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Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$

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Abstract We show that for each $\gamma \in (0, 2)$, there is a unique metric (i.e., distance function) associated with γ-Liouville quantum gravity (LQG). More precisely, we show that for the whole-plane Gaussian free field (GFF) h, there is a unique random metric D_h associated with the Riemannian metric tensor "$e^{\gamma h}(dx^2 + dy^2)$" on $\mathbb{C}$ which is characterized by a certain list of axioms: it is locally determined by h and it transforms appropriately when either adding a continuous function to h or applying a conformal automorphism of $\mathbb{C}$ (i.e., a complex affine transformation). Metrics associated with other variants of the GFF can be constructed using local absolute continuity. The γ-LQG metric can be constructed explicitly as the scaling limit of *Liouville first passage percolation* (LFPP), the random metric obtained by exponentiating a mollified version of the GFF. Earlier work by Ding et al. (Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$, 2019. arXiv:1904.08021) showed that LFPP admits non-trivial subsequential limits. This paper shows that the subsequential limit is unique and satisfies our list of axioms. In the case when $\gamma = \sqrt{8/3}$, our metric coincides with the $\sqrt{8/3}$-LQG metric constructed in previous work by Miller and Sheffield, which in turn is equivalent to the Brownian map for a certain variant of the GFF. For general $\gamma \in (0, 2)$, we conjecture that our metric is the Gromov–Hausdorff limit of appropriate weighted random planar map models, equipped with their graph distance. We include a substantial list of open problems.

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1 Introduction

1.1 Overview

Fix $\gamma \in (0, 2)$, let $U \subset \mathbb{C}$ be an open domain, and let h be the Gaussian free field (GFF) on U, or some minor variant thereof. The *γ-Liouville quantum gravity (LQG)* surface described by (U, h) is formally the random two-dimensional Riemannian manifold with metric tensor

$$e^{\gamma h}\,(dx^2 + dy^2), \tag{1.1}$$

where $dx^2 + dy^2$ is the Euclidean Riemannian metric tensor.

LQG surfaces were first introduced non-rigorously in the physics literature by Polyakov [73,74] as a canonical model of a random Riemannian metric on U. Another motivation to study LQG surfaces is that they describe the scaling limit of random planar maps. The special case when $\gamma = \sqrt{8/3}$ (called "pure gravity") corresponds to uniformly random planar maps, including uniform triangulations, quadrangulations, etc. Other values of γ (sometimes referred to as "gravity coupled to matter") correspond to random planar maps weighted by the partition function of an appropriate statistical mechanics model on the map, for example the uniform spanning tree for $\gamma = \sqrt{2}$ or the Ising model for $\gamma = \sqrt{3}$.

The definition (1.1) of LQG does not make literal sense since h is only a distribution, not a function, so it does not have well-defined pointwise values and cannot be exponentiated. Nevertheless, it is known that one can make sense of the associated volume form $\mu_h = e^{\gamma h(z)}\,dz$ (where dz denotes Lebesgue measure) as a random measure on U via various regularization procedures [27,51,76]. One such regularization procedure is as follows. For $s > 0$ and $z, w \in \mathbb{C}$, let $p_s(z,w) = \frac{1}{2\pi s}\exp\left(-\frac{|z-w|^2}{2s}\right)$ be the heat kernel, and note that $p_s(z,\cdot)$ approximates a point mass at z when s is small. For $\varepsilon > 0$, we define a mollified version of the GFF by

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_U h(w) p_{\varepsilon^2/2}(z,w)\,dw, \quad \forall z \in U, \tag{1.2}$$

where the integral is interpreted in the sense of distributional pairing. One can then define the γ-LQG measure μ_h as the a.s. weak limit [9,27,51,76,79]

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon^*(z)}\,dz. \tag{1.3}$$

By [27, Proposition 2.1], the measure μ_h is conformally covariant: if $\phi : \widetilde{U} \to U$ is a conformal map and we set

$$\widetilde{h} := h\circ\phi + Q\log|\phi'|, \quad \text{where} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}, \tag{1.4}$$

then a.s. $\mu_h(\phi(A)) = \mu_{\widetilde{h}}(A)$ for each Borel set $A \subset \mathbb{C}$. This leads one to define a *γ-LQG surface* as an equivalence class of pairs (U, h), with two such pairs (U, h) and $(\widetilde{U}, \widetilde{h})$ declared to be equivalent if there is a conformal map

$\phi : \widetilde{U} \to U$ for which h and $\widetilde{h}$ are related as in (1.4). We think of two equivalent pairs as representing different parameterizations of the same random surface. The conformal covariance property of μ_h says that this measure is intrinsic to the quantum surface—it does not depend on the particular equivalence class representative.

In order for γ-LQG to be a reasonable model of a "random two-dimensional Riemannian manifold", one also needs a random metric[1] (distance function) D_h on U which is in some sense obtained by exponentiating h and which satisfies a conformal covariance property analogous to that of the γ-LQG area measure. Moreover, this metric should be the scaling limit of the graph distance on random planar maps with respect to the Gromov–Hausdorff topology. Constructing a metric on γ-LQG is a much more difficult problem than constructing the measure μ_h. Indeed, any natural regularization scheme for LQG distances involves minimizing over a large collection of paths, which results in a substantial degree of non-linearity.

Prior to this work, a γ-LQG metric has only been constructed in the special case when $\gamma = \sqrt{8/3}$ in a series of works by Miller and Sheffield [64,65,72]. In this case, for certain special choices of the pair (U, h), the random metric space (U, D_h) agrees in law with a *Brownian surface*, such as the Brownian map [57,59] or the Brownian disk [10]. These Brownian surfaces are continuum random metric spaces which arise as the scaling limits of uniform random planar maps with respect to the Gromov–Hausdorff topology. Miller and Sheffield's construction of the $\sqrt{8/3}$-LQG metric does not use a direct regularization of the field h. Instead, they first construct a candidate for $\sqrt{8/3}$-LQG metric balls using a process called *quantum Loewner evolution*, which is built out of the Schramm-Loewner evolution with parameter $\kappa = 6$ (SLE_6), then show that there is a metric which corresponds to these balls.

In this paper, we will construct a γ-LQG metric for all $\gamma \in (0, 2)$ via an explicit regularization procedure analogous to (1.3). We will also show that this metric is uniquely characterized by a list of natural properties that any reasonable notion of a metric on γ-LQG should satisfy, so is in some sense the only "correct" metric on γ-LQG. For simplicity, we will mostly restrict attention to the whole-plane case, but metrics associated with GFF's on other domains can be easily constructed via restriction and/or absolute continuity (see Remark 1.5). In contrast to [64,65,72], the present work will make no use of SLE. Furthermore, we do not a priori have an ambient metric space to compare to (such as the Brownian map in the case $\gamma = \sqrt{8/3}$) and we do not have any sort of exact solvability, i.e., we do not know the exact laws of any observables related to the metric.

[1] Throughout this paper, the term "metric" will be used to mean a distance function, rather than a metric tensor. We will not prove anything rigorous about metric tensors. The metric tensor (1.1) is introduced only for context.

We now describe how our metric is constructed. It is shown in [20], building on [30,33], that for each $\gamma \in (0, 2)$, there is an exponent $d_\gamma > 2$ which describes distances in various discrete approximations of γ-LQG. *A posteriori*, once the γ-LQG metric is constructed, one can show that d_γ is its Hausdorff dimension [43]. The value of d_γ is not known explicitly except in the case when $\gamma = \sqrt{8/3}$, in which case we know that $d_{\sqrt{8/3}} = 4$ (see Problem 7.1). We refer to [20,42] for bounds for d_γ and some speculation about its possible value. For $\gamma \in (0, 2)$, we define

$$\xi = \xi_\gamma := \frac{\gamma}{d_\gamma}. \tag{1.5}$$

We say that a random distribution h on $\mathbb{C}$ is a *whole-plane GFF plus a continuous function* if there exists a coupling of h with a random continuous function $f : \mathbb{C} \to \mathbb{R}$ such that the law of $h - f$ is that of a whole-plane GFF. We similarly define a *whole-plane GFF plus a bounded continuous function*, except we require that f is bounded.[2] Note that the whole-plane GFF is defined only modulo a global additive constant, but these definitions do not depend on the choice of additive constant. By definition, a whole-plane GFF plus a continuous function is well-defined as a distribution, not just modulo additive constant. For example, a whole-plane GFF with a particular choice of additive constant can be viewed as a whole-plane GFF plus a continuous function.

If h is a whole-plane GFF plus a bounded continuous function, we define $h_\varepsilon^*(z)$ for $\varepsilon > 0$ and $z \in \mathbb{C}$ as in (1.2) for our given choice of h. For $z, w \in \mathbb{C}$ and $\varepsilon > 0$, we define the *ε-LFPP metric* by

$$D_h^\varepsilon(z, w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt \tag{1.6}$$

where the infimum is over all piecewise continuously differentiable paths from z to w. One should think of LFPP as the metric analog of the approximations of the LQG measure in (1.3).[3] The intuitive reason why we look at $e^{\xi h_\varepsilon^*(z)}$ instead of $e^{\gamma h_\varepsilon^*(z)}$ to define the metric is as follows. By (1.3), we can scale LQG areas by a factor of $C > 0$ by adding $\gamma^{-1} \log C$ to the field. By (1.6), this results in scaling distances by $C^{\xi/\gamma} = C^{1/d_\gamma}$, which is consistent with the

[2] The reason why we sometimes restrict to bounded continuous functions is to ensure that the convolution with the whole-plane heat kernel is finite (so D_h^ε is defined) and that the results about subsequential limits of LFPP in [16,18] are applicable.

[3] One can also consider other variants of LFPP, defined using different approximations of the GFF, but we consider h_ε^* here since this is the approximation for which tightness is proven in [16]. If we knew tightness and some basic properties of the subsequential limiting metrics for LFPP defined using a different approximation of the GFF, then Theorem 1.8 below would show that these variants of LFPP also converge to the γ-LQG metric.

fact that the "dimension" should be the exponent relating the scaling of areas and distances.

Let $\mathfrak{a}_\varepsilon$ be the median of the D_h^ε-distance between the left and right boundaries of the unit square in the case when h is a whole-plane GFF normalized so that its circle average[4] over $\partial\mathbb{D}$ is zero. We do not know the value of $\mathfrak{a}_\varepsilon$ explicitly, but see Corollary 1.11. It was shown by Ding, Dubédat, Dunlap, and Falconet [16] that the laws of the metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight w.r.t. the local uniform topology on $\mathbb{C} \times \mathbb{C}$, and every possible subsequential limit induces the Euclidean topology on $\mathbb{C}$ (see also the earlier tightness results for small $\gamma > 0$ [15,17] and for Liouville graph distance, a related model, for all $\gamma \in (0,2)$ [14]). Subsequently, it was shown by Dubédat, Falconet, Gwynne, Pfeffer, and Sun [18], using [38, Corollary 1.8] (a general criterion for a local metric to be determined by the GFF), that every subsequential limit can be realized as a measurable function of h, so in fact the metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ admit subsequential limits in probability. One of the main results of this paper gives the uniqueness of this subsequential limit.

Theorem 1.1 (Convergence of LFPP) *The random metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability w.r.t. the local uniform topology on $\mathbb{C} \times \mathbb{C}$ to a random metric on $\mathbb{C}$ which is a.s. determined by h.*

It is natural to define the limiting metric from Theorem 1.1 to be the γ-LQG metric associated with h. However, this definition is not entirely satisfactory since it is a priori possible that there are other natural ways to construct a metric on γ-LQG which do not yield the same result as the one in Theorem 1.1. For example, Theorem 1.1 does not yet tell us that the limit of LFPP coincides with the metric of [64,65,72] in the case when $\gamma = \sqrt{8/3}$.

We will therefore define a γ-LQG metric in terms of a list of axioms (see Sect. 1.2 just below). We will show that (a) the metric of Theorem 1.1 satisfies these axioms and (b) there is at most one metric satisfying these axioms for each $\gamma \in (0,2)$. Taken together, these statements tell us that the metric of Theorem 1.1 is the *only* reasonable metric that one can put on γ-LQG.

An important feature of our proofs is that they can be read with essentially no knowledge of the (substantial) existing literature on LQG. Aside from basic properties of the GFF (as discussed, e.g., in [80] and the introductory sections of [66,70,83]), the only prior works which this paper relies on are [16,18,36,38]. All of the results which we need from these papers are reviewed in Sect. 2.

Our results open up many important new research directions in the theory of LQG. We have included in Sect. 7 a substantial list of open problems related to the γ-LQG metric.

[4] See [27, Section 3.1] for the basic properties of the circle average process. Even though we define LFPP using truncation with the heat kernel, we will always fix the additive constant for the whole-plane GFF using the circle average.

1.2 Axiomatic characterization of the γ-LQG metric

To state our list of axioms precisely, we will need some preliminary definitions concerning metric spaces. In what follows, we let $(X, \mathfrak{d})$ be a metric space.

For $A, B \subset X$, we define

$$\mathfrak{d}(A, B) := \inf_{x\in A, y\in B} \mathfrak{d}(x, y).$$

A *curve* in X is a continuous function $P : [a, b] \to X$. For a curve P, the $\mathfrak{d}$-*length* of P is defined by

$$\operatorname{len}(P; \mathfrak{d}) := \sup_T \sum_{i=1}^{\#T} \mathfrak{d}(P(t_i), P(t_{i-1}))$$

where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$. Note that the $\mathfrak{d}$-length of a curve may be infinite.

For $Y \subset X$, the *internal metric of* $\mathfrak{d}$ *on* Y is defined by

$$\mathfrak{d}(x, y; Y) := \inf_{P\subset Y} \operatorname{len}(P; \mathfrak{d}), \quad \forall x, y \in Y \tag{1.7}$$

where the infimum is over all paths P in Y from x to y. Then $\mathfrak{d}(\cdot, \cdot; Y)$ is a metric on Y, except that it is allowed to take infinite values.

We say that $(X, \mathfrak{d})$ is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of $\mathfrak{d}$-length at most $\mathfrak{d}(x, y) + \varepsilon$ from x to y.

A *continuous metric* on an open domain $U \subset \mathbb{C}$ is a metric $\mathfrak{d}$ on U which induces the Euclidean topology on U, i.e., the identity map $(U, |\cdot|) \to (U, \mathfrak{d})$ is a homeomorphism. We equip the space of continuous metrics on U with the local uniform topology for functions from $U \times U$ to $[0, \infty)$ and the associated Borel σ-algebra. We allow a continuous metric to satisfy $\mathfrak{d}(u, v) = \infty$ if u and v are in different connected components of U. In this case, in order to have $\mathfrak{d}^n \to \mathfrak{d}$ w.r.t. the local uniform topology we require that for large enough n, $\mathfrak{d}^n(u, v) = \infty$ if and only if $\mathfrak{d}(u, v) = \infty$.

Let $\mathcal{D}'(\mathbb{C})$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For $\gamma \in (0, 2)$, a *(strong)* γ*-Liouville quantum gravity (LQG) metric* is a measurable function $h \mapsto D_h$ from $\mathcal{D}'(\mathbb{C})$ to the space of continuous metrics on $\mathbb{C}$ such that the following is true whenever h is a whole-plane GFF plus a continuous function.

I. **Length space** Almost surely, $(\mathbb{C}, D_h)$ is a length space, i.e., the D_h-distance between any two points of $\mathbb{C}$ is the infimum of the D_h-lengths of D_h-continuous paths (equivalently, Euclidean continuous paths) between the two points.

II. **Locality** Let $U \subset \mathbb{C}$ be a deterministic open set. The internal metric $D_h(\cdot, \cdot; U)$ is a.s. determined by $h|_U$.

III. **Weyl scaling** Let ξ be as in (1.5) and for each continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z, w) := \inf_{P: z \to w} \int_0^{\mathrm{len}(P; D_h)} e^{\xi f(P(t))} \, dt, \quad \forall z, w \in \mathbb{C}, \quad (1.8)$$

where the infimum is over all continuous paths from z to w parameterized by D_h-length. Then a.s. $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

IV. **Coordinate change for translation and scaling** For each fixed deterministic $r > 0$ and $z \in \mathbb{C}$, a.s.

$$D_h(ru + z, rv + z) = D_{h(r\cdot+z)+Q\log r}(u, v), \ \forall u, v \in \mathbb{C}$$
$$\text{where} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \quad (1.9)$$

Let us briefly discuss why the above axioms are natural. Recall that γ-LQG should be the random Riemannian metric with metric tensor $e^{\gamma h}(dx^2 + dy^2)$. Axiom I is simply the LQG analog of the statement that for a true Riemannian metric, the distance between two points can be defined as the infima of the lengths of paths connecting them. In a similar vein, Axiom II corresponds to the fact that for a smooth Riemannian metric, the lengths of paths are determined locally by the Riemannian metric tensor. Axiom III is just expressing the fact that the metric is obtained by exponentiating ξh, so adding a continuous function f to h results in re-scaling the metric length measure on paths by $e^{\xi f}$.

Axiom IV is the metric analog of the conformal coordinate change formula (1.4) for the γ-LQG area measure, but restricted to translations and scalings. This axiom together with Corollary 1.3 says that D_h depends only on the LQG surface $(\mathbb{C}, h)$, not on the particular choice of parameterization. We will prove a conformal covariance property for the γ-LQG metric w.r.t. conformal automorphisms between arbitrary domains, directly analogous to the conformal covariance of the γ-LQG area measure, in [37].

Theorem 1.2 (Existence and uniqueness of the LQG metric) *Fix $\gamma \in (0, 2)$. There is a γ-LQG metric D such that the limiting metric of Theorem 1.1 is a.s. equal to D_h whenever h is a whole-plane GFF plus a bounded continuous function. Furthermore, the γ-LQG metric is unique in the following sense. If D and $\widetilde{D}$ are two γ-LQG metrics, then there is a deterministic constant $C > 0$ such that if h is a whole-plane GFF plus a continuous function, then a.s. $D_h = C\widetilde{D}_h$.*

Theorem 1.2 justifies us in referring to *the* γ-LQG metric. Technically speaking there is a one-parameter family of such metrics, which differ by a global deterministic multiplicative constant. But, one can fix the constant in various ways to get a single canonically defined metric. For example, we can require that the median distance between the left and right boundaries of the unit square is 1 for the metric associated with a whole-plane GFF normalized so that its circle average over $\partial\mathbb{D}$ is zero (the limiting metric in Theorem 1.1 has this normalization).

Theorem 1.2 is related to Shamov's axiomatic characterization of Gaussian multiplicative chaos (GMC) measures, such as the γ-LQG measure [79, Corollary 5]. Shamov's result says that a subcritical GMC measure associated with a field X is uniquely characterized by how it transforms when we add to X a function in the Cameron–Martin space. Weyl scaling (Axiom III) is the metric analog of this property. Unlike in Shamov's characterization we need other properties besides just Weyl scaling to characterize the LQG metric, most notably some sort of uniform control of the metric at different Euclidean scales (in the above list of axioms this is provided by Axiom IV, but this axiom can be weakened, see Sect. 1.4).

In Axiom IV in the definition of a strong γ-LQG metric, we did not require that the metric is invariant under rotations of $\mathbb{C}$. It turns out that rotational invariance is implied by the other axioms. See Remark 1.6 below for an intuitive explanation of why this is the case.

Corollary 1.3 (Rotational invariance) *If $\gamma \in (0,2)$ and D is a γ-LQG metric then D is rotationally invariant, i.e., if $\omega \in \mathbb{C}$ with $|\omega| = 1$ and h is a whole-plane GFF plus a continuous function, then a.s. $D_h(u,v) = D_{h(\omega\cdot)}(\omega^{-1}u, \omega^{-1}v)$ for all $u, v \in \mathbb{C}$.*

Proof Define $D_h^{(\omega)}(u,v) := D_{h(\omega\cdot)}(\omega^{-1}u, \omega^{-1}v)$. It is easily verified that $D^{(\omega)}$ is a strong LQG metric, so Theorem 1.2 implies that there is a deterministic constant $C > 0$ such that a.s. $D_h^{(\omega)} = CD_h$ whenever h is a whole-plane GFF plus a continuous function. To check that $C = 1$, consider a whole-plane GFF h normalized so that its circle average over $\partial\mathbb{D}$ is 0. Then the law of h is rotationally invariant, so $\mathbb{P}[D_h(0,\partial\mathbb{D}) > R] = \mathbb{P}[D_h^{(\omega)}(0,\partial\mathbb{D}) > R]$ for every $R > 0$. Therefore $C = 1$. □

It is easy to check that the metric constructed in [64,65,72] satisfies the axioms for a $\sqrt{8/3}$-LQG metric; see [41, Section 2.5] for a careful explanation of why this is the case. Consequently, Theorem 1.2 implies the following.

Corollary 1.4 (Equivalence with the construction of [64,65,72]) *The $\sqrt{8/3}$-LQG metric constructed in [64,65,72] agrees with the limiting metric of Theorem 1.1 (equivalently, the metric of Theorem 1.2) up to a deterministic global scaling factor.*

The present work does not use the results of [64,65,72], but also does not supersede these results. Indeed, without these works it is not at all clear how to link the $\sqrt{8/3}$-LQG metric constructed in the present article to Brownian surfaces, and thereby to uniform random planar maps.

There are a number of properties of the γ-LQG metric which are already known. It is shown in [18, Section 3.1] that one has superpolynomial concentration for the D_h-distance between two disjoint compact, connected sets which are not singletons (e.g., the inner and outer boundaries of an annulus or two opposite sides of a rectangle). Building on this, [18] computes the optimal Hölder exponents between D_h and the Euclidean metric, in both directions, and establishes moment bounds for various distance quantities (see also Sect. 2.4). Confluence properties for D_h-geodesics analogous to the ones known for the Brownian map [56] are proven in [36] (see also Sect. 2.5). It is shown in [62] that D_h-geodesics are conformally removable and their laws are mutually singular with respect to Schramm-Loewner evolution curves. After the appearance of this paper, the work [43] proved that D_h satisfies a version of the KPZ formula [27,53] and the work [1] proved a concentration result for the LQG mass of a D_h-metric ball.

Remark 1.5 (Metrics associated with other fields) Theorem 1.2 gives us a canonical γ-LQG metric associated with a whole-plane GFF plus a continuous function. It is not hard to see that one can also define the metric if h is equal to a whole-plane GFF plus a continuous function plus a finite number of logarithmic singularities of the form $-\alpha \log|\cdot - z|$ for $z \in \mathbb{C}$ and $\alpha < Q$; see [18, Theorem 1.10 and Proposition 3.17].

We can also define metrics associated with GFF's on proper sub-domains of $\mathbb{C}$. To this end, let $U \subset \mathbb{C}$ be open and let h be a whole-plane GFF. Due to Axiom II, we can define for each open set $U \subset \mathbb{C}$ the metric $D_{h|_U} := D_h(\cdot, \cdot; U)$ as a measurable function of $h|_U$. We can write $h|_U = \mathring{h}^U + \mathfrak{h}^U$, where $\mathring{h}^U$ is a zero-boundary GFF on U and $\mathfrak{h}^U$ is a random harmonic function on U independent from $\mathring{h}^U$. In the notation (1.8), we define

$$D_{\mathring{h}^U} := e^{-\xi \mathfrak{h}^U} \cdot D_{h|_U}. \tag{1.10}$$

Note that this is well-defined even though $\mathfrak{h}^U$ does not extend continuously to ∂U, since the definition of $D_{h|_U}$ involves only paths contained in U. It is easily seen from Axioms II (locality) and III (Weyl scaling) that $D_{\mathring{h}^U}$ is a measurable function of $\mathring{h}^U$: indeed, if we are given an open set $V \subset U$ with $\overline{V} \subset U$, choose a smooth compactly supported bump $f : U \to [0, 1]$ which is identically equal to 1 on V. Then Axiom II applied to the field $h - f\mathfrak{h}^U$ implies that the internal metric of $D_{\mathring{h}^U}$ on V, which equals $D_{h - f\mathfrak{h}^U}(\cdot, \cdot; V)$, is determined by $(h - f\mathfrak{h}^U)|_V = \mathring{h}^U|_V$. Letting V increase to all of U gives the desired measurability of $D_{\mathring{h}^U}$ w.r.t. $\mathring{h}^U$. This defines the γ-LQG metric for a zero-boundary GFF.

By Axiom III, we can also define the metric $D_{\widetilde{h}}$ in the case when $\widetilde{h} = \mathring{h}^U + f$ is a zero-boundary GFF plus a continuous function on U, namely $D_{\widetilde{h}} := e^{\xi f} D_{\mathring{h}^U}$. It is shown in [37] that this metric satisfies a conformal coordinate change relation analogous to the one satisfied by the γ-LQG measure (as discussed just below (1.4)).

We expect that for a fixed proper subdomain $U \subset \mathbb{C}$ there is an analogous formulation and characterization of the LQG metric on U. However, we will not formulate such a result here. We emphasize that the LQG metric on U is determined by the LQG metric on $\mathbb{C}$, and moreover the LQG metric on U determines the LQG metric on $\mathbb{C}$ due to Axiom II (locality) and the local absolute continuity between GFF's on different domains. It is not hard to show using the results of [16] that for, say, a zero-boundary GFF $\mathring{h}^U$ on U, the metric $D_{\mathring{h}^U}$ is the limit in law of LFPP on U w.r.t. the topology of uniform convergence on compact subsets of $U \times U$: see, e.g., the arguments of [18, Section 2.2].

Remark 1.6 (Why rotational invariance is unnecessary) At a first glance, it may seem surprising that one does not need rotational invariance to uniquely characterize the LQG metric in Theorem 1.2. Indeed, one can define variants of LFPP which are not rotationally invariant by working with a stretched version of the Euclidean metric. For example, for a given $A > 1$ one can replace (1.6) by

$$D_{h,A}^{\varepsilon}(z, w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_{\varepsilon}^*(P(t))} \sqrt{P_1'(t)^2 + A P_2'(t)^2} \, dt \tag{1.11}$$

where the infimum is over all piecewise continuously differentiable paths $P = (P_1, P_2)$ from z to w. The arguments of this paper and its predecessors apply verbatim with $D_{h,A}^{\varepsilon}$ in place of D_h^{ε}. In particular, $D_{h,A}^{\varepsilon}$ converges in probability to (a deterministic constant times) the γ-LQG metric and hence satisfies the rotational invariance property of Corollary 1.3. This is despite the fact that the metrics (1.11) do *not* satisfy this rotational invariance property.

Here is an intuitive explanation for this phenomenon. First, we note that $D_{h,A}^{\varepsilon}$ is bi-Lipschitz equivalent with respect to $D_{h,1}^{\varepsilon} = D_h^{\varepsilon}$ for each $\varepsilon > 0$, with a deterministic bi-Lipschitz constants. Therefore in a subsequential limit as $\varepsilon \to 0$, we obtain two metrics $D_{h,A}$ and $D_h = D_{h,1}$ which are bi-Lipschitz equivalent with deterministic bi-Lipschitz constants. Suppose that P is a D_h-geodesic connecting z and w. Using the confluence of geodesics results from [36], one can show that (very roughly speaking) for distinct times $s, t \in [0, D_h(z, w)]$, the restrictions of h to small neighborhoods of $P(s)$ and $P(t)$ are approximately independent; see the outline of Sect. 4 in Sect. 1.5 below for details. Moreover, since P is a fractal type curve, it has no local notion of direction, so one expects that the law of h restricted to a small neighborhood of $P(t)$ does not depend very strongly on t or on the endpoints z, w of P. If we fix $n \in \mathbb{N}$ and let $0 = t_0 < \cdots t_n = D_h(z, w)$ be equally spaced times, we can approximate the $D_{h,A}$-length of P by

$$\sum_{j=1}^{n} D_{h,A}(P(t_{j-1}), P(t_j)).$$

The above considerations suggest that each of the random variables $D_{h,A}(P(t_{j-1}), P(t_j))$ has approximately the same distribution and is bounded above and below by deterministic constants times $t_j - t_{j-1}$. From law of large numbers type considerations, it follows that the $D_{h,A}$-length of P is a deterministic constant times the D_h-length of P, where the constant does not depend on the endpoints of P.

Knowing that the $D_{h,A}$-length of every D_h geodesic is a constant times its D_h-length (and vice-versa) does not immediately imply that D_h is equal to a constant times $D_{h,A}$. This is because if P_n is a sequence of paths which converge uniformly to P, then it is not necessarily true that $\mathrm{len}(P_n; D_{h,A})$ converges to $\mathrm{len}(P; D_{h,A})$. For this and other reasons, we will argue in a somewhat different manner than we have indicated above, though our arguments will still be based on the bi-Lipschitz equivalence of metrics and approximate independence statements for the local behavior of a geodesic at different times. We will explain the general strategy in Sect. 1.5 in more detail.

1.3 Conjectured random planar map connection

As noted above, the γ-LQG metric should describe the large scale behavior of the graph metric for random planar maps. Since our γ-LQG metric is in some sense canonical, it is natural to make the following conjecture.

Conjecture 1.7 *For each $\gamma \in (0, 2)$, random planar maps in the γ-LQG universality class, equipped with their graph distance, converge in the scaling limit with respect to the Gromov–Hausdorff topology to γ-LQG surfaces equipped with the γ-LQG metric constructed in Theorem 1.1 (see also Remark 1.5).*

Examples of planar map models to which Conjecture 1.7 should apply include random planar maps weighted by the number of spanning trees ($\gamma = \sqrt{2}$), the Ising model partition function ($\gamma = \sqrt{3}$), the number of bipolar orientations ($\gamma = \sqrt{4/3}$; [52]), or the Fortuin-Kasteleyn model partition function ($\gamma \in (\sqrt{2}, 2)$; [82]). Another class of models is the so-called *mated-CRT maps*, which are defined for all $\gamma \in (0, 2)$; see [23,33,40].

For $\gamma = \sqrt{8/3}$, Conjecture 1.7 has already been proven for many different uniform-type random planar maps. The reason for this is that we know that our

$\sqrt{8/3}$-LQG metric is equivalent to the metric of [64,65,72] (Corollary 1.4); which in turn is equivalent to a Brownian surface, such as the Brownian map, for certain special $\sqrt{8/3}$-LQG surfaces [64, Corollary 1.5]; which in turn is the scaling limit of uniform random planar maps of various types [57,59].

Conjecture 1.7 has not been proven for any random planar map model for $\gamma \neq \sqrt{8/3}$. However, we already have a relationship between the continuum LQG metric and graph distances in random planar maps at the level of exponents for all $\gamma \in (0, 2)$. Indeed, the quantity d_γ appearing in (1.5) describes several exponents associated with random planar maps, such as the ball volume exponent [20,33] and the displacement exponent for simple random walk on the map [31,35]. It is proven in [43] that d_γ is the Hausdorff dimension of D_h.

Conjecture 1.7 can be made somewhat more precise by specifying exactly what type of γ-LQG surface should arise in the scaling limit. For random planar maps with the topology of the sphere (resp. disk, plane, half-plane) this surface should be the quantum sphere (resp. quantum disk, γ-quantum cone, γ-quantum wedge). See [23] for precise definitions of these quantum surfaces. Equivalent definitions of the quantum sphere and quantum disk, respectively, can be found in [22,50] (see [2,12] for a proof of the equivalence). Some planar map models have been proven to converge to these quantum surfaces, for general $\gamma \in (0, 2)$, with respect to topologies which do not encode the metric structure explicitly. Examples of such topologies include convergence in the so-called *peanosphere sense* [23,82] and convergence of the counting measure on vertices to the γ-LQG measure when the planar map is embedded appropriately into the plane [40].

1.4 Weak LQG metrics and a stronger uniqueness statement

We will prove Theorem 1.1 and 1.2 simultaneously by establishing a uniqueness statement for metrics under a weaker list of axioms, which are satisfied for both the strong LQG metrics considered in Sect. 1.2 and for subsequential limits of LFPP (as is shown in [16,18]).

Let $\mathcal{D}'(\mathbb{C})$ be the space of distributions as in Sect. 1.2. A *weak γ-LQG metric* is a measurable function $h \mapsto D_h$ from $\mathcal{D}'(\mathbb{C})$ to the space of continuous metrics on $\mathbb{C}$ such that the following is true whenever h is a whole-plane GFF plus a continuous function.

I. **Length space** Almost surely, $(\mathbb{C}, D_h)$ is a length space, i.e., the D_h-distance between any two points of $\mathbb{C}$ is the infimum of the D_h-lengths of D_h-continuous paths (equivalently, Euclidean continuous paths) between the two points.
II. **Locality** Let $U \subset \mathbb{C}$ be a deterministic open set. The internal metric $D_h(\cdot, \cdot; U)$ is a.s. determined by $h|_U$.

III. **Weyl scaling** If we define $e^{\xi f} \cdot D_h$ as in (1.8), then a.s. $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.
IV. **Translation invariance** For each fixed deterministic $z \in \mathbb{C}$, a.s. $D_{h(\cdot+z)} = D_h(\cdot + z, \cdot + z)$.
V. **Tightness across scales** Suppose h is a whole-plane GFF and for $z \in \mathbb{C}$ and $r > 0$ let $h_r(z)$ be the average of h over the circle $\partial B_r(z)$. For each $r > 0$, there is a deterministic constant $\mathfrak{c}_r > 0$ such that the set of laws of the metrics $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot)$ for $r > 0$ is tight (w.r.t. the local uniform topology). Furthermore, the closure of this set of laws w.r.t. the Prokhorov topology is contained in the set of laws on continuous metrics on $\mathbb{C}$ (i.e., every subsequential limit of the laws of the metrics $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot)$ is supported on metrics which induce the Euclidean topology on $\mathbb{C}$). Finally, there exists $\Lambda > 1$ such that for each $\delta \in (0, 1)$,

$$\Lambda^{-1} \delta^{\Lambda} \le \frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} \le \Lambda \delta^{-\Lambda}, \quad \forall r > 0. \tag{1.12}$$

Axioms I through III for a weak LQG metric are identical to the corresponding axioms for a strong LQG metric. Axiom IV for a weak LQG metric is equivalent to Axiom IV (coordinate change) for a strong LQG metric with $r = 1$. Axiom V for a weak γ-LQG metric is a substitute for the exact scale invariance property given by Axiom IV for a strong LQG metric. This axiom implies the tightness of various functionals of D_h. For example, if $U \subset \mathbb{C}$ is open and $K \subset U$ is compact, then the laws of

$$\left(\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(rK, r\partial U) \right)^{-1} \quad \text{and} \quad \mathfrak{c}_r^{-1} e^{-\xi h_r(0)} \sup_{u,v \in rK} D_h(u, v; rU) \tag{1.13}$$

as r varies are tight. It is shown in [18, Theorem 1.5] that for any weak γ-LQG metric, one in fact has the following stronger version of (1.12):

$$\frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} = \delta^{\xi Q + o_\delta(1)}, \quad \text{uniformly over all } r > 0. \tag{1.14}$$

By the scale invariance of the law of the whole-plane GFF, modulo additive constant, Axiom IV for a strong LQG metric immediately implies Axiom V for a weak LQG metric with $\mathfrak{c}_r = r^{\xi Q}$, for Q as in (1.4). Indeed, using Axiom IV and then Axiom III for a strong γ-LQG metric shows that

$$r^{-\xi Q} e^{-\xi h_r(0)} D_h(r\cdot, r\cdot) = r^{-\xi Q} e^{-\xi h_r(0)} D_{h(r\cdot)+Q\log r} = D_{h(r\cdot)-h_r(0)} \overset{d}{=} D_h. \tag{1.15}$$

Hence every strong γ-LQG metric is a weak γ-LQG metric.

It is shown in [18, Theorem 1.2] that every subsequential limit in probability of the LFPP metrics D_h^ε of (1.6) is of the form D_h where D is a weak γ-LQG metric. Consequently, the following theorem contains both Theorem 1.1 and Theorem 1.2.

Theorem 1.8 (Strong uniqueness of weak LQG metrics) *Let $\gamma \in (0,2)$. Every weak γ-LQG metric is a strong γ-LQG metric. In particular, by Theorem 1.2, such a metric exists for each $\gamma \in (0,2)$ and if D and $\widetilde{D}$ are two weak γ-LQG metrics, then there is a deterministic constant $C > 0$ such that if h is a whole-plane GFF plus a continuous function, then a.s. $D_h = C\widetilde{D}_h$.*

It turns out that all of our main results are easy consequences of the following statement, which superficially seems to be weaker that Theorem 1.8.

Theorem 1.9 (Weak uniqueness of weak LQG metrics) *Let $\gamma \in (0,2)$ and let D and $\widetilde{D}$ be two weak γ-LQG metrics which have the* same *values of $\mathfrak{c}_r$ in Axiom V. There is a deterministic constant $C > 0$ such that if h is a whole-plane GFF plus a continuous function, then a.s. $D_h = C\widetilde{D}_h$.*

Most of the paper is devoted to the proof of Theorem 1.9. Let us now explain how Theorem 1.9 implies the other main theorems stated above. We first establish the first statement of Theorem 1.8.

Lemma 1.10 *Every weak γ-LQG metric is a strong γ-LQG metric.*

Proof of Lemma 1.10 assuming Theorem 1.9 Suppose that D is a weak γ-LQG metric. For $b > 0$, we define

$$D_h^{(b)}(\cdot,\cdot) := D_{h(\cdot/b)}(b\cdot, b\cdot). \tag{1.16}$$

We claim that $D^{(b)}$ is a weak γ-LQG metric with the same scaling constants $\mathfrak{c}_r$ as D. It is easily verified that $D^{(b)}$ satisfies Axioms I through IV in the definition of a weak γ-LQG metric. To check Axiom V (tightness across scales), we compute for $r > 0$:

$$\begin{aligned}\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h^{(b)}(r\cdot, r\cdot) &= \mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_{h(\cdot/b)}(br\cdot, br\cdot)\\ &= \left(\frac{\mathfrak{c}_{br}}{\mathfrak{c}_r} e^{-\xi(h_r(0)-h_{br}(0))}\right) \mathfrak{c}_{br}^{-1} e^{-\xi h_{br}(0)} D_{h(\cdot/b)}(br\cdot, br\cdot).\end{aligned}$$

In the case when h is a whole-plane GFF, the random variable $h_r(0) - h_{br}(0)$ is centered Gaussian with variance $\log b^{-1}$ [27, Section 3.1]. By (1.12), $\mathfrak{c}_{br}/\mathfrak{c}_r$ is bounded above by a constant depending only on b (not on r). Axiom V (tightness across scales) for D applied with $h(\cdot/b)$ in place of h and br in place of r therefore implies that the laws of the metrics $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h^{(b)}(r\cdot, r\cdot)$ are tight in the case when h is a whole-plane GFF, and that every subsequential limit of the laws of these metrics is supported on metrics (not pseudometrics).

Hence we can apply Theorem 1.9 with $\widetilde{D} = D^{(b)}$ to get that for each $b > 0$, there is a deterministic constant $\mathfrak{k}_b > 0$ such that whenever h is a whole-plane GFF plus a continuous function, a.s. $D_h^{(b)} = \mathfrak{k}_b D_h$. We now argue that $\mathfrak{k}_b$ is a power of b.

For $b_1, b_2 > 0$, we have $D^{(b_1 b_2)} = (D^{(b_1)})^{(b_2)}$, which implies that a.s. $D_h^{(b_1 b_2)} = \mathfrak{k}_{b_2} D_h^{(b_1)} = \mathfrak{k}_{b_1}\mathfrak{k}_{b_2} D_h$. Therefore,

$$\mathfrak{k}_{b_1 b_2} = \mathfrak{k}_{b_1}\mathfrak{k}_{b_2}. \tag{1.17}$$

It is also easy to see that $\mathfrak{k}_b$ depends continuously on b. Indeed, by Axiom III (Weyl scaling) and since $h(\cdot/b) - h_{1/b}(0) \overset{d}{=} h$, we have $e^{-\xi h_{1/b}(0)} D_h^{(b)}(\cdot/b, \cdot/b) \overset{d}{=} D_h$. By the continuity of $(z,w) \mapsto D_h(z,w)$ and $r \mapsto h_r(0)$, it follows that $D_h^{(b)} \to D_h$ in law as $b \to 1$. This gives the continuity of $b \mapsto \mathfrak{k}_b$ at $b = 1$. Using (1.17) then gives the desired continuity in general.

The relation (1.17) and the continuity of $b \mapsto \mathfrak{k}_b$ (actually, just Lebesgue measurability is enough) imply that $\mathfrak{k}_b = b^\alpha$ for some $\alpha \in \mathbb{R}$. Equivalently, for $b > 0$, a.s.

$$D_h(b\cdot, b\cdot) = b^{-\alpha} D_{h(b\cdot)}(\cdot,\cdot). \tag{1.18}$$

For a whole-plane GFF, $h(b\cdot) - h_b(0) \overset{d}{=} h$. By Axiom III (Weyl scaling) and the definition of $\mathfrak{k}_b$,

$$b^\alpha e^{-\xi h_b(0)} D_h(b\cdot, b\cdot) = D_{h(b\cdot)-h_b(0)} \overset{d}{=} D_h. \tag{1.19}$$

Therefore, Axiom V holds for D with $\mathfrak{c}_r = r^{-\alpha}$. By (1.14), we get that $\alpha = -\xi Q$. Hence for $b > 0$, we have (using Axiom III in the first equality)

$$D_{h(\cdot/b)+Q\log(1/b)}(b\cdot, b\cdot) = b^{-\xi Q} D_h^{(b)} = D_h. \tag{1.20}$$

Therefore, D is a strong LQG metric. $\square$

Proof of Theorems 1.1, 1.2, and 1.8 assuming Theorem 1.9 By Lemma 1.10, every weak γ-LQG metric is a strong γ-LQG metric. By (1.15), every strong LQG metric satisfies the axioms in the definition of a weak γ-LQG metric with $\mathfrak{c}_r = r^{\xi Q}$. We can therefore apply Theorem 1.9 to get that there is at most one strong LQG metric. This completes the proof of the uniqueness parts of Theorems 1.2 and 1.8.

As for existence, we recall that [18, Theorem 1.2] (building on [16]) shows that for every sequence of ε's tending to zero, there is a weak γ-LQG metric D and a subsequence along which the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability to D_h, whenever h is a whole-plane GFF plus a bounded continuous function. By the uniqueness part of Theorem 1.8, D is in fact a strong γ-LQG metric and any two different subsequential limiting metrics differ by a deterministic multiplicative constant factor. Recall that $\mathfrak{a}_\varepsilon$ is the median D_h^ε-distance between the left and right boundaries of the unit square in the case when h is a whole-plane GFF normalized so that $h_1(0) = 0$. Hence for any subsequential limiting metric the median D_h-distance between the left and right boundaries of the unit square is 1. Therefore, the multiplicative constant factor is 1, so the subsequential limit of D_h^ε in probability is unique. This gives Theorem 1.1 and the existence parts of Theorems 1.2 and 1.8 . □

Finally, we note that our results give non-trivial information about the approximating LFPP metrics from (1.6). Indeed, let $\{\mathfrak{a}_\varepsilon\}_{\varepsilon>0}$ be the scaling constants from Theorem 1.1. It is shown in [20, Theorem 1.5] that $\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)}$. Using Theorem 1.1, we obtain the following stronger form of this relation.

Corollary 1.11 *The function $\varepsilon \mapsto \mathfrak{a}_\varepsilon$ is regularly varying with exponent $1 - \xi Q$, i.e., for every $C > 0$ one has $\lim_{\varepsilon\to 0} \mathfrak{a}_{C\varepsilon}/\mathfrak{a}_\varepsilon = C^{1-\xi Q}$.*

We expect, but do not prove here, that in fact Theorem 1.1 holds with $\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q}$.

Proof of Corollary 1.11 It is shown in [18, Lemma 2.14] that for any sequence of ε's tending to zero along which the re-scaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in law, also $\mathfrak{a}_{C\varepsilon}/\mathfrak{a}_\varepsilon$ converges (the limit is $C\mathfrak{c}_{1/C}$, with $\mathfrak{c}_{1/C}$ as in Axiom V (tightness across scales) for the limiting weak γ-LQG metric). By Theorem 1.1, $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converges in probability as $\varepsilon \to 0$, so in fact $\mathfrak{a}_{C\varepsilon}/\mathfrak{a}_\varepsilon$ converges, not just subsequentially. This means that $\mathfrak{a}_{C\varepsilon}$ is regularly varying with some exponent $\alpha > 0$. Since $\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)}$, we must have $\alpha = 1 - \xi Q$. □

1.5 Outline

As explained above, to prove our main results it remains only to prove Theorem 1.9. We emphasize that unlike many results in the theory of LQG, this paper does not build on a large amount of external input. Rather, we will only use some results from the papers [16,18,36,38], which can be taken as black boxes. All of the externally proven results which we will use are reviewed in Sect. 2.

Throughout this outline and the rest of the paper, we will use (without comment) the following two basic facts about D_h-geodesics when D is a weak γ-LQG metric and h is a whole-plane GFF.

- Almost surely, for every $z, w \in \mathbb{C}$, there is at least one D_h-geodesic from z to w. This follows from [6, Corollary 2.5.20] and the fact that $(\mathbb{C}, D_h)$ is a boundedly compact length space (i.e., closed bounded subsets are compact; see [18, Lemma 3.8]).
- For each fixed $z, w \in \mathbb{C}$, the D_h-geodesic from z to w is a.s. unique. This follows from, e.g., the proof of [62, Theorem 1.2] (see also [36, Lemma 2.2]).

In the remainder of this section we give a very rough idea of the proof of Theorem 1.9. There are a number of technicalities involved, which we will gloss over in order to make the central ideas as transparent as possible. Consequently, some of the statements in this subsection are not exactly accurate without additional caveats. More detailed (and more precise) outlines can be found at the beginnings of the individual sections and subsections.

We first comment briefly on the role of the axioms in the proof. Axiom II (locality) shows that the metric is compatible with the long-range independence and domain Markov properties of the GFF. These properties will be used in several places of our proofs (see Sect. 2.3). Axiom III (Weyl scaling) has two main uses. First, it implies that adding a constant C to the field scales distances by a factor of $e^{\xi C}$. This is important since the law of the GFF is only scale and translation invariant modulo additive constant. Second, it allows us to show that certain distance-related events occur with positive probability by adding a smooth bump function h and noting that this affects the law of the GFF in an absolutely continuous way (see the outline of Section 5 below). Axioms IV (translation invariance) and V (tightness across scales) are often used together to get estimates for the restriction of the metric to the Euclidean ball of radius r centered at z which are uniform over all possible points z and radii r. We will sometimes also use Axiom IV by itself, with r fixed, when we need more precise information than just up-to-constants estimates.

Main idea of the proof Suppose D and $\widetilde{D}$ are two weak γ-LQG metrics as in Theorem 1.9 and let h be a whole-plane GFF. As explained in Proposition 2.2, it follows from a general theorem for local metrics of the Gaussian free field [38, Theorem 1.6] that D_h and $\widetilde{D}_h$ are bi-Lipschitz equivalent, i.e.,

$$c_* := \inf\left\{\frac{\widetilde{D}_h(u,v)}{D_h(u,v)} : u, v \in \mathbb{C},\ u \neq v\right\} > 0 \quad \text{and}$$
$$C_* := \sup\left\{\frac{\widetilde{D}_h(u,v)}{D_h(u,v)} : u, v \in \mathbb{C},\ u \neq v\right\} < \infty. \tag{1.21}$$

It is easily seen that c_* and C_* are a.s. equal to deterministic constants (Lemma 3.1). We identify c_* and C_* with these constants (which amounts to re-defining c_* and C_* on an event of probability zero). To prove Theorem 1.9 we will show that $c_* = C_*$.

The basic idea of the proof of this fact is as follows. Suppose by way of contradiction that $c_* < C_*$. Then for any $c' \in (c_*, C_*)$ there a.s. exist distinct points $u, v \in \mathbb{C}$ such that $\widetilde{D}_h(u, v) \leq c' D_h(u, v)$. In Sect. 3 (see outline below), using translation invariance of the GFF, modulo additive constant, and the local independence properties of the GFF, we will deduce from this that the following is true. There exists $\underline{\beta}, \underline{p} \in (0, 1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $c' \in (c_*, C_*)$ there are many small values of $r > 0$ (how small depends on c') for which

$$\mathbb{P}\left[\exists u, v \in B_r(0) \text{ s.t. } |u - v| \geq \underline{\beta} r \text{ and } \widetilde{D}_h(u, v) \leq c' D_h(u, v)\right] \geq \underline{p}, \tag{1.22}$$

where $B_r(0)$ is the Euclidean ball of radius r centered at 0. By interchanging the roles of D_h and $\widetilde{D}_h$, we can similarly find $\overline{\beta}, \overline{p} \in (0, 1)$, depending only on the laws of D_h and $\widetilde{D}_h$, such that for each $C' \in (c_*, C_*)$, there are many small values of $r > 0$ (how small depends on C') for which

$$\mathbb{P}\left[\exists u, v \in B_r(0) \text{ s.t. } |u - v| \geq \overline{\beta} r \text{ and } \widetilde{D}_h(u, v) \geq C' D_h(u, v)\right] \geq \overline{p}. \tag{1.23}$$

See Sect. 3 for precise statements. The reason why the bounds only hold for "many" choices of $r > 0$, instead of for all $r > 0$, is that we only have tightness across scales (Axiom V), not exact scale invariance. We will use (1.22) to deduce a contradiction to (1.23).

Consider a D_h-geodesic P between two fixed points $\mathbb{z}, \mathbb{w} \in \mathbb{C}$. Using (1.22) and a local independence argument for different segments of P (which is explained in the outlines of Sects. 4 and 5 below), one can show that it holds with superpolynomially high probability as $\delta \to 0$ (i.e., except on an event of probability decaying faster than any positive power of δ), at a rate which is uniform over the choice of $\mathbb{z}$ and $\mathbb{w}$, that the following is true. There are times $0 < s < t < D_h(\mathbb{z}, \mathbb{w})$ such that $\widetilde{D}_h(P(s), P(t)) \leq c'(t - s)$ and $D_h(P(s), P(t)) \geq \delta D_h(\mathbb{z}, \mathbb{w})$. By the definition (1.21) of C_*, the $\widetilde{D}_h$-distance from $\mathbb{z}$ to $P(s)$ is at most $C_* s$ and the $\widetilde{D}_h$-distance from $P(t)$ to $\mathbb{w}$ is at most $C_*(D_h(\mathbb{z}, \mathbb{w}) - t)$. Combining these facts shows that with superpolynomially high probability as $\delta \to 0$,

$$\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \leq (C_* - (C_* - c')\delta) D_h(\mathbb{z}, \mathbb{w}). \tag{1.24}$$

We now let $\overline{\beta}$ be as in (1.23) and fix a large constant $q > 1$. For any $r > 0$, we can take a union bound to get that with probability tending to 1 as

$\delta \to 0$, at a rate which is uniform in r, the bound (1.24) holds simultaneously for all $\mathbb{z}, \mathbb{w} \in \left(\delta^q r \mathbb{Z}^2\right) \cap B_r(0)$. Now consider an arbitrary pair of points $\mathbb{z}, \mathbb{w} \in B_r(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta} r$. Let $\mathbb{z}', \mathbb{w}' \in \left(r\delta^q \mathbb{Z}^2\right) \cap B_r(0)$ be the points closest to $\mathbb{z}$ and $\mathbb{w}$, respectively. By the bi-Hölder continuity of D_h and $\widetilde{D}_h$ w.r.t. the Euclidean metric [18, Theorem 1.7], if we choose q sufficiently large, in a manner depending only on the Hölder exponents (i.e., only on γ), then $|D_h(\mathbb{z}, \mathbb{w}) - D_h(\mathbb{z}', \mathbb{w}')|$ and $|\widetilde{D}_h(\mathbb{z}, \mathbb{w}) - \widetilde{D}_h(\mathbb{z}', \mathbb{w}')|$ are much smaller than $\delta D_h(\mathbb{z}, \mathbb{w})$. From this, we infer that with probability tending to 1 as $\delta \to 0$, at a rate which is uniform in r, the bound (1.24) holds simultaneously for all $\mathbb{z}, \mathbb{w} \in B_r(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta} r$. If δ is chosen sufficiently small so that this probability is at least $1 - \overline{p}/2$, we get a contradiction to (1.23) with $C' = C_* - (C_* - c')\delta$.

The purpose of Sects. 3, 4, and 5 is to fill in the details of the above argument. These three sections are mostly independent from one another: only the main theorem/proposition statements at the beginning of each section are used in later sections.

Section 3: bounds for ratios of distances at many scales The purpose of Sect. 3 is to prove (more quantitative versions of) the bounds (1.22) and (1.23) stated above. Since we are only working with a weak γ-LQG metric, not a strong γ-LQG metric, we do not have exact scale invariance, just tightness across scales (Axiom V). Consequently, if $c' \in (c_*, C_*)$, then we cannot necessarily say that pairs of points u, v for which $\widetilde{D}_h(u, v) \leq c' D_h(u, v)$ exist with uniformly positive probability over different Euclidean scales. That is, it could in principle be that for every small fixed $\underline{\beta} > 0$, the probability that there exists $u, v \in B_r(0)$ with $\widetilde{D}_h(u, v) \leq c' D_h(u, v)$ and $|u - v| \geq \underline{\beta} r$ is very small for some values of $r > 0$. However, we can say that such pairs of points exist with uniformly positive probability for a suitably "dense" set of scales r via an argument which proceeds (very roughly) as follows.

Let $\underline{\beta}, \underline{p} \in (0, 1)$ be small and suppose by way of contradiction that there is a sequence $r_k \to 0$ such that r_{k+1}/r_k is bounded above and below by deterministic constants and the following is true. For each k, it holds with probability at least $1 - \underline{p}$ that $\widetilde{D}_h(u, v) \geq c' D_h(u, v)$ for every pair of points $u, v \in B_{r_k}(0)$ for which $|u - v| \geq \underline{\beta} r_k$. Using the translation invariance of the metric (Axiom IV) and the local independence properties of the GFF (in particular, Lemma 2.6 below), we see that if $\underline{\beta}, \underline{p}$ are sufficiently small (how small depends only on the laws of D_h and $\widetilde{D}_h$, not on c' or r_k), then the following is true. We can cover any fixed compact subset of $\mathbb{C}$ by Euclidean balls of the form $B_{r_k}(z)$ with the property that $\widetilde{D}_h(u, v) \geq c' D_h(u, v)$ for every pair of points $u \in \partial B_{(1-\underline{\beta})r_k}(z)$ and $v \in \partial B_{r_k}(z)$. By considering the times when a $\widetilde{D}_h$-geodesic between two fixed points of $\mathbb{C}$ crosses an annulus $B_{r_k}(z) \setminus B_{(1-\underline{\beta})r_k}(z)$

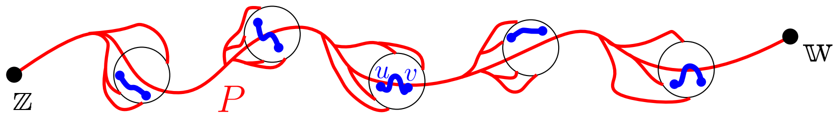

Fig. 1 Illustration of the main ideas in Sect. 4. Using results on confluence of geodesics from [36], we can show that there are many times t at which the D_h-geodesic P is *stable*, in the sense that changing the behavior of the field in a small Euclidean ball around $P(t)$ does not result in a macroscopic change to the D_h-geodesic (the precise condition is given in (4.11)). In particular, to produce such stable times we consider the metric ball growth started from $\mathbb{z}$ and use the confluence across a metric annulus from [36, Theorem 3.9] at a large number of evenly spaced radii. In fact, using the results of Sect. 3, we can arrange that there are many such stable times whose corresponding balls contain a pair of points u, v such that $\widetilde{D}_h(u, v) \le c' D_h(u, v)$ and $|u - v|$ is comparable to the Euclidean radius of the ball. These pairs of points and the $\widetilde{D}_h$-geodesics between them are shown in blue. Using the results of Sect. 5, we can show that for each of these stable times, it holds with positive conditional probability given the past that P gets close to the corresponding pair of points u, v. By a standard concentration inequality for Bernoulli sums, applied at the stable times, this shows that P has to get close to at least one such pair of points u, v with extremely high probability

for z as above, we get that a.s. $\inf_{z,w\in\mathbb{C}} \widetilde{D}_h(z, w)/D_h(z, w) \ge c''$ for a constant $c'' \in (c_*, c')$. This contradicts the definition (1.21) of c_*.

Hence the set of "bad" scales r for which points $u, v \in B_r(0)$ with $|u-v| \ge \underline{\beta} r$ and $\widetilde{D}_h(u, v) \le c' D_h(u, v)$ are unlikely to exist cannot be too large, which means that the complementary set of "good" scales for which such points exist with probability at least $\underline{p}$ has to be reasonably dense. This leads to (1.22). The bound (1.23) follows by interchanging the roles of D_h and $\widetilde{D}_h$.

Section 4: independence along an LQG geodesic Once we know that there are many pairs of points u, v with $\widetilde{D}_h(u, v) \le c' D_h(u, v)$, we want to use some sort of local independence to say that a D_h-geodesic P is extremely likely to get close to at least one such pair of points (i.e., we need the D_h-distance from P to each of u and v to be much smaller than $D_h(u, v)$). However, D_h-geodesics are highly non-local functionals of the field and do not satisfy any reasonable Markov property. So, techniques for obtaining local independence which may be familiar from the theory of SLE/GFF couplings [23,28,66–68,70,81,83] do not apply in our setting.

Instead we need to develop a new set of techniques to obtain local independence at different points of D_h-geodesics. See Fig. 1 for an illustration. In fact, we will prove a general theorem (Theorem 4.1) which roughly speaking says the following. Suppose we are given events $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ for $z, \mathbb{z}, \mathbb{w} \in \mathbb{C}$ and $r > 0$ with the following properties. The event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ is determined by $h|_{B_r(z)}$ and the part of the D_h-geodesic $P^{\mathbb{z},\mathbb{w}}$ from $\mathbb{z}$ to $\mathbb{w}$ which is contained in $B_r(z)$. Moreover, for each $z, \mathbb{z}, \mathbb{w} \in \mathbb{C}$, the conditional probability of $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ given $h|_{\mathbb{C}\setminus B_r(z)}$ and the event $\{P^{\mathbb{z},\mathbb{w}} \cap B_r(z) \neq \emptyset\}$ is a.s. bounded below by a deterministic constant. Then when r is small it is very likely that for nearly every choice of $\mathbb{z}, \mathbb{w} \in \mathbb{C}$, the event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs for at least one ball $B_r(z)$ hit by $P^{\mathbb{z},\mathbb{w}}$.

We will eventually apply this theorem with $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ given by, roughly speaking, the event that $P^{\mathbb{z},\mathbb{w}}$ gets close to a pair of points $u, v \in B_r(z)$ with $\widetilde{D}_h(u, v) \leq c' D_h(u, v)$ and $|u - v| \geq \text{const} \times r$. This together with the triangle inequality and the bi-Hölder continuity of D_h and $\widetilde{D}_h$ w.r.t. the Euclidean metric (to transfer from $|u - v| \geq \text{const} \times r$ to a lower bound for $D_h(u, v)$) will lead to (1.24).

We will prove the above "independence along a geodesic" theorem using the results on confluence of D_h-geodesics established in [36]. These results tell us that if $\mathbb{z} \in \mathbb{C}$ is fixed and $\mathbb{w}_1, \mathbb{w}_2 \in \mathbb{C}$ are close together, then the D_h-geodesics P_1 from $\mathbb{z}$ to $\mathbb{w}_1$ and P_2 from $\mathbb{z}$ to $\mathbb{w}_2$ typically agree until they get close to $\mathbb{w}_1$ and $\mathbb{w}_2$, i.e., $P_1|_{[0,\tau]} = P_2|_{[0,\tau]}$ for a time τ which is close to $D_h(\mathbb{z}, \mathbb{w}_1)$ (equivalently, to $D_h(\mathbb{z}, \mathbb{w}_2)$) when $D_h(\mathbb{w}_1, \mathbb{w}_2)$ is small. Note that this property is not true for geodesics for a smooth Riemannian metric, but it is true for geodesics in the Brownian map [56].

Now fix $\mathbb{z}, \mathbb{w}$ and consider the D_h-geodesic $P = P^{\mathbb{z},\mathbb{w}}$ from $\mathbb{z}$ to $\mathbb{w}$. The above confluence property applied with $\mathbb{w}_1 = P(t)$ for a typical time $t \in [0, D_h(\mathbb{z}, \mathbb{w})]$ and $\mathbb{w}_2$ a point near $P(t)$ will allow us to show that with extremely high probability, there are many times $t \in [0, D_h(\mathbb{z}, \mathbb{w})]$ at which P is "stable" in the following sense. If we make a small modification to h in a neighborhood of $P(t)$, then we will not change $P|_{[0,\tau]}$ for a time τ a little bit less than t. This allows us to say that events depending on the field in a small neighborhood of $P(t)$ have positive conditional probability given an initial segment of P. Applying this at a large number of evenly spaced times $t \in [0, D_h(\mathbb{z}, \mathbb{w})]$ will show that it is extremely likely that the event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ discussed above occurs for at least one Euclidean ball $B_r(z)$ hit by P.

Section 5: an LQG geodesic gets close to a shortcut with positive probability Fix $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ and let $P = P^{\mathbb{z},\mathbb{w}}$ be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. By (1.22) and translation invariance (Axiom IV) we know that there exists $\underline{\beta}, \underline{p} \in (0, 1)$ such that if $c' \in (c_*, C_*)$, then there are many values of $r > 0$ such that (1.22) holds with z in place of 0 (actually, we will use a variant of (1.22) which gives more precise information about the locations of u and v; see Proposition 3.5). In light of the results of Sect. 4, we want to show that if we condition on $\{P \cap B_r(z) \neq \emptyset\}$, then the conditional probability that P gets close to a pair of points u, v as in (1.22) (with z in place of 0) is bounded below by a positive deterministic constant which does not depend on r or z.

For a deterministic open set $U \subset \mathbb{C}$, one can prove that the D_h-geodesic P enters U with positive probability as follows. Consider a deterministic path from $\mathbb{z}$ to $\mathbb{w}$ and let ϕ be a smooth bump function which takes large values in a narrow "tube" around this path and which vanishes outside a slightly larger tube. By Weyl scaling (Axiom III), $D_{h-\phi}$ distances in the tube are much shorter than distances anywhere else. Hence the $D_{h-\phi}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$ has to stay in the tube and hence has to enter U. Since the laws of h and $h-\phi$ are absolutely continuous, we get that the D_h-geodesic enters U with positive probability.

We will use a similar strategy to show that P has positive conditional probability given $\{P \cap B_r(z) \neq \emptyset\}$ to get near a pair of points $u, v \in B_r(z)$ with $\widetilde{D}_h(u,v) \leq c' D_h(u,v)$ and $|u-v| \geq \underline{\beta} r$. However, additional complications arise. For example, the region we want P to enter (a small neighborhood of either u or v) is random, which will be resolved by choosing a deterministic region which contains the $\widetilde{D}_h$-geodesic between u and v with positive probability. We also need to ensure that the condition $\widetilde{D}_h(u,v) \leq c' D_h(u,v)$ is not destroyed when we add our bump function. To do this, we will need to make sure that the $\widetilde{D}_h$-geodesic between u and v is contained in the region where the bump function attains its largest possible value. Another issue is that we need the bump function ϕ to be supported on a region of diameter of order $r \approx |u-v|$, so that its Dirichlet energy is bounded independently of r. In particular, this support cannot contain the starting and ending points $\mathbb{z}$ and $\mathbb{w}$ of the D_h-geodesic. This will be resolved by growing the D_h-metric balls from $\mathbb{z}$ and $\mathbb{w}$ until they hit $B_{3r}(z)$ and choosing a bump function whose support approximates a path between the hitting points.

In Sect. 6, we combine all of the above ingredients to conclude the proof of Theorem 1.9, following the argument in the "main ideas" section above. Section 7 contains a list of open problems.

Remark 1.12 (Proof for strong LQG metrics) As explained above, we prove Theorem 1.9 instead of just proving Theorem 1.2 since subsequential limits of LFPP are only known to be weak LQG metrics, not strong LQG metrics. If we only wanted to prove Theorem 1.2, we could make only a few minor simplifications to our proofs. The most significant simplifications would be in Sect. 3. In particular, similar arguments to the ones in Sect. 3 would give points u, v such that $\widetilde{D}_h(u,v) = C_* D_h(u,v)$ instead of just $\widetilde{D}_h(u,v) \geq C' D_h(u,v)$ for C' slightly less than C_*. Additionally, all of the results in Sect. 3 which are currently only proven to hold for "at least $\mu \log_8 \varepsilon^{-1}$ scales" could instead be shown to hold for all scales. This would allow us to eliminate the parameters μ, ν, and C' throughout the paper. We could of course also replace $\mathfrak{c}_r$ by $r^{\xi Q}$ and eliminate the "scale parameter" $\mathbb{r}$ throughout. This results in cosmetic simplifications in Sects. 4, 5 and 6.

Remark 1.13 (Relationship to [57,59]) It is natural to ask how our proof compares to the proofs of the Gromov–Hausdorff convergence of uniform quadrangulations to the Brownian map in [57,59]. Both this paper and [57,59] start from a tightness result and seek to show that the limiting object is unique. Moreover, all three papers rely crucially on confluence of geodesics (in the

Brownian map setting, tightness is proven in [55] and confluence is proven in [56]). However, this is about the extent of the similarities.

In the Brownian map setting, one has an explicit a priori description of the conjectural limiting metric space $(X, \mathfrak{d})$ in terms of the Brownian snake. In particular, there is a marked point $x_* \in X$ (which is a uniform sample from the area measure on the Brownian map) such that $\mathfrak{d}(x_*, x)$ can be described explicitly in terms of the Brownian snake. Due to the convergence of discrete snakes to the Brownian snake and the Schaeffer bijection [13,78], one gets that any possible subsequential limit of uniform quadrangulations can be represented by a metric $\widetilde{\mathfrak{d}}$ on X such that $\widetilde{\mathfrak{d}} \leq \mathfrak{d}$ and $\widetilde{\mathfrak{d}}(x_*, x) = \mathfrak{d}(x_*, x)$ for every $x \in X$ (see [60]). The heart of the proof in each of [56,59] consists of using confluence to approximate a $\widetilde{\mathfrak{d}}$-geodesic by a concatenation of segments of $\widetilde{\mathfrak{d}}$-geodesics started from x_* (the method of approximation in the two papers is quite different).

In our setting, we do not have an a priori construction of the limiting object and we do not know a priori that any quantities related to two different weak LQG metrics are exactly equal. Instead, we have a coupling of our weak LQG metric to the GFF. We use confluence together with the Markov property of the GFF to get that far-away geodesic segments are nearly independent from each other.

2 Preliminaries

In this subsection, we first introduce some basic (mostly standard) notation. We then review all of the results from [18,36,38] which we will need for the proof of Theorem 1.9. On a first read, the reader may wish to read only Sects. 2.1 (which introduces notation) and 2.2 (which proves the bi-Lipschitz equivalence of the metrics D_h and $\widetilde{D}_h$ in Theorem 2.5) then refer back to the other subsections as needed.

2.1 Basic notation and terminology

Integers

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $a < b$, we define $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

Asymptotics

If $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to (0, \infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (resp. $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (resp. tends to zero) as $\varepsilon \to 0$. We say that

$$f(\varepsilon) = o_\varepsilon^\infty(\varepsilon) \quad \text{if and only if} \quad f(\varepsilon) = o_\varepsilon(\varepsilon^p),\ \forall p > 0. \tag{2.1}$$

We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

If $f, g : (0,\infty) \to [0,\infty)$, we say that $f(\varepsilon) \preceq g(\varepsilon)$ if there is a constant $C > 0$ (independent from ε and possibly from other parameters of interest) such that $f(\varepsilon) \le Cg(\varepsilon)$. We write $f(\varepsilon) \asymp g(\varepsilon)$ if $f(\varepsilon) \preceq g(\varepsilon)$ and $g(\varepsilon) \preceq f(\varepsilon)$.

We often specify requirements on the dependencies on rates of convergence in $O(\cdot)$ and $o(\cdot)$ errors, implicit constants in $\preceq$, etc., in the statements of lemmas/propositions/theorems, in which case we implicitly require that errors, implicit constants, etc., in the proof satisfy the same dependencies.

The parameter γ is fixed throughout the paper. All implicit constants and rates of convergence are allowed to depend on γ, and this will not be stated explicitly.

Balls and annuli

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the Euclidean ball of radius r centered at z. We also define the open annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}, \quad \forall 0 < r_r < r_2 < \infty. \tag{2.2}$$

For a metric space $(X, \mathfrak{d})$ and $r > 0$, we write $\mathcal{B}_r(A; \mathfrak{d})$ for the open ball consisting of the points $x \in X$ with $\mathfrak{d}(x, A) < r$. If $A = \{y\}$ is a singleton, we write $\mathcal{B}_r(\{y\}; \mathfrak{d}) = \mathcal{B}_r(y; \mathfrak{d})$.

For a metric $\mathfrak{d}$ on $\mathbb{C}$, $r > 0$, and $z \in \mathbb{C}$ we write $\mathcal{B}_r^\bullet(z; \mathfrak{d})$ for the *filled metric ball* which is the union of $\overline{\mathcal{B}_r(z; \mathfrak{d})}$ and the bounded connected components of $\mathbb{C} \setminus \overline{\mathcal{B}_r(z; \mathfrak{d})}$.

Local sets

Following [83, Lemma 3.9], if (h, A) is a coupling of a whole-plane GFF and random compact set $A \subset \mathbb{C}$, we say that A is a *local set* for h if for each open set $U \subset \mathbb{C}$, the event $\{A \cap U \neq \emptyset\}$ is conditionally independent from $h|_{\mathbb{C}\setminus U}$ given $h|_U$. If A is determined by h (which will be the case for all of the local sets we consider), this is equivalent to the statement that A is determined by $h|_U$ on the event $\{A \subset U\}$. The following lemma is a re-statement of [36, Lemma 2.1].

Lemma 2.1 [36] *Let D be a weak γ-LQG metric and let h be a whole-plane GFF. Also let $z \in \mathbb{C}$ and let τ be a stopping time for the filtration generated by*

$(\mathcal{B}_s^\bullet(z; D_h), h|_{\mathcal{B}_s^\bullet(z;D_h)})$. Then $\mathcal{B}_\tau^\bullet(z; D_h)$ is a local set for h. The same is true with closures of ordinary D_h-metric balls in place of filled D_h-metric balls.

General notational conventions

We make some comments about how various symbols are used in order to help the reader follow the paper (we will not make any precise definitions here).

We use the symbols $\mathbb{z}, \mathbb{w}, z, w, u, v$ for points in $\mathbb{C}$. Typically, $\mathbb{z}, \mathbb{w}$ are fixed (often the endpoints of a geodesic), z and w are allowed to vary (e.g., over some open set) or are random, and u, v are dummy variables appearing, e.g., in suprema/infima.

We use the symbols p and $\mathbb{p}$ for probabilities. Typically, $\mathbb{p}$ is fixed throughout several lemmas, whereas p is allowed to change more frequently.

The symbols r and $\mathbb{r}$ denote Euclidean radii. Typically, $\mathbb{r}$ represents a fixed Euclidean scale. The reason why we need this is that we do not have exact scale invariance, only tightness across scales, so we often need to prove things at an arbitrary Euclidean scale, rather than just considering a single scale and then re-scaling. The symbol r is used for other Euclidean radii, which may depend on $\mathbb{r}$ and/or be random. We use s and t for LQG radii.

The symbol ε typically denotes a small parameter which is independent from the Euclidean scale $\mathbb{r}$ (so $\varepsilon \to 0$ at a rate which does not depend on $\mathbb{r}$). The symbols μ and ν will always carry the same meaning as in the proposition statements in Sect. 3: namely, we require that for any fixed $\mathbb{r}$ and any small enough ε, there are at least $\mu \log_8 \varepsilon^{-1}$ "good" scales $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$.

2.2 Bi-Lipschitz equivalence of weak LQG metrics

In this subsection we explain why the results of [38] imply that any two weak γ-LQG metrics with the same scaling constants are bi-Lipschitz equivalent.

Proposition 2.2 *Let h be a whole-plane GFF, let $\gamma \in (0, 2)$, and let D and $\widetilde{D}$ be two weak γ-LQG metrics, with the same scaling constants $\mathfrak{c}_r$. There is a deterministic constant $C > 0$ such that a.s.*

$$C^{-1} D_h(z, w) \le \widetilde{D}_h(z, w) \le C D_h(z, w), \quad \forall z, w \in \mathbb{C}. \tag{2.3}$$

Proposition 2.2 is a special case of a general theorem from [38] which tells us when two random metrics coupled with the same GFF are bi-Lipschitz equivalent. To state the theorem, we first recall some definitions.

Definition 2.3 (*Jointly local metrics*) Let $(h, D_1, \dots, D_n)$ be a coupling of the GFF h with n random continuous length metrics. We say that $D_1, \dots, D_n$ are *jointly local metrics* for h if for any open set $V \subset \mathbb{C}$, the collection of internal metrics $\{D_j(\cdot,\cdot;V)\}_{j=1,\ldots,n}$ is conditionally independent from $(h|_{\mathbb{C}\setminus V}, \{D_j(\cdot,\cdot;U\setminus\overline{V})\}_{j=1,\ldots,n})$ given $h|_V$.

In the setting of Proposition 2.2, the metrics D_h and $\widetilde{D}_h$ are each local for h due to Axiom II. Since these metrics are each determined by h, they are conditionally independent given h. Therefore, we can apply [38, Lemma 1.4] to get that D_h and $\widetilde{D}_h$ are jointly local for h.

Definition 2.4 (*Additive local metrics*) Let $(h, D_1, \ldots, D_n)$ be a coupling of h with n random continuous length metric which are jointly local for h. For $\xi \in \mathbb{R}$, we say that $D_1, \ldots, D_n$ are *ξ-additive* for h if for each $z \in \mathbb{C}$ and each $r > 0$ such that $B_r(z) \subset U$, the metrics $(e^{-\xi h_r(z)} D_1, \ldots, e^{-\xi h_r(z)} D_n)$ are jointly local metrics for $h - h_r(z)$.

By Axiom III (Weyl scaling), it follows that our metrics D_h and $\widetilde{D}_h$ are jointly local for h. The following theorem is a special case of [38, Theorem 1.6].

Theorem 2.5 [38] *Let $\xi \in \mathbb{R}$, let h be a whole-plane GFF normalized so that $h_1(0) = 0$, and let $(h, D_h, \widetilde{D}_h)$ be a coupling of h with two random continuous metrics on $\mathbb{C}$ which are jointly local and ξ-additive for h. There is a universal constant $p \in (0,1)$ such that the following is true. Suppose there is a constant $C > 0$ such that (using the notation for annuli from (2.2)), we have*

$$\mathbb{P}\left[\sup_{u,v\in\partial B_r(z)} \widetilde{D}_h\left(u,v;\mathbb{A}_{r/2,2r}(z)\right) \le C D_h(\partial B_{r/2}(z), \partial B_r(z))\right] \ge p, \quad \forall z \in \mathbb{C}, \quad \forall r > 0. \tag{2.4}$$

Then a.s. $\widetilde{D}(z,w) \le C D(z,w)$ for all $z, w \in \mathbb{C}$.

Proof of Proposition 2.2 By Axioms IV and V for each of D_h and $\widetilde{D}_h$, for any $p \in (0,1)$ we can find a constant $C_p > 1$ such that for each $z \in \mathbb{C}$ and each $r > 0$, it holds with probability at least p that

$$\sup_{u,v\in\partial B_r(z)} D_h\left(u,v;\mathbb{A}_{r/2,2r}(z)\right) \le C_p \mathfrak{c}_r e^{\xi h_r(z)}, \quad D_h(\partial B_{r/2}(z), \partial B_r(z)) \ge C_p^{-1} \mathfrak{c}_r e^{\xi h_r(z)}, \tag{2.5}$$

and the same is true with $\widetilde{D}_h$ in place of h. Therefore, (2.4) holds with $C = C_p^2$ for each of the pairs $(D_h, \widetilde{D}_h)$ and $(\widetilde{D}_h, D_h)$. Theorem 2.5 therefore implies Proposition 2.2 with $C = C_p^2$, where p is as in Theorem 2.5. □

2.3 Local independence for the GFF

In many places throughout the paper, we will estimate various probabilities using the local independence properties of the GFF. We will do this using two different lemmas, which we state in this section. The first is a restatement of part of [38, Lemma 3.1].

Lemma 2.6 (Iterating events in nested annuli) *Fix $0 < s_1 < s_2 < 1$. Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $r_{k+1}/r_k \leq s_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k\in\mathbb{N}}$ be events such that $E_{r_k} \in \sigma\left((h - h_{r_k}(0))|_{\mathbb{A}_{s_1 r_k, s_2 r_k}(0)}\right)$ for each $k \in \mathbb{N}$. For $K \in \mathbb{N}$, let $N(K)$ be the number of $k \in [1, K]_{\mathbb{Z}}$ for which E_{r_k} occurs. For each $a > 0$ and each $b \in (0, 1)$, there exists $p = p(a, b, s_1, s_2) \in (0, 1)$ and $c = c(a, b, s_1, s_2) > 0$ such that if*

$$\mathbb{P}\left[E_{r_k}\right] \geq p, \quad \forall k \in \mathbb{N}, \tag{2.6}$$

then

$$\mathbb{P}\left[N(K) < bK\right] \leq ce^{-aK}, \quad \forall K \in \mathbb{N}. \tag{2.7}$$

We will only ever apply Lemma 2.6 to say that $N(K) \geq 1$ with high probability, i.e., the choice of b in (2.7) will not matter for our purposes.

Lemma 2.7 (Iterating events in disjoint balls) *Let h be a whole-plane GFF and fix $s > 0$. Let $n \in \mathbb{N}$ and let $\mathcal{Z}$ be a collection of $\#\mathcal{Z} = n$ points in $\mathbb{C}$ such that $|z - w| \geq 2(1 + s)$ for each distinct $z, w \in \mathcal{Z}$. For $z \in \mathcal{Z}$, let E_z be an event which is determined by $(h - h_{1+s}(z))|_{B_1(z)}$. For each $p, q \in (0, 1)$, there exists $n_* = n_*(s, p, q) \in \mathbb{N}$ such that if $\mathbb{P}[E_z] \geq p$ for each $z \in \mathcal{Z}$, then*

$$\mathbb{P}\left[\bigcup_{z\in\mathcal{Z}} E_z\right] \geq q, \quad \forall n \geq n_*.$$

Proof Let $U := \bigcup_{z\in\mathcal{Z}} B_{1+s}(z)$ and let $\mathfrak{h}$ be the harmonic part of $h|_U$. Since the balls $B_{1+s}(z)$ for $z \in \mathcal{Z}$ are disjoint, the Markov property of h implies that the fields $(h - h_{1+s}(z))|_{B_{1+s}(z)}$ for $z \in \mathcal{Z}$, and hence also the events E_z, are conditionally independent given $h|_{\mathbb{C}\setminus U}$ (equivalently, given $\mathfrak{h}$).

We will now compare the conditional law given $h|_{\mathbb{C}\setminus U}$ to the unconditional law. For $z \in \mathcal{Z}$, let

$$\mathfrak{M}_z := \sup_{u\in B_{1+s/2}(z)} |\mathfrak{h}(u) - \mathfrak{h}(z)|. \tag{2.8}$$

By a standard Radon-Nikodym derivative calculation for the GFF (see, e.g., [62, Lemma 4.1]) and the translation and scale invariance of the law of h, modulo additive constant, for each $\alpha > 0$ there is a constant $C = C(\alpha, s) > 0$ such that the following is true. The conditional law given of $(h - h_{1+s}(z))|_{B_1(z)}$ given $h|_{\mathbb{C}\setminus U}$ is absolutely continuous with respect to its marginal law and if H_z denotes the Radon-Nikodym derivative of the conditional law with respect to the marginal law, then a.s.

$$\max\left\{\mathbb{E}\left[H_z^{\alpha} \,|\, h|_{\mathbb{C}\setminus U}\right], \ \mathbb{E}\left[H_r^{-\alpha} \,|\, h|_{\mathbb{C}\setminus U}\right]\right\} \leq C \exp\left(C\mathfrak{M}_z^2\right). \tag{2.9}$$

Each $\mathfrak{M}_z$ is an a.s. finite random variable. By the translation invariance of the law of h, modulo additive constant, the law of $\mathfrak{M}_z$ does not depend on z. So, we can find a constant $A = A(s, q) > 0$ such that $\mathbb{P}[\mathfrak{M}_z \leq A] \geq 1-(1-q)/4$ for each $z \in \mathcal{Z}$. Then $\mathbb{E}[\#\{z \in \mathcal{Z} : \mathfrak{M}_z > A\}] \leq (1-q)n/4$ so

$$\mathbb{P}\left[\#\{z \in \mathcal{Z} : \mathfrak{M}_z \leq A\} \geq n/2\right] \geq 1 - \frac{1-q}{2}. \tag{2.10}$$

Since E_z is determined by $(h - h_{1+s}(z))|_{B_1(z)}$ and $\mathbb{P}[E_z] \geq p$ for each $z \in \mathcal{Z}$, (2.9) implies that there exists $\widetilde{p} = \widetilde{p}(p, A) > 0$ such that on the event $\{\mathfrak{M}_z \leq A\}$ (which is determined by $h|_{\mathbb{C}\setminus U}$), a.s.

$$\mathbb{P}\left[E_z \,|\, h|_{\mathbb{C}\setminus U}\right] \geq \widetilde{p}. \tag{2.11}$$

Since the E_z's are conditionally independent given $h|_{\mathbb{C}\setminus U}$, we see that a.s.

$$\mathbb{P}\left[\bigcup_{z\in\mathcal{Z}} E_z \,|\, h|_{\mathbb{C}\setminus U}\right] \geq 1 - \widetilde{p}^{\#\{z\in\mathcal{Z}:\mathfrak{M}_z\leq A\}}. \tag{2.12}$$

We now choose n_* large enough that $1 - \widetilde{p}^{n_*/2} \geq 1 - (1-q)/2$ and combine (2.10) with (2.12). □

2.4 Estimates for weak LQG metrics

In this subsection we review results from [18] which we will need for the proofs of our main theorems. Throughout, D denotes a weak γ-LQG metric and h denotes a whole-plane GFF. In particular, we state a bi-Hölder continuity bound for D_h and the Euclidean metric (Lemma 2.8), a bound for the D_h-diameters of squares (Lemma 2.9), and bounds which prevent a D_h-geodesic from spending a long time near a line (Lemma 2.10), a circle (Lemma 2.11), or the boundary of a D_h-metric ball (Lemma 2.12).

All of the results which we state in this subsection involve a parameter $\mathbb{r}$, which controls the "Euclidean scale" at which we are working. This parameter is necessary since we are only assuming tightness across scales (Axiom V) instead of exact scale invariance. All estimates are required to be uniform in the choice of $\mathbb{r}$. Our first result, which follows from [18, Lemmas 3.20 and 3.22], is a form of local Hölder continuity for the identity map $(\mathbb{C}, |\cdot|) \to (\mathbb{C}, D_h)$ and its inverse.

Lemma 2.8 (Hölder continuity) *Fix a compact set $K \subset \mathbb{C}$ and exponents $\chi \in (0, \xi(Q-2))$ and $\chi' > \xi(Q+2)$. For each $\mathbb{r} > 0$, it holds with probability tending to 1 as $a \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $u, v \in \mathbb{r}K$ with $|u-v| \le a\mathbb{r}$,*

$$D_h(u, v) \geq \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \left|\frac{u-v}{\mathbb{r}}\right|^{\chi'} \quad \text{and} \tag{2.13}$$

$$D_h\left(u, v; B_{2|u-v|}(u)\right) \leq \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \left|\frac{u-v}{\mathbb{r}}\right|^{\chi}. \tag{2.14}$$

We note that (2.14) gives an upper bound for the D_h-distance from u to v *along paths which stay in $B_{2|u-v|}(u)$*. This is slightly stronger than just an upper bound for $D_h(u, v)$. In Sect. 5, we will also need the following variant of (2.14) which gives an upper bound for the D_h-internal diameters of Euclidean squares and is proven in [18, Lemma 3.20].

Lemma 2.9 (Internal diameters of Euclidean squares) *Let K and χ be as in Lemma 2.8. For each $\chi \in (0, \xi(Q-2))$ and each $\mathbb{r} > 0$, it holds with probability tending to 1 as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that for each $k \in \mathbb{N}_0$ and each $2^{-k}\varepsilon\mathbb{r} \times 2^{-k}\varepsilon\mathbb{r}$ square S with corners in $2^{-k}\varepsilon\mathbb{r}\mathbb{Z}^2$ which intersects $\mathbb{r}K$,*

$$\sup_{u,v\in S} D_h(u, v; S) \leq \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} (2^{-k}\varepsilon)^{\chi}. \tag{2.15}$$

In several places throughout the paper, we will want to prevent a D_h-geodesic from staying in small neighborhood of a fixed Euclidean path. The following lemma, which is a restatement of [18, Proposition 4.1], will allow us to do this.

Lemma 2.10 (Lower bound for distances in a narrow tube) *Let $L \subset \mathbb{C}$ be a compact set which is either a line segment or an arc of a circle and fix $b > 0$. For each $\mathbb{r} > 0$ and each $q > 0$, it holds with probability at least*

$1 - \varepsilon^{q^2/(2\xi^2)+o_\varepsilon(1)}$ *that*

$$\begin{aligned}&\inf \{D_h (u, v; B_{\varepsilon\mathbb{r}}(\mathbb{r}L)) : u, v \in B_{\varepsilon\mathbb{r}}(\mathbb{r}L), |u - v| \geq b\mathbb{r}\}\\&\quad \geq \varepsilon^{q+\xi Q-1-\xi^2/2}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)},\end{aligned} \tag{2.16}$$

where the rate of the $o_\varepsilon(1)$ *depends on* L, b, q *but not on* $\mathbb{r}$.

By [4, Theorem 1.9], for each $\gamma \in (0, 2)$ we have $1 - \xi Q \geq 0$, and hence $\xi Q - 1 - \xi^2/2 < 0$. Therefore, the power of ε on the right side of (2.16) is negative for small enough q. Hence, Lemma 2.10 implies that when ε is small and $u, v \in B_{\varepsilon\mathbb{r}}(\mathbb{r}L)$ with $|u - v| \geq b\mathbb{r}$, it holds with high probability that $D_h (u, v; B_{\varepsilon\mathbb{r}}(\mathbb{r}L))$ is much larger than $D_h(u, v)$. In particular, a D_h-geodesic from u to v cannot stay in $B_{\varepsilon\mathbb{r}}(L)$. Lemma 2.10 has the following useful corollary. For the statement, we recall the notation for Euclidean annuli from (2.2).

Lemma 2.11 (Lower bound for distances in a narrow annulus) *For each* $S > s > 0$ *and each* $p \in (0, 1)$*, there exists* $\alpha_* = \alpha_*(s, S, p) \in (1/2, 1)$ *such that for each* $\alpha \in [\alpha_*, 1)$*, each* $z \in \mathbb{C}$*, and each* $\mathbb{r} > 0$*,*

$$\begin{aligned}&\mathbb{P}\Big[\inf \Big\{D_h \left(u, v; \mathbb{A}_{\alpha\mathbb{r},\mathbb{r}}(z)\right) : u, v \in \mathbb{A}_{\alpha\mathbb{r},\mathbb{r}}(z), D_h(u, v) \geq s\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(z)}\Big\}\\&\quad \geq S\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(z)}\Big] \geq p.\end{aligned} \tag{2.17}$$

Proof By Weyl scaling (Axiom III), the event in (2.17) does not depend on the choice of additive constant for h. By Axiom IV (translation invariance) and the translation invariance of the law of h modulo additive constant, the probability of this event does not depend on z. By Axiom V (tightness across scales), we can find $b = b(s) > 0$ such that with probability at least $1 - (1 - p)/2$, any points $u, v \in B_{\mathbb{r}}(0)$ with $D_h(u, v) \geq s\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$ satisfy $|u - v| \geq b\mathbb{r}$. Combining with Lemma 2.10 (with $\varepsilon = 1 - \alpha$ and $L = \partial\mathbb{D}$) concludes the proof. □

Finally, we record a lemma which prevents D_h-geodesics from spending a long time near the boundary of a D_h-metric ball which is needed in Sect. 4.2. The lemma is a re-statement of [18, Proposition 4.3].

Lemma 2.12 (Geodesics cannot spend a long time near metric ball boundary) *For each* $M > 0$ *and each* $\mathbb{r} > 0$*, it holds with probability* $1 - o_\varepsilon^\infty(\varepsilon)$ *as* $\varepsilon \to 0$*, at a rate which is uniform in the choice of* $\mathbb{r}$*, that the following is true. For each* $s > 0$ *for which* $\mathcal{B}_s(0; D_h) \subset B_{\varepsilon^{-M}\mathbb{r}}(0)$ *and each* D_h*-geodesic* P *from 0 to a point outside of* $\mathcal{B}_s(0; D_h)$*,*

$$\text{area}\left(B_{\varepsilon\mathbb{r}}(P) \cap B_{\varepsilon\mathbb{r}}\left(\partial\mathcal{B}_s(0; D_h)\right)\right) \leq \varepsilon^{2-1/M}\mathbb{r}^2, \tag{2.18}$$

where area *denotes 2-dimensional Lebesgue measure.*

2.5 Confluence of geodesics

In this subsection we will review some facts about D_h-geodesics which are proven in [36]. These facts are used only in Sect. 4.4. For $z \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{N}$ we define the radii $\rho_r^n(z)$ as in [36, Equation (3.13)]. The radius $\rho_r^n(z)$ is the nth smallest $t \in \{2^k r\}_{k\in\mathbb{N}}$ for which a certain event in $\sigma((h - h_{6r}(z))|_{\mathbb{A}_{2r,5r}(z)})$ occurs. Roughly speaking, the event in question tells us that if we fix $\mathbb{z} \in \mathbb{C}$ and $t > 0$ such that the filled LQG metric ball $\mathcal{B}_t^\bullet(\mathbb{z}; D_h)$ intersects $B_r(z)$, then with constant-order conditional probability given $(\mathcal{B}_t^\bullet(\mathbb{z}; D_h), h|_{\mathcal{B}_t^\bullet(\mathbb{z}; D_h)})$, no D_h-geodesic from outside of $\mathcal{B}_t^\bullet(\mathbb{z}; D_h) \cup B_{5r}(z)$ can enter $B_r(z)$ before hitting $\mathcal{B}_t^\bullet(\mathbb{z}; D_h)$ (the precise definition of the event is given in [36, Section 3.2]). We will not need the precise definition of $\rho_r^n(z)$ here, only a few facts which we will review in this subsection.

We have $\rho_r^n(z) \geq 6r$ and $\rho_r^n(z)$ is a stopping time for the filtration generated by $h|_{B_{6t}(z)}$ for $t \geq r$. The following is immediate from [36, Lemma 3.4], the translation invariance of the law of h, modulo additive constant, and Axiom IV (translation invariance).

Lemma 2.13 (Bounds for radii used to control geodesics) *There is a constant* $\eta > 0$ *depending only on the choice of metric such that the following is true. If we abbreviate*

$$\rho_{\mathbb{r},\varepsilon}(z) := \rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z), \tag{2.19}$$

then for each compact set $K \subset \mathbb{C}$*, each* $\mathbb{r} > 0$*, and each* $\mathbb{z} \in \mathbb{C}$*, it holds with probability* $1 - O_\varepsilon(\varepsilon^2)$ *(at a rate depending on* K*, but not on* $\mathbb{r}$ *or* $\mathbb{z}$*) that*

$$\rho_{\mathbb{r},\varepsilon}(z) \leq \varepsilon^{1/2}\mathbb{r}, \quad \forall z \in \left(\frac{\varepsilon\mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(\mathbb{r}K + \mathbb{z}). \tag{2.20}$$

Henceforth fix η as in Lemma 2.13 and let $\rho_{\mathbb{r},\varepsilon}(z)$ be as in (2.19). For $\mathbb{r} > 0$, $\varepsilon > 0$, and a compact set $K \subset \mathbb{C}$, we define

$$R_{\mathbb{r}}^{\varepsilon}(K) := 6 \sup\left\{\rho_{\mathbb{r},\varepsilon}(z) : z \in \left(\frac{\varepsilon\mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(K)\right\} + \varepsilon\mathbb{r}. \tag{2.21}$$

Since $\rho_{\mathbb{r},\varepsilon}(z)$ is a stopping time for the filtration generated by $h|_{B_{6t}(z)}$ for $t \geq r$, each $\rho_{\mathbb{r},\varepsilon}(z)$ for $z \in \left(\frac{\varepsilon\mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(K)$ is a.s. determined by $R_{\mathbb{r}}^{\varepsilon}(K)$ and the restriction of h to $B_{R_{\mathbb{r}}^{\varepsilon}(K)}(K)$. Lemma 2.13 shows that for each fixed choice of K, $\mathbb{P}[R_{\mathbb{r}}^{\varepsilon}(\mathbb{r}K + \mathbb{z}) \leq (6\varepsilon^{1/2} + \varepsilon)\mathbb{r}]$ tends to 1 as $\varepsilon \to 0$, uniformly over all $\mathbb{z} \in \mathbb{C}$ and $\mathbb{r} > 0$.

Recall from Sect. 2.1 that $\mathcal{B}_s^\bullet(\mathbb{z}; D_h)$ for $\mathbb{z} \in \mathbb{C}$ and $s > 0$ denotes the filled D_h-ball of radius s centered at $\mathbb{z}$. Throughout the rest of this subsection we fix $\mathbb{z} \in \mathbb{C}$ and abbreviate $\mathcal{B}_s^\bullet := \mathcal{B}_s^\bullet(\mathbb{z}; D_h)$. For $s > 0$, define

$$\sigma_{s,\mathbb{r}}^{\varepsilon} = \sigma_{s,\mathbb{r}}^{\varepsilon}(\mathbb{z}) := \inf\left\{s' > s : B_{R_{\mathbb{r}}^{\varepsilon}(\mathcal{B}_s^\bullet)}(\mathcal{B}_s^\bullet) \subset \mathcal{B}_{s'}^\bullet\right\}. \tag{2.22}$$

We observe that if τ is a stopping time for $\left\{\left(\mathcal{B}_t^\bullet, h|_{\mathcal{B}_t^\bullet}\right)\right\}_{t\geq 0}$, then so is $\sigma_{\tau,\mathbb{r}}^{\varepsilon}$. The following lemma is used to prevent D_h-geodesics from getting near a specified boundary point of a D_h-metric ball. It is an immediate consequence of [36, Lemma 3.6] (which is the case when $\mathbb{z} = 0$) together with the translation invariance of the law of h, modulo additive constant, and Axiom IV (translation invariance).

Lemma 2.14 (Geodesics are unlikely to get near a specified point of $\partial\mathcal{B}_\tau^\bullet$) *There exists $\alpha > 0$, depending only on the choice of metric, such that the following is true. Let $r > 0$, let τ be a stopping time for the filtration generated by $\left\{\left(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet}\right)\right\}_{s\geq 0}$, and let $x \in \partial\mathcal{B}_\tau^\bullet$ and $\varepsilon \in (0,1)$ be chosen in a manner depending only on $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$. There is an event $G_x^\varepsilon \in \sigma\left(\mathcal{B}_{\sigma_{\tau,r}^\varepsilon}^\bullet, h|_{B_{\sigma_{\tau,r}^\varepsilon}^\bullet}\right)$ with the following properties.*

A. If $R_r^\varepsilon(\mathcal{B}_\tau^\bullet) \leq \operatorname{diam} \mathcal{B}_\tau^\bullet$ and G_x^ε occurs, then no D_h-geodesic from $\mathbb{z}$ to a point in $\mathbb{C}\setminus\mathcal{B}_{\sigma_{\tau,\mathbb{r}}^\varepsilon}^\bullet$ can enter $B_{\varepsilon r}(x)\setminus\mathcal{B}_\tau^\bullet$.
B. There is a deterministic constant $C_0 > 1$ depending only on the choice of metric such that a.s. $\mathbb{P}\left[G_x^\varepsilon \mid \mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet}\right] \geq 1 - C_0\varepsilon^\alpha$.

We will now state a confluence property for LQG geodesics started from $\mathbb{z}$. Each point $x \in \partial\mathcal{B}_s^\bullet$ lies at D_h-distance exactly s from $\mathbb{z}$, so every D_h-geodesic from $\mathbb{z}$ to x stays in $\mathcal{B}_s^\bullet$. For some atypical points x there might be many such D_h-geodesics. But, it is shown in [36, Lemma 2.4] that there is always a distinguished D_h-geodesic from $\mathbb{z}$ to x, called the *leftmost geodesic*, which lies (weakly) to the left of every other D_h-geodesic from $\mathbb{z}$ to x if we stand at x and look outward from $\mathcal{B}_s^\bullet$. The following is [36, Theorem 1.4].

Theorem 2.15 (Confluence of geodesics across a metric annulus) *Almost surely, for each $0 < t < s < \infty$ there is a finite set of D_h-geodesics from $\mathbb{z}$ to $\partial\mathcal{B}_t^\bullet$ such that every leftmost D_h-geodesic from $\mathbb{z}$ to $\partial\mathcal{B}_s^\bullet$ coincides with one of these D_h-geodesics on the time interval $[0,t]$. In particular, there are a.s. only finitely many points of $\partial\mathcal{B}_t^\bullet$ which are hit by leftmost D_h-geodesics from $\mathbb{z}$ to $\partial\mathcal{B}_s^\bullet$.*

Combined with [36, Lemma 2.7], Theorem 2.15 tells us that we can decompose $\partial\mathcal{B}_s^\bullet$ into a finite union of boundary arcs such that for any points $x, y \in \partial\mathcal{B}_s^\bullet$ which lie in the same arc, the leftmost D_h-geodesics from $\mathbb{z}$ to

x and from $\mathbb{z}$ to y coincide in the time interval $[0, t]$. We will need a more quantitative version of Theorem 2.15 which gives us stretched exponential concentration for the number of such arcs if we truncate on a certain high-probability regularity event. To this end, we define

$$\tau_r(\mathbb{z}) := D_h(\mathbb{z}, \partial B_r(\mathbb{z})) = \inf\left\{s > 0 : \mathcal{B}_s^\bullet \not\subset B_r(\mathbb{z})\right\}, \quad \forall r > 0. \tag{2.23}$$

We also fix $\chi \in (0, \xi(Q-2))$, chosen in a manner depending only on ξ and Q, so that by Lemma 2.8 D_h is a.s. locally χ-Hölder continuous w.r.t. the Euclidean metric. For $\mathbb{r} > 0$ and $a \in (0,1)$, we define $\mathcal{E}_{\mathbb{r}}^{\mathbb{z}}(a)$ to be the event that the following is true.

1. *(Comparison of D_h-balls and Euclidean balls)* $B_{a\mathbb{r}}(\mathbb{z}) \subset \mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$ and $\tau_{3\mathbb{r}} - \tau_{2\mathbb{r}} \geq a \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$.
2. *(One-sided Hölder continuity)* $\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h(u, v) \leq \left(\frac{|u-v|}{\mathbb{r}}\right)^\chi$ for each $u, v \in B_{4\mathbb{r}}(0)$ with $|u-v|/\mathbb{r} \leq a$.
3. *(Bounds for radii used to control geodesics)* The radii of Lemma 2.13 satisfy $\rho_{\mathbb{r},\varepsilon}(z) \leq \varepsilon^{1/2}\mathbb{r}$ for each $z \in \left(\frac{\varepsilon \mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{4\mathbb{r}}(\mathbb{z})$ and each dyadic $\varepsilon \in (0, a]$.

It is easy to see that $\mathbb{P}[\mathcal{E}_{\mathbb{r}}^{\mathbb{z}}(a)] \to 1$ as $a \to 0$, uniformly over the choice of $\mathbb{r}$ and $\mathbb{z}$: in particular, this follows from [36, Lemma 3.8] (which is the case when $\mathbb{z} = 0$) and Axiom IV. We will in fact show in Sect. 4.3 that with high probability, $\mathcal{E}_{\mathbb{r}}^{\mathbb{z}}(a)$ occurs *simultaneously* for all $\mathbb{z}$ in a fixed bounded open subset of $\mathbb{C}$. The following more quantitative version of Theorem 2.15 is [36, Theorem 3.9].

Theorem 2.16 (Quantitative confluence of geodesics) *For each $a \in (0,1)$, there is a constant $b_0 > 0$ depending only on a and constants $b_1, \beta > 0$ depending only on the choice of metric D such that the following is true. For each $\mathbb{z} \in \mathbb{C}$, each $\mathbb{r} > 0$, each $N \in \mathbb{N}$, and each stopping time τ for $\{(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet})\}_{s \geq 0}$ with $\tau \in [\tau_{\mathbb{r}}(\mathbb{z}), \tau_{2\mathbb{r}}(\mathbb{z})]$ a.s., the probability that $\mathcal{E}_{\mathbb{r}}^{\mathbb{z}}(a)$ occurs and there are more than N points of $\partial \mathcal{B}_\tau^\bullet$ which are hit by leftmost D_h-geodesics from $\mathbb{z}$ to $\partial \mathcal{B}^\bullet_{\tau + N^{-\beta}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}}$ is at most $b_0 e^{-b_1 N^\beta}$.*

3 The optimal bi-Lipschitz constant

Throughout this section, we assume that we are in the setting of Theorem 1.9, so that D and $\widetilde{D}$ are two weak γ-LQG metrics with the same scaling constants. We also let h be a whole-plane GFF. We know from Proposition 2.2 that D_h and $\widetilde{D}_h$ are a.s. bi-Lipschitz equivalent. We define the optimal bi-Lipschitz constants c_* and C_* as in (1.21). Since D_h and $\widetilde{D}_h$ are a.s. bi-Lipschitz equivalent (Proposition 2.2), a.s. $0 < c_* \leq C_* < \infty$.

Lemma 3.1 *Each of c_* and C_* is a.s. equal to a deterministic constant.*

Proof We will prove the statement for C_*; the statement for c_* is proven in an identical manner. Suppose $C > 0$ is such that $\mathbb{P}[C_* > C] > 0$. We will show that in fact $\mathbb{P}[C_* > C] = 1$.

There is some large deterministic $R > 0$ such that with positive probability, there are points $u, v \in B_R(0)$ such that $\widetilde{D}_h(u, v)/D_h(u, v) > C$. Since each of D_h and $\widetilde{D}_h$ induces the Euclidean topology on $\mathbb{C}$, after possibly increasing R, we can arrange that with positive probability, there are points $u, v \in B_R(0)$ such that

$$\begin{aligned} &\widetilde{D}_h(u, v)/D_h(u, v) > C, \quad D_h(u, v) \le D_h(u, \partial B_R(0)), \\ &\quad \text{and} \quad \widetilde{D}_h(u, v) \le \widetilde{D}_h(u, \partial B_R(0)). \end{aligned} \tag{3.1}$$

The condition that $D_h(u, v) \le D_h(u, \partial B_R(0))$ is equivalent to the condition that v is contained in the D_h-metric ball of radius $D_h(u, \partial B_R(0))$ centered at u. By Axiom II (locality), it follows that $h|_{B_R(0)}$ a.s. determines $D_h(u, \partial B_R(0))$ for every $u \in B_R(0)$ and hence also $h|_{B_R(0)}$ determines all of the D_h-metric balls of radius $D_h(u, \partial B_R(0))$ centered at points of $B_R(0)$. Similar considerations hold with $\widetilde{D}_h$ in place of D_h. Therefore, the event that there exist $u, v \in B_R(0)$ such that (3.1) holds is determined by $h|_{B_R(0)}$. In fact, by Axiom III (Weyl scaling) this event is determined by $h|_{B_R(0)}$ *viewed modulo additive constant*, since adding a constant to h results in scaling D_h and $\widetilde{D}_h$ by the same constant factor.

For $z \in \mathbb{C}$, let $E(z)$ be the event that there exist points $u, v \in B_R(z)$ such that (3.1) holds with $B_R(z)$ in place of $B_R(0)$. Then $E(z)$ is determined by $h|_{B_R(z)}$, viewed modulo additive constant. By Axiom IV (translation invariance) and the translation invariance of the law of h, modulo additive constant, the probability of $E(z)$ does not depend on z. The event that $E(z)$ occurs for infinitely many $z \in \mathbb{Z}^2$ is determined by the tail σ-algebra generated by $h|_{\mathbb{C}\setminus B_r(z)}$, viewed modulo additive constant, as $r \to \infty$. This tail σ-algebra is trivial, so we get that a.s. $E(z)$ occurs for infinitely many $z \in \mathbb{C}$. This means that in fact $\mathbb{P}[C_* > C] = 1$, so C_* is a.s. equal to a deterministic constant. □

We henceforth re-define each of c_* and C_* on an event of probability zero so that they are deterministic. The main goal of this section is to show that there are many values of $r > 0$ for which it holds with uniformly positive probability that there are points $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ such that $|\mathbb{z}|$, $|\mathbb{w}|$, and $|\mathbb{z} - \mathbb{w}|$ are all of order r and $\widetilde{D}_h(\mathbb{z}, \mathbb{w})/D_h(\mathbb{z}, \mathbb{w})$ is close to C_* (resp. c_*). To quantify this, we introduce the following events. For $r > 0$, $C' \in (0, C_*]$, and $\beta \in (0, 1)$, define

$$\overline{G}_r(C', \beta) := \left\{ \exists \mathbb{z}, \mathbb{w} \in B_r(0) \text{ s.t. } |\mathbb{z} - \mathbb{w}| \geq \beta r \text{ and } \widetilde{D}_h(\mathbb{z}, \mathbb{w}) \geq C' D_h(\mathbb{z}, \mathbb{w}) \right\}. \tag{3.2}$$

For $c' \geq c_*$, we similarly define

$$\underline{G}_r(c', \beta) := \left\{ \exists \mathbb{z}, \mathbb{w} \in B_r(0) \text{ s.t. } |\mathbb{z} - \mathbb{w}| \geq \beta r \text{ and } \widetilde{D}_h(\mathbb{z}, \mathbb{w}) \leq c' D_h(\mathbb{z}, \mathbb{w}) \right\}. \tag{3.3}$$

It is easy to see from the definition (1.21) of C_* that for each fixed $r > 0$ and $C' \in (0, C_*)$, there exists $p, \beta \in (0, 1)$ (allowed to depend on C' and r) such that $\mathbb{P}[\overline{G}_r(C', \beta)] \geq p$.[5] Since we are working with weak LQG metrics, which are not known to be exactly invariant under spatial scaling, it is *not* clear a priori that p and β can be taken to be uniform in the choice of r. It is also not clear a priori that p and β can be chosen independently of C'. Similar considerations apply for $\underline{G}_r(c', \beta)$. We will establish that one can choose p and β independently of C' and r provided r is restricted to lie in a suitably "dense" subset of (0, 1), in the following sense.

Proposition 3.2 *For each $0 < \mu < \nu < 1$, there exists $\overline{\beta} = \overline{\beta}(\mu, \nu) \in (0, 1)$ and $\overline{p} = \overline{p}(\mu, \nu) \in (0, 1)$ such that for each $C' \in (0, C_*)$ and each sufficiently small $\varepsilon > 0$ (depending on C'), there are at least $\mu \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{8^{-k} : k \in \mathbb{N}\}$ for which $\mathbb{P}[\overline{G}_r(C', \overline{\beta})] \geq \overline{p}$.*

Proposition 3.3 *For each $0 < \mu < \nu < 1$, there exists $\underline{\beta} = \underline{\beta}(\mu, \nu) \in (1/2, 1)$ and $\underline{p} = \underline{p}(\mu, \nu) \in (0, 1)$ such that for each $c' > c_*$ and each sufficiently small $\varepsilon > 0$ (depending on c'), there are at least $\mu \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{8^{-k} : k \in \mathbb{N}\}$ for which $\mathbb{P}[\underline{G}_r(c', \underline{\beta})] \geq \underline{p}$.*

We emphasize that the parameters $\overline{\beta}, \overline{p}$ in Proposition 3.2 (resp. the parameters $\underline{\beta}, \underline{p}$ in Proposition 3.3) *do not* depend on C' (resp. c'). The only thing which depends on C' (resp. c') is how small ε has to be in order for the conclusion of the proposition statement to hold.

[5] By the definition (1.21) of C_*, there exists some $p, \beta \in (0, 1)$ and $R > 0$ (allowed to depend on r) such that with probability at least p, there exists $\mathbb{z}, \mathbb{w} \in B_{Rr}(0)$ such that $|\mathbb{z} - \mathbb{w}| \geq \beta r$ and $\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \geq C' D_h(\mathbb{z}, \mathbb{w})$. We need to replace $B_{Rr}(0)$ by $B_r(0)$. By possibly replacing $\mathbb{z}$ and $\mathbb{w}$ by a pair of points along a D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$, we can arrange that in fact $|\mathbb{z} - \mathbb{w}| = \beta r$. We can cover $B_{Rr}(0)$ by at most a β, R-dependent constant number N of Euclidean balls of the form $B_r(z)$ for $z \in B_{Rr}(0)$ such that any two points $\mathbb{z}, \mathbb{w} \in B_{Rr}(0)$ with $|\mathbb{z} - \mathbb{w}| = \beta r$ are contained in one of these balls. By Weyl scaling (Axiom III), the translation invariance of the law of h modulo additive constant, and Axiom IV, the probability that there exists $\mathbb{z}, \mathbb{w} \in B_r(z)$ with $|\mathbb{z} - \mathbb{w}| \geq \beta r$ and $\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \geq C' D_h(\mathbb{z}, \mathbb{w})$ does not depend on z. By a union bound, it therefore follows that $\mathbb{P}[\overline{G}_r(C', \beta)] \geq p/N$.

3.1 Quantitative versions of Propositions 3.2 and 3.3

We will need more quantitative versions of Propositions 3.2 and 3.3 which differ from the original proposition statements in two important ways. First, instead of starting at a constant-order scale, we will start at some given scale $\mathbb{r} > 0$ for which we have an a priori lower bound on $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)]$ for some $C'' \in (0, C_*)$ and $\beta \in (0,1)$ (or $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)]$ for some $c'' > c_*$ and $\beta \in (0,1)$). We will then produce many radii in $[\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ instead of in $[\varepsilon^{1+\nu}, \varepsilon]$. The reason for introducing $\mathbb{r}$ is that we only have tightness across scales (Axiom V) instead of true scale invariance. Second, instead of just lower bounding the probability of $\overline{G}_r(C', \beta)$ or $\underline{G}_r(c', \beta)$, we will obtain a lower bound for the probability of a smaller event which is more complicated, but also more useful. Let us begin by stating a more quantitative version of Proposition 3.2.

Proposition 3.4 *For each* $0 < \mu < \nu < 1$*, there exists* $\alpha_* = \alpha_*(\mu, \nu) \in (1/2, 1)$ *and* $p = p(\mu, \nu) \in (0,1)$ *such that for each* $\alpha \in [\alpha_*, 1)$ *and each* $C' \in (0, C_*)$*, there exists* $C'' = C''(\alpha, C', \mu, \nu) \in (C', C_*)$ *such that for each* $\beta \in (0,1)$*, there exists* $\varepsilon_0 = \varepsilon_0(\beta, \alpha, C', \mu, \nu) > 0$ *such that the following holds for each* $\mathbb{r} > 0$ *for which* $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)] \geq \beta$ *and each* $\varepsilon \in (0, \varepsilon_0]$.

(A) *There are at least* $\mu \log_8 \varepsilon^{-1}$ *values of* $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ *for which the following holds with probability at least* p*. There exists* $u \in \partial B_{\alpha r}(0)$ *and* $v \in \partial B_r(0)$ *such that*

$$\widetilde{D}_h(u, v) \geq C' D_h(u, v) \tag{3.4}$$

and the D_h*-geodesic from* u *to* v *is unique and is contained in* $\overline{\mathbb{A}_{\alpha r, r}(0)}$.

The event described in (A) is contained in $\overline{G}_r(C', 1-\alpha)$, so if (A) holds for some $\mathbb{r} > 0$ then there are at least $\mu \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k} : k \in \mathbb{N}\}$ such that

$$\mathbb{P}[\overline{G}_r(C', 1-\alpha)] \geq p.$$

Furthermore, as explained in Footnote 5, the definition (1.21) of C_* implies that for any $C'' \in (0, C_*)$, there exists some $\beta \in (0,1)$ such that $\mathbb{P}[\overline{G}_1(C'', \beta)] \geq \beta$. Therefore, Proposition 3.4 applied with $\mathbb{r} = 1$ implies Proposition 3.2 with $\overline{\beta} = 1 - \alpha$ and $\overline{p} = p$.

By the symmetry between our hypotheses on $\widetilde{D}_h$ and D_h, Proposition 3.4 implies the analogous statement with the roles of D_h and $\widetilde{D}_h$ interchanged, which reads as follows.

Proposition 3.5 *For each* $0 < \mu < \nu < 1$*, there exists* $\alpha_* = \alpha_*(\mu, \nu) \in (1/2, 1)$ *and* $p = p(\mu, \nu) \in (0,1)$ *such that for each* $\alpha \in [\alpha_*, 1)$ *and each*

$c' > c_*$, there exists $c'' = c''(\alpha, c', \mu, \nu) \in (c_*, c')$ such that for each $\beta \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0(\alpha, \beta, c', \mu, \nu) > 0$ such that the following holds for each $\mathbb{r} > 0$ for which $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \geq \beta$ and each $\varepsilon \in (0, \varepsilon_0]$.

(A') *There are at least $\mu \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ for which it holds with probability at least p that the following is true. There exists $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ such that*

$$\widetilde{D}_h(u, v) \leq c' D_h(u, v) \tag{3.5}$$

and the $\widetilde{D}_h$-geodesic from u to v is unique and is contained in $\overline{\mathbb{A}_{\alpha r, r}(0)}$.

As in the case of Proposition 3.4, Proposition 3.5 immediately implies Proposition 3.3.

To prove Proposition 3.4, we will (roughly speaking) prove the contrapositive.

Proposition 3.6 *For each $0 < \mu < \nu < 1$, there exists $\alpha_* = \alpha_*(\mu, \nu) \in (1/2, 1)$ and $p = p(\mu, \nu) \in (0, 1)$ such that for each $\alpha \in [\alpha_*, 1)$ and each $C' \in (0, C_*)$, there exists $C'' = C''(\alpha, C', \mu, \nu) \in (C', C_*)$ such that for each $\beta \in (0, 1)$, there exists $\varepsilon_0 = \varepsilon_0(\alpha, \beta, C', \mu, \nu) > 0$ such that if $\mathbb{r} > 0$ and there exists $\varepsilon \in (0, \varepsilon_0]$ satisfying the condition (B) just below, then $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)] < \beta$.*

(B) *There are at least $(\nu - \mu) \log_8 \varepsilon^{-1}$ values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ for which it holds with probability at least $1 - p$ that the following is true. For each $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ for which the D_h-geodesic from u to v is unique and is contained in $\overline{\mathbb{A}_{\alpha r, r}(0)}$, one has*

$$\widetilde{D}_h(u, v) \leq C' D_h(u, v). \tag{3.6}$$

Proof of Proposition 3.4, assuming Proposition 3.6 Assume we are given $0 < \mu < \nu < 1$ and let α_*, p be chosen as in Proposition 3.6. Also fix $\alpha \in [\alpha_*, 1)$, $C' \in (0, C_*)$, and $\beta \in (0, 1)$ and let C'' and ε_0 be chosen as in Proposition 3.6. For $\mathbb{r}, \varepsilon > 0$, let $\mathcal{K}_{\mathbb{r}}^{\varepsilon} := [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \left\{8^{-k}\mathbb{r} : k \in \mathbb{N}\right\}$ and note that $\#\mathcal{K}_{\mathbb{r}}^{\varepsilon} = \lfloor \nu \log_8 \varepsilon^{-1} \rfloor$.

If (A) does not hold for some $\mathbb{r} > 0$ and $\varepsilon \in (0, \varepsilon_0]$, then there are *fewer* than $\mu \log_8 \varepsilon^{-1}$ values of $k \in \mathcal{K}_{\mathbb{r}}^{\varepsilon}$ for which the last sentence of (A) holds with probability at least p. For such a choice of $\mathbb{r}$ and ε, there are at least $(\nu - \mu) \log_8 \varepsilon^{-1}$ values of $k \in \mathcal{K}_{\mathbb{r}}^{\varepsilon}$ for which the last sentence of (B) holds with probability at least $1 - p$. That is, (B) holds for the pair $(\mathbb{r}, \varepsilon)$. By Proposition 3.6, this means that $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)] < \beta$. Hence we have proven the contrapositive of Proposition 3.4. □

3.2 Proof of Proposition 3.6

As explained in Sect. 3.1, to prove all of the propositions statements from earlier in this section it remains only to prove Proposition 3.6. The basic idea of the proof is as follows. If we assume that (B) holds for a small enough choice of $p \in (0, 1)$ (depending only on μ and ν), then we can use Lemma 2.6 to cover space by Euclidean balls of the form $B_{r/2}(z)$ for $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ with the following property. For each $u \in \partial B_{\alpha r}(z)$ and each $v \in \partial B_r(z)$ such that the D_h-geodesic from u to v is unique and is contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$, we have $\widetilde{D}_h(u, v) \le C' D_h(u, v)$. By considering the times when a D_h-geodesic between two fixed points $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ crosses the annulus $\mathbb{A}_{\alpha r,r}(z)$ for such a z and r, we will be able to show that $\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \le C'' D_h(\mathbb{z}, \mathbb{w})$ for a suitable constant $C'' \in (C', C_*)$. Applying this to an appropriate β-dependent collection of pairs of points $(\mathbb{z}, \mathbb{w})$ will show that $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)] < \beta$. The reason why we need to make α close to 1 is to ensure that the events we consider depend on h in a sufficiently "local" manner (see the discussion just after the definition of $\mathsf{E}_r(z)$ below).

Let us now define the events to which we will apply Lemma 2.6. For $z \in \mathbb{C}$, $r > 0$, and parameters $\alpha \in (1/2, 1)$, $A > 1$ and $C' \in (0, C_*)$, let $\mathsf{E}_r(z) = \mathsf{E}_r(z; \alpha, A, C')$ be the event that the following is true.

1. *(Comparison of D_h and $\widetilde{D}_h$)* For each $u \in \partial B_{\alpha r}(z)$ and each $v \in \partial B_r(z)$ such that the D_h-geodesic from u to v is unique and is contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$, we have $\widetilde{D}_h(u, v) \le C' D_h(u, v)$.
2. *(Lower bound for paths in $\mathbb{A}_{\alpha r,r}(z)$)* If $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$ are such that either $D_h(u, v) > D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ or $\widetilde{D}_h(u, v) > \widetilde{D}_h(u, \partial \mathbb{A}_{r/2,2r}(z))$, then each path from u to v which stays in $\overline{\mathbb{A}_{\alpha r,r}(z)}$ has D_h-length strictly larger than $D_h\left(u, v; \mathbb{A}_{r/2,2r}(z)\right)$.
3. *(Distance around $\mathbb{A}_{\alpha r,r}(z)$) There is a path in $\mathbb{A}_{\alpha r,r}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{\alpha r,r}(z)$ and has D_h-length at most $A D_h\left(\partial B_{\alpha r}(z), \partial B_r(z)\right)$.*

Condition 1 is the main point of the event $\mathsf{E}_r(z)$, as discussed just above. The purpose of condition 2 is to ensure that $\mathsf{E}_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Without this condition, we would not necessarily be able to tell whether a path in $\overline{\mathbb{A}_{\alpha r,r}(z)}$ is a D_h-geodesic without seeing the field outside of $\mathbb{A}_{r/2,2r}(z)$ (see Lemma 3.7). The purpose of condition 3 is as follows. If a D_h-geodesic between two points outside of $B_r(z)$ enters $B_{\alpha r}(z)$, then it must cross the path from condition 3 twice. This means that it can spend at most $A D_h\left(\partial B_{\alpha r}(z), \partial B_r(z)\right)$ units of time in $B_{\alpha r}(z)$ since otherwise the path from condition 3 would provide a shortcut, which would contradict the definition of a geodesic. If we assume (B), this fact will eventually allow us to force a D_h-geodesic to spend a positive fraction of its time tracing segments between points u, v with $\widetilde{D}_h(u, v) \le C' D_h(u, v)$.

We want to use Lemma 2.6 to argue that if (B) holds, then with high probability there are many values of $z \in \mathbb{C}$ such that $\mathsf{E}_r(z)$ occurs for some $r \in [\varepsilon^{1+\nu}\mathbbm{r}, \varepsilon\mathbbm{r}]$. We first check the measurability condition in Lemma 2.6

Lemma 3.7 *For each $z \in \mathbb{C}$ and $r > 0$,*

$$\mathsf{E}_r(z) \in \sigma\left((h - h_{4r}(z))|_{\mathbb{A}_{r/2,2r}(z)}\right). \tag{3.7}$$

Proof By Axiom III (Weyl scaling) subtracting $h_{4r}(0)$ from h results in scaling D_h and $\widetilde{D}_h$ by the same factor, so does not affect the occurrence of $\mathsf{E}_r(z)$. Hence it suffices to prove (3.7) with $h|_{\mathbb{A}_{r/2,2r}(z)}$ in place of $(h - h_{4r}(z))|_{\mathbb{A}_{r/2,2r}(z)}$. From Axiom III, it is obvious that condition 3 in the definition of $\mathsf{E}_r(z)$ (distance around $\mathbb{A}_{\alpha r,r}(z)$) is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$.

For $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$, we can determine whether $D_h(u, v) > D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ from the internal metric $D_h\left(\cdot, \cdot; \mathbb{A}_{r/2,2r}(z)\right)$: indeed, $D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ is clearly determined by this internal metric and $D_h(u, v) \leq D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ if and only if v is contained in the D_h-ball of radius $D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ centered at v, which is contained in $\overline{\mathbb{A}_{r/2,2r}(z)}$. Similar considerations hold with $\widetilde{D}_h$ in place of D_h. Hence condition 2 in the definition of $\mathsf{E}_r(z)$ (lower bound for paths in $\mathbb{A}_{\alpha r,r}(z)$) is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$.

If P is a path from $u \in \partial B_{\alpha r}(z)$ to $v \in \partial B_r(z)$ which stays in $\overline{\mathbb{A}_{\alpha r,r}(z)}$, then P is a D_h-geodesic if and only if $\mathrm{len}(P; D_h) = D_h(u, v)$. Hence if condition 2 holds, then P cannot be a D_h-geodesic unless $D_h(u, v) \leq D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ and $\widetilde{D}_h(u, v) \leq \widetilde{D}_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ (note that $D_h(u, v; \mathbb{A}_{r/2,2r}(z)) \geq D_h(u, v)$), in which case we can tell whether P is a D_h-geodesic from the restriction of h to the D_h-metric ball of radius $D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ centered at u, which in turn is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Furthermore, on the event that $D_h(u, v) \leq D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ and $\widetilde{D}_h(u, v) \leq \widetilde{D}_h(u, \partial \mathbb{A}_{r/2,2r}(z))$, both $D_h(u, v)$ and $\widetilde{D}_h(u, v)$ are determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Therefore, the intersection of conditions 1 (comparison of D_h and $\widetilde{D}_h$) and 2 in the definition of $\mathsf{E}_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Hence we have proven (3.7). □

We now show that (B) implies a lower bound for $\mathbb{P}[\mathsf{E}_r(z)]$ for some values of $r \in [\varepsilon^{1+\nu}\mathbbm{r}, \varepsilon\mathbbm{r}]$.

Lemma 3.8 *For each $0 < \mu < \nu < 1$ and each $q > 0$, there exists $\alpha_* \in (1/2, 1)$ and $p \in (0, 1)$ depending only on q, μ, ν such that for each $\alpha \in [\alpha_*, 1)$, there exists $A = A(\alpha, q, \mu, \nu) > 1$ such that the following is true for each $C' \in (0, C_*)$. If $\mathbbm{r} > 0$ and $\varepsilon \in (0, 1)$ such that (B) holds for the above choice of p, α, C', then*

$$\begin{aligned}&\mathbb{P}\left[\mathsf{E}_r(z) \text{ occurs for at least one } r \in [\varepsilon^{1+\mu}\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \{8^{-k}\mathbbm{r} : k \in \mathbb{N}\}\right] \\ &\quad \geq 1 - O_\varepsilon(\varepsilon^q), \quad \forall z \in \mathbb{C},\end{aligned} \tag{3.8}$$

at a rate which is uniform over the choices of z and $\mathbb{r}$.

Proof Assume (B) is satisfied for some choice of $\mathbb{r}, \varepsilon, p, \alpha, C'$ and let $r_1, \ldots, r_K \in [\varepsilon^{1+\mu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ be the values of r from (B), enumerated in decreasing order. Note that $K \geq (\nu - \mu)\log_8 \varepsilon^{-1}$ by assumption. By Lemma 3.7, we can apply Lemma 2.6 to find that there exists $\widetilde{p} = \widetilde{p}(q, \mu, \nu) \in (0, 1)$ such that if

$$\mathbb{P}[\mathsf{E}_{r_k}(z)] \geq \widetilde{p}, \quad \forall z \in \mathbb{C}, \quad \forall k \in [1, K]_{\mathbb{Z}}, \tag{3.9}$$

then (3.8) holds. It therefore suffices to choose p, α_*, and A in an appropriate manner depending on $\widetilde{p}$ so that if (B) holds, then (3.9) holds.

By tightness across scales (Axiom V), we can find $S > s > 0$ depending on $\widetilde{p}$ such that for each $z \in \mathbb{C}$ and $r > 0$, it holds with probability at least $1 - (1 - \widetilde{p})/4$ that

$$\begin{aligned} &D_h\left(\partial B_r(z), \partial \mathbb{A}_{r/2,2r}(z)\right) \geq s\mathfrak{c}_r e^{\xi h_r(z)} \quad \text{and} \\ &\sup_{u,v \in \mathbb{A}_{3r/4,r}(z)} D_h\left(u, v; \mathbb{A}_{r/2,2r}(z)\right) \leq S\mathfrak{c}_r e^{\xi h_r(z)}, \end{aligned} \tag{3.10}$$

and the same is true with $\widetilde{D}_h$ in place of D_h. Since $\mathbb{A}_{\alpha r,r}(z) \subset \mathbb{A}_{3r/4,r}(z)$ for any choice of $\alpha \in [3/4, 1)$, Lemma 2.11 with the above choice of s and S gives an $\alpha_* \in [3/4, 1)$ depending on $\widetilde{p}$ such that for each $\alpha \in [\alpha_*, 1)$, $z \in \mathbb{C}$, and $r > 0$, condition 2 (lower bound for paths in $\mathbb{A}_{\alpha r,r}(z)$) in the definition of $\mathsf{E}_r(z)$ holds with probability at least $1 - (1 - \widetilde{p})/3$.

Now suppose $\alpha \in [\alpha_*, 1)$. We can again apply Axiom V (tightness across scales) to find that there exists $A > 1$ depending on α and $\widetilde{p}$ such that for each $z \in \mathbb{C}$ and $r > 0$, condition 3 (distance around $\mathbb{A}_{\alpha r,r}(z)$) in the definition of $\mathsf{E}_r(z)$ occurs with probability at least $1 - (1 - \widetilde{p})/3$.

If (B) holds for the above choice of α and with $p < (1 - \widetilde{p})/3$, then for each $z \in \mathbb{C}$ and each $k \in [1, K]_{\mathbb{Z}}$, condition 1 (comparison of D_h and $\widetilde{D}_h$) in the definition of $\mathsf{E}_{r_k}(z)$ holds with probability at least $1 - (1 - \widetilde{p})/3$. Combining the three preceding paragraphs shows that (3.9) holds. □

Lemma 3.9 *There is a $q > 1$ depending only on μ, ν such that if p, α_*, $\alpha \in [\alpha_*, 1)$, and A is chosen as in Lemma 3.8 for this choice of q, then the following is true for each $C' \in (0, C_*)$. If (B) holds for some $\mathbb{r} > 0$ and $\varepsilon \in (0, 1)$ and for this choice of p, α, C', then for each open set $U \subset \mathbb{C}$, it holds with probability tending to 1 as $\varepsilon \to 0$ (at a rate which is uniform in $\mathbb{r}$) that for $z \in \mathbb{r}U$, there exists $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ and $w \in \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{100}\mathbb{Z}^2\right) \cap (\mathbb{r}U)$ such that $z \in B_{r/2}(w)$ and $\mathsf{E}_r(w)$ occurs.*

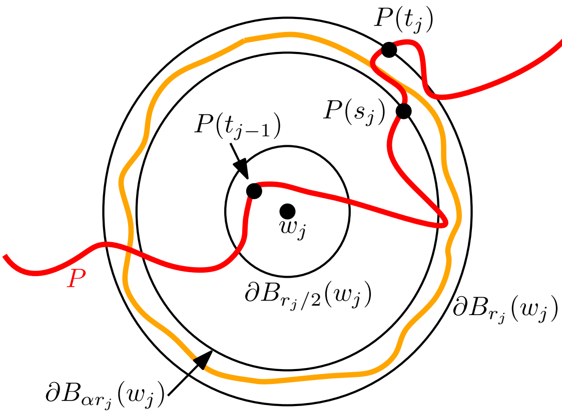

Fig. 2 Illustration of the proof of Proposition 3.6. The D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ along with one of the balls $B_{r_j}(w_j)$ hit by P for which $\mathsf{E}_{r_j}(w_j)$ occurs are shown. The time t_j is the first time that P exits $B_{r_j}(w_j)$ after time t_{j-1} and the time s_j is the last time before t_j at which P hits $\partial B_{\alpha r_j}(w_j)$. Condition 1 in the definition of $\mathsf{E}_{r_j}(w_j)$ shows that $\widetilde{D}_h(P(s_j), P(t_j)) \le C'(t_j - s_j)$. The orange path comes from condition 3 in the definition of $\mathsf{E}_{r_j}(w_j)$, and its D_h-length is at most $A D_h(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)) \le A(t_j - s_j)$. Since P crosses this orange path both before time t_{j-1} and after time s_j and P is a D_h-geodesic, we have that $s_j - t_{j-1} \le A(t_j - s_j)$. This shows that the intervals $[s_j, t_j]$ occupy a uniformly positive fraction of the total D_h-length of P, which in turn allows us to show that $\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \le C'' D_h(\mathbb{z}, \mathbb{w})$ for a constant $C'' \in (C', C_*)$ depending only on C', A

Proof Upon choosing q sufficiently large, this follows from Lemma 3.8 and a union bound over all $w \in \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{100}\mathbb{Z}^2\right) \cap (\mathbb{r}U)$. $\square$

Proof of Proposition 3.6 See Fig. 2 for an illustration of the proof.

Step 1: setup Let $p, \alpha_*, \alpha \in [\alpha_*, 1)$, and $A > 1$ be chosen as in Lemma 3.9. Also fix

$$C'' \in \left(C' + \frac{A}{A+1}(C_* - C'), C_*\right), \tag{3.11}$$

and note that we can choose C'' in a manner depending only on α, C', μ, ν (since A depends only on α, μ, ν).

We will show that there exists $\varepsilon_0 = \varepsilon_0(\beta, \alpha, C', \mu, \nu) > 0$ such that if $\mathbb{r} > 0$, $\varepsilon \in (0, \varepsilon_0]$, and (B) holds for these values of $\mathbb{r}, \varepsilon, p, \alpha$, then with probability greater than $1 - \beta$,

$$\widetilde{D}_h(\mathbb{z}, \mathbb{w}) \le C'' D_h(\mathbb{z}, \mathbb{w}) \quad \forall \mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0) \text{ with } |\mathbb{z} - \mathbb{w}| \ge \beta \mathbb{r}. \tag{3.12}$$

In other words, $\mathbb{P}[\overline{G}_{\mathbb{r}}(C'', \beta)^c] > 1 - \beta$, as required.

By Axiom V (tightness across scales), there is some large bounded open set $U \subset \mathbb{C}$ depending only on β such that for each $\mathbb{r} > 0$, it holds with probability at least $1 - \beta/2$, the D_h-diameter of $B_{\mathbb{r}}(0)$ is smaller than the D_h-distance from $B_{\mathbb{r}}(0)$ to $\partial(\mathbb{r}U)$, in which case every D_h-geodesic between points of $B_{\mathbb{r}}(0)$ is contained in $\mathbb{r}U$. Henceforth fix such a choice of U. Let $F_{\mathbb{r}}^{\varepsilon}$ be the event that every D_h-geodesic between points of $B_{\mathbb{r}}(0)$ is contained in $\mathbb{r}U$ and the event of Lemma 3.9 with the above choices of α, A, C', and U, so that $\mathbb{P}[F^{\varepsilon}] \geq 1 - \beta/2 - o_{\varepsilon}(1)$, uniformly in $\mathbb{r}$, under the assumption (B).

Step 2: covering a D_h-geodesic with paths of short $\widetilde{D}_h$-length To prove (3.12), we consider points $\mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0) \cap \mathbb{Q}^2$ with $|\mathbb{z} - \mathbb{w}| \geq \beta\mathbb{r}$ and let $P : [0, D_h(\mathbb{z}, \mathbb{w})] \to \mathbb{C}$ be the (a.s. unique) D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. Let $t_0 = 0$ and inductively let t_j for $j \in \mathbb{N}$ be the smallest time $t \geq t_{j-1}$ at which P exits a Euclidean ball of the form $B_r(w)$ for $w \in \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{100}\mathbb{Z}^2\right) \cap (\mathbb{r}U)$ and $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r} : k \in \mathbb{N}\}$ such that $P(t_{j-1}) \in B_{r/2}(w)$ and $\mathsf{E}_r(w)$ occurs; or let $t_j = D_h(\mathbb{z}, \mathbb{w})$ if no such t exists. If $t_j < D_h(\mathbb{z}, \mathbb{w})$, let w_j and r_j be the corresponding values of w and r. Also let s_j be the last time before t_j at which P hits $\partial B_{\alpha r_j}(w)$, so that $s_j \in [t_{j-1}, t_j]$ and $P([s_j, t_j]) \subset \overline{\mathbb{A}_{\alpha r_j, r_j}(w_j)}$. Finally, define

$$\underline{J} := \max\left\{j \in \mathbb{N} : |\mathbb{z} - P(t_{j-1})| < 2\varepsilon\mathbb{r}\right\} \quad \text{and}$$
$$\overline{J} := \min\left\{j \in \mathbb{N} : |\mathbb{w} - P(t_{j+1})| < 2\varepsilon\mathbb{r}\right\}. \tag{3.13}$$

The reason for the definitions of $\underline{J}$ and $\overline{J}$ is that $\mathbb{z}, \mathbb{w} \notin B_{r_j}(w_j)$ for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$ (since $r_j \leq \varepsilon\mathbb{r}$ and $P(t_j) \in B_{r_j}(w_j)$). By the definition of $F_{\mathbb{r}}^{\varepsilon}$, on this event we have $t_j < D_h(\mathbb{z}, \mathbb{w})$ and $|P(t_{j-1}) - P(t_j)| \leq 2\varepsilon\mathbb{r}$ whenever $|\mathbb{w} - P(t_{j-1})| \geq \varepsilon\mathbb{r}$. Therefore, on $F_{\mathbb{r}}^{\varepsilon}$,

$$P(t_{\underline{J}}) \in B_{4\varepsilon\mathbb{r}}(\mathbb{z}) \quad \text{and} \quad P(t_{\overline{J}}) \in B_{4\varepsilon\mathbb{r}}(\mathbb{w}). \tag{3.14}$$

Since P is a D_h-geodesic, for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$ also $P|_{[s_j,t_j]}$ is a D_h-geodesic from $P(s_j) \in \partial\mathcal{B}_{\alpha r_j}(w_j)$ to $P(t_j) \in \partial B_{r_j}(w_j)$ and by definition this D_h-geodesic stays in $\overline{\mathbb{A}_{\alpha r_j, r_j}(w_j)}$. Moreover, $P|_{[s_j,t_j]}$ is the only D_h-geodesic from $P(s_j)$ to $P(t_j)$ since otherwise we could re-route P along another such D_h-geodesic to contradict the uniqueness of the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$.

Combining this with condition 1 in the definition of $\mathsf{E}_{r_j}(w_j)$ (comparison of D_h and $\widetilde{D}_h$), applied with $u = P(s_j)$ and $v = P(t_j)$, and the definition (1.21) of C_*, we find that

$$\widetilde{D}_h\left(P(s_j), P(t_j)\right) \leq C'(t_j - s_j) \quad \text{and} \quad \widetilde{D}_h\left(P(t_{j-1}), P(s_j)\right)$$
$$\leq C_*(s_j - t_{j-1}), \quad \forall j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}. \tag{3.15}$$

We will now argue that $s_j - t_{j-1}$ is not too much larger than $t_j - s_j$. If $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$, then since $r_j \leq \varepsilon \mathbbm{r}$ and $|P(t_j) - \mathbbm{z}| \wedge |P(t_j) - \mathbbm{w}| \geq 2\varepsilon\mathbbm{r}$, the D_h-geodesic P must cross the annulus $\mathbb{A}_{\alpha r_j, r_j}(w_j)$ at least once before time t_{j-1} and at least once after time s_j. By condition 3 in the definition of $\mathsf{E}_{r_j}(w_j)$, there is a path disconnecting the inner and outer boundaries of this annulus with D_h-length at most $A D_h\left(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)\right)$. The geodesic P must hit this path at least once before time t_{j-1} and at least once after time s_j. Since P is a geodesic and $P(s_j) \in \partial B_{\alpha r_j}(w_j)$, $P(t_j) \in \partial B_{r_j}(w_j)$, it follows that

$$s_j - t_{j-1} \leq A D_h\left(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)\right) \leq A(t_j - s_j).$$

Adding $A(s_j - t_{j-1})$ to both sides of this inequality, then dividing by $A+1$, gives

$$s_j - t_{j-1} \leq \frac{A}{A+1}(t_j - t_{j-1}). \tag{3.16}$$

Step 3: upper bound for $\widetilde{D}_h$ By combining the above relations, we get that on $F_{\mathbbm{r}}^{\varepsilon}$,

$$\begin{aligned}
&\widetilde{D}_h\left(B_{4\varepsilon\mathbbm{r}}(\mathbbm{z}), B_{4\varepsilon\mathbbm{r}}(\mathbbm{w})\right) \\
&\quad \leq \sum_{j=\underline{J}+1}^{\overline{J}} \left(\widetilde{D}_h\left(P(t_{j-1}), P(s_j)\right) + \widetilde{D}_h\left(P(s_j), P(t_j)\right)\right) \quad \text{(by (3.14))} \\
&\quad \leq \sum_{j=\underline{J}+1}^{\overline{J}} \left(C_*(s_j - t_{j-1}) + C'(t_j - s_j)\right) \quad \text{(by (3.15))} \\
&\quad = \sum_{j=\underline{J}+1}^{\overline{J}} \left(C'(t_j - t_{j-1}) + (C_* - C')(s_j - t_{j-1})\right) \\
&\quad \leq \left(C' + \frac{A}{A+1}(C_* - C')\right) \sum_{j=\underline{J}+1}^{\overline{J}} (t_j - t_{j-1}) \quad \text{(by (3.16))} \\
&\quad \leq \left(C' + \frac{A}{A+1}(C_* - C')\right) D_h(\mathbbm{z}, \mathbbm{w}).
\end{aligned} \tag{3.17}$$

By (3.11), Axiom V (tightness across scales) for D and $\widetilde{D}$, and the triangle inequality, it holds with probability tending to 1 as $\varepsilon \to 0$, uniformly in r, that

$$\left|\widetilde{D}_h(\mathbbm{z}, \mathbbm{w}) - \widetilde{D}_h\left(B_{4\varepsilon\mathbbm{r}}(\mathbbm{z}), B_{4\varepsilon\mathbbm{r}}(\mathbbm{w})\right)\right|$$

$$\leq \frac{1}{100}\left(C'' - \left(C' + \frac{A}{A+1}(C_* - C')\right)\right) D_h(\mathbb{z}, \mathbb{w}) \qquad (3.18)$$

simultaneously for all $\mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \beta\mathbb{r}$. By combining this with (3.17) and recalling that $\mathbb{P}[F_{\mathbb{r}}^{\varepsilon}] = 1 - \beta/2 - o_\varepsilon(1)$ uniformly in $\mathbb{r}$ if (B) holds, we get that if ε_0 is chosen to be sufficiently small, in a manner which does not depend on $\mathbb{r}$, then if (B) holds for $\mathbb{r} > 0$ and $\varepsilon \in (0, \varepsilon_0]$, then it holds with probability at least $1 - \beta$ that (3.12) holds simultaneously for each $\mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0) \cap \mathbb{Q}^2$ with $|\mathbb{z} - \mathbb{w}| \geq \beta\mathbb{r}$. By the continuity of D_h and $\widetilde{D}_h$, we can remove the requirement that $\mathbb{z}, \mathbb{w} \in \mathbb{Q}^2$ (which was only used to get the uniqueness of the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$). □

4 Independence along a geodesic

Let h be a whole-plane GFF and let D be a weak γ-LQG metric. The goal of this section is to prove the following general "local independence" type result for events depending on a small segment of a D_h-geodesic. We will first state a simplified version of our result which is easier to parse (Theorem 4.1), then state the full version (Theorem 4.2).

Theorem 4.1 *Suppose we are given events $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z) \in \sigma(h)$ for $z \in \mathbb{C}$, $r > 0$, and $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ and a deterministic constant $\Lambda > 1$ which satisfy the following properties, where here $P = P^{\mathbb{z},\mathbb{w}}$ denotes the (a.s. unique) D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$.*

1. *(Measurability) The event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ is determined by $h|_{B_r(z)}$ and the geodesic P stopped at the last time it exists $B_r(z)$.*
2. *(Lower bound for $\mathbb{P}[\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)]$) If $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_r(z)$, then a.s.*

$$\mathbb{P}\left[\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z) \,\middle|\, h|_{\mathbb{C}\setminus B_r(z)}, \{P \cap B_r(z) \neq \emptyset\}\right] \geq \Lambda^{-1}. \qquad (4.1)$$

For each $\nu \in (0, 1)$, $q > 0$, $\ell \in (0, 1)$, and bounded open set $U \subset \mathbb{C}$, it holds with probability tending to 1 as $\varepsilon \to 0$, at a rate depending only on U, q, ℓ, Λ, that for each $\mathbb{z}, \mathbb{w} \in \left(\varepsilon^q \mathbb{Z}^2\right) \cap U$ with $|\mathbb{z} - \mathbb{w}| \geq \ell$, there exists $z \in \mathbb{C}$ and $r \in [\varepsilon^{1+\nu}, \varepsilon]$ such that $P^{\mathbb{z},\mathbb{w}} \cap B_r(z) \neq \emptyset$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs.

We think of the parameter $q > 0$ in Theorem 4.1 as large, so the conclusion of the theorem holds for all pairs $(\mathbb{z}, \mathbb{w})$ in a fine mesh of U.

Intuitively, the reason why Theorem 4.1 is true is as follows. The geodesic segments $P \cap B_r(z)$ and $P \cap B_r(w)$ are approximately independent from one another when $|z - w|$ is much larger than r. When r is small, we can cover P by a large number of balls $B_r(z)$ whose corresponding center points z lie at Euclidean distance much further than r from one another. Using (4.1) and a general concentration inequality for independent random variables, one gets that for each fixed pair $(\mathbb{z}, \mathbb{w})$, with high probability there exists $z \in \mathbb{C}$ such that $P^{\mathbb{z},\mathbb{w}} \cap B_r(z) \neq \emptyset$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs. One then takes a union bound over all pairs $\mathbb{z}, \mathbb{w} \in \left(r^q \mathbb{Z}^2\right) \cap U$.

The above heuristic is not quite right since D_h-geodesics do not depend locally on the field, so $P \cap B_r(z)$ and $P \cap B_r(w)$ are not approximately independent when $|z-w|$ is much greater than r. Indeed, it is possible that changing what happens in $B_r(z)$ could affect the behavior of P macroscopically even when r is very small. As a substitute for this lack of long-range independence, we will use the confluence of geodesics results from [36], as discussed in Sect. 1.5, and only make changes to the field at places where the geodesics are "stable" in the sense that a microscopic change does not lead to macroscopic changes to P. The reason why we only get a statement which holds with probability tending to 1 as $r \to 0$ at the end of Theorem 4.1 is that we need to truncate on a global regularity event in order to make confluence hold with high probability.

We will actually prove (and use) a more general version of Theorem 4.1 which differs from Theorem 4.1 in the following respects.

- We allow for more flexibility in the Euclidean radii involved in the various conditions, which is represented by constants $\{\lambda_i\}_{i=1,\dots,5}$ (for our particular application, the constants are chosen explicitly in (5.11)).
- We introduce events $E_r(z)$ which are determined by the restriction of h to an annulus $\mathbb{A}_{\lambda_1 r, \lambda_4 r}(z)$ (for constants $\lambda_1 < \lambda_4$) and which are required to have probability close to 1. We replace (4.1) by a comparison between the conditional probabilities of $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ and $E_r(z)$ given $h|_{\mathbb{C}\setminus B_{\lambda_3 r}(z)}$, for another constant λ_3. The occurrence of $E_r(z)$ can be thought of as the statement that "$h|_{\mathbb{A}_{\lambda_1 r, \lambda_4 r}(z)}$ is sufficiently well behaved that $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ has a chance to occur".
- We do not require our events to be defined for all $r > 0$, but rather only for values of r in a suitably "dense" set $\mathcal{R} \subset (0, \infty)$. The reason why we need to allow for this is that the results of Sect. 3 only hold for values of r in a suitably dense set.
- We work with a given "base scale" $\mathbb{r} > 0$ (e.g., we consider points in $\mathbb{r}U$ instead of in U) and we require our estimates to be uniform in the choice of $\mathbb{r}$. The reason for this is that we have only assumed tightness across scales (Axiom V) instead of exact scale invariance.

Theorem 4.2 *There exists $\nu_* \in (0, 1)$ depending only on the choice of metric D such that for each $0 < \mu < \nu \le \nu_*$ and each $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 < \lambda_5$, there exists $\mathbb{p} \in (0, 1)$ such that the following is true. Suppose $\mathbb{r} > 0$ and we are given a small number $\varepsilon_0 > 0$; a deterministic set of radii $\mathcal{R} \subset (0, \varepsilon_0]$; events $E_r(z) \in \sigma(h)$ for $z \in \mathbb{C}$ and $r \in \mathcal{R}$; events $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z) \in \sigma(h)$ for $z \in \mathbb{C}$, $r \in \mathcal{R}$, and $\mathbb{z}, \mathbb{w} \in \mathbb{C}$; and a deterministic constant $\Lambda > 1$ which satisfy the following properties.*

1. *(Density of $\mathcal{R}$) For each $\varepsilon \in (0, \varepsilon_0]$, there exist $\lfloor \mu \log_8 \varepsilon^{-1} \rfloor$ radii $r_1^\varepsilon, \ldots, r^\varepsilon_{\lfloor \mu \log_8 \varepsilon^{-1} \rfloor} \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \mathcal{R}$ such that $r_k^\varepsilon / r_{k-1}^\varepsilon \geq \lambda_4/\lambda_1$ for each $k = 2, \ldots, \lfloor \mu \log_8 \varepsilon^{-1} \rfloor$.*
2. *(Measurability) For each $z \in \mathbb{C}$ and $r \in \mathcal{R}$, $E_r(z)$ is determined by $(h - h_{\lambda_5 r}(z))|_{\mathbb{A}_{\lambda_1 r, \lambda_4 r}(z)}$ for each $\mathbb{z}, \mathbb{w} \in \mathbb{C}$, and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ is determined by $h|_{B_{\lambda_4 r}(z)}$ and the (a.s. unique) D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$ stopped at the last time it exists $B_{\lambda_4 r}(z)$.*
3. *(Lower bound for $\mathbb{P}[E_r(z)]$) For each $z \in \mathbb{C}$ and $r \in \mathcal{R}$, we have $\mathbb{P}[E_r(z)] \geq \mathbb{p}$.*
4. *(Comparison of $E_r(z)$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$) Suppose $z \in \mathbb{C}$, $r \in \mathcal{R}$, $\mathbb{z}, \mathbb{w}$ are distinct points of $\mathbb{C} \setminus B_{\lambda_4 r}(z)$, and $P = P^{\mathbb{z},\mathbb{w}}$ is the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. Then a.s.*

$$\Lambda^{-1} \mathbb{P}\left[E_r(z) \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\} \,\middle|\, h|_{\mathbb{C} \setminus B_{\lambda_3 r}(z)} \right] \\ \leq \mathbb{P}\left[\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z) \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\} \,\middle|\, h|_{\mathbb{C} \setminus B_{\lambda_3 r}(z)} \right]. \tag{4.2}$$

Under the above hypotheses, for each $q > 0$, $\ell \in (0,1)$, and bounded open set $U \subset \mathbb{C}$, it holds with probability tending to 1 as $\varepsilon \to 0$, at a rate depending only on $U, q, \ell, \mu, \nu, \{\lambda_i\}_{i=1,\ldots,5}, \varepsilon_0, \Lambda$, that for each $\mathbb{z}, \mathbb{w} \in \left(\varepsilon^q \mathbb{r} \mathbb{Z}^2\right) \cap (\mathbb{r} U)$ with $|\mathbb{z} - \mathbb{w}| \geq \ell \mathbb{r}$, there exists $z \in \mathbb{C}$ and $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ such that $P^{\mathbb{z},\mathbb{w}} \cap B_{\lambda_2 r}(z) \neq \emptyset$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs.

Theorem 4.1 is the special case of Theorem 4.2 where $\mathcal{R} = (0, \infty)$; $\lambda_2 = \lambda_3 = \lambda_4 = 1$; $E_r(z)$ is the whole probability space; and $\mathbb{r} = 1$. The parameter $\mathbb{p}$ in Theorem 4.2 will eventually be chosen to be sufficiently close to 1 that we can apply Lemma 2.6 to cover a large region of space by balls $B_{\lambda_1 r}(z)$ for pairs (z, r) such that $E_r(z)$ occurs (see Lemma 4.11). The events $E_r(z)$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ play very different roles in the statement of Theorem 4.2. The event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ is the main event that we are interested in, and concerns a segment of the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. The event $E_r(z)$ is locally determined by h, has probability close to 1, and can be thought of as the event that the restriction of h to the annulus $\mathbb{A}_{\lambda_1 r, \lambda_4 r}(z)$ is sufficiently regular that $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ has a chance to occur.

The statement of Theorem 4.2 is easier to understand if one thinks of the particular setting in which we will apply it. Recall the optimal bi-Lipschitz constants from (1.21). For us, $E_r(z)$ will be the event that there exists a pair of points $u, v \in \mathbb{A}_{\lambda_1 r, \lambda_2 r}(z)$ at Euclidean distance of order r from each other for which $\widetilde{D}_h(u, v) \leq c_2' D_h(u, v)$ for a constant $c_2' \in (c_*, C_*)$; and some regularity conditions hold which are needed to ensure that conditions 2 and 4 in the theorem statement are satisfied. We will only be able to show that $\mathbb{P}[E_r(z)]$ is bounded below for a "dense" set of scales $\mathcal{R}$ as in condition 1 due to the results in Sect. 3. The event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ will be the event that, roughly speaking, the D_h-geodesic $P^{\mathbb{z},\mathbb{w}}$ gets close to u, v and hence (by the triangle inequality) hits a pair of points $P(s)$, $P(t)$ at Euclidean distance of order r from each other for which $\widetilde{D}_h(P(s), P(t)) \le c_2' D_h(P(s), P(t))$. More precisely, we will prove the following statement in Sect. 5.

Proposition 4.3 *Assume (by way of eventual contradiction) that $c_* < C_*$. Let $0 < \mu < \nu \le \nu_*$ and $c_* < c_1' < c_2' < C_*$. There exist universal constants $\{\lambda_i\}_{i=1,\dots,5}$ and parameters $b, \rho \in (0,1)$ depending only on μ, ν such that the following is true. Let $\mathbb{p}$ be as in Theorem 4.2 for the above choice of $\mu, \nu, \{\lambda_i\}_{i=1,\dots,5}$ and let $c'' = c''(c_1', \mu, \nu) > c_*$ be as in Proposition 3.5 with $c' = c_1'$. If $\beta \in (0,1)$ and $\mathbb{r} > 0$ are such that $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \ge \beta$ (in the notation (3.3)), then there exists $\varepsilon_0 = \varepsilon_0(\beta, c_1', c_2', \mu, \nu) \in (0,1)$, a deterministic set of radii $\mathcal{R} \subset (0, \varepsilon_0]$, events $E_r(z)$ and $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$, and a deterministic constant $\Lambda = \Lambda(c_1', c_2', \mu, \nu) > 1$ which satisfy the hypotheses of Theorem 4.2 with $\rho^{-1}\mathbb{r}$ in place of $\mathbb{r}$ and have the following additional property. Suppose $z \in \mathbb{C}$, $r \in \mathcal{R}$, and $\mathbb{z}, \mathbb{w} \in \mathbb{C} \setminus B_{\lambda_4 r}(z)$, and let $P = P^{\mathbb{z},\mathbb{w}}$ be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. If $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs, then there are times $0 < s < t < |P|$ such that*

$$P([s,t]) \subset B_{\lambda_2 r}(z), \quad |P(s) - P(t)| \ge br, \quad \text{and} \quad \widetilde{D}_h(P(s), P(t)) \le c_2' D_h(P(s), P(t)). \tag{4.3}$$

Roughly speaking, Proposition 4.3 combined with Theorem 4.2 implies that the pairs of points (u, v) such that $\widetilde{D}_h(u, v) \le c_2' D_h(u, v)$ and $|u - v|$ is not too small are sufficiently dense that a typical D_h-geodesic is extremely likely to get close to such a pair of points. This will be applied in Sect. 6 to derive a contradiction to the definition (1.21) of C_* if we assume that $c_* < C_*$, and thereby to show that $c_* = C_*$.

Remark 4.4 The reason for the parameter ρ in Proposition 4.3 is as follows. If $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \ge \beta$, then Proposition 3.5 gives us a parameter $p = p(\mu, \nu) \in (0,1)$ such that there are many values of $r \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}]$ for which a certain event occurs with probability at least p. In Sect. 5, we will use the event of Proposition 3.5 to build the event $E_r(z)$. In order to make $E_r(z)$ occur with probability at least $\mathbb{p}$ (which can be arbitrarily close to 1) instead of just probability p, we will consider lots of small Euclidean balls and argue (using Lemma 2.7) that with probability at least $\mathbb{p}$ the event of Proposition 3.5 occurs for at least one of these balls. In order to do this, we need the radius of the annulus involved in the definition of $E_r(z)$ to be a large deterministic constant factor times the radius of the balls involved in the event of Proposition 3.5 (so that we can fit many such balls in the annulus). This factor is ρ^{-1}.

4.1 Setup and outline

Assume that we are in the setting of Theorem 4.2 for some $\mathbb{r} > 0$. To lighten notation, we will also impose the assumption that $\lambda_3 = 1$ (the proof when $\lambda_3 \neq 1$ is identical, just with extra factors of λ_3 in various subscripts). Let $U \subset \mathbb{C}$ be open and bounded and let $\ell > 0$, as in the conclusion of Theorem 4.2. Also fix $\varepsilon \in (0, \varepsilon_0]$ and distinct points $\mathbb{z}, \mathbb{w} \in \mathbb{r}U$ with $|\mathbb{z} - \mathbb{w}| \geq 4\ell\mathbb{r}$ (the reason for the factor of 4 here is to reduce factors of 4 elsewhere). Let

$$P = P^{\mathbb{z},\mathbb{w}} := (D_h\text{-geodesic from}\, \mathbb{z} \text{ to } \mathbb{w})\,. \tag{4.4}$$

To lighten notation, throughout the rest of this section we will not include the parameters $\mathbb{r}, \varepsilon, \mathbb{z}, \mathbb{w}$ in the notation. But, we will always require that all estimates are uniform in the choice of $\mathbb{r}$, $\mathbb{z}$, and $\mathbb{w}$ (we will typically be sending $\varepsilon \to 0$). Since we will commonly be growing metric balls starting from $\mathbb{z}$, we also introduce the following abbreviations for $z \in \mathbb{C}$ and $r, s > 0$:

$$\begin{aligned} &\mathfrak{E}_r(z) = \mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z), \quad \mathcal{B}_s^\bullet := \mathcal{B}_s^\bullet(\mathbb{z}; D_h) \quad \text{and} \\ &\tau_r := \tau_r(\mathbb{z}) = \inf\{s > 0 : \mathcal{B}_s^\bullet \not\subset B_r(\mathbb{z})\}, \end{aligned} \tag{4.5}$$

where here we recall that $\mathcal{B}_s^\bullet(\mathbb{z}; D_h)$ is the filled metric ball.

We now define several objects which we will work with throughout the rest of this section. See Fig. 3 for an illustration. Fix $\beta \in (0, 1)$ to be chosen later, in a manner depending only on D. Define

$$\begin{aligned} &s_k := \tau_{\ell\mathbb{r}} + k\varepsilon^\beta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} \quad \text{and} \quad t_k := s_k + \varepsilon^{2\beta} \\ &\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} \in [s_k, s_{k+1}], \quad \forall k \in \mathbb{N}_0. \end{aligned} \tag{4.6}$$

By Theorem 2.15, it is a.s. the case that for each $k \in \mathbb{N}_0$ there are only finitely many points of $\partial\mathcal{B}_{s_k}^\bullet$ which are hit by leftmost D_h-geodesics from $\mathbb{z}$ to $\partial\mathcal{B}_{t_k}^\bullet$. Let $\mathrm{Conf}_k \subset \partial\mathcal{B}_{s_k}^\bullet$ be the set of such points and let $\mathcal{I}_k$ be the set of subsets of $\partial\mathcal{B}_{t_k}^\bullet$ of the form

$$\begin{aligned} &\left\{y \in \partial\mathcal{B}_{t_k}^\bullet : \text{leftmost } D_h\text{-geodesic from } \mathbb{z} \text{ to } y \text{ passes through } x\right\} \\ &\quad \text{for} \quad x \in \mathrm{Conf}_k. \end{aligned} \tag{4.7}$$

By [36, Lemma 2.7], $\mathcal{I}_k$ is a collection of disjoint arcs of $\partial\mathcal{B}_{t_k}^\bullet$ whose union is all of $\partial\mathcal{B}_{t_k}^\bullet$. We also note that by Axiom II (locality), $\mathcal{I}_k$ is determined by $\mathcal{B}_{t_k}^\bullet$ and $h|_{\mathcal{B}_{t_k}^\bullet}$.

For much of this section, we will work with the increasing filtration

$$\mathcal{F}_k := \sigma\left(\mathcal{B}_{t_k}^\bullet, h|_{\mathcal{B}_{t_k}^\bullet}, P|_{[0,s_k]}\right), \quad \forall k \in \mathbb{N}_0. \tag{4.8}$$

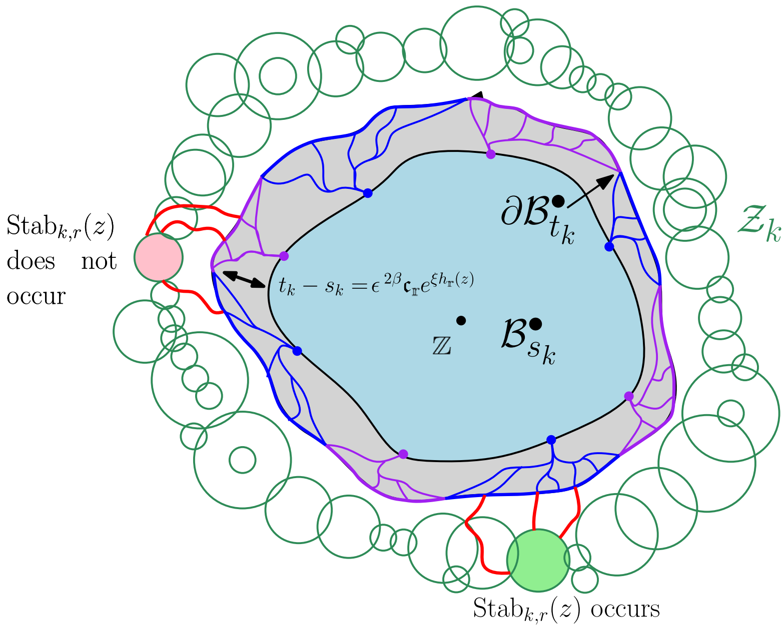

Fig. 3 Illustration of the objects defined in Sect. 4.1. The two filled LQG metric balls $\mathcal{B}^\bullet_{s_k} \subset \mathcal{B}^\bullet_{t_k}$ centered at $\mathbbm{z}$ are shown, along with the set of points $\mathrm{Conf}_k \subset \partial\mathcal{B}^\bullet_{s_k}$ hit by leftmost D_h-geodesics from $\mathbbm{z}$ to $\partial\mathcal{B}^\bullet_{t_k}$ (alternating blue and purple) and the set of arcs $\mathcal{I}_k$ of $\partial\mathcal{B}^\bullet_{t_k}$ consisting of points whose leftmost D_h-geodesics hit the same point of Conf_k. Several representative leftmost D_h-geodesics are shown for each such arc. We have also shown in green several of the balls $B_r(z)$ for $(z, r) \in \mathcal{Z}_k$. Each such ball has radius in $[\varepsilon^{1+\nu}\mathbbm{r}, \varepsilon\mathbbm{r}]$ and its Euclidean distance from $\mathcal{B}^\bullet_{t_k}$ is of order ε. We have highlighted examples of one such ball $B_r(z)$ for which the event $\mathrm{Stab}_{k,r}(z)$ of (4.11) occurs (light green), i.e., each of the red $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics from $\mathbbm{z}$ to points of $\partial B_r(z)$ hit the same arc of $\mathcal{I}_k$ (we have only shown the segments of these geodesics after they exit $\mathcal{B}^\bullet_{t_k}$). We have also highlighted one ball for which $\mathrm{Stab}_{k,r}(z)$ does not occur (pink) (color figure online)

Conditioning on all of $P|_{[0,s_k]}$ may seem rather extreme, but thanks to the confluence of geodesics this conditioning is a equivalent to a much tamer looking conditioning.

Lemma 4.5 *We have the equivalent representation*

$$\mathcal{F}_k = \sigma\left(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}, \text{arc of } \mathcal{I}_k \text{ which contains } P(t_k)\right). \tag{4.9}$$

Proof On the event that the target point $\mathbbm{w}$ of P lies in $\mathcal{B}^\bullet_{t_k}$, the path $P|_{[0,s_k]}$ is determined by $(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$. On the complementary event $\{\mathbbm{w} \notin \mathcal{B}^\bullet_{t_k}\}$, we have $P(s_k) \in \partial\mathcal{B}^\bullet_{s_k}$ and $P|_{[0,s_k]}$ is the a.s. unique $D_h(\cdot,\cdot;\mathcal{B}^\bullet_{t_k})$-geodesic from $\mathbbm{z}$ to $P(s_k)$. Hence, on this event $P|_{[0,s_k]}$ is determined by $(\mathcal{B}^\bullet_{s_k}, h|_{\mathcal{B}^\bullet_{s_k}}, P(s_k))$. Moreover, $P|_{[0,t_k]}$ is a.s. the unique (hence also leftmost) D_h-geodesic from

$\mathbb{z}$ to $P(t_k)$, hence $P(s_k)$ is one of the points of Conf_k. By the definition of $\mathcal{I}_k$, this point is determined by which arc of $\mathcal{I}_k$ contains $P(t_k)$. □

We now introduce the set of Euclidean balls $B_r(z)$ which we will consider when trying to produce a ball for which $\mathfrak{E}_r(z)$ occurs. With $r_1^\varepsilon, \ldots, r^\varepsilon_{\lfloor \mu \log_8 \varepsilon^{-1} \rfloor} \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \mathcal{R}$ as in condition 1 from Theorem 4.2, let $\mathcal{Z}_k$ for $k \in \mathbb{N}$ be the set of pairs (z, r) such that

$$z \in \left(\frac{\lambda_1 \varepsilon^{1+\nu}\mathbb{r}}{4} \mathbb{Z}^2 \right) \setminus \mathcal{B}^\bullet_{t_k}, \quad r \in \left\{ r_1^\varepsilon, \ldots, r^\varepsilon_{\lfloor \mu \log_8 \varepsilon^{-1} \rfloor} \right\},$$
$$\text{and} \quad \mathrm{dist}\left(z, \partial \mathcal{B}^\bullet_{t_k}\right) \in [\lambda_4 \varepsilon \mathbb{r}, 2\lambda_4 \varepsilon \mathbb{r}]. \tag{4.10}$$

Note that $\mathcal{Z}_k \in \sigma\left(\mathcal{B}^\bullet_{t_k}\right)$.

We want to say that with extremely high probability, there are many values of $k \in \mathbb{N}_0$ for which the event $\mathfrak{E}_r(z)$ occurs for some $(z, r) \in \mathcal{Z}_k$ such that $P \cap B_{\lambda_2 r}(z) \neq \emptyset$. We will do this by lower-bounding the conditional probability given $\mathcal{F}_k$ that $\mathfrak{E}_r(z)$ occurs for at least one $(z, r) \in \mathcal{Z}_k$, then considering a polynomial (in ε) number of values of k and applying a standard concentration inequality for binomial random variables.

In order to say something useful about the conditional law given $\mathcal{F}_k$ of what happens in one of the balls $B_r(z)$ for $(z, r) \in \mathcal{Z}_k$, we need to know that making a small change to what happens in $B_r(z)$ does not affect which arc of $\mathcal{I}_k$ contains $P(t_k)$. For $z \in \mathbb{C}$ and $r > 0$, we therefore let $\mathrm{Stab}_{k,r}(z)$ be the event that $(z, r) \in \mathcal{Z}_k$ and

$$\text{Each } D_h(\cdot, \cdot; \mathbb{C}\setminus\overline{B_r(z)})\text{-geodesic from } \mathbb{z} \text{ to a point of } \partial B_r(z) \text{ hits } \partial\mathcal{B}^\bullet_{t_k} \text{ in the same arc of } \mathcal{I}_k. \tag{4.11}$$

Here, by a $D_h(\cdot, \cdot; \mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point $x \in \partial B_r(z)$ we mean a path from $\mathbb{z}$ to x in $\mathbb{C}\setminus B_r(z)$ which has minimal D_h-length among all such paths and which does not hit $\partial B_r(z)$ except at x. Note that such a geodesic need not exist for every point of $\partial B_r(z)$. However, if P is a D_h geodesic started from $\mathbb{z}$ which enters $\overline{B_r(z)}$, then P, stopped at the first time when it enters $B_r(z)$, is a $D_h(\cdot, \cdot; \mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$.

In Sect. 4.4, we will use various quantitative results on confluence of geodesics from [36] to show that with high probability $\mathrm{Stab}_{k,r}(z)$ occurs for most of the pairs $(z, r) \in \mathcal{Z}_k$ such that P enters $B_{\lambda_2 r}(z)$. The reason why the events $\mathrm{Stab}_{k,r}(z)$ are useful is the following lemma, which is used only in Sect. 4.2.

Lemma 4.6 *For each $z \in \mathbb{C}$, $r > 0$, and $k \in \mathbb{N}_0$ the event $\mathrm{Stab}_{k,r}(z)$ of (4.11) is a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$. Furthermore, on the event $\mathrm{Stab}_{k,r}(z) \cap \{P \cap$*

$B_r(z) \neq \emptyset\}$, both $(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$ and the arc of $\mathcal{I}_k$ which contains $P(t_k)$ are a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$ and the indicator $\mathbb{1}_{\mathrm{Stab}_{k,r}(z)\cap\{P\cap B_r(z)\neq\emptyset\}}$. In particular, for any event $F \in \mathcal{F}_k$ the event $F \cap \mathrm{Stab}_{k,r}(z) \cap \{P \cap B_r(z) \neq \emptyset\}$ is a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$ and the indicator $\mathbb{1}_{\mathrm{Stab}_{k,r}(z)\cap\{P\cap B_r(z)\neq\emptyset\}}$.

Proof Since $\mathcal{B}^\bullet_{t_k}$ is a local set for h (Lemma 2.1) and since balls $B_r(z)$ for $(z,r) \in \mathcal{Z}_k$ are disjoint from $\mathcal{B}^\bullet_{t_k}$, we find that $\{(z,r) \in \mathcal{Z}_k\}$ is determined by $h|_{\mathbb{C}\setminus B_r(z)}$. Furthermore, $\left(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}\right)$ and hence also $\mathcal{I}_k$ is determined by $h|_{\mathbb{C}\setminus B_r(z)}$ on the event $\{(z,r) \in \mathcal{Z}_k\}$. By Axiom II (locality), it then follows that $\mathrm{Stab}_{k,r}(z)$ is determined by $h|_{\mathbb{C}\setminus B_r(z)}$.

Since $\mathrm{Stab}_{k,r}(z) \subset \{(z,r) \in \mathcal{Z}_k\}$, we already know that $\left(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}\right)$ is a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$ on the event $\mathrm{Stab}_{k,r}(z)$. On the event $\{P \cap B_r(z) \neq \emptyset\}$, the D_h-geodesic P stopped at the first time it enters $B_r(z)$ is a $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$. If $\mathrm{Stab}_{k,r}(z)$ occurs, then every such $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic passes through the same arc of $\mathcal{I}_k$, and we can see which arc this is by observing $h|_{\mathbb{C}\setminus B_r(z)}$. Therefore, on $\mathrm{Stab}_{k,r}(z) \cap \{P \cap B_r(z) \neq \emptyset\}$, the arc of $\mathcal{I}_k$ which contains $P(t_k)$ is a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$ and $\mathbb{1}_{\mathrm{Stab}_{k,r}(z)\cap\{P\cap B_r(z)\neq\emptyset\}}$.

The last statement of the lemma follows from the second statement and Lemma 4.5. □

We define the set of "good" pairs

$$\mathcal{Z}_k^E := \left\{(z,r) \in \mathcal{Z}_k : E_r(z) \cap \mathrm{Stab}_{k,r}(z) \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\} \text{ occurs}\right\} \tag{4.12}$$

and the set of "very good" pairs

$$\mathcal{Z}_k^{\mathfrak{E}} := \left\{(z,r) \in \mathcal{Z}_k : \mathfrak{E}_r(z) \cap \mathrm{Stab}_{k,r}(z) \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\} \text{ occurs}\right\}. \tag{4.13}$$

The proof of Theorem 4.1 is based on lower-bounding the conditional probability that $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$ given $\mathcal{F}_k$, which allows us to say that the number of k for which $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$ stochastically dominates a binomial random variable. To lower-bound $\mathbb{P}[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k]$, we will first establish a lower bound for $\mathbb{P}[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k]$ in terms of $\mathbb{P}[\mathcal{Z}_k^E \neq \emptyset \,|\, \mathcal{F}_k]$ using condition 4 in Theorem 4.2 (Sect. 4.2). We will then show that it is very likely that $\mathcal{Z}_k^E \neq \emptyset$ for many values of k (Sect. 4.4). This will imply that it is very likely that there are many values of k for which $\mathbb{P}[\mathcal{Z}_k^E \neq \emptyset \,|\, \mathcal{F}_k]$ is bounded below, and hence there are many values of k for which $\mathbb{P}[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k]$ is bounded below (Sect. 4.5). We will now outline the rest of the proof of Theorem 4.2. See Fig. 4 for a schematic illustration of how the various results in this section fit together.

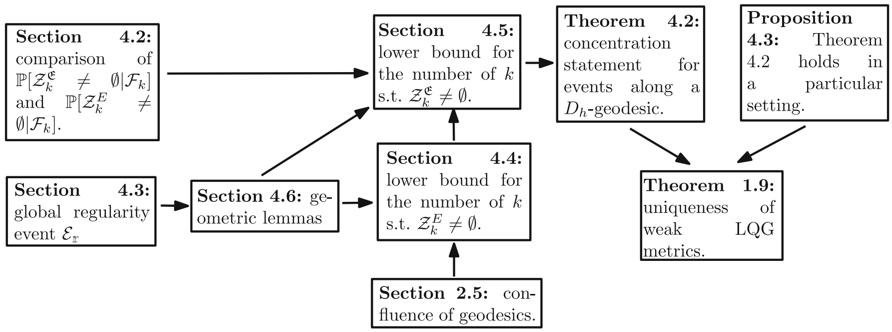

Fig. 4 Schematic outline of Sect. 4. An arrow between two sections/results means that the first is used in the proof of the second. Note that Proposition 4.3 is proven in Sect. 5 and Theorem 1.9 is proven in Sect. 6

In Sect. 4.2, we show that for each $k \in \mathbb{N}$, $\mathbb{P}[\mathcal{Z}_k^{\mathfrak{C}} \neq \emptyset|\mathcal{F}_k]$ is bounded below by $\varepsilon^{2\nu+o_\varepsilon(1)}\mathbb{P}[\mathcal{Z}_k^{E} \neq \emptyset|\mathcal{F}_k]$, minus a small error. The reason why this is true is that (4.2) together with Lemma 4.6 allows us to lower-bound $\mathbb{E}[\#\mathcal{Z}_k^{\mathfrak{C}} \mid \mathcal{F}_k]$ in terms of $\mathbb{E}[\#\mathcal{Z}_k^{E} \mid \mathcal{F}_k]$. Then, Lemma 2.12 along with a Paley-Zygmund type argument allows us to transfer from a lower bound for $\mathbb{E}[\#\mathcal{Z}_k^{\mathfrak{C}} \mid \mathcal{F}_k]$ to a lower bound for $\mathbb{P}[\mathcal{Z}_k^{\mathfrak{C}} \neq \emptyset \mid \mathcal{F}_k]$. Here, one should think of ν as being small (relative to β), so that $\varepsilon^{2\nu+o_\varepsilon(1)}$ is not too much different from $\varepsilon^{o_\varepsilon(1)}$.

In Sect. 4.3, we define a global regularity event $\mathcal{E}_{\mathbb{r}}$ which we will truncate on for most of the rest of the proof and show that it occurs with high probability. This event includes various bounds for D_h-distances (e.g., Hölder continuity), but the most important condition is that $E_r(z)$ occurs for many pairs (z, r). To make the latter condition occur with high probability, we will make $\mathbb{p}$ sufficiently close to 1 to allow us to apply Lemma 2.6 and a union bound.

In Sect. 4.4, we show that if we truncate on $\mathcal{E}_{\mathbb{r}}$, then with very high probability there are many values of k for which $\mathcal{Z}_k^{E} \neq \emptyset$. Since the definition of $\mathcal{E}_{\mathbb{r}}$ already includes the condition that $E_r(z)$ occurs for many pairs $(z, r) \in \mathcal{Z}_k$, the main difficulty here is showing that $\mathrm{Stab}_{k,r}(z)$ occurs for most of the pairs (z, r) such that $P \cap B_{\lambda_2 r}(z) \neq \emptyset$. This will be accomplished by applying the results on confluence of geodesics from [36], as reviewed in Sect. 2.5, and multiplying over k to get concentration. We will choose the parameter β from (4.6) to be small so that we have enough "room" between $\partial\mathcal{B}_{s_k}^{\bullet}$ and $\partial\mathcal{B}_{t_k}^{\bullet}$ for various confluence effects to occur.

In Sect. 4.5, we will transfer from the statement that "$\mathcal{Z}_k^{E} \neq \emptyset$ for many values of k" to the statement that "$\mathcal{Z}_k^{\mathfrak{C}} \neq \emptyset$ for many values of k". This will be accomplished using the result of Sect. 4.2 and an elementary probabilistic lemma (Lemma 4.18) which allows us to convert between conditional and unconditional probabilities. We will then complete the proof of Theorem 4.2

by truncating on $\mathcal{E}_{\mathbb{r}}$ and then taking a union bound over many pairs of initial and terminal points $\mathbb{z}$, $\mathbb{w}$.

In Sect. 4.6, we collect the proofs of some geometric lemmas which are stated in Sects. 4.4 and 4.5 , but whose proofs are postponed to avoid distracting from the core of the argument. These geometric lemmas are used to control the behavior of D_h-geodesics on the regularity event $\mathcal{E}_{\mathbb{r}}$.

4.2 Comparison of $E_r(z)$ and $\mathfrak{E}_r(z)$

Recall the definitions of the filtration $\{\mathcal{F}_k\}_{k\geq 0}$ from (4.8), the set of "good" pairs $\mathcal{Z}_k^E$ from (4.12), and the set of "very good" pairs $\mathcal{Z}_k^{\mathfrak{E}}$ from (4.13). The events $E_r(z)$ are easier to work with than the events $\mathfrak{E}_r(z)$ since $E_r(z)$ has high probability and is determined locally by h. The goal of this subsection is to prove the following lemma, which will eventually allow us to transfer from a lower bound for the probability that $\mathcal{Z}_k^E \neq \emptyset$ to a lower bound for the probability that $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$.

Lemma 4.7 *Let $M > 0$. On the event $\{\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}}(\mathbb{z})\} \cap \{\mathbb{w} \notin B_{3\lambda_4\varepsilon\mathbb{r}}(\mathcal{B}_{t_k}^\bullet)\}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that*

$$\mathbb{P}\left[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k\right] \geq \varepsilon^{2\nu+o_\varepsilon(1)}\mathbb{P}\left[\mathcal{Z}_k^E \neq \emptyset \,|\, \mathcal{F}_k\right] - o_\varepsilon^\infty(\varepsilon), \tag{4.14}$$

where the rates of the $o_\varepsilon(1)$ and $o_\varepsilon^\infty(\varepsilon)$ are deterministic and depend only on M, μ, ν, $\{\lambda_i\}_{i=1,\dots,5}$.

Nothing from this section besides Lemma 4.7 is used in subsequent subsections. Lemma 4.7 will eventually be a consequence of condition 4 of Theorem 4.2, which together with Lemma 4.6 allows us to compare the conditional expectations of $\#\mathcal{Z}_k^E$ and $\#\mathcal{Z}_k^{\mathfrak{E}}$ given $\mathcal{F}_k$. To transfer from a lower bound for the conditional expectation of $\#\mathcal{Z}_k^{\mathfrak{E}}$ to a lower bound for the probability that $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$, we will use a Paley-Zygmund type argument. For this purpose we need the following upper bound for $\#\mathcal{Z}_k^{\mathfrak{E}}$, which comes from Lemma 2.12 and Markov's inequality (to transfer from unconditional to conditional probability).

Lemma 4.8 *Let $M > 0$ and $\zeta \in (0,1)$. Also let*

$$\mathcal{Z}_k(P) := \{(z,r) \in \mathcal{Z}_k : P \cap B_r(z) \neq \emptyset\},$$

so that $\mathcal{Z}_k^{\mathfrak{E}} \subset \mathcal{Z}_k^E \subset \mathcal{Z}_k(P)$. On the event $\{\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})\}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ as $\varepsilon \to 0$ that

$$\mathbb{E}\left[\#\mathcal{Z}_k(P)\mathbb{1}_{(\#\mathcal{Z}_k(P)>\varepsilon^{-2\nu-\zeta})} \,|\, \mathcal{F}_k\right] = o_\varepsilon^\infty(\varepsilon), \tag{4.15}$$

where the rate of the $o_\varepsilon^\infty(\varepsilon)$ *depends only on* $M, \zeta, \mu, \nu, \{\lambda_i\}_{i=1,\ldots,5}$.

Proof By Lemma 2.12 (applied with $M \vee (2/\zeta)$ in place of M and $4\lambda_4\varepsilon$ in place of ε), on the event $\{\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})\}$ it is extremely unlikely that P spends a long time near $\partial\mathcal{B}_{t_k}^\bullet$: more precisely, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ as $\varepsilon \to 0$ that

$$\text{area}\left(B_{4\lambda_4\varepsilon\mathbb{r}}(P) \cap B_{4\lambda_4\varepsilon\mathbb{r}}\left(\partial\mathcal{B}_{t_k}^\bullet\right)\right) \le \varepsilon^{2-\zeta/2}\mathbb{r}^2. \tag{4.16}$$

By (4.10), each ball $B_r(z)$ for $(z, r) \in \mathcal{Z}_k$ is contained in $B_{4\lambda_4\varepsilon\mathbb{r}}\left(\partial\mathcal{B}_{t_k}^\bullet\right)$ and the maximal number of such balls which contain any given point of $\mathbb{C}$ is at most a constant (depending only on $M, \mu, \nu, \{\lambda_i\}_{i=1,\ldots,5}$) times $\varepsilon^{-2\nu}\log_8 \varepsilon^{-1}$. By the definition of $\mathcal{Z}_k(P)$, each ball $B_r(z)$ for $(z, r) \in \mathcal{Z}_k(P)$ is contained in $B_{4\lambda_4\varepsilon\mathbb{r}}(P)$. Therefore, the left side of (4.16) is at least a constant times $\varepsilon^{2+2\nu}(\log_8 \varepsilon^{-1})^{-1}\mathbb{r}^2\#\mathcal{Z}_k(P)$. From (4.16), we now get that

$$\mathbb{P}\left[\#\mathcal{Z}_k(P) > \varepsilon^{-2\nu-\zeta},\ \mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})\right] = o_\varepsilon^\infty(\varepsilon). \tag{4.17}$$

Since $\{\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})\} \subset \mathcal{F}_k$, we can apply Markov's inequality to deduce from (4.17) that with probability $1 - o_\varepsilon^\infty(\varepsilon)$,

$$\mathbb{P}\left[\#\mathcal{Z}_k(P) > \varepsilon^{-2\nu-\zeta} \mid \mathcal{F}_k\right] \mathbb{1}_{(\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z}))} = o_\varepsilon^\infty(\varepsilon). \tag{4.18}$$

If $\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})$, then (4.10) implies that for each $(z, r) \in \mathcal{Z}_k$,

$$z \in \mathcal{Z}_k \subset B_{(\varepsilon^{-M}+2\lambda_4\varepsilon)\mathbb{r}}(\mathbb{z}) \cap \left(\frac{\lambda_1\varepsilon^{1+\nu}}{4}\mathbb{Z}^2\right).$$

Since there are at most $\mu \log_8 \varepsilon^{-1}$ possibilities for r, on the event $\{\mathcal{B}_{t_k}^\bullet \subset B_{\varepsilon^{-M}\mathbb{r}}(\mathbb{z})\}$, we have the trivial upper bound

$$\#\mathcal{Z}_k(P) \le \#\mathcal{Z}_k \le O_\varepsilon\left(\varepsilon^{-2M(1+\nu)}\log_8 \varepsilon^{-1}\right). \tag{4.19}$$

Combining (4.18) and (4.19) gives (4.15). □

We will also need the following elementary probabilistic lemma which will be used in conjunction with Lemma 4.6 to transfer from conditional probabilities given $h|_{\mathbb{C}\setminus B_r(z)}$ to conditional probabilities given $\mathcal{F}_k$.

Lemma 4.9 *Let* $(\Omega, \mathcal{M}, \mathbb{P})$ *be a probability space. Let* $\mathcal{F}, \mathcal{G} \subset \mathcal{M}$ *be sub-*σ*-algebras. Let* $E \in \mathcal{M}$ *be an event such that* $F \cap E \in \mathcal{G} \vee \sigma(E)$ *for each*

$F \in \mathcal{F}$. Also let $G \in \mathcal{F} \cap \mathcal{G}$. Suppose $H_1, H_2 \in \mathcal{M}$ are events and $\Lambda > 0$ is a deterministic constant such that a.s.

$$\mathbb{P}\left[H_1 \cap E \,|\, \mathcal{G}\right] \mathbb{1}_G \le \Lambda \mathbb{P}\left[H_2 \cap E \,|\, \mathcal{G}\right] \mathbb{1}_G. \tag{4.20}$$

Then a.s.

$$\mathbb{P}\left[H_1 \cap E \,|\, \mathcal{F}\right] \mathbb{1}_G \le \Lambda \mathbb{P}\left[H_2 \cap E \,|\, \mathcal{F}\right] \mathbb{1}_G. \tag{4.21}$$

Proof Let $\mathcal{G}' := \mathcal{G} \vee \sigma(E)$. On the event that $\mathbb{P}[E \,|\, \mathcal{G}] > 0$, for any $H \in \mathcal{M}$,

$$\mathbb{P}\left[H \cap E \,|\, \mathcal{G}'\right] = \frac{\mathbb{P}\left[H \cap E \,|\, \mathcal{G}\right]}{\mathbb{P}\left[E \,|\, \mathcal{G}\right]} \mathbb{1}_E. \tag{4.22}$$

On the event that $\mathbb{P}[E \,|\, \mathcal{G}] = 0$, we instead have $\mathbb{P}\left[H \cap E \,|\, \mathcal{G}'\right] = 0$.

Applying (4.22) with $H = H_1$ and with $H = H_2$ and plugging the results into (4.20) shows that a.s.

$$\mathbb{P}\left[H_1 \cap E \,|\, \mathcal{G}'\right] \mathbb{1}_G \le \Lambda \mathbb{P}\left[H_2 \cap E \,|\, \mathcal{G}'\right] \mathbb{1}_G. \tag{4.23}$$

We claim that for any $H \in \mathcal{M}$, a.s.

$$\mathbb{E}\left[\mathbb{P}\left[H \cap E \,|\, \mathcal{G}'\right] \,|\, \mathcal{F}\right] \mathbb{1}_G = \mathbb{P}\left[H \cap E \,|\, \mathcal{F}\right] \mathbb{1}_G. \tag{4.24}$$

Once (4.24) is proven, we can take the conditional expectations given $\mathcal{F}$ of both sides of (4.23) to get (4.21). To prove (4.24), let $F \in \mathcal{F}$. By hypothesis, $F \cap E \in \mathcal{G}'$. Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}\left[\mathbb{P}\left[H \cap E \,|\, \mathcal{G}'\right] \,|\, \mathcal{F}\right] \mathbb{1}_G \mathbb{1}_F\right] \\
&\quad = \mathbb{E}\left[\mathbb{P}\left[H \cap E \,|\, \mathcal{G}'\right] \mathbb{1}_{F \cap G}\right] \quad (\text{since } F \cap G \in \mathcal{F}) \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{H \cap E} \,|\, \mathcal{G}'\right] \mathbb{1}_{F \cap G \cap E}\right] \quad (\text{since } E \in \mathcal{G}' \text{ and } \mathbb{1}_E \mathbb{1}_E = \mathbb{1}_E) \\
&\quad = \mathbb{E}\left[\mathbb{1}_{H \cap E} \mathbb{1}_{F \cap G \cap E}\right] \quad \text{since } F \cap G \cap E \in \mathcal{G}' \\
&\quad = \mathbb{P}\left[H \cap F \cap G \cap E\right].
\end{aligned} \tag{4.25}$$

By the definition of conditional expectation, this implies (4.24). □

Proof of Lemma 4.7 Recall that we are assuming that $\lambda_3 = 1$, so that our hypothesis (4.2) says that for $(z, r) \in \mathbb{C} \times \mathcal{R}$ such that $\mathbb{z}, \mathbb{w} \notin B_{\lambda_4 r}(z)$,

$$\begin{aligned}
&\mathbb{P}\left[E_r(z) \cap \left\{P \cap B_{\lambda_2 r}(z) \neq \emptyset\right\} \,\middle|\, h|_{\mathbb{C} \setminus B_r(z)}\right] \\
&\quad \le \Lambda \mathbb{P}\left[\mathfrak{E}_r(z) \cap \left\{P \cap B_{\lambda_2 r}(z) \neq \emptyset\right\} \,\middle|\, h|_{\mathbb{C} \setminus B_r(z)}\right].
\end{aligned} \tag{4.26}$$

Since $\mathrm{Stab}_{k,r}(z) \in \sigma\left(h|_{\mathbb{C}\setminus B_r(z)}\right)$ (Lemma 4.6), we infer from (4.26) and the definitions (4.12) and (4.13) of $\mathcal{Z}_k^E$ and $\mathcal{Z}_k^{\mathfrak{E}}$ that for each $(z,r) \in \mathbb{C} \times \mathcal{R}$ such that $\mathbb{z}, \mathbb{w} \notin B_{\lambda_4 r}(z)$, a.s.

$$\mathbb{P}\left[(z,r) \in \mathcal{Z}_k^E \,\middle|\, h|_{\mathbb{C}\setminus B_r(z)}\right] \leq \Lambda \mathbb{P}\left[(z,r) \in \mathcal{Z}_k^{\mathfrak{E}} \,\middle|\, h|_{\mathbb{C}\setminus B_r(z)}\right]. \quad (4.27)$$

We will now deduce from (4.27) and Lemma 4.9 that on $\{(z,r) \in \mathcal{Z}_k\} \cap \{\mathbb{w} \notin B_{3\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})\}$, a.s.

$$\mathbb{P}\left[(z,r) \in \mathcal{Z}_k^E \,\middle|\, \mathcal{F}_k\right] \leq \Lambda \mathbb{P}\left[(z,r) \in \mathcal{Z}_k^{\mathfrak{E}} \,\middle|\, \mathcal{F}_k\right]. \quad (4.28)$$

In particular, we will apply Lemma 4.9 with $\mathcal{F} = \mathcal{F}_k$, $\mathcal{G} = \sigma\left(h|_{\mathbb{C}\setminus B_r(z)}\right)$, $E = \mathrm{Stab}_{k,r}(z) \cap \{P \cap B_r(z) \neq \emptyset\}$, $G = \{(z,r) \in \mathcal{Z}_k\} \cap \{\mathbb{w} \notin B_{3\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})\}$, $H_1 = \{(z,r) \in \mathcal{Z}_k^E\}$, and $H_2 = \{(z,r) \in \mathcal{Z}_k^{\mathfrak{E}}\}$.

We check the hypotheses of Lemma 4.9 with the above choice of parameters, starting with the requirement that the event G defined above belongs to $\mathcal{F}_k \cap \sigma\left(h|_{\mathbb{C}\setminus B_r(z)}\right)$. Indeed, it is clear from the definition (4.10) of $\mathcal{Z}_k$ that $G \in \sigma(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$. By the definition (4.9) of $\mathcal{F}_k$, we have $G \in \mathcal{F}_k$. By the definition (4.10) of $\mathcal{Z}_k$ and the locality of $\mathcal{B}^\bullet_{t_k}$ (Lemma 2.1), also $G \in \sigma\left(h|_{\mathbb{C}\setminus B_r(z)}\right)$. By the definition (4.10) of $\mathcal{Z}_k$, if G occurs with positive probability then $\mathbb{z}, \mathbb{w} \notin B_{\lambda_4 r}(z)$, so in particular (4.27) holds a.s. on G. By Lemma 4.6, the intersection of any event in $\mathcal{F}_k$ with $\mathrm{Stab}_{k,r}(z) \cap \{P \cap B_r(z) \neq \emptyset\}$ is a.s. determined by $h|_{\mathbb{C}\setminus B_r(z)}$ and $\mathbb{1}_{\mathrm{Stab}_{k,r}(z) \cap \{P \cap B_r(z) \neq \emptyset\}}$. We may therefore apply Lemma 4.9 to deduce (4.28) from (4.27).

Summing (4.28) over all $(z,r) \in \mathcal{Z}_k$ gives that on $\{\mathbb{w} \notin B_{3\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})\}$,

$$\mathbb{E}\left[\#\mathcal{Z}_k^{\mathfrak{E}} \,|\, \mathcal{F}_k\right] \geq \Lambda^{-1} \mathbb{E}\left[\#\mathcal{Z}_k^E \,|\, \mathcal{F}_k\right] \geq \Lambda^{-1} \mathbb{P}\left[\mathcal{Z}_k^E \neq \emptyset \,|\, \mathcal{F}_k\right]. \quad (4.29)$$

By Lemma 4.8, for each $\zeta \in (0,1)$, on the event $\{\mathcal{B}^\bullet_{t_k} \subset B_{\varepsilon^{-M}}(\mathbb{z})\}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that

$$\begin{aligned} \mathbb{E}\left[\#\mathcal{Z}_k^{\mathfrak{E}} \,|\, \mathcal{F}_k\right] &\leq \varepsilon^{-2\nu-\zeta} \mathbb{P}\left[0 < \#\mathcal{Z}_k^{\mathfrak{E}} \leq \varepsilon^{-2\nu-\zeta} \,|\, \mathcal{F}_k\right] \\ &\quad + \mathbb{E}\left[\#\mathcal{Z}_k^{\mathfrak{E}} \mathbb{1}_{(\#\mathcal{Z}_k^{\mathfrak{E}} > \varepsilon^{-2\nu-\zeta})} \,|\, \mathcal{F}_k\right] \\ &\leq \varepsilon^{-2\nu-\zeta} \mathbb{P}\left[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k\right] + o_\varepsilon^\infty(\varepsilon). \end{aligned} \quad (4.30)$$

Combining (4.29) and (4.30) gives that on the event $\{\mathcal{B}^\bullet_{t_k} \subset B_{\varepsilon^{-M}}(\mathbb{z})\} \cap \{\mathbb{w} \notin B_{3\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})\}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that

$$\varepsilon^{-2\nu-\zeta}\mathbb{P}\left[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,|\, \mathcal{F}_k\right] + o_\varepsilon^\infty(\varepsilon) \geq \Lambda^{-1}\mathbb{P}\left[\mathcal{Z}_k^{E} \neq \emptyset \,|\, \mathcal{F}_k\right]. \tag{4.31}$$

Re-arranging this inequality and then sending $\zeta \to 0$ sufficiently slowly as $\varepsilon \to 0$ gives (4.14). □

4.3 Global regularity event

Throughout most of the rest of the proof of Theorem 4.2, we will truncate on a global regularity event which we define in this subsection. The parameter $\mathbb{p} \in (0,1)$ of Theorem 4.2 has to be chosen sufficiently close to 1 to allow us to apply Lemma 2.6 to make the probability of one of the conditions in the event as close to 1 as we like. We emphasize that our global regularity event does *not* depend on the particular choice of $\mathbb{z}$, $\mathbb{w}$ in (4.4).

Fix bounded, connected open sets $U \subset V \subset \mathbb{C}$ and parameters $\nu, \ell > 0$ (ν, U, and ℓ are the parameters from Theorem 4.2). Also fix, once and for all, parameters $\chi \in (0, \xi(Q-2))$ and $\chi' > \xi(Q+2)$ as in Lemma 2.8, chosen in a manner which depends only on γ (we will not make the dependence on these parameters explicit). For $\mathbb{r} > 0$ and $a \in (0,1)$, let $\mathcal{E}_{\mathbb{r}} = \mathcal{E}_{\mathbb{r}}(a, \nu, \ell, U, V)$ be the event that the following is true.

1. *(Comparison of domains)* $\sup_{z,w \in \mathbb{r}U} D_h(z,w) \leq D_h(\mathbb{r}U, \mathbb{r}\partial V)$.
2. *(Comparison of D_h-balls and Euclidean balls)* For each $z \in \mathbb{C}$ and $r > 0$, let $\tau_r(z)$ be the smallest $t > 0$ for which the filled D_h-metric ball $B_t^\bullet(z; D_h)$ intersects $\partial B_r(z)$, as in (2.23). Then for each $z \in B_{4\ell\mathbb{r}}(\mathbb{r}V)$, we have $B_{a\mathbb{r}}(z) \subset \mathcal{B}^\bullet_{\tau_{\ell\mathbb{r}}(z)}(z; D_h)$ and

$$\min\{\tau_{2\ell\mathbb{r}}(z) - \tau_{\ell\mathbb{r}}(z), \tau_{3\ell\mathbb{r}}(z) - \tau_{2\ell\mathbb{r}}(z)\} \geq a \max\left\{\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(z)}, \mathfrak{c}_{\ell\mathbb{r}} e^{\xi h_{\ell\mathbb{r}}(z)}\right\}. \tag{4.32}$$

3. *(Hölder continuity)* For each $z, w \in B_{4\ell\mathbb{r}}(\mathbb{r}V)$ with $|z-w| \leq a\mathbb{r}$,

$$\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h(z,w) \geq \left|\frac{z-w}{\mathbb{r}}\right|^{\chi'} \quad \text{and}$$
$$\mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h\left(z,w; B_{2|z-w|}(z)\right) \leq \left|\frac{z-w}{\mathbb{r}}\right|^{\chi}. \tag{4.33}$$

4. *(Comparison of circle averages)* We have

$$\sup_{z \in \mathbb{r} V} |h_{\mathbb{r}}(z) - h_{\mathbb{r}}(0)| \leq a^{-1}. \tag{4.34}$$

5. *(Existence of good annuli)* Define $r_1^\varepsilon, \ldots, r^\varepsilon_{\lfloor \mu \log_8 \varepsilon^{-1} \rfloor} \in [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon \mathbb{r}] \cap \mathcal{R}$ as in condition 1 from Theorem 4.2. For each $\varepsilon \in (0, a\mathbb{r}] \cap \{2^{-n}\mathbb{r}\}_{n\in\mathbb{N}}$ and each $z \in \left(\frac{\lambda_1 \varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{4\ell\mathbb{r}}(\mathbb{r}V)$, there exists at least one $r \in \{r_1^\varepsilon, \ldots, r_{\lfloor \mu \log_8 \varepsilon^{-1} \rfloor}\}$ for which $E_r(z)$ occurs.
6. *(Bounds for radii used to control geodesics)* Define the radii $\rho_{\mathbb{r},\varepsilon}(z)$ for $\varepsilon > 0$ and $z \in \mathbb{C}$ as in Lemma 2.13 and the discussion just preceding it. For each $\varepsilon \in (0, a] \cap \{2^{-n}\}_{n\in\mathbb{N}}$ and each $z \in \left(\frac{\varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{4\ell\mathbb{r}}(\mathbb{r}V)$, we have $\rho_{\mathbb{r},\varepsilon}(z) \leq \varepsilon^{1/2}\mathbb{r}$.

We note that the upper bound in (4.33) uses $D_h\left(z, w; B_{2|z-w|}(z)\right) \geq D_h(z, w)$ instead of $D_h(z, w)$. We will need this slightly stronger upper bound for D_h-distances in the proof of Lemma 4.19 below.

Remark 4.10 Due to conditions 2, 3, and 6 , and since $\ell \in (0, 1)$, for each $\mathbb{z} \in \mathbb{r}U$ the event $\mathcal{E}_{\mathbb{r}}$ defined just above is contained in the event $\mathcal{E}^{\mathbb{z}}_{\ell\mathbb{r}}(a)$ as defined just above Theorem 2.16 with $\ell\mathbb{r}$ in place of $\mathbb{r}$.

Lemma 4.11 *There exists* $\mathbb{p} \in (0, 1)$ *depending only on* $\mu, \nu, \{\lambda_i\}_{i=1,\ldots,5}$ *such that under the hypotheses of Theorem* 4.2*, the following is true. For each bounded open set* $U \subset \mathbb{C}$, $\nu, \ell \in (0, 1)$*, and* $p \in (0, 1)$*, there exists a bounded open set* $V \supset U$ *and a parameter* $a \in (0, 1)$*, depending only* U, ν, ℓ, p*, such that* $\mathbb{P}[\mathcal{E}_{\mathbb{r}}] \geq p$ *for each* $\mathbb{r} > 0$.

Proof By Axiom V (tightness across scales), we can find a bounded open set $V \supset U$, depending only on U, such that condition 1 (comparison of domains) in the definition of $\mathcal{E}_{\mathbb{r}}$ holds with probability at least $1 - (1 - p)/6$. Again using Axiom V, we can find a small enough $a \in (0, 1)$, depending on ℓ, V, p, such that condition 2 (comparison of balls) holds with probability at least $1 - (1 - p)/6$. By Lemma 2.8, after possibly shrinking a we can further arrange that condition 3 (Hölder continuity) holds with probability at least $1 - (1 - p)/6$. By the continuity of the circle average process and the scale invariance of the law of h, modulo additive constant, after possibly further shrinking a we can arrange that condition 4 (comparison of circle averages) holds with probability at least $1-(1-p)/6$. By Lemma 2.6, conditions 2 and 3 of Theorem 4.2, and a union bound over all $z \in \left(\frac{\lambda_1 \varepsilon^{1+\nu}\mathbb{r}}{4}\mathbb{Z}^2\right) \cap V$, if $\mathbb{p}$ is chosen sufficiently close to 1, in a manner depending only on μ, ν, and $\{\lambda_i\}_{i=1,\ldots,5}$, then the probability of condition 5 (existence of good annuli) in the definition of $\mathcal{E}_{\mathbb{r}}$ tends to 1 as $a \to 0$, uniformly over the choice of $\mathbb{r}$. Therefore, after possibly further shrinking a, we can arrange that condition 5 in the definition of $\mathcal{E}_{\mathbb{r}}$ holds with probability at least $1-(1-p)/6$. By Lemma 2.13 and a union bound over values of $\varepsilon \in (0, a] \cap \{2^{-n}\}_{n\in\mathbb{N}}$, after possibly further shrinking a we can also arrange that condition 6 (bounds for $\rho_{\mathbb{r},\varepsilon}(z)$) in the definition of $\mathcal{E}_{\mathbb{r}}$ holds with probability at least $1-(1-p)/6$. □

4.4 Geodesic stability event occurs at many times

Henceforth fix $p \in (0, 1)$ (which we will eventually send to 1), a bounded open set $U \subset \mathbb{C}$, and $\ell \in (0, 1)$ and let V, a be as in Lemma 4.11 for this choice of p, U, ℓ and the given values of μ, ν from Theorem 4.2. Let $\mathcal{E}_{\mathbb{r}}$ be the event of Sect. 4.3 with this choice of parameters, so that $\mathbb{P}[\mathcal{E}_{\mathbb{r}}] \geq p$. Define

$$K := \lfloor a\varepsilon^{-\beta} \rfloor - 1, \tag{4.35}$$

where β is as in (4.6). The significance of the value K is that condition 2 (comparison of balls) in the definition of $\mathcal{E}_{\mathbb{r}}$ implies that, in the notation (4.6),

$$s_{K+1} \leq \tau_{2\ell\mathbb{r}}, \quad \text{on } \mathcal{E}_{\mathbb{r}}. \tag{4.36}$$

Recalling the parameter β from (4.6) and the parameters $\chi < \chi'$ as in condition 3 (Hölder continuity) in the definition of $\mathcal{E}_{\mathbb{r}}$, we henceforth impose the requirement that

$$\beta \in (0, \chi/\chi'). \tag{4.37}$$

We will make our final choice of β in Proposition 4.12 just below.

Let $\mathcal{Z}_k^E$ be as in (4.12) and let K be as in (4.35). The goal of this section is to show that with high probability there are many values of $k \in [0, K]_{\mathbb{Z}}$ for which $\mathcal{Z}_k^E \neq \emptyset$. In the next subsection, we will combine this with Lemma 4.7 to show that there are many values of $k \in [0, K]_{\mathbb{Z}}$ for which $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$. The following proposition is the main result of this subsection and is the only statement from this subsection which is referenced in Sect. 4.5.

Proposition 4.12 *There are small constants $\beta, \theta \in (0, 1)$ depending only on the choice of metric D such that if we use this choice of β in (4.6), then on $\mathcal{E}_{\mathbb{r}}$ it holds except on an event of probability decaying faster than any positive power of ε, at a rate which is uniform in $\mathbb{r}, \mathbb{z}, \mathbb{w}$, that there are at least $(1-\varepsilon^{\theta})K$ values of $k \in [0, K]_{\mathbb{Z}}$ for which $\mathcal{Z}_k^E \neq \emptyset$.*

By condition 5 (existence of good annuli) in the definition of $\mathcal{E}_{\mathbb{r}}$, we already know that on this event, for each $k \in [0, K]_{\mathbb{Z}}$ there are many pairs $(z, r) \in \mathcal{Z}_k$ for which $P \cap B_{\lambda_2 r}(z) \neq \emptyset$ and $E_r(z)$ occurs. The main point of this subsection

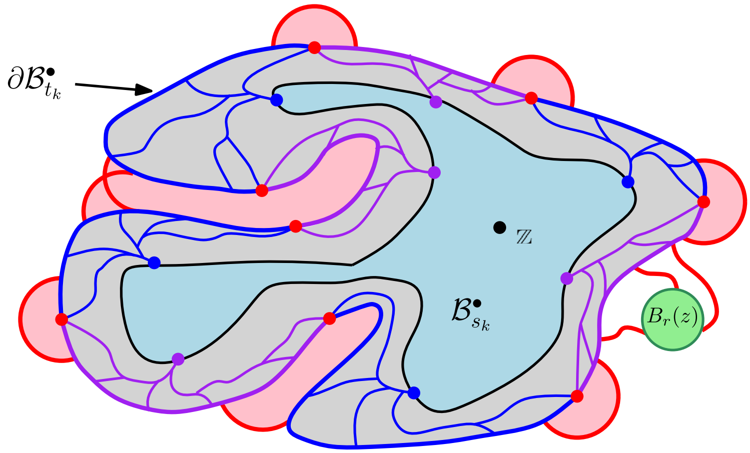

Fig. 5 Illustration of the proof of Proposition 4.12. The points in the set EndPts_k of endpoints of arcs in $\mathcal{I}_k$ are shown in red. We first use Theorem 2.16 to bound $\#\mathrm{EndPts}_k = \#\mathrm{Conf}_k$. Lemma 4.14 allows us to choose for each $y \in \mathrm{EndPts}_k$ a point $z_y \in \partial\mathcal{B}^\bullet_{t_k}$ (not shown) such that an arc of $B_{16\varepsilon^\kappa \mathbb{r}}(z_y)$ disconnects the set $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$ (defined just after Lemma 4.14) from ∞ in $\mathbb{C}\setminus\mathcal{B}^\bullet_{t_k}$. The set which this arc disconnects from ∞, which contains $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$, is shown in pink. Note that the sets $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$ for different choices of y are allowed to overlap. Lemma 2.14 and a union bound over $y \in \mathrm{EndPts}_k$ shows that with high probability, for each $y \in \mathrm{EndPts}_k$, no D_h-geodesic from $\mathbb{z}$ to $\partial\mathcal{B}^\bullet_{s_{k+1}}$ can enter any of the $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$'s. This together with Lemma 4.13 allows us to show that $\mathrm{Stab}_{k,r}(z)$ occurs for each $(z,r) \in \mathcal{Z}_k$ such that $P \cap B_r(z) \neq \emptyset$. One such ball $B_r(z)$ is shown in green and several segments of $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to points of $\partial B_r(z)$ are shown in red (color figure online)

is to show that there are many such pairs for which also the event $\mathrm{Stab}_{k,r}(z)$ of (4.11) occurs. Roughly speaking, the idea of the proof is as follows; see Fig. 5 for an illustration. If P enters $B_r(z)$ but $\mathrm{Stab}_{k,r}(z)$ fails to occur, then P has to get "close" in some sense to one of the endpoints of one of the arcs in $\mathcal{I}_k$.[6] Indeed, otherwise Hölder continuity allows us to force all of the $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to points of $\partial B_r(z)$ to hit the same arc of $\mathcal{I}_k$ as P. This is explained in Lemma 4.13.

On the other hand, if we choose β sufficiently small then results from [36] (in particular, Theorem 2.16) show that $\#\mathcal{I}_k$ is extremely likely to be of smaller order than $\varepsilon^{-\alpha}$, where α is the exponent from Lemma 2.14. We can therefore apply that lemma once for each of the endpoints of the $\mathcal{I}_k$'s and take a union bound to say that with polynomially high probability given $(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$, no D_h-geodesic from $\mathbb{z}$ to a point at macroscopic distance from $\partial\mathcal{B}^\bullet_{s_k}$ can get near any of the endpoints of the $\mathcal{I}_k$'s (Lemma 4.15). The claimed superpolynomial

[6] Technically speaking, we are only able to show (using Lemma 4.13 below) that P has to enter a region which has one of these endpoints on its boundary and which can be disconnected from ∞ in $\mathbb{C}\setminus\mathcal{B}^\bullet_{t_k}$ by a small set.

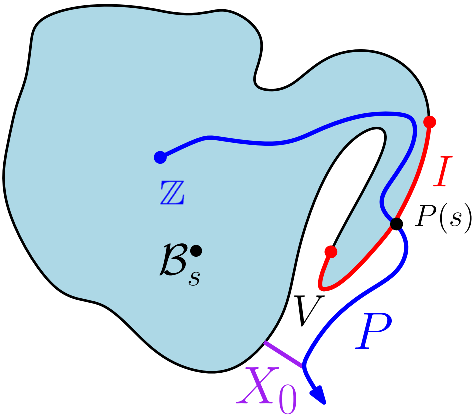

Fig. 6 Illustration of the statement and proof of Lemma 4.13. If P gets $d^{\mathbb{C}\setminus\mathcal{B}_s^\bullet}$-close to $\partial\mathcal{B}_s^\bullet\setminus I$, then there is a set X_0 of small Euclidean diameter which intersects P and $\partial\mathcal{B}_s^\bullet\setminus I$. Moreover the Hölder continuity condition in the definition of $\mathcal{E}_{\mathbb{r}}$ implies that the Euclidean diameter of the segment of P between s and the first time it hits X_0 is small. The union X of this segment and X_0 disconnects one of the endpoints of I from ∞

concentration when we truncate on $\mathcal{E}_{\mathbb{r}}$ comes from a standard concentration bound for independent Bernoulli random variables, provided we choose θ to be sufficiently small relative to α.

In order to quantify how close D_h-geodesics get to the endpoints of the $\mathcal{I}_k$'s, we will need some deterministic definitions. Let $U \subset \mathbb{C}$ be a connected domain such that $\mathbb{C}\setminus U$ is compact and connected. View ∂U as a collection of prime ends. If $X \subset U$, we define the *prime end closure* $\mathrm{Cl}'(X)$ to be the set of points in $z \in U \cup \partial U$ with the following property: if $\phi : U \cup \partial U \to \mathbb{C}\setminus\mathbb{D}$ is a conformal map, then $\phi(z)$ lies in $\overline{\phi(X)}$. Following [36, Equation (2.19)], for $z, w \in U \cup \partial U$ we define

$$d^U(z,w) = \inf\left\{\mathrm{diam}(X) : X \text{ is a connected subset of } U \text{ with } z, w \in \mathrm{Cl}'(X)\right\}, \tag{4.38}$$

where here diam denotes the Euclidean diameter. Then d^U is a metric on $U \cup \partial U$ which is bounded below by the Euclidean metric on $\mathbb{C}$ restricted to $U \cup \partial U$ and bounded above by the internal Euclidean metric on $U \cup \partial U$. Note that d^U is not a length metric.

Lemma 4.13 *Almost surely, if $\mathcal{E}_{\mathbb{r}}$ occurs then the following is true for every $s \in [0, \tau_{3\ell_{\mathbb{r}}}]$, every $\varepsilon \in (0, a]$, and every non-trivial proper connected arc $I \subset \partial\mathcal{B}_s^\bullet$ (i.e., I is the image of an arc of $\partial\mathbb{D}$ which is not a singleton or all of $\partial\mathbb{D}$ under a conformal map $\mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus\mathcal{B}_s^\bullet$). Let P be a D_h-geodesic* *from $\mathbb{z}$ to a point outside of $\mathcal{B}_s^\bullet$ which passes through I and suppose that in the notation (4.38), we have $d^{\mathbb{C}\setminus\mathcal{B}_s^\bullet}(P, \partial\mathcal{B}_s^\bullet\setminus I) \leq \varepsilon\mathbb{r}$. There is a connected set $X \subset \mathbb{C}\setminus\mathcal{B}_s^\bullet$ with Euclidean diameter at most $2\varepsilon^{\chi/\chi'}\mathbb{r}$ such that $P(s)$ and at least one of the two endpoints of I both lie in the prime end closure of the same bounded connected component of $\mathbb{C}\setminus(\mathcal{B}_s^\bullet \cup X)$.*

Proof See Fig. 6 for an illustration of the statement and proof of the lemma. Assume that $\mathcal{E}_{\mathbb{r}}$ occurs and let s, I, ε, and P be as in the lemma statement. By hypothesis, for each $\delta \in (0,1)$ there is a connected set $X_0 \subset \mathbb{C}\setminus\mathcal{B}_s^\bullet$ which has Euclidean diameter at most $(\varepsilon+\delta)\mathbb{r}$ and which satisfies $P \cap X_0 \neq \emptyset$ and $\mathrm{Cl}'(X_0)\cap(\partial\mathcal{B}_s^\bullet\setminus I) \neq \emptyset$. By possibly shrinking X_0, we can assume without loss of generality that $\mathrm{Cl}'(X_0)\cap(\partial\mathcal{B}_s^\bullet\setminus I)$ is a single prime end, which is necessarily in $\mathcal{B}_s^\bullet\setminus I$.

Let t be the first time after s at which P hits X_0. By the upper bound in condition 3 (Hölder continuity) in the definition of $\mathcal{E}_{\mathbb{r}}$, the D_h-diameter of X_0 is at most $(\varepsilon+\delta)^\chi \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. Since P is a D_h-geodesic, $P(t) \in X_0$, and $\mathrm{Cl}'(X_0)$ contains a point of $\partial\mathcal{B}_s^\bullet$ (which implies that $D_h(X_0, \partial\mathcal{B}_s^\bullet) = 0$), it follows that $t - s \leq (D_h\text{-diameter of } X_0) \leq (\varepsilon+\delta)^\chi \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. By the lower bound in condition 3 (Hölder continuity) in the definition of $\mathcal{E}_{\mathbb{r}}$, the Euclidean diameter of $P([s,t])$ is at most $(\varepsilon+\delta)^{\chi/\chi'}\mathbb{r}$. The set $X := X_0 \cup P((s,t])$ has Euclidean diameter at most $((\varepsilon+\delta)^{\chi/\chi'} + \varepsilon + \delta)\mathbb{r}$ and its prime end closure contains both the point $P(s) \in I$ and a point of $\partial\mathcal{B}_s^\bullet\setminus I$. Hence one of the connected components V of $\mathbb{C}\setminus(\mathcal{B}_s^\bullet \cup X)$ is bounded and contains an endpoint of I. Since $\mathrm{Cl}'(X)$ intersects I only at $P(s)$ (here we use that $\mathrm{Cl}'(X_0) \cap \partial\mathcal{B}_s^\bullet$ is a single point), it follows that also $P(s) \in \partial V$. We now conclude the proof by choosing δ to be sufficiently small (depending on ε) so that $(\varepsilon+\delta)^{\chi/\chi'} + \varepsilon + \delta \leq 2\varepsilon^{\chi/\chi'}$. □

We will eventually apply the contrapositive of Lemma 4.13, i.e., we will say that if P does *not* enter a region which contains one of the endpoints of I and which is disconnected from ∞ in $\mathbb{C}\setminus\mathcal{B}_s^\bullet$ by a set of small diameter, then $d^{\mathbb{C}\setminus\mathcal{B}_s^\bullet}(P; \partial\mathcal{B}_s^\bullet\setminus I)$ is bounded below. The following elementary deterministic lemma will be used in conjunction with Lemma 2.14 to prevent P from entering such a region (we will apply the lemma with $\mathcal{K} = \mathcal{B}_{t_k}^\bullet$).

Lemma 4.14 *Let $\mathcal{K} \subset \mathbb{C}$ be a compact connected set such that $\mathbb{C}\setminus\mathcal{K}$ is connected and view $\partial\mathcal{K}$ as a collection of prime ends. For $y \in \partial\mathcal{K}$ and $\varepsilon > 0$, let $\mathcal{C}_y^\varepsilon$ be the set of points in $z \in \mathbb{C}\setminus\mathcal{K}$ such that the following is true. There is a connected set $X \subset \mathbb{C}\setminus\mathcal{K}$ (allowed to depend on z and y, ε) with Euclidean diameter at most ε such that z and y lie in the prime end closure of the same bounded connected component of $\mathbb{C}\setminus(X \cup \mathcal{K})$. Then there is a compact connected set $Y_y^\varepsilon \subset \mathbb{C}\setminus\mathcal{K}$ of Euclidean diameter at most 16ε (depending only on y, ε) such that $\mathcal{C}_y^\varepsilon$ is contained in the prime end closure of a single bounded connected component of $\mathbb{C}\setminus(Y_y^\varepsilon \cup \mathcal{K})$.*

The proof of Lemma 4.14 is straightforward, but it takes a few paragraphs so we postpone it until Sect. 4.6 to avoid interrupting the proof of Theorem 4.2. The reader may want to refer to Fig. 7 for an illustration of the definition of $\mathcal{C}_y^\varepsilon$.

Returning now to the setting of Proposition 4.12, for $k \in [0, K]_{\mathbb{Z}}$ let EndPts_k be the set of endpoints of the arcs in $\mathcal{I}_k$. As in Lemma 4.14, for $\delta > 0$ and $y \in \partial \mathcal{B}_{t_k}^\bullet$, we let $\mathcal{C}_y^\delta$ be the set of points $z \in (\mathbb{C}\setminus\mathcal{B}_{t_k}^\bullet) \cup \partial\mathcal{B}_{t_k}^\bullet$ with the following property: there is a compact connected set $X \subset \overline{\mathbb{C}\setminus\mathcal{B}_{t_k}^\bullet}$ with Euclidean diameter at most δ such that z and y lie in the closure of the same bounded connected component of $\mathbb{C}\setminus(\mathcal{B}_{t_k}^\bullet \cup X)$.

Lemma 4.15 *Fix $\kappa \in (0,1)$. If $\beta, \theta \in (0,1)$ are chosen sufficiently small, in a manner depending only on κ and the choice of metric D, then on $\mathcal{E}_{\mathbb{r}}$ it holds except on an event of probability decaying faster than any positive power of ε, at a rate which is uniform in $\mathbb{r}$, that there are at least $(1-\varepsilon^\theta)K$ values of $k \in [0, K]_{\mathbb{Z}}$ for which the following is true. In the notation introduced just above, no D_h-geodesic from $\mathbb{z}$ to $\partial\mathcal{B}_{s_{k+1}}^\bullet$ can enter $\bigcup_{y\in\mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$.*

Proof Fix parameters $\theta, \omega \in (0,1)$ to be chosen later, in a manner depending only on D. We will first choose β in a manner depending on ω, D and then choose ω in a manner depending on κ, D, and then choose θ in a manner depending on β, ω. In particular, we will take $\omega < \alpha\kappa/2$ and $\theta < \min\{\omega, \beta/2\}$. The parameter κ will be chosen in a manner depending only on D in the proof of Proposition 4.12 below.

We will first show, using Theorem 2.16, that if β is chosen to be sufficiently small (depending on ω, D) then with extremely high probability on $\mathcal{E}_{\mathbb{r}}$ one has for each $k \in [0, K]_{\mathbb{Z}}$ that $\#\mathrm{Conf}_k \le \varepsilon^{-\omega}$, which implies that $\#\mathrm{EndPts}_k \le \varepsilon^{-\omega}$. We then show using Lemma 2.14 and a union bound over at most $\varepsilon^{-\omega}$ elements of EndPts that if ω is chosen to be sufficiently small (depending on the parameter α of Lemma 2.14, which depends only on D), then for each k it holds with conditional probability at least $1 - \varepsilon^\omega$ given $\mathcal{B}_{t_k}^\bullet, h|_{\mathcal{B}_{t_k}^\bullet}$ that no D_h-geodesic from $\mathbb{z}$ to $\partial\mathcal{B}_{s_{k+1}}^\bullet$ can enter $\bigcup_{y\in\mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$. Finally, we will use the Markovian structure of the GFF together with a standard concentration inequality for Bernoulli random variables to show that if θ is chosen to be sufficiently small then with extremely high probability this happens for at least $(1-\varepsilon^\theta)K$ values of $k \in [0, K]_{\mathbb{Z}}$.

Step 1: bounding the number of confluence points Recall from Sect. 4.1 that $t_k = s_k + \varepsilon^{2\beta} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$ and Conf_k is the set of points of $\partial\mathcal{B}_{s_k}^\bullet$ which are hit by leftmost D_h-geodesics from $\mathbb{z}$ to $\partial\mathcal{B}_{t_k}^\bullet$. Due to Remark 4.10, we can apply Theorem 2.16 (with $N = \lfloor \varepsilon^{-\omega} \rfloor$ and $\tau = s_k$) to get that if β is chosen sufficiently small, in a manner depending only on ω and D, then for each $k \in [0, K]_{\mathbb{Z}}$, the probability that $\mathcal{E}_{\mathbb{r}}$ occurs and $\#\mathrm{Conf}_k > \varepsilon^{-\omega}$ decays faster than any positive power of ε. By a union bound over k,

$$\mathbb{P}\left[\mathcal{E}_{\mathbb{r}},\ \max_{k\in[0,K]_{\mathbb{Z}}} \#\mathrm{Conf}_k > \varepsilon^{-\omega}\right] = o_\varepsilon^\infty(\varepsilon). \tag{4.39}$$

Step 2: bounding the parameters from Lemma 2.14 Recall the radii $\rho_{\mathbb{r},\varepsilon}(z)$, which appear in Lemma 2.13 and condition 6 (bounds for $\rho_{\mathbb{r},\varepsilon}(z)$) in the definition of $\mathcal{E}_{\mathbb{r}}$ (the precise definition of these radii is not needed here, only their role in Lemma 2.14). To lighten notation, for $k \in [0, K]_{\mathbb{Z}}$ we define

$$\begin{aligned}
R_k &:= R_{\mathbb{r}}^{\varepsilon^\kappa}(\mathcal{B}^\bullet_{t_k}) = 6 \max\left\{\rho_{\mathbb{r},\varepsilon^\kappa}(z) : z \in \left(\frac{\varepsilon^\kappa \mathbb{r}}{4}\mathbb{Z}^2\right) \cap B_{\varepsilon^\kappa \mathbb{r}}\left(\mathcal{B}^\bullet_{t_k}\right)\right\} \\
&\qquad + \varepsilon \mathbb{r}, \quad \text{as in (2.21) and} \\
\sigma_k &:= \sigma^{\varepsilon^\kappa}_{t_k,\mathbb{r}} = \inf\left\{s' > s : B_{R_k}(\mathcal{B}^\bullet_{t_k}) \subset \mathcal{B}^\bullet_{s'}\right\}, \quad \text{as in (2.22)}.
\end{aligned} \tag{4.40}$$

Note that we use (2.21) and (2.22) with ε^κ in place of ε.

On $\mathcal{E}_{\mathbb{r}}$, we have $\mathcal{B}^\bullet_{t_k} \subset \mathcal{B}^\bullet_{\tau_{2\ell\mathbb{r}}} \subset B_{2\ell\mathbb{r}}(\mathbb{z}) \subset B_{4\ell\mathbb{r}}(\mathbb{r}V)$ for each $k \in [0, K]_{\mathbb{Z}}$ (see (4.36)). Hence we can apply condition 6 (bounds for $\rho_{\mathbb{r},\varepsilon}(z)$) in the definition of $\mathcal{E}_{\mathbb{r}}$ and the definition (4.40) of R_k to get that if ε is chosen sufficiently small, depending on a and κ, then on $\mathcal{E}_{\mathbb{r}}$, we have $R_k \le (6\varepsilon^{\kappa/2} + \varepsilon^{\kappa/2})\mathbb{r} \le 7\varepsilon^{\kappa/2}\mathbb{r}$ for each $k \in [0, K]_{\mathbb{Z}}$. By combining this with the upper bound for D_h-distances from condition 3 (Hölder continuity) in the definition of $\mathcal{E}_{\mathbb{r}}$, we get that $B_{R_k}(\mathcal{B}^\bullet_{t_k}) \subset \mathcal{B}^\bullet_{t_k + 7^\chi \varepsilon^{\kappa\chi/2}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}}$. By this together with the definition of σ_k and condition 4 (comparison of circle averages) in the definition of $\mathcal{E}_{\mathbb{r}}$ (to replace $h_{\mathbb{r}}(0)$ with $h_{\mathbb{r}}(\mathbb{z})$), on $\mathcal{E}_{\mathbb{r}}$ we have $\sigma_k \le t_k + A\varepsilon^{\kappa\chi/2}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}$, where $A = 7^\chi e^{\xi/a}$ is an unimportant constant.

We henceforth assume that $\beta < \kappa\chi/2$, so that by the conclusion of the preceding paragraph and the definition (4.6) of t_k and s_{k+1}, for small enough $\varepsilon \in (0, 1)$ (how small depends only on a, β, κ),

$$\sigma_k \le t_k + (\varepsilon^\beta - \varepsilon^{2\beta})\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} = s_{k+1}, \quad \forall k \in [0, K]_{\mathbb{Z}}, \quad \text{on } \mathcal{E}_{\mathbb{r}}. \tag{4.41}$$

Step 3: killing off geodesics near the endpoints with polynomially high probability Recall that EndPts_k denotes the set of endpoints of arcs in $\mathcal{I}_k$. We have $\#\mathrm{EndPts}_k = \#\mathcal{I}_k$. By Lemma 4.14, each of the sets $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$ can be disconnected from ∞ in $\mathbb{C}\setminus\mathcal{B}^\bullet_{t_k}$ by a connected subset of $\mathbb{C}\setminus\mathcal{B}^\bullet_{t_k}$ of Euclidean diameter at most $16\varepsilon^\kappa\mathbb{r}$. By an argument as in (4.41), if $\varepsilon \in (0, 1)$ is chosen sufficiently small (how small depends only on β, κ), then $B_{16\varepsilon^\kappa \mathbb{r}}(\mathcal{B}^\bullet_{t_k}) \subset \mathcal{B}^\bullet_{s_{k+1}}$. We may therefore choose for each $y \in \mathrm{EndPts}_k$ a point $z_y \in \partial\mathcal{B}^\bullet_{t_k}$, in a manner depending only on $(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$, with the following property.

(*) Every path in $\mathbb{C}\setminus\mathcal{B}^\bullet_{t_k}$ from $\mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$ to $\mathbb{C}\setminus\mathcal{B}^\bullet_{s_{k+1}}$ must enter $B_{16\varepsilon^\kappa\mathbb{r}}(z_y)$.

By Lemma 2.14 (applied with $\tau = t_k$ and $16\varepsilon^\kappa$ in place of ε), there are constants $C_0 > 1$ and $\alpha > 0$, depending only on the choice of metric, and an event G_y for each $y \in \mathrm{EndPts}_k$ such that $G_y \in \sigma\left(\mathcal{B}^\bullet_{\sigma_k}, h|_{\mathcal{B}^\bullet_{\sigma_k}}\right)$ and the following is true.

A. If $R_k \leq \mathrm{diam}(\mathcal{B}^\bullet_{t_k})$ and G_y occurs, then no D_h-geodesic from $\mathbb{z}$ to a point of $\mathbb{C}\setminus\mathcal{B}^\bullet_{\sigma_k}$ can enter $B_{16\varepsilon^\kappa \mathbb{r}}(z_y)\setminus\mathcal{B}^\bullet_{t_k}$.
B. Almost surely, $\mathbb{P}[G_y | \mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}] \geq 1 - C_0\varepsilon^{\alpha\kappa}$.

Henceforth assume that $\omega \in (0, \alpha\kappa/2)$. On the event $\{\#\mathrm{Conf}_k \leq \varepsilon^{-\omega}\}$ (which is in $\sigma(\mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}})$ and has high probability by (4.39) and the fact that $\mathcal{E}_{\mathbb{r}}$ has high probability), we can take a union bound over at most $\varepsilon^{-\omega}$ elements of EndPts_k to get

$$\mathbb{P}\left[\bigcap_{y\in\mathrm{EndPts}_k} G_y \,\middle|\, \mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}\right] \geq 1 - C_0\varepsilon^{\alpha\kappa-\omega}. \tag{4.42}$$

Since $\omega < \alpha\kappa/2$, the right side of (4.42) is at least $1 - \varepsilon^\omega$ for small enough $\varepsilon \in (0, 1)$ (how small depends only on $\alpha, \kappa, \omega, C_0$). This implies that for each such ε,

$$\mathbb{P}\left[\left(\bigcap_{y\in\mathrm{EndPts}_k} G_y\right)^c, \ \sigma_k \leq s_{k+1}, \ \#\mathrm{Conf}_k \leq \varepsilon^{-\omega} \,\middle|\, \mathcal{B}^\bullet_{t_k}, h|_{\mathcal{B}^\bullet_{t_k}}\right] \leq \varepsilon^\omega. \tag{4.43}$$

Note that we have added the additional event $\{\sigma_k \leq s_{k+1}\}$, for reasons which will become apparent just below.

Step 4: independence across radii to get concentration The radius σ_k is a stopping time for $\{(\mathcal{B}^\bullet_s, h|_{\mathcal{B}^\bullet_s})\}_{s\geq 0}$, so the event inside the conditional probability in (4.43) belongs to $\sigma\left(\mathcal{B}^\bullet_{s_{k+1}}, h|_{\mathcal{B}^\bullet_{s_{k+1}}}\right)$. Since $t_{k+1} \geq s_{k+1}$, it therefore follows from (4.43) that the number of $k \in [0, K]_{\mathbb{Z}}$ for which either $\bigcap_{y\in\mathrm{EndPts}_k} G_y$ occurs, $\sigma_k > s_{k+1}$, or $\#\mathrm{Conf}_k > \varepsilon^{-\omega}$ stochastically dominates a binomial distribution with K trials and success probability $1 - \varepsilon^\omega$. By Hoeffding's inequality, for any choice of $\theta \in (0, 1)$ the probability that there are fewer than $(1 - \varepsilon^\theta)K$ such values of k is at most

$$\exp\left(-2(\varepsilon^\theta - \varepsilon^\omega)^2 K\right).$$

Since $K = \lfloor a\varepsilon^{-\beta}\rfloor - 1$ by (4.35), this last quantity decays faster than any positive power of ε provided we take $\theta \in (0, \min\{\omega, \beta/2\})$.

By (4.41), on $\mathcal{E}_{\mathbb{r}}$ we have $\sigma_k \leq s_{k+1}$ for each $k \in [0, K]_{\mathbb{Z}}$. By (4.39), if $\mathcal{E}_{\mathbb{r}}$ occurs then except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ we have $\#\mathrm{Conf}_k \leq \varepsilon^{-\omega}$ for each $K \in [0, K]_{\mathbb{Z}}$. Combining these observations with the preceding paragraph shows that

$$\mathbb{P}\left[\mathcal{E}_{\mathbb{r}}, \#\left\{k \in [0, K]_{\mathbb{Z}} : \bigcap_{y \in \mathrm{EndPts}_k} G_y \text{ occurs}\right\} < (1-\varepsilon^\theta)K\right] = o_\varepsilon^\infty(\varepsilon). \tag{4.44}$$

Recall that $R_k \leq 7\varepsilon^{\kappa/2}\mathbb{r}$ on $\mathcal{E}_{\mathbb{r}}$ (see this discussion just after (4.40)). As $t_k \geq \tau_{\ell\mathbb{r}}$ we have that $\operatorname{diam} \mathcal{B}_{t_k}^\bullet \geq \ell\mathbb{r}$. By choosing $\varepsilon > 0$ sufficiently small we can arrange so that $\ell\mathbb{r} \geq 7\varepsilon^{\kappa/2}\mathbb{r}$. That is, $R_k \leq \operatorname{diam} \mathcal{B}_{t_k}^\bullet$ on $\mathcal{E}_{\mathbb{r}}$ provided ε is chosen sufficiently small (in a manner depending only on κ and ℓ). Consequently, (4.44) together with property A of G_y show that on $\mathcal{E}_{\mathbb{r}}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that there are at least $(1-\varepsilon^\theta)K$ values of $k \in [0, K]_{\mathbb{Z}}$ for which no D_h-geodesic from $\mathbb{z}$ to a point outside of $\mathcal{B}_{\sigma_k}^\bullet$ can enter $\bigcup_{y \in \mathrm{EndPts}_k} B_{16\varepsilon^\kappa \mathbb{r}}(z_y) \backslash \mathcal{B}_{t_k}^\bullet$. By (4.41), this holds in particular for each D_h-geodesic from $\mathbb{z}$ to $\partial \mathcal{B}_{s_{k+1}}^\bullet$.

A D_h-geodesic started from $\mathbb{z}$ can hit $\partial \mathcal{B}_{t_k}^\bullet$ at most once. Therefore, the defining property $(*)$ of z_y, applied to the path $P|_{(t_k, |P|]}$, shows that for each k as in the preceding paragraph, no D_h-geodesic from $\mathbb{z}$ to $\partial \mathcal{B}_{s_{k+1}}^\bullet$ can enter $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$. □

To deduce Proposition 4.12 from Lemma 4.15, we need some quantitative control on the $D_h(\cdot, \cdot; \mathbb{C} \backslash \overline{B_r(z)})$-geodesics appearing in the definition (4.11) of $\mathrm{Stab}_{k,r}(z)$. The needed control is provided by the following lemma.

Lemma 4.16 *If $\mathcal{E}_{\mathbb{r}}$ occurs, then for each $k \in [0, K]_{\mathbb{Z}}$, each $(z, r) \in \mathcal{Z}_k$, and each $D_h(\cdot, \cdot; \mathbb{C} \backslash \overline{B_r(z)})$-geodesic $P' : [0, |P'|] \to \mathbb{C} \backslash B_r(z)$ from $\mathbb{z}$ to a point of $\partial B_r(z)$, we have*

$$\operatorname{diam} P'([t_k, |P'|]) \preceq \varepsilon^{\chi/\chi'}\mathbb{r}, \tag{4.45}$$

with a deterministic implicit constant depending only on a and λ_4, where diam *denotes Euclidean diameter.*

Lemma 4.16 is a straightforward consequence of the definition of $\mathcal{E}_{\mathbb{r}}$. We postpone the proof until Sect. 4.6.

Proof of Proposition 4.12 Let χ, χ' be the Hölder exponents from condition 3 in the definition of $\mathcal{E}_{\mathbb{r}}$ and let $\beta, \theta \in (0, 1)$ be chosen so that the conclusion of Lemma 4.15 holds with $\kappa = \frac{1}{2}(\chi/\chi')^2$. By Lemma 4.15, we only need to prove that if $\mathcal{E}_{\mathbb{r}}$ occurs and $\varepsilon \in (0,1)$ is chosen to be sufficiently small (in a deterministic manner which does not depend on k or $\mathbb{r}$), then the following is true. If $k \in [0,K]_{\mathbb{Z}}$ is such that no D_h-geodesic from $\mathbb{z}$ to $\partial \mathcal{B}^\bullet_{s_{k+1}}$ can enter $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$, then $\mathcal{Z}_k^E \neq \emptyset$.

Henceforth assume that $\mathcal{E}_{\mathbb{r}}$ occurs and k is as above. Recall the definition (4.10) of $\mathcal{Z}_k$. By condition 5 (existence of good annuli) in the definition of $\mathcal{E}_{\mathbb{r}}$, each point of $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})$ is contained in a Euclidean ball $B_{\lambda_2 r}(z)$ for some $(z,r) \in \mathcal{Z}_k$ for which $E_r(z)$ occurs. By the definition (4.10), each of these Euclidean balls has radius $r \leq \varepsilon \mathbb{r}$, so is contained in $B_{4\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})$.

Since $t_k \leq s_{k+1} \leq \tau_{3\ell \mathbb{r}}$ (by (4.36)) and $|\mathbb{z} - \mathbb{w}| \geq 4\ell \mathbb{r}$, if ε is sufficiently small then the union of these Euclidean balls disconnects $\partial \mathcal{B}^\bullet_{t_k}$ from $\mathbb{w}$. Therefore, P must enter $B_{\lambda_2 r}(z)$ for some $(z,r) \in \mathcal{Z}_k$ such that $E_r(z)$ occurs.

We will now conclude the proof by showing that, in the notation (4.11),

$$\mathrm{Stab}_{r,k}(z) \text{ occurs for every } (z,r) \in \mathcal{Z}_k \text{ with } P \cap B_r(z) \neq \emptyset. \tag{4.46}$$

Recall that we are assuming that k is such that no D_h-geodesic from $\mathbb{z}$ to $\partial \mathcal{B}^\bullet_{s_{k+1}}$ can enter $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$. Since $s_{k+1} \leq \tau_{3\ell \mathbb{r}} \leq \tau_{|\mathbb{z}-\mathbb{w}|}$ we must have $|P| \geq s_{k+1}$, so P passes through $\partial \mathcal{B}^\bullet_{s_{k+1}}$. Hence $P|_{[0,s_{k+1}]}$ cannot enter $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$. Since P does not re-enter $\mathcal{B}^\bullet_{s_{k+1}}$ after time s_{k+1} and $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}} \subset \mathcal{B}^\bullet_{s_{k+1}}$, also P cannot enter $\bigcup_{y \in \mathrm{EndPts}_k} \mathcal{C}_y^{\varepsilon^\kappa \mathbb{r}}$. From this and Lemma 4.13 (applied in the contrapositive direction with $(\varepsilon^\kappa/2)^{\chi'/\chi}$ in place of ε and I_k in place of I), we infer that if $I_k \in \mathcal{I}_k$ is chosen so that $P(t_k) \in I_k$, then

$$d^{\mathbb{C} \setminus \mathcal{B}^\bullet_{t_k}}\left(P, \partial \mathcal{B}^\bullet_{t_k} \setminus I_k\right) \geq (\varepsilon^\kappa/2)^{\chi'/\chi} \mathbb{r}. \tag{4.47}$$

Now let $(z,r) \in \mathcal{Z}_k$ with $P \cap B_r(z) \neq \emptyset$ and let $P' : [0,|P'|] \to \mathbb{C} \setminus B_r(z)$ be a $D_h(\cdot,\cdot; \mathbb{C} \setminus \overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$. We will show that $P'(t_k) \in I_k$ for any possible choice of P', which by definition implies that $\mathrm{Stab}_{r,k}(z)$ occurs. By Lemma 4.16,

$$\mathrm{diam}\left(P'([t_k, |P'|])\right) \preceq \varepsilon^{\chi/\chi'} \mathbb{r} \tag{4.48}$$

with a deterministic implicit constant depending only on a and λ_4.

Since $B_r(z) \subset \mathbb{C} \setminus \mathcal{B}^\bullet_{t_k}$, the definition (4.38) of $d^{\mathbb{C} \setminus \mathcal{B}^\bullet_{t_k}}$ implies that the $d^{\mathbb{C} \setminus \mathcal{B}^\bullet_{t_k}}$-diameter of $B_r(z)$ is the same as its Euclidean diameter, which is $2\varepsilon \mathbb{r}$. Since $P \cap B_r(z) \neq \emptyset$ and $P'(|P'|) \in \partial B_r(z)$, it follows from (4.48) and the triangle inequality that for small enough $\varepsilon \in (0,1)$,

$$d^{\mathbb{C} \setminus \mathcal{B}^\bullet_{t_k}}(P, P'(t_k)) \preceq \varepsilon \mathbb{r} + \varepsilon^{\chi/\chi'} \mathbb{r} \preceq \varepsilon^{\chi/\chi'} \mathbb{r}. \tag{4.49}$$

Since $\kappa < (\chi/\chi')^2$, we infer that the left side of (4.49) is strictly smaller than $(\varepsilon^\kappa/2)^{\chi'/\chi}\mathbbm{r}$ for small enough $\varepsilon \in (0,1)$ (depending only on a and λ_4). By combining (4.47) and (4.49) we infer that $P'(t_k) \notin \partial\mathcal{B}^\bullet_{t_k}\setminus I_k$. Hence $P'(t_k) \in I_k$. Since this holds for every choice of P', we get that $\mathrm{Stab}_{r,k}(z)$ occurs, as required. $\square$

4.5 Transferring from $E_r(z)$ to $\mathfrak{E}_r(z)$

We now want to combine Lemma 4.7 and Proposition 4.12 to say that with high probability, there are many values of $k \in [0,K]_{\mathbb{Z}}$ for which there exists $(z,r) \in \mathcal{Z}_k$ for which $\mathfrak{E}_r(z)$ occurs. In particular, we will establish the following statement.

Proposition 4.17 *Let $\beta, \theta \in (0,1)$ be as in Proposition 4.12 and suppose we have chosen ν sufficiently small that $4\nu < \beta \wedge \theta$. Also let $\zeta \in (0,1)$ be a small "error" parameter. If $\mathcal{E}_{\mathbbm{r}}$ occurs, then except on an event of probability $o^\infty_\varepsilon(\varepsilon)$, at a rate which is uniform in the choice of $\mathbbm{r}, \mathbbm{z}, \mathbbm{w}$, there are at least $\varepsilon^{2\nu+\zeta}K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which $\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset$.*

Lemma 4.7 gives a comparison of the conditional probabilities given $\mathcal{F}_k$ of $\{\mathcal{Z}_k^E \neq \emptyset\}$ and $\{\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset\}$ (the reason why we have this comparison is that condition 4 in Theorem 4.2 has a comparison of conditional probabilities). On the other hand, Propositions 4.12 and 4.17 give statements which hold with high unconditional probability. To transfer between conditional and unconditional probabilities we will use the following elementary lemma.

Lemma 4.18 *Let $K \in \mathbb{N}$ and let $E_0, \ldots, E_K$ be events (not necessarily independent). Also let $\mathcal{F}_1 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_K$ be σ-algebras such that $E_k \in \mathcal{F}_{k+1}$ for each $k \in [0,K-1]_{\mathbb{Z}}$. For $\alpha \in (0,1)$, $\delta \in (0,\alpha)$, and $m \in \mathbb{N}$,*

$$\mathbb{P}\left[\sum_{k=0}^K \mathbb{1}_{(\mathbb{P}[E_k \mid \mathcal{F}_k] \geq \alpha)} \leq K - m,\ \sum_{k=0}^K \mathbb{1}_{E_k} \geq K - (1-\alpha-\delta)m\right] \leq e^{-2\delta^2 m}. \tag{4.50}$$

Proof For $j \in \mathbb{N}$, let τ_j be the jth smallest $k \in [0,K]_{\mathbb{Z}}$ for which $\mathbb{P}[E_k \mid \mathcal{F}_k] < \alpha$, or $\tau_j = K+1$ if no such j exists. Then $\{\tau_j = k\} \in \mathcal{F}_k$ for each $k \in [0,K]_{\mathbb{Z}}$ and

$$\begin{aligned}\left\{\sum_{k=0}^K \mathbb{1}_{(\mathbb{P}[E_k \mid \mathcal{F}_k] \geq \alpha)} \leq K - m\right\} &= \left\{\sum_{k=0}^K \mathbb{1}_{(\mathbb{P}[E_k \mid \mathcal{F}_k] < \alpha)} \geq m+1\right\} \\ &= \{\tau_{m+1} \leq K\}.\end{aligned} \tag{4.51}$$

By the definition of the τ_j's, for each $j \in \mathbb{N}$,

$$\mathbb{P}\left[E_{\tau_j}^c \mid \mathcal{F}_{\tau_j}\right] \geq 1-\alpha. \tag{4.52}$$

Since $E_{\tau_{j'}} \in \mathcal{F}_{\tau_{j-1}}$ for each $j' \leq j-1$, it follows that $\sum_{j=1}^{m+1} \mathbb{1}_{E_{\tau_j}^c}$ stochastically dominates a binomial distribution with $m+1$ trials and success probability $1-\alpha$. By Hoeffding's inequality, for $m \in \mathbb{N}$ the probability that the number of $j \in [1, m+1]_{\mathbb{Z}}$ for which $E_{\tau_j}^c$ occurs is smaller than $(1-\alpha-\delta)m$ is at most $e^{-2\delta^2 m}$. Therefore,

$$\mathbb{P}\left[\tau_{m+1} \leq K, \sum_{j=0}^{K} \mathbb{1}_{E_j} \geq K-(1-\alpha-\delta)m\right] \leq e^{-2\delta^2 m}. \tag{4.53}$$

Combining this with (4.51) gives (4.50). □

We want to apply Lemma 4.18 to the events $\{\mathcal{Z}_k^E \neq \emptyset\}$ and $\{\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset\}$. However, these events are not $\mathcal{F}_{k+1}$-measurable since for $(z,r) \in \mathcal{Z}_k$, the ball $B_r(z)$ and the $D_h(\cdot,\cdot;\mathbb{C}\backslash\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to $\partial B_r(z)$ are not necessarily contained in $\mathcal{B}_{s_{k+1}}^\bullet$. To get around this, we need to instead work with a slightly modified event which is $\mathcal{F}_{k+1}$-measurable. In particular, we will intersect each of $\{\mathcal{Z}_k^E \neq \emptyset\}$ and $\{\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset\}$ with the event F_k of the following lemma.

Lemma 4.19 *For each $k \in [0, K]_{\mathbb{Z}}$, there is an event $F_k \in \sigma\left(\mathcal{B}_{s_{k+1}}^\bullet, h|_{\mathcal{B}_{s_{k+1}}^\bullet}\right)$ with the following properties. If ε is sufficiently small (how small depends only on a, λ_4), then whenever $\mathcal{E}_{\mathbb{r}}$ occurs also $\bigcap_{k=0}^K F_k$ occurs. Moreover, if F_k occurs then $s_{k+1} \leq \tau_{2\ell_{\mathbb{r}}}$ and for each $(z,r) \in \mathcal{Z}_k$ we have $B_{\lambda_4 r}(z) \subset \mathcal{B}_{s_{k+1}}^\bullet$ and the set of $D_h(\cdot,\cdot;\mathbb{C}\backslash\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to points of $\partial B_r(z)$ is determined by $(\mathcal{B}_{s_{k+1}}^\bullet, h|_{\mathcal{B}_{s_{k+1}}^\bullet})$.*

Lemma 4.19 is a relatively straightforward consequence of the definition of $\mathcal{E}_{\mathbb{r}}$. The proof is postponed until Sect. 4.6. The event F_k is defined explicitly in Lemma 4.22 below, but only the properties of the event given in Lemma 4.19 are important for our purposes.

Lemma 4.20 *Let F_k for $k \in [0, K]_{\mathbb{Z}}$ be the event of Lemma 4.19 and let $\mathcal{F}_k$ be the σ-algebra from (4.8). Then for $k \in \mathbb{N}$,*

$$\left\{\mathcal{Z}_k^E \neq \emptyset\right\} \cap F_k \in \mathcal{F}_{k+1} \quad \textit{and} \quad \left\{\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset\right\} \cap F_k \in \mathcal{F}_{k+1}. \tag{4.54}$$

Proof By Lemma 4.19, we have $F_k \in \mathcal{F}_{k+1}$. By the definition (4.10) we also have $\mathcal{Z}_k \in \mathcal{F}_k \subset \mathcal{F}_{k+1}$.

We now argue that on F_k, the set $\mathcal{Z}_k^E$ is determined by $\mathcal{F}_{k+1}$. Since there are only countably many pairs $(z,r) \in \mathbb{C} \times (0,\infty)$ which can possibly belong to $\mathcal{Z}_k^E$, it suffices to show that the event $\{(z,r) \in \mathcal{Z}_k^E\} \cap F_k$ is $\mathcal{F}_{k+1}$-measurable for each such pair (z,r). Recall from (4.12) that $\mathcal{Z}_k^E$ is the set of $(z,r) \in \mathcal{Z}_k$ for which $E_r(z) \cap \mathrm{Stab}_{r,k}(z) \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\}$ occurs. By Lemma 4.19, if F_k occurs then $B_{\lambda_4 r}(z) \subset \mathcal{B}^\bullet_{s_{k+1}}$ for each $(z,r) \in \mathcal{Z}_k$. Since $E_r(z)$ is determined by $h|_{B_{\lambda_4 r}(z)}$ (condition 2), it follows that $F_k \cap E_r(z) \cap \{(z,r) \in \mathcal{Z}_k\} \in \mathcal{F}_{k+1}$ for each $(z,r) \in \mathbb{C} \times (0,\infty)$. Moreover, since $P|_{[0,s_{k+1}]} \in \mathcal{F}_{k+1}$ and P does not re-enter $\mathcal{B}^\bullet_{s_{k+1}}$ after time s_{k+1}, we have $F_k \cap \{P \cap B_{\lambda_2 r}(z) \neq \emptyset\} \cap \{(z,r) \in \mathcal{Z}_k\} \in \mathcal{F}_{k+1}$ for each (z,r). By (4.11), each of the events $\mathrm{Stab}_{r,k}(z)$ for $(z,r) \in \mathcal{Z}_k$ is determined by $\mathcal{F}_k$ and the set of $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to points of $\partial B_r(z)$. By Lemma 4.19, it therefore follows that $F_k \cap \mathrm{Stab}_{r,k}(z) \in \mathcal{F}_{k+1}$ for each $(z,r) \in \mathcal{Z}_k$. Combining these statements shows that $\{\mathcal{Z}_k^E \neq \emptyset\} \cap F_k \in \mathcal{F}_{k+1}$.

Using condition 2 from Theorem 4.2, we similarly obtain that $\left\{\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset\right\} \cap F_k \in \mathcal{F}_{k+1}$. $\square$

Lemma 4.21 *Let θ be as in Proposition 4.12 and let F_k for $k \in \mathbb{N}$ be as in Lemma 4.19. If $\mathcal{E}_{\mathbb{r}}$ occurs, then except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ there are at least $(1-4\varepsilon^\theta)K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which*

$$\mathbb{P}\left[\left\{\mathcal{Z}_k^E \neq \emptyset\right\} \cap F_k \,\middle|\, \mathcal{F}_k\right] \geq \frac{1}{2} \tag{4.55}$$

and

$$\mathbb{P}\left[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,\middle|\, \mathcal{F}_k\right] \geq \varepsilon^{2\nu + o_\varepsilon(1)}, \tag{4.56}$$

where the rate of the $o_\varepsilon(1)$ in (4.56) is deterministic and depends only on ν and the choice of metric D.

Proof For $k \in \mathbb{N}$, let $E_k := \left\{\mathcal{Z}_k^E \neq \emptyset\right\} \cap F_k$. By Lemma 4.20, we have $E_k \in \mathcal{F}_{k+1}$. We may therefore apply Lemma 4.18 with $m = \lfloor 4\varepsilon^\theta K \rfloor$, $\alpha = 1/2$, and $\delta = 1/4$ to get that

$$\mathbb{P}\left[\sum_{k=0}^{K} \mathbb{1}_{(\mathbb{P}[E_k \,|\, \mathcal{F}_k] \geq 1/2)} \leq (1-4\varepsilon^\theta)K,\ \sum_{k=0}^{K} \mathbb{1}_{E_k} \geq (1-\varepsilon^\theta)K\right] = o_\varepsilon^\infty(\varepsilon). \tag{4.57}$$

By Proposition 4.12 and Lemma 4.19, on $\mathcal{E}_{\mathbb{r}}$ it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that $\sum_{k=1}^K \mathbb{1}_{E_k} \geq (1-\varepsilon^\theta)K$. Combining this with (4.57) shows that if $\mathcal{E}_{\mathbb{r}}$ occurs, then except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ there are at least $(1-4\varepsilon^\theta)K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which (4.55) holds.

On $\mathcal{E}_{\mathbb{r}}$, for each $k \in [0,K]_{\mathbb{Z}}$ we have $\mathcal{B}_{t_k}^\bullet \subset B_{3\ell\mathbb{r}}(\mathbb{z})$ (by (4.36)) and $\mathbb{w} \notin B_{3\lambda_4\varepsilon\mathbb{r}}(\mathcal{B}_{t_k}^\bullet)$ (since $|\mathbb{z}-\mathbb{w}| \geq 4\ell\mathbb{r}$). By Lemma 4.7, whenever these latter conditions hold it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ that

$$\mathbb{P}\left[\mathcal{Z}_k^{\mathfrak{E}} \neq \emptyset \,\middle|\, \mathcal{F}_k\right] \geq \varepsilon^{2\nu+o_\varepsilon(1)}\mathbb{P}\left[\mathcal{Z}_k^E \neq \emptyset \,\middle|\, \mathcal{F}_k\right] - o_\varepsilon^\infty(\varepsilon).$$

Combining this with (4.55) shows that if $\mathcal{E}_{\mathbb{r}}$ occurs, then except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ there are at least $(1-4\varepsilon^\theta)K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which (4.55) and (4.56) both hold. $\square$

We now apply the estimate (4.56) to deduce Proposition 4.17.

Proof of Proposition 4.17 Let F_k be the event of Lemma 4.19, so that by Lemma 4.20 we have $\left\{\mathcal{Z}_k^{\mathfrak{E}} = \emptyset\right\} \cap F_k \in \mathcal{F}_{k+1}$. By Lemma 4.21, if $\mathcal{E}_{\mathbb{r}}$ occurs then except on an event of probability $o_\varepsilon^\infty(\varepsilon)$ there are at least $(1-4\varepsilon^\theta)K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which

$$\mathbb{P}\left[\left\{\mathcal{Z}_k^{\mathfrak{E}} = \emptyset\right\} \cap F_k \mid \mathcal{F}_k\right] \leq 1 - \varepsilon^{2\nu+\zeta/2}, \tag{4.58}$$

equivalently, there are at most $4\varepsilon^\theta K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which

$$\mathbb{P}\left[\left\{\mathcal{Z}_k^{\mathfrak{E}} = \emptyset\right\} \cap F_k \mid \mathcal{F}_k\right] \geq 1 - \varepsilon^{2\nu+\zeta/2}.$$

By Lemma 4.18 applied with $E_k = \left\{\mathcal{Z}_k^{\mathfrak{E}} = \emptyset\right\} \cap F_k$, $m = \lfloor (1-4\varepsilon^\theta)K \rfloor$, $\alpha = 1-\varepsilon^{2\nu+\zeta/2}$, and $\delta = \varepsilon^{2\nu+\zeta/2}/2$, it follows that if $\mathcal{E}_{\mathbb{r}}$ occurs and ε is sufficiently small, then except on an event of probability at most

$$\exp\left(-\frac{1}{2}\varepsilon^{4\nu+\zeta}\lfloor (1-4\varepsilon^\theta)K \rfloor\right) \tag{4.59}$$

there are at most

$$K - (1-\alpha-\delta)m \leq \left(1 - \varepsilon^{2\nu+\zeta/2}(1-4\varepsilon^\theta)/2\right)K \leq (1-\varepsilon^{2\nu+\zeta})K$$

values of $k \in [0,K]_{\mathbb{Z}}$ for which E_k occurs. Equivalently, there are at least $\varepsilon^{2\nu+\zeta}K$ values of $k \in [0,K]_{\mathbb{Z}}$ for which either $\mathcal{Z}^{\mathfrak{E}} \neq \emptyset$ or F_k does not occur. By Lemma 4.19, on $\mathcal{E}_{\mathbb{r}}$ the event F_k occurs for every $k \in [0,K]_{\mathbb{Z}}$. Since $K \asymp \varepsilon^{-\beta}$ (by (4.35)), if $4\nu < \beta$ then for a small enough choice of $\zeta \in (0,1)$, the quantity (4.59) is of order $o_\varepsilon^\infty(\varepsilon)$. The proposition now follows. $\square$

Proof of Theorem 4.2 Assume we are in the setting of the theorem statement with $\nu_* = \frac{1}{8}(\beta \wedge \theta)$. Fix $q > 0$. Recall that we have been fixing $\mathbbm{z}, \mathbbm{w} \in \mathbbm{r}U$ with $|\mathbbm{z} - \mathbbm{w}| \geq 4\ell\mathbbm{r}$ throughout this section. Proposition 4.17 implies that if $\mathcal{E}_{\mathbbm{r}}$ occurs, then for each fixed choice of $\mathbbm{z}, \mathbbm{w} \in \left(\varepsilon^q \mathbbm{r}\mathbb{Z}^2\right) \cap (\mathbbm{r}U)$ with $|\mathbbm{z} - \mathbbm{w}| \geq 4\ell\mathbbm{r}$, it holds except on an event of probability $o_\varepsilon^\infty(\varepsilon)$, at a rate which does not depend on $\mathbbm{z}$, $\mathbbm{w}$, or $\mathbbm{r}$, that there exists $k \in [0, K]_{\mathbb{Z}}$ for which the corresponding set $\mathcal{Z}_k^{\mathfrak{E}}$ of (4.13) is non-empty. By (4.13), this means that there exists $z \in \mathbb{C}$ and $r \in [\varepsilon^{1+\nu}\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \mathcal{R}$ such that $P^{\mathbbm{z},\mathbbm{w}} \cap B_{\lambda_2 r}(z) \neq \emptyset$ and $\mathfrak{E}_r^{\mathbbm{z},\mathbbm{w}}(z)$ occurs.

Since the definition of $\mathcal{E}_{\mathbbm{r}}$ does not depend on $\mathbbm{z}$, $\mathbbm{w}$, we can truncate on $\mathcal{E}_{\mathbbm{r}}$, then take a union bound over all pairs $\mathbbm{z}, \mathbbm{w} \in \left(\varepsilon^q \mathbbm{r}\mathbb{Z}^2\right) \cap (\mathbbm{r}U)$ with $|\mathbbm{z} - \mathbbm{w}| \geq 4\ell\mathbbm{r}$, to get that if $\mathcal{E}_{\mathbbm{r}}$ occurs then the following is true except on an event of probability $o_\varepsilon^\infty(\varepsilon)$. For each such pair $\mathbbm{z}$, $\mathbbm{w}$ that there exists $z \in \mathbb{C}$ and $r \in [\varepsilon^{1+\nu}\mathbbm{r}, \varepsilon\mathbbm{r}] \cap \mathcal{R}$ such that $P^{\mathbbm{z},\mathbbm{w}} \cap B_{\lambda_2 r}(z) \neq \emptyset$ and $\mathfrak{E}_r^{\mathbbm{z},\mathbbm{w}}(z)$ occurs.

Since the parameters in the definition of $\mathcal{E}_{\mathbbm{r}}$ can be chosen so as to make $\mathbb{P}[\mathcal{E}_{\mathbbm{r}}]$ as close to 1 as we like (Lemma 4.11), we obtain the theorem statement with 4ℓ in place of ℓ, which is sufficient since ℓ is arbitrary. □

4.6 Proofs of geometric lemmas

In this section we prove the geometric lemmas stated in Sects. 4.4 and 4.5 whose proofs were postponed to avoid distracting from the main argument, namely Lemmas 4.14, 4.16, and 4.19 . The arguments in this section use only the definitions in Sects. 4.1 and 4.3 . In particular, we do not use any of the results in Sects. 4.4 or 4.5.

Proof of Lemma 4.14 See Fig. 7 for an illustration. The proof consists of two main steps.

1. We show that there is a *finite* collection of connected sets $X \subset \mathbb{C}\backslash\mathcal{K}$ with Euclidean diameter at most 4ε such that each point of $\mathcal{C}_y^\varepsilon$ is contained in the bounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup X)$ for one of these sets X. The sets X can be taken to be appropriate boundary arcs of Euclidean balls of radius 2ε.
2. We consider the maximal elements of our finite collection, i.e., those which do not lie in a bounded connected component of any other set in the collection. We show that any two maximal elements have to intersect, so the union of the maximal elements has Euclidean diameter at most 8ε. We then choose a single connected set (which can be taken to be an arc of a Euclidean ball of radius 8ε) which disconnects the union of the maximal elements from ∞ in $\mathbb{C}\backslash\mathcal{K}$.

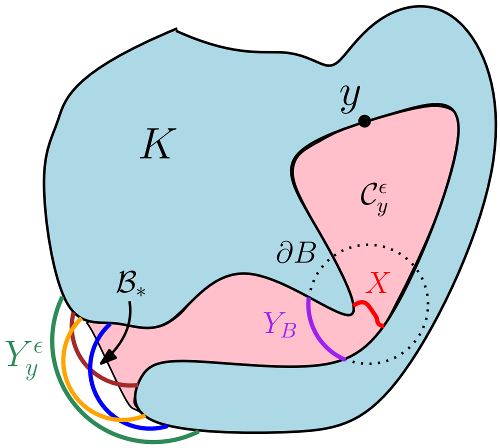

Fig. 7 Illustration of the proof of Lemma 4.14. The set $\mathcal{C}_y^\varepsilon$ is shown in pink. We have shown the boundary of a (non-maximal) ball $B \in \mathcal{B}$ as a dashed line and the associated arc $Y_B \subset \partial B \setminus K$ in purple. Each set X as in the lemma statement is contained in such a ball B and lies in the bounded connected component U_B of $\mathbb{C}\setminus(Y_B \cup K)$. Several arcs Y_B for maximal balls $B \in \mathcal{B}_*$ are shown in various colors. Any two such arcs must intersect each other, so the Euclidean diameter of their union is at most 8ε. The set Y_y^ε (green) in the lemma statement is chosen so as to disconnect this union from ∞ in $\mathbb{C}\setminus K$ (color figure online)

Step 1: reducing to finitely many arcs of Euclidean balls We will first reduce to considering only a finite collection of sets X as in the statement of the lemma by looking at arcs of Euclidean balls. Let $\mathcal{B}$ be the set of closed Euclidean balls of the form $B = \overline{B_{2\varepsilon}(z)}$ for $z \in \frac{\varepsilon}{4}\mathbb{Z}^2$ with the following properties: $B \cap \partial\mathcal{K} \neq \emptyset$ and every unbounded connected subset of $\mathbb{C}\setminus\mathcal{K}$ whose prime end closure contains y has to intersect B. Since $\mathcal{K}$ is compact, $\mathcal{B}$ is a finite set.

For $B \in \mathcal{B}$, the set $\partial B\setminus\mathcal{K}$ is a countable union of open arcs of ∂B. Each such arc divides $\mathbb{C}\setminus\mathcal{K}$ into a bounded connected component and an unbounded connected component. There is one such arc Y_B with the property that y lies on the boundary of the bounded connected component of $\mathbb{C}\setminus(\mathcal{K} \cup Y_B)$ and Y_B is not contained in the bounded connected component of $\mathbb{C}\setminus(\mathcal{K} \cup X)$ for any other such arc $X \neq Y_B$. Note that since B has radius 2ε, the arc Y_B is connected and has Euclidean diameter at most 4ε.

For $B \in \mathcal{B}$, let U_B be the bounded connected component of $\mathbb{C}\setminus(\mathcal{K} \cup Y_B)$ so that $y \in \partial U_B$. We claim that

$$\forall z \in \mathcal{C}_y^\varepsilon, \quad \exists B \in \mathcal{B} \quad \text{such that} \quad z \in U_B. \tag{4.60}$$

Indeed, let X be as in the definition of $\mathcal{C}_y^\varepsilon$ for our given z and let V_X be the bounded connected component of $\mathbb{C}\setminus X$ with y on its boundary. Since X has Euclidean diameter at most ε, we can find $B \in \mathcal{B}$ such that X is contained in the interior of B. We claim that $V_X \subset U_B$, and hence $z \in U_B$. Since X is connected and $X \cap Y_B \subset X \cap \partial B = \emptyset$, it follows that X is either entirely contained in U_B or X is entirely contained in the unbounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup Y_B)$. We claim that X cannot be entirely contained in the unbounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup U_B)$. Indeed, by the definition of X, each unbounded connected subset of $\mathbb{C}\backslash\mathcal{K}$ with y on its boundary must intersect X. Since $X \cap U_B = \emptyset$ and $y \in \partial U_B$, each unbounded connected subset of $\mathbb{C}\backslash\mathcal{K}$ which intersects U_B must intersect X. This implies that $U_B \subset V_X$, but this cannot happen since $X \subset B$ and by the definition of Y_B. Therefore $X \subset U_B$, so $V_X \subset U_B$, so (4.60) holds.

Step 2: maximal elements of $\mathcal{B}$ We define a partial order on $\mathcal{B}$ by declaring that $B \preceq B'$ if and only if $U_B \subset U_{B'}$. Let $\mathcal{B}_*$ be the set of maximal elements of $\mathcal{B}$, i.e., $B_* \in \mathcal{B}_*$ if and only if there is no $B \in \mathcal{B}\backslash\{B_*\}$ such that $B_* \preceq B$. Since $\mathcal{B}$ is a finite set, for every $B \in \mathcal{B}$ there exists $B_* \in \mathcal{B}_*$ satisfying $B \preceq B_*$.

We claim that if $B_1, B_2 \in \mathcal{B}_*$, then $Y_{B_1} \cap Y_{B_2} \neq \emptyset$. Indeed, if $Y_{B_1} \cap Y_{B_2} = \emptyset$ then Y_{B_1} is contained in either U_{B_2} or in the unbounded connected component of $\mathbb{C}\backslash(Y_{B_2} \cup \mathcal{K})$. By the maximality of B_1, Y_{B_1} must be contained in the unbounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup Y_{B_2})$. We will now argue that $U_{B_2} \subset U_{B_1}$, which will contradict the maximality of B_2. Indeed, by the definition of Y_{B_1}, every unbounded connected subset of $\mathbb{C}\backslash\mathcal{K}$ whose prime end closure contains y has to intersect Y_{B_1}. Since Y_{B_1} is disjoint from U_{B_2} and $y \in \mathrm{Cl}'(U_{B_2})$, it follows that every unbounded connected subset of $\mathbb{C}\backslash\mathcal{K}$ which intersects U_{B_2} has to intersect Y_{B_1}. Therefore, $U_{B_2} \subset U_{B_1}$, which gives the desired contradiction.

Since each set Y_B for $B \in \mathcal{B}$ has Euclidean diameter at most 4ε, the preceding paragraph implies that the set $\widetilde{Y}_y^\varepsilon := \overline{\bigcup_{B_* \in \mathcal{B}_*} Y_{B_*}}$ is connected and has Euclidean diameter at most 8ε. Choose a Euclidean ball $\widetilde{B}$ of radius at most 8ε which contains $\widetilde{Y}_y^\varepsilon$. As in Step 1, there is a unique connected arc Y_y^ε of $\partial\widetilde{B}\backslash\mathcal{K}$ with the property that y lies on the boundary of the bounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup Y_y^\varepsilon)$ and Y_y^ε is not contained in the bounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup X)$ for any other such arc X. This arc Y_y^ε has Euclidean diameter at most 16ε. Then each Y_{B_*} for $B_* \in \mathcal{B}_*$, and hence also each U_{B_*} for $B_* \in \mathcal{B}_*$, is contained in the bounded connected component of $\mathbb{C}\backslash(\mathcal{K} \cup Y_y^\varepsilon)$. Since each $z \in \mathcal{C}_y^\varepsilon$ is contained in U_B for some $B \in \mathcal{B}$, and hence in U_{B_*} for some $B_* \in \mathcal{B}_*$, we get that Y_y^ε satisfies the desired property. $\square$

We now turn our attention to Lemmas 4.16 and 4.19 . Both lemmas will be proven using the following statement, which in particular gives an explicit definition of the event F_k of Lemma 4.19.

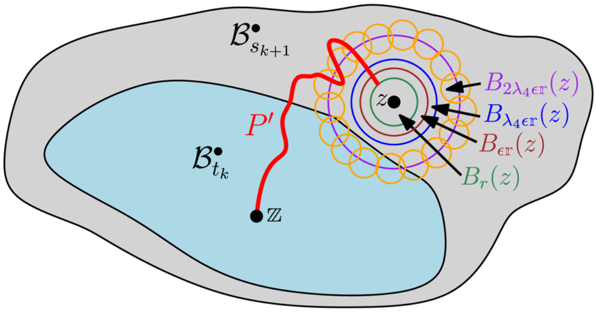

Fig. 8 Illustration of the statement and proof of Lemma 4.22. In order to upper-bound $\sup_{u\in\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)} D_h\left(\mathbb{z}, u; \mathcal{B}^\bullet_{s_{k+1}}\setminus\overline{B_{\varepsilon\mathbb{r}}(z)}\right)$, we cover $\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)$ by Euclidean balls of radius $\varepsilon\mathbb{r}/2$ (orange) and upper-bound the D_h-diameters of these balls using condition 3 (Hölder continuity) in the definition of $\mathcal{E}_\mathbb{r}$. Each of these balls is disjoint from $B_{\varepsilon\mathbb{r}}(z)$ and is contained in $\mathcal{B}^\bullet_{s_{k+1}}$, which leads to (4.61). Using Lemma 4.22 we get an upper bound for the D_h-length of the segment of a $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$ (such as the one shown in red) stopped at the last time it hits $\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)$. This upper bound allows us to prevent such a D_h-geodesic from exiting $\mathcal{B}^\bullet_{s_{k+1}}$. These considerations lead to the proofs of Lemmas 4.16 and 4.19 (color figure online)

Lemma 4.22 *For $k \in [0, K]_\mathbb{Z}$, let F_k be the event that the following is true. We have $s_{k+1} \le \tau_{2\ell\mathbb{r}}$ and for each $z \in B_{2\lambda_4\varepsilon\mathbb{r}}(\mathcal{B}^\bullet_{t_k})\setminus B_{\lambda_4\varepsilon\mathbb{r}}(\mathcal{B}^\bullet_{t_k})$,*

$$\sup_{u\in\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)} D_h\left(\mathbb{z}, u; \mathcal{B}^\bullet_{s_{k+1}}\setminus\overline{B_{\varepsilon\mathbb{r}}(z)}\right) \le t_k + c\varepsilon^\chi \mathfrak{c}_\mathbb{r} e^{\xi h_\mathbb{r}(\mathbb{z})}, \tag{4.61}$$

where λ_4 is the constant from Theorem 4.2, χ is as in condition 3 (Hölder continuity) in the definition of $\mathcal{E}_\mathbb{r}$, and $c > 0$ is constant depending only on a, λ_4 (which we do not make explicit). If $\mathcal{E}_\mathbb{r}$ occurs and ε is sufficiently small (how small depends only on a, λ_4), then F_k occurs for each $k \in [0, K]_\mathbb{Z}$.

The reason why we use internal distances in $\mathcal{B}^\bullet_{s_{k+1}}\setminus\overline{B_{\varepsilon\mathbb{r}}(z)}$ in (4.61) is as follows. Such distances are bounded above by $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-distances if $r \le \varepsilon\mathbb{r}$ (which is the case if $(z, r) \in \mathcal{Z}_k$), which will be important for controlling $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics. Furthermore, such distances are determined by $(\mathcal{B}^\bullet_{s_{k+1}}, h|_{\mathcal{B}^\bullet_{s_{k+1}}})$ by Axiom II (locality), which will be important for the proof of Lemma 4.19. We also emphasize that the right side of (4.61) is smaller than $s_{k+1} = t_k + (\varepsilon^\beta - \varepsilon^{2\beta})\mathfrak{c}_\mathbb{r} e^{\xi h_\mathbb{r}(\mathbb{z})}$ if ε is small since $\beta < \chi$.

Proof of Lemma 4.22 See Fig. 8 for an illustration of the statement and proof. Assume that $\mathcal{E}_\mathbb{r}$ occurs. By (4.36), we have $s_{k+1} \le \tau_{2\ell\mathbb{r}}$. Hence we just need to check (4.61). By the definition (4.6) of t_k and s_{k+1} and since $\beta < \chi/\chi' < \chi$ (by (4.37)), it holds for small enough $\varepsilon \in (0, 1)$ that

$$D_h(\partial \mathcal{B}^\bullet_{t_k}, \partial \mathcal{B}^\bullet_{s_{k+1}}) \geq (\varepsilon^\beta - \varepsilon^{2\beta}) \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} > \varepsilon^{\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}. \tag{4.62}$$

Note that $\chi' > \xi(Q+2) \geq 1$, where the last inequality follows, e.g., from the fact that $1 - \xi Q \leq 2\xi$, which is obvious from the definition of LFPP and an estimate for the maximum of h^*_ε on a bounded open set.

For $z \in B_{2\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k}) \backslash B_{\lambda_4 \varepsilon \mathbb{r}}(\mathcal{B}^\bullet_{t_k})$, the Euclidean circle $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$ intersects $\partial \mathcal{B}^\bullet_{t_k}$. We can cover $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$ by a λ_4-dependent constant number of Euclidean balls of the form $B_{\varepsilon \mathbb{r}/2}(w)$ for $w \in \partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$. Note that since $\lambda_4 \geq 1$, the corresponding balls $B_{\varepsilon \mathbb{r}}(w)$ are disjoint from $B_{\lambda_4 \varepsilon \mathbb{r}}(z) \supset B_{\varepsilon \mathbb{r}}(z)$. By the upper bound for D_h-distances from condition 3 in the definition of $\mathcal{E}_{\mathbb{r}}$ and then condition 4 (comparison of circle averages) in the definition of $\mathcal{E}_{\mathbb{r}}$, each such ball satisfies

$$\sup_{u,v \in B_{\varepsilon \mathbb{r}/2}(w)} D_h(u, v; B_{\varepsilon \mathbb{r}}(w)) \leq 2(\varepsilon/2)^{\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \preceq \varepsilon^{\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}, \tag{4.63}$$

with the implicit constant depending only on a.

By summing (4.63) over all such balls $B_{\varepsilon \mathbb{r}/2}(w)$, using that $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z) \cap \partial \mathcal{B}^\bullet_{t_k} \neq \emptyset$, and comparing to (4.62), we get that for small enough ε each such ball $B_{\varepsilon \mathbb{r}}(w)$ is contained in $\mathcal{B}^\bullet_{s_{k+1}} \backslash \overline{B_{\varepsilon \mathbb{r}}(z)}$. We deduce that the $D_h\left(\cdot, \cdot; \mathcal{B}^\bullet_{s_{k+1}} \backslash \overline{B_{\varepsilon \mathbb{r}}(z)}\right)$-diameter of $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$ is at most a a, λ_4-dependent constant times $\varepsilon^{\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}$. Since $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z) \cap \partial \mathcal{B}^\bullet_{t_k} \neq \emptyset$, we get that the left side of (4.61) is at most $t_k + c\varepsilon^{\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})}$ for an appropriate constant c. $\square$

Proof of Lemma 4.16 Assume that $\mathcal{E}_{\mathbb{r}}$ occurs and let P' be a $D_h(\cdot, \cdot; \overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$, as in the statement of the lemma. Let $t' \in [t_k, |P'|]_{\mathbb{Z}}$ be the last time that P' hits $\partial B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$. Since $\overline{B_r(z)}$ is disjoint from $\mathcal{B}^\bullet_{t_k}$, the segment $P'|_{[0,t_k]}$ is a D_h-geodesic and P' does not re-enter $\mathcal{B}^\bullet_{t_k}$ after time t_k. By (4.61) of Lemma 4.19 and since P' is $D_h(\cdot, \cdot; \mathbb{C} \backslash \overline{B_r(z)})$-geodesic, it follows that the D_h-length of $P'|_{[0,t']}$ (which equals t') is at most $t_k + c\varepsilon^{\chi} \mathfrak{c}_r e^{\xi h_r(\mathbb{z})}$. Therefore, the D_h-length of $P'([t_k, t'])$ is at most $c\varepsilon^{\chi} \mathfrak{c}_r e^{\xi h_r(\mathbb{z})}$. By conditions 3 (Hölder continuity) and 4 (comparison of circle averages) in the definition of $\mathcal{E}_{\mathbb{r}}$, the Euclidean diameter of $P'([t_k, t'])$ is at most a a, λ_4-dependent constant times $\varepsilon^{\chi/\chi'} \mathbb{r}$. Since $P'([t', |P'|]) \subset B_{2\lambda_4 \varepsilon \mathbb{r}}(z)$, we obtain (4.45). $\square$

Proof of Lemma 4.19 Define F_k as in Lemma 4.22. That lemma tells us that $\mathcal{E}_{\mathbb{r}} \subset \bigcap_{k=0}^K F_k$ for small enough $\varepsilon \in (0, 1)$ (depending only on a, λ_4). Furthermore, it is clear from the definition of F_k and Axiom II (locality) that $F_k \in \sigma\left(\mathcal{B}^\bullet_{s_{k+1}}, h|_{\mathcal{B}^\bullet_{s_{k+1}}}\right)$. Now assume that F_k occurs. By definition, we have $s_k \leq \tau_{2\ell \mathbb{r}}$. We consider $(z, r) \in \mathcal{Z}_k$ and check that if $\varepsilon \in (0, 1)$ is small enough, then $B_{\lambda_4 r}(z) \subset \mathcal{B}^\bullet_{s_{k+1}}$ and the set of $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesics from $\mathbb{z}$ to points of $\partial B_r(z)$ is determined by $(\mathcal{B}^\bullet_{s_{k+1}}, h|_{\mathcal{B}^\bullet_{s_{k+1}}})$.

Note that the right side of (4.61) satisfies $t_k + c\varepsilon^\chi \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} \le s_{k+1}$. Since the left side of (4.61) is an upper bound for $\sup_{u\in\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)} D_h(\mathbb{z},u)$, it follows that $\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z) \subset \mathcal{B}^\bullet_{s_{k+1}}$. Since $B_{\lambda_4 r}(z) \subset B_{\lambda_4\varepsilon\mathbb{r}}(z)$ (by (4.10)) and $\mathcal{B}^\bullet_{s_{k+1}}$ contains every point which it disconnects from ∞, we therefore have $B_{\lambda_4 r}(z) \subset \mathcal{B}^\bullet_{s_{k+1}}$.

Finally, we claim that a $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$ is the same as a $D_h(\cdot,\cdot;\mathcal{B}^\bullet_{s_{k+1}}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$, which gives the desired measurability statement due to Axiom II for D_h. To see this, it suffices to show that if P' is a $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic from $\mathbb{z}$ to a point of $\partial B_r(z)$, then $P' \subset \mathcal{B}^\bullet_{s_{k+1}}$.

To this end, let t be the last time that P' hits $\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)$. By (4.61) and since P' is a $D_h(\cdot,\cdot;\mathbb{C}\setminus\overline{B_r(z)})$-geodesic, it follows that the D_h-length of $P'|_{[0,t]}$ (which equals t) is at most $t_k + c\varepsilon^\chi \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(\mathbb{z})} < s_{k+1}$. Consequently, P' cannot exit $\mathcal{B}^\bullet_{s_{k+1}}$ before time t. Since t is the *last* time that P' hits $\partial B_{2\lambda_4\varepsilon\mathbb{r}}(z)$ and the terminal point of P' is contained in $\partial B_r(z) \subset B_{\varepsilon\mathbb{r}}(z)$, P' cannot exit $\mathcal{B}^\bullet_{s_{k+1}}$ after time t, either. □

5 Forcing a geodesic to take a shortcut

The goal of this section is to prove Proposition 4.3. Throughout, we assume that we are in the setting of Theorem 1.9, so D and $\widetilde{D}$ are two weak γ-LQG metrics with the same scaling constants. We also let h be a whole-plane GFF and we implicitly assume (by way of eventual contradiction) that the optimal bi-Lipschitz constants c_* and C_* of (1.21) satisfy $c_* < C_*$.

With ν_* as in Theorem 4.2, fix $0 < \mu < \nu \le \nu_*$ and let $\alpha_* \in (1/2, 1)$ and $p_0 \in (0,1)$ be the parameters from Proposition 3.5 for this choice of μ and ν (we write p_0 instead of p to avoid confusion with another parameter called p below). Also fix $\alpha \in [\alpha_*, 1)$ (to be chosen in Lemma 5.5 just below) and parameters c_1', c_2' such that $c_* < c_1' < c_2' < C_*$.

Let $\mathcal{R}_0$ be the set of $r > 0$ for which it holds with probability at least p_0 that the following is true. There exists $u \in \partial B_{\alpha r}(0)$ and $v \in \partial B_r(0)$ such that

$$\widetilde{D}_h(u,v) \le c_1' D_h(u,v) \tag{5.1}$$

and the $\widetilde{D}_h$-geodesic from u to v is unique and is contained in $\overline{\mathbb{A}_{\alpha r, r}(0)}$. We note that Proposition 3.4 implies in particular that for each $\mathbb{r} > 0$ one has $\#(\mathcal{R}_0 \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r}\}_{k\in\mathbb{N}}) \ge \mu \log_8 \varepsilon^{-1}$ for small enough $\varepsilon \in (0,1)$.

5.1 Outline of the proof of Proposition 4.3

The main task in the proof of Proposition 4.3 is to define the event $E_r(0)$ (which we abbreviate as E_r throughout most of this section). The other events $E_r(z)$ for $z \in \mathbb{C}$ will be defined by translation.

Main ideas The basic idea to define E_r is as follows. We will define for each pair of points $x', y' \in \partial B_{3r}(0)$ a deterministic smooth bump function ϕ which takes a large (but independent of r, x', y') value in a long, narrow "tube" contained in $B_{3r}(0)$ which (almost) contains a path from x' to y' and which vanishes outside of a small neighborhood of this tube. Roughly speaking, E_r will be the event that, simultaneously for every choice of x' and y', this tube contains a pair of points u, v such that $\widetilde{D}_h(u, v) \leq c_1' D_h(u, v)$ and $|u-v| \asymp r$; and several regularity conditions hold. We will show using Proposition 3.2 and basic estimates for LQG distances that when $\rho \in (0, 1)$ is small (but independent of r), $\mathbb{P}[E_r]$ is close to 1 for all $r \in \rho^{-1}\mathcal{R}_0$ (Lemma 5.10).

We will then consider a fixed pair of points $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_{4r}(0)$ and let x' and y' be the first points of $\partial B_{3r}(0)$ hit by the D_h-metric balls grown from $\mathbb{z}$ and $\mathbb{w}$, respectively. This choice of x' and y' (and hence also the corresponding bump function ϕ) are random, but are determined by $h|_{\mathbb{C}\setminus B_{3r}(0)}$. We will show that if E_r occurs and the D_h-geodesic between $\mathbb{z}$ and $\mathbb{w}$ enters $B_{2r}(0)$, then the $D_{h-\phi}$-geodesic between $\mathbb{z}$ and $\mathbb{w}$ has to stay close to the long narrow tube where ϕ is large, and hence has to get close to points u, v with $\widetilde{D}_h(u, v) \leq c_1' D_h(u, v)$ and $|u - v| \asymp r$. Essentially, this is because Axiom III (Weyl scaling) implies that subtracting ϕ makes distances inside the tube much shorter than distances outside. If we let $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(0)$ be the event that the D_h-geodesic gets close to such points u, v, then since the conditional laws of $h-\phi$ and h given $h|_{\mathbb{C}\setminus B_{3r}(0)}$ are mutually absolutely continuous (and we can add regularity conditions to E_r to control the Radon-Nikodym derivative), we get condition 4 in Theorem 4.2 (with $\lambda_3 = 3$).

We emphasize that the event E_r does *not* include the condition that P stays in the long narrow tube where ϕ is large. Indeed, E_r cannot include any conditions which depend on P since E_r needs to be locally determined by h. Rather, as explained in the preceding paragraph, if E_r occurs then we can force P to stay in the tube by subtracting the bump function ϕ from h.

Section 5.2. We give a precise statement of the properties that we need the event E_r and the bump function ϕ described above to satisfy. We then assume the existence of these objects and deduce Proposition 4.3. Condition 1 of Theorem 4.2 (with $\mathcal{R} = \rho^{-1}\mathcal{R}_0$) is true in our framework by the definition of $\mathcal{R}_0$ and Proposition 3.4. Conditions 2 and 3 are true by assumption (these conditions will be clear from the construction of E_r and ϕ). Condition 4 is proven by comparing the conditional laws of h and $h - \phi$ given $h|_{\mathbb{C}\setminus B_{3r}(0)}$, as discussed above. The rest of the section is devoted to constructing the event E_r and the bump functions ϕ.

Section 5.3. We first show that for any $z \in \mathbb{C}$ and $r \in \mathcal{R}_0$, we can find a *deterministic* open "tube" $V_r(z) \subset B_{3r}(z)$ such that with uniformly positive probability over the choice of z and r, there are points $u, v \in V_r(z)$ with the following properties. We have $\widetilde{D}_h(u, v) \le c_1' D_h(u, v)$, $|u - v| \asymp r$, the $\widetilde{D}_h$-geodesic from u to v is contained in $V_r(z)$, and any path in $V_r(z)$ between $z - 2r$ and $z + 2r$ has to get close to each of u and v (Lemma 5.6). This is illustrated in Fig. 10.

To do this, we start with a pair of points u, v as in the definition of $\mathcal{R}_0$, but with z in place of 0. Such a pair of points exists with probability at least p_0 by Axiom IV (translation invariance). We then extend the $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v to a path $\widetilde{P}'$ from $z - 2r$ to $z + 2r$ by concatenating $\widetilde{P}$ with smooth paths. For this purpose, the fact that $\widetilde{P}$ is contained in $\overline{\mathbb{A}_{\alpha r, r}(z)}$ is useful to ensure that the extra smooth paths intersect $\widetilde{P}$ only at u and v. We consider the set of squares in a fine grid which intersect $\widetilde{P}'$. Since there are only finitely many possibilities for this set of squares, there has to be a deterministic set of squares which equals the set of squares which intersect $\widetilde{P}'$ with uniformly positive probability. We define $V_r(z)$ to be the interior of the union of the squares in this set.

Section 5.4. We now have an event which satisfies many of the conditions which we are interested in, but it holds only with uniformly positive probability, not with probability close to 1. To get an event which holds with probability close to 1, we consider a small but fixed $\rho \in (0, 1)$ and a radius $r \in \rho^{-1}\mathcal{R}_0$. We can find a large number of disjoint balls of the form $B_{\rho r}(z)$ contained in $B_{2r}(0)$ (note that $\rho r \in \mathcal{R}_0$). By the spatial independence properties of the GFF (Lemma 2.7), if we make ρ sufficiently small then it holds with high probability that the event of the preceding subsection occurs for a large number of these balls $B_{\rho r}(z)$. We then link up the corresponding sets $V_{\rho r}(z)$ by deterministic paths of squares to find a deterministic open "tube" $U_r^{x,y}$ joining any two given points of $x, y \in \partial B_{2r}(0)$ with the following property. With probability close to 1, there are points $u, v \in U_r^{x,y}$ such that $\widetilde{D}_h(u, v) \le c_1' D_h(u, v)$, $|u - v| \asymp r$, the $\widetilde{D}_h$-geodesic from u to v is contained in $U_r^{x,y}$, and any path in $U_r^{x,y}$ between x and y has to get close to each of u and v (Lemma 5.8). See Fig. 11 for an illustration of this part of the argument.

Section 5.5. Taking Lemma 5.8 as our starting point, we then build the high-probability event E_r in Proposition 4.3 for $r \in \rho^{-1}\mathcal{R}_0$. In addition to the aforementioned conditions on the tube $U_r^{x,y}$, we also include extra regularity conditions which will eventually be used to prevent D_h-geodesics from staying close to the boundary of $U_r^{x,y}$ without entering it, to get geodesics from $\partial B_{3r}(0)$ to $\partial B_{2r}(0)$, and to control the Radon-Nikodym derivative between the conditional law of h and $h - \phi$ (where ϕ is the bump function mentioned above) given $h|_{\mathbb{C}\setminus B_{3r}(0)}$. We also give a precise definition of the bump function ϕ which we will subtract from the field: it is equal to a large positive constant on the long narrow tube $U_r^{x,y}$, it is equal to an even larger constant on even narrower tubes which approximate each of the segments $[x, 3x/2]$ and $[y, 3y/2]$, and it vanishes outside of a small neighborhood of the union of $U_r^{x,y}$ and these two narrower tubes. The definitions of these objects are illustrated in Fig. 12.

Section 5.6. We prove that a $D_{h-\phi}$-geodesic is likely to get near points u, v satisfying (5.1), using the definition of $E_r(0)$ and deterministic arguments to compare various distances. A key point here is that we have set things up so that on E_r, the $\widetilde{D}_h$-geodesic from u to v is contained in $U_r^{x,y}$ and is far away from the narrow tubes where ϕ is larger than it is on $U_r^{x,y}$. This allows us to show that subtracting ϕ does not change the fact that $\widetilde{D}_h(u, v) \le c_1' D_h(u, v)$.

Remark 5.1 Our proof only shows that the D_h-geodesic P gets close to each of the points u and v from (5.1) with positive probability (we then use the triangle inequality to compare the D_h-length of a segment of P to $D_h(u, v)$). We do *not* show that P actually merges into the D_h-geodesic from u to v. We believe that it should be possible to show that P merges into this D_h-geodesic, but doing so is highly non-trivial. Indeed, this is closely related to the problem of showing that there are no "ghost geodesics" for D_h which do not merge into any other D_h-geodesics; see [3, Section 1.4] for some discussion about the analogous problem in the setting of the Brownian map. Because we do not show that P merges into the D_h-geodesic from u to v, the arguments of this section do not immediately imply other statements of the form "if an event occurs for some (random) geodesic with high probability, then with high probability it occurs somewhere along the D_h-geodesic between two fixed points".

5.2 Proof of Proposition 4.3 assuming the existence of events and functions

In this subsection, we assume the existence of an event $E_r = E_r(0)$ and a collection $\mathcal{G}_r$ of smooth bump functions ϕ which satisfy a few simple properties and deduce Proposition 4.3 from the existence of these objects. The later subsections are devoted to constructing these objects. In particular, we will deduce Proposition 4.3 from the following proposition.

Proposition 5.2 *Let $0 < \mu < \nu \le \nu_*$ be as above and let $\mathbb{p} \in (0, 1)$. There exists $\rho \in (0, 1)$, depending only on $\mathbb{p}, \mu, \nu$, such that for each $r \in \rho^{-1}\mathcal{R}_0$, there is an event E_r and a finite collection $\mathcal{G}_r$ of smooth bump functions, each of which is supported on a compact subset of $\mathbb{A}_{r/4,3r}(0)$, with the following properties.*

(A) (Measurability and high probability) *We have $E_r \in \sigma((h - h_{5r}(0))|_{\mathbb{A}_{r/4,4r}(0)})$ and $\mathbb{P}[E_r] \ge \mathbb{p}$.*

(B) (Bound for Dirichlet inner products) *There is a deterministic constant* $\Lambda_0 > 0$ *depending only on* $\mathbb{p}, \mu, \nu, c_1', c_2'$ *such that, writing* $(\cdot,\cdot)_\nabla$ *for the Dirichlet inner product, it holds on* E_r *that*

$$|(h,\phi)_\nabla| + \frac{1}{2}(\phi,\phi)_\nabla \le \Lambda_0, \quad \forall \phi \in \mathcal{G}_r. \tag{5.2}$$

(C) (Subtracting a bump function forces a geodesic to take a shortcut) *Suppose we are given points* $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_{4r}(0)$. *There is a random* $\phi \in \mathcal{G}_r$ *depending only on* $\mathbb{z}$, $\mathbb{w}$, *and* $h|_{\mathbb{C}\setminus B_{3r}(0)}$ *such that the following is true. Let* P *(resp.* P^ϕ*) be the a.s. unique* D_h*- (resp.* $D_{h-\phi}$*-) geodesic from* $\mathbb{z}$ *to* $\mathbb{w}$. *There is a deterministic constant* $b_0 > 0$ *depending only on* $\mathbb{p}, \mu, \nu, c_1', c_2'$ *such that if* $P \cap B_{2r}(0) \neq \emptyset$ *and* E_r *occurs, then there are times* $0 < s < t < D_{h-\phi}(\mathbb{z},\mathbb{w})$ *and such that*

$$\begin{aligned} &P^\phi(s), P^\phi(t) \in B_{3r/2}(0), \quad |P^\phi(s) - P^\phi(t)| \ge b_0 r, \\ &\quad \widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \le c_2' D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right), \quad \textit{and} \\ &\quad \widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \le (c_*/C_*)\widetilde{D}_{h-\phi}\left(P^\phi(s), \partial B_{3r}(0)\right). \end{aligned} \tag{5.3}$$

The event E_r and the collection of functions $\mathcal{G}_r$ will be defined explicitly in Sect. 5.5; see Sect. 5.1 for an overview of the definitions. The reason why we are able to restrict to a finite collection $\mathcal{G}_r$ of bump functions ϕ is that we will break up space into a fine grid and require that the "tube" where ϕ is very large (as referred to in Sect. 5.1) is a finite union of squares in the grid. As explained in Lemma 5.4 just below, Properties (B) and (C) are used to check condition 4 in Theorem 4.2. The purpose of Property (B) is to control the Radon-Nikodym derivative between the conditional laws of h and $h-\phi$ given $h|_{\mathbb{C}\setminus B_{3r}(0)}$.

We now explain how to conclude the proof of Proposition 4.3 assuming Proposition 5.2. Fix points $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_{4r}(0)$ and let P be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$, as in Property (C). We first define the event $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(0)$ appearing in Proposition 4.3. Let $\mathfrak{E}_r = \mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(0)$ be the event that there are times $0 < s < t < D_h(\mathbb{z},\mathbb{w})$ such that

$$\begin{aligned} &P(s), P(t) \in B_{3r/2}(0), \quad |P(s) - P(t)| \ge b_0 r, \\ &\quad \widetilde{D}_h(P(s), P(t)) \le c_2' D_h(P(s), P(t)), \quad \text{and} \\ &\quad \widetilde{D}_h(P(s), P(t)) \le (c_*/C_*)\widetilde{D}_h(P(s), \partial B_{3r}(0)), \end{aligned} \tag{5.4}$$

and

$$\exp\left(-(h,\phi)_\nabla + \frac{1}{2}(\phi,\phi)_\nabla\right) \le \Lambda, \quad \forall \phi \in \mathcal{G}_r, \quad \text{where} \quad \Lambda := e^{\Lambda_0}, \tag{5.5}$$

where Λ_0 is the constant from Property (B) and b_0 is the constant from Property (C). We note that (5.4) is the same as (5.3) from Property (C), but with h instead of $h - \phi$. This condition is the main point of the definition of $\mathfrak{E}_r$. The extra condition (5.5) is only included to control a Radon-Nikodym derivative when we compare the conditional probabilities of $\mathfrak{E}_r$ and E_r given $h|_{\mathbb{C}\setminus B_{3r}(0)}$.

Lemma 5.3 *The event $\mathfrak{E}_r$ is a.s. determined by $h|_{B_{3r}(0)}$ and the D_h-geodesic P stopped at its last exit time from $B_{3r}(0)$.*

Proof Recall that each of the functions $\phi \in \mathcal{G}_r$ is supported on $\mathbb{A}_{r/4,3r}(0)$. Since $\mathcal{G}_r$ is a finite set, it is clear that the condition (5.5) is determined by $h|_{B_{3r}(0)}$.

To deal with (5.4), we first observe that the set of pairs of times $s, t \in [0, D_h(\mathbb{z}, \mathbb{w})]$ satisfying $\widetilde{D}_h(P(s), P(t)) \le (c_*/C_*)\widetilde{D}_h(P(s), \partial B_{3r}(0))$ is determined by P stopped at its last exit time from $B_{3r}(0)$ and the internal metric $\widetilde{D}_h(\cdot, \cdot; B_{3r}(0))$. Indeed, a pair (s, t) belongs to this set if and only if $P(t)$ is contained in the $\widetilde{D}_h$-metric ball of radius $(c_*/C_*)\widetilde{D}_h(P(s), \partial B_{3r}(0))$ centered at $P(s)$. For each such pair of times s, t, we have $\widetilde{D}_h(P(s), P(t)) = \widetilde{D}_h(P(s), P(t); B_{3r}(0))$. Since P is a D_h-geodesic, the points $P(s)$, $P(t)$ and the distance $D_h(P(s), P(t)) = t - s$ for each such pair of points s, t is determined by P stopped at its last exit time from $B_{3r}(0)$. Since $\widetilde{D}_h(\cdot, \cdot; B_{3r}(0))$ is determined by $h|_{B_{3r}(0)}$ (Axiom II) we get that the event that there exists times $s, t \in [0, D_h(\mathbb{z}, \mathbb{w})]$ satisfying (5.4) is determined by $h|_{B_{3r}(0)}$ and P stopped at its last exit time from $B_{3r}(0)$. $\square$

We can now check condition 4 of Theorem 4.2 for the above definitions of $E_r = E_r(0)$ and $\mathfrak{E}_r = \mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(0)$ using the mutual absolute continuity of the laws of h and $h - \phi$.

Lemma 5.4 *Assume Proposition 5.2. With Λ as in (5.5), it is a.s. the case that*

$$\begin{aligned}
&\mathbb{P}\left[E_r \cap \{P \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right] \\
&\quad \le \Lambda \mathbb{P}\left[\mathfrak{E}_r \cap \{P \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right]. \qquad (5.6)
\end{aligned}$$

Proof The occurrence of the events E_r and $\mathfrak{E}_r$ is unaffected by adding a constant to h, so we can assume without loss of generality that h is normalized so that its circle average over $\partial B_{4r}(0)$, say, is zero. By the Markov property of h, under the conditional law given $h|_{\mathbb{C}\setminus B_{3r}(0)}$, we can decompose $h|_{B_{3r}(0)}$ as the sum of a harmonic function which is determined by $h|_{\mathbb{C}\setminus B_{3r}(0)}$ and a zero-boundary GFF on $B_{3r}(0)$ which is independent from $h|_{\mathbb{C}\setminus B_{3r}(0)}$.

Let $\phi \in \mathcal{G}_r$ be the smooth bump function from Property (C), which is determined by $h|_{\mathbb{C}\setminus B_{3r}(0)}$. By a standard Radon-Nikodym derivative calculation for the GFF, if we condition $h|_{\mathbb{C}\setminus B_{3r}(0)}$ then the conditional law of $h - \phi$ is a.s. absolutely continuous with respect to the conditional law of h, and the Radon-Nikodym derivative of the former w.r.t. the latter is

$$M_h = \exp\left(-(h,\phi)_\nabla - \frac{1}{2}(\phi,\phi)_\nabla\right). \tag{5.7}$$

Note that since ϕ is supported on $B_{3r}(0)$, the Radon-Nikodym derivative M_h depends only on the zero-boundary part of $h|_{B_{3r}(0)}$.

Define the $D_{h-\phi}$-geodesic P^ϕ from $\mathbb{z}$ to $\mathbb{w}$ and the event $\mathfrak{E}_r^\phi$ in the same manner as P and $\mathfrak{E}_r$ but with $h-\phi$ in place of h. By (5.5), on $\mathfrak{E}_r$, we have $M_h \leq \Lambda$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left[\mathfrak{E}_r^\phi \cap \{P^\phi \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right] \\
&\quad = \mathbb{E}\left[M_h \mathbb{1}_{\mathfrak{E}_r \cap \{P\cap B_{2r}(0)\neq\emptyset\}} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right] \\
&\quad \leq \Lambda \mathbb{P}\left[\mathfrak{E}_r \cap \{P \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right].
\end{aligned} \tag{5.8}$$

We now claim that

$$E_r \cap \{P \cap B_{2r}(0) \neq \emptyset\} \subset \mathfrak{E}_r^\phi \cap \{P^\phi \cap B_{2r}(0) \neq \emptyset\}. \tag{5.9}$$

Indeed, Property (C) (subtracting a bump function) says that the main condition (5.4) in the definition of $\mathfrak{E}_r$ is satisfied with $h-\phi$ in place of h whenever $E_r \cap \{P \cap B_{2r}(0) \neq \emptyset\}$ occurs, which implies in particular that $P^\phi \cap B_{2r}(0) \neq \emptyset$ whenever $E_r \cap \{P \cap B_{2r}(0) \neq \emptyset\}$ occurs. Furthermore, Property (B) (bound for Dirichlet inner products) implies that the Dirichlet energy condition (5.5) in the definition of $\mathfrak{E}_r$ holds with $h-\phi$ in place of h whenever E_r occurs. Thus (5.9) holds.

As an immediate consequence of (5.9), a.s.

$$\begin{aligned}
&\mathbb{P}\left[E_r \cap \{P \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right] \\
&\quad \leq \mathbb{P}\left[\mathfrak{E}_r^\phi \cap \{P^\phi \cap B_{2r}(0) \neq \emptyset\} \,|\, h|_{\mathbb{C}\setminus B_{3r}(0)}\right].
\end{aligned} \tag{5.10}$$

Combining (5.8) and (5.10) gives (5.6). □

Proof of Proposition 4.3, assuming Proposition 5.2 Let $\mathbb{p}$ be as in Theorem 4.2 with our given choice of $0 < \mu < \nu \leq \nu_*$ and with the constants

$$\lambda_1 := 1/4, \quad \lambda_2 := 2, \quad \lambda_3 := 3, \quad \lambda_4 := 4, \quad \text{and} \quad \lambda_5 := 5. \tag{5.11}$$

For $z \in \mathbb{C}$, $r \in \rho^{-1}\mathcal{R}_0$, and $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_{4r}(z)$, let $E_r(z)$ (resp. $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$) be the event E_r of Proposition 5.2 (resp. the event and $\mathfrak{E}_r^{\mathbb{z}+z,\mathbb{w}+z}$ defined above) with the field $h(\cdot + z) - h_1(z) \overset{d}{=} h$ in place of h.

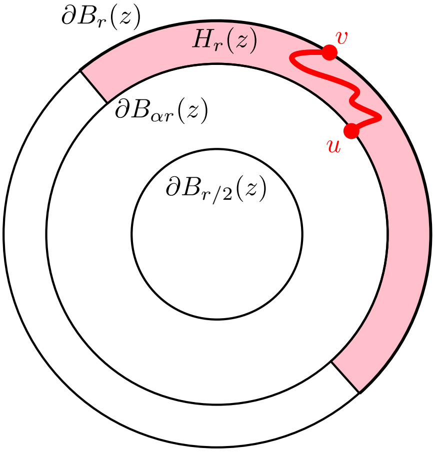

Fig. 9 Illustration of the statement of Lemma 5.5. The lemma asserts that with probability at least $p_0/8$, there is a D_h-geodesic (red) between points u and v in the inner and outer boundaries, resp., of the pink half-annulus $H_r(z)$ which is contained in $\overline{H_r(z)}$ and satisfies $\widetilde{D}_h(u,v) \le c_1' D_h(u,v)$. The main task of Sect. 5 is to force a D_h-geodesic between two far away points to get near a $\widetilde{D}_h$-geodesic like the red one in the picture (color figure online)

Let $c'' = c''(\alpha, c_1', \mu, \nu) \in (c_*, c_1')$ be chosen as in Proposition 3.5 with α as in Lemma 5.5 and c_1' in place of c'. Also let $\mathcal{R}_0$ be defined as in the discussion surrounding (5.1) and let $\mathcal{R} := \rho^{-1}\mathcal{R}_0$. By the definition of $\mathfrak{E}_r^{\mathbb{Z},\mathbb{W}}(z)$ (in particular, (5.4)), the conditions (4.3) hold on $\mathfrak{E}_r^{\mathbb{Z},\mathbb{W}}(z)$ with $b = b_0$.

If $\mathbb{r} > 0$ such that $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \ge \beta$, then Proposition 3.4 implies that there exists $\varepsilon_0 = \varepsilon_0(\beta, c_1', \mu, \nu) > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$,

$$\#(\mathcal{R}_0 \cap [\varepsilon^{1+\nu}\mathbb{r}, \varepsilon\mathbb{r}] \cap \{8^{-k}\mathbb{r}\}_{k\in\mathbb{N}}) \ge \mu \log_8 \varepsilon^{-1}, \tag{5.12}$$

equivalently,

$$\#(\mathcal{R} \cap [\varepsilon^{1+\nu}\rho^{-1}\mathbb{r}, \varepsilon\rho^{-1}\mathbb{r}] \cap \{8^{-k}\rho^{-1}\mathbb{r}\}_{k\in\mathbb{N}}) \ge \mu \log_8 \varepsilon^{-1}. \tag{5.13}$$

This shows that condition 1 of Theorem 4.2 is satisfied with $\rho^{-1}\mathbb{r}$ in place of $\mathbb{r}$. By Property (A) (measurability and high probability) and Lemma 5.3, conditions 2 and 3 of Theorem 4.2 are satisfied for the events $E_r(z)$ and $\mathfrak{E}_r^{\mathbb{Z},\mathbb{W}}(z)$ above. By Lemma 5.4, condition 4 of Theorem 4.2 is also satisfied. $\square$

5.3 Building a tube which contains a shortcut with positive probability

We now turn our attention to constructing the event E_r and the collection of functions $\mathcal{G}_r$ of Proposition 5.2, following the strategy outlined in Sect. 5.1.

Recall that $\alpha_* \in (1/2, 1)$ and $p_0 \in (0, 1)$ are the parameters from Proposition 3.5 with μ and ν as in Proposition 4.3.

Our goal is to define for each $z \in \mathbb{C}$ and each $r \in \mathcal{R}_0$ a deterministic open "tube" $V_r(z) \subset B_{3r}(z)$ and an event $F_r(z)$ such that $\mathbb{P}[F_r(z)]$ is bounded below uniformly over z and r, $F_r(z) \in \sigma\left((h - h_{4r}(z))|_{B_{3r}(z)}\right)$, and on $F_r(z)$ there are points $u, v \in V_r(z)$ which satisfy (5.1) plus some additional conditions. We will define $V_r(z)$ and $F_r(z)$ and prove a lower bound for $\mathbb{P}[F_r(z)]$ in Lemma 5.6, with Lemma 5.5 as an intermediate step. We will prove the required measurability in Lemma 5.7.

We define a *half-annulus* of an annulus A to be the intersection of A with a half-plane whose boundary passes through the center of A. It is easier for us to work with a $\widetilde{D}_h$-geodesic which is constrained to stay in a half-annulus rather than a whole annulus. The reason for this is that it allows us to easily find paths from each of the endpoints of the geodesic to points far away from the half-annulus which do not get near the geodesic except at their endpoints (this might be trickier if the geodesic wraps around the whole annulus). The following lemma, which is a slight improvement on the condition in the definition of $\mathcal{R}_0$, will allow us to work with a half-annulus rather than a whole annulus.

Lemma 5.5 *There exists $\alpha \in [\alpha_*, 1)$ depending only on μ, ν such that for each $r \in \mathcal{R}_0$ and each $z \in \mathbb{C}$, there is a deterministic half-annulus $H_r(z) \subset \mathbb{A}_{\alpha r, r}(z)$ such that with probability at least $p_0/8$, there exists $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$ with the following properties.*

1. $\widetilde{D}_h(u, v) \le c_1' D_h(u, v)$.
2. *The $\widetilde{D}_h$-geodesic from u to v is unique and is contained in $\overline{H_r(z)}$.*
3. $\widetilde{D}_h(u, v) \le (c_*/C_*)^2 \widetilde{D}_h\left(\mathbb{A}_{\alpha r, r}(z), \partial B_{2r}(z)\right)$, *where c_* and C_* are as in (1.21).*

Proof By Axioms IV and V , we can find $S > s > 0$ depending only on p_0 (and hence only on μ, ν) such that for each $r > 0$, it holds with probability at least $1 - p_0/2$ that the following is true.

- Any two points of $\mathbb{A}_{3r/4,r}(z)$ which are not contained in a single quarter-annulus of $\overline{\mathbb{A}_{3r/4,r}(z)}$ lie at $\widetilde{D}_h$-length at least $s\mathfrak{c}_r e^{\xi h_r(z)}$ from each other.
- $(c_*/C_*)^2 \widetilde{D}_h\left(\mathbb{A}_{3r/4,r}(z), \partial B_{2r}(z)\right) \ge s\mathfrak{c}_r e^{\xi h_r(z)}$.
- $\widetilde{D}_h(u, v) \le S\mathfrak{c}_r e^{\xi h_r(z)}$ for each $u, v \in \overline{\mathbb{A}_{3r/4,r}(z)}$.

Since $\mathbb{A}_{\alpha r, r}(z) \subset \mathbb{A}_{3r/4,r}(z)$ for each $\alpha \in [3/4, 1)$, Lemma 2.11 applied with the above choice of s and S gives an $\alpha \in [(3/4) \vee \alpha_*, 1)$ depending on p_0 and α_* such that for each $r > 0$ it holds with probability at least $1 - p_0/2$ that the following is true. For each pair of points $u, v \in \overline{\mathbb{A}_{\alpha r, r}(z)}$ such that $\widetilde{D}_h(u, v; \mathbb{A}_{\alpha r, r}(0)) = \widetilde{D}_h(u, v)$, it holds that u and v are contained in a single quarter-annulus of $\overline{\mathbb{A}_{\alpha r, r}(z)}$ and $\widetilde{D}(u, v) \le \widetilde{D}_h\left(\mathbb{A}_{\alpha r, r}(z), \partial B_{2r}(z)\right)$.

This happens in particular if there is a $\widetilde{D}_h$-geodesic from u to v contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$.

Combining this with translation invariance (Axiom IV) and the definition of $\mathcal{R}_0$ shows that for $r \in \mathcal{R}_0$, it holds with probability at least $p_0/2$ that the conditions in the lemma statement hold but with a *random* quarter-annulus in place of a deterministic half-annulus. This random quarter annulus is a.s. contained in one of four possible deterministic half-annuli, so must be contained in one of these four half-annuli with probability at least $p_0/8$. We therefore obtain that for an appropriate choice of $H_r(z)$, it holds with probability at least $p_0/8$ that all of the conditions in the lemma statement hold. □

We henceforth assume that $\alpha \in [\alpha_*, 1)$ is chosen so that the conclusion of Lemma 5.5 is satisfied. In order to construct deterministic "tubes" as described in Sect. 5.1, we will look at unions of squares in a fine grid. For $\varepsilon > 0$ and $X \subset \mathbb{C}$, let

$$\mathcal{S}_\varepsilon(X) := \left\{\text{closed } \varepsilon \times \varepsilon \text{ squares with corners in } \varepsilon\mathbb{Z}^2 \text{ which intersect } X\right\}. \tag{5.14}$$

Recall that we have fixed $c_2' > c_1' > c_*$. Choose, in a manner depending only on c_1', c_2', c_*, C_*, a small parameter $\eta \in (0, 1)$ such that

$$\frac{c_1'(1+2\eta)}{1 - 2c_*^{-1}C_*\eta} < c_2' \quad \text{and} \quad 1 + 2\eta < C_*/c_*. \tag{5.15}$$

The particular choice of η in (5.15) will not be used until (5.49) below. For now, the reader should just think of it as a small constant depending on c_1', c_2'. We also note that η is fixed in a way that depends only on c_1', c_2', c_*, C_* (hence only on c_1', c_2' and the choice of D, $\widetilde{D}$), so we do not need to explicitly mention the dependence on η in what follows. The following lemma gives us the basic "building blocks" which will be used to construct E_r in the next two subsections.

Lemma 5.6 *There exist small parameters $b_1, p_1 \in (0, 1/100)$ depending only on μ, ν and a parameter $\varepsilon_1 \in (0, b_1/100)$ depending only on c_1', c_2', μ, ν such that for each $z \in \mathbb{C}$ and each $r \in \mathcal{R}_0$, there exists a deterministic connected open set $V_r(z) \subset B_{(2+2\varepsilon_1)r}(z)$ with the following properties. The set $V_r(z)$ is the interior of a finite union of squares in $\mathcal{S}_{\varepsilon_1 r}(B_{2r}(z))$, $z - 2r, z + 2r \in V_r$, and we have $\mathbb{P}[F_r(z)] \geq p_1$, where $\underline{F_r(z)}$ is the event that the following is true. There are points $u, v \in V_r(z) \cap \overline{B_r(z)}$ with the following properties.*

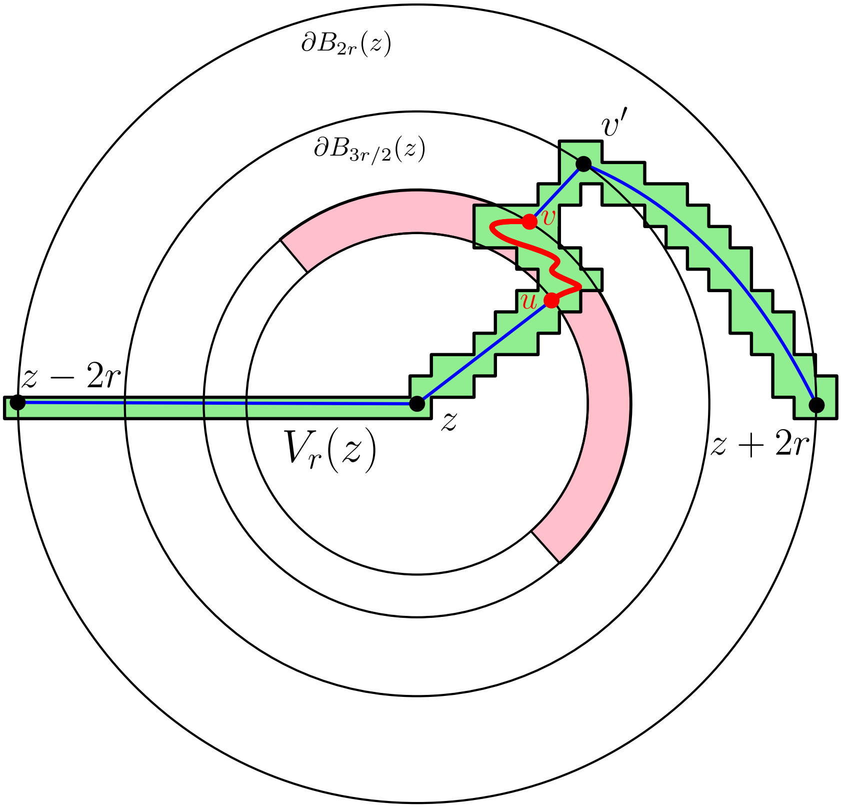

Fig. 10 Illustration of the statement and proof of Lemma 5.6. Building on the setting of Fig. 9, we show that there is a deterministic long narrow "tube" $V_r(z)$ (light green), which is the interior of the set of $\varepsilon_1 r \times \varepsilon_1 r$ squares with corners in $\varepsilon_1 r\mathbb{Z}^2$ which intersect a certain path from $z-2r$ to $z+2r$, with the following property. With positive probability, every path in the tube from a point near $z-2r$ to a point near $z+2r$ has to get near a pair of points u, v in the tube for which $\widetilde{D}_h(u,v) \le c_1' D_h(u,v)$. We will eventually add a bump function to h which takes a very negative value in such a tube in order to force a geodesic between points which are far away from $B_{2r}(z)$ to get near u and v (color figure online)

1. (Existence of a shortcut) *We have*

$$|u-v| \ge b_1 r, \quad \widetilde{D}_h(u,v) \le c_1' D_h(u,v), \quad \widetilde{D}_h(u,v) \le (c_*/C_*)^2 \widetilde{D}_h\left(u, \partial B_{2r}(z)\right), \tag{5.16}$$

and the $\widetilde{D}_h$-geodesic from u to v is unique and is contained in $V_r(z) \cap \overline{B_r(z)}$.
2. (Removing neighborhoods of u, v disconnects $V_r(z)$) *Let O_u be the connected component of $V_r(z) \cap B_{20\varepsilon_1 r}(u)$ which contains u and similarly define O_v with v in place of u. The connected component of $V_r(z) \setminus O_u$ which contains $z-2r$ lies at Euclidean distance at least $\varepsilon_1 r$ from the*

union of the other connected components of $V_r(z)\setminus O_u$. The same is true with v in place of u and $z+2r$ in place of $z-2r$.

3. (Upper bound for internal diameters of neighborhoods of u and v) *Each point of O_u lies at $\widetilde{D}_h(\cdot,\cdot;V_r(z))$-distance at most $\eta\widetilde{D}_h(u,v)$ from u, and the same is true with v in place of u (here η is as in (5.15)).*

Proof Let α be as in Lemma 5.5 and set $b_1 := 1-\alpha$. On the event that points $u\in\partial B_{\alpha r}(z)$ and $v\in\partial B_r(z)$ satisfying the conditions on Lemma 5.5 exist (which happens with probability at least $p_0/8$), choose one such pair of points (u,v) in some measurable manner. Otherwise, let $u=v=0$. On the event $\{u\neq 0\}$, let $\widetilde{P}$ be the unique $\widetilde{D}_h$-geodesic from u to v and let $H_r(z)\subset\mathbb{A}_{\alpha r,r}(z)$ be the half-annulus with $\widetilde{P}\subset\overline{H_r(z)}$ as in Lemma 5.5.

We will now extend $\widetilde{P}$ to a path $\widetilde{P}'$ in $B_{2r}(z)$ from $z-2r$ to $z+2r$ (which will no longer be a $\widetilde{D}_h$-geodesic). To this end, we first let $v' := (3/2)(v-z)+z\in\partial B_{3r/2}(z)$ and we let L_- (resp. L_+) be the linear segment from z to u (resp. v to v'). We note that the Euclidean distance between L_- and L_+ is at least $b_1 r$. We can choose a path π_- from $z-2r$ to z and a path π_+ from v' to $z+2r$ in $B_{2r}(z)$ such that the Euclidean distances from $\pi_-\cup\pi_+$ to $H_r(z)$ and from $\pi_-\cup L_-$ to $\pi_+\cup L_+$ are each at least $b_1 r$. Let $\widetilde{P}'$ be the concatenation of $\pi_-, L_-, \widetilde{P}, L_+, \pi_+$.

Since $|u-v|\geq b_1 r$ on the event $\{u\neq 0\}$, Axiom V (tightness across scales) together with Lemma 2.9 imply that we can find $\varepsilon_1\in(0,b_1/100)$ depending only on c_1', c_2', μ, ν such that with probability at least $p_0/9$, the event of Lemma 5.5 occurs (i.e., $u\neq 0$) and also

$$\sup_{S\in\mathcal{S}_{\varepsilon_1 r}(B_{2r}(z))}\ \sup_{w_1,w_2\in S}\widetilde{D}_h(w_1,w_2;S)\leq\frac{\eta}{100}\widetilde{D}_h(u,v). \tag{5.17}$$

The number of subsets of $\mathcal{S}_{\varepsilon_1 r}(B_{2r}(z))$ is bounded above by a deterministic constant depending only on ε_1. Consequently, we can choose $p_1\in(0,1)$ depending only on μ,ν,D and a deterministic $\mathcal{K}_r(z)\subset\mathcal{S}_{\varepsilon_1 r}(B_{2r}(z))$ such that with probability at least p_1, the events of Lemma 5.5 and (5.17) occur and also

$$\mathcal{K}_r(z)=\left\{S\in\mathcal{S}_{\varepsilon_1 r}(B_{2r}(z)) : S\cap\widetilde{P}'\neq\emptyset\right\}. \tag{5.18}$$

Let $V_r(z)$ be the interior of the union of the squares in $\mathcal{K}_r(z)$. Since $z-2r, z+2r\in\widetilde{P}'$ and $\widetilde{P}'$ is connected, it follows that $V_r(z)$ is connected and contains $z-2r$ and $z+2r$.

Henceforth assume that the events of Lemma 5.5, (5.17), and (5.18) occur. We will check the conditions in the lemma statement with $V_r(z)$ as above.

Condition 1 This is immediate from the conditions on u and v from Lemma 5.5.

Condition 2 By the above definitions of π_- and L_-, the Euclidean $\varepsilon_1 r$-neighborhood of each square of $\mathcal{S}_{\varepsilon_1 r}$ which intersects both $B_{2\varepsilon_1 r}(\pi_- \cup L_-)$ and $B_{2\varepsilon_1 r}(H_r(z))$ must be contained in $B_{10\varepsilon_1 r}(u)$. Furthermore, using that L_- is a linear segment, we get that the ε_1-neighborhood of each such square which intersects $B_{2\varepsilon_1 r}(\pi_- \cup L_-)$ and belongs to $\mathcal{K}_r(z)$ (as defined in (5.18)) must be contained in $\overline{O_u}$, with O_u as in the lemma statement. Since the Euclidean distance between $\pi_- \cup L_-$ and $\pi_+ \cup L_+$ is at least $b_1 r \geq 100\varepsilon_1 r$ and $\widetilde{P} \subset H_r(z)$, we see that removing O_u disconnects $V_r(z)$ into at least two connected components, and the Euclidean distance between the connected component which contains $z - 2r$ and the union of the other connected components is at least $\varepsilon_1 r$. A similar argument applies with v in place of u.

Condition 3 Each point of O_u is contained in a square of $\mathcal{K}_r(z)$ which lies at graph distance at most 40 from a square which contains u in the adjacency graph of squares of $\mathcal{K}_r(z)$. The same is true with v in place of u. It therefore follows from (5.17) that condition 3 in the lemma statement is satisfied. □

For $z \in \mathbb{C}$ and $r > 0$, let $F_r(z)$ be as in Lemma 5.6. In the next subsection, we will use the local independence properties of the GFF (in the form of Lemma 2.7) to argue that for a small enough $\rho \in (0, 1)$ and for all $r \in \rho^{-1}\mathcal{R}_0$, it is very likely that $F_{\rho r}(z)$ occurs for many points $z \in B_r(0)$. To apply the lemma, we will need the following measurability statement.

Lemma 5.7 *For each $z \in \mathbb{C}$ and $r > 0$, the event $F_r(z)$ is a.s. determined by $(h - h_{4r}(z))|_{B_{3r}(z)}$.*

Proof First note that the occurrence of $F_r(z)$ is unaffected by scaling each of D_h and $\widetilde{D}_h$ by the same constant factor. Therefore, Axiom III (Weyl scaling) implies that $F_r(z)$ is determined by h, viewed modulo additive constant. So, we only need to show that $F_r(z) \in \sigma\left(h|_{B_{3r}(z)}\right)$.

We first observe that for $u, v \in \overline{B_r(z)}$, we have $\widetilde{D}_h(u, v) \leq (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z))$ if and only if v is contained in the $\widetilde{D}_h$-metric ball of radius $(c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z))$ centered at v. Since this $\widetilde{D}_h$-metric ball is contained in $B_{2r}(z)$, we infer from the locality of $\widetilde{D}_h$ that the set of $u, v \in B_{2r}(z)$ for which $\widetilde{D}_h(u, v) \leq (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z))$ is determined by $h|_{B_{3r}(z)}$. If $\widetilde{D}_h(u, v) \leq (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z))$, then each $\widetilde{D}_h$-geodesic from u to v is contained in $B_{2r}(z)$, so the set of $\widetilde{D}_h$-geodesics from u to v is the same as the set of $\widetilde{D}_h(\cdot, \cdot; B_{2r}(z))$-geodesics from u to v.

Furthermore, by the definition (1.21) of c_* and C_*, we see that

$$\begin{aligned} \widetilde{D}_h(u, v) \leq (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z)) &\Rightarrow D_h(u, v) \\ &\leq (c_*/C_*) D_h(u, \partial B_{2r}(z)), \end{aligned} \tag{5.19}$$

so $D_h(u, v) = D_h(u, v; \partial B_{2r}(z))$ whenever $\widetilde{D}_h(u, v) \leq (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{2r}(z))$.

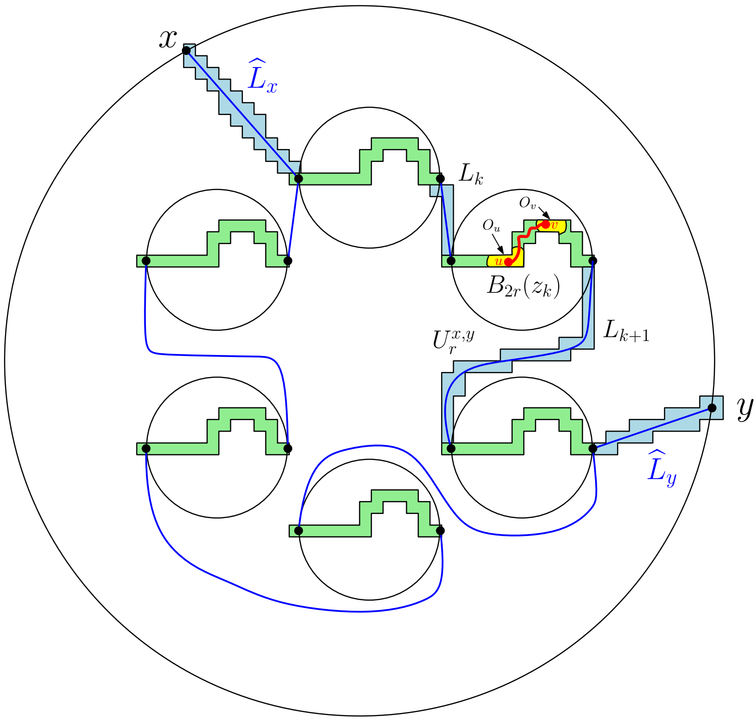

Fig. 11 Illustration of the statement and proof of Lemma 5.8. To get an event with probability close to 1, instead of just an event with uniformly positive probability, we consider a large number of disjoint balls $B_{\rho r}(z)$ centered at a finite set of points $\mathcal{Z} \subset \partial B_r(0)$ and use Lemma 2.7 to argue that with high probability, the event $F_{\rho r}(z)$ of Lemma 5.6 occurs for a suitably "dense" set of points $z \in \mathcal{Z}$. Then, we link up the tubes $V_{\rho r}(z)$ for $z \in \mathcal{Z}$ (light green) via deterministic paths L_k (blue). For a given choice of points $x, y \in \partial B_{2r}(0)$ with $|x - y| \geq \delta r$, we define $U_r^{x,y}$ to be the union of the sets $V_{\rho r}(z)$ for points $z \in \mathcal{Z}$ along the counterclockwise arc of $\partial B_r(0)$ from $x/2$ to $y/2$, the squares of $\mathcal{S}_{\varepsilon_1 \rho r}(B_r(0))$ which intersect the deterministic paths joining these sets $V_{\rho r}(z)$, and paths of squares starting from each of x and y (light blue). The sets O_u and O_v from assertion B are shown in yellow (color figure online)

By combining these observations with the locality of the metrics D_h and $\widetilde{D}_h$, it follows that $F_r(z)$ is determined by $h|_{B_{3r}(z)}$. □

5.4 Building a tube which contains a shortcut with high probability

In the rest of this section, unlike in Sect. 5.3, our events will no longer depend on a parameter z. Rather, we will only define events for Euclidean balls centered at 0. We will now prove a variant of Lemma 5.6 which holds with probability close to 1, not just with uniformly positive probability. This will be accomplished as follows. We fix a small parameter $\rho > 0$ and consider a large number of radius-ρr balls $B_{\rho r}(z)$ contained in $B_{2r}(0)$ for which the event $F_{\rho r}(z)$ of Lemma 5.6 occurs with positive probability. We join up the "tubes" $V_{\rho r}(z)$ for the individual balls into a single large tube, which we will denote by $U_r^{x,y}$. We use Lemma 2.7 to say that with high probability the event $F_{\rho r}(z)$ occurs for at least one of the small balls, which means that with high probability the tube $U_r^{x,y}$ contains a pair of points u, v as in (5.1). See Fig. 11 for an illustration.

Lemma 5.8 *For each $p, \delta \in (0,1)$, there exists $b, \rho \in (0, 1/100)$ depending only on p, δ, μ, ν and $\varepsilon_0 \in (0, b/100)$ depending only on $c_1', c_2', p, \delta, \mu, \nu$ such that for each $r \in \rho^{-1}\mathcal{R}_0$ and each $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$, there exists a deterministic connected open set $U_r^{x,y} \subset B_{3r}(0)$ with the following properties. The set $U_r^{x,y}$ is the interior of a finite union of squares in $\mathcal{S}_{\varepsilon_0 r}(\mathbb{A}_{r/2,2r}(0))$, $x, y \in U_r^{x,y}$. Moreover, with probability at least p, it holds simultaneously for each $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$ that there are points $u, v \in \mathbb{A}_{(1-4\rho)r,(1+4\rho)r}(0) \cap U_r^{x,y}$ with the following properties.*

A. (Existence of a shortcut) *We have*

$$|u-v| \geq br, \quad \widetilde{D}_h(u,v) \leq c_1' D_h(u,v), \quad \widetilde{D}_h(u,v) \\ \leq (c_*/C_*)^2 \widetilde{D}_h\left(u, \partial B_{4\rho r}(u)\right), \tag{5.20}$$

and the $\widetilde{D}_h$-geodesic from u to v is unique and is contained in $U_r^{x,y}$.

B. (Removing neighborhoods of u, v disconnects $U_r^{x,y}$) *Let O_u be the connected component of $U_r^{x,y} \cap B_{20\varepsilon_0 r}(u)$ which contains u and define O_v similarly with v in place of u. The connected component of $U_r^{x,y} \setminus O_u$ which contains x lies at Euclidean distance at least $\varepsilon_0 r$ from the union of the other connected components, and the same is true with v in place of u and y in place of x.*

C. (Upper bound for internal diameters of neighborhoods of u and v) *Each point of O_u lies at $\widetilde{D}_h(\cdot,\cdot; U_r^{x,y})$-distance at most $\eta \widetilde{D}_h(u,v)$ from u, and the same is true with v in place of u (here η is as in (5.15)).*

Proof Define the event $F_{\rho r}(z)$ for $z \in \mathbb{C}$ and $r \in \rho^{-1}\mathcal{R}_0$ as in Lemma 5.6.

Step 1: $F_{\rho r}(z)$ occurs for many points $z \in B_{2r}(0)$ Let $n_* \in \mathbb{N}$ be chosen so that the conclusion of Lemma 2.7 is satisfied with $s = 1/3$, p_1 in place of p, and $1 - \delta(1-p)/100$ in place of q. Let $\rho := (500 n_*)^{-1}\delta$ and define the set of points

$$\mathcal{Z} := \left\{ r \exp\left(\frac{2\pi \mathrm{i} \delta k}{100 n_*}\right) : k \in [1, 100 n_* \delta^{-1}]_{\mathbb{Z}} \right\} \subset \partial B_r(0). \tag{5.21}$$

Then the balls $B_{4\rho r}(z)$ for $z \in \mathcal{Z}$ are disjoint and each such ball is contained in $\mathbb{A}_{(1-4\rho)r,(1+4\rho)r}(0)$.

By Lemmas 5.6 and 5.7 , if $r \in \rho^{-1}\mathcal{R}_0$, then each of the events $F_{\rho r}(z)$ for $z \in \mathcal{Z}$ has probability at least p_1 and is determined by $(h - h_{4\rho r}(z))|_{B_{3\rho r}(z)}$. Each arc $I \subset \partial B_r(0)$ with Euclidean length at least $\delta r/4$ satisfies $\#(\mathcal{Z} \cap I) \geq n_*$. Therefore, Lemma 2.7 (applied with the whole-plane GFF $h(\cdot/(3\rho r))$ in place of h) implies that for each such arc I,

$$\mathbb{P}\left[\exists z \in \mathcal{Z} \cap I \text{ such that } F_{\rho r}(z) \text{ occurs}\right] \geq 1 - \frac{\delta(1-p)}{100}. \tag{5.22}$$

We can choose at most $4\pi\delta^{-1}$ arcs of $\partial B_r(0)$ with Euclidean length $\delta r/4$ in such a way that each arc of $\partial B_r(0)$ with Euclidean length at least $\delta r/2$ contains one of these arcs. By a union bound, we therefore get that with probability at least $1 - (1-p)/4$,

$$\text{Each arc of } \partial B_r(0) \text{ with length at least } \delta r/2 \text{ contains a point } z \in \mathcal{Z} \text{ s.t. } F_{\rho r}(z) \text{ occurs.} \tag{5.23}$$

We will show that the statement of the lemma is satisfied with

$$\varepsilon_0 = \varepsilon_1 \rho \quad \text{and} \quad b = b_1 \rho. \tag{5.24}$$

Step 2: defining $U_r^{x,y}$ Enumerate $\mathcal{Z} = \{z_1, \ldots, z_N\}$, where $N := \lfloor 100 n_* \delta^{-1} \rfloor$ and $z_k := r \exp\left(\frac{2\pi \mathbb{i} \delta k}{100 n_*}\right)$. Also set $z_0 := z_N$. We now join up the balls $B_{2\rho r}(z_k)$, in a manner which is illustrated in Fig. 11. For $k \in [1, N]_{\mathbb{Z}}$, choose in a deterministic manner a piecewise linear path L_k from $z_{k-1} + 2\rho r$ to $z_k - 2\rho r$ which is contained in $\mathbb{A}_{(1-4\rho)r,(1+4\rho)r}(0)$. We can choose the paths L_k in such a way that the L_k's do not intersect any of the balls $B_{2\rho r}(z)$ for $z \in \mathcal{Z}$ and lie at Euclidean distance at least ρr from one another.

Now consider points $x, y \in \partial B_{2r}(0)$ with $|x - y| \geq \delta r$. By possibly relabeling, we can assume without loss of generality that the counterclockwise arc of $\partial B_{2r}(0)$ from x to y is shorter than the clockwise arc. Let $J \subset \partial B_r(0)$ by the counterclockwise arc from $x/2$ to $y/2$, so that J has length at least $\delta r/2$. Let $k_x, k_y \in [1, N]_{\mathbb{Z}}$ be chosen so that $J \cap \mathcal{Z} = \{z_{k_x}, \ldots, z_{k_y}\}$. Let $\widehat{L}_x$ (resp. $\widetilde{L}_y$) be a smooth path from x to $z_{k_y} - 2r$ (resp. from $z_{k_y} + 2r$ to y) which does not intersect any of the $B_{2\rho r}(z)$'s for $z \in \mathcal{Z}$ and such that $\widehat{L}_x$ and $\widehat{L}_y$ lie Euclidean distance at least ρr from each other and from each L_k for $k \in [k_x + 1, k_y]_{\mathbb{Z}}$.

Recall that for $X \subset \mathbb{C}$, $\mathcal{S}_{\varepsilon_1 \rho r}(X)$ denotes the set of closed Euclidean squares of side length $\varepsilon_1 \rho r$ with corners in $\varepsilon_1 \rho r \mathbb{Z}^2$ which intersect X. With $V_{\rho r}(z)$ as in the definition of $F_{\rho r}(z)$, we define

$$\overline{U_r^{x,y}} := \bigcup_{k=k_x}^{k_y} \overline{V_{\rho r}(z_k)} \cup \bigcup \mathcal{S}_{\varepsilon_1 \rho r}\left(\widehat{L}_x \cup \widehat{L}_y \cup \bigcup_{k=k_x+1}^{k_y} L_k\right) \tag{5.25}$$

and we let $U_r^{x,y}$ be the interior of $\overline{U_r^{x,y}}$. Since each $V_r(z_k)$ is the interior of a finite union of squares in $\mathcal{S}_{\varepsilon_1 \rho r}(B_{\rho r}(z_k))$, it follows that $U_r^{x,y}$ is the interior of a finite union of squares $\mathcal{S}_{\varepsilon_1 \rho r}(\mathbb{A}_{r/2,2r}(0))$. Since the $V_r(z_k)$'s are connected, it is clear that $U_r^{x,y}$ is connected and contains x, y. We also note that $U_r^{x,y}$ is deterministic.

Step 3: checking the conditions for u and v On the event that (5.23) holds, there is a random $k \in [k_x, k_y]_{\mathbb{Z}}$ for which $F_{\rho r}(z_k)$ occurs. If this is the case, choose such a k and point $u, v \in V_{\rho r}(z_k) \cap \overline{B_{\rho r}(z_k)}$ as in the definition of $F_{\rho r}(z_k)$ in some measurable manner. We will show that for ε_0, b as in (5.24), the conditions in the lemma statement hold whenever (5.23) holds.

Condition A Since $B_{2\rho r}(z_k) \subset B_{4\rho r}(u)$ and $V_{\rho r}(z_k) \subset U_r^{x,y}$, it is immediate from Condition 1 in the definition of $F_{\rho r}(z_k)$ that this condition holds with $b = b_1 \rho$ whenever (5.23) holds.

Condition B Assume (5.23). Let

$$\overline{W_k(x)} := \bigcup_{j=k_x}^{k-1} \overline{V_{\rho r}(z_j)} \cup \bigcup \mathcal{S}_{\varepsilon_1 \rho r}\left(\widehat{L}_x \cup \bigcup_{j=k_x+1}^{k} L_j\right) \quad \text{and}$$
$$\overline{W_k(y)} := \bigcup_{j=k+1}^{k_y} \overline{V_{\rho r}(z_j)} \cup \bigcup \mathcal{S}_{\varepsilon_1 \rho r}\left(\widehat{L}_y \cup \bigcup_{j=k+1}^{k_y} L_j\right) \tag{5.26}$$

and let $W_k(x)$ and $W_k(y)$ be the interiors of $W_k(x)$ and $W_k(y)$, respectively. By (5.25), $\overline{U_r^{x,y}} = \overline{W_k(x)} \cup \overline{W_k(y)} \cup \overline{V_{\rho r}(z_k)}$. Since $\widehat{L}_x$, $\widehat{L}_y$, and the L_k's for $k \in [k_x+1, k_y]_{\mathbb{Z}}$ each lie at Euclidean distance at least ρr from one another and do not intersect the interiors of the balls $B_{\rho r}(z)$ for $z \in \mathcal{Z}$ and $\varepsilon_1 < 1/100$, the sets $W_k(x)$ and $W_k(y)$ lie at Euclidean distance at least $\rho r/2$ from each other and from $B_{\rho r}(z_k)$.

We have

$$U_r^{x,y} \cap B_{20\varepsilon_1 \rho r}(u) = V_{\rho r}(z_k) \cap B_{20\varepsilon_1 \rho r}(u), \tag{5.27}$$

so the definition of O_u is unaffected if we replace $U_r^{x,y}$ by $V_{\rho r}(z_k)$. Furthermore, the connected component of $U_r^{x,y} \setminus O_u$ which contains x is the same as the union of $W_k(x)$ and the connected component of $V_{\rho r}(z_k) \setminus O_u$ which contains $z_k - 2\rho r$; and the union of the other connected components of $U_r^{x,y} \setminus O_u$ is the same as the union of $W_k(y)$) and the connected components of $V_{\rho r}(z_k) \setminus O_u$

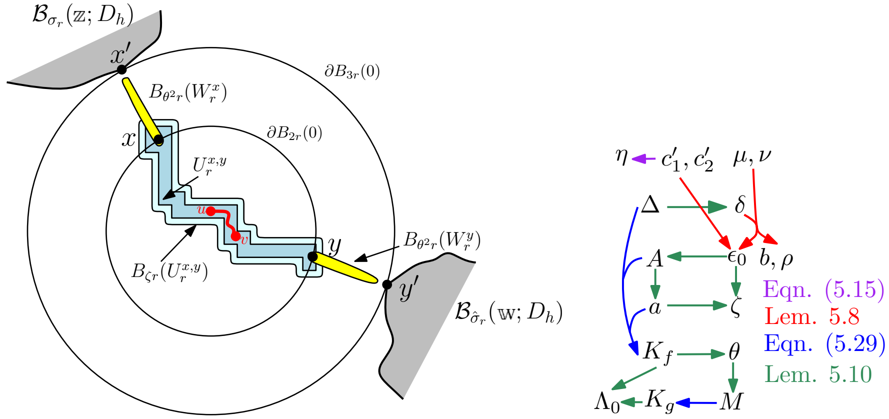

Fig. 12 Left: Illustration of the definition of the event E_r. The blue set in the middle is the set $U_r^{x,y}$ of Lemma 5.8. The light blue region surrounding it is $B_{\zeta r}(U_r^{x,y})$, which is the support of the bump function $f_r^{x,y}$. The yellow regions are the supports of the bump functions g_r^x and g_r^y, which are used to force D_h-geodesics started from points outside of $B_{3r}(0)$ to enter $B_{\zeta r}(U_r^{x,y})$. The figure shows the relevant set for one pair of points $x, y \in \partial B_{2r}(0)$, but all of the conditions in the event E_r are required to hold *simultaneously* for all pairs of points $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$. This is important since in Sect. 5.6 we will take $x' = (3/2)x$ and $y' = (3/2)y$ to be the *random* points where the metric balls based at the starting and ending points of a given geodesic (here shown in grey) first hit $\partial B_{3r}(0)$. Right: schematic diagram of how the various quantities in the definitions of E_r and $\mathcal{G}_r$ are chosen. An arrow between two parameters indicates that one is chosen in a way which depends directly on the other. The colors indicate where the choice is made. Most of the choices in the figure depend on $\mathbb{p}$, but this is not illustrated. In the end, all of the parameters depend only on $\mathbb{p}$, μ, ν (and the choice of metric) (color figure online)

which do not contain $z_k + 2\rho r$. By condition 2 in the definition of $F_{\rho r}(z_k)$, we find that these two sets lie at Euclidean distance at least $\varepsilon_1 \rho r$ from one another.

Condition C By (5.27), condition 3 in the definition of $F_{\rho r}(z_k)$ implies that each point of O_u lies at $D_h(\cdot, \cdot; U_r^{x,y})$-distance at most $\eta \widetilde{D}_h(u, v)$ from u. The same is true with v in place of u. □

5.5 Definition of the event E_r and the bump functions $\mathcal{G}_r$

The goal of this subsection is to define the event E_r and the collection of smooth bump functions $\mathcal{G}_r$ appearing in Proposition 5.2. We will also check Properties (A) and (B) from that proposition (measurability and high probability and bounds for Dirichlet inner products). Property (C) (subtracting a bump function) will be checked in Sect. 5.6.

The definitions in this section are illustrated in Fig. 12, left. Before proceeding with the details, we briefly discuss the main ideas involved. Following

Sect. 5.1, we want to define $\mathcal{G}_r$ to include for each $x, y \in \partial B_{3r}(0)$ a function ϕ which is equal to a large positive constant on the region $U_r^{x,y}$ of Lemma 5.8 and which is supported on the union of a small neighborhood of $U_r^{x,y}$ and two even narrower "tubes" which approximate the segments $[x, 3x/2]$ and $[y, 3y/2]$ (shown in yellow in the figure). The event E_r will consist of the conditions of Lemma 5.8 plus several regularity conditions discussed below.

We will eventually consider a fixed pair of points $\mathbb{z}, \mathbb{w} \in \mathbb{C}\setminus B_{4r}(0)$ and choose $x, y \in \partial B_{2r}(0)$ in such a way that $x' := 3x/2$ and $y' := 3y/2$ are the first points $\partial B_{3r}(0)$ hit by the D_h-metric balls grown from $\mathbb{z}$ and $\mathbb{w}$, respectively. Since these points are random, it is important that the conditions in our event hold simultaneously for all possible choices of x and y. We will show in Sect. 5.6 that on E_r, subtracting a suitable $\phi \in \mathcal{G}_r$ from the field makes distances in the support of ϕ much shorter than distances outside, so the $D_{h-\phi}$-geodesic has to travel through the support of ϕ and hence has to get close to the points u, v of Lemma 5.8.

There are several subtleties involved in this argument which are dealt with via regularity conditions in the definition of E_r. For example, Lemma 5.8 requires that $|x - y| \geq \delta r$, so we need to ensure that our random metric ball hitting points x', y' are separated. This is the purpose of condition 4 in the definition of E_r. Another difficulty is that it is relatively straightforward to get $D_{h-\phi}$-geodesics into the *support* of ϕ, but we want such geodesics to actually enter the region $U_r^{x,y}$ where ϕ is equal to a large positive constant. The reason for this is that we will be comparing ratios of distances via Weyl scaling (Axiom III) and it could be that ϕ is much smaller on some parts of its support than it is on $U_r^{x,y}$. To deal with this, we will include a condition to the effect that paths which stay in a small neighborhood of $\partial U_r^{x,y}$ without entering $U_r^{x,y}$ are very long (condition 6). We also need functions in $\mathcal{G}_r$ to be supported on $\mathbb{A}_{r,3r}(0)$ so we need to make the yellow tubes in Fig. 12 very close to x' and y' without actually allowing these tubes to contain x' and y' (condition 8). The choice of constants involved in these conditions is somewhat delicate, so the event E_r will include several parameters.

We now commence with the definitions. Fix a parameter $\delta \in (0, 1)$, to be chosen in a manner depending only on $\mathbb{p}$ in Lemma 5.10 below. Let ρ, b, ε_0 be as in Lemma 5.8 for this choice of δ and with $p = 1 - (1 - \mathbb{p})/2$, so that ρ, b, ε_0 depend only on $\delta, \mathbb{p}, \mu, \nu$. The definitions of E_r and $\mathcal{G}_r$ involve several additional small parameters $\Delta \in (0, 1)$ and $\zeta, a, \theta \in (0, \varepsilon_0)$ and large parameters $A, M, \Lambda_0 > 1$ which we will choose in Lemma 5.10 below, in a manner depending only on $\mathbb{p}, \mu, \nu$. See Fig. 12, right for a schematic illustration of how the parameters are chosen.

5.5.1 Definition of $\mathcal{G}_r$

We first give the definition of $\mathcal{G}_r$ in terms of the above parameters. For each $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$, choose in a deterministic manner depending only on $U_r^{x,y}$ (not on the particular values of x and y) a smooth, compactly supported bump function $f_r^{x,y} : \mathbb{C} \to [0, 1]$ which is identically equal to 1 on $U_r^{x,y}$ and vanishes outside of $B_{\zeta r}(U_r^{x,y})$.

Since each $U_r^{x,y}$ is the interior of a finite union of squares in $\mathcal{S}_{\varepsilon_0 r}(B_{2r}(0))$, there are at most a finite, r-independent number of possibilities for $U_r^{x,y}$ as x and y vary. From this and the scale invariance of Dirichlet energy (i.e., $(f(r\cdot), f(r\cdot))_\nabla = (f, f)_\nabla$) it follows that we can arrange that the Dirichlet energy $(f_r^{x,y}, f_r^{x,y})_\nabla$ is bounded above by a constant depending only on ε_0, ζ.

If we subtract a large constant multiple of $f_r^{x,y}$ from h, then LQG geodesics for the resulting field between points of $U_r^{x,y}$ will tend to stay in $U_r^{x,y}$. However, we also need to get geodesics between points of $\mathbb{C}\backslash B_{4r}(0)$ into $U_r^{x,y}$. For this purpose, we will also subtract even larger constant multiples of bump functions g_r^x and g_r^y which are supported in narrow tubes which approximate the segments $[x, 3x/2]$ and $[y, 3y/2]$. The supports of these bump functions are shown in yellow in Fig. 12.

To define these bump functions, we first define for $x \in \partial B_{2r}(0)$ the set

$$W_r^x = W_r^x(\theta) := \left(\text{Interior of} \bigcup_{S \in \mathcal{S}_{\theta r}([x,(3/2-\theta)x])} S\right) \subset \mathbb{A}_{r,3r}(0) \quad (5.28)$$

where here we recall from (5.14) that $\mathcal{S}_{\theta r}([x, (3/2-\theta)x])$ is the set of $\theta r \times \theta r$ squares with corners in $\theta r\mathbb{Z}^2$ which intersect $[x, (3/2-\theta)x]$. Let $g_r^x : \mathbb{C} \to [0, 1]$ be a smooth compactly supported function which is identically equal to 1 on W_r^x and is identically equal to 0 outside of $B_{\theta^2 r}(W_r^x) \subset B_{3r}(0)$. As in the case of $f_r^{x,y}$ (see the paragraph just above (5.29)), we can arrange that the Dirichlet energy of g_r^x is bounded above by a constant depending only on θ, $\mathbb{p}$.

We define the large constants

$$K_f := \frac{1}{\xi} \log\left(\frac{100A}{a\Delta}\right) \quad \text{and} \quad K_g := K_f + \frac{1}{\xi}\log(M). \quad (5.29)$$

For each $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$, we define

$$\phi_r^{x,y} := K_f f_r^{x,y} + K_g(g_r^x + g_r^y). \quad (5.30)$$

Since each of $f_r^{x,y}, g_r^x, g_r^y$ is supported on $\mathbb{A}_{r/4,3r}(0)$, so is $\phi_r^{x,y}$. We set

$$\mathcal{G}_r := \left\{\phi_r^{x,y} : x, y \in \partial B_{2r}(0),\ |x-y| \geq \delta r\right\} \cup \{\text{zero function}\}. \quad (5.31)$$

We emphasize that the definition of $\mathcal{G}_r$ does not depend on the parameter Λ_0. This will be important when we choose Λ_0 in Lemma 5.10 below.

Recall from the above discussion that the number of possibilities for each of $f_r^{x,y}$, g_r^x, g_r^y as x and y vary and the Dirichlet energies of each of these functions is bounded above by a constant which does not depend on r, x, or y. Consequently, each of

$$\#\mathcal{G}_r \quad \text{and} \quad \max_{\phi \in \mathcal{G}_r} (\phi, \phi)_\nabla \tag{5.32}$$

is bounded above by a constant which does not depend on r, x, or y.

5.5.2 *Definition of E_r*

We now define the event E_r appearing in Proposition 5.2.

We encourage the reader to skim the list of conditions on a first read and refer back to them as they are used while reading the proof of Lemma 5.11 below.

With the parameters $\delta, \Delta, A, \zeta, a, \theta, M, \Lambda_0$ as above, we define E_r to be the event that the following is true. For each $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$, there exists $u, v \in \mathbb{A}_{(1-4\rho)r,(1+4\rho)r}(0)$ satisfying the three numbered conditions of Lemma 5.8 and moreover the following additional conditions hold.

4. For each $x, y \in \partial B_{2r}(0)$ with $|x - y| < \delta r$,
$$\begin{aligned} &D_h\left(x', y'; \mathbb{A}_{r,4r}(0)\right) \leq \Delta \mathfrak{c}_r e^{\xi h_r(0)} \\ &\quad \leq D_h\left(\partial B_{2r}(0), \partial B_{3r}(0)\right), \quad \text{where} \quad x' = \frac{3}{2}x \text{ and } y' = \frac{3}{2}y. \end{aligned}$$
5. For each $x, y \in \partial B_{2r}(0)$ with $|x - y| \geq \delta r$ the D_h-internal diameter of $U_r^{x,y}$ satisfies
$$\sup_{w_1, w_2 \in U_r^{x,y}} D_h\left(w_1, w_2; U_r^{x,y}\right) \leq A \mathfrak{c}_r e^{\xi h_r(0)}.$$
6. For each $x, y \in \partial B_{2r}(0)$ with $|x - y| \geq \delta r$, the D_h-length of every continuous path of Euclidean diameter at least $\varepsilon_0 r/100$ which is contained in $B_{2\zeta r}(\partial U_r^{x,y})$ is at least $100 A \mathfrak{c}_r e^{\xi h_r(0)}$.
7. For each $z_1, z_2 \in \mathbb{A}_{r/4,4r}(0)$ such that $|z_1 - z_2| \geq \zeta r$,
$$D_h\left(z_1, z_2; \mathbb{A}_{r/4,4r}(0)\right) \geq a \mathfrak{c}_r e^{\xi h_r(0)}.$$
8. With K_f as in (5.29),
$$D_h\left(3x/2, (3/2 - \theta)x; \mathbb{A}_{r,4r}(0)\right) \leq e^{-\xi K_f} \mathfrak{c}_r e^{\xi h_r(0)}, \quad \forall x \in \partial B_{2r}(0).$$

9. If we let $W_r^x \subset \mathbb{A}_{r,3r}(0)$ be the long narrow tube as in (5.28), then

$$\sup_{w_1, w_2 \in W_r^x} D_h(w_1, w_2; W_r^x) \le M \mathfrak{c}_r e^{\xi h_r(0)}.$$

10. With $\mathcal{G}_r$ as in (5.31), we have $(h, \phi)_\nabla + \frac{1}{2}|(\phi, \phi)_\nabla| \le \Lambda_0$ for each $\phi \in \mathcal{G}_r$.

The conditions in the definition of E_r are numbered in such a way that the new parameters involved in each condition depend only on the parameters from the previous conditions. We now comment briefly on the purpose of each of the conditions. As discussed in Sect. 5.1, to prove Property (C) (subtracting a bump function) of Proposition 5.2, we will grow the D_h-metric balls started from $\mathbb{z}$ and $\mathbb{w}$ until they hit $B_{3r}(0)$. We will let $\mathbb{x}'$ and $\mathbb{y}'$ be their respective hitting points, and we will apply the above conditions with $x = \mathbb{x} := (2/3)\mathbb{x}'$ and $y = \mathbb{y} = (2/3)\mathbb{y}'$ (note that $\mathbb{x}, \mathbb{y} \in \partial B_{2r}(0)$).

Condition 4 is used to ensure that if P hits $B_{2r}(0)$, then $|\mathbb{x} - \mathbb{y}| \ge \delta r$ (see Lemma 5.12). Condition 5 gives us a deterministic upper bound for the D_h-diameter of $U_r^{\mathbb{x},\mathbb{y}}$ before we subtract the bump function ϕ. This allows us say that the $D_{h-\phi}$-diameter of $U_r^{x,y}$ is very small, which is what forces the $D_{h-\phi}$-geodesic P^ϕ to enter $U_r^{\mathbb{x},\mathbb{y}}$. Condition 6 prevents P^ϕ from staying close to $\partial U_r^{\mathbb{x},\mathbb{y}}$ (in the region where ϕ positive, but does not attain its largest possible value) without entering $U_r^{x,y}$ itself. Condition 7 is used to prevent P^ϕ from exiting $B_{\zeta r}(U_r^{\mathbb{x},\mathbb{y}})$ prematurely. Conditions 8 and 9 concern the yellow tubes in Fig. 12. These conditions are used to force P^ϕ to enter and exit $U_r^{\mathbb{x},\mathbb{y}}$ at points near $\mathbb{x}$ and $\mathbb{y}$, respectively. Condition 10 is used to prove Property (B) (bounds for Dirichlet inner products) of Proposition 5.2.

5.5.3 *Proof of Properties (A) and (B)*

It is immediate from condition 10 in the definition of E_r that Property (B) (bounds for Dirichlet inner products) of Proposition 5.2 is satisfied. In the next two lemmas we check the two assertions of Property (A) (measurability and high probability).

Lemma 5.9 *The event E_r is determined by $(h - h_{5r}(0))|_{\mathbb{A}_{r/4,4r}(0)}$*

Proof By Axiom III (Weyl scaling), the occurrence of E_r is unaffected by adding a real number to h, so we only need to show $E_r \in \sigma\left(h|_{\mathbb{A}_{r/4,4r}(0)}\right)$. The measurability of condition A follows from exactly the same argument used in the proof of Lemma 5.7 (this can also be seen from Lemma 5.7 and the proof of Lemma 5.6). Since $U_r^{x,y}, W_r^x, W_r^y \subset \mathbb{A}_{(1/2-2\varepsilon_0)r,3r}(0)$ and D_h and $\widetilde{D}_h$ are local metrics for h, the measurability of the other conditions in the definition of E_r follows by inspection and Axiom II (locality). $\square$

Lemma 5.10 *We can choose the parameters* $\delta, \Delta, A, \zeta, a, \theta, M, \Lambda_0$ *in a manner depending only on* $\mathbb{p}, \mu, \nu, c_1', c_2'$ *in such a way that* $\mathbb{P}[E_r] \geq \mathbb{p}$ *for each* $r \in \rho^{-1}\mathcal{R}_0$.

Proof By tightness across scales (Axiom V), we can choose Δ and then δ in such a way that condition 4 holds with probability at least $1-(1-\mathbb{p})/100$. As above, we choose b, ρ, ε_0 as in Lemma 5.8 with the above choice of δ and with $p = 1-(1-\mathbb{p})/100$ (so that b, ρ depend only on $\mathbb{p}, \mu, \nu$ and ε_0 depends only on $\mathbb{p}, \mu, \nu, c_1', c_2'$) and define $U_r^{x,y}$ for $x, y \in \partial B_{2r}(0)$ with $|x-y| \geq \delta r$ as in that lemma. Then the first four conditions (including the three from Lemma 5.8) in the definition of E_r occur simultaneously with probability at least $1-2(1-\mathbb{p})/100$.

We will now choose the parameters so as to lower-bound the probabilities of the other conditions in the definition of E_r in numerical order. By Lemma 2.9, we can find $C > 0$ depending only on ε_0 (and hence only on p, μ, ν, c_1', c_2') such that with probability at least $1-(1-\mathbb{p})/100$, we have, with $\mathcal{S}_{\varepsilon_0 r}(\cdot)$ as in (5.14),

$$\sup_{S \in \mathcal{S}_{\varepsilon_0 r}(B_{2r}(0))} \sup_{w_1, w_2 \in S} D_h(w_1, w_2; S) \leq C \mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.33}$$

The total number of squares of $\mathcal{S}_{\varepsilon_0 r}(B_{2r}(0))$ is at bounded above by a constant depending only on ε_0 (and hence only on $\mathbb{p}, \mu, \nu, c_1', c_2'$). Since each $U_r^{x,y}$ is connected and is the interior of a finite union of such squares, the triangle inequality shows that there is an $A > 1$ depending only on $\mathbb{p}, \mu, \nu$ such that whenever (5.33) holds, also condition 5 holds. Hence the probability of condition 5 is at least $1-(1-\mathbb{p})/100$.

The set $\partial U_r^{x,y}$ is the union of some subset of the set of sides of squares in $\mathcal{S}_{\varepsilon_0 r}(B_{2r}(0))$. By Lemma 2.10 (applied with ζ in place of ε) and a union bound over all of the sides of all of the squares in $\mathcal{S}_{\varepsilon_0 r}(B_{2r}(0))$, we can choose $\zeta \in (0, \varepsilon_0/100)$ depending only on $\mathbb{p}, \varepsilon_0, A$ (and hence only on $\mathbb{p}, \mu, \nu, c_1', c_2'$) such that condition 6 holds with probability at least $1-(1-\mathbb{p})/100$.

Since D_h induces the Euclidean topology, we can find $a \in (0,1)$ depending only on $\mathbb{p}, \zeta$ (and hence only on $\mathbb{p}, \mu, \nu, c_1', c_2'$) such that condition 7 holds with probability at least $1-(1-\mathbb{p})/100$.

Since the constant K_f of (5.29) depends only on A, Δ, a, which have already been chosen in a manner depending only on $\mathbb{p}, \mu, \nu, c_1', c_2'$, we can find a small enough $\theta \in (0, \zeta/100)$ depending only on $\mathbb{p}, \mu, \nu, c_1', c_2'$ such that condition 8 holds with probability at least $1-(1-\mathbb{p})/100$.

Recall from (5.28) that W_r^x is the interior of the union of a set of squares in $\mathcal{S}_{\theta r}(B_{3r}(0))$. By Axiom V (tightness across scalings) and Lemma 2.9, we can find a sufficiently large $M > 0$ depending only θ (hence only on $\mathbb{p}, \mu, \nu, c_1', c_2'$) such that condition 9 holds with probability at least $1-(1-\mathbb{p})/100$.

The definition of the set of bump functions $\mathcal{G}_r$ above does not use the parameter Λ_0. As discussed just after (5.32), the number of functions in $\mathcal{G}_r$ and the Dirichlet energies of these functions are each bounded above by constants which depend only on $\mathbb{p}$, μ, ν, c_1', c_2' and the other parameters which we have already chosen in a manner depending only on $\mathbb{p}$, μ, ν, c_1', c_2'. Consequently, we can find a constant $\Lambda_0 > 0$ depending only on $\mathbb{p}$, μ, ν, c_1', c_2' such that condition 10 holds with probability at least $1 - (1 - \mathbb{p})/100$. Combining our above estimates gives the statement of the lemma. □

5.6 Subtracting a bump function to move a geodesic

To prove Proposition 5.2, it remains to check Property (C) (subtracting a bump function) for the event E_r and the collection of smooth bump functions $\mathcal{G}_r$ defined above. To this end, fix distinct points $\mathbb{z}, \mathbb{w} \in \mathbb{C}\backslash B_{4r}(0)$ and let $P = P^{\mathbb{z},\mathbb{w}}$ be the (a.s. unique) D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. We first grow the D_h-metric balls until they hit $\partial B_{3r}(0)$. Let σ_r (resp. $\widehat{\sigma}_r$) be the smallest $s > 0$ for which the D_h-metric ball $\mathcal{B}_s(\mathbb{z}; D_h)$ (resp. $\mathcal{B}_s(\mathbb{w}; D_h)$) intersects $\overline{B_{3r}(0)}$. Also let $\mathbb{x}'$ (resp. $\mathbb{y}'$) be a point of $\partial B_{3r}(0) \cap \mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$ (resp. $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$), chosen in some manner depending only on the appropriate D_h-metric ball,[7] and define the points of $\partial B_{2r}(0)$

$$\mathbb{x} := (2/3)\mathbb{x}' \quad \text{and} \quad \mathbb{y} := (2/3)\mathbb{y}'. \tag{5.34}$$

Note that $\mathbb{x}, \mathbb{y} \in \sigma\left(h|_{\mathbb{C}\backslash B_{3r}(0)}\right)$.

In the notation (5.30), we set

$$\phi = \begin{cases} \phi_r^{\mathbb{x},\mathbb{y}}, & \text{if } |\mathbb{x} - \mathbb{y}| \geq \delta r \\ 0, & \text{otherwise.} \end{cases} \tag{5.35}$$

Then $\phi \in \mathcal{G}_r$, as defined in (5.31), and ϕ is determined by $\mathbb{x}, \mathbb{y}$ and hence by $h|_{\mathbb{C}\backslash B_{3r}(0)}$. Hence to prove Property (C) it remains only to prove the following.

Lemma 5.11 *Let P^ϕ be the (a.s. unique) $D_{h-\phi}$-geodesic from $\mathbb{z}$ to $\mathbb{w}$. If $P \cap B_{2r}(0) \neq \emptyset$ and E_r occurs, then there are times $0 < s < t < D_{h-\phi}(\mathbb{z}, \mathbb{w})$ such that*

$$\begin{aligned} &P^\phi(s), P^\phi(t) \in B_{3r/2}(0), \quad |P^\phi(s) - P^\phi(t)| \geq (b - 40\varepsilon_0)r, \\ &\quad \widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \leq c_2' D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right), \quad \textit{and} \\ &\quad \widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \leq (c_*/C_*)\widetilde{D}_{h-\phi}\left(P^\phi(s), \partial B_{3r}(0)\right). \end{aligned} \tag{5.36}$$

[7] It is in fact not difficult to see that there is a.s. a unique intersection point by repeating the argument of [62, Theorem 1.2].

The rest of this section is devoted to the proof of Lemma 5.11. To lighten notation, write

$$U = U_r^{\mathbb{x},\mathbb{y}} \quad \text{and} \quad \mathcal{W} = B_{\theta^2 r}(W_r^{\mathbb{x}}) \cup B_{\theta^2 r}(W_r^{\mathbb{y}}). \tag{5.37}$$

Throughout, we assume that E_r occurs and $P \cap B_{2r}(0) \neq \emptyset$. The proof is an elementary (though somewhat technical) deterministic argument using the conditions in the definition of E_r, and is divided into several lemmas.

Lemma 5.12 *We have* $|\mathbb{x} - \mathbb{y}| \geq \delta r$.

Lemma 5.12 allows us to apply all of the conditions in the definition of E_r with $x = \mathbb{x}$ and $y = \mathbb{y}$ (note that these conditions hold for all $x, y \in \partial B_{2r}(0)$ with $|x - y| \geq \delta r$ simultaneously). We will use this fact without comment throughout the rest of the proof.

Proof of Proposition 4.3, assuming Proposition 5.2 Since P is a D_h-geodesic, the D_h-distance between the metric balls $\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$ and $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$ is equal to the D_h-distance traveled by P between the times when it hits these two metric balls. Since P enters $B_{2r}(0)$, it must cross between the inner and outer boundaries of $\mathbb{A}_{2r,3r}(0)$ at least twice between hitting these two metric balls, so the D_h-distance between $\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$ and $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$ must be at least $2D_h(\partial B_{2r}(0), \partial B_{3r}(0))$. Condition 4 in the definition of E_r implies that if $|\mathbb{x} - \mathbb{y}| < \delta r$ then $D_h(\partial B_{2r}(0), \partial B_{3r}(0)) \geq D_h(\mathbb{x}', \mathbb{y}'; \mathbb{A}_{r,4r}(0))$ which is at least the D_h-distance between $\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$ and $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$. This is a contradiction and therefore $|\mathbb{x} - \mathbb{y}| \geq \delta r$. □

We now prove an upper bound for $D_{h-\phi}(\mathbb{x}', \mathbb{y}')$. Since P^ϕ is a $D_{h-\phi}$-geodesic, this upper bound will allow us to constrain the behavior of P^ϕ since P^ϕ cannot have any segment whose $D_{h-\phi}$-length is larger than $D_{h-\phi}(\mathbb{x}', \mathbb{y}')$ (see Lemma 5.14 below).

Lemma 5.13 *We have*

$$D_{h-\phi}\left(\mathbb{x}', \mathbb{y}'\right) \leq e^{-\xi K_f}(A+4)\mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.38}$$

Proof By condition 8 in the definition of E_r and since $D_{h-\phi} \leq D_h$,

$$D_{h-\phi}\left(\mathbb{x}', W_r^{\mathbb{x}}\right) \leq e^{-\xi K_f}\mathfrak{c}_r e^{\xi h_r(0)} \quad \text{and} \quad D_{h-\phi}\left(\mathbb{y}', W_r^{\mathbb{y}}\right) \leq e^{-\xi K_f}\mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.39}$$

By condition 9, Axiom III (Weyl scaling), and since $\phi \geq K_g$ on each of $W_r^{\mathbb{x}}$ and $W_r^{\mathbb{y}}$ (with K_g as in (5.29)),

The internal $D_{h-\phi}$-diameters of $W_r^{\mathbb{x}}$ and $W_r^{\mathbb{y}}$ are each $\le e^{-\xi K_g} M \mathfrak{c}_r e^{\xi h_r(0)}$
$$\le e^{-\xi K_f} \mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.40}$$

By condition 5, Axiom III, and since $\phi \ge K_f$ on U,

$$\sup_{w_1, w_2 \in U} D_{h-\phi}(w_1, w_2; U) \le e^{-\xi K_f} A \mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.41}$$

Since $W_r^{\mathbb{x}}$ and $W_r^{\mathbb{y}}$ each intersect U, we can combine (5.39), (5.40), and (5.41) and use the triangle inequality to get (5.38). □

Lemma 5.14 *To lighten notation, let*

$$\overline{P}^{\phi} := P^{\phi} \setminus \left(\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h) \cup \mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h) \right).$$

In the notation (5.37)*,* $\overline{P}^{\phi}$ *is contained in* $B_{2\zeta r}(U \cup \mathcal{W})$*. Furthermore, there is no segment of* $\overline{P}^{\phi}$ *of Euclidean diameter* $\ge \varepsilon_0 r/100$ *which is contained in* $B_{2\zeta r}(\partial U) \setminus \mathcal{W}$*.*

Proof Since ϕ is supported on $B_{3r}(0)$, the definitions of σ_r, $\widehat{\sigma}_r$, $\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$, and $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$ are unaffected if we replace h by $h - \phi$. Since $\overline{P}^{\phi}$ is the $D_{h-\phi}$-shortest path between these metric balls, Lemma 5.13 implies that

$$\left(D_{h-\phi}\text{-length of } \overline{P}^{\phi} \right) \le e^{-\xi K_f} (A+4) \mathfrak{c}_r e^{\xi h_r(0)}. \tag{5.42}$$

We will now explain how (5.42) together with the definition of E_r allows us to constrain the behavior of $\overline{P}^{\phi}$.

As in the proof of Lemma 5.12, condition 4 in the definition of E_r implies that the D_h-distance between $\mathcal{B}_{\sigma_r}(\mathbb{z}; D_h)$, and $\mathcal{B}_{\widehat{\sigma}_r}(\mathbb{w}; D_h)$ is at least $2\Delta \mathfrak{c}_r e^{\xi h_r(0)}$, which is larger than $e^{-\xi K_f}(A+4)\mathfrak{c}_r e^{\xi h_r(0)}$ by the definition (5.29) of K_f. If $\overline{P}^{\phi}$ did not enter the support $B_{\zeta r}(U) \cup \mathcal{W}$ of ϕ, then the $D_{h-\phi}$-length of $\overline{P}^{\phi}$ would be the same as its D_h-length, which must be at least $2\Delta \mathfrak{c}_r e^{\xi h_r(0)}$. Hence (5.42) implies that $\overline{P}^{\phi}$ must enter $B_{\zeta r}(U) \cup \mathcal{W}$.

Since $\phi \le K_f$ outside of $\mathcal{W}$, Axiom III (Weyl scaling) together with condition 6 in the definition of E_r implies that the $D_{h-\phi}$-length of every continuous path of Euclidean diameter at least $\varepsilon_0 r/100$ which is contained in $B_{2\zeta r}(\partial U) \setminus \mathcal{W}$ is at least $100 e^{-\xi K_f} A \mathfrak{c}_r e^{\xi h_r(0)}$.

It therefore follows from (5.42) that the second assertion of the lemma holds.

We now prove the first assertion of the lemma. Since ϕ is identically equal to 0 on $\mathbb{C} \setminus (B_{\zeta r}(U) \cup \mathcal{W})$, condition 7 in the definition of E_r implies that the $D_{h-\phi}$-length of any curve which is contained in $\mathbb{A}_{r/4,4r}(0) \setminus (B_{\zeta r}(U) \cup \mathcal{W})$

and has Euclidean diameter at least ζr is at least $a\mathfrak{c}_r e^{\xi h_r(0)}$. This last quantity is strictly larger than the right side of (5.42) by the definition (5.29) of K_f. It follows that there is no segment of $\overline{P}^\phi$ of Euclidean diameter at least ζr which is contained in $\mathbb{A}_{r/4,4r}(0)\setminus(B_{\zeta r}(U)\cup\mathcal{W})$. Each path from $B_{\zeta r}(U)\cup\mathcal{W}$ to a point outside of $B_{2\zeta r}(U\cup\mathcal{W})$ has a sub-path which is contained in $\mathbb{A}_{r/4,4r}(0)\setminus(B_{\zeta r}(U)\cup\mathcal{W})$ and has Euclidean diameter at least ζr. Since we know that $\overline{P}^\phi$ has to hit $B_{\zeta r}(U)\cup\mathcal{W}$, we infer that $\overline{P}^\phi$ is contained in $B_{2\zeta r}(U)\cup\mathcal{W})$. □

We now produce the points $0<s<t<D_{h-\phi}(\mathbbm{z},\mathbbm{w})$ from Lemma 5.11 and check all of the conditions of the lemma except $\widetilde{D}_{h-\phi}\left(P^\phi(s),P^\phi(t)\right)\le c_2' D_{h-\phi}\left(P^\phi(s),P^\phi(t)\right)$ (we will check this last condition in the proof of Lemma 5.11 just below).

Lemma 5.15 *There are times $0<s<t<D_{h-\phi}(\mathbbm{z},\mathbbm{w})$ such that $P^\phi(s),P^\phi(t)\in B_{3r/2}(0)$, $|P^\phi(s)-P^\phi(t)|\ge(b-40\varepsilon_0)r$, and*

$$\widetilde{D}_{h-\phi}(P^\phi(s),P^\phi(t))\le(c_*/C_*)\widetilde{D}_{h-\phi}\left(P^\phi(s),\partial B_{3r}(0)\right). \tag{5.43}$$

Proof Recall the points $u,v\in\mathbb{A}_{(1-4\rho)r,(1+4\rho)r}(0)$ from condition A in the definition of E_r. That condition says that the $\widetilde{D}_h$-geodesic $\widetilde{P}$ from u to v is contained in U and its $\widetilde{D}_h$-length is at most $(c_*/C_*)^2\widetilde{D}_h(u,\partial B_{4\rho r}(u))$. The idea of the proof is to use Lemma 5.14 to force P^ϕ to get close to each of u and v, and then to take s and t to be the times at which it does so. Since ϕ attains its largest possible value on $B_{4\rho r}(u)$ (namely, K_f) at every point of $B_{4\rho r}(u)\cap U$ (here we note that $\mathcal{W}$ is disjoint from $B_{3r/2}(0)\supset B_{4\rho r}(u)$), it follows that $\widetilde{P}(e^{\xi K_f}\cdot)$ is a $\widetilde{D}_{h-\phi}$-geodesic from u to v and

$$\widetilde{D}_{h-\phi}(u,v)=\widetilde{D}_{h-\phi}\left(u,v;U\cap B_{4\rho r}(u)\right)=e^{-\xi K_f}\widetilde{D}_h(u,v). \tag{5.44}$$

Recall from condition B in the definition of E_r that O_u (resp. O_v) is the connected component of $U\cap B_{20\varepsilon_0 r}(u)$ which contains u. Since $B_{20\varepsilon_0 r}(u)$ is contained in $B_{3r/2}(0)$, so is disjoint from $\mathcal{W}$, that condition tells us that the connected component of $(U\cup\mathcal{W})\setminus O_u$ which contains $\mathbbm{x}'$ lies at Euclidean distance at least $\varepsilon_0 r$ from the union of the other connected components of $(U\cup\mathcal{W})\setminus O_u$. Since $\zeta<\varepsilon_0/100$, the $2\zeta r$-neighborhoods of these two sets lie at Euclidean distance at least $\varepsilon_0 r/2$ from one another. By Lemma 5.14, $\overline{P}^\phi=P^\phi\setminus\left(\mathcal{B}_{\sigma_r}(\mathbbm{z};D_h)\cup\mathcal{B}_{\widehat{\sigma}_r}(\mathbbm{w};D_h)\right)$ cannot exit $B_{2\zeta r}(U\cup\mathcal{W})$, so $\overline{P}^\phi$ must have a segment of Euclidean diameter at least $\varepsilon_0 r/2$ which is contained in

$$B_{2\zeta r}(O_u)\subset O_u\cup\left(B_{2\zeta r}(\partial U)\setminus\mathcal{W}\right).$$

By the other assertion of Lemma 5.14, this segment cannot be entirely contained in $B_{2\zeta r}(\partial U)\backslash \mathcal{W}$, so P^ϕ must enter O_u. Similarly, P^ϕ must enter O_v (and must do so at some time after it enters O_u).

Choose times $0 < s < t < |P^\phi|$ such that $P^\phi(s) \in O_u$ and $P^\phi(t) \in O_v$. Then $|P^\phi(s) - u| \le 20\varepsilon_0 r$ and $|P^\phi(t) - v| \le 20\varepsilon_0 r$, By condition C in the definition of E_r, (5.44), and the fact that $\phi \equiv K_f$ on $U\backslash\mathcal{W}$, we get that

$$\begin{aligned} &\widetilde{D}_{h-\phi}\left(P^\phi(s), u; U\right) \le \eta \widetilde{D}_{h-\phi}(u, v) \quad \text{and} \\ &\widetilde{D}_{h-\phi}\left(P^\phi(t), v; U\right) \le \eta \widetilde{D}_{h-\phi}(u, v). \end{aligned} \tag{5.45}$$

Since $|u - v| \ge br$, we have $|P^\phi(s) - P^\phi(t)| \ge (b - 40\varepsilon_0)r$ and since $u, v \in \overline{B_r(0)}$ we have $P^\phi(s), P^\phi(t) \in B_{3r/2}(0)$.

It remains to check the condition (5.43). Recall that $\widetilde{D}_h(u, v) \le (c_*/C_*)^2 \widetilde{D}_h(u, \partial B_{4\rho r}(u))$ and the $\widetilde{D}_h$-geodesic from u to v is contained in U. Since $\phi \equiv K_f$ on U and $\phi \le K_f$ on $B_{4\rho r}(u)$, it follows that

$$\begin{aligned} \widetilde{D}_{h-\phi}(u, v) &\le (c_*/C_*)^2 \widetilde{D}_{h-\phi}(u, \partial B_{4\rho r}(u)) \\ &\le (c_*/C_*)^2 \widetilde{D}_{h-\phi}\left(P^\phi(s), \partial B_{3r}(0)\right). \end{aligned}$$

By (5.45) and the triangle inequality,

$$\begin{aligned} \widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) &\le (1 + 2\eta)\widetilde{D}_{h-\phi}(u, v) \\ &\le (1 + 2\eta)(c_*/C_*)^2 \widetilde{D}_{h-\phi}\left(P^\phi(s), \partial B_{3r}(0)\right), \end{aligned}$$

which is bounded above by the right side of (5.43) by the definition (5.15) of η. □

Proof of Lemma 5.11 Let s and t be as in Lemma 5.15. By that lemma, it remains only to check that

$$\widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \le c_2' D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right).$$

By (5.45) and the definitions of c_* and C_*,

$$\begin{aligned} D_{h-\phi}\left(P^\phi(s), u; U\right) \le c_*^{-1} C_* \eta D_{h-\phi}(u, v) \quad \text{and} \quad & D_{h-\phi}\left(P^\phi(t), v; U\right) \\ &\le c_*^{-1} C_* \eta D_{h-\phi}(u, v). \end{aligned} \tag{5.46}$$

By the triangle inequality, (5.46) implies that

$$\begin{aligned} D_{h-\phi}(u, v) &\le D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) + D_{h-\phi}\left(P^\phi(s), u\right) + D_{h-\phi}\left(P^\phi(t), v\right) \\ &\le D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) + 2c_*^{-1} C_* \eta D_{h-\phi}(u, v) \end{aligned}$$

which re-arranges to give

$$D_{h-\phi}(u,v) \le \left(1 - 2c_*^{-1} C_* \eta\right)^{-1} D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right). \tag{5.47}$$

Recall that $\phi \le K_f$ on $\mathbb{C} \setminus \mathcal{W}$ and by the last condition in (5.20) we have $D_h(u,v) \le D_h(u, \mathcal{W})$. It follows from this that each $D_{h-\phi}$-geodesic from u to v is disjoint from $\mathcal{W}$ and $D_{h-\phi}(u,v) \ge e^{-\xi K_f} D_h(u,v)$. By combining this with (5.44) and condition A in the definition of E_r, we get

$$\widetilde{D}_{h-\phi}(u,v) \le c_1' D_{h-\phi}(u,v). \tag{5.48}$$

By the triangle inequality,

$$\begin{aligned}
&\widetilde{D}_{h-\phi}\left(P^\phi(s), P^\phi(t); U\right) \\
&\quad \le \widetilde{D}_{h-\phi}(u,v;U) + \widetilde{D}_{h-\phi}\left(P^\phi(s), u; U\right) + \widetilde{D}_{h-\phi}\left(v, P^\phi(t); U\right) \\
&\quad \le (1+2\eta)\widetilde{D}_{h-\phi}(u,v) \quad \text{(by (5.44) and (5.45))} \\
&\quad \le c_1'(1+2\eta) D_{h-\phi}(u,v) \quad \text{(by (5.48))} \\
&\quad \le \frac{c_1'(1+2\eta)}{1-2c_*^{-1}C_*\eta} D_{h-\phi}\left(P^\phi(s), P^\phi(t)\right) \quad \text{(by (5.47))} \\
&\quad \le c_2' D_{h-\phi}(P^\phi(s), P^\phi(t)) \quad \text{(by the definition (5.15) of } \eta).
\end{aligned} \tag{5.49}$$

□

6 Proof of Theorem 1.9

Assume we are in the setting of Theorem 1.9 and let h be a whole-plane GFF. Also recall the definitions of the optimal bi-Lipschitz constants c_* and C_* from (1.21) and the events $\overline{G}_r(C', \beta)$ and $\underline{G}_r(c', \beta)$ from (3.2) and (3.3). We want to show that $c_* = C_*$. To do this we will assume that $c_* < C_*$ and derive a contradiction. The following proposition will be used in conjunction with Proposition 3.3 to tell us that there are many scales for which the following is true: the pairs (u,v) such that $\widetilde{D}_h(u,v)/D_h(u,v)$ is close to C_* are very sparse.

Proposition 6.1 *Assume that $c_* < C_*$. Then there exists $c'' > c_*$, depending only on the values of c_* and C_*, such that the following is true. If $\beta \in (0,1)$ and $\mathbb{r} > 0$ are such that $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \ge \beta$, then for every choice of $\overline{\beta} \in (0,1)$, one has*

$$\lim_{\delta \to 0} \mathbb{P}\left[\overline{G}_{\mathbb{r}}(C_* - \delta, \overline{\beta})\right] = 0 \tag{6.1}$$

at a rate depending only on $\beta, \overline{\beta}$ (not on $\mathbb{r}$).

Proof Assume $c_* < C_*$. Let ν_* be as in Theorem 4.2 and fix parameters $0 < \mu < \nu \leq \nu_*$ and $c_* < c_1' < c_2' < C_*$ chosen in a manner depending only on c_* and C_*. The proof follows the strategy outlined in the "main idea" part of the outline in Sect. 1.5. Theorem 4.2 and Proposition 4.3 will allow us to show that if $q > 0$ is fixed, then with probability tending to 1 as $\varepsilon \to 0$, the following is true. For every pair of points $\mathbb{z}, \mathbb{w} \in (\varepsilon^q \mathbb{r}\mathbb{Z}^2) \cap B_{\mathbb{r}}(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta}\mathbb{r}$, the D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ has to hit a pair of points $P(s), P(t)$ such that $|P(s) - P(t)| \geq \text{const} \times \varepsilon^{1+\nu}\mathbb{r}$ and $\widetilde{D}_h(P(s), P(t)) \leq c_2' D_h(P(s), P(t))$. This allows us to show that $\widetilde{D}_h(\mathbb{z}, \mathbb{w})/D_h(\mathbb{z}, \mathbb{w})$ is bounded above by C_* minus a γ-dependent power of ε for all such pairs of points $\mathbb{z}, \mathbb{w}$. We can then use Hölder continuity to get the same statement for all pairs of points $\mathbb{z}, \mathbb{w} \in B_{\mathbb{r}}(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta}\mathbb{r}$ simultaneously. Choosing ε to be an appropriate γ-dependent power of δ then gives (6.1).

Step 1: setup and regularity events Let $c'' = c''(c_1', \mu, \nu)$, $b = b(\mu, \nu) \in (0, 1)$, and $\rho = \rho(\mu, \nu) \in (0, 1)$ be as in Proposition 4.3 with the above choice of μ, ν, c_1', c_2'. Also fix $q > 0$ to be chosen later in a manner depending on $\beta, \overline{\beta}$.

By Theorem 4.2 applied to the objects of Proposition 4.3 and with the above choice of q, $\rho^{-1}\mathbb{r}$ in place of $\mathbb{r}$, $U = B_2(0)$, and $\ell = \rho\overline{\beta}$, we get the following. If $\mathbb{r} > 0$ is such that $\mathbb{P}[\underline{G}_{\mathbb{r}}(c'', \beta)] \geq \beta$, then it holds with probability tending to 1 as $\varepsilon \to 0$, at a rate depending only on $q, \overline{\beta}, \beta, c_1', c_2', \mu, \nu$, that the following is true. Let $\mathbb{z}, \mathbb{w} \in \left(\varepsilon^q \rho^{-1}\mathbb{r}\mathbb{Z}^2\right) \cap B_{2\mathbb{r}}(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta}\mathbb{r}$ and let $P = P^{\mathbb{z},\mathbb{w}}$ be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. Then there exists times $0 < s < t < |P|$ such that

$$|P(s) - P(t)| \geq b\varepsilon^{1+\nu}\rho^{-1}\mathbb{r} \quad \text{and} \quad \widetilde{D}_h\left(P(s), P(t)\right) \leq c_2' D_h\left(P(s), P(t)\right) \tag{6.2}$$

(in particular, the times s, t arise from a radius $r \in [\varepsilon^{1+\nu}\rho^{-1}\mathbb{r}, \varepsilon\rho^{-1}\mathbb{r}]$ and a point $z \in \mathbb{C}$ for which $\mathfrak{E}_r^{\mathbb{z},\mathbb{w}}(z)$ occurs). Henceforth assume that (6.2) holds for every $\mathbb{z}, \mathbb{w} \in \left(\varepsilon^q \rho^{-1}\mathbb{r}\mathbb{Z}^2\right) \cap B_{2\mathbb{r}}(0)$ with $|\mathbb{z} - \mathbb{w}| \geq \overline{\beta}\mathbb{r}$.

Fix $\chi \in (0, \xi(Q-2))$ and $\chi' > \xi(Q+2)$, as in Lemma 2.8. By Axiom V (tightness across scales), for each $p \in (0, 1)$ we can find a bounded open set $U \subset \mathbb{C}$ which contains $B_2(0)$ such that $\mathbb{P}[\sup_{u,v \in B_{2\mathbb{r}}(0)} D_h(u, v) < D_h(B_{2\mathbb{r}}(0), \mathbb{r}\partial U)] \geq p$ for every $\mathbb{r} > 0$. On the event of the preceding sentence, every D_h-geodesic between two points of $B_{2\mathbb{r}}(0)$ is contained in $\mathbb{r}U$. By applying Lemma 2.8 with $K = \overline{U}$ and then sending $p \to 1$, we get that with probability tending to 1 as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, for any two points $z, w \in \mathbb{C}$ with $|z - w| \leq (\varepsilon^q \vee (b\varepsilon^{1+\nu}))\rho^{-1}\mathbb{r}$ which are either contained in $B_{2\mathbb{r}}(0)$ or which lie on a D_h-geodesic between two points of $B_{2\mathbb{r}}(0)$,

$$\left|\frac{z-w}{\mathbb{r}}\right|^{\chi'} \le \mathfrak{c}_{\mathbb{r}}^{-1} e^{-\xi h_{\mathbb{r}}(0)} D_h(z,w) \le \left|\frac{z-w}{\mathbb{r}}\right|^{\chi}. \tag{6.3}$$

Henceforth assume that this is the case.

Step 2: bounding $\widetilde{D}_h(\mathbb{z},\mathbb{w})/D_h(\mathbb{z},\mathbb{w})$ for points in a fine mesh By (6.2) and (6.3), the times s and t from (6.2) satisfy

$$t-s = D_h(P(s),P(t)) \ge (b/\rho)^{\chi'} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \varepsilon^{(1+\nu)\chi'}. \tag{6.4}$$

By the definition (1.21) of C_*, the $\widetilde{D}_h$-lengths of the segments $P|_{[0,s]}$ and $P|_{[t,|P|]}$ are bounded above by $C_* s$ and $C_*(|P|-t)$, respectively. Therefore, for each $\mathbb{z},\mathbb{w} \in \left(\varepsilon^q \rho^{-1} \mathbb{r}\mathbb{Z}^2\right) \cap B_{2\mathbb{r}}(0)$ with $|\mathbb{z}-\mathbb{w}| \ge \overline{\beta}\mathbb{r}$,

$$\begin{aligned}
\widetilde{D}_h(\mathbb{z},\mathbb{w}) &\le C_*(|P|-t+s) + \widetilde{D}_h(P(s),P(t)) \\
&\le C_*(|P|-t+s) + c_2'(t-s) \quad (\text{by } (6.2)) \\
&\le C_* D_h(\mathbb{z},\mathbb{w}) - (C_*-c_2')(t-s) \\
&\le C_* D_h(\mathbb{z},\mathbb{w}) - (C_*-c_2')(b/\rho)^{\chi'} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \varepsilon^{(1+\nu)\chi'} \quad (\text{by } (6.4)).
\end{aligned} \tag{6.5}$$

Step 3: transferring from points in a fine mesh to general points If $z,w \in B_{\mathbb{r}}(0)$ with $|z-w| \ge \overline{\beta}\mathbb{r}$, then we can find $\mathbb{z},\mathbb{w} \in \left(\varepsilon^q \mathbb{r}\mathbb{Z}^2\right) \cap B_{2\mathbb{r}}(0)$ such that $|\mathbb{z}-\mathbb{w}| \ge \overline{\beta}\mathbb{r}$ and $\max\{|z-\mathbb{z}|,|w-\mathbb{w}|\} \le 2\varepsilon^q \rho^{-1}\mathbb{r}$. By (6.3) and the triangle inequality,

$$|D_h(\mathbb{z},\mathbb{w}) - D_h(z,w)| \le 2^{2+\chi} \rho^{-\chi} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \varepsilon^{q\chi}, \tag{6.6}$$

and the same is true with $\widetilde{D}_h$ in place of D_h. If we choose $q > \chi'(1+\nu)/\chi$, then (6.6) and (6.5) together imply that for each $z,w \in B_{\mathbb{r}}(0)$ with $|z-w| \ge \overline{\beta}\mathbb{r}$ and each small enough ε,

$$\begin{aligned}
&\widetilde{D}_h(z,w) \le C_* D_h(z,w) - a\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \varepsilon^{(1+\nu)\chi'}, \\
&\forall z,w \in B_{\mathbb{r}}(0) \quad \text{s.t.} \quad |z-w| \ge \overline{\beta}\mathbb{r}.
\end{aligned} \tag{6.7}$$

where $a>0$ is a constant depending only on $q, \overline{\beta}, c_1', c_2', \mu, \nu$.

Step 4: choosing ε By Axiom V (tightness across scales), it holds with probability tending to 1 as $\varepsilon \to 0$, uniformly over all $\mathbb{r} > 0$, that $D_h(z,w) \le \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \varepsilon^{-\chi'}$ for each $z,w \in B_{\mathbb{r}}(0)$. If this is the case then

$a\mathfrak{c}_{\mathbbm{r}} e^{\xi h_{\mathbbm{r}}(0)} \varepsilon^{(1+\nu)\chi'} \geq a\varepsilon^{(2+\nu)\chi'} D_h(z,w)$. Hence (6.7) implies that with probability tending to 1 as $\varepsilon \to 0$, at a rate depending only on $q, \overline{\beta}, c', \mu, \nu$,

$$\widetilde{D}_h(z,w) \leq \left(C_* - a\varepsilon^{(2+\nu)\chi'}\right) D_h(z,w),$$
$$\forall z, w \in B_{\mathbbm{r}}(0) \quad \text{s.t.} \quad |z-w| \geq \overline{\beta}\mathbbm{r}. \tag{6.8}$$

Recalling the definition (3.2) of $\overline{G}_{\mathbbm{r}}(C_* - \delta, \overline{\beta})$, we can choose ε so that $a\varepsilon^{(2+\nu)\chi'} = \delta$ to get the proposition statement. □

Proof of Theorem 1.9 Let D and $\widetilde{D}$ be as in Theorem 1.9, let h be a whole-plane GFF, and define the maximal and minimal ratios c_* and C_* as in (1.21). We claim that $c_* = C_*$, i.e., a.s. $\widetilde{D}_h = c_* D_h$. This gives the theorem statement in the case of a whole-plane GFF, which in turn implies the theorem statement for a whole-plane GFF plus a continuous function due to Axiom III (Weyl scaling).

It remains to prove that $c_* = C_*$. By Proposition 3.2 applied with $C' = C_* - \delta$, there exists $\overline{\beta} = \overline{\beta}(\mu,\nu) \in (0,1)$ and $\overline{p} = \overline{p}(\mu,\nu) \in (0,1)$ with the following property. For each $\delta \in (0,1)$, there exists $\varepsilon_0 = \varepsilon_0(\delta,\mu,\nu) > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, there are at least $\mu \log_8 \varepsilon^{-1}$ values of $\mathbbm{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{8^{-k} : k \in \mathbb{N}\}$ for which

$$\mathbb{P}\left[\overline{G}_{\mathbbm{r}}(C_* - \delta, \overline{\beta})\right] \geq \overline{p}. \tag{6.9}$$

We emphasize that $\overline{\beta}$ and $\overline{p}$ do not depend on δ.

We now assume by way of contradiction that $c_* < C_*$ and show that this assumption is incompatible with the conclusion of the preceding paragraph. To this end, let $c'' \in (c_*, C_*)$ be as in Proposition 6.1, so that c'' depends only on the choice of metrics D and $\widetilde{D}$. Proposition 3.3 applied with c'' in place of c' shows that there exists $\underline{\beta} = \underline{\beta}(\mu,\nu) \in (0,1)$, $\underline{p} = \underline{p}(\mu,\nu) \in (0,1)$, and $\varepsilon_1 = \varepsilon_1(\mu,\nu) > 0$ such that for each $\varepsilon \in (0,\varepsilon_1]$, there are at least $\mu \log_8 \varepsilon^{-1}$ values of $\mathbbm{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{8^{-k} : k \in \mathbb{N}\}$ for which $\mathbb{P}[\underline{G}_{\mathbbm{r}}(c'', \underline{\beta})] \geq \underline{p}$.

Proposition 6.1 applied with $\beta = \underline{\beta} \wedge \underline{p}$ therefore implies that there exists $\delta = \delta(\mu,\nu) \in (0,1)$ such that for each $\varepsilon \in (0,\varepsilon_1]$, there are at least $\mu \log_8 \varepsilon^{-1}$ values of $\mathbbm{r} \in [\varepsilon^{1+\nu}, \varepsilon] \cap \{8^{-k} : k \in \mathbb{N}\}$ for which $\mathbb{P}\left[\overline{G}_{\mathbbm{r}}(C_* - \delta, \overline{\beta})\right] \leq \overline{p}/2$. If we take $\mu > \nu/2$, then this is incompatible with (6.9) whenever $\varepsilon \in (0, \varepsilon_0 \wedge \varepsilon_1)$, so we have obtained the desired contradiction. □

7 Open problems

Dimension calculations

An important remaining question concerning the LQG metric is the following.

Problem 7.1 (Hausdorff dimension of γ-LQG) Compute the exponent d_γ appearing in (1.5), which is the Hausdorff dimension of $\mathbb{C}$ with respect to the γ-LQG metric (this is proven in [43]).

Since $\xi = \gamma/d_\gamma$ and $Q = 2/\gamma + \gamma/2$, Problem 7.1 is equivalent to determining the relationship between these two parameters. The only case in which d_γ is known is when $\gamma = \sqrt{8/3}$, in which case $d_{\sqrt{8/3}} = 4$. Due to existing results in the literature, d_γ can equivalently be defined in a large number of other equivalent ways, e.g., the following.

1. For a large class of infinite-volume random planar maps in the γ-LQG universality class, the number of vertices in the graph distance ball of radius r centered at the root vertex is of order $r^{d_\gamma + o_r(1)}$ [20, Theorem 1.6] and the graph distance traveled by a simple random walk started from the root vertex and run for n steps is of order $n^{1/d_\gamma + o_n(1)}$ [31,35].
2. For fixed distinct points $z, w \in \mathbb{C}$, the Liouville heat kernel (as constructed in [45]) satisfies $\mathsf{p}_t^\gamma(z,w) = \exp\left(-t^{-\frac{1}{d_\gamma - 1} + o_t(1)}\right)$ as $t \to 0$ [30, Theorem 1.1].
3. The optimal Hölder exponent for the γ-LQG metric w.r.t. the Euclidean metric is $\frac{\gamma}{d_\gamma}(Q-2)$ and the optimal Hölder exponent for the Euclidean metric w.r.t. the γ-LQG metric is $\frac{d_\gamma}{\gamma}(Q+2)^{-1}$ [18, Theorem 1.7].

The best-known physics prediction for the value of d_γ is the Watabiki prediction [85],

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2}. \tag{7.1}$$

However, this prediction is known to be false at least for small values of γ due to the results of Ding-Goswami [19]. See [20,42] for rigorous upper and lower bounds for d_γ as well as additional discussion about various possibilities for its value. In addition to d_γ, there are a number of other interesting dimensions related to γ-LQG which have not yet been computed, for example the following.

Problem 7.2 (Geodesic dimension) Compute the Euclidean Hausdorff dimension of the γ-LQG geodesic between two typical points of $\mathbb{C}$.

Problem 7.3 (Ball boundary dimension) Compute the γ-LQG Hausdorff dimension and the Euclidean Hausdorff dimension of the boundary of a filled γ-LQG metric ball $\mathcal{B}_s^\bullet(0; D_h)$.

In the setting of Problem 7.2, the γ-LQG Hausdorff dimension of a γ-LQG geodesic is trivially equal to 1. The Euclidean dimensions of γ-LQG geodesics and filled metric ball boundaries are unknown even for $\gamma = \sqrt{8/3}$ and there are not even any conjectures as to their values. The $\sqrt{8/3}$-LQG dimension of the outer boundary of a filled $\sqrt{8/3}$-LQG metric ball is 2 [64], but this quantity is not known (even heuristically) for any other value of γ. See [43] for upper bounds for the Euclidean Hausdorff dimension of a γ-LQG geodesic and for the outer boundary of a filled $\sqrt{8/3}$-LQG metric ball.

Currently, no explicit lower bounds for any of these quantities are known, although we expect it is not hard to show that they are strictly larger than 1; c.f. [29].

Another natural random fractal associated with the LQG metric is the boundary of a (non-filled) LQG metric ball (note that this boundary is typically not connected). It is shown in [44,48] that a.s. the Hausdorff dimension of the LQG metric ball boundary w.r.t. the Euclidean (resp. LQG) metric is $2 - \xi Q + \xi^2/2$ (resp. $d_\gamma - 1$). It is also shown in [44] that a.s. the Hausdorff dimension of the boundary of a filled metric ball w.r.t. the Euclidean (resp. LQG) metric is strictly smaller than this quantity.

The "quantum dimension" part of Problem 7.3 is closely related to the following question.

Problem 7.4 (γ-LQG boundary length of metric balls) Is there a natural LQG length measure on the boundary of a filled γ-LQG metric ball?

In the case when $\gamma = \sqrt{8/3}$, for $s > 0$ the field $h|_{\mathbb{C} \setminus \mathcal{B}_s(0; D_h)}$ locally looks like a free-boundary GFF near $\partial \mathcal{B}_s(0; D_h)$. This allows one to define the γ-LQG boundary length measure on $\partial \mathcal{B}_s(0; D_h)$ in the manner of [27, Section 6]. Alternatively, the length measure on $\partial \mathcal{B}_s(0; D_h)$ can equivalently be constructed using Brownian surface theory; see [58,63]. For general $\gamma \in (0, 2)$, it is not expected that $h|_{\mathbb{C} \setminus \mathcal{B}_s(0; D_h)}$ locally looks like a free-boundary GFF near $\partial \mathcal{B}_s(0; D_h)$. Indeed, if this were the case then the heuristic argument in [69, Section 3.3] would imply that the dimension of γ-LQG is given by Watabiki's prediction (7.1), which we know is false, at least for small γ, by the results of [19]. Hence new ideas are required to construct a natural length measure on $\partial \mathcal{B}_s(0; D_h)$ in this case.

Discrete approximations

Another interesting open problem is to connect the γ-LQG metric to its discrete counterparts.

Problem 7.5 (Scaling limit of random planar maps) Prove Conjecture 1.7, which asserts that random planar maps, equipped with their graph distance, converge to the γ-LQG surface, equipped with the γ-LQG metric, w.r.t. the Gromov–Hausdorff topology.

One possible approach to Problem 7.5 is to first prove a scaling limit result for the so-called *mated-CRT maps*, as studied, e.g., in [33,34,40] using their direct connection to Liouville quantum gravity. One could then try to transfer to other random planar map models by improving on the strong coupling techniques used in [33], which currently only give estimates for distances up to polylogarithmic multiplicative errors. We emphasize, however, that both of these steps are highly non-trivial and are likely to require substantial new ideas. Another possible approach would be to find some sort of "combinatorial miracle" which allows one to analyze distances in weighted random planar maps directly (analogous to the Schaeffer bijection [7,13,78] for uniform random planar maps).

A likely easier scaling limit problem is to show universality of the γ-LQG metric across different approximation schemes. One of the most natural approximation schemes is *Liouville graph distance (LGD)*, whereby the distance between two points $z, w \in \mathbb{C}$ is defined to be the minimal number of Euclidean balls of γ-LQG mass ε whose union contains a path from z to w.

Problem 7.6 (Other approximation schemes) Show that the LGD metrics, appropriately re-scaled, converge in law to the γ-LQG metric as $\varepsilon \to 0$.

We expect that the difficulties involved in solving Problem 7.6 are similar to the difficulties involved in showing that the mated-CRT map converges to γ-LQG in the metric sense, due to the SLE/LQG representation of the mated-CRT map (see [33,40]).

It is shown in [14] that LGD, re-scaled by the median distance across a square, is tight and each subsequential limit induces the Euclidean topology. We expect that it is not hard to check that these subsequential limits satisfy Axioms I, II, and IV in the definition of the γ-LQG metric (the latter is just a consequence of the coordinate change formula for the LQG area measure [27, Proposition 2.1]). One can also obtain a much weaker version of Weyl scaling analogous to the "tightness across scales" condition (Axiom V) used in our definition of a weak γ-LQG metric, where one requires that the metrics obtained by adding different constants to the field, then re-scaling appropriately, are tight.

Hence one possible approach to Problem 7.6 is to adapt the arguments of this paper and its predecessors to the case when we know that our metric satisfies the coordinate change formula for translations and scalings, but we do not know that it satisfies Weyl scaling. However, our arguments are in some ways optimized to work for subsequential limits of LFPP, so there may also be an entirely different argument which is more appropriate for subsequential limits of LGD.

Theorem 1.1 says that the LFPP metrics converge in probability, unlike the case of various approximations of the LQG measure which are known to converge a.s. [27,76,84].

Problem 7.7 (Almost sure convergence of LFPP) Can the convergence $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon \to D_h$ in Theorem 1.1 be improved from convergence in probability to a.s. convergence?

Metric space structure versus quantum surface structure

In [65], it is shown that a $\sqrt{8/3}$-LQG surface is a.s. determined by its structure as a metric measure space, i.e., the metric measure space $(\mathbb{C}, \mu_h, D_h)$ a.s. determines its embedding into $\mathbb{C}$ and the associated GFF h (modulo conformal automorphisms). Our next problem asks for an extension of this result to the case when $\gamma \in (0, 2)$.

Problem 7.8 (Metric measure space structure determines the field) Show that the field h is a.s. determined (modulo rotation and scaling) by the pointed γ-LQG metric measure space $(\mathbb{C}, 0, \mu_h, D_h)$.

Likely the easiest approach to Problem 7.8 is to adapt the arguments of [41], which gives for $\gamma = \sqrt{8/3}$ an explicit way of re-constructing h from $(\mathbb{C}, 0, \mu_h, D_h)$ using the adjacency graph of a fine mesh of Poisson-Voronoi cells. The arguments of [41] are not very specific to the case when $\gamma = \sqrt{8/3}$. The main missing ingredient to extend these arguments to general values of γ is the following estimate of independent interest.

Problem 7.9 (Concentration of areas of LQG metric balls) Show that the γ-LQG area of a γ-LQG metric ball has superpolynomial concentration, i.e., show that for $C > 1$,

$$\mathbb{P}\left[C^{-1} \leq \mu_h\left(\mathcal{B}_1(0; D_h)\right) \leq C\right] = 1 - O_C(C^{-p}), \quad \forall p > 0. \tag{7.2}$$

Problem 7.9 in the case when $\gamma = \sqrt{8/3}$ follows from known estimates for the Brownian map; see [56, Corollary 6.2] and [41, Section 4.3].

Update Problems 7.8 and 7.9 are solved in [1].

It is shown in [11] that the LQG measure a.s. determines the GFF. It is also natural to try to recover the LQG measure (and thereby the GFF) from the LQG metric.

Problem 7.10 Does the LQG metric a.s. determine the LQG measure? More concretely, can the LQG measure be recovered as some sort of Minkowski content measure w.r.t. the LQG metric?

In this paper, we gave a characterization of the γ-LQG metric in terms of its coupling with the GFF. In light of Problem 7.8, it is natural to ask if there is also a characterization solely in terms of the metric space structure, which does *not* require reference to the GFF. Such a characterization of the Brownian map (equivalently, the $\sqrt{8/3}$-LQG sphere) is proven in [71]. A purely metric characterization of γ-LQG could potentially play an important role in a solution to Problem 7.5.

Problem 7.11 (Metric space characterization) Is there a characterization of $(\mathbb{C}, D_h)$ as a metric space (or of $(\mathbb{C}, \mu_h, D_h)$ as a metric measure space), without reference to the GFF and the embedding of this metric space into $\mathbb{C}$?

It is likely that the most natural setting to consider in Problem 7.11 is the one where h the field corresponding to a quantum cone or quantum sphere (as defined in [23]) rather than a whole-plane GFF.

Additional properties of the LQG metric

The construction of the $\sqrt{8/3}$-LQG metric in [64,65,72] yields many special properties of the metric in this case which are not known (and in many cases not expected to hold) for general $\gamma \in (0, 2)$. For example, one has $d_{\sqrt{8/3}} = 4$. Moreover, in the case when h is the GFF associated with a quantum sphere or $\sqrt{8/3}$-quantum wedge, the quantum surfaces obtained by restricting h to the complementary connected components of a $\sqrt{8/3}$-LQG metric ball are conditionally independent quantum disks given their boundary lengths. Many further properties can be obtained using the equivalence of $\sqrt{8/3}$-LQG surfaces and Brownian surfaces. However, there is nothing obviously special about $\gamma = \sqrt{8/3}$ from either of the definitions of the LQG metric given in this paper (the limit of LFPP or the axiomatic definition).

Problem 7.12 Can one prove that $d_{\sqrt{8/3}} = 4$, the independence properties for complementary connected components of a $\sqrt{8/3}$-LQG metric ball, or any other special property of the $\sqrt{8/3}$-LQG metric directly from the LFPP definition or the axiomatic definition?

There has been a recent proliferation of exact formulas for quantities related to the γ-LQG area and boundary length measures for general $\gamma \in (0, 2)$, proven using ideas from conformal field theory: see, e.g., [54,75,77]. In the special case when $\gamma = \sqrt{8/3}$, exact formulas for various quantities associated with the $\sqrt{8/3}$-LQG metric can be obtained using its connection to the Brownian surfaces. Exact formulas for the γ-LQG metric, if they can be found, could be very useful in attempts to solve most of the other problems listed above.

Problem 7.13 (Exact formulas) Are there exact formulas for any objects related to the γ-LQG metric for general $\gamma \in (0, 2)$?

Problem 7.14 (Topology of geodesics) For a general value of $\gamma \in (0,2)$, what is the maximal possible number of γ-LQG geodesics joining two points in $\mathbb{C}$? Is this number finite, and, if so, does it depend on γ? More generally, can one prove results about the possible topologies of the set of γ-LQG geodesics joining two points in $\mathbb{C}$ analogous to the results for the Brownian map in [3]?

Update This problem is solved for $\gamma = \sqrt{8/3}$ in [61] using Brownian map based techniques. Substantial progress on Problem 7.14 is made in [47], where it is shown that the results about geodesic networks from [3] extend verbatim to the case of general $\gamma \in (0,2)$ and that the maximal number of LQG geodesics joining any two points is a.s. finite. It is also conjectured in [47] that the maximal number of geodesics is 9, regardless of the value of γ.

Liouville Brownian motion [8,46] is the natural "quantum time" parameterization of Brownian motion on an LQG surface. If we condition Liouville Brownian motion to travel a macroscopic distance (e.g., from the origin to the unit circle) in a short amount of time, then it is natural to expect that it would roughly follow a path of minimal LQG length.

Problem 7.15 (Liouville Brownian motion and LQG geodesics) Does Liouville Brownian motion conditioned to travel a macroscopic (Euclidean or quantum) distance in a short amount of time approximate an LQG geodesic?

There is a one-parameter family of infinite-volume γ-LQG surfaces with boundary called *quantum wedges*, which can be indexed by the *weight* parameter $\mathfrak{w} > 0$. See [23] for details. In [23], building on [81], it is shown that one can conformally weld together a weight-$\mathfrak{w}_1$ quantum wedge and a weight-$\mathfrak{w}_2$ quantum wedge according to the quantum length measure along their boundaries to get a weight-$\mathfrak{w}_1 + \mathfrak{w}_2$ quantum wedge decorated by an $\mathrm{SLE}_\kappa(\mathfrak{w}_1 - 2; \mathfrak{w}_2 - 2)$ curve which corresponds to the gluing interface. In [39], it is shown that in the special case when $\gamma = \sqrt{8/3}$, this conformal welding is compatible with the $\sqrt{8/3}$-LQG metric in the following sense: the weight-$(\mathfrak{w}_1 + \mathfrak{w}_2)$ quantum wedge, equipped with its $\sqrt{8/3}$-LQG metric, is the metric space quotient of the weight-$\mathfrak{w}_1$ and weight-$\mathfrak{w}_2$ quantum wedges, equipped with their $\sqrt{8/3}$-LQG metrics, under the same equivalence relation used to define the conformal welding.

Problem 7.16 (Metric gluing of γ-LQG surfaces) Prove metric gluing statements for quantum wedges analogous to the ones in [39] for general $\gamma \in (0,2)$.

The main missing ingredient needed to solve Problem 7.16 is suitable estimates for distances between points of $\partial\mathbb{D}$ with respect to the γ-LQG metric induced by a free-boundary GFF on $\mathbb{D}$ (or a variant thereof, like the quantum disk). For $\gamma = \sqrt{8/3}$, the needed estimates are proven in [39, Section 3.2] using results for the Brownian disk.

Extensions of the theory

Throughout this paper, we have neglected the critical case when $\gamma = 2$.

Problem 7.17 (Critical LQG metric) Construct a metric on γ-LQG when $\gamma = 2$.

See [25,26] for a construction of the γ-LQG measure for $\gamma = 2$. One possible approach to Problem 7.17 is to try to take a limit of the γ-LQG metrics as γ increases to 2 (it is shown that the 2-LQG measure is the $\gamma \nearrow 2$ limit of the γ-LQG measures, appropriately renormalized, in [5]). Another (likely more involved) possibility is to adapt the arguments of this paper and its predecessors [18,36,38] to the critical case, corresponding to LFPP with parameter $\xi = 2/d_2$. A major difficulty in the critical case is that the 2-LQG metric is not expected to be Hölder continuous w.r.t. the Euclidean metric (indeed, the optimal Hölder exponent from [18, Theorem 1.7] converges to zero as $\gamma \to 2^-$), so more refined estimates for the continuity of the metric and for LFPP are likely to be required.

Recall that our metric for $\gamma \in (0, 2)$ is constructed as the limit of LFPP with parameter $\xi = \gamma/d_\gamma$. Extending further, it is natural to ask what happens when $\xi > 2/d_2$ (it is shown in [20, Proposition 1.7] that $\gamma \mapsto \gamma/d_\gamma$ is increasing, so $\gamma/d_\gamma < 2/d_2$). Very recently, it was shown in [21] that LFPP is tight w.r.t. the topology on lower semicontinuous functions for all $\xi > 0$. For $\xi > 2/d_2$ every possible subsequential limit is a metric on $\mathbb{C}$ which does *not* induce the Euclidean topology. Rather, there is an uncountable, dense, fractal set of "singular points" whose distance to every other point is infinite. These singular points arise from the thick points of the GFF [49].

Problem 7.18 (LFPP with $\xi > 2/d_2$) Show that LFPP with parameter $\xi > 2/d_2$ converges in law to a limiting metric w.r.t. the topology of [21].

This metric of Problem 7.18 should be related to Liouville quantum gravity with central charge $\mathbf{c} \in (1, 25)$. Note that the central charge associated with γ-LQG for $\gamma \in (0, 2]$ is $\mathbf{c} = 25 - 6(2/\gamma + \gamma/2)^2 \in (-\infty, 1]$. We refer to [21,32] and the references therein for more on LQG with $\mathbf{c} \in (1, 25)$.

The γ-LQG measure is a special case of a more general theory of random measures called *Gaussian multiplicative chaos* (GMC) [51,76], which studies limits of regularized versions of "$e^{\gamma X}\, dz$" for certain Gaussian random distributions X. Here, X is a random distribution on $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and dz denotes Lebesgue measure on $\mathbb{R}^n$.

Problem 7.19 (More general random metrics) Is there a more general theory of random metrics associated with log-correlated random Gaussian distributions analogous to GMC? In particular, can one construct metrics with similar properties to the γ-LQG metric in higher dimensions?

Some of the arguments in the construction of the LQG metric, in this paper as well as [16,18,36,38] are specific to the two-dimensional case. The following seem to be the places where the use of two-dimensionality is the most fundamental.

- The construction of the LQG metric makes extensive use of the Markov property of the GFF: for an open set $U \subset \mathbb{C}$, $h|_U$ decomposes as a zero-boundary GFF in U plus an independent random harmonic function on U. This property is not satisfied for log-correlated fields in dimension ≥ 3, see, e.g., [24] (note that the GFF is only log-correlated in dimension 2).
- The proof of tightness in [16], as well as several proofs in [18], use RSW-type arguments which are based on the fact that one can force two paths to intersect each other in dimension 2.
- The proof of confluence in [36] is based on a decomposition of the boundary of a filled LQG metric ball into arcs of topological dimension 1, together with an iterative argument where one "kills off" all but one of the arcs by preventing LQG geodesics from passing through them. In higher dimensions, the boundary of an LQG metric ball cannot be decomposed into sets of dimension 1. In fact, it is plausible that confluence fails in higher dimensions since there is more "room" for geodesics to move around.

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