)^{-1}\cdot I(\iota f, \iota\chi_1, 1/2)) \\
&\quad \cdot (L(1/2, \pi\otimes\iota\chi_1)^{-1}\cdot I(\iota f^\vee, \iota\chi_1, 1/2)),
\end{aligned} \tag{4.2.4}$$

where $I(\cdot,\cdot,1/2)$ is the zeta integral of [34, § 5.2] for $\mathrm{GL}_2(F)\times\mathrm{GL}_1(F)$, and $\doteq$ denotes an equality up to constants in $L^\times$ depending on the choices of measures.

4.2.3 Special line in the unramified case

We study the first one in a short list of special cases in which there are ‘canonical’ lines in Π, $\Pi^\vee$, on which the value of the pairings Q can be explicitly computed.

Lemma 4.2.1 [105, Lemme 14]. *Suppose that B is split, E/F is unramified, and both π and χ are unramified. Let $K\subset(G\times E^\times)/F^\times$ be a maximal compact subgroup. Then*

$$Q_{(\,,\,),dt}(v,w)=\mathrm{vol}(\mathscr{O}_E^\times/\mathscr{O}_F^\times, dt)\cdot(v,w)$$

for all v, respectively w, in the lines Π^K, respectively $(\Pi^\vee)^K$.

4.2.4 Special line when B is nonsplit

Suppose now that B is nonsplit and that Π is an irreducible representation of $(G\times E^\times)/F^\times$ as above. Note that Π is finite-dimensional and H' is compact, so that $\Pi^\vee=\Pi^*$ and the natural maps $\Pi^{H'}\to\Pi_{H'}$ ($=H'$-coinvariants) and $\Pi^{*,H'}\to(\Pi^{H'})^*$ are isomorphisms. Moreover the non-degenerate pairing $(\,,\,)$ restricts to a non-degenerate pairing on $\Pi_{H'}\otimes\Pi^\vee_{H'}$. Then we may compare the restrictions of the pairings $Q_{(\,,\,)}$ of $(\,,\,)$ to the line $\Pi^{H'}\otimes\Pi^{\vee,H'}$.

Lemma 4.2.2 *In the situation of the previous paragraph, we have*

$$Q_{(\,,\,),dt}=\mathscr{L}(V_{(\pi,\chi),v},0)^{-1}\cdot\mathrm{vol}(E^\times/F^\times, dt)\cdot(\,,\,)$$

as elements of $(\Pi^{H'})^*\otimes(\Pi^{\vee,H'})^*$.

Proof This follows from the definition in (4.2.1), since in this case the integration over the compact set $E^\times/F^\times$ converges. □

4.3 Ordinary toric pairings

We define a variant for ordinary forms of the pairing Q.

4.3.1 Definition of the ordinary paring

Let $\Pi = \pi \otimes \chi$ be an ordinary automorphic representation of $(\mathrm{G} \times \mathrm{H})'(\mathbf{A})$ over L. When referring to local objects considered in the previous paragraphs or products thereof, we append subscripts as appropriate.

For each $v|p$, let

$$\mu_v^+ \colon E_v^\times \to L^\times$$

be the character by which $E_v^\times$ (or equivalently $\prod_{w|v} G_{E_w}^{\mathrm{ab}}$) acts on $V_{\pi,v}^+ \otimes \chi_v$, and let $\mathrm{j}_v \in E_v$ be the purely imaginary element fixed in (A.1.2). Define

$$\mu^+(\mathrm{j}) := \prod_{v|p} \mu_v^+(\mathrm{j}_v).$$

For measures $dt_v = dt_{v,p}$ on H'_v, $dt_{v,\infty}$ on $H'_{v,\infty}$ (the latter a merely formal notion as in the introduction), define

$$\begin{aligned}
\mathrm{vol}^\circ(H'_v, dt_v) &:= \frac{\mathrm{vol}(\mathscr{O}_{E,v}^\times/\mathscr{O}_{F,v}^\times, dt_v)}{e_v L(1,\eta_v)^{-1}}, \ \mathrm{vol}^\circ(H'_{v,\infty}, dt_{v,\infty}) \\
&:= \frac{\mathrm{vol}(H'_{v,\infty}, dt_{v,\infty})}{2^{[F_v:\mathbf{Q}_p]}}, \\
\mathrm{vol}^\circ(H'_{p\infty}, dt_{p\infty}) &:= \prod_{v|p} \mathrm{vol}^\circ(H'_v, dt_v) \cdot \mathrm{vol}^\circ(H'_{v,\infty}, dt_{v,\infty}). \qquad (4.3.1)
\end{aligned}$$

The denominators in the right-hand sides are the volumes of $\mathrm{vol}(\mathscr{O}_{E,v}^\times/\mathscr{O}_{F,v}^\times)$, respectively $\mathbf{C}^\times/\mathbf{R}^\times$, for the ratio of (rational normalisations of) selfdual measures, cf. [107, § 1.6.2] and the proof of Proposition A.3.4.

Definition 4.3.1 Let $dt = dt^{p\infty} dt_{p\infty}$ be a decomposition of the adèlic measure dt specified in (1.2.9). Then we define:

– for each $f_{1,p\infty}, f_{3,p\infty} \in \Pi^{\rm ord}_{p\infty}$, $f_{2,p\infty}, f_{4,p\infty} \in \Pi^{\vee,{\rm ord}}_{p\infty}$ with $f_{3,p\infty} \otimes f^{\rm ord}_{4,p\infty} \neq 0$,

$$Q^{\rm ord}_{dt_{p\infty}}\left(\frac{f_{1,p\infty}\otimes f_{2,p\infty}}{f_{3,p\infty}\otimes f_{4,p\infty}}\right) := \mu^+({\rm j})\,{\rm vol}^\circ(H'_{p\infty}, dt_{p\infty})\cdot \frac{f_{1,p\infty}\otimes f_{2,p\infty}}{f_{3,p\infty}\otimes f_{4,p\infty}}. \tag{4.3.2}$$

– for each $f_1, f_3 \in \Pi^{\rm ord}$, $f_2, f_4 \in \Pi^{\vee,{\rm ord}}$ with $(f_3, f_4)^{\rm ord} \neq 0$,

$$Q^{\rm ord}\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right) := Q^{p\infty}_{dt^{p\infty}}\left(\frac{f_1^{p\infty}\otimes f_2^{p\infty}}{f_3^{p\infty}\otimes f_4^{p\infty}}\right)\cdot Q^{\rm ord}_{dt_{p\infty}}\left(\frac{f_{1,p\infty}\otimes f_{2,p\infty}}{f_{3,p\infty}\otimes f_{4,p\infty}}\right). \tag{4.3.3}$$

The normalisation at $p\infty$ is justified by the clean formula of Proposition 4.3.4 below.

Remark 4.3.2 Suppose that Π is locally distinguished, so that as explained in the introduction the functional Q_{dt} is nonzero. Then the functional $Q^{\rm ord}_{dt}$ is also nonzero.

4.3.2 Decomposition

Fix a decomposition $dt = \prod_{v\nmid p\infty} dt_v dt_{p\infty}$ such that for all but finitely many v, ${\rm vol}(\mathscr{O}^\times_{E,v}/\mathscr{O}^\times_{F,v}) = 1$. Let Σ' be a finite set of finite places of F disjoint from Σ and S_p and containing the other places of ramification of Π, and those such that ${\rm vol}(\mathscr{O}^\times_{E,v}/\mathscr{O}^\times_{F,v}) \neq 1$. Let $K^p \subset ({\rm G}\times{\rm H})'(\mathbf{A}^{p\infty})$ be an open compact subgroup that is maximal away from $S := \Sigma \cup \Sigma'$ and such that $\Pi_v^{K_v} = \Pi_v$ for $v \in \Sigma$.

Lemma 4.3.3 *For all* $f_1, f_3 \in \Pi^{K^p,{\rm ord}}_{H'_\Sigma}$, $f_2, f_4 \in \Pi^{\vee,K^p,{\rm ord}}_{H'_\Sigma}$ *with* $(f_3, f_4)^{\rm ord} \neq 0$, *we have*

$$\begin{aligned} Q^{\rm ord}\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right) &= \prod_{v\in\Sigma'} Q_{v,dt_v}\left(\frac{f_{1,v}\otimes f_{2,v}}{f_{3,v}\otimes f_{4,v}}\right) \\ &\quad\cdot \prod_{v\in\Sigma} {\rm vol}(E_v^\times/F_v^\times, dt)\mathscr{L}(V_{(\pi,\chi),v},0)^{-1}\frac{f_{1,v}\otimes f_{2,v}}{f_{3,v}\otimes f_{4,v}} \\ &\quad\cdot \frac{f_1^{Sp\infty}\otimes f_2^{Sp\infty}}{f_3^{Sp\infty}\otimes f_4^{Sp\infty}}\cdot Q^{\rm ord}_{p\infty,dt_{p\infty}}\left(\frac{f_{1,p\infty}\otimes f_{2,p\infty}}{f_{3,p\infty}\otimes f_{4,p\infty}}\right). \end{aligned} \tag{4.3.4}$$

Proof This follows from the definitions and the results of Sect. 4.2. □

4.3.3 Relation to between the toric pairing and its ordinary variant

We gather the conclusion of the computations from the appendix.

Proposition 4.3.4 *Let $\Pi = \pi \otimes \chi = \Pi^{\infty} \otimes W$ be an ordinary representation of $(\mathrm{G} \times \mathrm{H})'(\mathbf{A})$. Let $w_{\mathrm{a}}^{\mathrm{ord}}$ and $\gamma_{H'}^{\mathrm{ord}}$ be the operators defined in Propositions A.2.1 and A.2.4. Let $e_p(V_{(\pi,\chi)}) = (1.4.6)$ be the interpolation factor of the p-adic L-function. For all $f_1, f_3 \in \Pi^{\mathrm{ord}}$, $f_2, f_4 \in \Pi^{\mathrm{ord}}$ with $(f_3, f_4)^{\mathrm{ord}} \neq 0$, we have*

$$Q\left(\frac{\gamma_{H'}^{\mathrm{ord}}(f_1) \otimes \gamma_{H'}^{\mathrm{ord}}(f_2)}{w_{\mathrm{a}}^{\mathrm{ord}}(f_3) \otimes f_4}\right) = e_p(V_{(\pi,\chi)}) \cdot \dim W \cdot Q^{\mathrm{ord}}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right).$$

Proof There is a decomposition $Q^{\mathrm{ord}}_{p\infty, dt_{p\infty}} = \prod_{v|p} Q^{\mathrm{ord}}_{v,dt_v} \cdot \prod_{v|p} Q^{\mathrm{ord}}_{v,\infty,dt_{v,\infty}}$, whose terms are defined in Sects. A.3–A.4. The only point worth stressing is that if μ_v^+, respectively $\mu_{v,\infty}^+$ is the character defined in Sect. A.3.3,[25] respectively Sect. A.4.3, then the decomposition $\mu^+ = \mu^{+,\mathrm{sm}}\mu^{+,\mathrm{alg}}$ of μ^+ into a product of a smooth and an algebraic character is given by $\mu^{+,\mathrm{sm}} = \prod_{v|p} \mu_v^+$, $\mu^{+,\mathrm{alg}} = \prod_{v|p} \mu_{v,\infty}^+$.

Then the result follows from Propositions A.3.4 and A.4.3. □

4.4 Interpolation of the toric pairings

We interpolate the pairings Q_{dt}^{ord} along Hida families

4.4.1 Interpolation of the local pairings

We use the same notation F, E of Sect. 4.2.

Lemma 4.4.1 *Let $\mathscr{X}$ be a scheme over $\mathbf{Q}$ and let $r' = (r, N)$ be a Weil–Deligne representation of W_F on a rank-2 locally free sheaf over $\mathscr{X}$. Suppose that $\mathscr{X}$ contains a dense subset $\mathscr{X}^{\mathrm{cl}}$ such that r'_x is pure for all $x \in \mathscr{X}^{\mathrm{cl}}$. Let $\mathrm{ad}(r')$ be the rank-3 adjoint representation. Then there exist an open subset $\mathscr{X}'' \subset \mathscr{X}$ containing $\mathscr{X}^{\mathrm{cl}}$ and functions*

$$L(0, r')^{-1}, \quad L(1, r', \mathrm{ad}) \in \mathscr{O}(\mathscr{X}'')$$

such that for every $x \in \mathscr{X}'$ we have $L(0, r')^{-1}(x) = L(0, r'_x)^{-1}$ and $L(1, r', \mathrm{ad})(x) = L(1, \mathrm{ad}(r'_x))$.

[25] Note that despite the similar notation, the character μ_v is defined using the Weil–Deligne representations rather than the continuous Galois representations.

Proof By [34, § 5.1], there exist an open set $\mathscr{X}''' \subset \mathscr{X}$ containing $\mathscr{X}^{\rm cl}$ and functions $L(0, r')^{-1}$, respectively $L(1, r', {\rm ad})^{-1}$, in $\mathscr{O}(\mathscr{X}''')$ interpolating $L(0, r'_x)^{-1}$, respectively $L(1, {\rm ad}(r'_x))^{-1}$, for all $x \in \mathscr{X}'$. By purity, $L(1, r, {\rm ad})^{-1}$ does not vanish on $\mathscr{X}^{\rm cl}$, hence it is invertible in an open neighbourhood $\mathscr{X}''$ of $\mathscr{X}^{\rm cl}$ in $\mathscr{X}'''$. □

Let $\mathscr{X}$ be an integral scheme, $\mathscr{F}^\times$ be a $\mathscr{K}_{\mathscr{X}}^\times$-module, then we define $\mathscr{F}^{\times,-1}$ to be the $\mathscr{K}_{\mathscr{X}}$-module such that for each open $\mathscr{U} \subset \mathscr{X}$,

$$\mathscr{F}^{\times,-1}(\mathscr{U}) := \{f^{-1} \mid \ f \in \mathscr{F}^\times(\mathscr{U})\}$$

with $\mathscr{K}_{\mathscr{X}}^\times$-action given by $a \cdot f^{-1} = (a^{-1} f)^{-1}$.

Proposition 4.4.2 *Consider the situation of Lemma 4.4.1. Let*

$$\pi = \pi(r')$$

be the $\mathscr{O}_{\mathscr{X}}[{\rm GL}_2(F)]$-module attached to r' by the local Langlands correspondence in families of [34]*, let $\omega\colon F^\times \to \mathscr{O}(\mathscr{X})^\times$ be its central character, and let $\chi\colon E^\times \to \mathscr{O}(\mathscr{X})^\times$ be a character such that $\omega \cdot \chi|_{F^\times} = \mathbf{1}$. Let $\pi^\vee := \pi(\rho^*(1))$ and let $\Pi = \pi \otimes \chi$, $\Pi^\vee = \pi^\vee \otimes \chi^{-1}$. Let $(\Pi \otimes_{\mathscr{O}_{\mathscr{X}}^\times} \Pi^\vee)^\times$ be the $\mathscr{O}_{\mathscr{X}}^\times$-submodule of those $f_3 \otimes f_4$ such that $(f_3, f_4) \neq 0$.*

Then there exist: an open subset $\mathscr{X}' \subset \mathscr{X}$ containing $\mathscr{X}^{\rm cl}$; letting $\mathscr{O} := \mathscr{O}_{\mathscr{X}'}$, $\mathscr{K} := \mathscr{K}_{\mathscr{X}'}$, an $\mathscr{O}^\times$-submodule $(\Pi \otimes_{\mathscr{O}_{\mathscr{X}}^\times} \Pi^\vee)^\times$ specialising at all $z \in \mathscr{X}^{cl}$ to the space of $f_{3,z} \otimes f_{4,z}$ such that $(f_{3,z}, f_{4,z})_z \neq 0$; and a map of $\mathscr{O}$-modules

$$\mathscr{Q}_{dt}\colon (\Pi \otimes_{\mathscr{O}} \Pi^\vee) \otimes_{\mathscr{O}^\times} (\Pi \otimes_{\mathscr{O}^\times} \Pi^\vee)^{\times,-1} \to \mathscr{K}$$

satisfying the following properties:

1. *For all $t_1, t_2 \in E^\times/F^\times$, $g \in ({\rm GL}_2(F) \times E^\times)/F$,*

$$\mathscr{Q}_{dt}\left(\frac{\Pi(t_1) f_1 \otimes \Pi^\vee(t_2) f_2}{\Pi(g) f_3 \otimes f_4}\right) = \mathscr{Q}_{dt}\left(\frac{f_1 \otimes f_2}{f_3 \otimes \Pi^\vee(g^{-1}) f_4}\right);$$

2. *For all $x \in \mathscr{X}^{\rm cl}$,*

$$\mathscr{Q}_{dt|x} = Q_{dt},$$

where Q_{dt} is the paring on $\Pi_x \otimes \Pi_x^\vee$ of (4.2.2).

Proof For each $x \in \mathscr{X}^{\mathrm{cl}}$, π_x corresponds to a pure Weil–Deligne representation under local Langlands, hence it is essentially unitarisable (and in fact tempered, see [99, Lemma 1.4 (3)]). Then by [34, Lemma 5.2.5] there is an open neighbourhood $\mathscr{X}'$ of $\mathscr{X}^{\mathrm{cl}}$ in $\mathscr{X}$ and an invariant pairing over $\mathscr{X}'$

$$(\ , \ ) \colon \pi \otimes \pi^\vee \to \mathscr{O}_{\mathscr{X}'} \tag{4.4.1}$$

specialising to the pairing $(\ , \ )_x$ defined by (4.2.3) at all $x \in \mathscr{X}^{\mathrm{cl}}$. It induces an invariant pairing $\Pi \otimes \Pi^\vee \to \mathscr{O}_{\mathscr{X}'}$ still denoted by $(\ , \ )$.

By Lemma 4.4.1, up to possibly shrinking $\mathscr{X}'$, we have regular functions on $\mathscr{X}'$ interpolating $z \mapsto L(1/2, \pi_{z,E} \otimes \chi_z)^{-1} = L(0, r_z|_{W'_E} \otimes \chi_z)^{-1}$ and $x \mapsto L(1, \pi_x, \mathrm{ad}) = L(1, r'_x, \mathrm{ad})$.

If E/F is split, [34, Proposition 5.2.4] applied to (4.2.4) gives an element $\mathscr{Q}_{(,),dt} \colon \Pi_{E^\times} \otimes \Pi^\vee_{E^\times} \to \mathscr{O}_{\mathscr{X}'}$ interpolating $Q_{(,)_x}$ for $x \in \mathscr{X}^{\mathrm{cl}}$, and we define

$$\mathscr{Q}_{dt}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right) := \frac{\mathscr{Q}_{(,),dt}(f_1, f_2)}{(f_3, f_4)} \tag{4.4.2}$$

If E/F is nonsplit, by the previous discussion we can interpolate all terms occurring in the definition (4.2.1) (note that the integral there is just a finite sum), to obtain a pairing $\mathscr{Q}_{(,)}$ over $\mathscr{X}'$ interpolating $\mathscr{Q}_{(,)_x}$ for $x \in \mathscr{X}^{\mathrm{cl}}$. Then we again define $\mathscr{Q}$ by (4.4.2). □

4.4.2 *Product of local pairings*

We consider the global situation, resuming with the setup of Sects. 3.3–4.1.

Let $\Pi := \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ over $\mathscr{X}^{(5)}$. Recall that we have a decomposition

$$\Pi \cong (\pi_{\mathrm{G},\Sigma'}(\mathscr{V}_{\mathrm{G}}) \otimes \chi_{\mathrm{H,univ},\Sigma'}) \otimes_{\mathscr{O}_{\mathscr{X}^{(5)}}} \Pi_{H'_\Sigma}^{K^{p\prime},S,\mathrm{ord}} \tag{4.4.3}$$

from Theorem 3.3.10.

Let $(\Pi \otimes_{\mathscr{K}^\times_{\mathscr{X}^{(5)}}} \Pi^\iota)^\times \subset \Pi \otimes_{\mathscr{K}^\times_{\mathscr{X}^{(5)}}} \Pi^\iota$ be the $\mathscr{O}^\times_{\mathscr{X}^{(5)}}$-submodule of sections $f_3 \otimes_{\mathscr{O}^\times_{\mathscr{X}^{(5)}}} f_4$ such that $f_3 \otimes f_4 \neq 0$ and $(f_{3,v}, f_{4,v})_v \neq 0$ for each the pairings $(,)_v = $ (4.4.1), $v \in \Sigma'$.

Theorem 4.4.3 *Let* $\Pi := \Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ *and* $\mathscr{X}^{(5)}$ *be as in Proposition 4.1.7, and let* $(\Pi \otimes_{\mathscr{K}_{\mathscr{X}^{(5)}}} \Pi^\iota)^\times \subset \Pi \otimes_{\mathscr{K}_{\mathscr{X}^{(5)}}} \Pi^\iota$ *be the submodule defined above. Then there exist an open subset* $\mathscr{X}^{(6)} \subset \mathscr{X}^{(5)}$ *containing* $\mathscr{X}^{\mathrm{cl}}$ *and, letting* $\mathscr{O} = \mathscr{O}_{\mathscr{X}^{(6)}}$, $\mathscr{K} := \mathscr{K}_{\mathscr{X}^{(6)}}$, *a map of* $\mathscr{O}^\times$*-modules*

$$\mathscr{Q} \colon (\Pi \otimes_{\mathscr{O}} \Pi^\iota) \otimes_{\mathscr{O}^\times} (\Pi \otimes_{\mathscr{O}^\times} \Pi^\iota)^{\times,-1} \to \mathscr{K}_{\mathscr{X}}$$

satisfying:

1. *For any* $t_1, t_2 \in E^\times_{\Sigma'}/F^\times_{\Sigma'} \subset (\mathrm{GL}_2(F_{\Sigma'}) \times E^\times_{\Sigma'})/F^\times_{\Sigma'}$, *any* $h \in \mathscr{H}_{S,K_{\Sigma'}}$ *and any section*

$$(f_1 \otimes f_2) \otimes (f_3 \otimes f_4)^{-1} \quad of \quad (\Pi \otimes_{\mathscr{K}} \Pi^\iota) \otimes_{\mathscr{K}^\times} (\Pi \otimes_{\mathscr{K}^\times} \Pi^\iota)^{\times,-1},$$

we have

$$\mathscr{Q}\left(\frac{\Pi_{\Sigma'}(t_1) f_1 \otimes \Pi^\iota_{\Sigma'}(t_2) f_2}{\Pi(h) f_3 \otimes f_4}\right) = \mathscr{Q}\left(\frac{f_1 \otimes f_2}{f_3 \otimes \Pi^\iota(h) f_4}\right);$$

in the left-hand side, $\Pi_{\Sigma'}$, *respectively* $\Pi^\iota_{\Sigma'}$ *denote the actions of the Hecke algebras at* S *on* Π, *respectively* Π^ι.

2. *For all* $x \in \mathscr{X}^{\mathrm{cl}}$,

$$\mathscr{Q}_{dt|x} = Q^{\mathrm{ord}},$$

where Q^{ord} *is the restriction of the pairing on* $\Pi_x^{\vee,\mathrm{ord}} \otimes \Pi_x^{\vee,\mathrm{ord}}$ *of Definition* 4.3.1.

Proof By (4.3.2), (4.3.3), (4.3.4), and (4.4.3), we need to interpolate:

- the terms $\mathscr{L}(V_{(\pi,\chi),v}$ for $v \in \Sigma$: this is Lemma 4.4.1;
- the characters μ_v^+ for $v|p$: this follows form the existence of the filtration (3.2.4) over an open subset of $\mathscr{X}$.
- the term $Q_{dt,\Sigma'} := \prod_{v\in\Sigma'} Q_{dt,v}$, According to the proof of [34, Theorem 4.4.1], the representation $\pi_{\mathrm{G},\Sigma'}(\mathscr{V}_{\mathrm{G}})$ is the maximal torsion-free quotient of $\otimes_{v\in\Sigma'}\pi_{\mathrm{G},v}(\mathscr{V}_{\mathrm{G}})$. For sections $f_{i,\Sigma'}$ that are images of $\otimes_{v\in\Sigma'} f_{i,v}$, with $f_{i,v}$ sections of $\pi_{\mathrm{G},v}(\mathscr{V}_{\mathrm{G}}) \otimes \chi_{\mathrm{H,univ},v}$ if $i = 1, 3$, or of $\pi_{\mathrm{G},v}(\mathscr{V}^\iota_{\mathrm{G}}) \otimes \chi^{-1}_{\mathrm{H,univ},v}$ if $i = 2, 4$, let

$$\mathscr{Q}_{\Sigma'}\left(\frac{f_{1,\Sigma'} \otimes f_{2,\Sigma'}}{f_{3,\Sigma'} \otimes f_{4,\Sigma'}}\right) := \prod_{v\in\Sigma'} \mathscr{Q}_v\left(\frac{f_{1,v} \otimes f_{2,v}}{f_{3,v} \otimes f_{4,v}}\right),$$

  where the factors in the right-hand side are provided by Proposition 4.4.2. This is well-defined independently of the choices of $f_{i,v}$ as $\mathscr{K}$ is torsion-free.

This completes the interpolation of (4.3.3) into a function $\mathscr{Q}$, that satisfies properties 1 and 2 by construction and the corresponding properties from Proposition 4.4.2. ∎

5 Selmer sheaves and p-adic heights

In this section we present the theory of Selmer complexes and p-adic heights needed in the rest of the paper. The foundational material is taken from the book of Nekovář [81].

5.1 Continuous cohomology

Let $(R^\circ, \mathfrak{m})$ be a complete Noetherian local ring, let G be a topological group.

5.1.1 Continuous cochains for (ind-) admissible $R[G]$-modules

Let M be an $R^\circ[G]$-module. We say that M is *admissible of finite type* if it is of finite type as an R°-module and the action $G \times M \to M$ is continuous (when M is given the $\mathfrak{m}$-adic topology). We say that M is *ind-admissible* if $M = \bigcup_\alpha M_\alpha$ where $\{M_\alpha\}$ is the set of finite-type admissible $R^\circ[G]$-submodules of M.

The complex of continuous cochains of M is denoted by $C^\bullet_{\rm cont}(G, M)$; it is defined in the usual way [81, (3.4.1)] when M is admissible of finite type, and by $C^i_{\rm cont}(G, M) := \varinjlim_\alpha C^i_{\rm cont}(G, M_\alpha)$ when we have a presentation $M = \bigcup_\alpha M_\alpha$ as above. The image of $C^\bullet_{\rm cont}(G, M)$ in the derived category of $D({}_R\mathrm{Mod})$ of R°-modules is denoted by

$$\mathrm{R}\Gamma(G, M)$$

and its cohomology groups by

$$H^i(G, M)$$

(we omit the subscript 'cont' as we will only be working with continuous cohomology).

5.1.2 Localisation

Let

$$R = R^\circ[\mathcal{S}^{-1}]$$

for some multiplicative subset $\mathcal{S} \subset R^\circ$, and let M be an $R[G]$-module. We say that M is ind-admissible if it is ind-admissible as an $R^\circ[G]$-module, and that it is of finite type if it is of finite type as an R-module. Suppose that

$M := M^\circ \otimes_{R^\circ} R$ for an ind-admissible $R^\circ[G]$-module M°. Then M is ind-admissible as an $R^\circ[G]$-module and there is a canonical isomorphism

$$C^\bullet_{\rm cont}(G, M) \cong C^\bullet_{\rm cont}(G, M^\circ) \otimes_{R^\circ} R \tag{5.1.1}$$

([81, (3.7.4)]).

Remark 5.1.1 Let

$$\mathscr{C} = \mathscr{C}_{R^\circ}$$

be the category of schemes isomorphic to open subschemes of Spec R°, with maps being open immersions. It follows from the previous paragraph that, for any object X of $\mathscr{C}$, the condition of ind-admissibility is defined for all quasicoherent $\mathscr{O}_X[G]$-modules, and the functors $\mathrm{R}\Gamma(G, -)$ are well-defined on ind-admissible $\mathscr{O}_X[G]$-modules. Moreover, both the ind-admissibility condition and the functors $\mathrm{R}\Gamma(G, -)$ are compatible with restriction along open immersions in $\mathscr{C}$.

In the following, we will not further comment on the generalisation indicated in the previous remark when referring to sources only considering $R^\circ[G]$-modules.

5.1.3 Completed product

For $i = 1, 2$, let R_i° be complete noetherian local rings, and let $R^\circ := R_1^\circ \hat{\otimes} R_2^\circ$. We have a functor

$$\hat{\times} : \mathscr{C}_{R_1^\circ} \times \mathscr{C}_{R_2^\circ} \to \mathscr{C}_{R^\circ} \tag{5.1.2}$$

defined on objects by Spec $R_1^\circ[1/f_1] \hat{\times}$ Spec $R_1^\circ[1/f_2] :=$ Spec $R_1^\circ \hat{\otimes} R_2^\circ[1/f_1 \otimes 1, 1/1 \otimes f_2]$ and glueing.

5.1.4 Notation

Throughout the rest of this section, X will denote an object of $\mathscr{C}_{R^\circ}$. If $\mathscr{A} = \mathscr{O}_X, \mathscr{O}_X[G]$, we denote by $D({}_{\mathscr{A}}\mathrm{Mod})$ the derived category of $\mathscr{A}$-moduels. We use sub- or superscripts

$$\text{ft, ind-adm}, +, -, \text{b}, [a, b], \text{perf},$$

to denote the full subcategory of objects quasi-isomorphic to complexes of $\mathscr{A}$-modules that are respectively termwise of finite type, termwise ind-admissible, bounded below, bounded above, bounded, concentrated in degrees $[a, b]$, bounded, perfect (= bounded and termwise projective and of finite type).

Proposition 5.1.2 [81, (3.5.6)]. *The functor* $\mathrm{R}\Gamma(G, -)$ *can be extended to a functor on the category of bounded-below complexes of ind-admissible* $\mathscr{O}_X[G]$*-modules, with values in bounded-below complexes of* $\mathscr{O}_X$*-modules* [81, (3.4.1.3), (3.5.1.1)]. *It descends to an exact functor*

$$\mathrm{R}\Gamma(G, -) \colon D^+({}^{\text{ind-adm}}_{\mathscr{O}_X[G]}\mathrm{Mod}) \to D^+({}_{\mathscr{O}_X}\mathrm{Mod}).$$

5.1.5 Base-change

Suppose that $R \twoheadrightarrow R' = R/I$ is a surjective map of rings. Let $j \colon R^\circ \to R = R^\circ[\mathscr{S}^{-1}]$ be the natural map and let $I^\circ := j^{-1}(I)$. Then $R^{\circ\prime} := R^\circ/I^\circ$ is also complete local Noetherian, and we may write $R' = R^{\circ\prime}[\mathscr{S}']^{-1}$ where $\mathscr{S}'$ is the image of $\mathscr{S}$ in $R^{\circ\prime}$. Let M' be an ind-admissible R'-module, then $C^\bullet_{\mathrm{cont}}(G, M')$ is the same whether we consider M' as an R'-module or as an R-module: in the special case $R = R^\circ$ this follows from the fact that the maximal ideal of $R^{\circ\prime}$ is the image of the maximal ideal of R°, so that the $\mathfrak{m}$-adic and $\mathfrak{m}'$-adic topologies on finitely generated $R^{\circ\prime}$-modules coincide; the general case follows from the special case by localisation (5.1.1).

More generally, if $Y \subset X$ is a closed subset, the functor $\mathrm{R}\Gamma(G, -)$ on $\mathscr{O}_Y[G]$-modules coincides with the restriction of the functor on $\mathscr{O}_X[G]$-modules of the same name.

Proposition 5.1.3 *Let M be an ind-admissible $\mathscr{O}_X[G]$-module and let N be an $\mathscr{O}_X$-module of finite projective dimension. Then there is a natural isomorphism in* $D^{\mathrm{b}}({}_{\mathscr{O}_X}\mathrm{Mod})$

$$\mathrm{R}\Gamma(G, M) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} N \cong \mathrm{R}\Gamma(G, M \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} N).$$

Proof Let $P^\bullet$ be a finite projective resolution of N. The natural map of complexes of $\mathscr{O}_X$-modules

$$C^\bullet_{\mathrm{cont}}(G, M) \otimes_{\mathscr{O}_X} P^\bullet \to C^\bullet_{\mathrm{cont}}(G, M \otimes_{\mathscr{O}_X} P^\bullet)$$

is an isomorphism by [81, (3.4.4)].[26] The desired result follows from the definition of derived tensor product. □

The proposition applies when $N = \mathscr{O}_Y$ with $Y \subset X$ a local complete intersection, or when X is regular and N is any coherent $\mathscr{O}_X$-module. We highlight the following case.

[26] In *loc. cit.*, the ring denoted by R is our R°, but as our X is open in Spec R°, the $\mathscr{O}_X$-modules P^n are also flat as R°-modules and the cited result applies.

Corollary 5.1.4 *Let M be an ind-admissible $\mathscr{O}_X[G]$-module that is flat as an $\mathscr{O}_X$-module, and let $x \in X$ be a nonsingular point. Then there is an isomorphism in $D^{\mathrm{b}}({}_{\kappa(x)}\mathrm{Mod})$*

$$\mathrm{R}\Gamma(G, M) \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \kappa(x) \cong \mathrm{R}\Gamma(G, M \otimes_{\mathscr{O}_X} \kappa(x))$$

hence a second-quadrant spectral sequence

$$\mathrm{Tor}_{-p}(H^q(G, M), \kappa(x)) \Rightarrow H^{q-p}(G, M \otimes_{\mathscr{O}_X} \kappa(x)).$$

Proof After possibly localising at x, we may assume that $X = \mathrm{Spec}\, R$ is the spectrum of a local ring, which by assumption will be regular. Then $\kappa(x)$ has finite projective dimension over R, and the result follows from the previous proposition. □

5.1.6 Continuous cohomology as derived functor

For $i = 0, 1$, the functors $M \mapsto H^i(G, M)$ on the category of ind-admissible R-modules coincide with the ith derived functors of $M \mapsto M^G$ [81, (3.6.2)(v)].

5.2 Specialisations

From here on we further assume that R° has finite residue field of characteristic p.

5.2.1 Finiteness conditions

Let G be a profinite group. We consider the condition

(F) $H^i(G, M)$ is finite for all finite discrete $\mathbf{F}_p[G]$-modules and all $i \geq 0$

and define the p-cohomological dimension of G to be

$$\mathrm{cd}_p(G) := \sup\{i : \exists \text{ a finite discrete } \mathbf{F}_p[G]\text{-module } M \text{ with } H^i(G, M) \neq 0\}.$$

Lemma 5.2.1 *If G satisfies (F) then the cohomology groups of ind-admissible $\mathscr{O}_X[G]$-modules of finite type are $\mathscr{O}_X$-modules of finite type* [81, (4.2.5), (4.2.10)]. *The cohomology of any ind-admissible $\mathscr{O}_X[G]$-module vanishes in degrees* $> \mathrm{cd}_p(G)$ [81, (4.26)].

When E is a number field, S is a finite set of places of E and $G = G_{E,S}$, condition (F) is satisfied and $\mathrm{cd}_p(G) = 3$. When E_w is a local field and

$G = G_{E_w}$, condition (F) is satisfied and $\mathrm{cd}_p(G) = 2$. In the latter case we use the notation $H^i(E_w, M)$ for $H^i(G, M)$.

5.2.2 Projective limits, specialisations

We give two results on the compatibility of G-cohomology with other functors.

Lemma 5.2.2 *Let G be a profinite group satisfying (F) and let $M = \varprojlim_n M_n$ be the limit of a countable projective system of admissible R°-modules of finite type. Then for all i the natural map*

$$H^i(G, M) \to \varprojlim_n H^i(G, M_n)$$

is an isomorphism.

Proof In the special case $M_n = M/\mathfrak{m}^n M$, it is shown in [81, Corollary 4.1.3] that the map under consideration is surjective with kernel $\lim_n^{(1)} H^{i-1}(G, M_n)$; this vanishes since by (F) those cohomology groups are finite, hence the projective system they form satisfies the Mittag-Leffler condition. The general case follows from applying the special case to M and the $M_n = \varprojlim_r M_n/\mathfrak{m}^r M_n$. □

Proposition 5.2.3 *Let G be a profinite group satisfying (F) and $\mathrm{cd}_p(G) = e < \infty$. Let M be an ind-admissible $\mathscr{O}_X[G]$-module of finite type. Let $x \in X$ be a nonsingular point, let $i_0 \geq 0$ and suppose that*

$$H^i(G, M \otimes_R \kappa(x)) = 0$$

for all $i \geq i_0 + 1$.

1. *For all $i \geq i_0+1$, the support of the finitely generated R-module $H^i(G, M)$ is a proper closed subset not containing x.*
2. *The natural map*

$$H^{i_0}(G, M) \otimes_{\mathscr{O}_X} \kappa(x) \to H^{i_0}(G, M \otimes_R \kappa(x))$$

   is an isomorphism.
3. *Suppose further that $i_0 = 1$, and that for y in some dense open subset of X, $\dim_{\kappa(y)} H^1(G, M \otimes \kappa(y)) = \dim_{\kappa(x)} H^1(G, M \otimes \kappa(x))$. Then the natural map*

$$H^0(G, M) \otimes_R \kappa(x) \to H^0(G, M \otimes_{\mathscr{O}_X} \kappa(x))$$

   is an isomorphism.

Proof By Nakayama's lemma and the vanishing assumption, the first statement is equivalent to

$$H^i(G, M) \otimes_{\mathscr{O}_X} \kappa(x) \cong H^i(G, M \otimes_{\mathscr{O}_X} \kappa(x)). \tag{5.2.1}$$

Therefore, for the proof of the first and second statements it is enough to prove (5.2.1) for all $i \geq i_0$, which we do by decreasing induction on i.

For $i \geq e+1$ the result is automatic. In general, Corollary 5.1.4 gives a second-quadrant spectral sequence

$$E_2^{p,q} = \mathrm{Tor}_{-p}^{\mathscr{O}_X}(H^q(G, M), \kappa(x)) \Rightarrow H^{q-p}(G, M \otimes_{\mathscr{O}_X} \kappa(x)). \tag{5.2.2}$$

By induction hypothesis, all terms on the diagonal $q - p = i$ vanish except possibly the one with $p = 0$, and the differentials with source and target such term are 0. It follows that $H^i(G, M \otimes_{\mathscr{O}_X} \kappa(x)) = E_\infty^{0,i} = E_2^{0,i} = H^i(G, M) \otimes_{\mathscr{O}_X} \kappa(x)$.

Finally, under the assumptions of part 3, the finitely generated R-module $H^1(G, M)$ is locally free of constant rank in a neighbourhood of x. Hence in the exact sequence

$$\begin{aligned} 0 \to H^0(G, M) \otimes_R \kappa(x) \to H^0(G, M \otimes_{\mathscr{O}_X} \kappa(x)) \\ \to \mathrm{Tor}_1^{\mathscr{O}_X}(H^1(G, M), \kappa(x)) \end{aligned}$$

deduced from (5.2.2), the last term vanishes. □

5.3 Selmer complexes and height pairings

As in the preceding subsection, let R° be a Noetherian local ring with finite residue field of characteristic p, X an object of $\mathscr{C}_R$.

When E_w is a local field, we write $\mathrm{R}\Gamma(E_w, -) := \mathrm{R}\Gamma(G_{E_w}, -)$ and similarly for its cohomology groups. For number fields, we will only use the analogous shortened notation for Selmer groups.

5.3.1 Greenberg data

Let E be a number field, Sp a finite set of finite places of E containing those above p. Fix for every $w|p$ an embedding $\overline{E} \hookrightarrow \overline{E}_w$ inducing an embedding $G_w := G_{E_w} \hookrightarrow G_{E,Sp}$. If M is an $\mathscr{O}_X[G_{E,Sp}]$-module, we denote by M_w the module M considered as an $\mathscr{O}_X[G_w]$-module.

Definition 5.3.1 A *Greenberg datum* $(M, (M_w^+)_{w \in Sp})$ (often abusively abbreviated by M in what follows) over X consists of

- an ind-admissible $\mathscr{O}_X[G_{E,Sp}]$-module M, finite and locally free as an $\mathscr{O}_X$-module;
- for every $w \in Sp$ a *Greenberg local condition*, that is a short exact sequence

$$0 \to M_w^+ \overset{i_w^+}{\to} M_w \to M_w^- \to 0$$

of ind-admissible $\mathscr{O}_X[G_w]$-modules, finite and locally free as $\mathscr{O}_X$-modules.

In this paper, at places $w \nmid p$ we will only consider the *strict* Greenberg conditions $M_w^+ = 0$.

5.3.2 Selmer complexes

Given a Greenberg datum $M = (M, (M_w^+)_{w \in Sp})$, the *Selmer complex*

$$\widetilde{\mathrm{R}\Gamma}_f(E, M)$$

is the image of the complex

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{E,Sp}, M) \oplus \bigoplus_{w \in Sp} C^\bullet_{\mathrm{cont}}(E_w, M_w^+) \right.$$
$$\left. \xrightarrow{\oplus_w \mathrm{res}_w - i^+_{w,*}} \bigoplus_{w \in Sp} C^\bullet_{\mathrm{cont}}(E_w, M_w) \right)[-1]$$

in $D({}^{\mathrm{ft}}_{\mathscr{O}_X}\mathrm{Mod})$. Its cohomology groups are denoted by $\widetilde{H}^i_f(E, M)$. We have an exact triangle

$$\widetilde{\mathrm{R}\Gamma}_f(E, M) \to \mathrm{R}\Gamma(E, M) \to \oplus_{w \in Sp} \mathrm{R}\Gamma(E_w, M_w^-). \tag{5.3.1}$$

Proposition 5.3.2 *The Selmer complex* $\widetilde{\mathrm{R}\Gamma}_f(E, M)$ *and all terms of* (5.3.1) *belong to* $D^{[0,3]}_{\mathrm{perf}}({}_{\mathscr{O}_X}\mathrm{Mod})$.

Proof As in [81, Proposition 9.7.2 (ii)]. □

From the triangle (5.3.1) we extract an exact sequence

$$0 \to H^0(G_{E,Sp}, M) \to \bigoplus_{w \in Sp} H^0(E_w, M_w^-) \to \widetilde{H}^1_f(E, M)$$
$$\to H^1_f(E, M) \to 0 \tag{5.3.2}$$

where the last term is the (Greenberg) Selmer group

$$H^1_f(E,M) := \mathrm{Ker}\left(H^1(G_{E,Sp},M) \to \bigoplus_{w\in Sp} H^1(E_w, M_w^-)\right). \quad (5.3.3)$$

5.3.3 Height pairings

For $? = \emptyset, \iota$, let $M^? = (M^?, (M_w^{?,+}))$ be a strict Greenberg datum for $G_{E,Sp}$ over X. Suppose given a perfect pairing of $\mathscr{O}_X[G_{E,Sp}]$-modules

$$M \otimes_{\mathscr{O}_X} M^\iota \to \mathscr{O}_X(1) \quad (5.3.4)$$

such that M_w^+ and $M_w^{+,\iota}$ are exact orthogonal of each other. Let Γ_F be a profinite abelian group.

For every pair of Greenberg data M, M^ι as above, there is a height pairing

$$h_M\colon \widetilde{H}^1_f(E,M) \otimes_{\mathscr{O}_X} \widetilde{H}^1_f(E,M^\iota) \to \mathscr{O}_X\hat{\otimes}\Gamma_F \quad (5.3.5)$$

constructed in [81, §11.1]. The following is a special case of [103, Appendix C, Lemma 0.16].

Proposition 5.3.3 *For each regular point $x \in X$ and $P_1 \otimes P_2 \in \widetilde{H}^1_f(E,M) \otimes_{\mathscr{O}_X} \widetilde{H}^1_f(E,M^\iota)$, we have*

$$h_{M\otimes\kappa(x)}(P_{1,x},P_{2,x}) = (h_M(P_1,P_2))(x).$$

Venerucci has defined height pairings in an even more general context. Let M_X be a strict Greenberg datum over X as above, let $Y \subset X$ be a local complete intersection, and let $M_Y^?$ be the restriction of $M_X^?$. Let $\mathscr{N}^*_{Y/X}$ be the conormal sheaf of $Y \to X$. Then there is a height pairing

$$h_{M_Y/M_X}\colon \widetilde{H}^1_f(E,M_Y) \otimes_{\mathscr{O}_Y} \widetilde{H}^1_f(E,M_Y^\iota) \to \mathscr{N}^*_{Y/X}, \quad (5.3.6)$$

constructed in [103, Appendix C, § 0.21].

We note its relation to (5.3.5) in a special case, and its symmetry properties in a conjugate-self-dual case.

Proposition 5.3.4 *The pairing (5.3.6) satisfies the following properties.*

1. *Let Γ_F be a profinite abelian quotient of $G_{E,Sp}$, let $X = Y \hat{\times}_{\mathrm{Spec}\,\mathbf{Q}_p} \mathrm{Spec}\,\mathbf{Z}_p[\![\Gamma_F]\!]_{\mathbf{Q}_p}$ (where $\hat{\times}$ =(5.1.2)), and assume that $M_X^? = M_Y^? \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\![\Gamma_F]\!]$*

for $? = \emptyset, \iota$, *where if* $? = \emptyset$ *(respectively* $? = \iota$*) then* $G_{E,Sp}$ *acts on* Γ_F *through the tautological character (respectively its inverse). Then*

$$h_{M_Y/M_X} = h_{M_Y} = (5.3.5).$$

2. *Suppose that there is an involution* $\iota\colon X \to X$ *stabilising* Y *and such that* $M_X^\iota = M_X \otimes_{\mathscr{O}_X,\iota} \mathscr{O}_X$, $M_{X,w}^{+,\iota} = M_{X,w}^{+} \otimes_{\mathscr{O}_X,\iota} \mathscr{O}_X$.
Let $\epsilon, \epsilon' \in \{\pm 1\}$. *Assume that the pairing* (5.3.4) *is* ϵ*-hermitian (Sect.* 4.1.1*), that* $\mathrm{d}_{Y/X}\iota = \epsilon' \mathrm{id}$ *on* $\mathscr{N}_{Y/X}^*$*, and that there is an* $\mathscr{O}_Y$*-linear isomorphism*

$$\mathrm{c}\colon \widetilde{H}_f^1(E, M_Y^\iota) = \widetilde{H}_f^1(E, M_Y)^\iota \to \widetilde{H}_f^1(E, M_Y).$$

Then the pairing

$$\begin{aligned} h_{M_Y/M_X}^{\square}\colon \widetilde{H}_f^1(E, M_Y) \otimes_{\mathscr{O}_Y} \widetilde{H}_f^1(E, M_Y) &\to \mathscr{N}_{Y/X}^* \\ (z, z') &\mapsto h_{M_Y/M_X}(z, \mathrm{c}z') \end{aligned} \tag{5.3.7}$$

is $\epsilon\epsilon'$*-symmetric.*

In our main application in Theorem D, we have $Y = \mathscr{X}$ (or an open subset), the Hida family for $(\mathrm{G}\times\mathrm{H})'$; $X = \mathscr{X}^\sharp$, the Hida family for $\mathrm{G}\times\mathrm{H}$ containing X; and $M_Y = \mathscr{V}$, $M_X = \mathscr{V}^\sharp$, the corresponding universal G_E-representations. In that case, the height pairing $h_{\mathscr{V}/\mathscr{V}^\sharp}$ is simply 1/2 of the pairing $h_{\mathscr{V}}$ of Proposition 5.3.3.

Proof Part 1 follows from the construction. (We omit further details since, by the remark preceding the present proof, we do not actually need it in this paper). We prove the symmetry properties. Let I be the ideal sheaf of Y; up to restricting to some open subset of X we may assume that I is generated by a regular sequence $x = (x_1, \ldots, x_r)$. Then $\mathscr{N}_{Y/X}^* = I/I^2$ is finite locally free generated by $([x_1], \ldots, [x_r])$. Let $\partial_i \in N = (I/I^2)^\vee$ be the map $\partial_i([x_j]) = \delta_{ij}$. It suffices to show that $h_i := \partial_i \circ h$ is $\epsilon\epsilon'$-hermitian for all i. By [103, Appendix C, Proposition 0.5], h_i is identified with $h_{M_Y/M_{X_i}}$, where $X_i = V_X((x_j)_{j\neq I})$ so that $Y = V_{X_i}(x_i)$. Hence it suffices to prove the claim for $r = 1$.

We argue similarly to [103, Proof of Corollary 10.10]. Assume thus $r = 1$, write x in place of x_1, and let K be the fraction field of X. By [81] and [103, Appendix A, § 0.7] we have an ϵ-hermitian Cassels–Tate pairing

$$\cup\colon \widetilde{H}_f^2(E, M_X)_{\mathscr{O}_X-\mathrm{tors}} \otimes_{\mathscr{O}_X} \widetilde{H}_f^2(E, M_X^\iota)_{\mathscr{O}_X-\mathrm{tors}} \to K/\mathscr{O}_X,$$

and by [103, Appendix C, Proposition 0.17], we have a map

$$i_x \colon \widetilde{H}^1_f(E, M_Y) \to \widetilde{H}^2_f(E, M_X)[x]$$

such that h_{M_Y/M_X} coincides with

$$\widetilde{H}^1_f(E, M_Y) \otimes \widetilde{H}^1_f(E, M_Y^{\iota}) \xrightarrow{i_x \otimes i_x^{\iota}} \widetilde{H}^2_f(E, M_X)[x] \otimes \widetilde{H}^2_f(E, M_X^{\iota})[x]$$
$$\xrightarrow{\cup} x^{-1}\mathscr{O}_X/\mathscr{O}_X \xrightarrow{[\cdot x^2]} I/I^2 = \mathscr{N}^*_{Y/X}.$$

Since all the above maps are ι-equivariant, we find that h_{M_Y/M_X} is ϵ-hermitian as well. The desired assertion follows from this and the fact that ι acts by ε' on $\mathscr{N}^*_{Y/X}$. □

6 Universal Heegner class

6.1 Tate cycles and Abel–Jacobi maps

Let X/E be an algebraic variety over a number field, and let R be a finite extension of $\mathbf{Q}_p$, or its ring of integers, or a finite quotient of its ring of integers.

6.1.1 Tate cycles

If $\mathscr{W}$ is an étale local system of $R[G_E]$-modules on X, the R-module of *Tate (0)-cycles* is the space

$$\mathscr{Z}_0(X, \mathscr{W}) := \bigoplus_{x \in X} H^0(x, \mathscr{W}),$$

where the sum runs over the closed points of X and, if $x \in X$ and $\overline{x} := x \times_{\operatorname{Spec} E} \operatorname{Spec} \overline{E}$, we define $H^0(x, \mathscr{W}) := H^0(\overline{x}, \mathscr{W})^{G_E}$. Elements of the latter space are written $\sum_{\overline{x}'}[\overline{x}'] \otimes \xi_{\overline{x}'}$, where $\overline{x}'$ runs through the points of $\overline{x}$. When $\mathscr{W} = R$, the module $\mathscr{Z}_0(X, R)$ is simply the usual R-module of 0-cycles with coefficients in R. Its quotient by the relation of rational equivalence is denoted $\mathrm{CH}_0(X, R)$.

When X has dimension 0, its *fundamental class* is the Tate cycle with trivial coefficients

$$[X] := \sum_{\overline{x}' \in Z(\overline{E}} [\overline{x}'] \otimes 1 \in \mathscr{Z}_0(X, \mathbf{Z}_p).$$

If $a = \sum_{\overline{x}'}[\overline{x}'] \otimes \xi_{\overline{x}'} \in \mathscr{Z}_0(X, \mathscr{W})$ its support $|a| \subset \overline{X}$ is the support of the divisor $\sum[\overline{x}']$, where the sum extends to those $\overline{x}'$ such that $\xi_{\overline{x}'} \neq 0$.

6.1.2 Abel–Jacobi map

A Tate cycle $a \in \mathscr{Z}_0(X, \mathscr{W})$ yields a map $R \to H^0(|a|, \mathscr{W})^{G_E}$ and, if X has dimension 1, the latter cohomology group maps to $H^2_{|a|}(\overline{X}, \mathscr{W}(1))$. The image of $1 \in R$ under the compostion

$$R \to H^0(|a|, \mathscr{W}) \to H^2_{|a|}(\overline{X}, \mathscr{W}(1)) \to H^2(\overline{X}, \mathscr{W}(1))$$

is denoted by $\overline{\mathrm{cl}}(a)$. Consider the exact sequence

$$\begin{aligned}0 \to H^1(\overline{X}, \mathscr{W}(1)) \to H^1(\overline{X} - |a|, \mathscr{W}(1)) \to H^2_{|a|}(\overline{X}, \mathscr{W}(1))\\ \to H^2(\overline{X}, \mathscr{W}(1)).\end{aligned} \tag{6.1.1}$$

Let e be a Galois-equivariant idempotent acting on the right on $H^*(\overline{X}, \mathscr{W}(1))$, such that $\overline{\mathrm{cl}}(a)e = 0$. Then we may apply the idempotent e to (6.1.1) and pull back the resulting exact sequence via the map $R \to H^2_{|a|}(\overline{X}, \mathscr{W}(1))$ given by a, obtaining an extension

$$0 \to H^1(\overline{X}, \mathscr{W}(1))e \to E_a \to R \to 0 \tag{6.1.2}$$

in the category of G_E-representations over R. The map sending a to the class $\mathrm{AJ}(a)e$ of this extension is called the e-Abel–Jacobi map,

$$\mathrm{AJ}e\colon \mathscr{Z}_0(X, \mathscr{W}) \to H^1(G_E, H^1(\overline{X}, \mathscr{W}(1))e) = H^1(G_E, H_1(\overline{X}, \mathscr{W})e),$$

where the last equality is just a reminder of our notational conventions. When $e = \mathrm{id}$, it is omitted from the notation. When $\mathscr{W} = R$ and e acts via correspondences, the map $\mathrm{AJ}e$ factors through $\mathrm{CH}_0(X, R)e$.

6.2 Heegner cycles

We use the notation from Sect. 2.1 for compact subgroups $U_{*,p,\underline{r}} \subset U_{*,p}(p^{\underline{r}}) \subset \mathrm{G}_*(\mathbf{Q}_p)$ and let $X_{*,U^{p\prime}_*,\underline{r}} \to X_{*,U^{p\prime}}(p^{\underline{r}})$ be the associated Shimura varieties; the level $U^{p\prime}_*$ will be fixed and often omitted from the notation. If $p\mathscr{O}_{F,p} = \prod_{v|p} \varpi_v^{e_v}\mathscr{O}_{F,v}$ we use r as a shorthand for $\underline{r} = (e_v r)_{v|p}$.

6.2.1 Embeddings of Shimura varieties

For any pair of subgroups $V' \subset \mathrm{H}'(\mathbf{A}^\infty)$, $K \subset (\mathrm{G}\times\mathrm{H})'(\mathbf{A}^\infty)$ such that $K \cap \mathrm{H}'(\mathbf{A}^\infty) \supset V$, we define the diagonal embedding

$$\begin{aligned} \mathrm{e}' = \mathrm{e}'_{V',K} \colon Y'_{V'} &\to Z_K \\ y &\mapsto [(\mathrm{e}(\tilde{y}), \tilde{y})] \end{aligned}$$

if $\tilde{y}$ is any lift of y to Y_V for some $V \subset \mathrm{H}(\mathbf{A}^\infty)$ such that $V F^\times_{\mathbf{A}^\infty} \subset V'$.

Let $W = W_\mathrm{G} \otimes W_\mathrm{H}$ be an irreducible right algebraic representation of $(\mathrm{G}\times\mathrm{H})'$ over $L \supset \mathbf{Q}_p$. If W satisfies (wt), the space $W^{H'}$ is 1-dimensional over L. Let $\mathscr{W}$ be the étale sheaf on the Shimura tower Z associated with W; any $\xi \in W^{H'}$ induces a map $\mathbf{Q}_p \to \mathrm{e}'^*\mathscr{W}$ of étale sheaves on the tower Y'; by adjunction we obtain a canonical map $\mathbf{Q}_p \to \mathrm{e}'^*\mathscr{W} \otimes W^\vee_{H'}$ where the second factor is simply an L-line.

We let

$$\begin{aligned} \mathrm{e}'_{W,K,V',*} \colon \mathscr{Z}_0(Y'_{V'}, \mathbf{Z}_p) &\to \mathscr{Z}_0(Y'_{V'}, \mathrm{e}'^*\mathscr{W}^\circ) \otimes W^\vee_{H'} \to \mathscr{Z}_0(Z_K, \mathscr{W}^\circ) \otimes W^\vee_{H'} \\ \mathrm{e}'_{W,\underline{r},*} \colon \mathscr{Z}_0(Y'_{\underline{r}}, \mathbf{Z}_p) &\to \mathscr{Z}_0(Y'_{\underline{r}}, \mathrm{e}'^*\mathscr{W}^\circ) \otimes W^\vee_{H'} \to \mathscr{Z}_0(Z(p^{\underline{r}}), \mathscr{W}^\circ) \otimes W^\vee_{H'} \\ &\to \mathscr{Z}_0(Z_{\underline{r}}, \mathscr{W}^\circ) \otimes W^\vee_{H'} \end{aligned}$$

be the compositions of the maps described above and, respectively, $\mathrm{e}'_{W,*}$ or $\mathrm{e}'_{\underline{r},*}$.

6.2.2 CM cycles

Let $[Y'_{V'}] \in \mathscr{Z}_0(Y'_{V'}, \mathbf{Z}_p)$ be the fundamental class. For any pair of levels K, V such that $\mathrm{e}_{W,(K,V')}$ is defined, let

$$\Delta_{W,(K,V')} := \mathrm{e}'_{W,K,V,*}[Y'_{V'}] \in \mathscr{Z}_0(Z_K, \mathscr{W}^\circ),$$

When $W \neq \mathbf{Q}_p$, we consider the elements

$$\Delta^\circ_{W,(K,V')} := \frac{1}{|Y'_{V'}(\overline{E})|} \cdot \Delta_{W,(K,V')} \in \mathscr{Z}_0(Z_K, \mathscr{W}^\circ)$$

When $W = \mathbf{Q}_p$, we consider the modification

$$\Delta^\circ_{(K,V')} := \frac{1}{|Y'_{V'}(\overline{E})|} \cdot (\Delta_{(K,V')} - \deg(\Delta_{(K,V')}) \cdot \xi_{\mathrm{Hodge}}) \in \mathrm{CH}_0(Z_K)_{\mathbf{Q}_p}, \tag{6.2.1}$$

where ξ_{Hodge} is the Hodge class of [107, §3.1.3], whose introduction is motivated by the following lemma.

Lemma 6.2.1 *The image under pushforward of $\Delta^{\circ}_{W,(K'',V'')}$ in $\mathscr{Z}_0(Z_K, \mathscr{W})$ (if $W \neq \mathbf{Q}_p$) or $\mathrm{CH}_0(Z_K)_{\mathbf{Q}_p}$ (if $W = \mathbf{Q}_p$) is independent of V'', K'' such that $V'' \subset K'' \cap \mathrm{H}'(\mathbf{A}^{\infty})$ and $K' \subset K$. We have*

$$\overline{\mathrm{cl}}(\Delta^{\circ}_{W,(K,V')}) = 0 \quad \textit{in } H^2(\overline{Z}_K, \mathscr{W}(1)).$$

Proof If $W \neq \mathbf{Q}_p$, the first assertion is clear; the second one is automatic as $H^2(\overline{Z}_K, \mathscr{W}(1)) = 0$ (see the argument in [91, bottom of p. 1089]). If $W = \mathbf{Q}_p$, the assertions amount, respectively, to the compatibility of the Hodge classes under pushforward and the fact that, by construction, the 0-cycle $\Delta^{\circ}_{K,V}$ has degree zero; both facts are explained in [107, §3.1.3]. □

6.2.3 Cycles, Selmer classes, and functionals

Let

$$P_{W,(K,V')} := \mathrm{AJ}(\Delta^{\circ}_{W,(K,V')}) \in H^1(G_{E,Sp}, H_1(\overline{Z}_K, \mathscr{W})). \qquad (6.2.2)$$

The classes $P_{W,(K,V')}$ are also compatible under pushforward and yields elements

$$P_W := \lim_{K \cap \mathrm{H}'(\mathbf{A}^{\infty}) \supset V'} P_{W,(K,V')} \in \varprojlim_K H^1(G_{E,Sp}, H_1(\overline{Z}_K, \mathscr{W})).$$

The space in the right-hand side has a right action by $(\mathrm{G} \times \mathrm{H})'(\mathbf{A}^{\infty})$, and P_W is invariant under $\mathrm{H}'(\mathbf{A}^{\infty})$. Via (2.5.3) and the biduality $W^{\vee\vee} = W$, P_W yields, for each ordinary representation Π of weight W, a map

$$P_{\Pi} \colon \Pi \to H^1(G_{E,Sp}, V_{\Pi}).$$

Using the map $\gamma^{\mathrm{ord}}_{H'} \colon \Pi^{\mathrm{ord}} \to \Pi_{H'}$ from Proposition A.2.4, we also obtain a map

$$P^{\mathrm{ord}}_{\Pi} := P_{\Pi} \gamma^{\mathrm{ord}}_{H'} \colon \Pi^{\mathrm{ord}} \to H^1(G_{E,Sp}, V_{\Pi}). \qquad (6.2.3)$$

Remark 6.2.2 We conjecture that (i) there exist an algebraic variety $N_{W,(K,V')}/E$ of odd dimension $2d_W + 1$, a homologically trivial cycle $\mathfrak{z}_{W,(K,V')} \in \mathrm{CH}_{d_W}(N_{W,(K,V')})_0$, and a map

$$\lambda \colon H^{2d_W+1}(\overline{N}_{W,(K,V')}, \mathbf{Q}_p(d+1)) \to H_1(\overline{Z}_K, \mathscr{W})$$

such that $P_{W,(K,V')} = \lambda(\mathrm{AJ}(\mathfrak{Z}_{W,(,V')}))$; (ii) the elements $P_{W,(K,V')}$ belong to $H^1_f(E, H_1(\overline{Z}_K, \mathscr{W}))$, so that the maps P_Π take values in $H^1_f(E, V_\Pi)$.

When $\mathrm{G} = \mathrm{GL}_{2/\mathbf{Q}}$, one can prove (i) with $N_{W,(K,V')}$ a Kuga–Sato variety for Z_K, generalising [78, Proposition II.2.4]. The (probably not insurmountable) difficulty in the general case is that, if $F \neq \mathbf{Q}$, the Shimura variety Z is not of PEL-type. Part (ii) should essentially be a consequence of either (a) part (i), via [79,82], or (b) granted a generalisation of the theory of *locc. citt.* to nontrivial coefficients system, of the weaker assertion that, for a finite place w of E, the image of $\Delta_{W,(K,V')}$ in $H_0(Z_{K,E_w}, \mathscr{W})$ comes from a corresponding class in the syntomic cohomology of Z_{K,E_w} with coefficients in $\mathscr{W}$.

6.3 Universal Heegner class

We use the local construction described in Sect. A.2.2 to turn the $\mathrm{H}'(\mathbf{A})$-invariant class P_W into an $\mathrm{H}'(\mathbf{A}^{p\infty})$-invariant class $\mathscr{P}_W$ with values in the ordinary completed homology. Then we show that $\mathscr{P}_W$ is independent of W and it interpolates P°_Π at all representations Π satisfying (ord), (n-exc).

6.3.1 Construction

Let $d_{\underline{r}} := |Y'_{\underline{r}}(\overline{E})|$ and let $d^\circ = d_{\underline{r}} \prod_{v|p} q_v^{-r_v} \in \mathbf{Z}_{\geq 1}$, which is the limit of an eventually constant sequence. Recall that for the tame level $K^{p'} \subset (\mathrm{G} \times \mathrm{H})'(\mathbf{A}^{p\infty})$, we denote $M^\circ_{K^{p'},W} := \varprojlim_r \mathrm{H}_1(\overline{Z}_{W,r}, \mathscr{W}^\circ)^{\mathrm{ord}}$.

Definition 6.3.1 The *universal Heegner point* of weight W is the element

$$\mathscr{P}_W := P_W \gamma^{\mathrm{ord}}_{H'} \in d^{\circ,-1} H^1(G_{E,Sp}, M^\circ_{K^p,W}), \tag{6.3.1}$$

where we still denote by $\gamma^{\mathrm{ord}}_{H'}$ the map induced by the map

$$\gamma^{\mathrm{ord}}_{H'} \colon \varprojlim_{K_p} \mathrm{H}_1(\overline{Z}_{K^{p'}K_p}, \mathscr{W})^{\mathrm{H}'(\mathbf{A})} \to M_{K^{p'},W}$$

of Proposition A.2.4. As usual, we simply write $\mathscr{P} := \mathscr{P}_{\mathbf{Q}_p}$. When we want to emphasise the choice of $K^{p'}$ we write $\mathscr{P}_{K^{p'},W}$ instead of $\mathscr{P}_W$.

6.3.2 Independence of weight

The class $\mathscr{P}_W$ does not depend on W.

Proposition 6.3.2 *Under the identification*

$$H^1(E, M^\circ_{K^{p'}} \otimes_{\mathbf{Z}_p} \mathscr{O}_L) \overset{j_{W,*}}{\cong} H^1(E, M^\circ_{K^{p'},W}) \tag{6.3.2}$$

induced from the isomorphism j_W *of Proposition* 3.1.2.2*, we have*

$$j_{W,*}(\mathscr{P}) = \mathscr{P}_W.$$

Proof We show that the difference $j_{W,*}(\mathscr{P}) - \mathscr{P}_W$ is p-divisible. Since $H^1(G_{E,Sp}, M^\circ_{K^{p\prime},W})$ is a finitely generated module over the ring $\Lambda^\circ_{K^{p\prime}}$ by Lemma 5.2.1, any p-divisible element is zero. We will use some of the notation and results of the Appendix, in particular the matrices γ defined in Sect. A.1, the involution $\iota = (-)^{\mathrm{T},-1}$ on GL_2, and the operator $\gamma^{\mathrm{ord}}_{H'}$ of Proposition A.2.4.

We tacitly multiply both sides by d°, so that they belong to the lattices (6.3.2). By the definitions of $\mathscr{P}_W$ and j_W, we need to show the following. Denote by $[-]_r$ the reduction modulo p^r, and by $c(W)$ the constant (A.2.3); then we should have

$$[p^{r[F:\mathbf{Q}]}\Delta_{\mathbf{Q}_p,\underline{r}}\gamma_{r,p}\mathrm{U}_p^{-r}\gamma^\iota_{0,\infty}]_r \mapsto [c(W)^{-1}p^{r[F:\mathbf{Q}]}\Delta_{W,\underline{r}}\gamma_{r,p}\mathrm{U}_p^{-r}\gamma^\iota_{0,\infty}]_r,$$

under the map

$$\begin{aligned} j'_W\colon & H^1(G_{E,Sp}, H_1(\overline{Z}_{K^pK_p(p^r)}, \mathbf{Z}/p^r)) \\ & \to H^1(G_{E,Sp}, H_1(\overline{Z}_{K^p,K_p(p^r)}, \mathbf{Z}/p^r) \otimes_{\mathbf{Z}/p^r} (W^\circ/p^r)^{N_{0,r}} \\ & \otimes_{\mathscr{O}_L/p^r} (W^{\vee,\circ}/p^r)_{N_{0,r}} \ c \mapsto c \otimes \zeta_r \otimes \zeta_r^\vee, \end{aligned} \tag{6.3.3}$$

where $\zeta_r \otimes \zeta_r^\vee$ is the unique element pairing to 1.

As the local system $\mathscr{W}^\circ/p^r\mathscr{W}^\circ$ is trivial on $\overline{Z}_{K^pK_p(p^r)}$, we have

$$[p^{r[F:\mathbf{Q}]}\Delta_{W,\underline{r}}]_r = [p^{r[F:\mathbf{Q}]}\Delta_{\mathbf{Q}_p,\underline{r}} \otimes \xi \otimes \xi^\vee]_r$$

in

$$\begin{aligned} & H^1(G_{E,Sp}, H_1(\overline{Z}_{K^p,K_p(p^r)}, \mathbf{Z}/p^r)) \otimes_{\mathbf{Z}/p^r} (W^\circ/p^rW^\circ)^{H'} \\ & \otimes_{\mathscr{O}_L/p^r} (W^{\vee,\circ}/p^rW^{\vee,\circ})_{H'}, \end{aligned}$$

where $\xi \otimes \xi^\vee$ is the unique element pairing to 1. Note first that the image of $[p^{r[F:\mathbf{Q}]}\Delta_{W,\underline{r}}]_r$ under $\gamma_{r,p}\mathrm{U}_p^{-r}\gamma^\iota_{0,\infty}$ belongs to the right-hand side of (6.3.3): indeed it suffices to show that for any $\xi \in W^\circ$, the class $[\xi\gamma_r]_r$ is fixed by $N_{0,r}$, which follows from the congruence

$$\gamma_r n - \gamma_r \equiv 0 \pmod{p^r M_2(\mathbf{Z}_p)}$$

valid for any $n \in N_{0,r}$.

It remains to see that if $\xi \otimes \xi^\vee$ pairs to 1, then so does $c(W)^{-1}\cdot\xi\gamma_{r,p}\otimes\xi^\vee\gamma^\iota_{0,\infty}$ in the limit $r \to \infty$. This is proved in Lemma A.4.2. ∎

6.4 Local properties of the universal Heegner class

Recall that $\mathscr{X}$ is an irreducible component of $\mathscr{E}_{K^p}$ hence of the form Spec R with $R = R^\circ[1/p]$ and $R^\circ = \mathbf{T}^{\mathrm{sph,ord}}_{(\mathrm{G}\times\mathrm{H})', K^p, \mathfrak{m}}/\mathfrak{a}$ for some maximal and minimal ideals $\mathfrak{m} \subset \mathfrak{a} \subset \mathbf{T}_{(\mathrm{G}\times\mathrm{H})', K^p}$. The ring R° satisfies the assumptions of Sect. 5.3, hence Greenberg data over open subsets of $\mathscr{X}$ give rise to sheaves of Selmer complexes.

Let $\mathscr{X}^{(i)} \subset \mathscr{X} \subset \mathscr{E}_{K^{p\prime}}$ be the open sets defined in Sect. 3. Proposition 3.2.4 provides a strict Greenberg datum $(\mathscr{V}, (\mathscr{V}_w^+)_{w|p}, (0)_{w\in S})$ over $\mathscr{X}$. Via Proposition 3.3.8.2 we obtain a strict Greenberg datum $(\mathscr{M}^{H'_\Sigma}_{K^{p\prime}}, (\mathscr{M}^{H'_\Sigma,+}_{K^{p\prime},w})_{w|p}, (0)_{w\in S})$ over $\mathscr{X}^{(3)}$ with

$$\mathscr{M}^{H'_\Sigma,\pm}_{K^{p\prime},w} = \mathscr{V}_w^\pm \otimes (\Pi^{K^{p\prime},\mathrm{ord}}_{H'_\Sigma})^\vee.$$

We begin the study of the Selmer complexes attached to the above Greenberg data, with the goal to promote $\mathscr{P}$ to a section of $\widetilde{H}^1_f(E, \mathscr{M}^{H'_\Sigma}_{K^{p\prime}})$ over a suitable open subset of $\mathscr{X}$.

6.4.1 *Comparison of Bloch–Kato and Greenberg Selmer groups*

Let $z \in \mathscr{X}^{\mathrm{cl}}$ and let $V = \mathscr{V}_{|z}$. We compare two notions of Selmer groups for V.

Lemma 6.4.1 *Let $w \nmid p$ be a place of E. Then, for all i,*

$$H^i(E_w, V) = 0.$$

Proof As observed in [80, Proposition 2.5], this is implied by the prediction from the weight-monodromy conjecture that the monodromy filtration on $\mathscr{V}_z$ is pure of weight -1. Writing $z = (x, y) \in \mathscr{E}^{\mathrm{ord,cl}} \subset \mathscr{E}^{\mathrm{ord,cl}}_{\mathrm{G}} \times \mathscr{E}^{\mathrm{ord,cl}}_{\mathrm{H}}$, the weight-mondromy conjecture for $\mathscr{V}_z$ follows from the corresponding statement for $\mathscr{V}_{\mathrm{G},x}$, that is Theorem 2.5.1.2. □

Let $H^1_{f,\mathrm{Gr}}(E, V)$ be the Greenberg Selmer group. Bloch and Kato [11] have defined subspaces $H^1_f(E_w, V) \subset H^1(E_w, V)$ and a Selmer group

$$H^1_{f,\mathrm{BK}}(E, V) := \{s \in H^1(E, V) : \forall w \in Sp,\ \mathrm{loc}_w(s) \in H^1_f(E_w, V)\}.$$

Lemma 6.4.2 *Suppose that Π_z satisfies* (wt). *We have*

$$H^1_{f,\mathrm{BK}}(E, V) = H^1_{f,\mathrm{Gr}}(E, V),$$

where the right-hand side is the Greenberg Selmer group as in (5.3.3).

Proof We need to show that for all $w \in Sp$, $H^1_f(E_w, V) = \mathrm{Ker}\,\big(H^1(E_w, V) \to H^1_f(E_w, V_w^-)\big)$. This is automatic for $w \nmid p$ by Lemma 6.4.1. For $w|p$ this is [81, (12.5.8)]: the context of *loc. cit* is more restricted but the proof still applies, the key point being that (12.5.7)(1)(i) *ibid.* still holds for all w under the weight condition (wt). □

Lemma 6.4.3 *Let $w \nmid p$ be a finite place of E. Then $H^1(E_w, \mathscr{V})$ and $H^1(E_w, \mathscr{M}_{K^{p\prime}}^{H'_\Sigma})$ are supported in a closed subset of $\mathscr{X}$ (respectively $\mathscr{X}^{(3)}$) disjoint from $\mathscr{X}^{\mathrm{cl}}$.*

Proof This follows from Proposition 5.2.3 and Lemma 6.4.1. □

6.4.2 Local Selmer properties of $\mathscr{P}$

Let $w \nmid p$ be a place of E as above. As

$$\mathscr{M}_{K^{p\prime}}^{H'_\Sigma} = \mathscr{V} \otimes (\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})^\vee$$

over $\mathscr{X}^{(3)}$, the support of $H^1(E_w, \mathscr{M}_{K^{p\prime}}^{H'_\Sigma})$ is in fact the intersection of $\mathscr{X}^{(3)}$ and of the support of $H^1(E_w, \mathscr{V})$. We denote by

$$\mathscr{X}^{(3,w)} \supset \mathscr{X}^{\mathrm{cl}} \tag{6.4.1}$$

the open complement in $\mathscr{X}^{(3)}$ of the support of $H^1(E_w, \mathscr{V})$.

Lemma 6.4.4 *Let $w|p$ be a place of E, with underlying place v of F. The image $\mathrm{loc}_w^-(\mathscr{P})$ of $\mathscr{P}$ in*

$$H^1(E_w, \mathscr{M}_{K^{p\prime},w}^{H'_\Sigma,-})$$

vanishes over $\mathscr{X}^{(3)}$.

Proof We lighten the notation by dropping form the notation the superscript '(3)' and all decorations from $\mathscr{M}$, $\mathscr{M}_w^-$. Let $\tilde{\mathscr{X}} := \mathrm{Spec}\,_{\mathscr{X}}\mathscr{O}_{\mathscr{X}}[([\sqrt{z}])_{z\in A}]$, where A is a (finite) set of topological generators for $F^\times\backslash\mathbf{A}_F^{\infty\times}/(K^{p\prime} \cap \mathrm{Z}(\mathbf{A}_F^{p\infty\times}))$. As $\tilde{\mathscr{X}} \to \mathscr{X}$ is faithfully flat, we may prove the statement after a base-change to $\tilde{\mathscr{X}}$; we denote base-changed sheaves and sections thereof with a tilde $\tilde{\square}$.

Let $\chi\colon E^\times\backslash\mathbf{A}_E^{\infty\times}\to G_E^{\rm ab}\to\mathscr{O}(\mathscr{X})^\times$ be the universal character, and let $\omega=\chi_{|F^\times\backslash\mathbf{A}_F^{\infty\times}}$, so that $\det\mathscr{V}_{{\rm G}|\mathscr{X}}(-1)=\omega$. Let $\omega^{1/2}\colon F^\times\backslash\mathbf{A}_F^{\infty\times}\to\mathscr{O}(\tilde{\mathscr{X}})^\times$ be a square root of ω. We may write

$$\tilde{\mathscr{V}}=(\tilde{\mathscr{V}}_{\rm G}\otimes\omega^{-1/2})_{|G_E}\otimes\tilde{\chi}',\quad \tilde{\chi}':=\tilde{\chi}\otimes\omega_{|G_E}^{-1/2},$$

where now $\tilde{\chi}'\colon G_E\to{\rm Gal}(E_\infty/E)$ is the projection for the abelian extension E_∞/E such that ${\rm Gal}(E_\infty/E)$ is the maximal pro-p quotient of $E^\times\mathbf{A}_F^{\infty\times}\backslash\mathbf{A}_E^{\infty\times}/V^{p\prime}$.

Write $E_\infty=\bigcup_{n\geq 0}E_n$ as an increasing union of finite extensions, where $E_0=E$ and eventually E_{n+1}/E_n is totally ramified at each prime above p, and let $\tilde{\chi}_n'\colon G_E\to{\rm Gal}(E_n/E)$ be the natural projection. Let α_v° be the character giving the G_{F_v}-action on $\mathscr{V}_{\rm G}^+(-1)$, so that

$$\tilde{\mathscr{V}}_w^-=\omega_w\alpha_w^{\circ,-1}\otimes\tilde{\chi}_w=\omega_w^{1/2}\alpha_w^{\circ,-1}\tilde{\chi}_w',$$

where for a character ω_v' of G_{F_v}, we denote $\omega_w':=\omega'_{v|{\rm G}_{E_w}}$. Let

$$\tilde{\mathscr{V}}_{n,w}^-:=\omega_w^{1/2}\alpha_w^{\circ,-1}\tilde{\chi}_{n,w}',\quad \tilde{\mathscr{M}}_{n,w}^-:=\tilde{\mathscr{V}}_{n,w}^-\otimes(\Pi_{H_\Sigma'}^{K^{p\prime},{\rm ord}})^\vee\subset\tilde{\mathscr{M}}_w.$$

Then the same argument as in [58, proof of Proposition 2.4.5, primes $v|p$] shows that the image ${\rm loc}_w^-(\tilde{\mathscr{P}})$ vanishes in $H^1(E_w,\mathscr{M}_{n,w}^-)=\prod_{w'|w}H^1(E_n,\tilde{\mathscr{M}}_{0,w}^-)$ for each n; here w' runs through the (eventually constant) set of primes of E_n above w. Since $H^1(E_w,\tilde{\mathscr{M}}_w^-)=\varprojlim_n H^1(E_w,\tilde{\mathscr{M}}_{n,w}^-)$ by Proposition 5.2.2, the lemma is proved. □

Corollary 6.4.5 *Let $\mathscr{X}^{(3,f)}:=\bigcap_{w\in S}\mathscr{X}^{(3,w)}\supset\mathscr{X}^{\rm cl}$, where the sets $\mathscr{X}^{(3,w)}$ are as defined in (6.4.1). Then $\mathscr{P}$ defines a section*

$$\mathscr{P}\in\widetilde{H}_f^1(E,\mathscr{M}_{K^{p\prime}}^{H_\Sigma'})(\mathscr{X}^{(3,f)})=H_f^1(E,\mathscr{M}_{K^{p\prime}}^{H_\Sigma'})(\mathscr{X}^{(3,f)}).$$

Proof This follows from Lemmas 6.4.3 and 6.4.4. The displayed equality is a consequence of (5.3.2). □

6.4.3 Proof of Theorem C

Via Proposition 3.3.8.2, we may view the class $\mathscr{P}=\mathscr{P}_{K^p}=\mathscr{P}_{K^p,\mathbf{Q}_p}$ (Definition 6.3.1) as an $\mathscr{H}_K^{p\Sigma}$-equivariant functional

$$\mathscr{P}_{K^{p\prime}}\colon\Pi_{{\rm H}_{\Sigma'}}^{K^{p\prime},{\rm ord}}\to H^1(E,\mathscr{V})(\mathscr{X}^{(3,f)}).\tag{6.4.2}$$

By the results of Sect. 6.4.2, $\mathscr{P}$ takes values in the Selmer group $\widetilde{H}^1_f(E, \mathscr{V})(\mathscr{X}^{(3,f)})$. It satisfies the asserted interpolation properties by the definitions of the classes $\mathscr{P}_W$ in Sect. 6.3 and Proposition 6.3.2.

6.4.4 Exceptional locus of $\mathscr{X}$

Let $w|v$ be places of E and F above p, and let $\mu_w^{\pm}\colon E_w^\times \to \mathscr{O}(\mathscr{X})^\times$ be the characters giving the Galois action on $\mathscr{V}^{\pm}$. Let $\mathscr{X}^{\mathrm{exc},v} \subset \mathscr{X}$ be the closed subset defined by $\mu_w^- = 1$ for some (hence automatically all) places $w|v$ of E. We let

$$\mathscr{X}^{\mathrm{exc}} := \bigcup_{v|p} \mathscr{X}^{\mathrm{exc},v}, \quad \mathscr{X}^{\mathrm{cl,exc},(v)} := \mathscr{X}^{\mathrm{cl}} \cap \mathscr{X}^{\mathrm{exc},(v)},$$
$$\mathscr{X}^{\mathrm{cl,n\text{-}exc},(v)} := \mathscr{X}^{\mathrm{cl}} - \mathscr{X}^{\mathrm{cl,exc},(v)}. \tag{6.4.3}$$

We say that an ordinary automorphic representation of $\Pi = \Pi_{|z}$ over a p-adic field is *exceptional at the place* $v|p$ if $z \in \mathscr{X}^{\mathrm{exc},v}$.

We may characterise the exceptional representations, and seize the opportunity to collect some useful results; see also Lemma A.2.5.

Lemma 6.4.6 *Let $\Pi = \pi \otimes \chi$ be an ordinary automorphic representation of* $(\mathrm{G} \times \mathrm{H})'(\mathbf{A})$ *over a p-adic field L, of numerical weights $\underline{w}$, $\underline{l}$.*

1. *Let $v|p$ be a place of F. The following are equivalent:*
   (a) *the representation Π is exceptional at v;*
   (b) $e_v(V_{(\pi,\chi)}) = (1.4.4) = 0$*;*
   (c) *the following conditions hold:*
      – *the smooth representation π_v of G_v is special of the form* $\mathrm{St} \otimes \alpha_v$*;*
      – *for some (equivalently all) places $w|v$ of E, we have $\chi_w \cdot \alpha_v \circ N_{E_w/F_v} = \mathbf{1}$.*
      – *$w_\sigma = 2$ and $l_\sigma = 0$ for all $\sigma : F \hookrightarrow \overline{L}$ inducing the place v.*
2. *Let $S_p^{\mathrm{exc}} = S_p^{\mathrm{exc,\,s}} \cup S_p^{\mathrm{exc,\,ns}}$ be the set of places $v|p$ (respectively those that moreover are split, nonsplit in E) where Π is exceptional.*
   (a) *The kernel of the natural surjective map*

$$\widetilde{H}^1_f(E, V_\Pi) \to H^1_f(E, V_\Pi)$$

   has dimension $2|S_p^{\mathrm{exc,\,s}}| + |S_p^{\mathrm{exc,\,ns}}|$.
   (b) *Assume that* G *is associated with a quaternion algebra* $\mathbf{B}$ *that is split at all places $v|p$. Then*

$$\varepsilon_v^{\mathrm{G}}(V_\Pi) = -1 \iff v \in S_p^{\mathrm{exc,\,ns}}.$$

Proof Consider part 1. We first prove the equivalence of the first two conditions. The adjoint gamma factor in the denominator of each $e_v(V_{(\pi,\chi)})$ is always defined and nonzero, whereas the gamma factor in the numerator is never zero and it has a pole if and only if, for some $w|v$, V_w^+ is the cyclotomic character of $E_w^\times$. This happens precisely when, for some $w|v$, V_w^- is the trivial character – that is, when Π is exceptional at v.

Now let us prove the equivalence to (c). Let $V = V_\Pi = V_{\pi|G_E} \otimes V_\chi$. By the weight-monodromy conjecture (Theorem 2.5.1.2), the 1-dimensional representations $V_{\pi,v}^\pm$ are both of motivic weight -1, thus have no G_{E_w}-invariants for any $w|v$, unless π_v is a special representation. In the latter case $V_{w|z}^+$ (respectively V_w^-) is of weight -2 (respectively 0). This is compatible with the ordinariness requirement only when the weight $\underline{w}$ is 2 at v as in the statement of the lemma. The second condition in (c) is immediate from the definition of (a).

Consider now part 2. The first statement follows directly from (5.3.2). Let us prove the second one. By the results recalled in Sect. 1.2.5 and [105, Lemme 10], the condition $\varepsilon_v^{\mathrm{G}}(V_\Pi) = -1$ is equivalent to the vanishing of the functional $Q = Q_{\Pi_v} =$(4.2.1) and of the space $\Pi_v^{*,H_v'}$. These conditions are never met if v splits in E or π_v is a principal series, and otherwise they are equivalent to the nonvanishing of $(\Pi_v')^{*,H_v'}$, where $\Pi_v' = \pi_v' \otimes \chi_v$ and π_v' denotes the Jacquet–Langlands transfer of π_v to the nonsplit quaternion algebra $B_v'^\times$ over F_v.

Assume that v is nonsplit in E. If π_v is exceptional, then by part 1 we have $\pi_v' = \chi_v \circ \mathrm{Nm}$, where Nm is the reduced norm of B_v', so that obviously $(\Pi_v')^{*,H_v'} \neq 0$. If π_v is not exceptional, then by the explicit computation of Proposition A.3.4 we have $Q_{\Pi_v'} \neq 0$ (see also [35, Corollary A.2.3] for a variant of the last argument). □

6.4.5 *Heegner classes belong to the Bloch–Kato Selmer group*

We can now prove the first assertion of Theorem B.

Proposition 6.4.7 *If Π is not exceptional or has trivial weight, the map P_Π of* (1.2.4) *takes values in $H^1_f(E, V_\Pi) \subset H^1(G_{E,Sp}, V_\Pi)$.*

Proof If Π has trivial weight this is clear. Assume that Π is not exceptional. Let $\partial\colon H^1(E, V)/H^1_f(E, V) \to L$ be any linear map. Then we need to show that the $\mathrm{H}'(\mathbf{A})$-invariant map $\partial P_\Pi \colon \Pi \to L$ is zero. By Corollary 6.4.5 and Theorem C, whose proof we have just completed, the map P_Π^{ord} takes values in $H^1_f(E, V)$; equivalently, $\partial P_\Pi \gamma_{H'}^{\mathrm{ord}} = 0$. Since Π is not exceptional, by Lemma A.2.5 this means that $\partial P_\Pi = 0$. □

6.4.6 Enhanced ordinary Heegner classes for exceptional representations

For any $z = (x, y) \in \mathscr{X}^{\rm cl}$ corresponding to a representation Π, define the *enhanced Heegner class*

$$\widetilde{P}_{\Pi}^{\rm ord} := \mathscr{P}_{|z} \in \widetilde{H}^1_f(E, V_\Pi). \qquad (6.4.4)$$

By the results established so far, $\widetilde{P}_{\Pi}^{\rm ord}$ has image $P_{\Pi}^{\rm ord}$ under the natural map $\widetilde{H}^1_f(E, V_\Pi) \to H^1_f(E, V_\Pi)$; as noted in Lemma 6.4.6, this map fails to be an isomorphism precisely when Π is exceptional.

7 The main theorems, and a conjecture

In this section, we prove our main theorems (Sect. 7.1, or Sect. 7.3.6 for Theorem G), as well as a universal Waldspurger formula for families of 'sign $+1$' (Sect. 7.2). Then, we discuss a conjecture on the leading terms of universal Heegner points (and toric periods) at classical points (Sect. 7.3).

7.1 Proofs of the main theorems

Both of our central theorems (Theorems B and D) ultimately follow from [32,35], where Theorem B is established when W is trivial, by an argument combining interpolation and multiplicity-one principles.

7.1.1 p-adic Gross–Zagier formula for ordinary forms

We start by stating a variant of Theorem B, valid under the same assumptions.

Theorem $B^{\rm ord}$. *Let* $\Pi = \pi \otimes \chi$ *be an ordinary, locally distinguished automorphic representation of* $(\mathrm{G} \times \mathrm{H})'(\mathbf{A})$ *over* L. *Let* $V = V_\Pi$, *and let* $\widetilde{P}_{\Pi}^{\rm ord} \in \widetilde{H}^1_f(E, V_\Pi)$ *be the enhanced Heegner class defined in* (6.4.4).

Then for all $f_1 \in \Pi^{\rm ord}_{H'_\infty}$, $f_2 \in \Pi^{\vee,{\rm ord}}_{H'_\infty}$, $f_3 \in \Pi^{\rm ord}$, $f_4 \in \Pi^{\vee,{\rm ord}}$ *with* $(f_3, f_4)^{\rm ord} \neq 0$, *we have*

$$\frac{h_V(\widetilde{P}_{\Pi}^{\rm ord}(f_1), \widetilde{P}_{\Pi^\vee}^{\rm ord}(f_2))}{(f_3, f_4)_{\Pi}^{\rm ord}} = \mathscr{L}'_p(V_{(\pi,\chi)}, 0) \cdot Q^{\rm ord}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right). \qquad (7.1.1)$$

Remark 7.1.1 In contrast to Theorem B:
- Theorem $\mathrm{B}^{\rm ord}$ also holds for exceptional Π;
- we have only included the Gross–Zagier formula and omitted an analogue to the first statement of Theorem B, that is that $\widetilde{P}_{\Pi}^{\rm ord}$ takes values in $\widetilde{H}^1_f(E, V)$, as that has already been established.

Lemma 7.1.2 *Suppose that Π is not exceptional. Theorem B$^{\rm ord}$ is equivalent to Theorem B.*

Proof Using freely the notation and results of "Appendix A", we first show that Theorem B$^{\rm ord}$ for f_1, f_2, f_3, f_4 is equivalent to Theorem B for

$$f_1' = \gamma_{H'}^{\rm ord} f_1 \in \Pi_{H'_{p\infty}}, \quad f_2' = \gamma_{H'}^{\rm ord} f_2 \in \Pi^\vee_{H'_{p\infty}},$$
$$f_3' = w_{\rm a}^{\rm ord} f_3 \in \Pi^{\rm a}, \quad f_4' = f_4 \in \Pi^{\vee,{\rm ord}};$$

let us call such (f_1', f_2', f_3', f_4') a 'special quadruple'.

Indeed, by the definitions (6.2.3), (4.1.8), the left hand side of (7.1.1) equals

$$\frac{h_V(P_\Pi(f_1'), P^{\rm ord}_{\Pi^\vee}(f_2'))}{\dim W \cdot (f_3', f_4')};$$

whereas by Proposition 4.3.4,

$$Q^{\rm ord}\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right) \doteq \frac{e_p(V_{(\pi,\chi)})^{-1}}{\dim W}\cdot Q\left(\frac{f_1'\otimes f_2'}{f_3'\otimes f_4'}\right).$$

By the multiplicity-one principle, Theorem B for special quadruples implies Theorem B in general, since under our assumptions the functional Q is non-vanishing on special quadruples: this again follows from Proposition 4.3.4 and Lemma 6.4.6.1.(a)–(b). □

7.1.2 Comparison of p-adic L-functions

We describe how, upon restricting $\mathscr{L}_p(\mathscr{V}^\sharp)$ to the cyclotomic line through a point of trivial weight, we recover the p-adic L-function of [32,35].

Lemma 7.1.3 *Let $z = (x, y) \in \mathscr{E}_{\rm G_0}^{\rm ord,cl} \times \mathscr{E}_{\rm H}^{\rm ord,cl}$ be a point corresponding to a representation $\pi_{0,x}\otimes\chi_y$ of weights* $(0; (2, \ldots, 2))$, $(0; 0, \ldots, 0)$. *Let A denote the modular abelian variety attached to $\pi_{0,x}$, and let*

$$\mathscr{L}_p(V_{(A,\chi)}) \in \mathscr{K}(\mathscr{E}_{\rm Z})$$

be the p-adic L-function of [35, Theorem A]. *Consider the map*

$$j_{x,y}\colon \mathscr{E}_{\rm Z} \to \{x\}\times\mathscr{E}_{\rm H}^{\rm ord} \subset \mathscr{E}_{\rm G_0}^{\rm ord,cl}\times\mathscr{E}_{\rm H}^{\rm ord}$$
$$\chi_F \mapsto \pi_{0,x}\otimes\chi_y\cdot\chi_{F\circ N_{E^\times_{\mathbf{A}}/F^\times_{\mathbf{A}}}}.$$

Then

$$\mathscr{L}_p(V_{(\pi_0,\chi)}) := j^*_{(x,y)}\mathscr{L}_p(\mathscr{V}^\sharp) = \mathscr{L}_p(V_{(A,\chi)}). \tag{7.1.2}$$

Proof This is immediate from the respective interpolation properties. (Note that the first equality in (7.1.2) is just a reminder of (1.1.1).) □

7.1.3 Interpolation argument and proof of the main theorems

Let $\mathscr{X} \subset \mathscr{E}^{\mathrm{ord}}_{K^p}$ be a locally distinguished Hida family for $(\mathrm{G}\times\mathrm{H})'$, as in Definition 1.3.1. Fix a level $K^{p\prime} \subset K^p$. Let

$$\mathscr{X}' = \mathscr{X}'_{K^{p\prime}} := \mathscr{X}^{(6)} \cap \mathscr{X}^{(3,f)} \supset \mathscr{X}^{\mathrm{cl}}$$

be the intersection of the open subsets of $\mathscr{X}$ of Theorem 4.4.3 and Corollary 6.4.5.

Recall that we denote by $\mathscr{X}^{\mathrm{cl},W}$ the set of classical points of weight W, and by $\mathscr{X}^{\mathrm{cl,\ n\text{-}exc}}$ the set of non-exceptional classical points. When $W=\mathbf{Q}_p$ is the trivial weight, we also define

$$\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p\mathrm{,ram}} \subset \mathscr{X}^{\mathrm{cl,p\text{-}crys,\ }\mathbf{Q}_p} \subset \mathscr{X}^{\mathrm{cl,n\text{-}exc}}$$

by the following conditions on the classical point $z=(x,y)$ (equivalently, on the representation Π_z):

(p-crys) for all $v|p$, the representation $V_{x|G_{F_v}}$ is potentially crystalline (equivalently, $\pi_{x,v}$ is a principal series; the second inclusion above follows from Lemma 6.4.6);

(ram) $\chi_{y,p}$ is *sufficiently ramified* in the following sense: let $r_v\geq 1$ be minimal such that $1+\varpi_v^{r_v}\mathscr{O}_{F_v}$ is contained in the kernel of $\omega_{x,v}$, and let $U^\circ_{F,p} = \prod_{v|p}(1+\varpi_v^{r_v}\mathscr{O}_{F_v})$; then $\chi_{y,p}$ is is nontrivial on $N^{-1}_{E_p/F_p}(U^\circ_{F,p})\cap\mathscr{O}^\times_{E_p}$.

Lemma 7.1.4 *The subset* $\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p\mathrm{,ram}} \subset \mathscr{X}'$ *is dense.*

Proof Denote by $\mathrm{p}_\mathrm{G}\colon \mathscr{X}'\to\mathscr{E}_\mathrm{G}$ the natural projection. If '?' is any relevant decoration, let $\mathscr{X}^?_\mathrm{G} := \mathrm{p}_\mathrm{G}(\mathscr{X}^?)$; for $x\in\mathscr{X}^{\mathrm{cl}}_\mathrm{G}$, let $\mathscr{X}^?_{x,\mathrm{H}} := \mathrm{p}_\mathrm{G}^{-1}(x)\cap\mathscr{X}^?$.

For each $x\in\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p}_\mathrm{G}$, the set $\mathscr{X}^{\mathrm{cl,\ }\mathbf{Q}_p\mathrm{,\ ram}}_{x,\mathrm{H}}$ contains contains all but finitely many points in $\mathscr{X}^{\mathrm{cl,\ }\mathbf{Q}_p}_{x,\mathrm{H}}$, which is dense in $\mathscr{X}_{x,\mathrm{H}}$. Thus the closure of $\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p\mathrm{,\ ram}}$ contains all of $\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p}$.

Now we observe that $\mathscr{X}^{\mathrm{cl,\ p\text{-}crys,\ }\mathbf{Q}_p}_\mathrm{G} = \mathscr{Y}_\mathrm{G}\cap\mathscr{X}^{\mathrm{cl,\ }\mathbf{Q}_p}_\mathrm{G}$ for the open subset

$$\mathscr{Y}_\mathrm{G} := \mathscr{X}_\mathrm{G} - \{x\,|\mathscr{V}^+_{\mathrm{G},v|x}(-1)\cong\mathscr{V}^-_{\mathrm{G},v|x}\text{for some } v|p\},$$

which is non-empty as it contains $\mathscr{X}_{\mathrm{G}}^{\mathrm{cl}, W_{\mathrm{G}}}$ for any representation W_{G} whose partial weights are all ≥ 3 (cf. the proof of Lemma 6.4.6.1 (c)). Therefore $\mathscr{X}^{\text{cl, p-crys, }\mathbf{Q}_p}$ is the intersection of the non-empty open $\mathrm{p}_{\mathrm{G}}^{-1}(\mathscr{Y}_{\mathrm{G}})$ with $\mathscr{X}^{\mathrm{cl}, \mathbf{Q}_p}$, which is dense in $\mathscr{X}'$ by Lemma 3.1.4. We conclude that $\mathscr{X}^{\text{cl, p-crys, }\mathbf{Q}_p}$ and $\mathscr{X}^{\text{cl, p-crys, }\mathbf{Q}_p\text{, ram}}$ are dense in $\mathscr{X}'$. □

Proposition 7.1.5 *The following are equivalent.*

1. *Theorem* D *holds over* $\mathscr{X}'_{K^{p\prime}}$ *for all* $K^{p\prime} \subset K^p$.

2a. *Theorem* B^{ord} *holds for all representations* Π *corresponding to points of* $\mathscr{X}^{\mathrm{cl}}$ *satisfying* (wt).

2b. *Theorem* B^{ord} *holds for all representations* Π *corresponding to points of* $\mathscr{X}^{\text{cl, p-crys, }\mathbf{Q}_p\text{, ram}}$.

3a. *Theorem* B *holds for all representations* Π *corresponding to points of* $\mathscr{X}^{\text{cl, n-exc}}$ *satisfying* (wt).

3b. *Theorem* B *holds for all representations* Π *corresponding to points of* $\mathscr{X}^{\text{cl, p-crys, }\mathbf{Q}_p\text{, ram}}$.

Proof For any point $z \in \mathscr{X}^{\text{cl,n-exc}}$ satisfying (wt), denote by Π_z the associated automorphic representation, by V_z the associated Galois representation . We have proved the following specialisation-at-z properties of objects defined over (open subsets of) $\mathscr{X}$:

- the $\mathscr{O}_{\mathscr{X}'}$-module $\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ (respectively $(\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord}})^\iota$ specialises to $\Pi_{z,H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ (respectively $\Pi_{z,H'_\Sigma}^{\vee,K^{p\prime},\mathrm{ord}}$), by Proposition 3.3.8.4 (and the definition of the involution ι);
- the Galois representation $\mathscr{V}$ (respectively $\mathscr{V}^\iota$) and its ordinary filtrations specialise to $V = V_\Pi$ (respectively $V_{\Pi^\vee}$) with its ordinary filtrations, by construction (Proposition 3.2.4);
- there is a natural map $\widetilde{H}^1_f(E, \mathscr{V})_{|z} \to \widetilde{H}^1_f(E, V)$;
- the class $\mathscr{P}_{K^{p\prime}}$ specialises to the restriction of $P_{\Pi_z}^{\mathrm{ord}}$ to $\Pi_{z,H'_\Sigma}^{K^{p\prime},\mathrm{ord}}$ under the above map, by Theorem C whose proof is completed in Sect. 6.4.3;
- the product of local terms $\mathscr{Q}$ specialises to the restriction of Q^{ord} to the spaces of H'_Σ-coinvariants, $K^{p\prime}$-invariants in Π_z^{ord}, $\Pi_z^{\vee,\mathrm{ord}}$, by Theorem 4.4.3.

Let us complete the proof that either side of Theorem D specialises to $1/2$ times the corresponding side of Theorem $\mathrm{B}^{\mathrm{ord}}$. Consider the diagram $\mathscr{X}_0 \to \mathscr{X}_0^\sharp \to \mathscr{E}_{\mathrm{Z}}$. It is *not* a product, even Zariski-locally; however the conormal sheaf is trivial. (This is dual to the fact that $\mathrm{G} \times \mathrm{H} \to (\mathrm{G} \times \mathrm{H})'$ is a Z-torsor for the étale topology but not for the Zariski topology.) The immersion $\mathscr{X} \hat{\times} \mathscr{E}_{\mathrm{Z}} \to \mathscr{X}^\sharp$ given by '$(\Pi, \chi_F) \mapsto \Pi \otimes \chi_F$' induces the map on conormal sheaves

$$\mathscr{N}^*_{\mathscr{X}/\mathscr{X}^\sharp} = \mathscr{O}_{\mathscr{X}}\hat{\otimes}\Gamma_F \to \mathscr{N}^*_{\mathscr{X}/\mathscr{X}\times\mathscr{E}_Z} = \mathscr{O}_{\mathscr{X}}\hat{\otimes}\Gamma_F$$

that is multiplication by $1/2$ under the canonical identifications. Hence:

- the p-adic height pairing $h_{\mathscr{V}} = h_{\mathscr{V}^\sharp|\mathscr{X}}$ specialises to $\frac{1}{2}h_V = \frac{1}{2}h_{V\otimes\chi_{F,\mathrm{univ}|G_E}|z}$, by Sect. 5.3.3;
- the derivative $\mathrm{d}^\sharp\mathscr{L}_p(\mathscr{V})$ specialises to $\frac{1}{2}\mathscr{L}'_p(V,0)$ in $\mathbf{Q}_p(z)\hat{\otimes}\Gamma_F$, by the definition in (1.1.1).

We may now complete the proof. By the specialisation properties summarised above, we have 1. $\Rightarrow$ 2.a ($\Rightarrow$ 2.b). By Lemma 7.1.2, we have 2.a $\Rightarrow$ 3.a, 2.b $\Leftrightarrow$ 3.b. By Lemma 7.1.4 and the specialisation properties, 2.b $\Rightarrow$ 1. □

Proof of Theorems B, D, *and* B$^{\mathrm{ord}}$ The first assertion of Theorem B was proved in Proposition 6.4.7. For a representation Π of trivial weight satisfying the conditions (p-crys), (ram), the formula of Theorem B is [35, Theorem B] (cf. Lemma 7.1.3). By Proposition 7.1.5, this implies Theorem D and the general case of Theorems B, B$^{\mathrm{ord}}$. □

7.1.4 Applications to non-vanishing/1: self-dual CM families

We prove the generic non-vanishing result of Theorem F. Recall that $\mathscr{Y}$ is a component of the subvariety $\mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}} \subset \mathscr{E}_{\mathrm{H}}^{\mathrm{ord}}$ cut out by the condition $\chi_{F^\times_{\mathbf{A}}} = \eta\chi_{\mathrm{cyc},F}$, and such that $\varepsilon(\chi_y, 1) = -1$ generically along $\mathscr{Y}$.

Proof of Theorem F Recall that a p-adic CM type of E over $\overline{\mathbf{Q}}_p$ is a choice Σ of a place $w|v$ of E for each place $v|p$ of F (we identify primes above p with embeddings into $\overline{\mathbf{Q}}_p$). For each of the p-adic CM types Σ of E and each connected component $\mathscr{Y}^\sharp$ of $\mathscr{E}_{\mathrm{H}}^{\mathrm{ord}}$, there is a Katz p-adic L-function

$$L_\Sigma \in \mathscr{O}(\mathscr{Y}^\sharp).$$

It is characterised (see [52,64]) by its values at the subset $\mathscr{Y}^{\sharp,\mathrm{cl}.\Sigma} \subset \mathscr{Y}^{\sharp,\mathrm{cl}}$ of those y such that the algebraic part of χ_y is $t \mapsto \prod_{\sigma\in\Sigma}\sigma(t)^w\sigma(t/t^c)^{k_\sigma}$ for integers w, k_σ such that either $w \geq 1$, or $w < 1$ and $w + k_\sigma > 0$ for all $\sigma \in \Sigma$. The interpolation property relates $L_\Sigma(y)$ to $L(1, \chi_y)$. It is easy to see that for a given $y \in \mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}}$, there is a unique CM type Σ such that y belongs to the interpolation subset of L_Σ. For such y and Σ, we denote by $L_p(\chi_y, s) \in \mathscr{O}(\mathscr{E}_{Z/\mathbf{Q}_p(y)})$ the function $s \mapsto L_\Sigma(y(s))$ where $\chi_{y(s)} = \chi\cdot\chi_{F,s}\circ N_{E/F})$.

Consider now the setup of the theorem, and let $\mathscr{Y}^\sharp \subset \mathscr{E}_{\mathrm{H}}^{\mathrm{ord}}$ be the component containing $\mathscr{Y}$. By [13], under our assumptions the normal derivative $d^\sharp L_\Sigma \in \mathscr{N}^*_{\mathscr{Y}/\mathscr{Y}^\sharp}$ is non-vanishing. Let $\mathscr{Y}'_\Sigma$ be the complement of its zero locus, and let

$$\mathscr{Y}' := \bigcap_\Sigma \mathscr{Y}'_\Sigma.$$

Let $y \in \mathscr{Y}^{\mathrm{cl}} \cap \mathscr{Y}'$, and let $\chi := \chi_y$. It is easy to see that there is a unique Σ such that $y \in \mathscr{Y}^{\sharp,\mathrm{cl},\Sigma}$.

We *claim* that there exists a a finite-order character $\chi_0 \in \mathscr{E}_{\mathrm{H}}^{\mathrm{ord,cl}}$, such that the character

$$\chi' = \chi^c \chi_0^c \chi_0^{-1}$$

(that has the same algebraic part as $\chi^c := \chi \circ c$, and defines a point $y' \in \mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd,cl}}$) satisfies the following properties:

– $L(1, \chi') \doteq L_p(\chi', 0) \neq 0$, where $\doteq$ denotes equality up to a nonzero constant;
– $H^1_f(E, \chi') = 0$.

Granted the claim, we have a decomposition of G_E-representations

$$(\chi\chi_0 \oplus \chi^c\chi_0^c) \otimes \chi_0^{-1} = \chi \oplus \chi'$$

and a corresponding factorisation

$$\mathscr{L}_p(V_{(\pi_0, \chi_0^{-1})}, s) \doteq L_p(\chi, s) L_p(\chi', s),$$

where $\pi_0 = \theta(\chi\chi_0)$ (the theta lift), and $\pi_0 \otimes \chi_0^{-1}$ descends to a representation of $(\mathrm{G}_0 \times \mathrm{H})'(\mathbf{A})$. It follows that $\mathscr{L}'_p(V_{(\pi_0, \chi_0^{-1})}, 0) \neq 0$. By Theorem A, we have a class

$$Z \in H^1_f(E, V_{(\pi_0,\chi_0^{-1})}) \otimes H^1_f(E, V_{(\pi_0,\chi_0^{-1})}) = (H^1_f(E, \chi) \otimes H^1_f(E, \chi^c)) \oplus (H^1_f(E, \chi') \otimes H^1_f(E, \chi'^c))$$

whose p-adic height is non-vanishing. Since $H^1_f(E, \chi') = H^1_f(E, \chi'^c)) = 0$, the class Z is as desired.

It remains to prove the claim. Let $\mathscr{Y}_1 \subset \mathscr{E}_{\mathrm{H}}^{\mathrm{ord,sd}}$ be a component over which the anticyclotomic Main Conjecture is known—that is, one containing a finite-order character satisfying the properties of [56]). By applying [14, Lemma 2.5] to any character corresponding to a point of $\mathscr{Y}_1^{\rm cl}$, we find another component $\mathscr{Y}_2 \subset \mathscr{E}_{\rm H}^{\rm ord,sd}$, whose classical points correspond to characters χ_2 with

$$\varepsilon(1, \chi_2) = 1;$$

moreover from the proof in *loc. cit.* one sees that $\mathscr{Y}_2$ may be taken to still satisfy the assumptions of [56]. Then the function $L_{\Sigma c|\mathscr{Y}_2}$ is non-vanishing by [59]; hence, by the density of classical points with a given weight, we may find $y' \in \mathscr{Y}_2^{\rm cl}$ corresponding to a character χ' satisfying the first among the required conditions. By the anticylcotomic Main Conjecture for $\mathscr{Y}_2$ proved in [54,56], that is equivalent to the second condition. Finally, the ratio χ'/χ^c is an anticyclotomic character (that is, trivial on $F_{\mathbf{A}}^\times$), hence [53, Lemma 5.3.1] of the form $\chi_0^c\chi_0^{-1}$ for some finite-order character χ_0. □

7.2 A universal Waldspurger formula

We describe the complementary picture over locally distinguished families attached to *coherent* quaternion algebras over F. Unexplained notions and notation will be entirely parallel to what defined in the introduction.

Let B be a totally definite quaternion algebra *over* F, let Σ be the set of finite places where B is ramified, and let $\mathrm{G}_{/\mathbf{Q}}$ be the algebraic group with $\mathrm{G}(R) = (B \otimes R)^\times$ for any $\mathbf{Q}$-algebra R. Let E be a CM quadratic extension of F, admitting an F-algebra embedding $\mathrm{e}\colon E \hookrightarrow B$ which we fix. We use the same symbols as in the introduction for the towers of Shimura varieties associated to the groups as in (1.2.1). Here, all those Shimura varieties are 0-dimensional.

Let L be a p-adic field and let Π be an automorphic representation of $(\mathrm{G}\times\mathrm{H})'(\mathbf{A}) := (\mathrm{G}\times\mathrm{H})'(\mathbf{A}^\infty) \times (\mathrm{G}\times\mathrm{H}')_{/\mathbf{Q}_p}$ over L of weight W, by which we mean one occurring in $\mathrm{H}^0(\overline{Z}, \mathscr{W}^\vee)\otimes W$. The normalised fundamental class of Y gives rise to an element $P \in \mathrm{H}_0(Z, \mathscr{W})^{\mathrm{H}'(\mathbf{A})}$ and to an $\mathrm{H}'(\mathbf{A})$-invariant functional

$$P_\Pi\colon \Pi \to L,$$

which may be nonzero only if Π is locally distinguished by H'. If Π is ordinary, we again define $P^{\rm ord} := P\gamma_{H'}^{\rm ord}$.

Let $\mathscr{X}$ be a locally distinguished Hida family for $(\mathrm{G}\times\mathrm{H})'$, which via a Jacquet–Langlands map is isomorphic to a Hida family $\mathscr{X}_0$ for $(\mathrm{G}_0\times\mathrm{H})'$. For each compact open subgroup $K^p\subset(\mathrm{G}\times\mathrm{H})'(\mathbf{A}^p)$, there is a 'universal ordinary representation' $\Pi_{H'_\Sigma}^{K^p,\mathrm{ord}}$ of $(\mathrm{G}\times\mathrm{H})'(\mathbf{A})$ over $\mathscr{X}$. As in Theorem C, there exists an $\mathrm{H}'(\mathbf{A}^{p\infty})$-invariant, $\mathscr{O}_{\mathscr{X}}$-linear functional

$$\mathscr{P}\colon \Pi_{H'_\Sigma}^{K^p,\mathrm{ord}}\to\mathscr{O}_{\mathscr{X}} \tag{7.2.1}$$

interpolating (the restrictions of) $P^{\mathrm{ord}}_{\Pi_z}$ at all $z\in\mathscr{X}^{\mathrm{cl}}$ satisfying (wt).

Starting from the natural pairings $\mathrm{H}_0(\overline{Z}_K,\mathscr{W})\otimes\mathrm{H}_0(\overline{Z}_K,\mathscr{W}^\vee)\to L$, we may define pairings $(\ ,\ )_\Pi$ on each representation Π over a field by the formula (4.1.7) (using the counting measure for $\mathrm{v}(K)$); then we obtain modified pairings $(\ ,\ )^{\mathrm{ord}}_\Pi$ on each $\Pi^{\mathrm{ord}}\otimes\Pi^{\vee,\mathrm{ord}}$, and a pairing $((\ ,\ ))$ on the universal representations over $\mathscr{X}$, interpolating modified pairings $(\ ,\ )^{\mathrm{ord}}_{\Pi_z}$.

Finally, the functional $\mathscr{Q}$, over an open $\mathscr{X}'\subset\mathscr{X}$ containing $\mathscr{X}^{\mathrm{cl}}$, is also constructed as in Sect. 4.4; in the argument using the local Langlands correspondence, we use the rank-2 family of Galois representations pulled back from $\mathscr{X}_0$.

Theorem H *Let* $\mathscr{X}$ *be a locally distinguished Hida family for* $(\mathrm{G}\times\mathrm{H})'$. *Abbreviate* $\Pi^{(\iota)}:=\Pi_{H'_\Sigma}^{K^{p\prime},\mathrm{ord},(\iota)}$, $\mathscr{O}:=\mathscr{O}_{\mathscr{X}}$, $\mathscr{K}:=\mathscr{K}_{\mathscr{X}}$.

There is an open subset $\mathscr{X}'\subset\mathscr{X}$ *containing* $\mathscr{X}^{\mathrm{cl}}$, *such that*

$$\frac{\mathscr{P}(f_1)\cdot\mathscr{P}^\iota(f_2)}{((f_3,f_4))}=\mathscr{L}_p(\mathscr{V}^\sharp)_{|\mathscr{X}}\cdot\mathscr{Q}\left(\frac{f_1\otimes f_2}{f_3\otimes f_4}\right),$$

an equality of $\mathscr{K}$*-valued* $\mathscr{O}$*-linear functionals on* $(\Pi\otimes_{\mathscr{O}}\Pi^\iota)\otimes_{\mathscr{O}^\times}(\Pi\otimes_{\mathscr{O}}\Pi^\iota)^{\times,-1}$.

Similarly to Sect. 7.1, this universal formula follows from its specialisations at all classical points satisfying (wt); those are known by modifying the main result of [105] as in Theorem B $^{\mathrm{ord}}$.

The formula essentially reduces the study of $\mathscr{L}_p(\mathscr{V}^\sharp)_{|\mathscr{X}}$ to the study of the universal Waldspurger periods $\mathscr{P}$. This is particularly interesting in the case of exceptional zeros, as we discuss next.

7.3 Bertolini–Darmon conjectures and exceptional zeros

We first formulate a conjecture on the behaviour of $\mathscr{P}$ at a point $z\in\mathscr{X}^{\mathrm{cl}}$ and gather some old and new evidence in its favour. In view of Sect. 6.4.6, the conjecture is often particularly interesting when z is exceptional. Then we deduce from our constructions and a known exceptional case of the conjecture a proof of Theorem G.

The conjecture requires some algebraic preliminaries.

7.3.1 *Pfaffian regulators*

Let L be a field of characteristic 0, and let M, T be a finite-dimensional L-vector spaces. Let $h\colon M \otimes M \to T$ be a skew-symmetric pairing, and let $r = \dim_L M$.

If r is even, we define the Pfaffian regulator

$$\mathrm{Pf}^+(M, h) \in (\mathrm{Sym}^{r/2} T)/L^\times$$

to be the Pfaffian of the skew-symmetric matrix $h(x_i, x_j)_{ij}$ for any L-basis x_i of M. It is well-defined modulo $L^\times$.

If r is odd, we define an enhanced Pfaffian regulator

$$\mathrm{Pf}^-(M, h) \in (M \otimes \mathrm{Sym}^{(r-1)/2} T)/L^\times$$

as follows. It suffices to define $\partial^e \mathrm{Pf}^-(h)$ for any basis $\partial_1, \ldots, \partial_d$ of $T^\vee$ and all tuples $e = (e_i)_{i=1}^d$ with $\sum_{i=1}^d e_i = (r-1)/2$; here $\partial^e := \prod_{i=1}^d \partial_i^{e_i}$. Let $I \subset \{1, \ldots, d\}$ be the support of the tuple e. Let $M_I \subset M$ be the sum, over $i \in I$, of the radicals of the pairings $\partial_i h$. If $\dim M_I \geq 2$, we define $\partial^e \mathrm{Pf}^-(h) := 0$. If $\dim M_I = 1$ and $x \in M_i$ is a generator, denote by $\overline{h}$ the induced pairing on $\overline{M} := M/M_I$; we define

$$\partial^e \mathrm{Pf}^-(M, h) = x \otimes \partial^e \mathrm{Pf}^+(\overline{M}, \overline{h}).$$

Remark 7.3.1 In the even case, we have of course $\mathrm{Pf}^+(M, h)^2 = R(M, h) \in (\mathrm{Sym}^r T)/L^{\times,2}$, where $R(M, h)$ is the discriminant of the pairing h. In the odd case, assume further given a symmetric bilinear pairing $h^\sharp\colon M \otimes M \to T^\sharp$. Let $h' = h \oplus h^\sharp\colon M \otimes M \to T' : -T \oplus T^\sharp$, and let $R(M, h') \in \mathrm{Sym}^r(T')/L^{\times,2}$ be its discriminant. Then it is easy to verify that

$$h^\sharp(\mathrm{Pf}^-(M, h), \mathrm{Pf}^-(M, h)) \in (T^\sharp \otimes \mathrm{Sym}^{r-1} T/)L^{\times,2}$$

is the image of $R(M, h')$ under the natural projection.

Remark 7.3.2 If L is the fraction field of a domain $\mathscr{O}$ and M is endowed with an $\mathscr{O}$-lattice, it is possible to lift the ambiguity in the definitions to an element of $\mathscr{O}^\times$.

7.3.2 A conjecture à la Bertolini–Darmon

Let G be either as in the introduction or as in Sect. 7.2. We define a sign $\epsilon := -1$ in the former case and $\epsilon := +1$ in the latter case. Let $\mathscr{X}$ be a locally distinguished Hida family for $(\mathrm{G} \times \mathrm{H})'$, and let

$$z \in \mathscr{X}^{\mathrm{cl}}$$

be a classical point. We denote by $\mathscr{P}$ the universal Heegner class (if $\epsilon = -1$) or toric period (if $\epsilon = +1$), viewed as a family of functionals as in (6.4.2), (7.2.1), parametrised by a subset $\mathscr{X}' \subset \mathscr{X}$. Assume that $\mathscr{X}'$ can be taken to be a neighbourhood of $z \in \mathscr{X}$.

Let $T_z^*\mathscr{X} = \mathfrak{m}_z/\mathfrak{m}_z^2$ be the cotangent space to $\mathscr{X}$ at z (where $\mathfrak{m}_z \subset \mathscr{O}_{\mathscr{X},z}$ is the maximal ideal), and for any $r \in \mathbf{N}$ let

$$(\mathrm{d}_z^{\|})^r : \mathfrak{m}_z^r \subset \mathscr{O}_{\mathscr{X},z} \to \mathfrak{m}_z^r/\mathfrak{m}_z^{r+1} = \mathrm{Sym}^r T_z^*\mathscr{X}$$

be the natural projection. It is easy to see that the involution ι of Sect. 2.1.4 satisfies $\mathrm{d}_z^{\|}\iota = \mathrm{id}$ (whereas $\mathrm{d}_z^{\sharp}\iota = -\mathrm{id}$).

Let $V = \mathscr{V}_{|z}$, and let $\mathrm{c}\colon \widetilde{H}^1_f(E, V^\iota) \to \widetilde{H}^1_f(E, V)$ be the isomorphism induced by the adjoint action, on $G_{E,S}$, of a lift of the complex conjugation in $\mathrm{Gal}(E/F)$. Let

$$h^{\|} := h^{\square}_{z/\mathscr{X}} : \widetilde{H}^1_f(E, V) \otimes \widetilde{H}^1_f(E, V) \to T_z^*\mathscr{X} \tag{7.3.1}$$

be the Nekovář–Venerucci height pairing as in (5.3.7). Since the pairing of Proposition 4.1.7 is skew-hermitian, by Proposition 5.3.4 the pairing $h^{\|}$ is skew-symmetric. Define

$$\mathrm{Pf}^{\|,\pm}(V) := \mathrm{Pf}^{\pm}(\widetilde{H}^1_f(E, V), h^{\|}).$$

Conjecture Pf *Let $\widetilde{r} := \dim_L \widetilde{H}^1_f(E, V)$. We have $(-1)^{\widetilde{r}} = \epsilon$, the universal element $\mathscr{P}$ vanishes to order at least $\lfloor \widetilde{r}/2 \rfloor$ at z, and for any generator $\wp \in (\Pi^{\mathrm{ord}})^{*,\mathrm{H}'(\mathbf{A}^{p\infty})}$ and all $f \in \Pi^{\mathrm{ord}}$, we have*

$$(\mathrm{d}_z^{\|})^{\lfloor \widetilde{r}/2 \rfloor} \mathscr{P}(f) = \mathrm{Pf}^{\|,\epsilon}(V) \cdot \wp(f) \tag{7.3.2}$$

in $[\widetilde{H}^1_f(E, V)^{(1+\epsilon)/2} \otimes_{\mathbf{Q}_p(z)} \mathrm{Sym}^{\lfloor \widetilde{r}/2 \rfloor}_{\mathbf{Q}_p(z)} T_z^*\mathscr{X}]/\mathbf{Q}_p(z)^\times$.

7.3.3 Relation to the original conjectures of Bertolini–Darmon

Define $\mathscr{X}^{\mathrm{a}} := \mathscr{X} \cap (\{x\} \times \mathscr{E}_{\mathrm{H}'})$ (the anticyclotomic family), $\mathscr{X}^{\mathrm{wt}} := \mathscr{X} \cap (\mathscr{E}_{\mathrm{G}} \times \{y\})$ (the weight family). In the classical case when E is imaginary quadratic, π is associated with an elliptic curve over $\mathbf{Q}$, $\chi = \mathbf{1}$, and we restrict to the anticyclotomic variable $\mathscr{X}^{\mathrm{a}}$, Conjecture Pf is a variant of conjectures of Bertolini–Darmon, surveyed in [6] and (in a form slightly closer to the one of the present work) in [33, § 4]. The Bertolini–Darmon conjectures were partly generalised to higher-weight modular forms in [71].

Remark 7.3.3 By using natural $G_{E,S}$-stable lattices in V, or better lattices in $\widetilde{H}^1_f(E, V)$ spanned by motivic elements, it is possible to define the Pfaffian regulators up to an ambiguity that is a unit in the ring of integers of a local field or of a number field (recall Remark 7.3.2). It should then possible to refine the conjecture up to such ambiguity (by including the appropriate constants), as in the original works of Bertolini–Darmon (see [33, Conjecture 4.2.1]).

In view of Remark 7.3.1, inserting Conjecture Pf into Theorem D would yield a multivariable formula relating higher partial derivatives of p-adic L-functions with suitable height regulators, in the spirit of the Birch and Swinnerton-Dyer conjecture; the argument is the same as that of [33, Proposition 5.1.1]. We plan to return to formulate such conjectural formulas in the appropriate generality in future work.

7.3.4 Evidence for Conjecture Pf in low rank

The preliminary parity conjecture

$$(-1)^{\widetilde{r}} = \epsilon \tag{7.3.3}$$

is known in many cases as a consequence of the work of Nekovář (see [81, Theorem 12.2.3]. Indeed, the statement proved in *loc. cit.*, in view of the functional equation of $\mathscr{L}(V_{(\pi,\chi)}, s)$, is that

$$(-1)^{r} = \varepsilon \tag{7.3.4}$$

where $r = \dim H^1_f(E, V)$ and $\varepsilon = \varepsilon(V)$. Now if $r^{\mathrm{exc}} = r^{\mathrm{exc,\,s}} + r^{\mathrm{exc,\,ns}}$ denotes the number of exceptional primes of F above p (respectively, the number of those that moreover are split or nonsplit in E), by Lemma 6.4.6.2 we have $\epsilon = \varepsilon \cdot (-1)^{r^{\mathrm{exc,\,ns}}}$, and $\widetilde{r} = r + 2r^{\mathrm{exc,\,s}} + r^{\mathrm{exc,\,ns}}$. Thus (7.3.4) is equivalent to (7.3.3).

Let us now review the conjectural vanishing and leading-term formula. Most of the available evidence is concentrated in the case where V arises from the classical context of Sect. 7.3.3, to which we restrict unless otherwise noted for the rest of this discussion. (All the results mentioned below hold under various additional assumptions, which we will not recall.)

If $\mathscr{X}$ is replaced with $\mathscr{X}^{\mathrm{a}}$ (the original Bertolini–Darmon case), the conjecture is known if $\mathscr{L}(V_{(\pi,\chi)}, s)$ vanishes to order 0 or 1 at $s = 0$, see [33, Theorem

4.2.5] and references therein, as well as [70]. Some of those results have been generalised to higher weight [62,72] or to totally real fields [4,61,75].

When $\mathscr{X}$ is replaced by $\mathscr{X}^{\rm wt}$, p is inert in E, V is exceptional, and $\epsilon = +1$, Bertolini–Darmon [7] proved a formula for $\mathrm{d}_z^{\|}\mathscr{P}_{|\mathscr{X}^{\rm wt}}$, which implies the projection of (7.3.2) to $T_z^*\mathscr{X}^{\rm wt}$ when $\mathrm{ord}_{s=0}\mathscr{L}(V_{(\pi,\chi)}, s) = 1$. The interpretation of the formula of Bertolini–Darmon in terms of height pairings was observed by Venerucci (see [104, Theorem 2.1 and Theorem 4.2.2]), whose work was a second important influence in the formulation of Conjecture Pf. The Bertolini–Darmon formula was generalised to higher-weight modular forms by Seveso [95] and to elliptic curves over totally real fields by Mok [74].

7.3.5 Evidence for Conjecture Pf in higher rank

Lower bounds for the order of vanishing of $\mathscr{P}_{|\mathscr{X}^{\rm a}}$ have been obtained in two recent works for $\epsilon = 1$. In the context of elliptic curves over totally real fields, [4, Theorem 5.5] gives a bound (that is coarser than predicted by Conjecture Pf) in terms of the number of exceptional primes. In a classical context (and if if p splits in E), Agboola–Castella [1, Corollary 6.5] prove a bound that is finer than that of Conjecture Pf. (That refined bound is predicted by Bertolini–Darmon; it is related to some trivial degeneracies of the anticyclotomic height pairing, as touched upon also in the paragraphs preceding (7.3.7) below.)

Regarding the formula of Conjecture Pf, an interesting anticyclotomic case can be deduced from the recent work [39], as we now explain.

The work of Fornea–Gehrmann Suppose that A is a modular elliptic curve over the totally real field F, such that the set $S_p^{\rm exc}$ of places $v|p$ where A has multiplicative reduction consists of exactly r primes $v_1, \ldots, v_r$ inert in E. Let $\phi\colon \bigotimes_{i=}^r E_{v_i}^\times \to \bigotimes_{i=1}^r A(E_{v_i})$ be the product of Tate unifomisations, and let $\hat{\phi}$ be the induced map on p-adic completions.

One of the main constructions of [39] produces an explicit element $Q_A \in \bigotimes_{i=1}^r E_{v_i}^\times \hat{\otimes} \mathbf{Q}_p$, which carries a precise conjectural relation to the arithmetic of A. Partition $S_p^{\rm exc} = S_p^{{\rm exc},+} \sqcup S_p^{{\rm exc},-}$ according to whether the multiplicative reduction is, respectively, split or nonsplit, and let $r^+ := |S_p^{{\rm exc},+}|$. Denote $\hat{A}(E_v) = A(E_v)\hat{\otimes}\mathbf{Q}_p$ and let $\hat{A}(E_v)^\pm$ be its $\pm$-eigenspaces for the conjugation $c_v \in \mathrm{Gal}(E_v/F_v)$.

Assume that A_E has rank $r_A \geq r$ and let r_A^+ be the rank of A. According to [40, Conjectures 1.3, 1.5], we have[27]

$$
\begin{aligned}
r_A > r \text{ or } r_A^+ + r^+ \neq r &\Longrightarrow \hat{\phi}(Q_A) = 0,\\
r_A = r \text{ and } r_A^+ + r^+ = r &\Longrightarrow \hat{\phi}(Q_A) \doteq \det((\hat{x}^{\mathrm{a}}_{i,v_j})_{1\leq i,j\leq r}),
\end{aligned}
\tag{7.3.5}
$$

where $(x_1, \dots, x_r)$ is a basis of $A(E)_{\mathbf{Q}}$, and for $v \in S_p^{\mathrm{exc},\pm}$ we denote by $\hat{x}^{\mathrm{a}}_v$ the eigen-projection of $x \in A(E)$ to $\hat{A}(E_v)^{\mp}$. The symbol $\doteq$ denotes equality up to a constant in $\mathbf{Q}^\times$.

For $V = V_p A_E$, assuming the finiteness of $\text{Ш}(A_E)[p^\infty]$ we have $\widetilde{r} := \dim \widetilde{H}^1_f(E, V) = r + r_A$ (see Lemma 6.4.6.1(a) or (7.3.7) below), and the parity conjecture is known. Hence if $r_A \equiv r \pmod 2$, which we henceforth assume, V corresponds to a point z of a locally distinguished Hida family $\mathscr{X}$ of sign $\epsilon = +1$ (for a unique choice of the coherent quaternionic group G). Let

$$
\Gamma^{\mathrm{a}} := (E^\times_{\mathbf{A}^\infty}/F^\times_{\mathbf{A}^\infty}\widehat{\mathscr{O}}_E^{p,\times})\hat{\otimes}\mathbf{Q}_p \cong T_z^*\mathscr{X}^{\mathrm{a}}, \quad \ell^{\mathrm{a}} = \prod_w \ell^{\mathrm{a}}_w \colon E^\times_{\mathbf{A}^\infty}/E^\times \to \Gamma^{\mathrm{a}}
$$

be the natural projection, and let $\ell^{\mathrm{a}}_{\mathrm{exc},\otimes} := \bigotimes_{i=1}^r \ell^{\mathrm{a}}_{v_i} \colon \bigotimes_{i=1}^r E^\times_{v_i}\hat{\otimes}\mathbf{Q}_p \to \mathrm{Sym}^r\,\Gamma^{\mathrm{a}}$. Denoting by $\mathrm{d}^{\mathrm{a}}_z$ the component of $\mathrm{d}^{\|}_z$ along $\mathscr{X}^{\mathrm{a}}$, Theorem A of [39] (which relies on the aforementioned lower bound from [4]) proves

$$
(\mathrm{d}^{\mathrm{a}}_z)^r \mathscr{P}(f^\circ) \doteq \ell^{\mathrm{a}}_{\mathrm{exc},\otimes}(Q_A) \tag{7.3.6}
$$

for a suitable test vector $f^\circ \in \Pi^{\mathrm{ord}}$.

Comparison with Conjecture Pf We show that granted (7.3.5) and the finiteness of $\text{Ш}(A_E)[p^\infty]$, the formula (7.3.6) is equivalent to the conjectured (7.3.2).

Let $h^{\mathrm{a}} \colon \widetilde{H}^1_f(E, V) \otimes \widetilde{H}^1_f(E, V) \to T_z^*\mathscr{X}^{\mathrm{a}} = \Gamma^{\mathrm{a}}$ be the projection of $h^{\|}$, and let

$$
\mathrm{Pf}^{\,\mathrm{a},+}(V) = \mathrm{Pf}^+(\widetilde{H}^1_f(E, V), h^{\mathrm{a}}) \in \mathrm{Sym}^r\,\Gamma^{\mathrm{a}}.
$$

[27] In *locc. citt.*, some restrictive assumptions are made (in particular that E is not CM), but the conjectures make sense even without those and indeed closely related conjectures appear in [39] without those assumptions. Moreover, our statement slightly differs from the ones of [39], which instead of postulating that $(x_1, \dots, x_r)$ is a basis, postulates that $\hat{\phi}(Q_A) \neq 0$ under the extra assumption that $\mathrm{Res}_{F/\mathbf{Q}}A$ is simple (equivalently $L(A, s)$ is primitive). Our slight reformulation appears more uniform and still addresses [39, Remark 1.1] (cf. the comment following [39, Conjecture 1.5]).

Let $\widetilde{H}^1_f(E, V)^{\pm}$ be the eigenspaces for the complex conjugation $c \in \mathrm{Gal}(E/F)$. Since $h_{z/\mathscr{X}^{\mathrm{a}}}$ enjoys the c-equivariance property $h_{z/\mathscr{X}^{\mathrm{a}}}(cx, cx') = c.h_{z/\mathscr{X}^{\mathrm{a}}}(x, x')$ and c acts by -1 on Γ^{a}, by construction the pairing h^{a} satisfies $h(cx, cx') = -h(x, x')$, so that each of $\widetilde{H}^1_f(E, V)^{\pm}$ is h^{a}-isotropic. In particular, Conjecture Pf agrees with (7.3.6) and (7.3.5) that, in the first case of the latter, we have $(\mathrm{d}^{\mathrm{a}}_z)^r \mathscr{P} = 0$.

Assume now we are in the second case of (7.3.5), so that each of $\widetilde{H}^1_f(E, V)^{\pm}$ has dimension r and h^{a} need not be degenerate. For $1 \le i \le r$, let $q_i = q_{v_i} \in E^{\times}_{v_i}$ be a Tate parameter for A/E_v. By [77, § 7.14] we explicitly have

$$\widetilde{H}^1_f(E, V) = H^1_f(E, V) \oplus \bigoplus_{i=1}^{r} \mathbf{Q}_p \cdot [q_{v_i}]$$
$$h^{\mathrm{a}}(q_v, q_{v'}) = 0, \quad h^{\mathrm{a}}(x, q_v) = \log^{\mathrm{a}}_{A,v}(\hat{x}_v), \quad x \in A(E), \quad v \neq v', \qquad (7.3.7)$$

where $\log^{\mathrm{a}}_{A,v} \colon \hat{A}(E_v) \otimes \mathbf{Q}_p \cong E^{\times}_v / q_v^{\mathbf{Z}} \hat{\otimes} \mathbf{Q}_p \cong \mathscr{O}^{\times}_{E_v} \hat{\otimes} \mathbf{Q}_p \xrightarrow{\ell^{\mathrm{a}}_v} \Gamma^{\mathrm{a}}$. Note that $\log^{\mathrm{a}}_{A,v}$ factors through $x \mapsto \hat{x}^{\mathrm{a}}_v$.

Up to changing the basis x_i of $A(E)_{\mathbf{Q}}$ and reordering the v_i, we may assume that the basis

$$x_{r^+ +1}, \ldots, x_r, q_1, \ldots, q_{r^+}, \quad x_1, \ldots, x_{r^+}, q_{r^+ +1}, \ldots, q_r$$

of $\widetilde{H}^1_f(E, V)$ is the concatenation of a basis of $\widetilde{H}^1_f(E, V)^+$ and a basis of $\widetilde{H}^1_f(E, V)^-$, respectively. Using this basis, (7.3.7), and the identity $\mathrm{pf}\begin{pmatrix} & M \\ -M^{\mathrm{t}} & \end{pmatrix} = \pm \det M$, we have[28]

$$\mathrm{Pf}^{\,\mathrm{a},+}(V) = \det M = \det M_1 \det M_2,$$

where the $r \times r$ matrix M is block-left-upper-triangular with anti-diagonal blocks $M_k = (\log^{\mathrm{a}}_{A,v_j}(x_i))_{i,j \in I_k}$ for $I_1 = \{r^+ + 1, \ldots, r\}$, $I_2 = \{1, \ldots, r^+\}$.

On the other hand, we note that under (7.3.5), we have $\ell^{\mathrm{a}}_{\mathrm{exc},\otimes}(Q_A) = \det N$ where the $r \times r$ matrix $N_{ij} = \log^{\mathrm{a}}_{A,v_j}(\hat{x}_{i,v}) = h^{\mathrm{a}}(x_i, q_j)$ is block-diagonal with blocks $N_1 = M_2$, $N_2 = M_1$. Thus $\ell^{\mathrm{a}}_{\mathrm{exc},\otimes}(Q_A) = \mathrm{Pf}^{\,\mathrm{a},+}(V)$, and (7.3.6) is equivalent to (7.3.2).

[28] All the equalities to follow ignore signs and in fact, by our coarse definitions, only make sense at best up to $\mathbf{Q}^{\times}$.

7.3.6 Applications to non-vanishing/2: exceptional families

We prove Theorem G.[29] Recall that $\mathscr{X}_0$ is a Hida family for $\mathrm{PGL}_{2/\mathbf{Q}}$, that contains a classical point $x_0 \in \mathscr{X}(\mathbf{Q}_p)$ corresponding to an elliptic curve A with split multiplicative reduction at p satisfying $L(A, 1) = L(V_p A, 0) \neq 0$.

Proof of Theorem G Let E be an imaginary quadratic field, with associated quadratic character η, satisfying the following: p is inert in A, all other primes dividing the conductor of A split in E, and the twisted L-value $L(A, \eta, 1) \neq 0$. Then A has split multiplicative reduction over E with Tate parameter

$$q = q_A \in E_p^\times.$$

Let $\Omega_{A_E} \in \mathbf{C}^\times$ be the Néron period, and let let $\mathrm{H} := \mathrm{Res}_{E/\mathbf{Q}}\mathbf{G}_m$.

By construction, $\varepsilon_v(V_p A_E) = 1$ for all finite $v \nmid p$, hence the Hida family $\mathscr{X} \subset \mathscr{E}^{\mathrm{ord}}_{(\mathrm{GL}_2\times\mathrm{H})'}$ containing the image of $\mathscr{X}_0 \times \{\mathbf{1}\}$ is locally distinguished. Let $\mathscr{X}^\sharp \subset \mathscr{E}^{\mathrm{ord}}_{(\mathrm{GL}_2\times\mathrm{H})}$ be the Hida family containing $\mathscr{X}$. Let Π be the universal ordinary representation over $\mathscr{X}$ and let $f \in \Pi$ be such that $f_{|x_0}$ is a test vector (that is, a vector not annihilated by any $\mathrm{H}'(\mathbf{A}^p)$-invariant functional $\lambda\colon \Pi_{|x_0} \to \mathbf{Q}_p$). Let $\mathscr{P}_{0,E}$ be the pullback of $\mathscr{P}(f)$ to $\mathscr{X}_0$, and let $\mathscr{P}_0 := \frac{1}{2}\mathrm{Tr}_{E/\mathbf{Q}}\mathscr{P}_{0,E}$. By Corollary 6.4.5,

$$\mathscr{P}_{0,E} \in \widetilde{H}^1_f(G_E, \mathscr{V}_0), \quad \mathscr{P}_0 \in \widetilde{H}^1_f(G_{\mathbf{Q}}, \mathscr{V}_0).$$

By the main result of [5] (as reformulated in [6, Theorem 5.4, § 5.2]),[30] there is a constant $c \in \mathbf{Q}_p^\times$ such that

$$\mathscr{P}_{0,E}(x_0) \otimes \mathscr{P}^\iota_{0,E}(x_0) = c \cdot \frac{L(A_E, 1)}{\Omega_{A_E}} \cdot [q] \otimes [q] \quad \text{in } \widetilde{H}^1_f(\mathbf{Q}, V_p A_E) \otimes \widetilde{H}^1_f(\mathbf{Q}, V_p A_E)$$

using the description $\widetilde{H}^1_f(\mathbf{Q}, V_p A_E) = \mathbf{Q}_p \cdot [q] \oplus H^1_f(\mathbf{Q}, V_p A_E)$ as in (7.3.7).

In particular, $\mathscr{P}_{0,E}(x_0) = \mathscr{P}_0(x_0)$ is a nonzero multiple of $[q]$, which is $\mathrm{Gal}(E/\mathbf{Q})$-invariant. Hence $\mathscr{P}_{0,E}$ and $\mathscr{P}_0$ are non-vanishing. Then by [76], $\widetilde{H}^1_f(\mathbf{Q}, \mathscr{V}_0)$ has generic rank 1.

Moreover,

$$h_{\mathscr{V}_0/\mathscr{V}_0^\sharp}(\mathscr{P}_0, \mathscr{P}_0^\iota)(x_0) = c \cdot \frac{L(A_E, 1)}{\Omega_{A_E}} \cdot h([q], [q])$$

[29] A less interesting variant of it was sketched in [35].

[30] In the works of Bertolini–Darmon, an explicit test vector f is chosen; cf. [33] for more details on bridging the setups.

$$= c \cdot \ell(q) \cdot \frac{L(A_E, 1)}{\Omega_{A_E}} \in \Gamma_{\mathbf{Q}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \tag{7.3.8}$$

where $\ell\colon \mathbf{Q}_p^\times \to \Gamma_{\mathbf{Q}}$ is the universal logarithm (see again [77, § 7.14] for the second equality). By [2], the right-hand side is nonzero, hence $h_{\mathscr{V}_0/\mathscr{V}_0^\sharp}(\mathscr{P}_0, \mathscr{P}_0^\iota) \neq 0$. □

Remark 7.3.4 As noted in [33,35], the combination of Theorem D (or rather Theorem B $^{\rm ord}$) and a precise form of (7.3.8) gives a new proof ot the following theorem of Greenberg–Stevens [43]: for $A_{/\mathbf{Q}_p}$ an elliptic curve of split multiplicative reduction at p and $L_p(V_pA) \in \mathbf{Z}_p[\![\Gamma_{\mathbf{Q}}]\!]_{\mathbf{Q}_p}$ its p-adic L-function,

$$L_p'(V_pA, 0) = \frac{\ell(q)}{\mathrm{ord}_p(q)} \cdot \frac{L(A, 1)}{\Omega_A}.$$

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Appendix A. *p*-adic semilocal constructions

A.1 Preliminaries

Throughout this appendix, unless otherwise noted L denotes a field of characteristic zero (admitting embeddings into $\mathbf{C}$).

A.1.1 Admissible and coadmissible representations

Let G be a reductive group over $\mathbf{Q}_p$. We denote

$$\begin{aligned} &G_p := \mathrm{G}(\mathbf{Q}_p), \quad G_\infty := \mathrm{G}(\mathbf{Q}_p), \quad G = G_{p\infty} := G_p \times G_\infty, \\ &G_\Delta := \Delta(\mathrm{G}(\mathbf{Q}_p)) \subset G, \end{aligned} \tag{A.1.1}$$

where G_p and G_Δ have the p-adic topology, G_∞ has the Zariski topology, and Δ is the (continuous) diagonal embedding. The difference between G_p, G_∞, G_Δ will be in the category of modules we choose to consider. Namely, we consider the categories of smooth admissible representations of G_p over L, of algebraic representations of G_∞ over L, and the products of such for G; we call the latter locally algebraic representations of G over L.

Definition A.1.1 Suppose that L is a finite extension of $\mathbf{Q}_p$. A *p-adic* locally algebraic admissible representation Π of G over L is one such that for each compact open subgroup $K \subset G_\Delta$, there exists a family of $\mathscr{O}_L$-lattices $\Pi^{K,\circ} \subset \Pi^K$, for $K \subset G_\Delta$, with the property that $\Pi^{K',\circ} \cap \Pi^K = \Pi^{K,\circ}$ for all $K' \subset K$.

The typical example of a p-adic locally algebraic admissible representation is $\varinjlim_{K_p \subset G_p} H^i(Y_{K^pK_p}, \mathscr{W}) \otimes W^\vee$, where Y_K is the system of locally symmetric spaces attached to a model $\mathrm{G}_{\mathbf{Q}}$ of G over $\mathbf{Q}$, and $\mathscr{W}$ is the automorphic local system attached to the algebraic representation W of G_∞.

There is a dual notion, introduced in [93, p. 152], see also [94]. Assume that L is endowed with a discrete valuation (possibly trivial), giving it a norm $|\cdot|$. Let G' be one of the groups (A.1.1) or an open subgroup. For $K \subset G'$ a compact open subgroup, let $\mathscr{D}_{G',K} = \mathscr{H}_{G',K} := C_c^\infty(K\backslash G'/K, L)$ and $\mathscr{D}_{G'} = \varprojlim \mathscr{D}_{G',K}$ be the Hecke algebras of distributions; they are endowed with a natural topology as L-vector space, respectively as the inverse limit. A *coadmissible* G'-representation M over $(L, |\cdot|)$ is a topological right $\mathscr{D}_{G'}$-module such that, for any compact subgroup $G^\circ \subset G'$, the $\mathscr{D}_{G^\circ}$-module M admits a presentation of the following form: there exists a system of topological $\mathscr{D}_{G^\circ,K}$-modules M_K and isomorphisms $M_K \cong M_{K'} \otimes_{\mathscr{D}_{G^\circ,K'}} \mathscr{D}_{G^\circ,K}$ for $K' \subset K \subset G^\circ$, such that $M \cong \varprojlim_K M_K$.

Considering first a field L as endowed with a trivial valuation, we shall consider coadmissible representations M of G_p over L that are *smooth* in the sense the Lie algebra $\mathfrak{g}$ of G_p acts trivially; coadmissible representations W of G_∞ that are algebraic (those are just algebraic representations); and the products of such as representations of G, which we call locally algebraic coadmissible representations of G.

Definition A.1.2 Suppose that L is a finite extension of $\mathbf{Q}_p$; denote by $|\cdot|$ the p-adic norm and by $|\cdot|_{\mathrm{triv}}$ the trivial norm on L. A *p-adic* locally algebraic coadmissible representation M of G over L is one as above for $(L, |\cdot|_{\mathrm{triv}})$, whose restriction to G_Δ is coadmissible for $(L, |\cdot|)$.

The typical example of a p-adic locally algebraic coadmissible representation is $\varprojlim_{K_p} H_i(Y_{K^pK_p}, \mathscr{W}) \otimes W^\vee$, where the notation is as after Definition A.1.1.

A.1.2 Notation

Consider the groups (1.2.1). For a place $v|p$ of F, we let

$$G_v := \mathbf{B}_v^\times, \quad H_v := E_v^\times, \quad H'_v := E_v^\times/F_v^\times, \quad (G \times H)'_v := (G_v \times H_v)/F_v^\times$$

as topological groups. We use the parallel notation $G_{*,v,\infty}$ for $G_{*,v}$ viewed as the group of points of an algebraic group over F_v.

We assume from now on that $\mathbf{B}_p$ is split and fix an isomorphism $\mathrm{G}_{\mathbf{Q}_p} \cong \mathrm{Res}_{F_p/\mathbf{Q}_p}\mathrm{GL}_2$, giving a model of G_* over $\mathbf{Z}_p$. We define involutions

$$g^\iota := g^{\mathrm{T},-1} \quad \text{on } \mathrm{G}(\mathbf{Q}_p), \quad h^\iota := h^{c,-1} \quad \text{on } \mathrm{H}(\mathbf{Q}_p),$$

that induce involutions ι on all our groups. The embedding $\mathrm{H}' \hookrightarrow (\mathrm{G} \times \mathrm{H})'$ is compatible with the involutions.

For $t \in T_{\mathrm{G}_*,p}$, let $\mathrm{U}_t := K_{p,r} t K_{p,r} \in \mathscr{H}_{G_{*,p}}^{K_{p,r}}$ for any $r \geq 1$, and

$$\mathrm{U}_{t,p\infty} := \mathrm{U}_t \otimes t_\infty.$$

When $x \in F_p^\times$, we abuse notation by writing $\mathrm{U}_x = \mathrm{U}_{\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)}$; we also write

$$\mathrm{U}_{p\infty} := \mathrm{U}_{\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right),p\infty}$$

for short.

A.1.3 Ordinary parts of admissible or coadmissible G_*-modules

Let L be a finite extension of $\mathbf{Q}_p$. Let $\Pi = \Pi_p \otimes W$ be a p-adic locally algebraic admissible representation of G_* Let us write

$$\Pi^{N_{0,(r)}} := \Pi^{N_{0,(r)}} \otimes W^N,$$

where $N_{0,r} := K_{p,r}$. Choose $\mathscr{O}_L$-lattices $W^\circ \subset W$, $\Pi_p^{\circ,K} \subset \Pi_p^K$, stable under the Hecke action, and compatibly with the transition maps associated with $K' \subset K$. Then $\Pi^{\circ,N_0} := \Pi_p^{\circ,N_0} \otimes W^{\circ,N} = \varinjlim_r \Pi_p^{\circ,K_{p,r}} \otimes W^{\circ,N}$ is stable under the action of $\mathrm{U}_{p\infty}$. As shown by Hida, the idempotent

$$e^{\mathrm{ord}} := \lim_n \mathrm{U}_{p\infty}^{n!} \colon \Pi^{\circ,N_0} \to \Pi^{\circ,N_0}$$

is then well-defined and its image is denoted by $\Pi^{\circ,\mathrm{ord}}$. The space $\Pi^{\circ,\mathrm{ord}}$ is the maximal split $\mathscr{O}_L$-submodule of Π°,N_0} over which $\mathrm{U}_{p\infty}$ acts invertibly. We also write e^{ord} for $e^{\mathrm{ord}} \otimes 1 \colon \Pi^{N_0} = \Pi^{\circ,N_0} \otimes L \to \Pi^{N_0}$, and we let $\Pi^{\mathrm{ord}} = e^{\mathrm{ord}}\Pi^{N_0}$ be its image. If Π_p and W are irreducible, then Π^{ord} has dimension either 0 or 1; in the latter case we say that Π is *ordinary*. (This notion is independent of the choice of lattices.)

Let $\mathrm{M} = \mathrm{M}_p \otimes W^\vee$ be a p-adic locally algebraic coadmissible right module for G_* over L. By definition of coadmissibility, the system $(\mathrm{M}_{p,K})_{K \subset G_{*,p}}$ is endowed with a compatible system $\mathscr{H}_{\mathrm{G}_*(\mathbf{Z}_p)),K}$-stable lattices $\mathrm{M}_{p,K}^\circ$, so that for some $\mathrm{G}_*(\mathbf{Z}_p)$-stable lattice $W^{\vee,\circ}$, $\mathrm{M}^\circ_{N_0} := \varprojlim \mathrm{M}^\circ_{p,K_{p,r}} \otimes W^{\vee,\circ}_{N_0}$ is stable under $\mathrm{U}_{p\infty}$. Then we can again define $e^{\mathrm{ord}}\colon \mathrm{M}^{(\circ)}_{N_0} \to \mathrm{M}^{(\circ)}_{N_0}$. Its image

$$\mathrm{M}^{(\circ),\mathrm{ord}} := \mathrm{M}^{(\circ)}_{N_0} e^{\mathrm{ord}}$$

is called the *ordinary part* of $\mathrm{M}^{(\circ)}_{N_0}$.

The ordinary parts Π^{ord}, $\mathrm{M}^{\mathrm{ord}}$ retain an action of the operators $\mathrm{U}_{t,p\infty}$.

A.1.4 Special group elements, and further notation

The following notation will be in use throughout this appendix. Let $v|p$ be a place of F. We denote by e_v be the ramification degree of E_v/F_v, and fix a uniformiser $\varpi_v \in F_v$ chosen so that $\prod_{v|p} \varpi_v^{e_v} = p$. Let $\mathrm{Tr}_v = \mathrm{Tr}_{E_v/F_v}$ and $\mathrm{Nm}_v := \mathrm{Nm}_{E_v/F_v}$ be the trace and norm. Fix an isomorphism $\mathscr{O}_{E,v} = \mathscr{O}_{F,v} \times \mathscr{O}_{F,v}$ if v is split. If v is nonsplit, let c be the Galois conjugation of E_v/F_v, and fix an element $\theta_v \in \mathscr{O}_{E,v}$ such that $\mathscr{O}_{E,v} = \mathscr{O}_{F,v}[\theta_v]$ (thus θ_v is a unit if v is inert and a uniformiser if v is ramified). We define a purely imaginary $\mathrm{j}_v \in E_v^\times$ to be

$$\mathrm{j}_v := \begin{cases} (-1_w, 1_{w^c}) & \text{if } E_v = E_w^\times \times E_{w^c}^\times, \\ \theta_v^c - \theta_v & \text{if } E_v \text{ is a field.} \end{cases} \tag{A.1.2}$$

We assume that E_v embeds in $\mathbf{B}_v$ and fix the embedding $E_v \to \mathbf{B}_v$ to be

$$t = (t_w, t_{w^c}) \mapsto \begin{pmatrix} t_w & \\ & t_{w^c} \end{pmatrix} \quad \text{if } E_v = E_w^\times \times E_{w^c}^\times,$$

$$t = a + \theta b \mapsto \begin{pmatrix} a + b\mathrm{Tr}_v\theta_v & b\mathrm{Nm}_v\theta_v \\ -b & a \end{pmatrix} \quad \text{if } E_v \text{ is a field.}$$

For $r \geq 0$, let

$$w_{r,v} := \begin{pmatrix} & 1 \\ -p^r & \end{pmatrix} \in \mathrm{GL}_2(F_v),$$

$$\gamma_{r,v} := \begin{cases} \begin{pmatrix} p^r & 1 \\ & 1 \end{pmatrix} & \text{if } v \text{ splits} \\ \begin{pmatrix} p^r\mathrm{Nm}_v(\theta_v) & \\ & 1 \end{pmatrix} & \text{if } v \text{ is nonsplit} \end{cases} \in (G \times H)'_v$$

and

$$w_r := \prod_{v|p} w_{r,v} \in \mathrm{G}(\mathbf{Q}_p), \quad \gamma_r := \prod_{v|p} \gamma_{r,v} \in (\mathrm{G} \times \mathrm{H})'(\mathbf{Q}_p).$$

A.2 Toric, ordinary, and anti-ordinary parts

Let L be a finite extension of $\mathbf{Q}_p$. We perform some twists.

A.2.1 Ordinary and anti-ordinary parts

Let $w := \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in G_{*,\Delta}$ and let π^w be the representation on the same space as π but with G-action given by $\pi^w(g)v := \pi(w^{-1}gw)v$. Let $N^- := w^{-1}Nw$, and $U^-_{p\infty} := U_{w^{-1}\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)w, p\infty}$.

Let $\pi = \pi_p \otimes W$ be a p-adic admissible locally algebraic representation of G over L. The *anti-ordinary part* of π is the space

$$\pi^{\mathrm{a}} := \pi_p^{\mathrm{a}} \otimes W^{N^-} \subset \pi$$

of 'ordinary' elements with respect to N^- and $U^-_{p\infty}$. Because π^w is isomorphic to π, the spaces π^{a} and π^{ord} have the same dimension.

Let $M = M_p \otimes W$ be a p-adic coadmissible locally algebraic representation of G over L. The *anti-ordinary part* of M is the quotient

$$M^{\mathrm{a}} := M_p^{\mathrm{a}} \otimes W_{N^-}$$

of M that is its 'ordinary' part with respect to N_0^- and and $U^-_{p\infty}$.

Proposition A.2.1 *Let W be an algebraic representation of G_∞.*

1. *Let π be a p-adic locally algebraic admissible representation of G. There is an isomorphism*

$$\begin{aligned} w_{\mathrm{a}}^{\mathrm{ord}} \colon \pi^{\mathrm{ord}} &\to \pi^{\mathrm{a}} \\ f &\mapsto \lim_{r\to\infty} p^{r[F:\mathbf{Q}]} w_{r,p} w_{0,\infty}^{\iota} \mathrm{U}_p^{-r} f, \end{aligned}$$

   where the sequence stabilises as soon as $r \geq 1$ is such that $f_p \in \pi_p^{U_1^1(p^r)}$.
2. *Let M be a p-adic locally algebraic coadmissible representation of G. There is a map*

$$w_{\mathrm{a}}^{\mathrm{ord}} \colon M^{\mathrm{ord}} \to M^{\mathrm{a}}$$

$$m = m_p \otimes m_\infty \mapsto \lim_{r\to\infty} p^{r[F:\mathbf{Q}]}[m(U_p)^{-r} w_{r,p}]_{N_0^-} \otimes [m_\infty w^\iota_{0,\infty}]_{N^-},$$

where before applying w_*, we take arbitrary lifts from N_0-coinvariants to the module M.

Proof That the maps are well-defined is a standard result left to the reader. At least for admissible representations, the map is an isomorphism (equivalently, nonzero) because of Lemma A.3.3 below. □

Let π_v^{ord} (respectively π_v^{a}) denote the preimage of π^{ord} (respectively π^{a}) in π_v, and let W_v be the $G_{v,\infty}$-component of W. The following local components of the above map are similarly well-defined:

$$\begin{aligned} w^{\mathrm{ord}}_{\mathrm{a},v}\colon \pi_v^{\mathrm{ord}} &\to \pi_v^{\mathrm{a}}, & w^{\mathrm{ord}}_{\mathrm{a},v,\infty}\colon W_v^{N_v} &\to W_v^{N_v^-} \\ f_v &\mapsto \lim_{r\to\infty} p^{r[F_v:\mathbf{Q}_p]} w_{r,v} \mathrm{U}_{p,v}^{-r} f_v, & f_{v,\infty} &\mapsto w^\iota_{0,v,\infty} f_{v,\infty}. \end{aligned} \tag{A.2.1}$$

Lemma A.2.2 *Let π be an ordinary representation of G. If $(\,,\,)\colon \pi\otimes\pi^\vee \to L$ is a nondegenerate G-invariant pairing, then the pairing*

$$\begin{aligned} (\,,\,)^{\mathrm{ord}}\colon \pi^{\mathrm{ord}} \otimes \pi^{\vee,\mathrm{ord}} &\to L \\ (f_1, f_2)^{\mathrm{ord}} &:= (w^{\mathrm{ord}}_{\mathrm{a}} f_1, f_2) \end{aligned}$$

is a nondegenerate pairing.

Proof It suffices to see this for a specific pairing $(\,,\,)$: we may take the product of the pairings (A.3.2) below, that are known to be nondegenerate, and any nondegenerate pairing on $W \otimes W^\vee$. Then the result follows from Lemmas A.3.3 and A.4.1 below. □

A.2.2 Ordinary and toric parts

We construct a map from the ordinary part of a representation of $(G \times H)'$ to its toric coinvariants, as well as a dual map in the opposite direction for coadmissibe modules. These map are the key to the interpolation of toric periods.

Suppose that $W_{(v)}$ (respectively $W = \bigotimes_{v|p} W_v$ is an algebraic representation of $(G \times H)_{(v),\infty}$ (respectively $(G \times H)'_\infty$) over L such that, for a field extension L'/L splitting E, $W_{(v),\infty} \otimes_L L' = \bigotimes_{\sigma\colon F_{(v)}\hookrightarrow L} W_\sigma$ with

$$W_\sigma = W_{\sigma,w,k,l} := \mathrm{Sym}^{k_\sigma - 2}\mathrm{Std}\cdot \det^{\frac{w-k_\sigma+2}{2}} \otimes \sigma^{\frac{l_\sigma - w}{2}} (\sigma^c)^{\frac{-l_\sigma - w}{2}} \tag{A.2.2}$$

for some integers $k_\sigma \geq 2$, $|l_\sigma| < k_\sigma$, w of the same parity. (Here we have chosen, for each $\sigma\colon F \hookrightarrow L$, an extension $\sigma\colon E \hookrightarrow L$.) Then we define a constant

$$c(W_\sigma) := \mathrm{j}_v^{-w-k_\sigma+2} \cdot \binom{k_\sigma - 2}{(k_\sigma - 2 - l_\sigma)/2} \cdot \begin{cases} 1 & \text{if } v \text{ splits in } E \\ \theta^{c,(k-2-l)/2}\theta^{(k-2+l)/2} & \text{if } v \text{ does not split in } E, \end{cases}$$

$$c(W_{(v)}) := \prod_{\sigma\colon F_{(v)} \hookrightarrow L} c(W_\sigma). \tag{A.2.3}$$

(Note that $\mathrm{j}_v^{-w-k_\sigma+2} = 1$ if v splits in E, as $w + k_\sigma - 2$ is even.)

Lemma A.2.3 *Recall the congruence subgroups $V'_{v,r}$, $K_{v,r}$ defined in Sect. 2.1.5. For all $r \geq 1$, we have the identity of Hecke operators in the Hecke algebra for $(G \times H)'_v$:*

$$V'_{v,r+1}\left(\sum_{t \in V'_{v,r}/V'_{v,r+1}} t\right) \cdot \gamma_{r+1,v} K_{v,r} = V'_{v,r+1}\gamma_{r,v} \cdot \mathrm{U}_{\varpi_v} K_{v,r}.$$

Proof This is a consequence of the following matrix identity.

Let $v|p$ be a prime of F. For $r \in \mathbf{Z}_{\geq 1}$, $j \in \mathscr{O}_{F,v}$ let $b_{j,v} := \begin{pmatrix} \varpi & j \\ & 1 \end{pmatrix}$. In the split case, let

$$t_{j,r,v} = k_{j,r,v} := \begin{pmatrix} 1 + j\varpi^r & \\ & 1 \end{pmatrix} \in E_v^\times$$

In the nonsplit case, let

$$t_{j,r,v} = 1 + \theta_v \varpi_v^r, \quad k_{j,r,v} = \begin{pmatrix} 1 + j\mathrm{Tr}_v(\theta_v)\varpi_v^r & \mathrm{Tr}_v(\theta_v) - j\varpi_v^r \\ -j\mathrm{N}_v(\theta_v)\varpi_v^{2r} & 1 + j^2\mathrm{N}_v(\theta_v)\varpi_v^r \end{pmatrix}.$$

Then

$$t_{j,r,v}\gamma_{r+1,v} = \gamma_{r,v} b_{j,v} k_{j,r,v}$$

in $\mathrm{GL}_2(F_v)$. □

Proposition A.2.4 *Let W be an algebraic representation of $(G \times H)'_\infty$.*

1. *Let* $\Pi = \Pi_p \otimes W$ *be a p-adic locally algebraic admissible representation of* $(G \times H)'$. *There is a map*

$$\begin{aligned}\gamma_{H'}^{\text{ord}} \colon \Pi^{\text{ord}} &\to \Pi_{H'}\\ f &\mapsto \lim_r \; H'[p^{r[F:\mathbf{Q}]} \cdot c(W)^{-1} \cdot \gamma_{r,p\infty} \mathrm{U}_{p\infty}^{-r} f],\end{aligned} \tag{A.2.4}$$

where $H'[-] \colon \Pi \to \Pi_{H'}$ *is the natural projection.*
The sequence in the right hand side of (A.2.4) *stabilises as soon as* $f_p \in \Pi^{K_{p,r}}$, *where* $K_{p,r} \subset (G \times H)'_p$ *is defined at the end of Sect.* 2.1.
2. *Let* $\mathrm{M} := \mathrm{M}_p \otimes W^\vee$ *be a p-adic locally algebraic coadmissible representation of* $(G \times H)'$. *There is a map*

$$\begin{aligned}\gamma_{H'}^{\text{ord}} \colon \mathrm{M}^{H'} &\to \mathrm{M}^{\text{ord}}\\ m &\mapsto \lim_r \; [p^{r[F:\mathbf{Q}]} \cdot c(W)^{-1} \cdot m\gamma_{r,p\infty}]N_{0,r} e^{\text{ord}} \mathrm{U}_{p\infty}^{-r},\end{aligned}$$

where $[-]N_{0,r} \colon \mathrm{M} \to \mathrm{M}_{N_0}$ *is the natural projection.*

The constant $c(W)$ is justified by Lemma A.4.2 below.

Proof For part 1, let $f \in \Pi_p^{K_{p,r}}$. Then it follows from Lemma A.2.3 that, denoting by $[f_r]_{H'}$ the sequence in the right hand side of (A.2.4), we have

$$\frac{1}{p^{[F:\mathbf{Q}]}} \sum_{t \in V'_{p,r}/V'_{p,r+1}} \Pi(t) f_{r+1} = f_r,$$

hence $[f_{r+1} - f_r]_{H'} = 0$ and the sequence stabilises.

For part 2, Lemma A.2.3 similarly implies (the boundedness and) the convergence of the sequence in $\varprojlim_r \mathrm{M}_{N_{0,r}}^{\text{ord}}$. □

Let Π_v^{ord} denote the preimage of Π^{ord} in Π_v, and let W_v be the $(G \times H)'_{v,\infty}$-component of W. The following local components of the above maps are similarly well-defined:

$$\begin{aligned}\gamma_{H',v}^{\text{ord}} \colon \Pi_v^{\text{ord}} &\to \Pi_{v,H'_v}\\ \gamma_{H',v,\infty}^{\text{ord}} \colon W_v^N &\to W_{v,H'_v}\\ f_v &\mapsto \lim [p^{r[F_v:\mathbf{Q}_p]} \gamma_{r,v} \mathrm{U}_{\varpi_v}^{-r} f_v]_{H'_v},\\ f_{v,\infty} &\mapsto c(W_v)^{-1} \cdot \gamma_{0,v,\infty}^{\iota} f_{v,\infty}.\end{aligned} \tag{A.2.5}$$

A.2.3 Exceptional representations and vanishing of P^{ord}

We show that $\gamma_{H'}^{\mathrm{ord}}$ acts by zero precisely on those representations that are exceptional.

Lemma A.2.5 *Let $\Pi = \pi \otimes \chi$ be an ordinary, distinguished, irreducible representation of $(G \times H)'$. The following are equivalent:*

1. *Π is exceptional;*
2. *$e_v(V_{(\pi,\chi)}) = 0$;*
3. *there exists $P \in \Pi^{*,H'} - \{0\}$ such that $P^{\mathrm{ord}} := P\gamma_{H'}^{\mathrm{ord}} = 0$;*
4. *for all $P \in \Pi^{*,H'}$, we have $P^{\mathrm{ord}} = 0$;*

Proof The equivalence of 1. and 2. is a reminder from Lemma 6.4.6. The equivalence of 3. and 4. is a consequence of multiplicity-one. Consider 3. Let $P \in \Pi^{*,H'}$. Identify $\Pi^\vee = \Pi^\iota$ (the representation on the same space as Π, with group action twisted by the involution ι). Then the identity map on spaces yields isomorphisms $\Pi^{*,H'} \cong \Pi^{\vee,*,H'}$ and $\Pi^{\mathrm{ord},*} \cong \Pi^{\vee,\mathrm{ord},*}$, and it follows from the explicit description of $\gamma_{H'}^{\mathrm{ord}}$ that if $P^\vee$ denotes the image of P, then the image of $P^\vee$ is $P^{\vee,\mathrm{ord}}$. Hence, P^{ord} is zero if and only if so is $P^{\vee,\mathrm{ord}}$, if and only if so is $P \otimes P^\vee \circ \gamma_{H'}^{\mathrm{ord}} \otimes \gamma_{H'}^{\mathrm{ord}}$. Now by the theory recalled in Sect. 1.2.6 , $P \otimes P^\vee$ is necessarily a multiple of the explicit functional $Q_{dt,(,)}$ defined there. Therefore it suffices to show that $Q_{dt,(,)}$ *vanishes on the line $\gamma_{H'}^{\mathrm{ord}}\Pi^{\mathrm{ord}} \otimes \gamma_{H'}^{\mathrm{ord}}\Pi^{\vee,\mathrm{ord}}$ if and only if* $e_v(V_{(\pi,\chi)}) = 0$. This follows from the explicit computations of Propositions A.3.4 and A.4.3 below, cf. also Proposition 4.3.4. ◻

A.3 Pairings at *p*

The goal of this subsection is to relate the p-components of the toric terms Q and their ordinary variants Q^{ord}, as defined in Sects. 4.2–4.3.

Let $v|p$ be a place of F.

A.3.1 Integrals and gamma factors

If π (respectively χ) is an irreducible representation of G_v over L, we denote by V_π (respectively V_χ) the associated 2- (respectively 1-) dimensional Frobenius-semisimple representation of WD_{F_v} (respectively of $\mathrm{WD}_{E_v} := \prod_{w|v} \mathrm{WD}_{E_w}$; we choose the "Hecke" normalisation, so that $\det V_\pi$ is the cyclotomic character if π is self-dual. If $\Pi = \pi \otimes \chi$ is an irreducible representation of $(G \times H)'_v$, we denote by $V_\Pi = V_{\pi|\mathrm{WD}_{E_v}} \otimes V_\chi$ the associated 2-dimensional representation of WD_{E_v}. If E_* is F or E, $w|p$ is a prime of E_* and V is any representation of $\mathrm{WD}_{E_{*,v}}$ as above, we let $V_w := V_{|\mathrm{WD}_{E_{*,w}}}$.

If $\psi\colon F_v \to \mathbf{C}^\times$ is a nontrivial character, we denote by $d_\psi y$ the selfdual Haar measure on F_v and $d^\times_\psi y := d_\psi y/|y|$. The *level* of ψ is the largest n such that $\psi_{|\varpi^{-n}\mathscr{O}_{F,v}} = 1$. We recall that if ψ has level 0, then $\mathrm{vol}(\mathscr{O}_{F,v}, d_\psi y) = 1$.

Recall the Deligne–Langlands γ-factor of (1.4.3).

Lemma A.3.1 [35, Lemma A.1.1]. *Let $\mu\colon F_v^\times \to \mathbf{C}^\times$ and $\psi\colon F_v \to \mathbf{C}^\times$ be characters, with $\psi_v \neq 1$. Let $d^\times y$ be a Haar measure on $F_v^\times$. Then*

$$\int_{F_v^\times} \mu(y)\psi(y)d^\times y = \frac{d^\times y}{d^\times_\psi y}\cdot \mu(-1)\cdot \gamma(\mu,\psi)^{-1}.$$

A.3.2 Local pairing

The following isolates those representations that can be components of an ordinary representation.

Definition A.3.2 A *refined* representation (π, α) of G_v over a field L consists of a smooth irreducible infinite-dimensional representation π and a character $\alpha\colon F_v \to L^\times$, such that π embeds into the un-normalised induction $\mathrm{Ind}(|\,|\alpha, |\,|^{-1}\omega\alpha^{-1}))$ for some other character $\omega\colon F_v^\times \to L^\times$.[31] Sometimes we abusively simply write π instead of (π, α). A refined representation $\Pi = \pi\otimes\chi$ $(G\times H)'_v$ is the product of a refined representation $\pi = (\pi, \alpha)$ of G and a character χ of H, such that $\omega\chi_{|F_v^\times} = \mathbf{1}$.

If (π, α) is a refined representation of G_v, we let $\pi^{\mathrm{ord}} \subset \pi^{N_0}$ be the unique line on which the operator U_t acts by $\alpha(t)$. If $\Pi = \pi\otimes\chi$ is a refined representation of $(G\times H)'_v$, we let $\Pi^{\mathrm{ord}} := \pi^{\mathrm{ord}}\otimes\chi$. The associated Weil–Deligne representation V_π is reducible, and we have a unique filtration

$$0 \to V_\pi^+ \to V_\pi \to V_\pi^- \to 0$$

such that WD_{F_v} acts on V_π^+ through the character $\alpha|\cdot|$.

Let π be a refined representation of G_v over L, and let $(\ ,\ )_\pi : \pi\otimes\pi^\vee \to L$ be a G-invariant pairing. Then we define

$$\begin{aligned}(\ ,\ )_\pi^{\mathrm{ord}}\colon \pi^{\mathrm{ord}}\otimes(\pi^\vee)^{\mathrm{ord}} &\to L\\ f\otimes f^\vee &\mapsto (w_{\mathrm{a}}^{\mathrm{ord}} f, f^\vee),\end{aligned}$$

where $w_{\mathrm{a}}^{\mathrm{ord}}$ is the operator denoted $w_{\mathrm{a},v}^{\mathrm{ord}}$ in (A.2.1). If Π is a refined representation of $(G\times H)'_v$ over L and $(\ ,\ ) = (\ ,\ )_\pi(\ ,\ )_\chi : \Pi\otimes\Pi^\vee \to L$ is a pairing, we define $(\ ,\ )^{\mathrm{ord}} := (\ ,\ )_\pi^{\mathrm{ord}}(\ ,\ )_\chi$, a pairing on $\Pi^{\mathrm{ord}}\otimes\Pi^{\vee,\mathrm{ord}}$.

[31] Note that π admits a refinement if and only if it is neither supercuspidal nor 1-dimensional.

Lemma A.3.3 *Let (π, α) be a refined representation of G_p over $\mathbf{C}$, with central character ω as in Definition A.3.2. Let $\alpha^\vee = \alpha\omega^{-1}$. Let*

$$\mathrm{ad}(V_\pi)^{++}(1) = \mathrm{Hom}\,(V_\pi^-, V_\pi^+)(1).$$

Fix a character $\psi\colon F_v \to \mathbf{C}^\times$ of level 0, and Kirillov models of $\iota\pi_v$, $\pi_v^\vee$ with respect to ψ_v, $-\psi_v$. Let

$$f_v^{(\vee)}(y) := \mathbf{1}_{\mathscr{O}_{F,v}}(y)\alpha_v^{(\vee)}|\ |_v(y) \in \pi_v^{\mathrm{ord}}. \tag{A.3.1}$$

Suppose that $(\ ,\ )_{\pi,v}$ is, in the Kirillov models, the pairing

$$(f, f^\vee)_\pi := \int_{F^\times} f(y) f^\vee(y) d_\psi^\times y. \tag{A.3.2}$$

Then

$$(f, f^\vee)_{\pi,v}^{\mathrm{ord}} = \omega_v(-1)\cdot\gamma(\mathrm{ad}(V_\pi)^{++}(1), \psi)^{-1}.$$

Proof We omit all remaining subscripts v and argue similarly to [60, Lemma 2.8]. The inner product $(f, f^\vee)_\pi^{\mathrm{ord}}$ is the value at $s=0$ of

$$\begin{aligned}&\alpha|\ |(\varpi)^{-r} Z(s+1/2, w_r f, \alpha^\vee|\ |), \quad Z(s+1/2, w_r f, \alpha^\vee|\ |)\\ &:= \int_{F^\times} w_r f(y)\alpha^\vee|\ |(y)|y|^s d_\psi^\times y.\end{aligned}$$

By the functional equation for GL_2, this equals

$$\begin{aligned}&\omega(-1)\cdot\gamma(s+1/2, \pi\otimes\alpha^\vee|\ |, \psi)^{-1}\cdot\int_{p^{-r}\mathscr{O}_F-\{0\}} \alpha\alpha^{\vee,-1}\omega^{-1}|\ |^{-s}(y) d_\psi^\times y\\ &\quad= \omega(-1)\cdot\gamma(s, \alpha\alpha^\vee|\ |^2, \psi)^{-1}\cdot\gamma(s, |\ |, \psi)^{-1}\cdot\zeta_F(1)^{-1}\zeta_F(-s),\end{aligned}$$

using the fact that the domain of integration can be replaced with $F^\times$, the additivity of gamma factors, and the relation $\alpha^\vee = \alpha\omega^{-1}$. Evaluating at $s=0$ we find $\gamma(\mathrm{ad}(V_\pi)^{++}(1), \psi)^{-1}$ as desired. □

A.3.3 Local toric period

We compute the value of the local toric periods on the lines of interest to us. Let $\Pi = \pi\otimes\chi$ be a refined representation of $(G\times H)'_v$. Let dt be a measure on H'_v, and set as in (4.3.1)

$$\mathrm{vol}^\circ(H'_v, dt) := \frac{\mathrm{vol}(\mathscr{O}_{E,v}^\times/\mathscr{O}_{F,v}^\times, dt)}{e_v L(1, \eta_v)^{-1}}.$$

Then for all $f_1, f_3 \in \Pi^{\mathrm{ord}}$, $f_2, f_4 \in \Pi^{\vee,\mathrm{ord}}$ with $f_3, f_4 \neq 0$, we define

$$Q_{dt}^{\mathrm{ord}}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right) := \mu^+(\mathrm{j}_v) \cdot \mathrm{vol}^\circ(H'_v, dt) \cdot \frac{f_1 \otimes f_2}{f_3 \otimes f_4}, \tag{A.3.3}$$

where $\mathrm{j}_v =$(A.1.2) and $\mu^+ = \chi_v \cdot \alpha|\cdot| \circ N_{E_v/F_v}$ is the character giving the action of $E_v^\times$ on $V^+ := V_\pi^+ \otimes \chi$.

Proposition A.3.4 *Let $\Pi = \pi \otimes \chi$ be a refined representation of $(G \times H)'_v$ over L, with associated Weil–Deligne representation $V = V_{\pi|\mathrm{WD}_{E_v}} \otimes \chi$. Let $\gamma_{H'}^{\mathrm{ord}} = \gamma_{H',v}^{\mathrm{ord}}$ be as defined in* (A.2.5)*. Then*

$$Q_{dt}\left(\frac{\gamma_{H'}^{\mathrm{ord}} f_1 \otimes \gamma_{H'}^{\mathrm{ord}} f_2}{w_{\mathrm{a}}^{\mathrm{ord}} f_3 \otimes f_4}\right) = e_v(V_{(\pi,\chi)}) \cdot Q_{dt}^{\mathrm{ord}}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right).$$

Here

$$e_v(V_{(\pi,\chi)}) = \mathscr{L}(V_{(\pi,\chi)}, 0)^{-1} \cdot \iota^{-1}\Bigg(|d|_v^{-1/2}\gamma(\mathrm{ad}(\iota V_\pi^{++})(1), \psi_v) \cdot \prod_{w|v} \gamma(\iota V^+_{|\mathrm{WD}_{E_w}}, \psi_{E_w})^{-1}\Bigg)$$

is defined independently of any choice of an embedding $\iota\colon L \hookrightarrow \mathbf{C}$ and non-trivial character $\psi\colon F_v \to \mathbf{C}^\times$.

Proof Identify $\chi^{\pm 1}$ with L and assume that $f_i = f_{i,\pi} f_{i,\chi}$ with $f_{i,\chi}$ identified with 1. Fix $\iota\colon L \hookrightarrow \mathbf{C}$ (omitted from the notation) and $\mathbf{1} \neq \psi\colon F_v \to \mathbf{C}^\times$. Identify $\pi, \pi^\vee$ with Kirillov models with respect to $\psi, -\psi$. Let $(\ ,\ ) = (\ ,\ )_\pi \cdot (\ )_\chi$ be the invariant pairing on $\Pi \otimes \Pi^\vee$ such that $(\ ,\ )_\pi =$ (4.2.3) and $(1\ , 1\ )_\chi = 1$. Assume, after a harmless extension of scalars, that $dt = |D_v|^{-1/2} d^\times_{\psi_E} z / d^\times_\psi y$, which gives $\mathrm{vol}^\circ(H', dt) = 1$. Let $f_1 = f_3 = f_\pi$, $f_2 = f_4 = f_\pi^\vee$ with $f_\pi^{(\vee)}$ as in (A.3.1).

In view of the definitions (4.2.2), (A.3.3) and of Lemma A.3.3, it suffices to show that

$$\begin{aligned} Q^\sharp(\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^\vee) &:= \int_{H'_v} (\pi(t)\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^\vee)\chi(t)\, dt \\ &= \omega(-1) \cdot \mu^+(\mathrm{j}_v) \cdot \prod_{w|v} \gamma(V^+_{|\mathrm{WD}_{E_w}}, \psi_{E_w})^{-1}. \end{aligned}$$

We denote by α the refinement of π, and we fix $r \geq 1$ to be larger than the valuations of the conductors of π and of the norm of the conductor of χ.

Split case. Suppose first that E_v/F_v is split and identify $E_v^\times = F_v^\times \times F_v^\times$ as usual. Then as in [32, Lemma 10.12] we find

$$\begin{aligned} Q^\sharp(\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^\vee) &= \prod_{w|v} \int_{E_w^\times} \alpha\chi_w|\ |_w(y_w)\psi_w(y_w) d^\times y_w \\ &\quad \cdot \int_{E_{w^c}^\times} \alpha\chi_{w^c}|\ |_{w^c}(y_{w^c})\psi_{w^c}(-y_{w^c}) d^\times y_{w^c} \\ &= \omega_v(-1)\cdot \mu^+(\mathrm{j}_v)\cdot \gamma(V_v^+, \psi_v)^{-1}, \end{aligned}$$

where we have used Lemma A.3.1.

Nonsplit case. Now suppose that $E_v = E_w$ is a field and drop all subscripts v, w. We abbreviate $\mathrm{T} := \mathrm{Tr}(\theta)$, $\mathrm{N} := \mathrm{Nm}(\theta)$.

We have

$$Q^\sharp(\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^\vee) = \int_{H'} \alpha\alpha^\vee|\ |^2(\varpi)^{-r}\cdot (\pi(\gamma_r^{-1} t\gamma_r) f_\pi, f_\pi^\vee)\chi(t)\, dt. \tag{A.3.4}$$

There is a decomposition

$$\begin{aligned} &H' = H_1' \sqcup H_2', \quad H_1' = \{1 + b\theta \mid b \in \mathscr{O}_F\}, \\ &H_2' = \{a\mathrm{N} + \theta \mid a \in \mathrm{N}^{-1}\varpi\mathscr{O}_F\}, \end{aligned}$$

that is an isometry when H_1', H_2' are endowed with the measures $d_\psi b$, $d_\psi a$.

Let $r' := r + e - 1$ and let us redefine, for the purposes of this proof, $w_{r'} := \begin{pmatrix} & 1 \\ -\mathrm{N}^{-1}\varpi^{-r} & \end{pmatrix}$. Let $\sim_{r'}$ denote the relation in $\mathrm{GL}(2, F)$ of equality up to right multiplication by an element of $U_1^1(\varpi^{r'})$, and let $t^{(r)} := \gamma_r^{-1} t\gamma_r$.

Contribution from H_1'. For $t = 1 + b\theta \in H_1'$, we have

$$t^{(r)} = \begin{pmatrix} 1 + b\mathrm{T} & b\varpi^{-r} \\ -b\mathrm{N}\varpi^r & 1 \end{pmatrix} \sim_{r'} \begin{pmatrix} 1 + b\mathrm{T} + b^2\mathrm{N} & b\varpi^{-r} \\ & 1 \end{pmatrix}.$$

Hence the integral over H_1' equals

$$\begin{aligned} &\omega^{-1}\alpha^2|\ |^2(\varpi)^{-r} \int_{\mathscr{O}_F}\int_{\mathscr{O}_F - \{0\}} \psi(by\varpi^{-r})\alpha|\ |(\mathrm{Nm}(1 + b\theta)y)\alpha\omega^{-1}|\ |(y) \\ &\quad \cdot \chi(1 + b\theta)\, d_\psi^\times y\, d_\psi b \\ &= \int_{\mathscr{O}_F}\int_{\varpi^{-r}\mathscr{O}_F - \{0\}} \chi\cdot\alpha|\ |\circ \mathrm{Nm}((1 + b\theta)y)\cdot \psi(by)\, d_\psi^\times y\, d_\psi b. \end{aligned}$$

We show that the domain of integration in y can be harmlessly extended to $F^\times$, i.e. that

$$\int_{\mathscr{O}_F}\int_{v(y)\leq -r-1}\mu^+((1+b\theta)y)\psi(by)\,d^\times_\psi y\,db$$

vanishes. Consider first the contribution from $v(b)\geq r$. On this domain, $\mu^+(1+b\theta)=1$ and integration in db yields $\int_{\varpi^r\mathscr{O}_F}\psi(by)\,db=\mathbf{1}_{\varpi^{-r}\mathscr{O}_F}(y)$, that vanishes on $v(y)\leq -r-1$. Consider now the contribution from $v(b)\leq r-1$

$$\int_{0\leq v(b)\leq r-1}\mu^+(1+b\theta)\int_{v(y)\leq -r-1}\mu^+(y)\psi(by)\,d^\times_\psi y db. \quad \text{(A.3.5)}$$

Let n be the conductor of $\mu^+_{|F^\times}$. Then (A.3.5) vanishes if $n=0$; otherwise only the annulus $v(y)=-n-1$ contributes, and after a change of variable $y'=by$ we obtain

$$\varepsilon(\mu^+_{|F^\times},\psi)^{-1}\cdot\int_{r-n\leq v(b)\leq r-1}\mu^+(1+b\theta)\mu^+(b)^{-1}db.$$

On our domain $\mu^+(1+b\theta)=1$, and $\int\mu^+(b)^{-1}=0$ as $\mu^+_{|F^\times}$ is ramified.

We conclude that the contribution from H_1' is

$$\int_{\mathscr{O}_F}\int_{F^\times}\mu^+((1+b\theta)y)\psi(by)\,d^\times_\psi y\,db\int_{H_1'}\int_{F^\times}\mu^+(ty)$$
$$\cdot\,\psi_E(ty/(\theta-\theta^c))\,d^\times_\psi y\,dt.$$

Contribution from H_2'. For $t=a\mathrm{N}+\theta\in H_2'$, we have

$$t^{(r)}=\begin{pmatrix}a\mathrm{N}+\mathrm{T} & \varpi^{-r}\\ -\mathrm{N}\varpi^r & a\mathrm{N}\end{pmatrix}=w_r'\begin{pmatrix}1 & -a\varpi^{-r}\\ a\mathrm{N}+\mathrm{T} & \varpi^{-r}\end{pmatrix}$$
$$\sim_r w_{r'}\begin{pmatrix}1+a\mathrm{T}+a^2\mathrm{N} & -a\varpi^{-r}\\ & \varpi^{-r}\end{pmatrix}.$$

Then the integral over H_2' is

$$\omega^{-1}\alpha^2|\ |^2(\varpi)^{-r}\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\left(\pi\left(\left(\begin{pmatrix}\mathrm{Nm}(1+a\theta) & -a\varpi^{-r}\\ & \varpi^{-r}\end{pmatrix}\right)\right)f,\pi^\vee(w_{r'}^{-1}f^\vee)\right)$$
$$\cdot\,\chi(a\mathrm{N}+\theta)\,d_\psi a$$

$$
\begin{aligned}
&= \omega^{-1}\alpha^2|\ |^2(\varpi)^{-r}\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\int_{\mathscr{O}_F-\{0\}}\omega(\varpi)^{-r}\\
&\quad\cdot\psi(-ay)\alpha|\ |(y\varpi^r\mathrm{Nm}(1+a\theta))\\
&\quad\cdot\pi^\vee(w_{r'}^{-1})f^\vee(y)\cdot\chi(a\mathrm{N}+\theta)\,d_\psi^\times y\ d_\psi a\\
&= \alpha|\ |(\varpi)^{-r}\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\int_{F^\times}\alpha|\ |(y\mathrm{Nm}(1+a\theta))\cdot\pi^\vee(w_{r'}^{-1})f^\vee(y)\\
&\quad\cdot\chi(a\mathrm{N}+\theta)\,d_\psi^\times y\,d_\psi a\\
&= \alpha|\ |(\varpi^{-r}\mathrm{N}^{-1})\cdot Z(1/2,\pi^\vee(w_{r'}^{-1})f^\vee,\alpha|\ |)\\
&\quad\cdot\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\alpha|\ |(\mathrm{Nm}(a\mathrm{N}+\theta))\cdot\chi(a\mathrm{N}+\theta)\,d_\psi a,
\end{aligned}
$$

where we have observed that $w_{r'}f^\vee$ vanishes outside $\mathscr{O}_F$, and that $\psi(-ay)=1$ for $y\in\mathscr{O}_F$. Applying first the same argument as in the proof of Lemma A.3.3, then Lemma A.3.1, this equals

$$
\begin{aligned}
&\gamma(\mathrm{ad}(V_\pi)^{++}(1),-\psi)^{-1}\cdot\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\alpha|\ |\circ\mathrm{Nm}\cdot\chi(a\mathrm{N}+\theta)d_\psi a\\
&\quad=\int_{F^\times}\mu^+(y)\psi(y)\,d_\psi^\times y\cdot\int_{\mathrm{N}^{-1}\varpi\mathscr{O}_F}\mu^+(a\mathrm{N}+\theta)d_\psi a\\
&\quad=\int_{H_2'}\int_{F^\times}\mu^+(ty)\psi_E(ty/(\theta-\theta^c))\,d_\psi^\times y.
\end{aligned}
$$

Conclusion. Summing up the two contributions to (A.3.4) yields

$$
\mu^+(\theta^c-\theta)\cdot\int_{H'}\int_{F^\times}\mu^+(u)\psi_E(u)\,d^\times u=\omega(-1)\cdot\mu^+(\mathrm{j})\cdot\gamma(\mu^+,\psi_E)^{-1},
$$

as desired. □

A.4 Pairings at infinity

Fix a place $v|p$ of F.

A.4.1 Models for algebraic representations and pairings

Suppose that W is the representation (A.2.2) of $(G\times H)'_{v,\infty}$ over $L \overset{\sigma}{\hookleftarrow} E$. We identify W with the space of homogeneous polynomials $p(x,y)$ of degree $k-2$ in $L[x,y]$, where x and y are considered as the components of a column (respectively row) vector if W is viewed as a right (respectively left) representation. In those two cases, the action is respectively:

$$p.(g,h)(x,y) = \det(g)^{\frac{w-k+2}{2}}\sigma(h)^{\frac{l-w}{2}}\sigma^c(h)^{\frac{-l-w}{2}}\cdot p(g(x,y)^{\mathrm{T}})$$
$$(g,h).p(x,y) = \det(g)^{\frac{w-k+2}{2}}\sigma(h)^{\frac{l-w}{2}}\sigma^c(h)^{\frac{-l-w}{2}}\cdot p((x,y)g). \quad \text{(A.4.1)}$$

In either case, we fix the invariant pairing

$$(x^{k-2-a}y^a, x^{a'}y^{k-2-a'}) = (-1)^a\binom{k-2}{a}^{-1}\delta_{a,a'}. \quad \text{(A.4.2)}$$

Lemma A.4.1 *Let $W=$(A.2.2), viewed as a left representation of $G_{v,\infty}$ only. Let $w_{\mathrm{a}}^{\mathrm{ord}}\colon W^N\to W_N$ be the map denoted by $w_{\mathrm{a},v,\infty}^{\mathrm{ord}}$ of (A.2.1). Fix the models and pairing described above. Then W^N is spanned by x^{k-2} and W_N is spanned by the image of y^{k-2}, and*

$$(w_{\mathrm{a}}^{\mathrm{ord}}(x^{k-2}), x^{k-2}) = 1.$$

A.4.2 The map $\gamma_{H'}^{\mathrm{ord}}$ is unitary on algebraic representations

We start with a lemma completing the proof of Proposition 6.3.2.

Suppose that $M_p = M_{p,0}\otimes W_p$ is a decomposition of a locally algebraic coadmissible right $(G\times H)'_p$-representation over L, into the product of a smooth and an irreducible algebraic representation, respectively. Let $W_\infty^\vee$ be the dual representation to W_p, viewed as a right representation of $(G\times H)'_\infty$. Assume that L is a p-adic field and that the $(G\times H)'$-module $M_p\otimes W_\infty^\vee$ is p-adic coadmissible. Then the operator $\gamma_{H'}^{\mathrm{ord}}$ on it (whose definition of Proposition A.2.4 extends *verbatim* to the case where M_p is only locally algebraic) decomposes as

$$\gamma_{H'}^{\mathrm{ord}} = \lim_{r\to\infty}(p^{r[F:\mathbf{Q}]}\cdot\gamma_{r,p}\mathrm{U}_p^{-r})\otimes\gamma_{r,p}\mathrm{U}_p^{-r}\otimes c(W)^{-1}\gamma_{0,\infty}^{\iota}.$$

according to the decomposition $M_p\otimes W_\infty^\vee = M_{p,0}\otimes W_p\otimes W_\infty^\vee$

Lemma A.4.2 *In relation to the situation just described, the operator*

$${}^{\mathrm{alg}}\gamma_{H'}^{\mathrm{ord}} := \lim_{r\to\infty}\gamma_{r,p}\mathrm{U}_p^{-r}\otimes c(W)^{-1}\gamma_{0,\infty}^{\iota}\colon W^{H'}\otimes W^{\vee,H'}\to W^N\otimes W_N$$

is unitary. That is, for any invariant pairing (,) on $W\otimes W^\vee$ and $\xi\in W^{H'}$, $\xi^\vee\in W^{\vee,H'}$, the images of $\xi\otimes\xi^\vee$ and ${}^{\mathrm{alg}}\gamma_{H'}^{\mathrm{ord}}(\xi\otimes\xi^\vee)$ under the pairings induced by (,) coincide.

Proof We may fix a place $v|p$, and consider the factor representations $W_v \otimes W^\vee_{v,\infty}$ of $(G\times H)'_v \times (G\times H)'_{v,\infty}$. After extension of scalars, we may decompose $W_v = \bigotimes_{\sigma\colon F\to \overline{Q}_p} W_v^\sigma$ where each W_v^σ is one of the representations (A.2.2) for suitable integers w, k, l. Thus we are reduced to proving the unitarity of the relevant component of ${}^{\mathrm{alg}}\gamma_{H'}^{\mathrm{ord}}$ on the representation $W_v^\sigma \otimes W_{v,\infty}^{\vee,\sigma}$. We omit all subscripts.

Split case. Suppose first that v splits in E. Then $W^{H'} = Lx^{(k-2-l)/2}y^{(k-2+l)/2}$, and if

$$\xi := x^{(k-2-l)/2}y^{(k-2+l)/2}$$

then

$$\xi^\vee := (-1)^{(k-2+l)/2}\binom{k-2}{(k-2-l)/2} x^{(k-2+l)/2}y^{(k-2-l)/2}$$

satisfies $(\xi,\xi^\vee)=1$. We have

$$\xi\gamma^{\mathrm{ord}}_{H',p} := \lim_{r\to\infty}\xi\gamma_{r,p}\mathrm{U}_p^{-r} = y^{k-2},$$

and

$$\begin{aligned}\xi^\vee\gamma^{\mathrm{ord}}_{H',\infty} &= (-1)^{(k-2+l)/2}c(W)^{-1}\binom{k-2}{(k-2-l)/2}x^{(k-2+l)/2}\\ &\quad\cdot(-x+y)^{(k-2-l)/2}\end{aligned}$$

projects into $W_N^\vee \stackrel{\cong}{\leftarrow} Lx^{k-2}$ to

$$\xi^\vee\gamma^{\mathrm{ord}}_{H',\infty} = (-1)^{k-2}c(W)^{-1}\binom{k-2}{(k-2-l)/2}x^{k-2}.$$

Hence

$$(\xi\gamma^{\mathrm{ord}}_{H',p},\xi^\vee\gamma^{\mathrm{ord}}_{H',\infty}) = c(W)^{-1}\binom{k-2}{(k-2-l)/2} = 1.$$

Nonsplit case. Suppose now that v does not split in E. Let $z := x+\theta^c y$, $\overline{z} := x+\theta y$. Then $W^{H'} = Lz^{(k-2-l)/2}\overline{z}^{(k-2+l)/2}$, and if

$$\begin{aligned}\xi &:= z^{(k-2-l)/2}\overline{z}^{(k-2+l)/2}\\ &= x^{(k-2-l)/2}y^{(k-2+l)/2}\cdot\begin{pmatrix}1 & \theta^c\\ 1 & \theta\end{pmatrix}(-\mathrm{j})^{(w+k-2)/2}\in W^{H'}\end{aligned}$$

then

$$\xi^\vee := (-1)^{(k-2+l)/2}\binom{k-2}{(k-2-l)/2}x^{(k-2+l)/2}y^{(k-2-l)/2} \cdot \begin{pmatrix} 1 & \theta^c \\ 1 & \theta \end{pmatrix}(-\mathrm{j})^{(-w-k+2)/2} \in W^{\vee,H'}$$

satisfies $(\xi, \xi^\vee) = 1$. We have

$$\xi\gamma^{\mathrm{ord}}_{H',p} = \mathrm{N}^{(w-k+2)/2}\theta^{c,(k-2-l)/2}\theta^{(k-2+l)/2}y^{k-2},$$

and

$$\xi^\vee\gamma^{\mathrm{ord}}_{H',\infty} = (-1)^{(k-2-l)/2}(-\mathrm{j})^{-(w+k-2)/2}c(W)^{-1} \cdot \binom{k-2}{(k-2-l)/2}x^{(k-2+l)/2}y^{(k-2-l)/2}\begin{pmatrix} 1 & \mathrm{N}^{-1}\theta^c \\ 1 & \mathrm{N}^{-1}\theta \end{pmatrix}$$

projects into $W_N^\vee \overset{\cong}{\leftarrow} Lx^{k-2}$ to

$$\xi^\vee\gamma^{\mathrm{ord}}_{H',\infty} = (-1)^{(k-2-l)/2}(-\mathrm{j})^{-w-k+2}c(W)^{-1} \cdot \binom{k-2}{(k-2-l)/2}\mathrm{N}^{(w+k-2)/2}x^{k-2}.$$

Then

$$(\xi\gamma^{\mathrm{ord}}_{H',p}, \xi^\vee\gamma^{\mathrm{ord}}_{H',\infty}) = (-\mathrm{j})^{-w-k+2}\theta^{c,(k-2-l)/2}\theta^{(k-2+l)/2} \cdot \binom{k-2}{(k-2-l)/2}c(W)^{-1} = 1.$$

□

A.4.3 Algebraic toric period

Let $W = W_G \otimes W_H$ be an algebraic representation of $(G\times H)'_{v,\infty}$ over L. For any $\iota\colon L \hookrightarrow \mathbf{C}$, let ιV (respectively ιV_G) be the Hodge structure associated with W (respectively W_G), and let[32]

$$\mathscr{L}(V_{(W_G,W_H)}, 0) := \iota^{-1}\left(\frac{\pi^{-[F_v:\mathbf{Q}_p]}L(\iota V, 0)}{L(\mathrm{ad}(\iota V_{\mathrm{G}),\infty}), 1)}\right).$$

[32] To compare with (1.2.7), we have $\zeta_{\mathbf{R}}(2)/L(1, \eta_{\mathbf{C}/\mathbf{R}}) = 1$.

Let dt be a 'measure' on $H'_{v,\infty}$, by which we simply mean a value $\mathrm{vol}(H'_{v,\infty}, dt)$ similarly to Sect. 1.2.6, and set as in (4.3.1)

$$\mathrm{vol}^\circ(H'_v, dt) := 2^{-[F_v:\mathbf{Q}]}\mathrm{vol}(H'_{v,\infty}, dt).$$

Let $(\ ,\ ) = (\ ,\ )_{W_G} \cdot (\ )_{W_H}$ be a nondegenerate invariant pairing on $W. \otimes W^\vee$.

Then for all $f_1, f_3 \in W$, $f_2, f_4 \in W^\vee$ with $(f_3, f_4) \neq 0$, we define

$$Q_{dt}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right) := \mathscr{L}(V_{(W_G,W_H)}, 0)^{-1} \cdot \mathrm{vol}(H'_{v,\infty}, dt) \cdot \frac{(\mathrm{p}_{H'}(f_1), \mathrm{p}_{H'}(f_2))}{(f_3, f_4)}, \tag{A.4.3}$$

where $\mathrm{p}_{H'}$ denotes the idempotent projector onto $H'_{v,\infty}$-invariants.

Let $\sigma_{W_G} : F_v^\times \to L^\times$ be the character giving the action of $\begin{pmatrix} F_v^\times & \\ & 1 \end{pmatrix}$ on W_G^N, let $\chi : E_v^\times \to L^\times$ be the algebraic character attached to W_H, and let

$$\mu^+ = \chi \cdot \sigma_{W_G} \circ N_{E_v/F_v}.$$

Let $\mathrm{j}_v =$(A.1.2). Then for all $f_1, f_3 \in W^N := W_G^N \otimes W_H$, $f_2, f_4 \in W^{\vee,N}$ with $f_3, f_4 \neq 0$, we define

$$Q_{dt}^{\mathrm{ord}}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right) := \mu^+(\mathrm{j}_v) \cdot \mathrm{vol}^\circ(H'_v, dt) \cdot \frac{f_1 \otimes f_2}{f_3 \otimes f_4}. \tag{A.4.4}$$

Proposition A.4.3 *Let W be a representation of $(G \times H)'_{v,\infty}$ over L. Let $\gamma_{H'}^{\mathrm{ord}} = \gamma_{H'\ ,v,\infty}^{\mathrm{ord}}$ be as defined in (A.2.5), and let $w_{\mathrm{a}}^{\mathrm{ord}} = w_{\mathrm{a},v,\infty}^{\mathrm{ord}}$ be as defined in (A.2.1). Then for all $f_1, f_3 \in W^N$, $f_2, f_4 \in W^{\vee,N}$ with $f_3, f_4 \neq 0$,*

$$Q_{dt}\left(\frac{\gamma_{H'}^{\mathrm{ord}} f_1 \otimes \gamma_{H'}^{\mathrm{ord}} f_2}{w_{\mathrm{a}}^{\mathrm{ord}} f_3 \otimes f_4}\right) = \dim W \cdot Q_{dt}^{\mathrm{ord}}\left(\frac{f_1 \otimes f_2}{f_3 \otimes f_4}\right).$$

Proof After possibly extending scalars we may assume that L splits E and pick an extensions of each $\sigma : F \hookrightarrow L$ to a $\sigma : E \hookrightarrow L$. We then have $W = \bigotimes_{\sigma : F \hookrightarrow L} W_\sigma$ with $W_\sigma =$(A.2.2) for suitable integers w, k_σ, l_σ, and analogously $\mu^+(t) = \prod_{\sigma : F \hookrightarrow L} \mu_\sigma^+$ with

$$\mu_\sigma^+(t) = \sigma(t)^{(k_\sigma - 2 + l_\sigma)/2}\sigma(t^c)^{(k_\sigma - 2 - l_\sigma)/2}, \quad \mu_\sigma^+(\mathrm{j}) = (-1)^{(k_\sigma - 2 - l_\sigma)/2} \cdot \mathrm{j}_v^{k_\sigma - 2}. \tag{A.4.5}$$

If v splits in E, this simplifies to $\mu_\sigma^+(\mathrm{j}) = (-1)^{(k_\sigma - 2 + l_\sigma)/2}$.

Moreover, $\mathscr{L}(V,0)=\prod_\sigma \mathscr{L}(V_\sigma,0)$ with

$$\mathscr{L}(V_\sigma,0)=\frac{\pi^{-1}\Gamma_{\mathbf{C}}(\frac{k_\sigma+l_\sigma}{2})\Gamma_{\mathbf{C}}(\frac{k_\sigma-l_\sigma}{2})}{\Gamma_{\mathbf{C}}(k_\sigma)\Gamma_{\mathbf{R}}(2)}=\frac{2}{k_\sigma-1}\cdot\binom{k_\sigma-2}{\frac{k_\sigma-2+l_\sigma}{2}}^{-1}.$$

Fix a $\sigma: F\hookrightarrow L$ for the rest of this proof, work with W_σ only, and we drop σ from the notation. We may assume that $f:=f_1$, $f^\vee:=f_2$ both equal x^{k-2} in the models (A.4.1), and that $\mathrm{vol}(H',dt)=1$. By the definitions above and Lemma A.4.1, we then need to prove that

$$\begin{aligned}Q(\gamma_{H'}^{\mathrm{ord}}f,\gamma_{H'}^{\mathrm{ord}}f^\vee):=\frac{k-1}{2}\cdot\binom{k-2}{\frac{k-2+l}{2}}\cdot(\mathrm{p}_{H'}(\gamma_{H'}^{\mathrm{ord}}f_1),\\ \mathrm{p}_{H'}(\gamma_{H'}^{\mathrm{ord}}f_2))=\frac{k-1}{2}\cdot\mu^+(\mathrm{j}).\end{aligned}$$

Recall in what follows that $\gamma_{H'}^{\mathrm{ord}}$ contains the factor $c(W)=$(A.2.3).
Split case. Suppose first that v splits in E. Then $W^{H'}=Lx^{(k-2-l)/2}y^{(k-2+l)/2}$, and $c(W)^{-1}\gamma_0^\iota f=c(W)^{-1}(x-y)^{k-2}$ projects to

$$\gamma_{H'}^{\mathrm{ord}}f=(-1)^{(k-2+l)/2}c(W)^{-1}\binom{k-2}{(k-2-l)/2}x^{(k-2-l)/2}y^{(k-2+l)/2}\in W^{H'}.$$

It follows that

$$\begin{aligned}Q(\gamma_{H'}^{\mathrm{ord}}f,\gamma_{H'}^{\mathrm{ord}}f^\vee)&=\frac{k-1}{2}\cdot\binom{k-2}{(k-2+l)/2}^2\cdot(-1)^{\frac{k-2+l}{2}}\cdot c(W)^{-1}c(W^\vee)^{-1}\\&=\frac{k-1}{2}\cdot\mu^+(\mathrm{j}).\end{aligned}$$

Nonsplit case. Suppose now that v is nonsplit in E. Let $z:=x-\theta^{c,-1}y$, $\overline{z}:=x-\theta^{-1}y$, then $W^{H'}=Lz^{(k-2-l)/2}\overline{z}^{(k-2+l)/2}$ and

$$\gamma_0^\iota f=c(W)^{-1}\mathrm{N}^{(w+k-2)/2}x^{k-2}=c(W)^{-1}\mathrm{N}^{(w+k-2)/2}\mathrm{j}^{2-k}(\theta^c z-\theta\overline{z})^{k-2}$$

projects to

$$\begin{aligned}\gamma_{H'}^{\mathrm{ord}}f&=c(W^\vee)^{-1}\binom{k-2}{\frac{k-2-l}{2}}(-1)^{(k-2+l)/2}\\&\quad\cdot\mathrm{N}^{(w+k-2)/2}\mathrm{j}^{2-k}\theta^{c,(k-l-2)/2}\theta^{(k+l-2)/2}\cdot z^{(k-2-l)/2}\overline{z}^{(k-2+l)/2}\\&=c(W^\vee)^{-1}\binom{k-2}{\frac{k-2-l}{2}}(-1)^{(k-2+l)/2}\end{aligned}$$

$$\cdot\, \mathrm{j}^{(-w-k+2)/2}\theta^{c,(k-l-2)/2}\theta^{(k+l-2)/2}$$
$$\cdot \begin{pmatrix} 1 & 1 \\ -\theta^{c,-1} & -\theta^{-1} \end{pmatrix} \cdot x^{(k-2-l)/2} y^{(k-2+l)/2}.$$

By the invariance of the pairing,

$$Q(\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^{\vee}) = (-1)^{(k-2+l)/2} c(W)^{-1} c(W^{\vee})^{-1} \binom{k-2}{\frac{k-2-l}{2}} (-\mathrm{j})^{2-k} \mathrm{N}^{k-2}$$
$$= (-1)^{k-2} \binom{k-2}{\frac{k-2-l}{2}}^{-1} \mu^{+}(\mathrm{j})$$

so that again

$$Q(\gamma_{H'}^{\mathrm{ord}} f, \gamma_{H'}^{\mathrm{ord}} f^{\vee}) = \dim W \cdot \mu^{+}(\mathrm{j}).$$

□

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