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New curvature conditions for the Bochner Technique

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Abstract We show that manifolds with $\lceil \frac{n}{2} \rceil$-positive curvature operators are rational homology spheres. This follows from a more general vanishing and estimation theorem for the pth Betti number of closed n-dimensional Riemannian manifolds with a lower bound on the average of the lowest $n-p$ eigenvalues of the curvature operator. This generalizes results due to D. Meyer, Gallot–Meyer, and Gallot.

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Introduction

A fundamental theme in Riemannian geometry is to understand the relationship between the curvature and the topology of a Riemannian manifold. The Bochner technique addresses this question by studying the existence of harmonic tensors on closed Riemannian manifolds. This is motivated by Hodge's theorem which asserts that every de Rham cohomology class is represented by a harmonic form.

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Bochner [5] opened up a link to geometry and proved that the first Betti number of compact manifolds with positive Ricci curvature vanishes. Berger [1] and Meyer [26] established vanishing results for the Betti numbers of manifolds with positive curvature operators and in particular Meyer showed that they are rational (co)homology spheres. Furthermore, Micallef–Wang [29] proved that the second Betti number of even dimensional manifolds with positive isotropic curvature vanishes.

The Ricci flow has been used extensively to obtain classification results, which in particular imply Bochner vanishing-type theorems. For example, Hamilton [20,21], Chen [13] and Böhm–Wilking [11] showed that manifolds with positive, in fact 2-positive, curvature operators are space forms. Brendle–Schoen [10] and Brendle [6] showed that this is more generally the case for manifolds whose product with $\mathbb{R}^2$ and $\mathbb{R}$, respectively, have positive isotropic curvature. As a consequence, Brendle–Schoen [10] proved the differentiable sphere theorem.

Based on Ricci flow with surgery, compact manifolds with positive isotropic curvature have been classified by Hamilton [22], Chen–Zhu [15] and Chen–Tang–Zhu [14] in dimension $n = 4$ and by Brendle [8] and Huang [24] in dimensions $n \geq 12$.

Using different techniques, Micallef–Moore [27] proved that simply connected compact manifolds with positive isotropic curvature are homotopy spheres.

Our first main theorem introduces nested curvature conditions that give rise to different vanishing results for the Betti numbers $b_p(M)$. Recall that the curvature operator of a Riemannian manifold is called l-positive if the sum of its lowest l eigenvalues is positive.

Theorem A *Let $n \geq 3$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. If (M, g) is a closed n-dimensional Riemannian manifold with $(n-p)$-positive curvature operator, then $b_1(M) = \ldots = b_p(M) = 0$ and $b_{n-p}(M) = \ldots = b_{n-1}(M) = 0$.*

Theorem A follows from Bochner's [5] theorem on manifolds with positive Ricci curvature for $n = 3$ or $p = 1$ and from the work of Böhm–Wilking [11] for $n = 4$.

In dimensions $n \geq 4$, the class of manifolds with k-positive curvature operator, $3 \leq k \leq n-1$, is different from the class of manifolds with positive isotropic curvature. The classes overlap but neither is contained in the other. In particular, Example 4.3(b) exhibits a 3-positive algebraic curvature operator with negative isotropic curvatures.

Furthermore, Böhm–Wilking [11] remarked that the set of 3-positive curvature operators is not Ricci flow invariant in dimensions $n \geq 4$. In contrast, positive and 2-positive curvature operator or positive isotropic curvature are curvature conditions preserved by the Ricci flow, as shown by Hamilton

[21,22], Chen [13], Brendle–Schoen [10] and Nguyen [30]. This observation is crucial for the Ricci flow results discussed above.

Theorem A raises the following question:

Question Are there closed, simply connected Riemannian manifolds with $(n-1)$-positive curvature operator and large second Betti number?

Similar questions can be asked for higher Betti numbers to address whether or not the conditions on the curvature operator in Theorem A are optimal.

Notice that $\mathbb{C}P^2$ is 3-positive with $b_2 = 1$ and that Sha–Yang [37] exhibited metrics of positive Ricci curvature on the connected sums $\left(S^2 \times S^2\right) \# \ldots \# \left(S^2 \times S^2\right)$.

In [23] Hoelzel established a surgery procedure for manifolds that satisfy a pointwise curvature condition. This, for instance, generalizes Micallef–Wang's [29] result that positive isotropic curvature is preserved under connected sums.

Example 4.3 (c) exhibits an $(n-1)$-positive algebraic curvature operator and a 2-form which yield a negative curvature term in the Bochner formula.

Theorem A immediately implies:

Corollary *Let $n \geq 3$ and let (M, g) be a closed n-dimensional Riemannian manifold. If the curvature operator is $\lceil \frac{n}{2} \rceil$-positive, then $b_p(M) = 0$ for $0 < p < n$.*

In view of the results of Böhm–Wilking [11], Brendle–Schoen [10] and Brendle [6], it is natural to ask:

Question Are closed manifolds with $\lceil \frac{n}{2} \rceil$-positive curvature operators space forms?

Many of the above mentioned results also have rigidity analogues in case of the corresponding nonnegativity conditions. In the context of the Bochner technique this goes back to Gallot–Meyer [17] who considered manifolds with nonnegative curvature operator. The more general results due to Ni–Wu [31], Brendle–Schoen [9], Seshadri [34] and Brendle [7] again rely on Ricci flow techniques. The rigidity result corresponding to Theorem A is:

Theorem B *Let $n \geq 3$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. If (M, g) is a closed n-dimensional Riemannian manifold with $(n-p)$-nonnegative curvature operator, then every harmonic p-form is parallel. Similarly every harmonic $(n-p)$-form is parallel.*

For $n = 3$ or $p = 1$ the result follows again from Bochner's [5] work and in dimension $n = 4$ from the results of Ni–Wu [31].

With regard to classification results for manifolds with a nonnegativity condition on the sum of the lowest eigenvalues, Theorem B is mainly interesting in the case of generic holonomy. Otherwise it reduces to previous results due to Gallot–Meyer [17], Böhm–Wilking [11] and Mok [28]:

Remark Suppose that (M, g) is n-dimensional and locally reducible. If the curvature operator is $(n-1)$-nonnegative, then the curvature operator is nonnegative. Similarly, if $\lambda_n > 0$, then $\lambda_1 = \ldots = \lambda_{n-1} = 0$.

Suppose that (M, g) is n-dimensional, locally irreducible, and has special holonomy. If the curvature operator is $\left(\frac{1}{4}n(n-2)\right)$-nonnegative, then the curvature operator is nonnegative. Similarly, if $\lambda_{\frac{1}{4}n(n-2)+1} > 0$, then $\lambda_1 = \ldots = \lambda_{\frac{1}{4}n(n-2)} = 0$.

Combined with Theorem A these observations lead to the following result:

Corollary *Let (M, g) be a closed connected n-dimensional Riemannian manifold with restricted holonomy $SO(n)$. If the curvature operator is $\lceil \frac{n}{2} \rceil$-nonnegative, then $b_p(M) = 0$ for $0 < p < n$.*

Cheeger [12] adapted the Bochner technique to singular spaces and proved a vanishing theorem for spaces with positive piecewise constant curvature, as well as the corresponding rigidity theorem. As Cheeger points out, these results indicate that spaces with nonnegative piecewise constant curvature may be regarded as a non-smooth analogue of manifolds with nonnegative curvature operator.

Based on work of Li [25], Gallot [16] further generalized the Bochner technique and obtained estimation results for the Betti numbers in case the curvature operator is bounded from below by $\kappa \leq 0$ and the diameter is bounded above by $D > 0$.

Theorem C *Let (M, g) be a closed connected n-dimensional Riemannian manifold, $n \geq 3$, and let $\lambda_1 \leq \ldots \leq \lambda_{\binom{n}{2}}$ denote the of the curvature operator of (M, g). Fix $\kappa \leq 0$, $D > 0$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$. If*

$$\frac{\lambda_1 + \ldots + \lambda_{n-p}}{n-p} \geq \kappa \text{ and } \operatorname{diam}(M) \leq D,$$

then there is a constant $C\left(n, \kappa D^2\right) > 0$ such that

$$b_p(M) \leq \binom{n}{p} \exp\left(C\left(n, \kappa D^2\right) \cdot \sqrt{-\kappa D^2 p(n-p)} \right).$$

In particular, there exists $\varepsilon(n) > 0$ such that $\kappa D^2 \geq -\varepsilon(n)$ implies $b_p(M) \leq \binom{n}{p}$.

Theorem C is due to Gallot [16] if instead $\lambda_1 \geq \kappa$ is assumed.

Remarkably, in the context of sectional curvature Gromov [19] established similar bounds on the Betti numbers, with coefficients in an arbitrary field, using purely geometric ideas.

Another application of our method yields a generalization of a theorem due to Tachibana [38].

Theorem D *Let $n \geq 5$ and let (M, g) be a closed n-dimensional Einstein manifold. If the curvature operator is $\lfloor \frac{n-1}{2} \rfloor$-nonnegative, then the curvature tensor is parallel.*

If M is connected and the curvature operator is $\lfloor \frac{n-1}{2} \rfloor$-positive, then (M, g) has constant sectional curvature.

In dimension $n = 4$ the classification results for manifolds with 2-nonnegative curvature operators due to Böhm–Wilking [11] and Ni–Wu [31] apply since Einstein metrics are fixed points of the Ricci flow.

The rigidity results due to Brendle–Schoen [9] and Seshadri [34] also yield Tachibana-type theorems. Brendle [7] specifically considers Einstein manifolds and shows that Einstein manifolds with nonnegative isotropic curvature are locally symmetric. In dimension $n = 4$ this was observed by Micallef–Wang [29].

The proofs of Theorems A–D rely on a new method to control the curvature term of Lichnerowicz Laplacians on $(0, k)$-tensors T,

$$\Delta_L T = \nabla^* \nabla T + c \operatorname{Ric}(T)$$

where $c > 0$ is a constant. In case of 1-forms, $\operatorname{Ric}(\omega)$ is indeed given by the Ricci curvature. In general $\operatorname{Ric}(T)$ depends on the entire Riemannian curvature tensor. The basic principle of the Bochner technique asserts that every harmonic tensor with $\operatorname{Ric}(T) \geq 0$ is parallel.

More concretely, the new method is based on a slight generalization of Poor's [33] approach to the Hodge Laplacian. Poor used the derivative of the regular representation on tensors to obtain a simple formula for the curvature term. Specifically, Poor studied p-forms and showed that

$$g(\operatorname{Ric}(\omega), \omega) = \sum_{\alpha} \lambda_\alpha |\Xi_\alpha \omega|^2$$

where $\{\Xi_\alpha\}$ is an orthonormal eigenbasis of the curvature operator and $\{\lambda_\alpha\}$ denote the corresponding eigenvalues.

The key new observation is that for $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$ every p-form ω satisfies $|\Xi_\alpha \omega|^2 \leq p|\omega|^2$ while $\sum_\alpha |\Xi_\alpha \omega|^2 = p(n-p)|\omega|^2$. This implies $g(\operatorname{Ric}(\omega), \omega) \geq 0$ provided $\lambda_1 + \ldots + \lambda_{n-p} \geq 0$.

Lemma 2.1 extends this idea and more generally explains how to control the curvature term based on an understanding how elements of $\mathfrak{so}(n)$ interact with tensors of a specific type. E.g. Theorem D considers the case of algebraic curvature tensors. The required estimates to apply Lemma 2.1 are established in Lemma 2.2 and Proposition 2.5. The work of Li [25] and Gallot [16] then implies a bound on the dimension of the kernel of the Lichnerowicz Laplacian.

Section 1 reviews the relevant background material. The key technical lemmas are given in Section 2. The proofs of the main theorems follow in section 3. Section 4 contains examples that show that the estimates in section 2 are optimal and considers algebraic curvature operators to discuss the optimality of the eigenvalue assumptions in Theorems A–D.

General references for background on the Bochner technique are Bérard [4], Goldberg [18] and Petersen [32]. For an account of the early developments of the Bochner technique the reader is referred to Yano–Bochner [39].

1 Preliminaries

1.1 Tensors

Let (V, g) be an n-dimensional Euclidean vector space. The vector space of $(0, k)$-tensors on V will be denoted by $\mathcal{T}^{(0,k)}(V)$ and the vector space of symmetric $(0, 2)$-tensors by $\mathrm{Sym}^2(V)$.

Recall that there is an orthogonal decomposition

$$\mathrm{Sym}^2(\Lambda^2 V) = \mathrm{Sym}^2_B(\Lambda^2 V) \oplus \Lambda^4 V,$$

where the vector space $\mathrm{Sym}^2_B(\Lambda^2 V)$ consists of all tensors $T \in \mathrm{Sym}^2(\Lambda^2 V)$ that also satisfy the first Bianchi identity. Any $R \in \mathrm{Sym}^2_B(\Lambda^2 V)$ is called an algebraic curvature tensor.

The following norms and inner products for tensors, whose components are with respect to an arbitrary choice of an orthonormal basis, will be used throughout: When $T \in \mathcal{T}^{(0,k)}(V)$ define

$$|T|^2 = \sum_{i_1,\dots,i_k} \left(T_{i_1\dots i_k}\right)^2$$

whereas for a p-form $\omega \in \Lambda^p V^*$ set

$$|\omega|^2 = \sum_{i_1<\dots<i_p} \left(\omega_{i_1\dots i_p}\right)^2 .$$

Similarly, if $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis for V, then $\left\{e_{i_1} \wedge \dots \wedge e_{i_p}\right\}_{1\le i_1<\dots<i_p\le n}$ is an orthonormal basis for $\Lambda^p V$. This also induces an inner product on $\mathfrak{so}(V)$ via its identification with $\Lambda^2 V$.

The Kulkarni-Nomizu product of $S, T \in \mathrm{Sym}^2(V)$ is given by

$$\begin{aligned}(S \owedge T)(X, Y, Z, W) = \; & S(X, Z)T(Y, W) - S(X, W)T(Y, Z)\\ & + S(Y, W)T(X, Z) - S(Y, Z)T(X, W).\end{aligned}$$

In particular, the tensor

$$(g \owedge g)(X, Y, Z, W) = 2\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}$$

corresponds to the curvature tensor of the sphere of radius $1/\sqrt{2}$.

Proposition 1.1 *If* $h \in \mathrm{Sym}^2(V)$*, then*

$$|g \owedge h|^2 = 4(n-2)|h|^2 + 4\,\mathrm{tr}(h)^2.$$

In particular, $|g \owedge g|^2 = 8(n-1)n$.

Proof By using an orthonormal basis $\{e_i\}$ for V that diagonalizes h one obtains:

$$(g \owedge h)_{ijkl} = \begin{cases} h_{ii} + h_{jj} & \text{if } i = k \neq j = l,\\ -h_{ii} - h_{jj} & \text{if } i = l \neq j = k,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$|g \owedge h|^2 = 2\sum_{i,j}(h_{ii} + h_{jj})^2 - 2\sum_i (2h_{ii})^2 = 4(n-2)|h|^2 + 4\,\mathrm{tr}(h)^2$$

as claimed. □

Recall that every algebraic (0, 4)-curvature tensor Rm satisfies the orthogonal decomposition

$$\mathrm{Rm} = \frac{\mathrm{scal}}{2(n-1)n} g \owedge g + \frac{1}{n-2} g \owedge \mathring{\mathrm{Ric}} + W,$$

where $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{scal}}{n} g$ is the trace-free Ricci tensor and W denotes the Weyl part. The associated algebraic curvature operator $\mathfrak{R}\colon \Lambda^2 V \to \Lambda^2 V$ is defined by

$$g(\mathfrak{R}(x \wedge y), z \wedge w) = \mathrm{Rm}(x, y, z, w).$$

Note that the induced algebraic curvature tensor $R \in \mathrm{Sym}^2_B(\Lambda^2 V)$ satisfies

$$|\,\mathrm{Rm}\,|^2 = 4|R|^2.$$

1.2 The regular representation

The derivative of the regular representation of $O(n)$ on (V, g) induces a derivation on tensors: If $T \in \mathcal{T}^{(0,k)}(V)$ and $L \in \mathfrak{so}(V)$, then

$$(LT)(X_1, \ldots, X_k) = -\sum_{i=1}^{k} T(X_1, \ldots, LX_i, \ldots, X_k).$$

Notice that the metric g satisfies $Lg = 0$ for all $L \in \mathfrak{so}(V)$ since

$$(Lg)(X, Y) = -g(LX, Y) - g(X, LY) = -g(LX, Y) + g(LX, Y) = 0.$$

The information on how all $L \in \mathfrak{so}(V)$ interact with a fixed $T \in \mathcal{T}^{(0,k)}$ can be encoded in a tensor $\hat{T}$ with values in $\Lambda^2 V$.

Definition 1.2 For $T \in \mathcal{T}^{(0,k)}(V)$ define $\hat{T} \in \Lambda^2 V \otimes \mathcal{T}^{(0,k)}(V)$ implicitly by

$$g(L, \hat{T}(X_1, \ldots, X_k)) = (LT)(X_1, \ldots, X_k)$$

for all $L \in \mathfrak{so}(V) = \Lambda^2 V$.

Notice that if $\{\Xi_\alpha\}$ is an orthonormal basis for $\mathfrak{so}(V) = \Lambda^2 V$, then

$$\hat{T} = \sum_\alpha \Xi_\alpha \otimes \Xi_\alpha T.$$

Consequently,

$$|\hat{T}|^2 = \sum_\alpha |\Xi_\alpha T|^2.$$

Example 1.3 Let $e_1, \ldots, e_n$ be an orthonormal basis for V with dual basis $e^1, \ldots, e^n$ and let $1 \leq i_1 < \ldots < i_p \leq n$. It is simple to verify that

$$\widehat{e^{i_1} \wedge \ldots \wedge e^{i_p}} = \sum_{\substack{j=1,\ldots,p \\ k \notin \{i_1,\ldots,i_p\}}} (-1)^j \, e_{\min\{k,i_j\}} \wedge e_{\max\{k,i_j\}} \, e^k \wedge e^{i_1} \wedge \ldots \wedge \widehat{e^{i_j}} \wedge \ldots \wedge e^{i_p}.$$

The following observation will be crucial for applications to the Bochner technique.

Proposition 1.4 *Let* $\mathfrak{R}\colon \Lambda^2 V \to \Lambda^2 V$ *be an algebraic curvature operator and* $\{\Xi_\alpha\}$ *an orthonormal basis for* $\Lambda^2 V$*. It follows that*

$$\mathfrak{R}(\hat{T}) = \mathfrak{R} \circ \hat{T} = \sum_\alpha \mathfrak{R}(\Xi_\alpha) \otimes \Xi_\alpha T.$$

Furthermore, if $\{\Xi_\alpha\}$ *is an eigenbasis of* $\mathfrak{R}$ *and* $\{\lambda_\alpha\}$ *denote the corresponding eigenvalues, then*

$$g(\mathfrak{R}(\hat{T}), \hat{T}) = \sum_{\alpha,\beta} g(\mathfrak{R}(\Xi_\alpha), \Xi_\beta) g(\Xi_\alpha T, \Xi_\beta T) = \sum_\alpha \lambda_\alpha |\Xi_\alpha T|^2.$$

The following proposition will be useful for the computation of examples:

Proposition 1.5 *If* $\{\Xi_\alpha\}$ *is an orthonormal basis for* $\Lambda^2 V$ *that diagonalizes* $R \in \mathrm{Sym}^2(\Lambda^2 V)$ *and* $\{\lambda_\alpha\}$ *denote the corresponding eigenvalues, then*

$$|LR|^2 = 2 \sum_{\alpha<\beta} (\lambda_\alpha - \lambda_\beta)^2 g(L\Xi_\alpha, \Xi_\beta)^2$$

for every $L \in \mathfrak{so}(V)$.

Proof This is a straightforward calculation:

$$\begin{aligned} |LR|^2 &= \sum_{\alpha,\beta} \big((LR)(\Xi_\alpha, \Xi_\beta)\big)^2 \\ &= \sum_{\alpha,\beta} \big(-R(L\Xi_\alpha, \Xi_\beta) - R(\Xi_\alpha, L\Xi_\beta)\big)^2 \\ &= \sum_{\alpha,\beta} \big(-\lambda_\beta g(L\Xi_\alpha, \Xi_\beta) - \lambda_\alpha g(\Xi_\alpha, L\Xi_\beta)\big)^2 \\ &= \sum_{\alpha,\beta} (\lambda_\alpha - \lambda_\beta)^2 g(L\Xi_\alpha, \Xi_\beta)^2. \end{aligned}$$

□

1.3 The Bochner technique

Let (M, g) be a closed n-dimensional Riemannian manifold and let $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ denote its curvature tensor. For $T \in \mathcal{T}^{(0,k)}(M)$ set

$$\mathrm{Ric}(T)(X_1, \ldots, X_k) = \sum_{i=1}^{k} \sum_{j=1}^{n} (R(X_i, e_j)T)(X_1, \ldots, e_j, \ldots, X_k).$$

Remark 1.6 Recall that the Ricci identity asserts

$$R(X, Y)T(X_1, \ldots, X_k) = -\sum_{i=1}^{k} T(X_1, \ldots, R(X, Y)X_i, \ldots, X_k),$$

which is in agreement with the effect of $R(X, Y) \in \mathfrak{so}(TM)$ on $T \in \mathcal{T}^{(0,k)}(M)$ defined in Sect. 1.2. In particular the above definition of $\mathrm{Ric}(T)$ carries over to algebraic curvature tensors. The notation $\mathrm{Ric}_R(T)$ will be used to specify the algebraic curvature tensor R.

Let $E \to M$ be a subbundle of $\mathcal{T}^{(0,k)}(M)$. For $c > 0$ the *Lichnerowicz Laplacian* on E is given by

$$\Delta_L = \nabla^*\nabla + c\,\mathrm{Ric}\,.$$

A tensor T is called *harmonic* if $\Delta_L T = 0$.

Example 1.7 There are various important examples of Lichnerowicz Laplacians for different $c > 0$.

(a) The Hodge Laplacian is a Lichnerowicz Laplacian for $c = 1$ and a p-form ω is harmonic if and only if it is closed and divergence free.
(b) The natural definition of the Lichnerowicz Laplacian for symmetric (0, 2)-tensors uses $c = \frac{1}{2}$. With this choice $h \in \mathrm{Sym}^2(M)$ is harmonic if and only if h is a Codazzi tensor and divergence free. This is equivalent to h being Codazzi and having constant trace. This has been used by Berger [2,3] in the case of Einstein metrics and by Simons [35] in the case of constant mean curvature hypersurfaces.
(c) The Lichnerowicz Laplacian for algebraic curvature tensors Rm on a Riemannian manifold also uses $c = \frac{1}{2}$. With this choice Rm is harmonic if it satisfies the second Bianchi identity and it is divergence free. If Rm satisfies the second Bianchi identity, then it is divergence free if and only if its Ricci tensor is a Codazzi tensor, and in this case its scalar curvature is constant. This was used by Tachibana [38].

The next proposition is established in [32, lemmas 9.3.3 and 9.4.3].

Proposition 1.8 *If* $S, T \in \mathcal{T}^{(0,k)}(M)$*, then*

$$g(\mathrm{Ric}(S), T) = g(\mathfrak{R}(\hat{S}), \hat{T}).$$

In particular, Ric *is self-adjoint.*

The Bochner technique relies on the following principle. Every harmonic $(0, k)$-tensor T satisfies

$$\Delta \frac{1}{2}|T|^2 = |\nabla T|^2 + c \cdot g(\mathfrak{R}(\hat{T}), \hat{T}).$$

If $g(\mathfrak{R}(\hat{T}), \hat{T}) \geq 0$, then the maximum principle implies that $\nabla T = 0$. If in addition $g(\mathfrak{R}(\hat{S}), \hat{S}) > 0$ for all tensors S of the same type as T and with

$\hat{S} \neq 0$, then $\hat{T} = 0$. For example, on 1-forms the curvature term is given by

$$g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) = \mathrm{Ric}(\omega^{\#}, \omega^{\#}).$$

Thus, if $\mathrm{Ric} \geq 0$ on M and $\mathrm{Ric}_p > 0$ for some $p \in M$, then every harmonic 1-form vanishes. Hence, by Hodge theory, $b_1(M) = 0$. This is Bochner's [5] original theorem.

Based on the work of Li [25], Gallot [16] showed that the Bochner technique implies estimation theorems if a negative lower bound on the curvature term and an upper diameter bound are assumed. The following theorem summarizes the framework of the Bochner technique for general Lichnerowicz Laplacians, see [32, Chapter 9] for a detailed introduction.

Theorem 1.9 *Let $n \geq 3$, $\kappa \leq 0$ and $D > 0$, and let (M, g) be a closed connected n-dimensional Riemannian manifold with* $\mathrm{Ric}(M) \geq (n-1)\kappa$ *and* $\mathrm{diam}(M) \leq D$.

Let $E \to M$ be a subbundle of $\mathcal{T}^{(0,k)}(M)$ with m-dimensional fiber and assume there is $C > 0$ such that

$$g(\mathfrak{R}(\hat{T}), \hat{T}) \geq \kappa C |T|^2$$

for all $T \in \Gamma(E)$.

If $\kappa = 0$, then all $T \in \ker(\Delta_L)$ are parallel. If in addition there is $p \in M$ such that $g(\mathfrak{R}_p(\hat{T}), \hat{T}) > 0$ for all $T \in \Gamma(E_p)$ with $\hat{T} \neq 0$, then

$$\ker(\Delta_L) = \{T \in \Gamma(E) \mid T \textit{ parallel}, \ \hat{T} = 0\}.$$

If $\kappa < 0$, then the dimension of the kernel of the associated Lichnerowicz Laplacian

$$\ker(\Delta_L) = \left\{T \in \Gamma(E) \mid \Delta_L T = \nabla^* \nabla T + c\,\mathrm{Ric}(T) = 0\right\}$$

is bounded by

$$m \cdot \exp\left(C\left(n, \kappa D^2\right) \cdot \sqrt{-\kappa D^2 c C}\right).$$

Moreover, there is $\varepsilon(n, cC) > 0$ such that $\kappa D^2 \geq -\varepsilon(n, cC)$ implies $\dim \ker(\Delta_L) \leq m$.

Remark 1.10 The condition $\mathrm{Ric}(M) \geq (n-1)\kappa$ is always satisfied in the situation of Theorems A - D since the Ricci curvature is bounded from below by the sum of the lowest $(n-1)$ eigenvalues of the curvature operator.

2 Controlling the curvature term of Lichnerowicz Laplacians

The following lemma provides a general method of controlling the curvature term of the Lichnerowicz Laplacian on tensors.

Lemma 2.1 *Let* $\mathfrak{R}\colon \Lambda^2 V \to \Lambda^2 V$ *be an algebraic curvature operator with eigenvalues* $\lambda_1 \leq \ldots \leq \lambda_{\binom{n}{2}}$ *and let* $T \in \mathcal{T}^{(0,k)}(V)$.

Suppose there is $C \geq 1$ *such that*

$$|LT|^2 \leq \frac{1}{C}|\hat{T}|^2|L|^2$$

for all $L \in \mathfrak{so}(V)$.

Let $\kappa \leq 0$. *If* $\frac{1}{\lfloor C \rfloor}\left(\lambda_1 + \ldots + \lambda_{\lfloor C \rfloor}\right) \geq \kappa$, *then* $g(\mathfrak{R}(\hat{T}), \hat{T}) \geq \kappa |\hat{T}|^2$ *and if* $\lambda_1 + \ldots + \lambda_{\lfloor C \rfloor} > 0$, *then* $g(\mathfrak{R}(\hat{T}), \hat{T}) > 0$ *unless* $\hat{T} = 0$.

Proof Choose an orthonormal basis $\{\Xi_\alpha\}$ for $\Lambda^2 V$ such that $\mathfrak{R}(\Xi_\alpha) = \lambda_\alpha \Xi_\alpha$. Notice that $\lambda_{\lfloor C \rfloor + 1} \geq \kappa$, which in turn implies

$$\begin{aligned}
g(\mathfrak{R}(\hat{T}), \hat{T}) &= \sum_{\alpha=1}^{\binom{n}{2}} \lambda_\alpha |\Xi_\alpha T|^2 \\
&= \sum_{\alpha=\lfloor C \rfloor+1}^{\binom{n}{2}} \lambda_\alpha |\Xi_\alpha T|^2 + \sum_{\alpha=1}^{\lfloor C \rfloor} \lambda_\alpha |\Xi_\alpha T|^2 \\
&\geq \lambda_{\lfloor C \rfloor+1} \sum_{\alpha=\lfloor C \rfloor+1}^{\binom{n}{2}} |\Xi_\alpha T|^2 + \sum_{\alpha=1}^{\lfloor C \rfloor} \lambda_\alpha |\Xi_\alpha T|^2 \\
&= \lambda_{\lfloor C \rfloor+1} |\hat{T}|^2 + \sum_{\alpha=1}^{\lfloor C \rfloor} \left(\lambda_\alpha - \lambda_{\lfloor C \rfloor+1}\right) |\Xi_\alpha T|^2 \\
&\geq \lambda_{\lfloor C \rfloor+1} |\hat{T}|^2 + \frac{1}{C} \sum_{\alpha=1}^{\lfloor C \rfloor} \left(\lambda_\alpha - \lambda_{\lfloor C \rfloor+1}\right) |\hat{T}|^2 \\
&= \lambda_{\lfloor C \rfloor+1} \left(1 - \frac{\lfloor C \rfloor}{C}\right) |\hat{T}|^2 + \frac{|\hat{T}|^2}{C} \sum_{\alpha=1}^{\lfloor C \rfloor} \lambda_\alpha \\
&\geq \kappa |\hat{T}|^2.
\end{aligned}$$

The last claim follows from the observation that for $\lambda_{\lfloor C \rfloor+1} \geq 0$ the above calculation implies $g(\mathfrak{R}(\hat{T}), \hat{T}) \geq \frac{|\hat{T}|^2}{C} \sum_{\alpha=1}^{\lfloor C \rfloor} \lambda_\alpha$. □

In the following, $|LT|^2$ will be estimated for various types of tensors:

Lemma 2.2 *Let (V, g) be an n-dimensional Euclidean vector space and $L \in \mathfrak{so}(V)$. The following hold:*

(a) *Every $T \in \mathcal{T}^{(0,k)}(V)$ satisfies*

$$|LT|^2 \leq k^2 |T|^2 |L|^2.$$

(b) *Every p-form ω satisfies*

$$|L\omega|^2 \leq \min\{p, n-p\} |\omega|^2 |L|^2.$$

(c) *Every $R \in \mathrm{Sym}^2(\Lambda^2 V)$ satisfies*

$$|LR|^2 \leq 8 |\mathring{R}|^2 |L|^2$$

and the associated $(0, 4)$-tensor Rm also satisfies

$$|L \,\mathrm{Rm}\,|^2 \leq 8 |\mathring{\mathrm{Rm}}|^2 |L|^2.$$

Proof Choose an orthonormal basis $\{e_i\}$ for V so that

$$L = \sum_{i=1}^{\lfloor n/2 \rfloor} \alpha_{2i - \frac{1}{2}} e_{2i-1} \wedge e_{2i}$$

and observe that $Le_i = (-1)^{i+1} \alpha_{i + \frac{(-1)^{i+1}}{2}} e_{i+(-1)^{i+1}}$.

In case (a) this yields

$$\begin{aligned} &T\left(e_{i_1}, \ldots, Le_{i_j}, \ldots, e_{i_k}\right) \\ &= (-1)^{i_j+1} \alpha_{i_j + \frac{(-1)^{i_j+1}}{2}} \, T\left(e_{i_1}, \ldots, e_{i_j+(-1)^{i_j+1}}, \ldots, e_{i_k}\right) \end{aligned}$$

and

$$\begin{aligned} |(LT)(e_{i_1}, \ldots, e_{i_k})|^2 &= \left| - \sum_{j=1}^{k} T\left(e_{i_1}, \ldots, Le_{i_j}, \ldots, e_{i_k}\right) \right|^2 \\ &= \left| - \sum_{j=1}^{k} (-1)^{i_j+1} \alpha_{i_j + \frac{(-1)^{i_j+1}}{2}} \, T_{i_1 \ldots i_j + (-1)^{i_j+1} \ldots i_k} \right|^2 \end{aligned}$$

$$\le \left(\sum_{j=1}^{k} \left(\alpha_{i_j + \frac{(-1)^{i_j+1}}{2}} \right)^2 \right) \left(\sum_{j=1}^{k} \left(T_{i_1 \ldots i_j + (-1)^{i_j+1} \ldots i_k} \right)^2 \right)$$

$$\le k|L|^2 \sum_{j=1}^{k} \left(T_{i_1 \ldots i_j + (-1)^{i_j+1} \ldots i_k} \right)^2 .$$

Summation over $i_1, \ldots, i_k$ implies

$$|LT|^2 \le k|L|^2 \sum_{i_1,\ldots,i_k} \sum_{j=1}^{k} \left(T_{i_1 \ldots i_j + (-1)^{i_j+1} \ldots i_k} \right)^2 \le k^2 |L|^2 |T|^2.$$

It suffices to prove (b) for $p \le \lfloor \frac{n}{2} \rfloor$ due to Hodge duality. Furthermore, assume $i_1 < \ldots < i_p$ in the above calculation. It follows that the coefficients $\alpha_{i_j + \frac{(-1)^{i_j+1}}{2}}$ that are summed over all correspond to different coefficients of L. Indeed, a coefficient can only occur twice if there are consecutive indices k, l such that

$$i_k + \frac{1}{2} = i_l - \frac{1}{2}.$$

However, in this case observe that

$$\begin{aligned} &\alpha_{i_k + \frac{1}{2}} \omega \left(e_{i_1}, \ldots, e_{i_k+1}, e_{i_l}, \ldots, e_{i_p} \right) \\ &\quad = 0 \ \text{ and } \ \alpha_{i_l - \frac{1}{2}} \omega \left(e_{i_1}, \ldots, e_{i_k}, e_{i_l-1}, \ldots, e_{i_p} \right) = 0 \end{aligned}$$

and hence these terms do not occur in the summation. Thus

$$|(L\omega) \left(e_{i_1}, \ldots, e_{i_p} \right)|^2 \le |L|^2 \sum_{j=1}^{p} \left(\omega \left(e_{i_1}, \ldots, e_{i_j + (-1)^{i_j+1}}, \ldots, e_{i_p} \right) \right)^2$$

and summation over $i_1 < \ldots < i_p$ yields the claim.

Case (c) follows as in (b) by using the symmetries of Rm . □

Remark 2.3 The estimates in Lemma 2.2 are optimal. For (a) this is easy to see in case of symmetric (0, 2)-tensors and the examples in section 4 show that the estimates in (b) and (c) cannot be improved without further assumptions either.

Let $\mathrm{id} \wedge \mathrm{id}$ denote the curvature tensor of the unit sphere. The computation of $|\hat{T}|^2$ in the propositions below relies on the observation that $|\hat{T}|^2 = g(\mathrm{Ric}_{\mathrm{id} \wedge \mathrm{id}}(T), T)$.

Proposition 2.4 *Let (V, g) be an n-dimensional Euclidean vector space and $T \in \mathcal{T}^{(0,k)}(V)$. It follows that*

$$\begin{aligned}
&\mathrm{Ric}_{\mathrm{id}\wedge\mathrm{id}}(T)(X_1, \ldots, X_k) \\
&= k(n-1)T(X_1, \ldots, X_k) + \sum_{i\neq j}(T \circ \tau_{ij})(X_1, \ldots, X_k) \\
&\quad - \sum_{i\neq j} g(X_i, X_j)c_{ij}(T)(X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),
\end{aligned}$$

where τ_{ij} denotes the transposition of the i^{th} and j^{th} entries and $c_{ij}(T)$ is the contraction of T in the i^{th} and j^{th} entries.

Proof Recall that the curvature tensor of the unit sphere satisfies

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X = (X \wedge Y)(Z).$$

Let $\{e_i\}$ be an orthonormal basis for V. The claim now follows from the calculation:

$$\begin{aligned}
&\mathrm{Ric}_{\mathrm{id}\wedge\mathrm{id}}(T)(X_1, \ldots, X_k) \\
&= \sum_{i=1}^{k}\sum_{a=1}^{n}(R(X_i, e_a)T)(X_1, \ldots, e_a, \ldots, X_k) \\
&= \sum_{i\neq j}\sum_{a=1}^{n} T(X_1, \ldots, (e_a \wedge X_i)X_j, \ldots, e_a, \ldots, X_k) \\
&\quad + \sum_{i=1}^{k}\sum_{a=1}^{n} T(X_1, \ldots, (e_a \wedge X_i)e_a, \ldots, X_k) \\
&= \sum_{i\neq j}\sum_{a=1}^{n} T(X_1, \ldots, g(e_a, X_j)X_i - g(X_i, X_j)e_a, \ldots, e_a, \ldots, X_k) \\
&\quad + \sum_{i=1}^{k}\sum_{a=1}^{n} T(X_1, \ldots, X_i - g(e_a, X_i)e_a, \ldots, X_k) \\
&= \sum_{i\neq j}\sum_{a=1}^{n} T(X_1, \ldots, X_i, \ldots, g(e_a, X_j)e_a, \ldots, X_k) \\
&\quad - \sum_{i\neq j}\sum_{a=1}^{n} g(X_i, X_j)T(X_1, \ldots, e_a, \ldots, e_a, \ldots, X_k) \\
&\quad + k(n-1)T(X_1, \ldots, X_k).
\end{aligned}$$

□

Proposition 2.5 *Let (V, g) be an n-dimensional Euclidean vector space and let $\operatorname{id} \wedge \operatorname{id}$ denote the curvature tensor of the unit sphere. The following hold:*

(a) *Every p-form ω satisfies*

$$\begin{aligned}\operatorname{Ric}_{\operatorname{id}\wedge\operatorname{id}}(\omega) &= p(n-p)\omega,\\ |\hat{\omega}|^2 &= p(n-p)|\omega|^2.\end{aligned}$$

(b) *Every algebraic $(0,4)$-curvature tensor Rm and every $R \in \operatorname{Sym}^2_B(\Lambda^2 V)$ satisfies*

$$\begin{aligned}\operatorname{Ric}_{\operatorname{id}\wedge\operatorname{id}}(\operatorname{Rm}) &= 4(n-1)\operatorname{Rm} - 2g \owedge \operatorname{Ric},\\ |\widehat{\operatorname{Rm}}|^2 = |\widehat{\mathring{\operatorname{Rm}}}|^2 &= 4(n-1)|\mathring{\operatorname{Rm}}|^2 - 8|\mathring{\operatorname{Ric}}|^2,\\ |\hat{R}|^2 = |\hat{\mathring{R}}|^2 &= 4(n-1)|\mathring{R}|^2 - 2|\mathring{\operatorname{Ric}}|^2.\end{aligned}$$

In particular $\widehat{\operatorname{Rm}} = 0$ if and only if $\operatorname{Rm} = \frac{\kappa}{2} g \owedge g$ for some $\kappa \in \mathbb{R}$.

Proof (a) Notice that $\omega \circ \tau_{ij} = -\omega$ for every transposition τ_{ij} and thus $c_{ij}(T) = 0$ for all $i \neq j$. This implies

$$\operatorname{Ric}_{\operatorname{id}\wedge\operatorname{id}}(\omega) = -\sum_{i\neq j} \omega + p(n-1)\omega = p(n-p)\omega.$$

(b) Due to the symmetries of the curvature tensor

$$\begin{aligned}&\sum_{i\neq j} \operatorname{Rm} \circ \tau_{ij}\\ &= 2(\operatorname{Rm}\circ\tau_{12} + \operatorname{Rm}\circ\tau_{13} + \operatorname{Rm}\circ\tau_{14} + \operatorname{Rm}\circ\tau_{23} + \operatorname{Rm}\circ\tau_{24} + \operatorname{Rm}\circ\tau_{34})\\ &= -4\operatorname{Rm} + 2(\operatorname{Rm}\circ\tau_{13} + \operatorname{Rm}\circ\tau_{14} + \operatorname{Rm}\circ\tau_{23} + \operatorname{Rm}\circ\tau_{24})\end{aligned}$$

which implies

$$\begin{aligned}&\sum_{i\neq j} \left(\operatorname{Rm}\circ\tau_{ij}\right)(X, Y, Z, W)\\ &= -4\operatorname{Rm}(X, Y, Z, W) + 2\{\operatorname{Rm}(Z, Y, X, W) + \operatorname{Rm}(W, Y, Z, X)\\ &\quad + \operatorname{Rm}(X, Z, Y, W) + \operatorname{Rm}(X, W, Z, Y)\}\\ &= -4\{\operatorname{Rm}(X, Y, Z, W) + \operatorname{Rm}(Y, Z, X, W) + \operatorname{Rm}(Z, X, Y, W)\}\\ &= 0\end{aligned}$$

due to the first Bianchi identity. For the remaining term one computes

$$\begin{aligned}
&\sum_{i\neq j}(g(\cdot,\cdot)c_{ij}(\mathrm{Rm}))(X,Y,Z,W)\\
&= 2\sum_{i=1}^{n}\{g(X,Z)\,\mathrm{Rm}(e_i,Y,e_i,W)+g(X,W)\,\mathrm{Rm}(e_i,Y,Z,e_i)\\
&\quad + g(Y,Z)\,\mathrm{Rm}(X,e_i,e_i,W)+g(Y,W)\,\mathrm{Rm}(X,e_i,Z,e_i)\}\\
&= 2\sum_{i=1}^{n}\{g(X,Z)\,\mathrm{Rm}(Y,e_i,W,e_i)-g(X,W)\,\mathrm{Rm}(Y,e_i,Z,e_i)\\
&\quad - g(Y,Z)\,\mathrm{Rm}(X,e_i,W,e_i)+g(Y,W)\,\mathrm{Rm}(X,e_i,Z,e_i)\}\\
&= 2(g \owedge \mathrm{Ric})(X,Y,Z,W).
\end{aligned}$$

To calculate $|\widehat{\mathrm{Rm}}|^2$ observe that

$$g(\mathrm{Rm}, g \owedge \mathrm{Ric}) = \frac{\mathrm{scal}^2}{2n^2(n-1)}|g \owedge g|^2 + \frac{1}{n-2}|\mathring{\mathrm{Ric}} \owedge g|^2 = 4\frac{\mathrm{scal}^2}{n} + 4|\mathring{\mathrm{Ric}}|^2$$

due to Proposition 1.1. For the last claim observe that

$$\begin{aligned}
|\widehat{\mathrm{Rm}}|^2 &= 4(n-1)|\mathring{\mathrm{Rm}}|^2 - 8|\mathring{\mathrm{Ric}}|^2\\
&= 4(n-1)\left(\frac{1}{(n-2)^2}|g \owedge \mathring{\mathrm{Ric}}|^2 + |W|^2\right) - 8|\mathring{\mathrm{Ric}}|^2\\
&= \left(16\frac{n-1}{n-2} - 8\right)|\mathring{\mathrm{Ric}}|^2 + 4(n-1)|W|^2.
\end{aligned}$$

In particular, $|\widehat{\mathrm{Rm}}|^2 = 0$ is equivalent to $|\mathring{\mathrm{Rm}}|^2 = 0$. □

3 Geometric applications

This section contains the proofs of the main theorems.

Proof of Theorems A–C By replacing M with its orientation double cover, if necessary, it may be assumed that M is orientable. Due to Poincaré duality it suffices to consider $p \leq \lfloor \frac{n}{2} \rfloor$. Let ω be a p-form. Lemma 2.2 and Proposition 2.5 imply

$$|L\omega|^2 \leq p|\omega|^2|L|^2 = \frac{1}{n-p}|\hat{\omega}|^2|L|^2$$

for all $L \in \mathfrak{so}(TM)$.

If the eigenvalues of the curvature operator satisfy $\frac{1}{n-p}(\lambda_1+\ldots+\lambda_{n-p}) \geq \kappa$, then Lemma 2.1 yields

$$g(\mathfrak{R}(\hat{w}), \hat{\omega}) \geq \kappa |\hat{\omega}|^2 = \kappa p(n-p)|\omega|^2.$$

An application of the Bochner technique as in Theorem 1.9 to the Hodge Laplacian completes the proof. □

Remark 3.1 Example 4.3(c) constructs an $(n-1)$-positive algebraic curvature operator $\mathfrak{R}\colon \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ and a 2-form ω such that $g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) < 0$.

Proof of Theorem D. Recall from Example 1.7(c) that the curvature tensor of an Einstein manifold is harmonic. Hence it satisfies the Bochner formula

$$\nabla^*\nabla \operatorname{Rm} + \frac{1}{2}\operatorname{Ric}(\operatorname{Rm}) = 0.$$

Moreover, since $\mathring{\operatorname{Ric}} = 0$, Proposition 2.5 shows that $|\widehat{\operatorname{Rm}}|^2 = 4(n-1)|\mathring{\operatorname{Rm}}|^2$ and Lemma 2.2 implies

$$|L \operatorname{Rm}|^2 \leq 8|\mathring{\operatorname{Rm}}|^2|L|^2 = \frac{2}{n-1}|\widehat{\operatorname{Rm}}|^2|L|^2$$

for all $L \in \mathfrak{so}(TM)$.

By assumption the eigenvalues of the curvature operator satisfy $\lambda_1 + \ldots + \lambda_{\lfloor \frac{n-1}{2} \rfloor} \geq 0$ and thus Lemma 2.1 implies

$$g(\mathfrak{R}(\widehat{\operatorname{Rm}}), \widehat{\operatorname{Rm}}) \geq 0.$$

An application of the Bochner technique discussed before Theorem 1.9 shows that Rm is parallel. Moreover, if $\lambda_1+\ldots+\lambda_{\lfloor \frac{n-1}{2} \rfloor} > 0$, then $|\widehat{\operatorname{Rm}}|^2 = 0$ and thus Rm has constant sectional curvature due to Proposition 2.5. □

Remark 3.2 (a) Recall that every irreducible Riemannian manifold with parallel Ricci tensor is Einstein, and hence Theorem D applies.

(b) Recall that in dimension $n = 4$ Theorem D follows from the work of Böhm–Wilking [11] and Ni–Wu [31] assuming $\lambda_1 + \lambda_2 \geq 0$. Using a Singer–Thorpe basis [36] for the curvature operator of orientable Einstein manifolds in dimension $n = 4$, it is possible to compute $g(\mathfrak{R}(\widehat{\operatorname{Rm}}), \widehat{\operatorname{Rm}})$ explicitly and conclude that also $g(\mathfrak{R}(\widehat{\operatorname{Rm}}), \widehat{\operatorname{Rm}}) \geq 0$ provided $\lambda_1+\lambda_2 \geq 0$. In contrast, Example 4.3 (d) exhibits an Einstein, 3-nonnegative algebraic curvature operator $\mathfrak{R}$ with $g(\mathfrak{R}(\widehat{\operatorname{Rm}}), \widehat{\operatorname{Rm}}) < 0$.

4 Examples

Examples 4.1–4.3 (a) show that the estimates in Lemma 2.2 are sharp in the cases of forms and algebraic curvature tensors.

Example 4.3(b) exhibits a 3-positive algebraic curvature operator with negative isotropic curvatures. By considering algebraic curvature operators, the optimality of the eigenvalue assumptions is discussed in examples 4.3(c) for Theorems A–C and in Example 4.3(d) for Theorem D in dimension $n = 4$.

Example 4.1 Consider the 2-forms $\omega_1 = e^1 \wedge e^3 - e^2 \wedge e^4$ and $\omega_2 = e^1 \wedge e^4 + e^2 \wedge e^3$ and the bivector $L = e_1 \wedge e_2 + e_3 \wedge e_4$. It follows that $L\omega_1 = -2\omega_2$ and $L\omega_2 = 2\omega_1$. In particular, $|L\omega_1|^2 = |L\omega_2|^2 = 8$, $|\omega_1|^2 = |\omega_2|^2 = 2$ and $|L|^2 = 2$. Thus the estimate in Lemma 2.2 is optimal for 2-forms.

Example 4.2 For p-forms on $\mathbb{R}^{2p}$ consider $L = e_1 \wedge e_2 + \ldots + e_{2p-1} \wedge e_{2p}$. There are 2^p forms ω of the form $e^I = e^{i_1} \wedge \ldots \wedge e^{i_p}$ with $i_1 < \ldots < i_p$ such that $L\omega$ is a linear combination of exactly p forms e^I. Notice from the proof of Lemma 2.2 that this happens precisely when $i_1 \in \{1, 2\}, \ldots, i_p \in \{2p-1, 2p\}$. The span of these e^I is a subspace which is invariant under L. Furthermore, there is a choice of $\alpha_I, \beta_I \in \{\pm 1\}$ such that $L \sum \alpha_I e^I = \pm p \sum \beta_I e^I$. The signs can be predicted in the following way:

The basis elements will be grouped into $p+1$ groups $B_0, \ldots, B_p$ where B_k consists of $\binom{p}{k}$ basis elements. The coefficients of the basis elements in each group will have the same sign but the coefficients of the basis elements in B_k and B_{k+2} must have opposite signs. Set $B_0 = \{e^1 \wedge e^3 \wedge \ldots \wedge e^{2p-1}\}$. Suppose that $B_0, \ldots, B_k$ have already been constructed. Apply L to the elements in B_k. This produces $\binom{p}{k+1}$ basis elements which have not occurred in $B_0, \ldots, B_k$. These elements form B_{k+1}. Note that, e.g., $B_p = \{e_2 \wedge e_4 \wedge \ldots \wedge e_{2p}\}$. Define

$$\omega_1 = +\sum_{B_0} e^I - \sum_{B_2} e^I + \sum_{B_4} e^I - \ldots,$$
$$\omega_2 = +\sum_{B_1} e^I - \sum_{B_3} e^I + \sum_{B_5} e^I - \ldots.$$

It follows that $L\omega_1 = -p\omega_2$ and $L\omega_2 = p\omega_1$. Notice that $\Omega = \omega_1 \pm \omega_2$ indeed uses 2^p basis elements.

In the case $p = 3$ one obtains

$$\omega_1 = e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 - e^2 \wedge e^3 \wedge e^6 - e^2 \wedge e^4 \wedge e^5,$$
$$\omega_2 = e^1 \wedge e^3 \wedge e^6 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^3 \wedge e^5 - e^2 \wedge e^4 \wedge e^6.$$

For dimensions $n \geq 2p$ notice that $\Lambda^2(\mathbb{R}^{2p})^* \subseteq \Lambda^2(\mathbb{R}^n)^*$. This shows that the estimate of Lemma 2.2 is optimal for p-forms.

Example 4.3 Consider the basis $\Xi_{1,\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$, $\Xi_{2,\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2)$, $\Xi_{3,\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3)$ for $\Lambda^2\mathbb{R}^4$. Note that if $\{i, j, k\} = \{1, 2, 3\}$ and all signs agree, then $|g\left((\Xi_{i,\pm})\Xi_{j,\pm}, \Xi_{k,\pm}\right)| = \sqrt{2}$ and otherwise zero.

The self-adjoint operator $\mathfrak{R}\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ defined by $\mathfrak{R}(\Xi_{i,\pm}) = \lambda_{i,\pm}\Xi_{i,\pm}$ satisfies the first Bianchi identity if and only if $\lambda_{1,+} + \lambda_{2,+} + \lambda_{3,+} = \lambda_{1,-} + \lambda_{2,-} + \lambda_{3,-}$. In this case $\mathfrak{R}$ is Einstein.

Note that $\mathfrak{R}$ can be extended to an algebraic curvature operator $\mathfrak{R}\colon \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ by setting $\mathfrak{R}(e_k \wedge e_l) = \lambda_{kl}\, e_k \wedge e_l$ for $k \in \{1, \ldots, 4\}$, $l \in \{5, \ldots, n\}$ and $k, l \in \{5, \ldots, n\}$ with $k < l$.

(a) Setting $\lambda_{1,+} = -1$, $\lambda_{2,+} = 1$, $\lambda_{3,+} = 3$ and $\lambda_{i,-} = 1$ for $i = 1, 2, 3$ one obtains an Einstein, 2-nonnegative curvature operator which satisfies $|\mathring{R}|^2 = 8$. Moreover, Proposition 1.5 implies that $|\Xi_{2,+}R|^2 = 8|\mathring{R}|^2$ and hence this example achieves equality in the estimate for algebraic curvature tensors in Lemma 2.2.

(b) Let $\varepsilon > 0$. Setting $\lambda_{1,+} = \lambda_{2,+} = -\varepsilon$, $\lambda_{3,+} = 3 + 3\varepsilon$ and $\lambda_{1,-} = \lambda_{2,-} = 1+3\varepsilon$, $\lambda_{3,-} = 1-8\varepsilon$ one obtains an Einstein, 3-positive curvature operator with negative isotropic curvatures. Note that for $\varepsilon = 0$ this is the curvature operator of $\mathbb{C}P^2$. By setting $\lambda_{kl} = 3\varepsilon$ this example extends to dimensions $n > 4$.

(c) Let $\omega = e^1 \wedge e^4 + e^2 \wedge e^3$. It follows that $|\Xi_{1,+}\omega|^2 = |\Xi_{2,+}\omega|^2 = 2|\omega|^2$ and $\Xi_{3,+}\omega = \Xi_{i,-}\omega = 0$ for $i = 1, 2, 3$. Furthermore, Proposition 1.4 yields $g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) = 2(\lambda_{1,+}+\lambda_{2,+})|\omega|^2$. In particular, the above curvature term vanishes on the Kähler form of $\mathbb{C}P^2$ with the Fubini Study metric. For $n > 4$, set $\lambda_{1,+} = \lambda_{2,+} = -(n-3)$, $\lambda_{3,+} = 2n$ and $\lambda_{i,-} = \lambda_{kl} = 2$ and note that $|(e_i \wedge e_j)\omega|^2 = \frac{1}{2}|\omega|^2$ if $i \in \{1, \ldots, 4\}$ and $j \in \{5, \ldots, n\}$ but $|(e_i \wedge e_j)\omega|^2 = 0$ if $i, j \in \{5, \ldots, n\}$. Proposition 1.5 shows that $\mathfrak{R}$ is an $(n-1)$-nonnegative algebraic curvature operator with $g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) = -4|\omega|^2 < 0$. Thus there also is an $(n-1)$-positive algebraic curvature operator $\widetilde{\mathfrak{R}}$ with $g(\widetilde{\mathfrak{R}}(\hat{\omega}), \hat{\omega}) < 0$.

(d) Recall that according to Remark 3.2 every Einstein, 2-nonnegative algebraic curvature operator $\mathfrak{R}\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ satisfies $g(\mathfrak{R}(\hat{R}), \hat{R}) \geq 0$. Setting $\lambda_{1,+} = \lambda_{2,+} = -1$, $\lambda_{3,+} = 8$, and $\lambda_{i,-} = 2$ for $i = 1, 2, 3$ one obtains an Einstein, 3-nonnegative algebraic curvature operator such that $g(\mathfrak{R}(\hat{R}), \hat{R}) < 0$. However, notice that the method used in the proof of Theorem D in section 3 requires $\lambda_1 \geq 0$ in dimension $n = 4$.

Remark 4.4 (a) For $p, q \geq 2$ there are doubly warped product metrics on S^{p+q+1}, $\mathcal{C}^1$-close to the round metric, of the following types:

(1) Metrics that have positive isotropic curvature but do not induce positive isotropic curvature on $S^{p+q+1} \times \mathbb{R}$, and have $(p+1)$-positive curvature operator but do not have p-positive curvature operator.

(2) Metrics with negative isotropic curvatures at some tangent space whose curvature operator is k-positive but not $(k-1)$-positive for $k = 5, \dots, pq + p + q + 1$.

In particular, Brendle's [6] convergence theorem for the Ricci flow, Micallef–Moore's [27] theorem on simply connected manifolds with positive isotropic curvature, and Theorem A indeed make different assumptions on curvature.

(b) Let $\mathfrak{R}\colon \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ be a self-adjoint operator and $2p \leq n$. If $\mathfrak{R}$ is $(n-p)$-positive, then Lemmas 2.1, 2.2 and Proposition 2.5 show that $g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) > 0$ for every non-zero $\omega \in \Lambda^p(\mathbb{R}^n)^*$. In particular, $\mathfrak{R}$ does not need to satisfy the first Bianchi identity.

Given the examples of $\omega \in \Lambda^p(\mathbb{R}^{2p})^*$ and $\Xi \in \mathfrak{so}(\mathbb{R}^{2p})$ with $|\Xi\omega|^2 = p|\omega|^2$ in Example 4.2, for every $n \geq 2p$ it is easy to find a self-adjoint, $(n-p+1)$-positive operator $\mathfrak{R}\colon \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ with $g(\mathfrak{R}(\hat{\omega}), \hat{\omega}) < 0$.

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