Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem

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Abstract It is well-known that quantitative, scale invariant absolute continuity (more precisely, the weak-A_∞ property) of harmonic measure with respect to surface measure, on the boundary of an open set $\Omega \subset \mathbb{R}^{n+1}$ withAhlfors–David

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regular boundary, is equivalent to the solvability of the Dirichlet problem in Ω, with data in $L^p(\partial\Omega)$ for some $p < \infty$. In this paper, we give a geometric characterization of the weak-A_∞ property, of harmonic measure, and hence of solvability of the L^p Dirichlet problem for some finite p. This characterization is obtained under background hypotheses (an interior corkscrew condition, along with Ahlfors–David regularity of the boundary) that are natural, and in a certain sense optimal: we provide counter-examples in the absence of either of them (or even one of the two, upper or lower, Ahlfors–David bounds); moreover, the examples show that the upper and lower Ahlfors–David bounds are each quantitatively sharp.

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1 Introduction

A classical criterion of Wiener characterizes the domains in which one can solve the Dirichlet problem for Laplace's equation with continuous boundary data, and with continuity of the solution up to the boundary. In this paper, we address the analogous issue in the case of singular data. To be more precise, the present work provides a purely geometric characterization of the open sets for which L^p solvability holds, for some $p < \infty$, and with non-tangential convergence to the data a.e., thus allowing for singular boundary data. We establish this characterization in the presence of background hypotheses (an interior corkscrew condition [see Definition 2.4 below], and Ahlfors–David regularity of the boundary [Definition 2.1]) that are in the nature of best possible, in the sense that there are counter-examples in the absence of either of them (or of even one of the two, upper or lower, Ahlfors–David bounds); moreover, the examples show that the upper and lower Ahlfors–David bounds are each quantitatively sharp (see the discussion following Theorem 1.2, as well as "Appendix A", for more details).

Solvability of the L^p Dirichlet problem is fundamentally tied to quantitative absolute continuity of harmonic measure with respect to surface measure on the boundary: indeed, *it is equivalent* to the so-called "weak-A_∞" property of the harmonic measure (see Definitions 2.14 and 2.15). It is through this connection to quantitative absolute continuity of harmonic measure that we shall obtain our geometric characterization of L^p solvability.

The study of the relationship between the geometry of a domain, and absolute continuity properties of its harmonic measure, has a long history. A classical result of F. and M. Riesz [47] states that for a simply connected domain Ω in the complex plane, rectifiability of $\partial\Omega$ implies that harmonic measure for Ω is absolutely continuous with respect to arclength measure on the boundary. A quantitative version of this theorem was later proved by Lavrentiev [41]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones [12] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion. They also present a counter-example to show that the result of [47] may fail in the absence of some connectivity hypothesis (e.g., simple connectedness).

In dimensions greater than 2, a fundamental result of Dahlberg [18] establishes a quantitative version of absolute continuity, namely that harmonic measure belongs to the class A_∞ in an appropriate local sense (see Definitions 2.14 and 2.15 below), with respect to surface measure on the boundary of a Lipschitz domain.

The result of Dahlberg was extended to the class of Chord-arc domains (see Definition 2.7) by David and Jerison [21], and independently by Semmes [48]. The Chord-arc hypothesis was weakened to that of a two-sided corkscrew condition (Definition 2.4) by Bennewitz and Lewis [11], who then drew the conclusion that harmonic measure is weak-A_∞ (in an appropriate local sense, see Definitions 2.14 and 2.15) with respect to surface measure on the boundary; the latter condition is similar to the A_∞ condition, but without the doubling property, and is the best conclusion that can be obtained under the weakened geometric conditions considered in [11]. We note that weak-A_∞ is still a quantitative, scale invariant version of absolute continuity.

More recently, one of us (Azzam) has given in [6] a geometric characterization of the A_∞ property of harmonic measure with respect to surface measure for domains with n-dimensional Ahlfors–David regular (n-ADR) boundary (see Definition 2.1). Azzam's results are related to those of the present paper, so let us describe them in a bit more detail. Specifically, he shows that for a domain Ω with n-ADR boundary, harmonic measure is in A_∞ with respect to surface measure, if and only if 1) $\partial\Omega$ is uniformly rectifiable (n-UR)[1], and 2) Ω is semi-uniform in the sense of Aikawa and Hirata [1]. The semi-uniform condition is a connectivity condition which states that for some uniform constant M, every pair of points $x \in \Omega$ and $y \in \partial\Omega$ may be connected by a rectifiable curve $\gamma = \gamma(y, x)$, with $\gamma \setminus \{y\} \subset \Omega$, with length $\ell(\gamma) \leq M|x - y|$, and which satisfies the "cigar path" condition

$$\min\left\{\ell\big(\gamma(y, z)\big), \ell\big(\gamma(z, x)\big)\right\} \leq M \operatorname{dist}(z, \partial\Omega), \quad \forall z \in \gamma. \qquad (1.1)$$

Semi-uniformity is a weak version of the well known uniform condition, whose definition is similar, except that it applies to all pairs of points $x, y \in \Omega$. For example, the unit disk centered at the origin, with the slit $\{-1/2 \leq x \leq 1/2, y = 0\}$ removed, is semi-uniform, but not uniform. It was shown in [1] that for a domain satisfying a John condition and the Capacity Density Condition (in particular, for a domain with an n-ADR boundary), semi-uniformity characterizes the doubling property of harmonic measure. The method of [6] is, broadly speaking, related to that of [21], and of [11]. In [21], the authors show that a Chord-arc domain Ω may be approximated in a "Big Pieces" sense (see [21] or [11] for a precise statement; also cf. Definition 2.12 below) by Lipschitz subdomains $\Omega' \subset \Omega$; this fact allows one to reduce matters to the result of Dahlberg via the maximum principle (a method which, to the present authors' knowledge, first appears in [39] in the context of BMO_1 domains). The same strategy, i.e., Big Piece approximation by Lipschitz subdomains, is employed in [11]. Similarly, in [6], matters are reduced to the result of [21],

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[1] This is a quantitative, scale-invariant version of rectifiability, see Definition 2.2 and the ensuing comments.

by showing that for a domain Ω with an n-ADR boundary, Ω is semi-uniform with a uniformly rectifiable boundary if and only if it has "Very Big Pieces" of Chord-arc subdomains (see [6] for a precise statement of the latter condition). As mentioned above, the converse direction is also treated in [6]. In that case, given an interior corkscrew condition (which holds automatically in the presence of the doubling property of harmonic measure), and provided that $\partial\Omega$ is n-ADR, the A_∞ (or even weak-A_∞) property of harmonic measure was already known to imply uniform rectifiability of the boundary [32] (although the published version appears in [28]; see also [44] for an alternative proof, and a somewhat more general result); as in [1], semi-uniformity follows from the doubling property, although in [6], the author manages to show this while dispensing with the John domain background assumption (given a harmlessly strengthened version of the doubling property).

Thus, in [6], the connectivity condition (semi-uniformity), is tied to the doubling property of harmonic measure, and not to absolute continuity. On the other hand, in light of the example of [12], and on account of the aforementioned connection to solvability of the Dirichlet problem, it has been an important open problem to determine the minimal connectivity assumption which, in conjunction with uniform rectifiability of the boundary, yields quantitative absolute continuity of harmonic measure with respect to surface measure. In the present work, we present a connectivity condition, significantly milder than semi-uniformity, which we call the *weak local John condition* (see Definition 2.11 below), and which solves this problem. Thus, we obtain a geometric characterization of the domains for which one has quantitative absolute continuity of harmonic measure; equivalently, for which one has solvability of the Dirichlet problem with singular (L^p) data (see Theorem 1.1 below). In fact, we provide two geometric characterizations of such domains, one in terms of uniform rectifiability combined with the weak local John condition, the other in terms of approximation of the boundary in a big pieces sense, by boundaries of Chord-arc subdomains.

Let us now describe the weak local John condition, which says, roughly speaking, that from each point $x \in \Omega$, there is local non-tangential access to an ample portion of a surface ball at a scale on the order of $\delta_\Omega(x) := \text{dist}(x, \partial\Omega)$. Let us make this a bit more precise. A "carrot path" (aka non-tangential path) joining a point $x \in \Omega$, and a point $y \in \partial\Omega$, is a connected rectifiable path $\gamma = \gamma(y, x)$, with endpoints y and x, such that for some $\lambda \in (0, 1)$ and for all $z \in \gamma$,

$$\lambda\, \ell\big(\gamma(y, z)\big) \ \leq \ \delta_\Omega(z), \tag{1.2}$$

where $\ell\big(\gamma(y, z)\big)$ denotes the arc-length of the portion of the original path with endpoints y and z. For $x \in \Omega$, and $N \geq 2$, set

$$\Delta_x = \Delta_x^N := B\big(x, N\delta_\Omega(x)\big) \cap \partial\Omega.$$

We assume that every point $x \in \Omega$ may be joined by a carrot path to each y in a "Big Piece" of Δ_x, i.e., to each y in a Borel subset $F \subset \Delta_x$, with $\sigma(F) \geq \theta\sigma(\Delta_x)$, where σ denotes surface measure on $\partial\Omega$, and where the parameters $N \geq 2$, $\lambda \in (0, 1)$, and $\theta \in (0, 1]$ are uniformly controlled. We refer to this condition as a "weak local John condition", although "weak local semi-uniformity" would be equally appropriate. See Definitions 2.8, 2.9 and 2.11 for more details. We remark that a strong version of the local John condition (i.e., with $\theta = 1$) has appeared in [37], in connection with boundary Poincaré inequalities for non-smooth domains.

Let us observe that the weak local John condition is strictly weaker than semi-uniformity: for example, the unit disk centered a the origin, with either the cross $\{-1/2 \leq x \leq 1/2, y = 0\} \cup \{-1/2 \leq y \leq 1/2, x = 0\}$ removed, or with the slit $\{0 \leq x \leq 1, y = 0\}$ removed, satisfies the weak local John condition, although semi-uniformity fails in each case.

The main result in the present work is the following geometric characterization of quantitative absolute continuity of harmonic measure, and of the L^p solvability of the Dirichlet problem. The terminology used here will be defined in the sequel.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set satisfying an interior corkscrew condition (see Definition 2.4 below), and suppose that $\partial\Omega$ is n-dimensional Ahlfors–David regular (n-ADR; see Definition 2.1 below). Then the following are equivalent:*

(1) *$\partial\Omega$ is Uniformly Rectifiable (n-UR; see Definition 2.2 below) and Ω satisfies the weak local John condition (see Definition 2.11 below).*
(2) *Ω satisfies an Interior Big Pieces of Chord-Arc Domains (IBPCAD) condition (see Definition 2.12 below).*
(3) *Harmonic measure ω is locally in weak-A_∞ (see Definition 2.15 below) with respect to surface measure σ on $\partial\Omega$.*
(4) *The L^p Dirichlet problem is solvable for some $p < \infty$, i.e., for some $p < \infty$, there is a constant C such that if $g \in L^p(\partial\Omega)$, then the solution to the Dirichlet problem with data g, is well defined as $u(x) := \int_{\partial\Omega} g\, d\omega^x$ for each $x \in \Omega$, converges to g non-tangentially, and enjoys the estimate*

$$\|N_*u\|_{L^p(\partial\Omega)} \leq C \|g\|_{L^p(\partial\Omega)}, \tag{1.3}$$

where N_*u is a suitable version of the non-tangential maximal function of u.

Some explanatory comments are in order. The proof has two main new ingredients: the implication (1) implies (2), and the fact that the weak-A_∞ property of harmonic measure implies the weak local John condition (this is the new part of (3) implies (1)). In turn, we split these main new results into

two theorems: the first implication is the content of Theorem 1.2 below, and the second is the content of Theorem 1.3. We remark that the interior corkscrew condition is not needed for (1) implies (2) (nor for (2) implies (3) if and only if (4)). Rather, it is crucial for (3) implies (1) (see "Appendix A").

As regards the other implications, the fact that (2) implies (3) follows by a well-known argument using the maximum principle and the result of [21] and [48] for Chord-arc domains[2], along with the criterion for weak-A_∞ obtained in [11]; the equivalence of (3) and (4) is well known, and we refer the reader to, e.g., [27, Section 4], and to [26] for details. The implication (3) implies (1) has two parts. As mentioned above, the fact that weak-A_∞ implies weak local John is new, and is the content of Theorem 1.3. The remaining implication, namely that weak-A_∞ implies n-UR, is the main result of [32]; an alternative proof, with a more general result, appears in [44], and see also [28] for the final published version of the results of [32], along with an extension to the p-harmonic setting.

We note that our background hypotheses (upper and lower n-ADR, and interior corkscrew) are in the nature of best possible: one may construct a counter-example in the absence of any one of them, for at least one direction of this chain of implications, as we shall discuss in "Appendix A". In addition, in the case of the n-ADR condition, given any $\varepsilon > 0$, the counter-examples for the upper (respectively, lower) n-ADR property can be constructed in such a way as to show that no weaker condition of the form $H^n(B(x, r) \cap \partial\Omega) \lesssim r^{n-\varepsilon}$ (resp., $H^n(B(x, r) \cap \partial\Omega) \gtrsim r^{n+\varepsilon}$), with $r < 1$, may be substituted for a true n-ADR upper or lower bound. Moreover, the first example shows that one cannot substitute the Capacity Density Condition (CDC)[3] in place of the n-ADR condition: indeed, the example is an NTA domain, in particular, it satisfies an exterior corkscrew condition, and thus also the CDC.

As regards our assumption of the interior corkscrew condition, we point out that, as is well known, the n-ADR condition implies that the open set $\mathbb{R}^{n+1} \setminus \partial\Omega$ satisfies a corkscrew condition, with constants depending only on n and ADR, i.e., at every scale r, and for every point $x \in \partial\Omega$, there is at least one component of $\mathbb{R}^{n+1} \setminus \partial\Omega$ containing a corkscrew point relative to the ball $B(x, r)$. Our last example shows that such a component should lie inside Ω itself, for each x and r; i.e., that Ω should enjoy an interior corkscrew condition.

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[2] See, e.g., [26, Proposition 13] for the details in this context, but the proof originates in [39].

[3] The CDC is a scale invariant potential theoretic "thickness" condition, i.e., a quantitative version of Weiner regularity; see, e.g., [1].

As explained above, the main new contributions of the present work are contained in the following pair of theorems:

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set, not necessarily connected, with an n-dimensional Ahlfors–David regular (n-ADR) boundary. Then the following are equivalent:*

(i) *$\partial\Omega$ is uniformly rectifiable (n-UR), and Ω satisfies the weak local John condition.*

(ii) *Ω satisfies an Interior Big Pieces of Chord-Arc Domains (IBPCAD) condition.*

Only the direction (i) implies (ii) is new. For the converse, the fact that IBP-CAD implies the weak local John condition is immediate from the definitions. Moreover, the boundary of a Chord-arc domain is n-UR, and an n-ADR set with big pieces of n-UR is also n-UR (see [23]). As noted above, that (ii) implies the weak-A_∞ property follows by well known arguments.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set satisfying an interior corkscrew condition and suppose that $\partial\Omega$ is n-dimensional Ahlfors–David regular (n-ADR). If the harmonic measure for Ω satisfies the weak-A_∞ condition, then Ω satisfies the weak local John condition.*

Let us mention that the present paper is a combination of unpublished work of two different subsets of the present authors: Theorem 1.2 is due to the second and third authors, and was first posted in the draft manuscript [34][4]; Theorem 1.3 is due to the first, fourth and fifth authors, and appeared first in the draft manuscript [9].

The paper is organized as follows. In the next section, we set notation and give some definitions. In Part 1 of the paper (Sects. 3–8), we give the proof of Theorem 1.2. In Part 2 of the paper (Sects. 9–16) we give the proof of Theorem 1.3. Finally, in "Appendix A", we discuss some counter-examples which show that our background hypotheses are in the nature of best possible.

We thank the referee for a careful reading of the paper, and for several helpful suggestions that have led us to clarify certain matters, and to make improvements in the presentation.

2 Notation and definitions

- Unless otherwise stated, we use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also

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[4] An earlier version of this work [33] gave a direct proof of the fact that (1) implies (3) in Theorem 1.1, without passing through condition (2).

sometimes write $a \lesssim b$, $a \gtrsim b$, and $a \approx b$ to mean, respectively, that $a \leq Cb$, $a \geq cb$, and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. In some occasions we will employ the notation $a \lesssim_\lambda b$, $a \gtrsim_\lambda b$ and $a \approx_\lambda b$ to emphasize that the previous implicit constants c and/or C may depend on some relevant parameter λ. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Our ambient space is $\mathbb{R}^{n+1}$, $n \geq 1$.
- Ω will always denote an open set in $\mathbb{R}^{n+1}$, not necessarily connected unless otherwise specified.
- We use the notation $\gamma(x, y)$ to denote a rectifiable path with endpoints x and y, and its arc-length will be denoted $\ell(\gamma(x, y))$. Given such a path, if $z \in \gamma(x, y)$, we use the notation $\gamma(z, y)$ to denote the portion of the original path with endpoints z and y.
- We let e_j, $j = 1, 2, \ldots, n + 1$, denote the standard unit basis vectors in $\mathbb{R}^{n+1}$.
- The open $(n + 1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$. For $x \in \partial\Omega$, a *surface ball* is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$.
- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.
- Given a Euclidean or surface ball $B = B(x, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(x, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.
- Given an open set $\Omega \subset \mathbb{R}^{n+1}$, for $x \in \Omega$, we set $\delta_\Omega(x) := \text{dist}(x, \partial\Omega)$.
- We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\big\lfloor_{\partial\Omega}$ denote the surface measure on $\partial\Omega$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let χ_A denote the usual indicator function of A, i.e. $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\text{int}(A)$ denote the interior of A.
- Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $\fint_A f \, d\mu := \mu(A)^{-1} \int_A f \, d\mu$.
- We shall use the letter I (and sometimes J) to denote a closed $(n + 1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$. Given an n-ADR set $E \subset \mathbb{R}^{n+1}$, we use Q (or sometimes P or R) to denote a dyadic "cube" on E. The latter exist (see [17,22,38]), and enjoy certain properties which we enumerate in Lemma 2.16 below.

Definition 2.1 (*n*-**ADR**) (aka *n*-*Ahlfors–David regular*). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is n-ADR if it is closed, and if there is some uniform constant C such that

$$\frac{1}{C} r^n \le \sigma\big(\Delta(x,r)\big) \le C\, r^n, \quad \forall r \in (0, \text{diam}(E)), \ x \in E, \qquad (2.1)$$

where $\text{diam}(E)$ may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the *surface ball* of radius r, and as above, $\sigma := H^n \lfloor_E$ is the "surface measure" on E.

Definition 2.2 (*n-UR*) (*aka n-uniformly rectifiable*). An n-ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is n-UR if and only if it contains "Big Pieces of Lipschitz Images" of $\mathbb{R}^n$ ("BPLI"). This means that there are positive constants c_1 and C_1, such that for each $x \in E$ and each $r \in (0, \text{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than C_1, such that

$$H^n\Big(E \cap B(x,r) \cap \rho\left(\{z \in \mathbb{R}^n : |z| < r\}\right)\Big) \ge c_1 r^n.$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of n-ADR sets, the n-UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [22]. In fact, for n-ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [43] for the case $n = 1$, and [45] in general). We further remark that there exist sets that are n-ADR (and that even form the boundary of a domain satisfying interior corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [23, Chapter 1]). Finally, we mention that there are numerous other characterizations of n-UR sets (many of which remain valid in higher co-dimensions); cf. [22,23].

Definition 2.3 ("*UR character*"). Given an n-UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants (c_1, C_1) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 2.4 (*Corkscrew condition*). Following [39], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *corkscrew condition* if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \text{diam}(\partial\Omega)$, there is a ball $B(x_\Delta, cr) \subset B(x,r) \cap \Omega$. The point $x_\Delta \subset \Omega$ is called a *corkscrew point* relative to Δ. We note that we may allow $r < C\,\text{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c. In order to emphasize that $B(x_\Delta, cr) \subset \Omega$, we shall sometimes refer to this property as the *interior corkscrew condition*.

Definition 2.5 (*Harnack Chains, and the Harnack Chain condition* [39]). Given two points $x, x' \in \Omega$, and a pair of numbers $M, N \geq 1$, an (M, N)-*Harnack Chain connecting x to x'*, is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, with $x \in B_1$, $x' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $M^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq M \operatorname{diam}(B_k)$. We say that Ω satisfies the *Harnack Chain condition* if there is a uniform constant M such that for any two points $x, x' \in \Omega$, there is an (M, N)-Harnack Chain connecting them, with N depending only on M and the ratio $|x - x'| / \left(\min \left(\delta_\Omega(x), \delta_\Omega(x') \right) \right)$.

Definition 2.6 (*NTA*). Again following [39], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is NTA (*Non-tangentially accessible*) if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \backslash \overline{\Omega}$ satisfy the corkscrew condition.

Definition 2.7 (*CAD*). We say that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a CAD (*Chord-arc domain*), if it is NTA, and if $\partial\Omega$ is n-ADR.

Definition 2.8 (*Carrot path*). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Given a point $x \in \Omega$, and a point $y \in \partial\Omega$, we say that a connected rectifiable path $\gamma = \gamma(y, x)$, with endpoints y and x, is a *carrot path* (more precisely, a λ-*carrot path*) connecting y to x, if $\gamma \backslash \{y\} \subset \Omega$, and if for some $\lambda \in (0, 1)$ and for all $z \in \gamma$,

$$\lambda \, \ell \left(\gamma(y, z) \right) \leq \delta_\Omega(z). \tag{2.2}$$

With a slight abuse of terminology, we shall sometimes refer to such a path as a λ-*carrot path in Ω*, although of course the endpoint y lies on $\partial\Omega$.

A carrot path is sometimes referred to as a non-tangential path.

Definition 2.9 (*(θ, λ, N)-weak local John point*). Let $x \in \Omega$, and for constants $\theta \in (0, 1]$, $\lambda \in (0, 1)$, and $N \geq 2$, set

$$\Delta_x = \Delta_x^N := B\left(x, N\delta_\Omega(x)\right) \cap \partial\Omega.$$

We say that a point $x \in \Omega$ is a (θ, λ, N)-*weak local John point* if there is a Borel set $F \subset \Delta_x^N$, with $\sigma(F) \geq \theta\sigma(\Delta_x^N)$, such that for every $y \in F$, there is a λ-carrot path connecting y to x.

Thus, a weak local John point is non-tangentially connected to an ample portion of the boundary, locally. We observe that one can always choose N smaller, for possibly different values of θ and λ, by moving from x to a point x' on a line segment joining x to the boundary.

Remark 2.10 We observe that it is a slight abuse of notation to write Δ_x, since the latter is not centered on $\partial\Omega$, and thus it is not a true surface ball; on the other hand, there are true surface balls, $\Delta_x' := \Delta(\hat{x}, (N-1)\delta_\Omega(x))$ and $\Delta_x'' := \Delta(\hat{x}, (N+1)\delta_\Omega(x))$, centered at a "touching point" $\hat{x} \in \partial\Omega$ with $\delta_\Omega(x) = |x - \hat{x}|$, which, respectively, are contained in, and contain, Δ_x.

Definition 2.11 (*Weak local John condition*). We say that Ω satisfies a *weak local John condition* if there are constants $\lambda \in (0, 1)$, $\theta \in (0, 1]$, and $N \geq 2$, such that every $x \in \Omega$ is a (θ, λ, N)-weak local John point.

Definition 2.12 (*IBPCAD*). We say that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ has *Interior Big Pieces of Chord-Arc Domains* (IBPCAD) if there exist positive constants η and C, and $N \geq 2$, such that for every $x \in \Omega$, with $\delta_\Omega(x) < \text{diam}(\partial\Omega)$, there is a Chord-arc domain $\Omega_x \subset \Omega$ satisfying

- $x \in \Omega_x$.
- $\text{dist}(x, \partial\Omega_x) \geq \eta\delta_\Omega(x)$.
- $\text{diam}(\Omega_x) \leq C\delta_\Omega(x)$.
- $\sigma(\partial\Omega_x \cap \Delta_x^N) \geq \eta \sigma(\Delta_x^N) \approx_N \eta \delta_\Omega(x)^n$.
- The Chord-arc constants of the domains Ω_x are uniform in x.

Remark 2.13 In the presence of an interior corkscrew condition, Definition 2.12 is easily seen to be essentially equivalent to the following more standard "Big Pieces" condition: there are positive constants η and C (perhaps slightly different to that in Definition 2.12), such that for each surface ball $\Delta := \Delta(x, r) = B(x, r) \cap \partial\Omega$, $x \in \partial\Omega$ and $r < \text{diam}(\partial\Omega)$, and for any corkscrew point x_Δ relative to Δ there is a Chord-arc domain Ω_Δ satisfying

- $x_\Delta \in \Omega_\Delta$
- $\text{dist}(x_\Delta, \partial\Omega_\Delta) \geq \eta r$.
- $\Omega_\Delta \subset B(x, Cr) \cap \Omega$.
- $\sigma(\partial\Omega_\Delta \cap \Delta(x, Cr)) \geq \eta \sigma(\Delta(x, Cr)) \approx \eta r^n$.
- The Chord-arc constants of the domains Ω_Δ are uniform in Δ.

Definition 2.14 (A_∞, *weak-*A_∞, *and weak-*RH_q). Given an n-ADR set $E \subset \mathbb{R}^{n+1}$, and a surface ball $\Delta_0 := B_0 \cap E$ centered on E, we say that a Borel measure μ defined on E belongs to $A_\infty(\Delta_0)$ if there are positive constants C and s such that for each surface ball $\Delta = B \cap E$ centered on E, with $B \subseteq B_0$, we have

$$\mu(A) \leq C \left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(\Delta), \qquad \text{for every Borel set } A \subset \Delta. \qquad (2.3)$$

Similarly, we say that $\mu \in$ weak-$A_\infty(\Delta_0)$ if for each surface ball $\Delta = B \cap E$ centered on E, with $2B \subseteq B_0$,

$$\mu(A) \leq C \left(\frac{\sigma(A)}{\sigma(\Delta)}\right)^s \mu(2\Delta), \qquad \text{for every Borel set } A \subset \Delta. \qquad (2.4)$$

We recall that, as is well known, the condition $\mu \in$ weak-$A_\infty(\Delta_0)$ is equivalent to the property that $\mu \ll \sigma$ in Δ_0, and that for some $q > 1$, the Radon–Nikodym derivative $k := d\mu/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_{\Delta} k^q \, d\sigma \right)^{1/q} \lesssim \fint_{2\Delta} k \, d\sigma \approx \frac{\mu(2\Delta)}{\sigma(\Delta)}, \quad \forall \Delta = B \cap E, \quad \text{with} \quad 2B \subseteq B_0,$$
(2.5)

with B centered on E. We shall refer to the inequality in (2.5) as a "weak-RH_q" estimate, and we shall say that $k \in$ weak-$RH_q(\Delta_0)$ if k satisfies (2.5).

Definition 2.15 (*Local A_∞ and local weak-A_∞*). We say that harmonic measure ω is locally in A_∞ (resp., locally in weak-A_∞) on $\partial\Omega$, if there are uniform positive constants C and s such that for every ball $B = B(x, r)$ centered on $\partial\Omega$, with radius $r < \text{diam}(\partial\Omega)/4$, and associated surface ball $\Delta = B \cap \partial\Omega$,

$$\omega^p(A) \le C \left(\frac{\sigma(A)}{\sigma(\Delta)} \right)^s \omega^p(\Delta), \qquad \forall \, p \in \Omega \backslash 4B, \quad \forall \, \text{Borel } A \subset \Delta, \quad (2.6)$$

or, respectively, that

$$\omega^p(A) \le C \left(\frac{\sigma(A)}{\sigma(\Delta)} \right)^s \omega^p(2\Delta), \qquad \forall \, p \in \Omega \backslash 4B, \quad \forall \, \text{Borel } A \subset \Delta; \quad (2.7)$$

equivalently, if for every ball B and surface ball $\Delta = B \cap \partial\Omega$ as above, and for each point $p \in \Omega \backslash 4B$, $\omega^p \in A_\infty(\Delta)$ (resp., $\omega^p \in$ weak-$A_\infty(\Delta)$) with uniformly controlled A_∞ (resp., weak-A_∞) constants.

Lemma 2.16 (*Existence and properties of the "dyadic grid"*) [17,22,23]. *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-ADR set. Then there exist constants $a_0 > 0$, $s > 0$ and $C_1 < \infty$, depending only on n and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathcal{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

  (i) *$E = \cup_j Q_j^k$ for each $k \in \mathbb{Z}$.*
  (ii) *If $m \ge k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \emptyset$.*
  (iii) *For each (j, k) and each $m < k$, there is a unique i such that $Q_j^k \subset Q_i^m$.*
  (iv) *$\text{diam}\left(Q_j^k\right) \le C_1 2^{-k}$.*
  (v) *Each Q_j^k contains some "surface ball" $\Delta\left(x_j^k, a_0 2^{-k}\right) := B\left(x_j^k, a_0 2^{-k}\right) \cap E$.*
  (vi) *$H^n\left(\{x \in Q_j^k : \text{dist}(x, E \backslash Q_j^k) \le \vartheta \, 2^{-k}\}\right) \le C_1 \vartheta^s H^n\left(Q_j^k\right)$, for all k, j and for all $\vartheta \in (0, a_0)$.*

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [17] (see also [38]), with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = 1/2$ (see [36, Proof of Proposition 2.12]). In the presence of the Ahlfors–David property (2.1), the result already appears in [22,23]. Some predecessors of this construction have appeared in [19] and [20].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We shall denote by $\mathcal{D} = \mathcal{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathcal{D} := \cup_k \mathcal{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.
- Properties (iv) and (v) imply that for each cube $Q \in \mathcal{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$ such that $r_Q \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Delta(x_Q, r_Q) \subset Q \subset \Delta(x_Q, Cr_Q), \tag{2.8}$$

for some uniform constant C. We shall refer to the point x_Q as the "center" of Q.
- For a dyadic cube $Q \in \mathcal{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, by adjusting if necessary some parameters, we can assume that

$$\operatorname{diam}(Q) \le \ell(Q) \lesssim \operatorname{diam}(Q).$$

We shall denote

$$B_Q := B(x_Q, 4\ell(Q)), \qquad \Delta_Q := \Delta(x_Q, 4\ell(Q)). \tag{2.9}$$

Notice that $Q \subset \Delta_Q \subset B_Q$.
- For a dyadic cube $Q \in \mathcal{D}$, we let $k(Q)$ denote the dyadic generation to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathcal{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- Given $R \in \mathcal{D}$, we set

$$\mathcal{D}(R) := \{Q \in \mathcal{D} : Q \subset R\}. \tag{2.10}$$

For $j \ge 1$, we also let

$$\mathcal{D}_j(R) := \left\{ Q \in \mathcal{D}(R) : \ell(Q) = 2^{-j}\ell(R) \right\}. \tag{2.11}$$

- For a pair of cubes Q', $Q \in \mathcal{D}$, if Q' is a dyadic child of Q, i.e., if $Q' \subset Q$, and $\ell(Q) = 2\ell(Q')$, then we write $Q' \lhd Q$.
- For $\lambda > 1$, we write

$$\lambda Q = \left\{ x \in E : \operatorname{dist}(x, Q) \le (\lambda - 1)\,\ell(Q) \right\}.$$

With the dyadic cubes in hand, we may now define the notion of a corkscrew point relative to a cube Q.

Definition 2.17 (*Corkscrew point relative to Q*). Let Ω satisfy the corkscrew condition (Definition 2.4), suppose that $\partial\Omega$ is n-ADR, and let $Q \in \mathcal{D}(\partial\Omega)$. A *corkscrew point relative to Q* is simply a corkscrew point relative to the surface ball $\Delta(x_Q, r_Q)$ defined in (2.8).

Definition 2.18 (*Coherency and Semi-coherency*). [23]. Let $E \subset \mathbb{R}^{n+1}$ be an n-ADR set. Let $\mathsf{T} \subset \mathcal{D}(E)$. We say that T is *coherent* if the following conditions hold:

(a) T contains a unique maximal element $Q(\mathsf{T})$ which contains all other elements of T as subsets.
(b) If Q belongs to T, and if $Q \subset \widetilde{Q} \subset Q(\mathsf{T})$, then $\widetilde{Q} \in \mathsf{T}$.
(c) Given a cube $Q \in \mathsf{T}$, either all of its children belong to T, or none of them do.

We say that T is *semi-coherent* if conditions (a) and (b) hold. We shall refer to a coherent or semi-coherent collection T as a *tree*.

Part 1: Proof of Theorem 1.2

3 Preliminaries for the Proof of Theorem 1.2

We begin by recalling a bilateral version of the David-Semmes "Corona decomposition" of an n-UR set. We refer the reader to [35] for the proof.

Lemma 3.1 ([35, Lemma 2.2]) *Let $E \subset \mathbb{R}^{n+1}$ be an n-UR set. Then given any positive constants $\eta \ll 1$ and $K \gg 1$, there is a disjoint decomposition $\mathcal{D}(E) = \mathcal{G} \cup \mathcal{B}$, satisfying the following properties.*

(1) *The "Good" collection $\mathcal{G}$ is further subdivided into disjoint trees, such that each such tree T is coherent (Definition 2.18).*
(2) *The "Bad" cubes, as well as the maximal cubes $Q(\mathsf{T})$, $\mathsf{T} \subset \mathcal{G}$, satisfy a Carleson packing condition:*

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') + \sum_{\mathsf{T} \subset \mathcal{G}: Q(\mathsf{T}) \subset Q} \sigma\big(Q(\mathsf{T})\big) \le C_{\eta, K}\, \sigma(Q), \quad \forall Q \in \mathcal{D}(E).$$

(3) *For each* $\mathsf{T} \subset \mathcal{G}$, *there is a Lipschitz graph* Γ_{T}, *with Lipschitz constant at most* η, *such that, for every* $Q \in \mathsf{T}$,

$$\sup_{x \in \Delta_Q^*} \operatorname{dist}(x, \Gamma_{\mathsf{T}}) + \sup_{y \in B_Q^* \cap \Gamma_{\mathsf{T}}} \operatorname{dist}(y, E) < \eta\, \ell(Q), \qquad (3.1)$$

where $B_Q^* := B(x_Q, K\ell(Q))$ *and* $\Delta_Q^* := B_Q^* \cap E$, *and* x_Q *is the "center" of* Q *as in* (2.8)–(2.9).

We remark that in [35], the trees T were denoted by **S**, and were called "stopping time regimes" rather than trees.

We mention that David and Semmes, in [22], had previously proved a unilateral version of Lemma 3.1, in which the bilateral estimate (3.1) is replaced by the unilateral bound

$$\sup_{x \in \Delta_Q^*} \operatorname{dist}(x, \Gamma_{\mathsf{T}}) < \eta\, \ell(Q), \qquad \forall\, Q \in \mathsf{T}. \qquad (3.2)$$

Next, we make a standard Whitney decomposition of $\Omega_E := \mathbb{R}^{n+1} \setminus E$, for a given n-UR set E (in particular, Ω_E is open, since n-UR sets are closed by definition). Let $\mathcal{W} = \mathcal{W}(\Omega_E)$ denote a collection of (closed) dyadic Whitney cubes of Ω_E, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω_E, which satisfy

$$4\operatorname{diam}(I) \leq \operatorname{dist}(4I, E) \leq \operatorname{dist}(I, E) \leq 40\operatorname{diam}(I), \qquad \forall\, I \in \mathcal{W} \quad (3.3)$$

(just dyadically divide the standard Whitney cubes, as constructed in [49, Chapter VI], into cubes with side length 1/8 as large) and also

$$\tfrac{1}{4}\operatorname{diam}(I_1) \leq \operatorname{diam}(I_2) \leq 4\operatorname{diam}(I_1),$$

whenever I_1 and I_2 touch.

We fix a small parameter $\tau_0 > 0$, so that for any $I \in \mathcal{W}$, and any $\tau \in (0, \tau_0]$, the concentric dilate

$$I^*(\tau) := (1 + \tau)I \qquad (3.4)$$

still satisfies the Whitney property

$$\operatorname{diam} I \approx \operatorname{diam} I^*(\tau) \approx \operatorname{dist}\big(I^*(\tau), E\big) \approx \operatorname{dist}(I, E), \quad 0 < \tau \leq \tau_0. \quad (3.5)$$

Moreover, for $\tau \leq \tau_0$ small enough, and for any $I, J \in \mathcal{W}$, we have that $I^*(\tau)$ meets $J^*(\tau)$ if and only if I and J have a boundary point in common, and that, if $I \neq J$, then $I^*(\tau)$ misses $(3/4)J$.

Pick two parameters $\eta \ll 1$ and $K \gg 1$ (eventually, we shall take $K = \eta^{-3/4}$). For $Q \in \mathcal{D}(E)$, define

$$\mathcal{W}_Q^0 := \left\{ I \in \mathcal{W} : \eta^{1/4}\ell(Q) \leq \ell(I) \leq K^{1/2}\ell(Q), \ \mathrm{dist}(I, Q) \leq K^{1/2}\ell(Q) \right\}. \tag{3.6}$$

Remark 3.2 We note that $\mathcal{W}_Q^0$ is non-empty, provided that we choose η small enough, and K large enough, depending only on dimension and ADR, since the n-ADR condition implies that Ω_E satisfies a corkscrew condition. In the sequel, we shall always assume that η and K have been so chosen.

Next, we recall a construction in [35, Section 3], leading up to and including in particular [35, Lemma 3.24]. We summarize this construction as follows.

Lemma 3.3 *Let $E \subset \mathbb{R}^{n+1}$ be n-UR, and set $\Omega_E := \mathbb{R}^{n+1}\backslash E$. Given positive constants $\eta \ll 1$ and $K \gg 1$, as in (3.6) and Remark 3.2, let $\mathcal{D}(E) = \mathcal{G} \cup \mathcal{B}$, be the corresponding bilateral Corona decomposition of Lemma 3.1. Then for each $\mathsf{T} \subset \mathcal{G}$, and for each $Q \in \mathsf{T}$, the collection $\mathcal{W}_Q^0$ in (3.6) has an augmentation $\mathcal{W}_Q^* \subset \mathcal{W}$ satisfying the following properties.*

(1) $\mathcal{W}_Q^0 \subset \mathcal{W}_Q^* = \mathcal{W}_Q^{*,+} \cup \mathcal{W}_Q^{*,-}$, *where (after a suitable rotation of coordinates) each $I \in \mathcal{W}_Q^{*,+}$ lies above the Lipschitz graph Γ_T of Lemma 3.1, each $I \in \mathcal{W}_Q^{*,-}$ lies below Γ_T. Moreover, if Q' is a child of Q, also belonging to T, then $\mathcal{W}_Q^{*,+}$ (resp. $\mathcal{W}_Q^{*,-}$) belongs to the same connected component of Ω_E as does $\mathcal{W}_{Q'}^{*,+}$ (resp. $\mathcal{W}_{Q'}^{*,-}$) and $\mathcal{W}_{Q'}^{*,+} \cap \mathcal{W}_Q^{*,+} \neq \emptyset$ (resp., $\mathcal{W}_{Q'}^{*,-} \cap \mathcal{W}_Q^{*,-} \neq \emptyset$).*

(2) *There are uniform constants c and C such that*

$$\begin{aligned} c\eta^{1/2}\ell(Q) \leq \ell(I) &\leq CK^{1/2}\ell(Q), \quad \forall I \in \mathcal{W}_Q^*, \\ \mathrm{dist}(I, Q) &\leq CK^{1/2}\ell(Q), \quad \forall I \in \mathcal{W}_Q^*, \\ c\eta^{1/2}\ell(Q) \leq \mathrm{dist}(I^*(\tau), \Gamma_\mathsf{T}), &\quad \forall I \in \mathcal{W}_Q^*, \quad \forall \tau \in (0, \tau_0]. \end{aligned} \tag{3.7}$$

Moreover, given $\tau \in (0, \tau_0]$, set

$$U_Q^\pm = U_{Q,\tau}^\pm := \bigcup_{I \in \mathcal{W}_Q^{*,\pm}} \mathrm{int}\left(I^*(\tau)\right), \qquad U_Q := U_Q^+ \cup U_Q^-, \tag{3.8}$$

and given T', a semi-coherent subtree of T, define

$$\Omega_{\mathsf{T}'}^\pm = \Omega_{\mathsf{T}'}^\pm(\tau) := \bigcup_{Q \in \mathsf{T}'} U_Q^\pm. \tag{3.9}$$
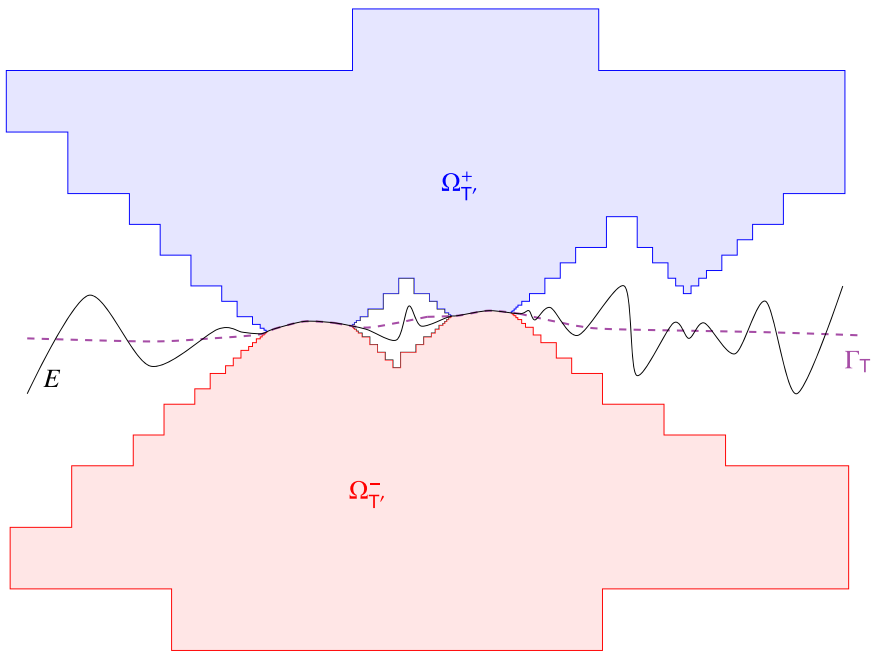
Fig. 1 The domains $\Omega_{\mathsf{T}'}^{\pm}$

Then each of $\Omega_{\mathsf{T}'}^{\pm}$ is a CAD, with Chord-arc constants depending only on n, τ, η, K, and the ADR/UR constants for $\partial\Omega$ (see Fig. 1).

Remark 3.4 In particular, for each $\mathsf{T} \subset \mathcal{G}$, if Q' and Q belong to T, and if Q' is a dyadic child of Q, then $U_{Q'}^{+} \cup U_{Q}^{+}$ is Harnack Chain connected, and every pair of points $x, y \in U_{Q'}^{+} \cup U_{Q}^{+}$ may be connected by a Harnack Chain in Ω_E of length at most $C = C(n, \tau, \eta, K, \text{ADR/UR})$. The same is true for $U_{Q'}^{-} \cup U_{Q}^{-}$.

Remark 3.5 Let $0 < \tau \leq \tau_0/2$. Given any $\mathsf{T} \subset \mathcal{G}$, and any semi-coherent subtree $\mathsf{T}' \subset \mathsf{T}$, define $\Omega_{\mathsf{T}'}^{\pm} = \Omega_{\mathsf{T}'}^{\pm}(\tau)$ as in (3.9), and similarly set $\widehat{\Omega}_{\mathsf{T}'}^{\pm} = \Omega_{\mathsf{T}'}^{\pm}(2\tau)$. Then by construction, for any $x \in \overline{\widehat{\Omega}_{\mathsf{T}'}^{\pm}}$,

$$\text{dist}(x, E) \approx \text{dist}(x, \partial\widehat{\Omega}_{\mathsf{T}'}^{\pm}),$$

where of course the implicit constants depend on τ.

As in [35], it will be useful for us to extend the definition of the Whitney region U_Q to the case that $Q \in \mathcal{B}$, the "bad" collection of Lemma 3.1. Let $\mathcal{W}_Q^*$ be the augmentation of $\mathcal{W}_Q^0$ as constructed in Lemma 3.3, and set

$$\mathcal{W}_Q := \begin{cases} \mathcal{W}_Q^*, & Q \in \mathcal{G}, \\ \mathcal{W}_Q^0, & Q \in \mathcal{B} \end{cases}. \tag{3.10}$$

For $Q \in \mathcal{G}$ we shall henceforth simply write $\mathcal{W}_Q^\pm$ in place of $\mathcal{W}_Q^{*,\pm}$. For arbitrary $Q \in \mathcal{D}(E)$, good or bad, we may then define

$$U_Q = U_{Q,\tau} := \bigcup_{I \in \mathcal{W}_Q} \mathrm{int}\left(I^*(\tau)\right). \tag{3.11}$$

Let us note that for $Q \in \mathcal{G}$, the latter definition agrees with that in (3.8). Note that by construction

$$U_Q \subset \{y \in \Omega_E : \mathrm{dist}(y, E) > c\eta^{1/2}\ell(Q)\} \cap B(x_Q, CK^{1/2}\ell(Q)), \quad (3.12)$$

for some uniform constants $C \geq 1$ and $0 < c < 1$ (see (3.3), (3.6), and (3.7)). In particular, for every $Q \in \mathcal{D}$ if follows that

$$\bigcup_{Q' \in \mathcal{D}(Q)} U_{Q'} \subset B(x_Q, K\ell(Q)) =: B_Q^*. \tag{3.13}$$

where we recall that $\mathcal{D}(Q)$ is defined in (2.10).

For future reference, we introduce dyadic sawtooth regions as follows. Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathcal{D}$, we define the *global discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathcal{D}_\mathcal{F} := \mathcal{D} \setminus \bigcup_{Q_j \in \mathcal{F}} \mathcal{D}(Q_j), \tag{3.14}$$

i.e., $\mathcal{D}_\mathcal{F}$ is the collection of all $Q \in \mathcal{D}$ that are not contained in any $Q_j \in \mathcal{F}$. We may allow $\mathcal{F}$ to be empty, in which case $\mathcal{D}_\mathcal{F} = \mathcal{D}$. Given some fixed cube Q, we also define the *local discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathcal{D}_\mathcal{F}(Q) := \mathcal{D}(Q) \setminus \bigcup_{Q_j \in \mathcal{F}} \mathcal{D}(Q_j) = \mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q). \tag{3.15}$$

Note that with this convention, $\mathcal{D}(Q) = \mathcal{D}_\emptyset(Q)$ (i.e., if one takes $\mathcal{F} = \emptyset$ in (3.15)).

4 Step 1: the set-up

In the proof of Theorem 1.2, we shall employ a two-parameter induction argument, which is a refinement of the method of "extrapolation" of Carleson measures. The latter is a bootstrapping scheme for lifting the Carleson measure constant, developed by J. L. Lewis [42], and based on the corona construction of Carleson [15] and Carleson and Garnett [16] (see also [4,5,29–31,35]).

4.1 Reduction to a dyadic setting

To set the stage for the induction procedure, let us begin by making a preliminary reduction. It will be convenient to work with a certain dyadic version of Definition 2.12. To this end, let $x \in \Omega$, with $\delta_\Omega(x) < \text{diam}(\partial\Omega)$, and set $\Delta_x = \Delta_x^N = B(x, N\delta_\Omega(x)) \cap \partial\Omega$, for some fixed $N \geq 2$ as in Definition 2.9.

Let $\hat{x} \in \partial\Omega$ be a touching point for x, i.e., $|x - \hat{x}| = \delta_\Omega(x)$. Choose x_1 on the line segment joining x to $\hat{x}$, with $\delta_\Omega(x_1) = \delta_\Omega(x)/2$, and set $\Delta_{x_1} = B(x_1, N\delta_\Omega(x)/2) \cap \partial\Omega$. Note that $B(x_1, N\delta_\Omega(x)/2) \subset B(x, N\delta_\Omega(x))$, and furthermore,

$$\text{dist}\left(B(x_1, N\delta_\Omega(x)/2), \partial B(x, N\delta_\Omega(x))\right) > \frac{N-1}{2}\delta_\Omega(x) \geq \frac{1}{2}\delta_\Omega(x).$$

We may therefore cover Δ_{x_1} by a disjoint collection $\{Q_i\}_{i=1}^M \subset \mathcal{D}(\partial\Omega)$, of equal length $\ell(Q_i) \approx \delta_\Omega(x)$, such that each $Q_i \subset \Delta_x$, and such that the implicit constants depend only on n and ADR, and thus the cardinality M of the collection depends on n, ADR, and N. With $E = \partial\Omega$, we make the Whitney decomposition of the set $\Omega_E = \mathbb{R}^{n+1} \setminus E$ as in Sect. 3 (thus, $\Omega \subset \Omega_E$). Moreover, for sufficiently small η and sufficiently large K in (3.6), we then have that $x \in U_{Q_i}$ for each $i = 1, 2, \ldots, M$. By hypothesis, there are constants $\theta_0 \in (0, 1]$, $\lambda_0 \in (0, 1)$, and $N \geq 2$ as above, such that every $z \in \Omega$ is a (θ_0, λ_0, N)-weak local John point (Definition 2.9). In particular, this is true for x_1, hence there is a Borel set $F \subset \Delta_{x_1}$, with $\sigma(F) \geq \theta_0\sigma(\Delta_{x_1})$, such that every $y \in F$ may be connected to x_1 via a λ_0-carrot path. By n-ADR, $\sigma(\Delta_{x_1}) \approx \sum_{i=1}^M \sigma(Q_i)$ and thus by pigeon-holing, there is at least one $Q_i =: Q$ such that $\sigma(F \cap Q) \geq \theta_1\sigma(Q)$, with θ_1 depending only on θ_0, n and ADR. Moreover, the λ_0-carrot path connecting each $y \in F$ to x_1 may be extended to a λ_1-carrot path connecting y to x, where λ_1 depends only on λ_0.

We have thus reduced matters to the following dyadic scenario: let $Q \in \mathcal{D}(\partial\Omega)$, let $U_Q = U_{Q,\tau}$ be the associated Whitney region as in (3.11), with $\tau \leq \tau_0/2$ fixed, and suppose that U_Q meets Ω (recall that by construction $U_Q \subset \Omega_E = \mathbb{R}^{n+1} \setminus E$, with $E = \partial\Omega$). For $x \in U_Q \cap \Omega$, and for a constant $\lambda \in (0, 1)$, let

$$F_{car}(x, Q) = F_{car}(x, Q, \lambda) \tag{4.1}$$

denote the set of $y \in Q$ which may be joined to x by a λ-carrot path $\gamma(y, x)$, and for $\theta \in (0, 1]$, set

$$T_Q = T_Q(\theta, \lambda) := \left\{ x \in U_Q \cap \Omega : \sigma(F_{car}(x, Q, \lambda)) \geq \theta \sigma(Q) \right\}. \tag{4.2}$$

Remark 4.1 Our goal is to prove that, given $\lambda \in (0, 1)$ and $\theta \in (0, 1]$, there are positive constants η and C, depending on θ, λ, and the allowable parameters, such that for each $Q \in \mathcal{D}(\partial\Omega)$, and for each $x \in T_Q(\theta, \lambda)$, there is a Chord-arc domain Ω_x, with uniformly controlled Chord-arc constants, constructed as a union $\cup_k I_k^*$ of fattened Whitney boxes I_k^*, such that

$$U_Q^i \subset \Omega_x \subset \Omega \cap B(x, C\delta_\Omega(x)),$$

where U_Q^i is the particular connected component of U_Q containing x, and

$$\sigma(\partial\Omega_x \cap Q) \geq \eta\sigma(Q). \tag{4.3}$$

For some $Q \in \mathcal{D}(\partial\Omega)$, it may be that T_Q is empty. On the other hand, by the preceding discussion, each $x \in \Omega$ belongs to $T_Q(\theta_1, \lambda_1)$ for suitable Q, θ_1 and λ_1, so that (4.3) (with $\theta = \theta_1, \lambda = \lambda_1$) implies

$$\sigma(\partial\Omega_x \cap \Delta_x) \geq \eta_1\sigma(\Delta_x),$$

with $\eta_1 \approx \eta$, where Q is the particular Q_i selected in the previous paragraph. Moreover, since $x \in T_Q \subset U_Q$, we can modify Ω_x if necessary, by adjoining to it one or more fattened Whitney boxes I^* with $\ell(I) \approx \ell(Q)$, to ensure that for the modified Ω_x, it holds in addition that $\text{dist}(x, \partial\Omega_x) \gtrsim \ell(Q) \approx \delta_\Omega(x)$, and therefore Ω_x verifies all the conditions in Definition 2.12.

The rest of this section is therefore devoted to proving that there exists, for a given Q and for each $x \in T_Q(\theta, \lambda)$, a Chord-arc domain Ω_x satisfying the stated properties (when the set $T_Q(\theta, \lambda)$ is not vacuous). To this end, we let $\lambda \in (0, 1)$ (by Remark 4.1, any fixed $\lambda \leq \lambda_1$ will suffice). We also fix positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, and for these values of η and K, we make the bilateral Corona decomposition of Lemma 3.1, so that $\mathcal{D}(\partial\Omega) = \mathcal{G} \cup \mathcal{B}$. We also construct the Whitney collections $\mathcal{W}_Q^0$ in (3.6), and $\mathcal{W}_Q^*$ of Lemma 3.3 for this same choice of η and K.

Given a cube $Q \in \mathcal{D}(\partial\Omega)$, we set

$$\mathcal{D}_*(Q) := \left\{ Q' \subset Q : \ell(Q)/4 \leq \ell(Q') \leq \ell(Q) \right\}. \tag{4.4}$$

Thus, $\mathcal{D}_*(Q)$ consists of the cube Q itself, along with its dyadic children and grandchildren. Let

$$\mathcal{M} := \{Q(\mathsf{T})\}_{\mathsf{T} \subset \mathcal{G}}$$

denote the collection of cubes which are the maximal elements of the trees T in $\mathcal{G}$. We define

$$\alpha_Q := \begin{cases} \sigma(Q), & \text{if } (\mathcal{M} \cup \mathcal{B}) \cap \mathcal{D}_*(Q) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \qquad (4.5)$$

Given any collection $\mathcal{D}' \subset \mathcal{D}(\partial\Omega)$, we set

$$\mathfrak{m}(\mathcal{D}') := \sum_{Q \in \mathcal{D}'} \alpha_Q. \qquad (4.6)$$

Then $\mathfrak{m}$ is a discrete Carleson measure, i.e., recalling that $\mathcal{D}(R)$ is the discrete Carleson region relative to R defined in (2.10), we claim that there is a uniform constant C such that

$$\mathfrak{m}(\mathcal{D}(R)) = \sum_{Q \subset R} \alpha_Q \leq C\sigma(R), \qquad \forall\, R \in \mathcal{D}(\partial\Omega). \qquad (4.7)$$

Indeed, note that for any $Q' \in \mathcal{D}$, there are at most 3 cubes Q such that $Q' \in \mathcal{D}_*(Q)$ (namely, Q' itself, its dyadic parent, and its dyadic grandparent), and that by n-ADR, $\sigma(Q) \approx \sigma(Q')$, if $Q' \in \mathcal{D}_*(Q)$. Thus, given any $R \in \mathcal{D}(\partial\Omega)$,

$$\begin{aligned}
\mathfrak{m}(\mathcal{D}(R)) = \sum_{Q \subset R} \alpha_Q &\leq \sum_{Q' \in \mathcal{M} \cup \mathcal{B}} \sum_{Q \subset R:\, Q' \in \mathcal{D}_*(Q)} \sigma(Q) \\
&\lesssim \sum_{Q' \in \mathcal{M} \cup \mathcal{B}:\, Q' \subset R} \sigma(Q') \leq C\sigma(R),
\end{aligned}$$

by Lemma 3.1 part (2). Here, and throughout the remainder of this section, a generic constant C, and implicit constants, are allowed to depend upon the choice of the parameters η and K that we have fixed, along with the usual allowable parameters.

With (4.7) in hand, we therefore have

$$M_0 := \sup_{Q \in \mathcal{D}(\Omega)} \frac{\mathfrak{m}(\mathcal{D}(Q))}{\sigma(Q)} \leq C < \infty. \qquad (4.8)$$

4.2 Induction hypothesis and outline of proof

As mentioned above, our proof will be based on a two parameter induction scheme. Given $\lambda \in (0, \lambda_1]$ fixed as above, we recall that the set $F_{car}(x, Q, \lambda)$ is defined in (4.1). The induction hypothesis, which we formulate for any $a \geq 0$, and any $\theta \in (0, 1]$ is as follows:

$\boxed{H[a, \theta]}$

> *There is a positive constant $c_a = c_a(\theta) < 1$ such that for any given $Q \in \mathcal{D}(\partial\Omega)$, if*
>
> $$\mathfrak{m}(\mathcal{D}(Q)) \leq a\sigma(Q), \qquad (4.10)$$
>
> *and if there is a subset $V_Q \subset U_Q \cap \Omega$ for which*
>
> $$\sigma\left(\bigcup_{x \in V_Q} F_{car}(x, Q, \lambda)\right) \geq \theta\sigma(Q), \qquad (4.11)$$
>
> *then there is a subset $V_Q^* \subset V_Q$, such that for each connected component U_Q^i of U_Q which meets V_Q^*, there is a Chord-arc domain Ω_Q^i which is the interior of the union of a collection of fattened Whitney cubes I^*, and whose Chord-arc constants depend only on dimension, λ, a, θ, and the ADR constants for Ω. Moreover, $U_Q^i \subset \Omega_Q^i \subset B_Q^* \cap \Omega = B(x_Q, K\ell(Q)) \cap \Omega$, and $\sum_i \sigma(\partial\Omega_Q^i \cap Q) \geq c_a\sigma(Q)$, where the sum runs over those i such that U_Q^i meets V_Q^*.*

Let us briefly sketch the strategy of the proof. We first fix $\theta = 1$, and by induction on a, establish $H[M_0, 1]$. We then show that there is a fixed $\zeta \in (0, 1)$ such that $H[M_0, \theta]$ implies $H[M_0, \zeta\theta]$, for every $\theta \in (0, 1]$. Iterating, we then obtain $H[M_0, \theta_1]$ for any $\theta_1 \in (0, 1]$. Now, by (4.8), we have (4.10) with $a = M_0$, for every $Q \in \mathcal{D}(\partial\Omega)$. Thus, $H[M_0, \theta_1]$ may be applied in every cube Q such that $T_Q(\theta_1, \lambda)$ (see (4.2)) is non-empty, with $V_Q = \{x\}$, for any $x \in T_Q(\theta_1, \lambda)$. For $\lambda \leq \lambda_1$, and an appropriate choice of θ_1, by Remark 4.1, we obtain the existence of a Chord-arc domain Ω_x verifying the conditions of Definition 2.12, and thus that Theorem 1.2 holds, as desired.

5 Some geometric observations

We begin with some preliminary observations. In what follows we have fixed $\lambda \in (0, \lambda_1]$ and two positive numbers $K \gg \lambda^{-4}$, and $\eta \leq K^{-4/3} \ll \lambda^4$, for which the bilateral Corona decomposition of $\mathcal{D}(\partial\Omega)$ in Lemma 3.1 is applied.

We now fix $k_0 \in \mathbb{N}$, $k_0 \geq 4$, such that

$$2^{-k_0} \leq \frac{\eta}{K} < 2^{-k_0+1}. \tag{5.1}$$

Lemma 5.1 *Let* $Q \in \mathcal{D}(\partial\Omega)$, *and suppose that* $Q' \subset Q$, *with* $\ell(Q') \leq 2^{-k_0}\ell(Q)$. *Suppose that there are points* $x \in U_Q \cap \Omega$ *and* $y \in Q'$, *that are connected by a* λ-*carrot path* $\gamma = \gamma(y, x)$ *in* Ω. *Then* γ *meets* $U_{Q'} \cap \Omega$.

Proof By construction (see (3.6), Lemma 3.3, (3.10) and (3.11)), $x \in U_Q$ implies that

$$\eta^{1/2}\ell(Q) \lesssim \delta_\Omega(x) \lesssim K^{1/2}\ell(Q).$$

Since $2^{-k_0} \ll \eta$, and $\ell(Q') \leq 2^{-k_0}\ell(Q)$, we then have that $x \in \Omega \backslash B(y, 2\ell(Q'))$, so $\gamma(y, x)$ meets $B(y, 2\ell(Q'))\backslash B(y, \ell(Q'))$, say at a point z. Since $\gamma(y, x)$ is a λ-carrot path, and since we have previously specified that $\eta \ll \lambda^4$,

$$\delta_\Omega(z) \geq \lambda\ell(\gamma(y, z)) \geq \lambda|y - z| \geq \lambda\ell(Q') \gg \eta^{1/4}\ell(Q').$$

On the other hand

$$\delta_\Omega(z) \leq \text{dist}(z, Q') \leq |z - y| \leq 2\ell(Q') \ll K^{1/2}\ell(Q').$$

In particular then, the Whitney box I containing z must belong to $\mathcal{W}^0_{Q'}$ (see (3.6)), so $z \in U_{Q'}$. Note that $z \in \Omega$ since $\gamma \subset \Omega$. $\qquad\square$

We shall also require the following. We recall that by Lemma 3.3, for $Q \in \mathsf{T} \subset \mathcal{G}$, the Whitney region U_Q has the splitting $U_Q = U_Q^+ \cup U_Q^-$, with U_Q^+ (resp. U_Q^-) lying above (resp., below) the Lipschitz graph Γ_T of Lemma 3.1.

Lemma 5.2 *Let* $Q' \subset Q$, *and suppose that* Q' *and* Q *both belong to* $\mathcal{G}$, *and moreover that both* Q' *and* Q *belong to the same tree* $\mathsf{T} \subset \mathcal{G}$. *Suppose that* $y \in Q'$ *and* $x \in U_Q \cap \Omega$ *are connected via a* λ-*carrot path* $\gamma(y, x)$ *in* Ω, *and assume that there is a point* $z \in \gamma(y, x) \cap U_{Q'} \cap \Omega$ *(by Lemma 5.1 we know that such a* z *exists provided* $\ell(Q') \leq 2^{-k_0}\ell(Q)$*). Then* $x \in U_Q^+$ *if and only if* $z \in U_{Q'}^+$ *(thus,* $x \in U_Q^-$ *if and only if* $z \in U_{Q'}^-$*).*

Proof We suppose for the sake of contradiction that, e.g., $x \in U_Q^+$, and that $z \in U_{Q'}^-$. Thus, in traveling from y to z and then to x along the path $\gamma(y, x)$, one must cross the Lipschitz graph Γ_T at least once between z and x. Let y_1

be the first point on $\gamma(y, x) \cap \Gamma_T$ that one encounters *after* z, when traveling toward x. By Lemma 3.3,

$$K^{1/2}\ell(Q) \gtrsim \delta_\Omega(x) \geq \lambda\ell\big(\gamma(y, x)\big) \gg K^{-1/4}\ell\big(\gamma(y, x)\big),$$

where we recall that we have fixed $K \gg \lambda^{-4}$. Consequently, $\ell\big(\gamma(y, x)\big) \ll K^{3/4}\ell(Q)$, so in particular, $\gamma(y, x) \subset B_Q^* := B\big(x_Q, K\ell(Q)\big)$, as in Lemma 3.1. On the other hand, $y_1 \notin B_{Q'}^*$. Indeed, $y_1 \in \Gamma_T$, so if $y_1 \in B_{Q'}^*$, then by (3.1), $\delta_\Omega(y_1) \leq \eta\ell(Q')$. However,

$$\delta_\Omega(y_1) \geq \lambda\ell\big(\gamma(y, y_1)\big) \geq \lambda\ell\big(\gamma(y, z)\big) \geq \lambda|y - z| \geq \lambda \operatorname{dist}(z, Q') \gtrsim \lambda\eta^{1/2}\ell(Q'),$$

where in the last step we have used Lemma 3.3. This contradicts our choice of $\eta \ll \lambda^4$.

We now form a chain of consecutive dyadic cubes $\{P_i\} \subset \mathcal{D}(Q)$, connecting Q' to Q, i.e.,

$$Q' = P_0 \lhd P_1 \lhd P_2 \lhd \cdots \lhd P_M \lhd P_{M+1} = Q,$$

where the introduced notation $P_i \lhd P_{i+1}$ means that P_i is the dyadic child of P_{i+1}, that is, $P_i \subset P_{i+1}$ and $\ell(P_{i+1}) = 2\ell(P_i)$. Let $P := P_{i_0}, 1 \leq i_0 \leq M+1$, be the smallest of the cubes P_i such that $y_1 \in B_{P_i}^*$. Setting $P' := P_{i_0-1}$, we then have that $y_1 \in B_P^*$, and $y_1 \notin B_{P'}^*$. By the coherency of T, it follows that $P \in T$, so by (3.1),

$$\delta_\Omega(y_1) \leq \eta\ell(P). \tag{5.2}$$

On the other hand,

$$\operatorname{dist}(y_1, P') \gtrsim K\ell(P') \approx K\ell(P),$$

and therefore, since $y \in Q' \subset P'$,

$$\delta_\Omega(y_1) \geq \lambda\ell\big(\gamma(y, y_1)\big) \geq \lambda|y - y_1| \geq \lambda \operatorname{dist}(y_1, P') \gtrsim \lambda K\ell(P). \tag{5.3}$$

Combining (5.2) and (5.3), we see that $\lambda \lesssim \eta/K$, which contradicts that we have fixed $\eta \ll \lambda^4$, and $K \gg \lambda^{-4}$. $\square$

Lemma 5.3 *Fix $\lambda \in (0, 1)$. Given $Q \in \mathcal{D}(\partial\Omega)$ and a non-empty set $V_Q \subset U_Q \cap \Omega$, such that each $x \in V_Q$ may be connected by a λ-carrot path to some $y \in Q$, set*

$$F_Q := \bigcup_{x \in V_Q} F_{car}(x, Q, \lambda), \tag{5.4}$$

where we recall that $F_{car}(x, Q, \lambda)$ is the set of $y \in Q$ that are connected via a λ-carrot path to x (see (4.1)). Let $Q' \subset Q$ be such that $\ell(Q') \leq 2^{-k_0}\ell(Q)$ and $F_Q \cap Q' \neq \emptyset$. Then, there exists a non-empty set $V_{Q'} \subset U_{Q'} \cap \Omega$ such that if we define $F_{Q'}$ as in (5.4) with Q' replacing Q, then $F_Q \cap Q' \subset F_{Q'}$. Moreover, for every $y \in V_{Q'}$, there exist $x \in V_Q$, $y \in Q'$ (indeed $y \in F_Q \cap Q'$) and a λ-carrot path $\gamma = \gamma(y, x)$ such that $y \in \gamma$.

Proof For every $y \in F_Q \cap Q'$, by definition of F_Q, there exist $x \in V_Q$ and a λ-carrot path $\gamma = \gamma(y, x)$. By Lemma 5.1, there is a point $y' = y'(y) \in \gamma \cap U_{Q'} \cap \Omega$ (there can be more than one y', but we just pick one). Note that the sub-path $\gamma(y, y') \subset \gamma(y, x)$ is also a λ-carrot path, for the same constant λ. All the conclusions in the lemma follow easily from the construction by letting $V_{Q'} = \bigcup_{y \in F_Q \cap Q'} y'(y)$.   □

Remark 5.4 It follows easily from the previous proof that under the same assumptions, if one further assumes that $\ell(Q') < 2^{-k_0}\ell(Q)$, we can then repeat the argument with both Q' and $(Q')^*$ (the dyadic parent of Q') to obtain respectively $V_{Q'}$ and $V_{(Q')^*}$. Moreover, this can be done in such a way that every point in $V_{Q'}$ (resp. $V_{(Q')^*}$) belongs to a λ-carrot path which also meets $V_{(Q')^*}$ (resp. $V_{Q'}$), connecting U_Q and Q'.

Given a family $\mathcal{F} := \{Q_j\} \subset \mathcal{D}(\partial\Omega)$ of pairwise disjoint cubes, we recall that the "discrete sawtooth" $\mathcal{D}_\mathcal{F}$ is the collection of all cubes in $\mathcal{D}(\partial\Omega)$ that are not contained in any $Q_j \in \mathcal{F}$ (see (3.14)), and we define the restriction of $\mathfrak{m}$ (cf. (4.5), (4.6)) to the sawtooth $\mathcal{D}_\mathcal{F}$ by

$$\mathfrak{m}_\mathcal{F}(\mathcal{D}') := \mathfrak{m}(\mathcal{D}' \cap \mathcal{D}_\mathcal{F}) = \sum_{Q \in \mathcal{D}' \setminus \left(\cup_{Q_j \in \mathcal{F}} \mathcal{D}(Q_j)\right)} \alpha_Q. \qquad (5.5)$$

We then set

$$\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} := \sup_{Q' \subset Q} \frac{\mathfrak{m}_\mathcal{F}(\mathcal{D}(Q'))}{\sigma(Q')}.$$

Let us note that we may allow $\mathcal{F}$ to be empty, in which case $\mathcal{D}_\mathcal{F} = \mathcal{D}$ and $\mathfrak{m}_\mathcal{F}$ is simply $\mathfrak{m}$. We note that the following claim, and others in the sequel, remain true when $\mathcal{F}$ is empty; sometimes trivially so, and sometimes with some straightforward changes that are left to the interested reader.

Claim 5.5 *Given $Q \in \mathcal{D}(\partial\Omega)$, and a family $\mathcal{F} = \mathcal{F}_Q := \{Q_j\} \subset \mathcal{D}(Q)\setminus\{Q\}$ of pairwise disjoint sub-cubes of Q, if $\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} \leq 1/2$, then each $Q' \in \mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q)$, each $Q_j \in \mathcal{F}$, and every dyadic child Q'_j of any $Q_j \in \mathcal{F}$, belong to the good collection $\mathcal{G}$, and moreover, every such cube belongs to the **same***

tree $\mathsf{T} \subset \mathcal{G}$. In particular, $\mathsf{T}' := \mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q)$ is a semi-coherent subtree of T, and so is $\mathsf{T}'' := (\mathcal{D}_\mathcal{F} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathcal{D}(Q)$, where $\mathcal{F}'$ denotes the collection of all dyadic children of cubes in $\mathcal{F}$.

Indeed, if any $Q' \in \mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q)$ were in $\mathcal{M} \cup \mathcal{B}$ (recall that $\mathcal{M} :=$ $\{Q(\mathsf{T})\}_{\mathsf{T} \subset \mathcal{G}}$ is the collection of cubes which are the maximal elements of the trees T in $\mathcal{G}$), then by construction $\alpha_{Q'} = \sigma(Q')$ for that cube (see (4.5)), so by definition of $\mathfrak{m}$ and $\mathfrak{m}_\mathcal{F}$, we would have

$$1 = \frac{\sigma(Q')}{\sigma(Q')} \leq \frac{\mathfrak{m}_\mathcal{F}(\mathcal{D}(Q'))}{\sigma(Q')} \leq \|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} \leq \frac{1}{2},$$

a contradiction. Similarly, if some $Q_j \in \mathcal{F}$ (respectively, $Q'_j \in \mathcal{F}'$) were in $\mathcal{M} \cup \mathcal{B}$, then its dyadic parent (respectively, dyadic grandparent) Q^*_j would belong to $\mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q)$, and by definition $\alpha_{Q^*_j} = \sigma(Q^*_j)$, so again we reach a contradiction. Consequently, $\mathcal{F} \cup \mathcal{F}' \cup (\mathcal{D}_\mathcal{F} \cap \mathcal{D}(Q))$ does not meet $\mathcal{M} \cup \mathcal{B}$, and the claim follows.

6 Construction of chord-arc subdomains

For future reference, we now prove the following. Recall that for $Q \in \mathcal{G}$, U_Q has precisely two connected components $U_Q^\pm$ in $\mathbb{R}^{n+1} \backslash \partial\Omega$.

Lemma 6.1 *Let $Q \in \mathcal{D}(\partial\Omega)$, let k_1 be such that $2^{k_1} > 2^{k_0} \gg 100K$, see (5.1), and suppose that there is a family $\mathcal{F} = \mathcal{F}_Q := \{Q_j\} \subset \mathcal{D}(Q) \backslash \{Q\}$ of pairwise disjoint sub-cubes of Q, with $\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} \leq 1/2$ (hence by Claim 5.5, there is some $\mathsf{T} \subset \mathcal{G}$ with $\mathsf{T} \supset (\mathcal{D}_\mathcal{F} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathcal{D}(Q)$), and a non-empty subcollection $\mathcal{F}^* \subset \mathcal{F}$, such that:*

(i) *$\ell(Q_j) \leq 2^{-k_1} \ell(Q)$, for each cube $Q_j \in \mathcal{F}^*$;*
(ii) *the collection of balls $\{\kappa B^*_{Q_j} := B(x_{Q_j}, \kappa K \ell(Q_j)) : Q_j \in \mathcal{F}^*\}$ is pairwise disjoint, where $\kappa \gg K^4$ is a sufficiently large positive constant; and*
(iii) *$\mathcal{F}^*$ has a disjoint decomposition $\mathcal{F}^* = \mathcal{F}^*_+ \cup \mathcal{F}^*_-$, where for each $Q_j \in \mathcal{F}^*_\pm$, there is a Chord-arc subdomain $\Omega^\pm_{Q_j} \subset \Omega$, consisting of a union of fattened Whitney cubes I^*, with $U^\pm_{Q_j} \subset \Omega^\pm_{Q_j} \subset B^*_{Q_j} := B(x_{Q_j}, K\ell(Q_j))$, and with uniform control of the Chord-arc constants.*

Define a semi-coherent subtree $\mathsf{T}^* \subset \mathsf{T}$ by

$$\mathsf{T}^* = \{Q' \in \mathcal{D}(Q) : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}^*\},$$

and for each choice of ± for which $\mathcal{F}_{\pm}^*$ is non-empty, set

$$\Omega_Q^\pm := \Omega_{\mathsf{T}^*}^\pm \bigcup \left(\bigcup_{Q_j \in \mathcal{F}_\pm^*} \Omega_{Q_j}^\pm \right) \tag{6.1}$$

Then for κ large enough, depending only on allowable parameters, $\Omega_Q^\pm$ is a Chord-arc domain, with chord arc constants depending only on the uniformly controlled Chord-arc constants of $\Omega_{Q_j}^\pm$ and on the other allowable parameters. Moreover, $\Omega_Q^\pm \subset B_Q^* \cap \Omega = B(x_Q, K\ell(Q)) \cap \Omega$, and $\Omega_Q^\pm$ is a union of fattened Whitney cubes.

Remark 6.2 Note that we define $\Omega_Q^\pm$ if and only if $\mathcal{F}_\pm^*$ is non-empty. It may be that one of $\mathcal{F}_+^*$, $\mathcal{F}_-^*$ is empty, but $\mathcal{F}_+^*$ and $\mathcal{F}_-^*$ cannot both be empty, since $\mathcal{F}^*$ is non-empty by assumption.

Proof of Lemma 6.1 Without loss of generality we may assume that $\Omega_{Q_j}^\pm$ is not contained in $\Omega_{\mathsf{T}^*}^\pm$ for all $Q_j \in \mathcal{F}^*$ (otherwise we can drop those cubes from $\mathcal{F}^*$). On the other hand, we notice that $\Omega_Q^\pm$ is a union of (open) fattened Whitney cubes (assuming that it is non-empty): each $\Omega_{Q_j}^\pm$ has this property by assumption, as does $\Omega_{\mathsf{T}^*}^\pm$ by construction.

We next observe that if Ω_Q^+ (resp. Ω_Q^-) is non-empty, then it is contained in Ω. Indeed, by construction, Ω_Q^+ is non-empty if and only if $\mathcal{F}_+^*$ is non-empty. In turn, $\mathcal{F}_+^*$ is non-empty if and only if there is some $Q_j \in \mathcal{F}^*$ such that $U_{Q_j}^+ \subset \Omega_{Q_j}^+ \subset \Omega$, and moreover, the latter is true for every $Q_j \in \mathcal{F}_+^*$, by definition. But each such Q_j belongs to T^*, hence $U_{Q_j}^+ \subset \Omega_{\mathsf{T}^*}^+$, again by construction (see (3.9)). Thus, $\Omega_{\mathsf{T}^*}^+$ meets Ω, and since $\Omega_{\mathsf{T}^*}^+ \subset \mathbb{R}^{n+1} \setminus \partial\Omega$, therefore $\Omega_{\mathsf{T}^*}^+ \subset \Omega$. Combining these observations, we see that $\Omega_Q^+ \subset \Omega$. Of course, the same reasoning applies to Ω_Q^-, provided it is non-empty.

In addition, since $\mathsf{T}^* \subset \mathsf{T}$, and since $K \gg K^{1/2}$, by Lemma 3.3 we have $\Omega_{\mathsf{T}^*}^\pm \subset B_Q^* = B(x_Q, K\ell(Q))$. Furthermore, $\Omega_{Q_j}^\pm \subset B_{Q_j}^* := B(x_{Q_j}, K\ell(Q_j))$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q) \leq (100K)^{-1}\ell(Q)$, we obtain

$$\operatorname{dist}(\Omega_{Q_j}^\pm, Q) + \operatorname{diam}(\Omega_{Q_j}^\pm) \leq 3K\ell(Q_j) \leq 3K2^{-k_1}\ell(Q) \ll \ell(Q).$$

Thus, in particular, $\Omega_{Q_j}^\pm \subset B_Q^*$, and therefore also $\Omega_Q^\pm \subset B_Q^*$.

It therefore remains to establish the Chord-arc properties. It is straightforward to prove the interior corkscrew condition and the upper n-ADR bound, and we omit the details. Thus, we must verify the Harnack Chain condition, the lower n-ADR bound, and the exterior corkscrew condition.

6.1 Harnack Chains

Suppose, without loss of generality, that Ω_Q^+ is non-empty, and let x, $y \in \Omega_Q^+$, with $|x - y| = r$. If x and y both lie in $\Omega_{T^*}^+$, or in the same $\Omega_{Q_j}^+$, then we can connect x and y by a suitable Harnack path, since each of these domains is Chord-arc. Thus, we may suppose either that 1) $x \in \Omega_{T^*}^+$ and y lies in some $\Omega_{Q_j}^+$, or that 2) x and y lie in two distinct $\Omega_{Q_{j_1}}^+$ and $\Omega_{Q_{j_2}}^+$. We may reduce the latter case to the former case: by the separation property (ii) in Lemma 6.1, we must have $r \gtrsim \kappa \max \left(\operatorname{diam}(\Omega_{Q_{j_1}}^+), \operatorname{diam}(\Omega_{Q_{j_2}}^+) \right)$, so given case 1), we can connect $x \in \Omega_{Q_{j_1}}^+$ to the center z_1 of some $I_1^* \subset U_{Q_1}^+$, and $y \in \Omega_{Q_{j_2}}^+$ to the center z_2 of some $I_2 \subset U_{Q_2}^+$, where $Q_1, Q_2 \in \mathsf{T}^*$, with $Q_{j_i} \subset Q_i \subset Q$, and $\ell(Q_i) \approx r$, $i = 1, 2$. Finally, we can connect z_1 and z_2 using that $\Omega_{T^*}^+$ is Chord-arc.

Hence, we need only construct a suitable Harnack Chain in Case 1). We note that by assumption and construction, $U_{Q_j}^+ \subset \Omega_{T^*}^+ \cap \Omega_{Q_j}^+$.

Suppose first that

$$|x - y| = r \leq c'\ell(Q_j), \tag{6.2}$$

where $c' \leq 1$ is a sufficiently small positive constant to be chosen. Since $y \in \Omega_{Q_j}^+ \subset B_{Q_j}^*$, we then have that $x \in 2B_{Q_j}^*$, so by the construction of $\Omega_{T^*}^+$ and the separation property (ii), it follows that $\delta_\Omega(x) \geq c\ell(Q_j)$, where c is a uniform constant depending only on the allowable parameters (in particular, this fact is true for all $x \in \Omega_{T^*}^+ \cap 2B_{Q_j}^*$, so it does not depend on the choice of $c' < 1$). Now choosing $c' \leq c/2$ (eventually, it may be even smaller), we find that $\delta_\Omega(y) \geq (c/2)\ell(Q_j)$. Moreover, $y \in \Omega_{Q_j}^+ \subset B_{Q_j}^*$ implies that $\delta_\Omega(y) \leq K\ell(Q_j)$. Also, since $x \in 2B_{Q_j}^*$ we have that $\delta_\Omega(x) \leq 2K\ell(Q_j)$. Since $\Omega_{Q_j}^+$ and $\Omega_{T^*}^+$ are each the interior of a union of fattened Whitney cubes, it follows that there are Whitney cubes I and J, with $x \in I^*$, $y \in J^*$, and

$$\ell(I) \approx \ell(J) \approx \ell(Q_j),$$

where the implicit constants depend on K. For c' small enough in (6.2), depending on the implicit constants in the last display, and on the parameter τ in (3.4), this can happen only if I^* and J^* overlap (recall that we have fixed τ small enough that I^* and J^* overlap if and only if I and J have a boundary point in common), in which case we may trivially connect x and y by a suitable Harnack Chain.

On the other hand, suppose that

$$|x - y| = r \geq c'\ell(Q_j).$$

Let $z \in U_{Q_j}^+ \subset \Omega_{\mathsf{T}^*}^+ \cap \Omega_{Q_j}^+$, with $\text{dist}(z, \partial \Omega_Q^+) \gtrsim \ell(Q_j)$ (we may find such a z, since $U_{Q_j}^+$ is a union of fattened Whitney cubes, all of length $\ell(I^*) \approx \ell(Q_j)$; just take z to be the center of such an I^*). We may then construct an appropriate Harnack Chain from y to x by connecting y to z via a Harnack Chain in the Chord-arc domain $\Omega_{Q_j}^+$, and z to x via a Harnack Chain in the Chord-arc domain $\Omega_{\mathsf{T}^*}^+$.

6.2 Lower n-ADR and exterior corkscrews

We will establish these two properties essentially simultaneously. Again suppose that, e.g., Ω_Q^+ is non-empty. Let $x \in \partial \Omega_Q^+$, and consider $B(x, r)$, with $r < \text{diam} \, \Omega_Q^+ \approx_K \ell(Q)$. Our main goal at this stage is to prove the following:

$$\left| B(x, r) \backslash \overline{\Omega_Q^+} \right| \geq cr^{n+1}, \tag{6.3}$$

with c a uniform positive constant depending only upon allowable parameters (including κ). Indeed, momentarily taking this estimate for granted, we may combine (6.3) with the interior corkscrew condition to deduce the lower n-ADR bound via the relative isoperimetric inequality [24, p. 190]. In turn, with both the lower and upper n-ADR bounds in hand, (6.3) implies the existence of exterior corkscrews (see, e.g., [31, Lemma 5.7]).

Thus, it is enough to prove (6.3). We consider the following cases.

Case 1: $B(x, r/2)$ does not meet $\partial \Omega_{Q_j}^+$ for any $Q_j \in \mathcal{F}_+^*$. In this case, the exterior corkscrew for $\Omega_{\mathsf{T}^*}^+$ associated with $B(x, r/2)$ easily implies (6.3).

Case 2: $B(x, r/2)$ meets $\partial \Omega_{Q_j}^+$ for at least one $Q_j \in \mathcal{F}_+^*$, and $r \leq \kappa^{1/2} \ell(Q_{j_0})$, where Q_{j_0} is chosen to have the largest length $\ell(Q_{j_0})$ among those Q_j such that $\partial \Omega_{Q_j}^+$ meets $B(x, r/2)$. We now further split the present case into subcases.

Subcase 2a: $B(x, r/2)$ meets $\partial \Omega_{Q_{j_0}}^+$ at a point z with $\delta_\Omega(z) \leq (M\kappa^{1/2})^{-1} \ell(Q_{j_0})$, where M is a large number to be chosen. Then $B(z, (M\kappa^{1/2})^{-1} r) \subset B(x, r)$, for M large enough. In addition, we claim that $B(z, (M\kappa^{1/2})^{-1} r)$ misses $\Omega_{\mathsf{T}^*}^+ \cup \left(\cup_{j \neq j_0} \Omega_{Q_j}^+ \right)$. The fact that $B(z, (M\kappa^{1/2})^{-1} r)$ misses every other $\Omega_{Q_j}^+$, $j \neq j_0$, follows immediately from the restriction $r \leq \kappa^{1/2} \ell(Q_{j_0})$, and the separation property (ii). To see that $B(z, (M\kappa^{1/2})^{-1} r)$ misses $\Omega_{\mathsf{T}^*}^+$, note that if $|z - y| < (M\kappa^{1/2})^{-1} r$, then

$$\delta_\Omega(y) \leq \delta_\Omega(z) + (M\kappa^{1/2})^{-1} r \leq \left((M\kappa^{1/2})^{-1} + M^{-1} \right) \ell(Q_{j_0}) \ll \ell(Q_{j_0}),$$

for M large. On the other hand,

$$\delta_\Omega(y) \gtrsim \ell(Q_{j_0}), \qquad \forall\, y \in \Omega^+_{T^*} \cap B\big(z, \kappa^{1/2}\ell(Q_{j_0})\big),$$

by the construction of $\Omega^+_{T^*}$ and the separation property (ii). Thus, the claim follows, for a sufficiently large (fixed) choice of M. Since $B(z, (M\kappa^{1/2})^{-1}r)$ misses $\Omega^+_{T^*}$ and all other $\Omega^+_{Q_j}$, we inherit an exterior corkscrew point in the present case (depending on M and κ) from the Chord-arc domain $\Omega^+_{Q_{j_0}}$. Again (6.3) follows.

Subcase 2b: $\delta_\Omega(z) \geq (M\kappa^{1/2})^{-1}\ell(Q_{j_0})$, for every $z \in B(x, r/2) \cap \partial\Omega^+_{Q_{j_0}}$ (hence $\delta_\Omega(z) \approx_{\kappa,K} \ell(Q_{j_0})$, since $\Omega^+_{Q_{j_0}} \subset B^*_{Q_{j_0}}$). We claim that consequently, $x \in \partial I^*$, for some I with $\ell(I) \approx \ell(Q_{j_0}) \gtrsim r$, such that int $I^* \subset \Omega^+_Q$. To see this, observe that it is clear if $x \in \partial\Omega^+_{Q_{j_0}}$ (just take $z = x$). Otherwise, by the separation property (ii), the remaining possibility in the present scenario is that $x \in \partial U^+_{Q'} \cap \partial\Omega^+_{T^*}$, for some $Q' \in T^*$ with $Q_{j_0} \subset Q'$, in which case $\delta_\Omega(x) \approx \ell(Q') \geq \ell(Q_{j_0})$. Since also $\delta_\Omega(x) \leq |x - z| + \delta_\Omega(z) \lesssim_{\kappa,K} \ell(Q_{j_0})$, for any $z \in B(x, r/2) \cap \partial\Omega^+_{Q_{j_0}}$, the claim follows.

On the other hand, since $x \in \partial\Omega^+_Q$, there is a $J \in \mathcal{W}$ with $\ell(J) \approx \ell(Q_{j_0})$, such that J^* is not contained in Ω^+_Q. We then have an exterior corkscrew point in $J^* \cap B(x, r)$, and (6.3) follows in this case.

Case 3: $B(x, r/2)$ meets $\partial\Omega^+_{Q_j}$ for at least one $Q_j \in \mathcal{F}^*_+$, and $r > \kappa^{1/2}\ell(Q_{j_0})$, where as above Q_{j_0} has the largest length $\ell(Q_{j_0})$ among those Q_j such that $\partial\Omega^+_{Q_j}$ meets $B(x, r/2)$. In particular then, $r \gg 2K\ell(Q_{j_0}) = \mathrm{diam}(B^*_{Q_{j_0}}) \geq \mathrm{diam}(\Omega^+_{Q_{j_0}})$, since we assume $\kappa \gg K^4$.

We next claim that $B(x, r/4)$ contains some $x_1 \in \partial\Omega^+_{T^*} \cap \partial\Omega^+_Q$. This is clear if $x \in \partial\Omega^+_{T^*}$ by taking $x_1 = x$. Otherwise, $x \in \partial\Omega^+_{Q_j}$ for some $Q_j \in \mathcal{F}^*$. Note that $U^\pm_{Q_j} \subset B(x_{Q_j}, K\ell(Q_j)) \subset B(x, 2K\ell(Q_j))$. Also, $U^\pm_{Q_j} \subset \Omega^+_{T^*}$, by construction. On the other hand we note that if $z \in U^\pm_Q$ we have by (3.12)

$$|z - x_{Q_j}| \geq \delta_\Omega(z) \gtrsim \eta^{1/2}\ell(Q) \geq \eta^{1/2}2^{k_1}\ell(Q_j) \gg K\ell(Q_j)$$

by our choice of k_1. By this fact, and the definition of Ω_{T^*}, we have

$$U^\pm_Q \subset \Omega^\pm_{T^*} \backslash B(x, 3K\ell(Q_j)).$$

Using then that $\Omega^\pm_{T^*}$ is connected, we see that a path within $\Omega^\pm_{T^*}$ joining $U^\pm_{Q_j}$ with $U^\pm_Q$ must meet $\partial B(x, 3K\ell(Q_j))$. Hence we can find $y^\pm \in \Omega^\pm_{T^*} \cap \partial B(x, 3K\ell(Q_j))$. By Lemma 3.3, $\Omega^+_{T^*}$ and $\Omega^-_{T^*}$ are disjoint (they live

respectively above and below the graph Γ_T), so a path joining y^+ and y^- within $\partial B(x, 3K\ell(Q_j))$ meets some $x_1 \in \partial\Omega^+_{T*} \cap \partial B(x, 3K\ell(Q_j))$. On the other hand, $x_1 \notin \overline{\Omega^+_{Q_j}}$, since $\overline{\Omega^+_{Q_j}} \subset \overline{B^*_{Q_j}} \subset B(x, 3K\ell(Q_j))$. Furthermore, $x_1 \in \partial B(x, 3K\ell(Q_j)) \subset \kappa B^*_{Q_j}$, so by assumption (ii), we necessarily have that $x_1 \notin \overline{\Omega^+_{Q_k}}$ for $k \neq j$. Thus, $x_1 \in \partial\Omega^+_Q$, and moreover, since $B(x, r/2)$ meets $\partial\Omega^+_{Q_j}$ (at x) we have $\ell(Q_j) \leq \ell(Q_{j_0})$. Therefore, x_1 is the claimed point, since in the current case $3K\ell(Q_j) \leq 3K\ell(Q_{j_0}) \ll r$.

With the point x_1 in hand, we note that

$$B(x_1, r/4) \subset B(x, r/2) \quad \text{and} \quad B(x_1, r/2) \subset B(x, r). \tag{6.4}$$

By the exterior corkscrew condition for Ω^+_{T*},

$$\left| B(x_1, r/4) \backslash \overline{\Omega^+_{T*}} \right| \geq c_1 r^{n+1}, \tag{6.5}$$

for some constant c_1 depending only on n and the ADR/UR constants for $\partial\Omega$, by Lemma 3.3. Also, for each $\Omega^+_{Q_j}$ whose boundary meets $B(x_1, r/4) \backslash \overline{\Omega^+_{T*}}$ (and thus meets $B(x, r/2)$),

$$\kappa^{1/4} \operatorname{diam}(B^*_{Q_j}) \leq \kappa^{1/4} \operatorname{diam}(B^*_{Q_{j_0}}) \leq 2K\kappa^{1/4}\ell(Q_{j_0}) \leq \frac{2Kr}{\kappa^{1/4}} \ll r, \tag{6.6}$$

in the present scenario. Consequently, $\kappa^{1/4} B^*_{Q_j} \subset B(x_1, r/2)$, for all such Q_j.

We now make the following claim.

Claim 6.3 *On has*
$$\left| B(x_1, r/2) \backslash \overline{\Omega^+_Q} \right| \geq c_2 r^{n+1}, \tag{6.7}$$

for some $c_2 > 0$ depending only on allowable parameters.

Observe that by the second containment in (6.4), we obtain (6.3) as an immediate consequence of (6.7), and thus the proof will be complete once we have established Claim 6.3.

Proof of Claim 6.3 To prove the claim, we suppose first that

$$\sum \left| B^*_{Q_j} \backslash \overline{\Omega^+_{T*}} \right| \leq \frac{c_1}{2} r^{n+1}, \tag{6.8}$$

where the sum runs over those j such that $\overline{B^*_{Q_j}}$ meets $B(x_1, r/4) \backslash \overline{\Omega^+_{T*}}$, and c_1 is the constant in (6.5). In that case, (6.7) holds with $c_2 = c_1/2$ (and even with $B(x_1, r/4)$), by definition of Ω^+_Q (see (6.1)), and the fact that $\Omega_{Q_j} \subset B^*_{Q_j}$. On

the other hand, if (6.8) fails, then summing over the same subset of indices j, we have

$$CK \sum \ell(Q_j)^{n+1} \geq \sum \left| B_{Q_j}^* \backslash \overline{\Omega_{T^*}^+} \right| \geq \frac{c_1}{2} r^{n+1} \tag{6.9}$$

We now make a second claim:

Claim 6.4 *For j appearing in the previous sum, we have*

$$\left| \left(\kappa^{1/4} B_{Q_j}^* \backslash B_{Q_j}^* \right) \backslash \overline{\Omega_{T^*}^+} \right| \geq c \, \ell(Q_j)^{n+1}, \tag{6.10}$$

for some uniform $c > 0$.

Taking the latter claim for granted momentarily, we insert estimate (6.10) into (6.9) and sum, to obtain

$$\sum \left| \left(\kappa^{1/4} B_{Q_j}^* \backslash B_{Q_j}^* \right) \backslash \overline{\Omega_{T^*}^+} \right| \gtrsim r^{n+1}. \tag{6.11}$$

By the separation property (ii), the balls $\kappa^{1/4} B_{Q_j}^*$ are pairwise disjoint, and by assumption $\Omega_{Q_j}^+ \subset B_{Q_j}^*$. Thus, for any given j_1, $\kappa^{1/4} B_{Q_{j_1}}^* \backslash \overline{B_{Q_{j_1}}^*}$ misses $\cup_j \overline{\Omega_{Q_j}^+}$. Moreover, as noted above (see (6.6) and the ensuing comment), $\kappa^{1/4} B_{Q_j}^* \subset B(x_1, r/2)$ for each j under consideration in (6.8)–(6.11). Claim 6.3 now follows. □

Proof of Claim 6.4 There are two cases: if $\frac{1}{2}\kappa^{1/4} B_{Q_j}^* \subset \mathbb{R}^{n+1} \backslash \overline{\Omega_{T^*}^+}$, then (6.10) is trivial, since $\kappa \gg 1$. Otherwise, $\frac{1}{2}\kappa^{1/4} B_{Q_j}^*$ contains a point $z \in \partial \Omega_{T^*}^+$. In the latter case, by the exterior corkscrew condition for $\Omega_{T^*}^+$,

$$\left| B\left(z, 2^{-1}\kappa^{1/4} K \ell(Q_j)\right) \backslash \overline{\Omega_{T^*}^+} \right| \gtrsim \kappa^{(n+1)/4} \left(K\ell(Q_j)\right)^{n+1} \gg |B_{Q_j}^*|,$$

since $\kappa \gg 1$. On the other hand, $B\left(z, 2^{-1}\kappa^{1/4} K\ell(Q_j)\right) \subset \kappa^{1/4} B_{Q_j}^*$, and (6.10) follows, finishing the proof of Claim 6.4. □

Next, (6.4) and (6.7) yield (6.3) in the present case and hence the proof of Lemma 6.1 is complete. □

7 Step 2: Proof of $H[M_0, 1]$

We shall deduce $H[M_0, 1]$ (see Sect. 4.2) from the following pair of claims.

Claim 7.1 $H[0, \theta]$ *holds for every $\theta \in (0, 1]$.*

Proof of Claim 7.1 If $a = 0$ in (4.10), then $\|\mathfrak{m}\|_{\mathcal{C}(Q)} = 0$, whence it follows by Claim 5.5, with $\mathcal{F} = \emptyset$, that there is a tree $\mathsf{T} \subset \mathcal{G}$, with $\mathcal{D}(Q) \subset \mathsf{T}$. Hence $\mathsf{T}' := \mathcal{D}(Q)$ is a coherent subtree of T, so by Lemma 3.3, each of $\Omega_{\mathsf{T}'}^{\pm}$ is a CAD, containing $U_Q^{\pm}$, respectively, with $\Omega_{\mathsf{T}'}^{\pm} \subset B_Q^*$ by (3.13). Moreover, by [35, Proposition A.14]

$$Q \subset \partial\Omega_{\mathsf{T}'}^{\pm} \cap \partial\Omega,$$

so that $\sigma(Q) \leq \sigma(\partial\Omega_{\mathsf{T}'}^{\pm} \cap \partial\Omega)$. Thus, $H[0, \theta]$ holds trivially.   □

Claim 7.2 *There is a uniform constant $b > 0$ such that $H[a, 1] \implies H[a + b, 1]$, for all $a \in [0, M_0)$.*

Combining Claims 7.1 and 7.2, we find that $H[M_0, 1]$ holds.
To prove Claim 7.2, we shall require the following.

Lemma 7.3 [31, Lemma 7.2] *Suppose that E is an n-ADR set, and let $\mathfrak{m}$ be a discrete Carleson measure, as in (4.6)–(4.8) above. Fix $Q \in \mathcal{D}(E)$. Let $a \geq 0$ and $b > 0$, and suppose that $\mathfrak{m}(\mathcal{D}(Q)) \leq (a+b)\sigma(Q)$. Then there is a family $\mathcal{F} = \{Q_j\} \subset \mathcal{D}(Q)$ of pairwise disjoint cubes, and a constant C depending only on n and the ADR constant such that*

$$\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq Cb, \tag{7.1}$$

$$\sigma\left(\bigcup_{\mathcal{F}_{bad}} Q_j\right) \leq \frac{a+b}{a+2b}\sigma(Q), \tag{7.2}$$

where $\mathcal{F}_{bad} := \{Q_j \in \mathcal{F} : \mathfrak{m}(\mathcal{D}(Q_j)\backslash\{Q_j\}) > a\sigma(Q_j)\}$.

We refer the reader to [31, Lemma 7.2] for the proof. We remark that the lemma is stated in [31] in the case that E is the boundary of a connected domain, but the proof actually requires only that E have a dyadic cube structure, and that σ be a non-negative, dyadically doubling Borel measure on E. In our case, we shall of course apply the lemma with $E = \partial\Omega$, where Ω is open, but not necessarily connected.

Proof of Claim 7.2 We assume that $H[a, 1]$ holds, for some $a \in [0, M_0)$. Let us set $b = 1/(2C)$, where C is the constant in (7.1). Consider a cube $Q \in \mathcal{D}(\partial\Omega)$ with $\mathfrak{m}(\mathcal{D}(Q)) \leq (a + b)\sigma(Q)$. Suppose that there is a set $V_Q \subset U_Q \cap \Omega$ such that (4.11) holds with $\theta = 1$. We fix $k_1 > k_0$ (see (5.1)) large enough so that $2^{k_1} > 100K$.
Case 1: There exists $Q' \in \mathcal{D}_{k_1}(Q)$ (see (2.11)) with $\mathfrak{m}(\mathcal{D}(Q')) \leq a\sigma(Q')$.
    In the present scenario $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (4.11) and (5.4)), which implies $\sigma(F_Q \cap Q') = \sigma(Q')$. We apply Lemma 5.3 to obtain

$V_{Q'} \subset U_{Q'} \cap \Omega$ and the corresponding $F_{Q'}$ which satisfies $\sigma(F_{Q'}) = \sigma(Q')$. That is, (4.11) holds for Q', with $\theta = 1$. Consequently, we may apply the induction hypothesis to Q', to find $V^*_{Q'} \subset V_{Q'}$, such that for each $U^i_{Q'}$ meeting $V^*_{Q'}$, there is a Chord-arc domain $\Omega^i_{Q'} \supset U^i_{Q'}$ formed by a union of fattened Whitney cubes with $\Omega^i_{Q'} \subset B(x'_Q, K\ell(Q')) \cap \Omega$, and

$$\sum_{i:U^i_{Q'} \text{ meets } V^*_{Q'}} \sigma(\partial\Omega^i_{Q'} \cap Q') \geq c_a\sigma(Q'). \tag{7.3}$$

By Lemma 5.3, and since $k_1 > k_0$, each $y \in V^*_{Q'}$ lies on a λ-carrot path connecting some $y \in Q'$ to some $x \in V_Q$; let V^{**}_Q denote the set of all such x, and let $\mathbf{U}^{**}_Q$ (respectively, $\mathbf{U}^*_{Q'}$) denote the collection of connected components of U_Q (resp., of $U_{Q'}$) which meet V^{**}_Q (resp., $V^*_{Q'}$). By construction, each component $U^i_{Q'} \in \mathbf{U}^*_{Q'}$ may be joined to some corresponding component in $\mathbf{U}^{**}_Q$, via one of the carrot paths. After possible renumbering, we designate this component as U^i_Q, we let x_i, y_i denote the points in $V^{**}_Q \cap U^i_Q$ and in $V^*_Q \cap U^i_{Q'}$, respectively, that are joined by this carrot path, and we let γ_i be the portion of the carrot path joining x_i to y_i (if there is more than one such path or component, we just pick one). We also let $V^*_Q = \{x_i\}_i$ be the collection of all of the selected points x_i. We let $\mathcal{W}_i$ be the collection of Whitney cubes meeting γ_i, and we then define

$$\Omega^i_Q := \Omega^i_{Q'} \bigcup \text{int}\left(\bigcup_{I\in\mathcal{W}_i} I^*\right) \bigcup U^i_Q.$$

By the definition of a λ-carrot path, since $\ell(Q') \approx_{k_1} \ell(Q)$, and since $\Omega^i_{Q'}$ is a CAD, one may readily verify that Ω^i_Q is also a CAD consisting of a union $\cup_k I^*_k$ of fattened Whitney cubes I^*_k. We omit the details. Moreover, by construction,

$$\partial\Omega^i_Q \cap Q \supset \partial\Omega^i_{Q'} \cap Q',$$

so that the analogue of (7.3) holds with Q' replaced by Q, and with c_a replaced by $c_{k_1}c_a$.

It remains to verify that $\Omega^i_Q \subset B^*_Q = B(x_Q, K\ell(Q))$. By the induction hypothesis, and our choice of k_1, since $\ell(Q') = 2^{-k_1}\ell(Q)$ we have

$$\Omega^i_{Q'} \subset B^*_{Q'} \cap \Omega = B(x_{Q'}, K\ell(Q')) \cap \Omega \subset B^*_Q \cap \Omega.$$

Moreover, $U_Q \subset B_Q^*$, by (3.13). We therefore need only to consider I^* with $I \in \mathcal{W}_i$. For such an I, by definition there is a point $z_i \in I \cap \gamma_i$ and $y_i \in Q'$, so that $z_i \in \gamma(y_i, x_i)$ and thus,

$$\delta_\Omega(z_i) \leq |z_i - y_i| \leq \ell(y_i, z_i) \leq \ell(y_i, x_i) \leq \lambda^{-1}\delta_\Omega(x_i)$$
$$\leq \lambda^{-1}|x_i - x_Q| \leq \lambda^{-1}CK^{1/2}\ell(Q),$$

where in the last inequality we have used (3.12) and the fact that $x_i \in U_Q$. Hence, for every $z \in I^*$ by (3.3)

$$|z - x_Q| \leq \operatorname{diam}(2I) + |z_i - y_i| + |y_i - x_Q| \leq C|z_i - y_i|$$
$$+ \operatorname{diam}(Q) < K\ell(Q),$$

by our choice of the parameters K and λ.

We then obtain the conclusion of $H[a + b, 1]$ in the present case.

Case 2: $\mathfrak{m}(\mathcal{D}(Q')) > a\sigma(Q')$ for every $Q' \in \mathcal{D}_{k_1}(Q)$.

In this case, we apply Lemma 7.3 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathcal{D}(Q)$ such that (7.1) and (7.2) hold. In particular, by our choice of $b = 1/(2C)$,

$$\|\mathfrak{m}_\mathcal{F}\|_{\mathcal{C}(Q)} \leq 1/2, \tag{7.4}$$

so that the conclusions of Claim 5.5 hold.

We set

$$F_0 := Q \setminus \left(\bigcup_\mathcal{F} Q_j \right), \tag{7.5}$$

define

$$\mathcal{F}_{good} := \mathcal{F} \setminus \mathcal{F}_{bad} = \left\{ Q_j \in \mathcal{F} : \mathfrak{m}\left(\mathcal{D}(Q_j) \setminus \{Q_j\}\right) \leq a\sigma(Q_j) \right\}, \tag{7.6}$$

and let

$$G_0 := \bigcup_{\mathcal{F}_{good}} Q_j.$$

Then by (7.2)

$$\sigma(F_0 \cup G_0) \geq \rho\sigma(Q), \tag{7.7}$$

where $\rho \in (0, 1)$ is defined by

$$\frac{a + b}{a + 2b} \leq \frac{M_0 + b}{M_0 + 2b} =: 1 - \rho \in (0, 1). \tag{7.8}$$

We claim that

$$\ell(Q_j) \leq 2^{-k_1}\ell(Q), \quad \forall Q_j \in \mathcal{F}_{good}. \tag{7.9}$$

Indeed, if this were not true for some Q_j, then by definition of $\mathcal{F}_{good}$ and pigeon-holing there will be $Q'_j \in \mathcal{D}(Q_j)$ with $\ell(Q'_j) = 2^{-k_1} \ell(Q)$ such that $\mathfrak{m}(\mathcal{D}(Q'_j)) \leq a\,\sigma(Q'_j)$. This contradicts the assumptions of the current case.

Note also that $Q \notin \mathcal{F}_{good}$ by (7.9) and $Q \notin \mathcal{F}_{bad}$ by (7.2), hence $\mathcal{F} \subset \mathcal{D}(Q) \setminus \{Q\}$. By (7.4) and Claim 5.5, there is some tree $\mathsf{T} \subset \mathcal{G}$ so that $\mathsf{T}'' = (\mathcal{D}_{\mathcal{F}} \cup \mathcal{F} \cup \mathcal{F}') \cap \mathcal{D}(Q)$ is a semi-coherent subtree of T, where $\mathcal{F}'$ denotes the collection of all dyadic children of cubes in $\mathcal{F}$.

Case 2a: $\sigma(F_0) \geq \frac{1}{2}\rho\sigma(Q)$.

In this case, Q has an ample overlap with the boundary of a Chord-arc domain with controlled Chord-arc constants. Indeed, let $\mathsf{T}' = \mathcal{D}_{\mathcal{F}} \cap \mathcal{D}(Q)$ which, by (7.4) and Claim 5.5, is a semi-coherent subtree of some $\mathsf{T} \subset \mathcal{G}$. Hence, by Lemma 3.3, each of $\Omega^{\pm}_{\mathsf{T}'}$ is a CAD with constants depending on the allowable parameters, formed by the union of fattened Whitney boxes, which satisfies $\Omega^{\pm}_{\mathsf{T}'} \subset B^*_Q \cap \Omega$ (see (3.8), (3.9), and (3.13)). Moreover, by [35, Proposition A.14] and [31, Proposition 6.3] and our current assumptions,

$$\sigma(Q \cap \partial\Omega^{\pm}_{\mathsf{T}'}) = \sigma(F_0) \geq \frac{\rho}{2}\sigma(Q).$$

Recall that in establishing $H[a+b, 1]$, we assume that there is a set $V_Q \subset U_Q \cap \Omega$ for which (4.11) holds with $\theta = 1$. Pick then $x \in V_Q$ and set $V^*_Q := \{x\} \subset V_Q$. Note that since $U_Q = U^+_Q \cup U^-_Q$ it follows that x belongs to either $U^+_Q \cap \Omega$ or $U^-_Q \cap \Omega$. For the sake of specificity assume that $x \in U^+_Q \cap \Omega$ hence, in particular, $U^+_Q \subset \Omega^+_{\mathsf{T}'} \subset \Omega$. Note also that U^+_Q is the only component of U_Q meeting V^*_Q. All these together give at once that the conclusion of $H[a+b, 1]$ holds in the present case.

Case 2b: $\sigma(F_0) < \frac{1}{2}\rho\sigma(Q)$.

In this case by (7.7)

$$\sigma(G_0) \geq \frac{\rho}{2}\sigma(Q). \tag{7.10}$$

In addition, by the definition of $\mathcal{F}_{good}$ (7.6), and pigeon-holing, every $Q_j \in \mathcal{F}_{good}$ has a dyadic child Q'_j (there could be more children satisfying this, but we just pick one) so that

$$\mathfrak{m}(\mathcal{D}(Q'_j)) \leq a\sigma(Q'_j). \tag{7.11}$$

Under the present assumptions $\theta = 1$, that is, $\sigma(F_Q) = \sigma(Q)$ (see (4.11) and (5.4)), hence $\sigma(F_Q \cap Q'_j) = \sigma(Q'_j)$. We apply Lemma 5.3 (recall (7.9)) to obtain $V_{Q'_j} \subset U_{Q'_j} \cap \Omega$ and $F_{Q'_j}$ which satisfies $\sigma(F_{Q'_j}) = \sigma(Q'_j)$. That is, (4.11) holds for Q'_j, with $\theta = 1$. Consequently, recalling that $Q'_j \in \mathsf{T} \subset \mathcal{G}$ (see Claim 5.5), and applying the induction hypothesis to Q'_j, we find $V^*_{Q'_j} \subset V_{Q'_j}$,

such that for each $U^{\pm}_{Q'_j}$ meeting $V^*_{Q'_j}$, there is a Chord-arc domain $\Omega^{\pm}_{Q'_j} \supset U^{\pm}_{Q'_j}$ formed by a union of fattened Whitney cubes with $\Omega^{\pm}_{Q'_j} \subset B^*_{Q'_j} \cap \Omega$. Moreover, since in particular, the cubes in $\mathcal{F}$ along with all of their children belong to the same tree T (see Claim 5.5), the connected component $U^{\pm}_{Q_j}$ overlaps with the corresponding component $U^{\pm}_{Q'_j}$ for its child, so we may augment $\Omega^{\pm}_{Q'_j}$ by adjoining to it the appropriate component $U^{\pm}_{Q_j}$, to form a chord arc domain

$$\Omega^{\pm}_{Q_j} := \Omega^{\pm}_{Q'_j} \cup U^{\pm}_{Q_j}. \tag{7.12}$$

Moreover, since $K \gg 1$, and since $Q'_j \subset Q_j$, we have that $B^*_{Q'_j} \subset B^*_{Q_j}$, hence $\Omega^{\pm}_{Q_j} \subset B^*_{Q_j}$ by construction.

By a covering lemma argument, for a sufficiently large constant $\kappa \gg K^4$, we may extract a subcollection $\mathcal{F}^*_{good} \subset \mathcal{F}_{good}$ so that $\{\kappa B^*_{Q_j}\}_{Q_j \in \mathcal{F}^*_{good}}$ is a pairwise disjoint family, and

$$\bigcup_{Q_j \in \mathcal{F}_{good}} Q_j \subset \bigcup_{Q_j \in \mathcal{F}^*_{good}} 5\kappa B^*_{Q_j}.$$

In particular, by (7.10),

$$\sum_{Q_j \in \mathcal{F}^*_{good}} \sigma(Q_j) \gtrsim_{\kappa, K} \sum_{Q_j \in \mathcal{F}_{good}} \sigma(Q_j) = \sigma(G_0) \gtrsim \rho\sigma(Q), \tag{7.13}$$

where the implicit constants depend on ADR, K, and the dilation factor κ.

By the induction hypothesis, and by construction (7.12) and n-ADR,

$$\sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q'_j) \gtrsim \sigma(Q_j), \tag{7.14}$$

where Ω_{Q_j} is equal either to $\Omega^+_{Q_j}$ or to $\Omega^-_{Q_j}$ (if (7.14) holds for both choices, we arbitrarily set $\Omega_{Q_j} = \Omega^+_{Q_j}$).

Combining (7.14) with (7.13), we obtain

$$\sum_{Q_j \in \mathcal{F}^*_{good}} \sigma(Q_j \cap \partial\Omega_{Q_j}) \gtrsim \sigma(Q). \tag{7.15}$$

We now assign each $Q_j \in \mathcal{F}^*_{good}$ either to $\mathcal{F}^*_+$ or to $\mathcal{F}^*_-$, depending on whether we chose Ω_{Q_j} satisfying (7.14) to be $\Omega^+_{Q_j}$, or $\Omega^-_{Q_j}$. We note that at least one of the sub-collections $\mathcal{F}^*_{\pm}$ is non-empty, since for each j, there was at

least one choice of "+' or "-" such that (7.14) holds for the corresponding choice of Ω_{Q_j}. Moreover, the two collections are disjoint, since we have arbitrarily designated $\Omega_{Q_j} = \Omega_{Q_j}^+$ in the case that there were two choices for a particular Q_j.

To proceed, as in Lemma 6.1 we set

$$\mathsf{T}^* = \left\{ Q' \in \mathcal{D}(Q) : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}_{good}^* \right\}$$

which is semi-coherent by construction.

For $\mathcal{F}_{\pm}^*$ non-empty, we now define

$$\Omega_Q^{\pm} = \Omega_{\mathsf{T}^*}^{\pm} \bigcup \left(\bigcup_{Q_j \in \mathcal{F}_{\pm}^*} \Omega_{Q_j} \right). \tag{7.16}$$

Observe that by the induction hypothesis, and our construction (see (7.12) and the ensuing comment), for an appropriate choice of $\pm$, $U_{Q_j}^{\pm} \subset \Omega_{Q_j} \subset B_{Q_j}^*$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q)$, by (7.15) and Lemma 6.1, with $\mathcal{F}^* = \mathcal{F}_{good}^*$, each (non-empty) choice of $\Omega_Q^{\pm}$ defines a Chord-arc domain with the requisite properties.

Thus, we have proved Claim 7.2 and therefore, as noted above, it follows that $H[M_0, 1]$ holds. $\qquad\square$

8 Step 3: Bootstrapping θ

In this last step, we shall prove that there is a uniform constant $\zeta \in (0, 1)$ such that for each $\theta \in (0, 1]$, $H[M_0, \theta] \implies H[M_0, \zeta\theta]$. Since we have already established $H[M_0, 1]$, we then conclude that $H[M_0, \theta_1]$ holds for any given $\theta_1 \in (0, 1]$. As noted above, it then follows that Theorem 1.2 holds, as desired.

In turn, it will be enough to verify the following.

Claim 8.1 *There is a uniform constant* $\beta \in (0, 1)$ *such that for every* $a \in [0, M_0)$, $\theta \in (0, 1]$, $\vartheta \in (0, 1)$, *and* $b = 1/(2C)$ *as in Step 2/Proof of Claim 7.2, if* $H[M_0, \theta]$ *holds, then*

$$H[a, (1 - \vartheta)\theta] \implies H[a + b, (1 - \vartheta\beta)\theta].$$

Let us momentarily take Claim 8.1 for granted. Recall that by Claim 7.1, $H[0, \theta]$ holds for all $\theta \in (0, 1]$. In particular, given $\theta \in (0, 1]$ fixed, for which $H[M_0, \theta]$ holds, we have that $H[0, \theta/2]$ holds. Combining the latter fact with Claim 8.1, and iterating, we obtain that $H[kb, (1 - 2^{-1}\beta^k)\theta]$ holds. We eventually reach $H[M_0, (1 - 2^{-1}\beta^\nu)\theta]$, with $\nu \approx M_0/b$. The conclusion of Step 3 now follows, with $\zeta := 1 - 2^{-1}\beta^\nu$.

Proof of Claim 8.1 The proof will be a refinement of that of Claim 7.2. We are given some $\theta \in (0, 1]$ such that $H[M_0, \theta]$ holds, and we assume that $H[a, (1 - \vartheta)\theta]$ holds, for some $a \in [0, M_0)$ and $\vartheta \in (0, 1)$. Set $b = 1/(2C)$, where as before C is the constant in (7.1). Consider a cube $Q \in \mathcal{D}(\partial\Omega)$ with $\mathfrak{m}(\mathcal{D}(Q)) \leq (a + b)\sigma(Q)$. Suppose that there is a set $V_Q \subset U_Q \cap \Omega$ such that (4.11) holds with θ replaced by $(1 - \vartheta\beta)\theta$, for some $\beta \in (0, 1)$ to be determined. Our goal is to show that for a sufficiently small, but uniform choice of β, we may deduce the conclusion of the induction hypothesis, with C_{a+b}, c_{a+b} in place of C_a, c_a.

By assumption, and recalling the definition of F_Q in (5.4), we have that (4.11) holds with constant $(1 - \vartheta\beta)\theta$, i.e.,

$$\sigma(F_Q) \geq (1 - \vartheta\beta)\theta\sigma(Q). \tag{8.1}$$

As in the proof of Claim 7.2, we fix $k_1 > k_0$ (see (5.1)) large enough so that $2^{k_1} > 100K$. There are two principal cases. The first is as follows.

Case 1: There exists $Q' \in \mathcal{D}_{k_1}(Q)$ (see (2.11)) with $\mathfrak{m}(\mathcal{D}(Q')) \leq a\sigma(Q')$.

We split **Case 1** into two subcases.

Case 1a: $\sigma(F_Q \cap Q') \geq (1 - \vartheta)\theta\sigma(Q')$.

In this case, we follow the **Case 1** argument for $\theta = 1$ in Sect. 7 *mutatis mutandis*, so we merely sketch the proof. By Lemma 5.3, we may construct $V_{Q'}$ and $F_{Q'}$ so that $F_Q \cap Q' = F_{Q'}$ and hence $\sigma(F_{Q'}) \geq (1 - \vartheta)\theta\sigma(Q')$. We may then apply the induction hypothesis $H[a, (1 - \vartheta)\theta]$ in Q', and then proceed exactly as in **Case 1** in Section 7 to construct a subset $V_Q^* \subset V_Q$ and a family of Chord-arc domains Ω_Q^i satisfying the various desired properties, and such that

$$\sum_{i:U_Q^i \text{ meets } V_Q^*} \sigma(\partial\Omega_Q^i \cap Q) \geq c_a\sigma(Q') \gtrsim_{k_1} c_a\sigma(Q).$$

The conclusion of $H[a + b, (1 - \vartheta\beta)\theta]$ then holds in the present scenario.

Case 1b: $\sigma(F_Q \cap Q') < (1 - \vartheta)\theta\sigma(Q')$.

By (8.1)

$$(1 - \vartheta\beta)\theta\sigma(Q) \leq \sigma(F_Q) = \sigma(F_Q \cap Q') + \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(F_Q \cap Q'').$$

In the scenario of **Case 1b**, this leads to

$$(1 - \vartheta\beta)\theta\sigma(Q') + (1 - \vartheta\beta)\theta \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(Q'') = (1 - \vartheta\beta)\theta\sigma(Q)$$

$$\leq (1 - \vartheta)\theta\sigma(Q') + \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(F_Q \cap Q''),$$

that is,

$$(1 - \beta)\vartheta\theta\sigma(Q') + (1 - \vartheta\beta)\theta \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(Q'')$$

$$\leq \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(F_Q \cap Q''). \tag{8.2}$$

Note that we have the dyadic doubling estimate

$$\sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(Q'') \leq \sigma(Q) \leq M_1\sigma(Q'),$$

where $M_1 = M_1(k_1, n, ADR)$. Combining this estimate with (8.2), we obtain

$$\left[(1 - \beta)\frac{\vartheta}{M_1} + (1 - \vartheta\beta)\right]\theta \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(Q'')$$

$$\leq \sum_{Q'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}} \sigma(F_Q \cap Q'').$$

We now choose $\beta \leq 1/(M_1 + 1)$, so that $(1 - \beta)/M_1 \geq \beta$, and therefore the expression in square brackets is at least 1. Consequently, by pigeon-holing, there exists a particular $Q_0'' \in \mathcal{D}_{k_1}(Q)\setminus\{Q'\}$ such that

$$\theta\sigma(Q_0'') \leq \sigma(F_Q \cap Q_0''). \tag{8.3}$$

By Lemma 5.3, we can find $V_{Q_0''}$ such that $F_Q \cap Q_0'' = F_{Q_0''}$, where the latter is defined as in (5.4), with Q_0'' in place of Q. By assumption, $H[M_0, \theta]$ holds, so combining (8.3) with the fact that (4.10) holds with $a = M_0$ for every $Q \in \mathcal{D}(\partial\Omega)$, we find that there exists a subset $V_{Q_0''}^* \subset V_{Q_0''}$, along with a family of Chord-arc domains $\{\Omega_{Q_0''}^i\}_i$ enjoying all the appropriate properties relative to Q_0''. Using that $\ell(Q_0'') \approx_{k_1} \ell(Q)$, we may now proceed exactly as in **Case 1a** above, and also **Case 1** in Section 7, to construct V_Q^* and $\{\Omega_Q^i\}_i$ such that the conclusion of $H[a + b, (1 - \vartheta\beta)\theta]$ holds in the present case also.

Case 2: $\mathfrak{m}(\mathcal{D}(Q')) > a\sigma(Q')$ for every $Q' \in \mathcal{D}_{k_1}(Q)$.

In this case, we apply Lemma 7.3 to obtain a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathcal{D}(Q)$ such that (7.1) and (7.2) hold. In particular, by our choice of $b = 1/(2C)$, $\|\mathfrak{m}_{\mathcal{F}}\|_{\mathcal{C}(Q)} \leq 1/2$.

Recall that F_Q is defined in (5.4), and satisfies (8.1). We define $F_0 = Q\setminus(\bigcup_{\mathcal{F}} Q_j)$ as in (7.5), and $\mathcal{F}_{good} := \mathcal{F}\setminus\mathcal{F}_{bad}$ as in (7.6). Let $G_0 :=$

$\bigcup_{\mathcal{F}_{good}} Q_j$. Then as above (see (7.7)),

$$\sigma(F_0 \cup G_0) \geq \rho\sigma(Q), \tag{8.4}$$

where again $\rho = \rho(M_0, b) \in (0, 1)$ is defined as in (7.8). Just as in **Case 2** for $\theta = 1$ in Sect. 7, we have that

$$\ell(Q_j) \leq 2^{-k_1} \ell(Q), \quad \forall Q_j \in \mathcal{F}_{good}, \quad \text{and} \quad \mathcal{F} \subset \mathcal{D}(Q)\backslash\{Q\} \tag{8.5}$$

(see (7.9)). Hence, the conclusions of Claim 5.5 hold.

We first observe that if $\sigma(F_0) \geq \varepsilon\sigma(Q)$, for some $\varepsilon > 0$ to be chosen (depending on allowable parameters), then the desired conclusion holds. Indeed, in this case, we may proceed exactly as in the analogous scenario in **Case 2a** in Sect. 7: the promised Chord-arc domain is again simply one of $\Omega_{T'}^{\pm}$, since at least one of these contains a point in V_Q and hence in particular is a subdomain of Ω. The constant c_{a+b} in our conclusion will depend on ε, but in the end this will be harmless, since ε will be chosen to depend only on allowable parameters.

We may therefore suppose that

$$\sigma(F_0) < \varepsilon\sigma(Q). \tag{8.6}$$

Next, we refine the decomposition $\mathcal{F} = \mathcal{F}_{good} \cup \mathcal{F}_{bad}$. With ρ as in (7.8) and (8.4), we choose $\beta < \rho/4$. Set

$$\mathcal{F}_{good}^{(1)} := \left\{ Q_j \in \mathcal{F}_{good} : \sigma(F_Q \cap Q_j) \geq \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q_j) \right\},$$

and define $\mathcal{F}_{good}^{(2)} := \mathcal{F}_{good}\backslash\mathcal{F}_{good}^{(1)}$. Let

$$\mathcal{F}_{bad}^{(1)} := \left\{ Q_j \in \mathcal{F}_{bad} : \sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j) \right\},$$

and define $\mathcal{F}_{bad}^{(2)} := \mathcal{F}_{bad}\backslash\mathcal{F}_{bad}^{(1)}$.

We split the remaining part of **Case 2** into two subcases. The first of these will be easy, based on our previous arguments.

Case 2a: There is $Q_j \in \mathcal{F}_{bad}^{(1)}$ such that $\ell(Q_j) > 2^{-k_1} \ell(Q)$.

By definition of $\mathcal{F}_{bad}^{(1)}$, one has $\sigma(F_Q \cap Q_j) \geq \theta\sigma(Q_j)$. By pigeon-holing, Q_j has a descendant Q' with $\ell(Q') = 2^{-k_1}\ell(Q)$, such that $\sigma(F_Q \cap Q') \geq \theta\sigma(Q')$. We may then apply $H[M_0, \theta]$ in Q', and proceed exactly as we did in **Case 1b** above with the cube Q_0'', which enjoyed precisely the same properties as does our current Q'. Thus, we draw the desired conclusion in the present case.

The main case is the following.

Case 2b: Every $Q_j \in \mathcal{F}_{bad}^{(1)}$ satisfies $\ell(Q_j) \leq 2^{-k_1} \ell(Q)$.

Observe that by definition,

$$\sigma(F_Q \cap Q_j) \leq \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q_j), \qquad \forall\, Q_j \in \mathcal{F}_{good}^{(2)}, \tag{8.7}$$

and also

$$\sigma(F_Q \cap Q_j) \leq \theta\sigma(Q_j), \qquad \forall\, Q_j \in \mathcal{F}_{bad}^{(2)}, \tag{8.8}$$

Set $\mathcal{F}_* := \mathcal{F}\backslash\mathcal{F}_{good}^{(2)}$. For future reference, we shall derive a certain ample-ness estimate for the cubes in $\mathcal{F}_*$. By (8.1),

$$(1 - \vartheta\beta)\theta\sigma(Q) \leq \sigma(F_Q) \leq \sigma(F_0) + \sum_{\mathcal{F}_*} \sigma(Q_j) + \sum_{\mathcal{F}_{good}^{(2)}} \sigma(F_Q \cap Q_j)$$

$$\leq \varepsilon\sigma(Q) + \sum_{\mathcal{F}_*} \sigma(Q_j) + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q), \tag{8.9}$$

where in the last step have used (8.6) and (8.7). Observe that

$$(1 - \vartheta\beta)\theta = \left(4\rho^{-1} - 1\right)\vartheta\beta\theta + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta. \tag{8.10}$$

Using (8.9) and (8.10), for $\varepsilon \ll \left(4\rho^{-1} - 1\right)\vartheta\beta\theta$, we obtain

$$2^{-1}\left(4\rho^{-1} - 1\right)\vartheta\beta\theta\sigma(Q) \leq \sum_{\mathcal{F}_*} \sigma(Q_j)$$

and thus

$$\sigma(Q) \leq C(\vartheta, \rho, \beta, \theta) \sum_{\mathcal{F}_*} \sigma(Q_j). \tag{8.11}$$

We now make the following claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Claim 8.2 *For ε chosen sufficiently small,*

$$\max\left(\sum_{\mathcal{F}_{good}^{(1)}} \sigma(Q_j), \sum_{\mathcal{F}_{bad}^{(1)}} \sigma(Q_j)\right) \geq \varepsilon\sigma(Q).$$

Proof of Claim 8.2 If $\sum_{\mathcal{F}_{good}^{(1)}} \sigma(Q_j) \geq \varepsilon\sigma(Q)$, then we are done. Therefore, suppose that

$$\sum_{\mathcal{F}_{good}^{(1)}} \sigma(Q_j) < \varepsilon\sigma(Q). \tag{8.12}$$

We have made the decomposition

$$\mathcal{F} = \mathcal{F}_{good}^{(1)} \cup \mathcal{F}_{good}^{(2)} \cup \mathcal{F}_{bad}^{(1)} \cup \mathcal{F}_{bad}^{(2)}. \tag{8.13}$$

Consequently

$$\sigma(F_Q) \leq \sum_{\mathcal{F}_{good}^{(2)}} \sigma(F_Q \cap Q_j) + \sum_{\mathcal{F}_{bad}} \sigma(F_Q \cap Q_j) + O\left(\varepsilon\sigma(Q)\right),$$

where we have used (8.6), and (8.12) to estimate the contributions of F_0, and of $\mathcal{F}_{good}^{(1)}$, respectively. This, (8.1), (8.7), and (8.8) yield

$$(1 - \vartheta\beta)\theta \left(\sum_{\mathcal{F}_{good}^{(2)}} \sigma(Q_j) + \sum_{\mathcal{F}_{bad}^{(2)}} \sigma(Q_j) \right) \leq (1 - \vartheta\beta)\theta\sigma(Q) \leq \sigma(F_Q)$$

$$\leq \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta \sum_{\mathcal{F}_{good}^{(2)}} \sigma(Q_j) + \sum_{\mathcal{F}_{bad}^{(1)}} \sigma(Q_j)$$

$$+\theta \sum_{\mathcal{F}_{bad}^{(2)}} \sigma(Q_j) + O\left(\varepsilon\sigma(Q)\right).$$

In turn, applying (8.10) in the latter estimate, and rearranging terms, we obtain

$$(4\rho^{-1} - 1)\vartheta\beta\theta \sum_{\mathcal{F}_{good}^{(2)}} \sigma(Q_j) - \vartheta\beta\theta \sum_{\mathcal{F}_{bad}^{(2)}} \sigma(Q_j) \leq \sum_{\mathcal{F}_{bad}^{(1)}} \sigma(Q_j) + O\left(\varepsilon\sigma(Q)\right). \tag{8.14}$$

Recalling that $G_0 = \cup_{\mathcal{F}_{good}} Q_j$, and that $\mathcal{F}_{good} = \mathcal{F}_{good}^{(1)} \cup \mathcal{F}_{good}^{(2)}$, we further note that by (8.4), choosing $\varepsilon \ll \rho$, and using (8.6) and (8.12), we find in particular that

$$\sum_{\mathcal{F}_{good}^{(2)}} \sigma(Q_j) \geq \frac{\rho}{2}\sigma(Q). \tag{8.15}$$

Applying (8.15) and the trivial estimate $\sum_{\mathcal{F}_{bad}^{(2)}} \sigma(Q_j) \le \sigma(Q)$ in (8.14), we then have

$$\vartheta\beta\theta\left[1 - \frac{\rho}{2}\right]\sigma(Q) = \left[(4\rho^{-1} - 1)\vartheta\beta\theta\frac{\rho}{2} - \vartheta\beta\theta\right]\sigma(Q)$$
$$\le (4\rho^{-1} - 1)\vartheta\beta\theta\sum_{\mathcal{F}_{good}^{(2)}}\sigma(Q_j) - \vartheta\beta\theta\sum_{\mathcal{F}_{nbad}^{(2)}}\sigma(Q_j)$$
$$\le \sum_{\mathcal{F}_{bad}^{(1)}}\sigma(Q_j) + O\left(\varepsilon\sigma(Q)\right).$$

Since $\rho < 1$, we conclude, for $\varepsilon \le (4C)^{-1}\vartheta\beta\theta$, that

$$\frac{1}{4}\vartheta\beta\theta\,\sigma(Q) \le \sum_{\mathcal{F}_{bad}^{(1)}}\sigma(Q_j),$$

and Claim 8.2 follows.

With Claim 8.2 in hand, let us return to the proof of **Case 2b** of Claim 8.1. We begin by noting that by definition of $\mathcal{F}_{bad}^{(1)}$, and Lemma 5.3, we can apply $H[M_0, \theta]$ to any $Q_j \in \mathcal{F}_{bad}^{(1)}$, hence for each such Q_j there is a family of Chord-arc domains $\{\Omega_{Q_j}^i\}_i$ satisfying the desired properties.

Now consider $Q_j \in \mathcal{F}_{good}^{(1)}$. Since $\mathcal{F}_{good}^{(1)} \subset \mathcal{F}_{good}$, by pigeon-holing Q_j has a dyadic child Q_j' satisfying

$$\mathfrak{m}\big(\mathcal{D}(Q_j')\big) \le a\sigma(Q_j'), \tag{8.16}$$

(there may be more than one such child, but we just pick one). Our immediate goal is to find a child Q_j'' of Q_j, which may or may not equal Q_j', for which we may construct a family of Chord-arc domains $\{\Omega_{Q_j''}^i\}_i$ satisfying the desired properties. To this end, we assume first that Q_j' satisfies

$$\sigma(F_Q \cap Q_j') \ge (1 - \vartheta)\theta\sigma(Q_j'). \tag{8.17}$$

In this case, we set $Q_j'' := Q_j'$, and using Lemma 5.3, by the induction hypothesis $H[a, (1 - \vartheta)\theta]$, we obtain the desired family of Chord-arc domains.

We therefore consider the case

$$\sigma(F_Q \cap Q_j') < (1 - \vartheta)\theta\sigma(Q_j'). \tag{8.18}$$

In this case, we shall select $Q_j'' \neq Q_j'$. Recall that we use the notation $Q'' \lhd Q$ to mean that Q'' is a dyadic child of Q. Set

$$\mathcal{F}_j'' := \left\{ Q_j'' \lhd Q_j : Q_j'' \neq Q_j' \right\}.$$

Note that we have the dyadic doubling estimate

$$\sum_{Q_j'' \in \mathcal{F}_j''} \sigma(Q_j'') \leq \sigma(Q_j) \leq M_1 \sigma(Q_j'), \tag{8.19}$$

where $M_1 = M_1(n, ADR)$. We also note that

$$\left(1 - 4\vartheta\beta\rho^{-1}\right)\theta = \left(1 - 4\beta\rho^{-1}\right)\vartheta\theta + (1 - \vartheta)\theta. \tag{8.20}$$

By definition of $\mathcal{F}_{good}^{(1)}$,

$$\left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q_j) \leq \sigma(F_Q \cap Q_j) = \sigma(F_Q \cap Q_j')$$
$$+ \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(F_Q \cap Q_j'').$$

By (8.18), it follows that

$$\left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q_j') + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(Q_j'')$$

$$= \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta\sigma(Q_j)$$
$$\leq (1 - \vartheta)\theta\sigma(Q_j') + \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(F_Q \cap Q_j'').$$

In turn, using (8.20), we obtain

$$\left(1 - 4\beta\rho^{-1}\right)\vartheta\theta\sigma(Q_j') + \left(1 - 4\vartheta\beta\rho^{-1}\right)\theta \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(Q_j'')$$

$$\leq \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(F_Q \cap Q_j'').$$

By the dyadic doubling estimate (8.19), this leads to

$$\left[\left(1 - 4\beta\rho^{-1}\right)\vartheta M_1^{-1} + \left(1 - 4\vartheta\beta\rho^{-1}\right)\right]\theta \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(Q_j'')$$

$$\leq \sum_{Q_j'' \in \mathcal{F}_j''} \sigma(F_Q \cap Q_j'').$$

Choosing $\beta \leq \rho/(4(M_1 + 1))$, we find that the expression in square brackets is at least 1, and therefore, by pigeon holing, we can pick $Q_j'' \in \mathcal{F}_j''$ satisfying

$$\sigma(F_Q \cap Q_j'') \geq \theta\sigma(Q_j''). \tag{8.21}$$

Hence, using Lemma 5.3, we see that the induction hypothesis $H[M_0, \theta]$ holds for $Q_j'' \in \mathcal{F}_j''$, and once again we obtain the desired family of Chord-arc domains.

Recall that we have constructed our packing measure $\mathfrak{m}$ in such a way that each $Q_j \in \mathcal{F}$, as well as all of its children, along with the cubes in $\mathcal{D}_{\mathcal{F}} \cap \mathcal{D}(Q)$, belong to the same tree T; see Claim 5.5. This means in particular that for each such Q_j, the Whitney region U_{Q_j} has exactly two components $U_{Q_j}^{\pm} \subset \Omega_{\mathsf{T}}^{\pm}$, and the analogous statement is true for each child of Q_j. This fact has the following consequences:

Remark 8.3 For each $Q_j \in \mathcal{F}_{bad}^{(1)}$, and for the selected child Q_j'' of each $Q_j \in \mathcal{F}_{good}^{(1)}$, the conclusion of the induction hypothesis produces at most two Chord-arc domains $\Omega_{Q_j}^{\pm} \supset U_{Q_j}^{\pm}$ (resp. $\Omega_{Q_j''}^{\pm} \supset U_{Q_j''}^{\pm}$), which we enumerate as $\Omega_{Q_j}^i$ (resp. $\Omega_{Q_j''}^i$), $i = 1, 2$, with $i = 1$ corresponding "+", and $i = 2$ corresponding to "-", respectively.

Remark 8.4 For each $Q_j \in \mathcal{F}_{good}^{(1)}$, the connected component $U_{Q_j}^{\pm}$ overlaps with the corresponding component $U_{Q_j''}^{\pm}$ for its child, so we may augment $\Omega_{Q_j''}^i$ by adjoining to it the appropriate component $U_{Q_j}^{\pm}$, to form a chord arc domain

$$\Omega_{Q_j}^i := \Omega_{Q_j''}^i \cup U_{Q_j}^i.$$

By the induction hypothesis, for each $Q_j \in \mathcal{F}_{bad}^{(1)} \cup \mathcal{F}_{good}^{(1)}$ (and by n-ADR, in the case of $\mathcal{F}_{good}^{(1)}$), the Chord-arc domains $\Omega_{Q_j}^i$ that we have constructed satisfy

$$\sum_i \sigma(Q_j \cap \partial \Omega^i_{Q_j}) \gtrsim \sigma(Q_j),$$

where the sum has either one or two terms, and where the implicit constant depends either on M_0 and θ, or on a and $(1 - \vartheta)\theta$, depending on which part of the induction hypothesis we have used. In particular, for each such Q_j, there is at least one choice of index i such that $\Omega^i_{Q_j} =: \Omega_{Q_j}$ satisfies

$$\sigma(Q_j \cap \partial \Omega_{Q_j}) \gtrsim \sigma(Q_j) \qquad (8.22)$$

(if the latter is true for both choices $i = 1, 2$, we arbitrarily choose $i = 1$, which we recall corresponds to "+"). Combining the latter bound with Claim 8.2, and recalling that ε has now been fixed depending only on allowable parameters, we see that

$$\sum_{Q_j \in \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}} \sigma(Q_j \cap \partial \Omega_{Q_j}) \gtrsim \sigma(Q)$$

For $Q_j \in \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}$, as above set $B^*_{Q_j} := B(x_{Q_j}, K\ell(Q_j))$. By a covering lemma argument, we may extract a subfamily $\mathcal{F}^* \subset \mathcal{F}^{(1)}_{bad} \cup \mathcal{F}^{(1)}_{good}$ such that $\{\kappa B^*_{Q_j}\}_{Q_j \in \mathcal{F}^*}$ is pairwise disjoint, where again $\kappa \gg K^4$ is a large dilation factor, and such that

$$\sum_{Q_j \in \mathcal{F}^*} \sigma(Q_j \cap \partial \Omega_{Q_j}) \gtrsim_\kappa \sigma(Q) \qquad (8.23)$$

Let us now build (at most two) Chord-arc domains Ω^i_Q satisfying the desired properties. Recall that for each $Q_j \in \mathcal{F}^*$, we defined the corresponding Chord-arc domain $\Omega_{Q_j} := \Omega^i_{Q_j}$, where the choice of index i (if there was a choice), was made so that (8.22) holds. We then assign each $Q_j \in \mathcal{F}^*$ either to $\mathcal{F}^*_+$ or to $\mathcal{F}^*_-$, depending on whether we chose Ω_{Q_j} satisfying (8.22) to be $\Omega^1_{Q_j} = \Omega^+_{Q_j}$, or $\Omega^2_{Q_j} = \Omega^-_{Q_j}$. We note that at least one of the sub-collections $\mathcal{F}^*_\pm$ is non-empty, since for each j, there was at least one choice of index i such that (8.22) holds with $\Omega_{Q_j} := \Omega^i_{Q_j}$. Moreover, the two collections are disjoint, since we have arbitrarily designated $\Omega_{Q_j} = \Omega^1_{Q_j}$ (corresponding to "+") in the case that there were two choices for a particular Q_j. We further note that if $Q_j \in \mathcal{F}^*_\pm$, then $\Omega_{Q_j} = \Omega^\pm_{Q_j} \supset U^\pm_{Q_j}$.

We are now in position to apply Lemma 6.1. Set

$$\mathsf{T}^* = \left\{ Q' \in \mathcal{D}(Q) : Q_j \subset Q' \text{ for some } Q_j \in \mathcal{F}^* \right\},$$

which is a semi-coherent subtree of T, with maximal cube Q. Without loss of generality, we may suppose that $\mathcal{F}_+^*$ is non-empty, and we then define

$$\Omega_Q^+ := \Omega_{\mathsf{T}^*}^+ \bigcup \left(\bigcup_{Q_j \in \mathcal{F}_+^*} \Omega_{Q_j} \right),$$

and similarly with "+" replaced by "−", provided that $\mathcal{F}_-^*$ is also non-empty. Observe that by the induction hypothesis, and our construction (see Remarks 8.3 and 8.4, and Lemma 3.3), for an appropriate choice of "±", $U_{Q_j}^\pm \subset \Omega_{Q_j} \subset B_{Q_j}^*$, and since $\ell(Q_j) \leq 2^{-k_1}\ell(Q)$, by (8.23) and Lemma 6.1, each (non-empty) choice defines a Chord-arc domain with the requisite properties. This completes the proof of **Case 2b** of Claim 8.1 and hence that of Theorem 1.2.

$\square$

Part 2: Proof of Theorem 1.3

9 Preliminaries for the Proof of Theorem 1.3

9.1 Uniform rectifiability

Recall the definition of n-uniform rectifiable (n-UR) sets in Definition 2.2. Given a ball $B \subset \mathbb{R}^{n+1}$, we denote

$$b\beta_E(B) = \inf_L \frac{1}{r(B)} \left(\sup_{y \in E \cap B} \text{dist}(y, L) + \sup_{y \in L \cap B} \text{dist}(y, E) \right), \qquad (9.1)$$

where the infimum is taken over all the affine n-planes that intersect B. The following result is due to David and Semmes:

Theorem 9.1 *Let $E \subset \mathbb{R}^{n+1}$ be n-ADR. Denote $\sigma = H^n \lfloor_E$ and let $\mathcal{D}$ be the associated dyadic lattice. Then, E is n-UR if and only if, for any $\varepsilon > 0$,*

$$\sum_{\substack{Q \in \mathcal{D}:\, Q \subset R, \\ b\beta(3B_Q) > \varepsilon}} \sigma(Q) \leq C(\varepsilon)\,\sigma(R) \quad \textit{for all } R \in \mathcal{D}.$$

For the proof, see [23, Theorem 2.4, p.32] (this provides a slight variant of Theorem 9.1, and it is straightforward to check that both formulations are equivalent). Remark that the constant 3 multiplying B_Q in the estimate above can be replaced by any number larger than 1.

Recall also the following result (see [28] or [44]).

Theorem 9.2 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, be an open set satisfying an interior corkscrew condition, with n-ADR boundary, such that the harmonic measure in Ω belongs to weak-A_∞. Then $\partial\Omega$ is n-UR.*

9.2 Harmonic measure

From now on we assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with n-ADR boundary such that the harmonic measure in Ω belongs to weak-A_∞. We denote by σ the surface measure in $\partial\Omega$, that is, $\sigma = H^n \lfloor_{\partial\Omega}$. We also consider the dyadic lattice $\mathcal{D}$ associated with σ as in Lemma 2.16. The AD-regularity constant of $\partial\Omega$ is denoted by C_0.

We denote by ω^p the harmonic measure with pole at p of Ω, and by $g(\cdot, \cdot)$ the Green function. Much as before we write $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$.

The following well known result is sometimes called "Bourgain's estimate":

Lemma 9.3 [13]. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-ADR boundary, $x \in \partial\Omega$, and $0 < r \leq \text{diam}(\partial\Omega)/2$. Then*

$$\omega^y(B(x, 2r)) \geq c > 0, \quad \text{for all } y \in \Omega \cap \overline{B}(x, r) \tag{9.2}$$

where c depends on n and the n-ADRity constant of $\partial\Omega$.

The following is also well known.

Lemma 9.4 *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-ADR boundary. Let $p, q \in \Omega$ be such $|p - q| \geq 4\,\delta_\Omega(q)$. Then,*

$$g(p, q) \leq C \frac{\omega^p(B(q, 4\delta_\Omega(q)))}{\delta_\Omega(q)^{n-1}}.$$

We remark that the previous lemma is also valid in the case $n > 1$ without the n-ADR assumption. In the case $n = 1$ this holds under the 1-ADR assumption, and also in the more general situation where Ω satisfies the CDC. This follows easily from [1, Lemmas 3.4 and 3.5]. Notice that n-ADR implies the CDC in $\mathbb{R}^{n+1}$ (for any n), by standard arguments.

The following lemma is also known. See [28, Lemma 3.14], for example.

Lemma 9.5 *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-ADR boundary and let $p \in \Omega$. Let B be a ball centered at $\partial\Omega$ such that $p \notin 8B$. Then*

$$\fint_B g(p, x)\, dx \leq C \frac{\omega^p(4B)}{r(B)^{n-1}}.$$

Lemma 9.6 *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-ADR boundary. Let $x \in \partial\Omega$ and $0 < r < \text{diam}(\Omega)$. Let u be a non-negative harmonic function in $B(x, 4r) \cap \Omega$*

and continuous in $B(x, 4r) \cap \overline{\Omega}$ such that $u \equiv 0$ in $\partial\Omega \cap B(x, 4r)$. Then extending u by 0 in $B(x, 4r) \backslash \overline{\Omega}$, there exists a constant $\alpha > 0$ such that, for all $y, z \in B(x, r)$,

$$|u(y) - u(z)| \leq C \left(\frac{|y - z|}{r}\right)^{\alpha} \sup_{B(x,2r)} u \leq C \left(\frac{|y - z|}{r}\right)^{\alpha} \fint_{B(x,4r)} u,$$

where C and α depend on n and the AD-regularity of $\partial\Omega$. In particular,

$$u(y) \leq C \left(\frac{\delta_{\Omega}(y)}{r}\right)^{\alpha} \sup_{B(x,2r)} u \leq C \left(\frac{\delta_{\Omega}(y)}{r}\right)^{\alpha} \fint_{B(x,4r)} u.$$

The next result provides a partial converse to Lemma 9.4.

Lemma 9.7 *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-ADR boundary. Let $p \in \Omega$ and let $Q \in \mathcal{D}$ be such that $p \notin 2B_Q$. Suppose that $\omega^p(Q) \approx \omega^p(2Q)$. Then there exists some $q \in \Omega$ such that*

$$\ell(Q) \lesssim \delta_{\Omega}(q) \approx \text{dist}(q, Q) \leq 4 \, \text{diam}(Q)$$

and

$$\frac{\omega^p(2Q)}{\ell(Q)^{n-1}} \leq c \, g(p, q).$$

Proof For a given $k_0 \geq 2$ to be fixed below, we can pick $P \in \mathcal{D}(Q)$ with $\ell(P) = 2^{-k_0}\ell(Q)$ such that

$$\omega^p(P) \approx_{k_0} \omega^p(Q).$$

Let φ_P be a C^{∞} function supported in B_P, $\varphi_P \equiv 1$ on P, and such that $\|\nabla\varphi_P\|_{\infty} \lesssim 1/\ell(P)$. Then, choosing k_0 small enough so that $p \notin 50B_P$, say, and applying Caccioppoli's inequality,

$$\omega^p(2Q) \approx \omega^p(Q) \approx_{k_0} \omega^p(P) \leq \int \varphi_P \, d\omega^p = -\int \nabla_y g(p, y) \, \nabla\varphi_P(y) \, dy$$

$$\lesssim \frac{1}{\ell(P)} \int_{B_P} |\nabla_y g(p, y)| \, dy \lesssim \ell(P)^n \left(\fint_{B_P} |\nabla_y g(p, y)|^2 \, dy\right)^{1/2}$$

$$\lesssim \ell(P)^{n-1} \left(\fint_{2B_P} |g(p, y)|^2 \, dy\right)^{1/2} \lesssim \ell(P)^{n-1} \fint_{3B_P} g(p, y) \, dy.$$

Applying now Lemmas 9.6 and 9.5 and taking k_0 small enough so that $24B_P \cap \partial\Omega \subset 2Q$, for any $a \in (0, 1)$ we get

$$\fint_{y \in 3B_P : \delta_\Omega(y) \leq a\ell(P)} g(p, y)\, dy \lesssim a^\alpha \fint_{6B_P} g(p, y)\, dy$$
$$\lesssim a^\alpha \frac{\omega^P(24B_P)}{\ell(P)^{n-1}} \lesssim a^\alpha \frac{\omega^P(2Q)}{\ell(P)^{n-1}}.$$

From the estimates above we infer that

$$\omega^P(2Q) \lesssim_{k_0} \ell(P)^{n-1} \fint_{y \in 3B_P : \delta_\Omega(y) \geq a\ell(P)} g(p, y)\, dy + a^\alpha\, \omega^P(2Q).$$

Hence, for a small enough, we derive

$$\omega^P(2Q) \lesssim_{k_0} \ell(P)^{n-1} \fint_{y \in 3B_P : \delta_\Omega(y) \geq a\ell(P)} g(p, y)\, dy,$$

which implies the existence of the point q required in the lemma. □

9.3 Harnack chains and carrots

It will be more convenient for us to work with Harnack chains instead of curves. The existence of a carrot curve is equivalent to having what we call a good chain between points.

Let $x \in \Omega$, $y \in \overline{\Omega}$ be such that $\delta_\Omega(y) \leq \delta_\Omega(x)$, and let $C > 1$. A C-*good chain* (or C-good Harnack chain) from x to y is a sequence of balls $B_1, B_2, \ldots$ (finite or infinite) contained in Ω such that $x \in B_1$ and either

- $\lim_{j \to \infty} \text{dist}(y, B_j) = 0$ if $y \in \partial\Omega$, or
- $y \in B_N$ if $y \in \Omega$, where N is the number of elements of the sequence if this is finite,

and moreover the following hold:

- $B_j \cap B_{j+1} \neq \varnothing$ for all j,
- $C^{-1}\, \text{dist}(B_j, \partial\Omega) \leq r(B_j) \leq C\, \text{dist}(B_j, \partial\Omega)$ for all j,
- $r(B_j) \leq C\, r(B_i)$ if $j > i$,
- for each $t > 0$ there are at most C balls B_j such that $t < r(B_j) \leq 2t$.

Abusing language, sometimes we will omit the constant C and we will just say "good chain" or "good Harnack chain".

Observe that in the definitions of carrot curves and good chains, the order of x and y is important: having a carrot curve from x to y is not equivalent to having one from y to x, and similarly with good chains.

Lemma 9.8 *There is a carrot curve from $x \in \Omega$ to $y \in \overline{\Omega}$ if and only if there is a good Harnack chain from x to y.*

Proof Let γ be a carrot curve from x to y. We can assume $y \in \Omega$, since if $y \in \partial\Omega$, we can obtain this case by taking a limit of points $y_j \in \Omega$ converging to y. Let $\{B_j\}_{j=1}^N$ be a Vitali subcovering of the family $\{B(z, \delta_\Omega(z)/10) : z \in \gamma\}$ and let r_{B_j} stand for the radius and x_{B_j} for the center of B_j. So the balls B_j are disjoint and $5B_j$ cover γ. Note that for $t > 0$, if $t < r_{B_j} \leq 2t$,

$$|x_{B_j} - y| \leq H^1(\gamma(x_{B_j}, y)) \lesssim \delta_\Omega(x_{B_j}) \approx r_{B_j} \leq 2t.$$

In particular, since the B_j's are disjoint, by volume considerations, there can only be boundedly many B_j of radius between $t/2$ and t, say. Moreover, we may order the balls B_j so that $x \in 5B_1$ and B_{j+1} is a ball B_k such that $5B_k \cap 5B_j \neq \varnothing$ and $5\overline{B_k}$ contains the point from $\gamma \cap \bigcup_{h:5B_h \cap 5B_j \neq \varnothing} 5\overline{B_h}$ which is maximal in the natural order induced by γ (so that x is the minimal point in γ). Then for $j > i$,

$$r_{B_j} \approx \delta_\Omega(x_{B_j}) \leq |x_{B_j} - x_{B_i}| + \delta_\Omega(x_{B_i}) \leq H^1(\gamma(x_{B_i}, y)) + \delta_\Omega(x_{B_i}) \lesssim r_{B_i}.$$

This implies $5B_1, 5B_2, \ldots$ is a C-good chain for a sufficiently big C.

Now suppose that we can find a good chain from x to y, call it $B_1, \ldots, B_N$. Let γ be the path obtained by connecting their centers in order. Let $z \in \gamma$. Then there is a j such that $z \in [x_{B_j}, x_{B_{j+1}}]$, the segment joining x_{B_j} and $x_{B_{j+1}}$. Since $\{B_i\}_i$ is a good chain,

$$H^1(\gamma(z, y)) \leq |z - x_{B_{j+1}}| + H^1(\gamma(x_{B_{j+1}}, y)) \leq r_{B_{j+1}} + \sum_{i=j}^N 2r_{B_i} \lesssim r_{B_j} \approx \delta_\Omega(z).$$

We would like to note that the implicit constants do not depend on N. Indeed, from the properties of the good chain it easily follows that

$$\sum_{i=j}^N r_{B_i} \leq \sum_{k=0}^\infty \sum_{i:2^{-k-1} < \frac{r_{B_i}}{Cr_{B_j}} \leq 2^{-k}} r_{B_i} \leq 2\,C^2\,r_{B_j}.$$

Thus, γ is a carrot curve from x to y. $\qquad\square$

10 The main lemma for the Proof of Theorem 1.3

Because of the absence of doubling conditions on harmonic measure under the weak-A_∞ assumption, to prove Theorem 1.3 we cannot use arguments similar

to the ones in [1] or [6]. Instead, we prove a local result which involves only one pole and one ball which has its own interest. This is the Main Lemma 10.2 below.

Let $B \subset \mathbb{R}^{n+1}$ be a ball centered at $\partial\Omega$ and let $p \in \Omega$. We restate Definition 2.15 in the following form: ω^p satisfies the *weak-A_∞ condition in B* if for every $\varepsilon_0 \in (0, 1)$ there exists $\delta_0 \in (0, 1)$ such that the following holds: for any subset $E \subset B \cap \partial\Omega$,

$$\sigma(E) \leq \delta_0\, \sigma(B \cap \partial\Omega) \quad \Longrightarrow \quad \omega^p(E) \leq \varepsilon_0\, \omega^p(2B). \tag{10.1}$$

In the next sections we will prove the following.

Main Theorem 10.2 Let $\Omega \subset \mathbb{R}^{n+1}$ have n-uniformly rectifiable boundary. Let $R_0 \in \mathcal{D}$ and let $p \in \Omega \backslash 4B_{R_0}$ be a point such that

$$c\,\ell(R_0) \leq \mathrm{dist}(p, \partial\Omega) \leq \mathrm{dist}(p, R_0) \leq c^{-1}\,\ell(R_0)$$

and $\omega^p(R_0) \geq c' > 0$. Suppose that ω^p satisfies the weak-A_∞ condition in B_{R_0}. Then there exists a subset $\mathrm{Con}(R_0) \subset R_0$ and a constant $c'' > 0$ with $\sigma(\mathrm{Con}(R_0)) \geq c''\sigma(R_0)$ such that each point $x \in \mathrm{Con}(R_0)$ can be joined to p by a carrot curve. The constant c'' and the constants involved in the carrot condition only depend on c, c', n, the weak-A_∞ condition, and the n-UR character of $\partial\Omega$.

The notation $\mathrm{Con}(\cdot)$ stands for "connectable".

It is easy to check that Theorem 1.3 follows from this result. First notice that the assumptions of the theorem imply that $\partial\Omega$ is n-uniformly rectifiable by Theorem 9.2. Consider now any $x \in \Omega$ and take a point $\xi \in \partial\Omega$ such that $|x - \xi| = \delta_\Omega(x)$. Then we consider the point p in the segment $[x, \xi]$ such that $|p - \xi| = \frac{1}{16}\,\delta_\Omega(x)$. By Lemma 9.3, we have

$$\omega^p(B(\xi, \tfrac{1}{8}\delta_\Omega(x))) \gtrsim 1,$$

because $p \in \overline{\tfrac{1}{2}B(\xi, \tfrac{1}{8}\delta_\Omega(x))}$. Hence, by covering $B(\xi, \tfrac{1}{8}\delta_\Omega(x)) \cap \partial\Omega$ with cubes $R \in \mathcal{D}$ contained in $B(\xi, \tfrac{1}{4}\delta_\Omega(x)) \cap \partial\Omega$ with side length comparable to $\delta_\Omega(x)$ we deduce that at least one these cubes, call it R_0, satisfies $\omega^p(R_0) \gtrsim 1$. Further, by taking the side length small enough, we may also assume that $p \notin 4B_{R_0}$. Since ω^p satisfies the weak-A_∞ property in B_{R_0} (by the assumptions in Theorem 1.3), we can apply the Main Lemma 10.2 above and infer that there exists a subset $F := \mathrm{Con}(R_0) \subset R_0$ with $\sigma(F) \geq c'\sigma(R_0) \gtrsim \delta_\Omega(x)^n$ such that all $y \in F$ can be joined to x by a carrot curve, which proves that Ω satisfies the weak local John condition and concludes the proof of Theorem 1.3.

Two essential ingredients of the proof of the Main Lemma 10.2 are a corona type decomposition (whose existence is ensured by the n-uniform rectifiability of the boundary) and the Alt–Caffarelli–Friedman monotonicity formula [3]. This formula is used in some of the connectivity arguments below. This allows to connect by carrot curves corkscrew points where the Green function is not too small to other corkscrew points at a larger distance from the boundary where the Green function is still not too small (see Lemma 11.6 for the precise statement). The use of the Alt–Caffarelli–Friedman formula is not new to problems involving harmonic measure and connectivity (see, for example, [7]). However, the way it is applied here is new, as far as we know.

Two important steps of the proof of the Main Lemma 10.2 (and so of Theorem 1.3) are the Geometric Lemma 14.3 and the Key Lemma 15.1. An essential idea consists of distinguishing cubes with "two well separated big corkscrews" (see Sect. 13.4 for the precise definition). In the Geometric Lemma 10.2 we construct two disjoint open sets satisfying a John condition associated to trees involving this type of cubes, so that the boundaries of the open sets are located in places where the Green function is very small. This construction is only possible because the associated tree involves only cubes with two well separated big corkscrews. The existence of these cubes is an obstacle for the construction of carrot curves. However, in a sense, in the Key Lemma 15.1 we take advantage of their existence to obtain some delicate estimates for the Green function on some corkscrew points.

We claim now that to prove he Main Lemma 10.2 **we can assume that** $\Omega = \mathbb{R}^{n+1} \backslash \partial\Omega$. To check this, let Ω, p, and R_0 satisfy the assumptions in the Main Lemma. Consider the open set $V = \mathbb{R}^{n+1} \backslash \partial\Omega$. Then the harmonic measure ω^p in Ω coincides with the harmonic measure ω_V^p in V (the fact that V is not connected does not disturb us). Thus V, p, and R_0 satisfy the assumptions in the Main Lemma, and moreover $V = \mathbb{R}^{n+1} \backslash \partial\Omega = \mathbb{R}^{n+1} \backslash \partial V$. Assuming the Main Lemma to be valid in this particular case, we deduce that there exists a subset $\mathrm{Con}(R_0) \subset R_0$ and a constant $c'' > 0$ with $\sigma(\mathrm{Con}(R_0)) \geq c'' \sigma(R_0)$ such that each point $x \in \mathrm{Con}(R_0)$ can be joined to p by a carrot curve in V. Now just observe that if γ is one of this carrot curves and it joints p and $x \in \mathrm{Con}(R_0) \subset \partial V = \partial\Omega$, then γ is contained in V except for its end-point x. By connectivity, since $p \in \Omega \cap \gamma$, γ must be contained in Ω, except for the end-point x. Hence, γ is a carrot curve with respect to Ω.

Sections 11–16 are devoted to the proof of Main Lemma 10.2. To this end, we will assume that $\Omega = \mathbb{R}^{n+1} \backslash \partial\Omega$.

11 The Alt–Caffarelli–Friedman formula and the existence of short paths

11.1 The Alt–Caffarelli–Friedman formula

Recall the following well known result of Alt–Caffarelli–Friedman (see [14, Theorems 12.1 and 12.3]):

Theorem 11.1 *Let $B(x, R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x, R)) \cap C(B(x, R))$ be nonnegative subharmonic functions. Suppose that $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Set*

$$J_i(x, r) = \frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_i(y)|^2}{|y - x|^{n-1}} dy,$$

and

$$J(x, r) = J_1(x, r) J_2(x, r). \tag{11.1}$$

Then $J(x, r)$ is a non-decreasing function of $r \in (0, R)$ and $J(x, r) < \infty$ for all $r \in (0, R)$. That is,

$$J(x, r_1) \leq J(x, r_2) < \infty \quad for \quad 0 < r_1 \leq r_2 < R. \tag{11.2}$$

Further,

$$J_i(x, r) \lesssim \frac{1}{r^2} \|u_i\|_{\infty, B(x,2r)}^2. \tag{11.3}$$

In the case of equality we have the following result (see [46, Theorem 2.9]).

Theorem 11.2 *Let $B(x, R)$ and u_1, u_2 be as in Theorem 11.1. Suppose that $J(x, r_a) = J(x, r_b)$ for some $0 < r_a < r_b < R$. Then either one or the other of the following holds:*

(a) *$u_1 = 0$ in $B(x, r_b)$ or $u_2 = 0$ in $B(x, r_b)$;*
(b) *there exists a unit vector e and constants $k_1, k_2 > 0$ such that*

$$u_1(y) = k_1 ((y - x) \cdot e)^+, \quad u_2(y) = k_2 ((y - x) \cdot e)^-, \quad in \ B(x, r_b).$$

We will also need the following auxiliary lemma.

Lemma 11.3 *Let $B(x, R) \subset \mathbb{R}^{n+1}$, and let $\{u_i\}_{i \geq 1} \subset W^{1,2}(B(x, R)) \cap C(B(x, R))$ a sequence of functions which are nonnegative, subharmonic, such that each u_i is harmonic in $\{y \in B(x, R) : u_i(y) > 0\}$ and $u_i(x) = 0$. Suppose also that*

$$\|u_i\|_{\infty, B(x,R)} \leq C_1 R \quad and \quad \|u_i\|_{\text{Lip}^\alpha, B(x,R)} \leq C_1 R^{1-\alpha}$$

for all $i \geq 1$. Then, for every $0 < r < R$ there exists a subsequence $\{u_{i_k}\}_{k \geq 1}$ which converges uniformly in $B(x, r)$ and weakly in $W^{1,2}(B(x, r))$ to some function $u \in W^{1,2}(B(x, r)) \cap C(B(x, r))$, and moreover,

$$\lim_{k \to \infty} \int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y - x|^{n-1}} dy = \int_{B(x,r)} \frac{|\nabla u(y)|^2}{|y - x|^{n-1}} dy. \qquad (11.4)$$

Proof The existence of a subsequence $\{u_{i_k}\}_{k \geq 1}$ converging weakly in $W^{1,2}(B(x, r))$ and uniformly in $B(x, r)$ to some function $u \in W^{1,2}(B(x, r)) \cap C(B(x, r))$ is an immediate consequence of the Arzelà–Ascoli and the Banach–Alaoglu theorems. The identity (11.4) is clear when $n = 1$, and quite likely, for $n > 1$ this is also well known. However, for completeness, we will show the details (for $n > 1$).

Consider a non-negative subharmonic function $v \in W^{1,2}(B(x, R)) \cap C(B(x, R))$ which is harmonic in $\{y \in B(x, R) : v(y) > 0\}$ so that $v(x) = 0$. For $0 < r < R$ and $0 < \delta < R - r$, let φ be a radial C^∞ function such that $\chi_{B(x,r)} \leq \varphi \leq \chi_{B(x,r+\delta)}$. Let $\mathcal{E}(y) = c_n^{-1} |y|^{1-n}$ be the fundamental solution of the Laplacian. For $\varepsilon > 0$, denote $v_\varepsilon = \max(v, \varepsilon) - \varepsilon$. Then we have

$$\int \frac{|\nabla v_\varepsilon(y)|^2}{|y - x|^{n-1}} \varphi(y) \, dy = c_n \int \nabla v_\varepsilon(y) \, \nabla(\mathcal{E}(x - \cdot) \, v_\varepsilon \, \varphi)(y) \, dy$$
$$- c_n \int \nabla v_\varepsilon(y) \, \mathcal{E}(x - y) \, v_\varepsilon(y) \, \nabla\varphi(y) \, dy$$
$$- c_n \int \nabla v_\varepsilon(y) \, \nabla_y \mathcal{E}(x - y) \, v_\varepsilon(y) \, \varphi(y) \, dy = c_n(I_1 - I_2 - I_3).$$

Using the fact that v_ε is harmonic in $\{v_\varepsilon > 0\}$ and that $\mathcal{E}(x - \cdot) \, v_\varepsilon \, \varphi \in W_0^{1,2}(\{v_\varepsilon > 0\} \cap B(x, R))$ since φ is compactly supported in $B(x, R)$, $v_\varepsilon = 0$ on $\partial\{v_\varepsilon > 0\}$, and x is far away from $\overline{\{v_\varepsilon > 0\}}$, it follows easily that $I_1 = 0$. On the other hand, we have

$$2 I_3 = \int \nabla(v_\varepsilon^2 \, \varphi)(y) \, \nabla_y \mathcal{E}(x - y) \, dy - \int v_\varepsilon(y)^2 \, \nabla_y \mathcal{E}(x - y) \, \nabla\varphi(y) \, dy$$
$$= -v_\varepsilon(x)^2 - \int v_\varepsilon(y)^2 \, \nabla_y \mathcal{E}(x - y) \, \nabla\varphi(y) \, dy.$$

Thus,

$$\int \frac{|\nabla v_\varepsilon(y)|^2}{|y - x|^{n-1}} \varphi(y) \, dy = -c_n \int \nabla v_\varepsilon(y) \, \mathcal{E}(x - y) \, v_\varepsilon(y) \, \nabla\varphi(y) \, dy$$
$$+ \frac{c_n}{2} \int v_\varepsilon(y)^2 \, \nabla_y \mathcal{E}(x - y) \, \nabla\varphi(y) \, dy.$$

Taking into account that supp $\nabla\varphi$ is far away from x, letting $\varepsilon \to 0$, we obtain

$$\int \frac{|\nabla v(y)|^2}{|y-x|^{n-1}}\varphi(y)\,dy = -c_n \int \nabla v(y)\,\mathcal{E}(x-y)\,v(y)\,\nabla\varphi(y)\,dy$$
$$+ \frac{c_n}{2}\int v(y)^2\,\nabla_y\mathcal{E}(x-y)\,\nabla\varphi(y)\,dy.$$

Using the preceding identity, it follows easily that

$$\lim_{k\to\infty}\int \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}}\varphi(y)\,dy = \int \frac{|\nabla u(y)|^2}{|y-x|^{n-1}}\varphi(y)\,dy.$$

Indeed, $\lim_{k\to\infty} u_{i_k}(x)^2 = u(x)^2$. Also, it is clear that

$$\lim_{k\to\infty}\int u_{i_k}(y)^2\,\nabla_y\mathcal{E}(x-y)\,\nabla\varphi(y)\,dy = \int u(y)^2\,\nabla_y\mathcal{E}(x-y)\,\nabla\varphi(y)\,dy.$$

Further,

$$\int \nabla u_{i_k}(y)\,\mathcal{E}(x-y)\,u_{i_k}(y)\,\nabla\varphi(y)\,dy$$
$$= \int \nabla u_{i_k}(y)\,\mathcal{E}(x-y)\,u(y)\,\nabla\varphi(y)\,dy$$
$$+ \int \nabla u_{i_k}(y)\,\mathcal{E}(x-y)\,(u_{i_k}(y)-u(y))\,\nabla\varphi(y)\,dy$$
$$\xrightarrow{k\to\infty} \int \nabla u(y)\,\mathcal{E}(x-y)\,u(y)\,\nabla\varphi(y)\,dy,$$

by the weak convergence of u_{i_k} in $W^{1,2}(B(x,R))$ and the uniform convergence in $B(x, r+\delta)$, since supp $\nabla\varphi$ is far away from x.

Let ψ be a radial C^∞ function such that $\chi_{B(x,r-\delta)} \leq \psi \leq \chi_{B(x,r)}$. The same argument as above shows that

$$\lim_{k\to\infty}\int \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}}\psi(y)\,dy = \int \frac{|\nabla u(y)|^2}{|y-x|^{n-1}}\psi(y)\,dy.$$

Consequently,

$$\limsup_{k\to\infty}\int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}}\,dy \leq \lim_{k\to\infty}\int \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}}\varphi(y)\,dy$$
$$= \int \frac{|\nabla u(y)|^2}{|y-x|^{n-1}}\varphi(y)\,dy,$$

and also

$$\liminf_{k\to\infty} \int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}} dy \geq \lim_{k\to\infty} \int \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}} \psi(y)\, dy$$

$$= \int \frac{|\nabla u(y)|^2}{|y-x|^{n-1}} \psi(y)\, dy.$$

Since $\delta > 0$ can be taken arbitrarily small, (11.4) follows. □

Lemma 11.4 *Let $B(x, 2R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x, 2R)) \cap C(B(x, 2R))$ be nonnegative subharmonic functions such that each u_i is harmonic in $\{y \in B(x, 2R) : u_i(y) > 0\}$. Suppose that $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Assume also that*

$$\|u_i\|_{\infty, B(x,2R)} \leq C_1\, R \quad \text{and} \quad \|u_i\|_{\mathrm{Lip}^\alpha, B(x,2R)} \leq C_1\, R^{1-\alpha} \quad \text{for } i = 1, 2.$$

For any $\varepsilon > 0$, there exists some $\delta > 0$ such that if

$$J(x, R) \leq (1 + \delta)\, J(x, \tfrac{1}{2}R),$$

with $J(\cdot, \cdot)$ defined in (11.1), then either one or the other of the following holds:

(a) *$\|u_1\|_{\infty, B(x,R)} \leq \varepsilon\, R$ or $\|u_2\|_{\infty, B(x,R)} \leq \varepsilon\, R$;*
(b) *there exists a unit vector e and constants $k_1, k_2 > 0$ such that*

$$\|u_1 - k_1\, ((\cdot - x) \cdot e)^+\|_{\infty, B(x,R)} \leq \varepsilon\, R,$$
$$\|u_2 - k_2\, ((\cdot - x) \cdot e)^-\|_{\infty, B(x,R)} \leq \varepsilon\, R.$$

The constant δ depends only on $n, \alpha, C_1, \varepsilon$.

Proof Suppose that the conclusion of the lemma fails. Then, by replacing $u_i(y)$ by $\frac{1}{R} u_i(Ry + x)$, we can assume that $x = 0$ and $R = 1$. Let $\varepsilon > 0$, and for each $\delta = 1/k$ and $i = 1, 2$, consider functions $u_{i,k}$ satisfying the assumptions of the lemma and such that neither (a) nor (b) holds for them. By Lemma 11.3, there exist subsequences (which we still denote by $\{u_{i,k}\}_k$) which converge uniformly in $B(0, \frac{3}{2})$ and weakly in $W^{1,2}(B(0, \frac{3}{2}))$ to some functions $u_i \in W^{1,2}(B(0, \frac{3}{2})) \cap C(B(0, \frac{3}{2}))$, and moreover,

$$\lim_{k\to\infty} \int_{B(0,r)} \frac{|\nabla u_{i,k}(y)|^2}{|y|^{n-1}} dy = \int_{B(0,r)} \frac{|\nabla u_i(y)|^2}{|y|^{n-1}} dy$$

both for $r = 1$ and $r = 1/2$. Clearly, the functions u_i are non-negative, subharmonic, and $u_1 \cdot u_2 = 0$. Hence, by Theorem 11.2, one of the following holds:

(a') $u_1 = 0$ in $B(0, 1)$ or $u_2 = 0$ in $B(0, 1)$;

(b') there exists a unit vector e and constants $k_1, k_2 > 0$ such that

$$u_1(y) = k_1 (y \cdot e)^+, \qquad u_2(y) = k_2 (y \cdot e)^-, \qquad \text{in } B(0, 1).$$

However, the fact that neither (a) nor (b) holds for any pair $u_{1,k}, u_{2,k}$, together with the uniform convergence of $\{u_{i,k}\}_k$, implies that neither (a') nor (b') can hold, and thus we get a contradiction. $\qquad\square$

11.2 Existence of short paths

Let $p \in \Omega$ and $\Lambda > 1$. For $x \in \partial\Omega$, we write $x \in \mathsf{WA}(p, \Lambda)$ if for all $0 < r \leq \delta_\Omega(p)$,

$$\Lambda^{-1} \frac{\sigma(B(x, r))}{\sigma(B(x, \delta_\Omega(p)))} \leq \omega^p(B(x, r)) \leq \Lambda \frac{\sigma(B(x, r))}{\sigma(B(x, \delta_\Omega(p)))}.$$

We will see in Sect. 12 that, under the assumptions of the Main Lemma 10.2, for some Λ big enough,

$$\sigma(\mathsf{WA}(p, \Lambda) \cap R_0) \gtrsim \sigma(R_0). \tag{11.5}$$

Lemma 11.5 *Let $p \in \Omega$, $x_0 \in \mathsf{WA}(p, \Lambda)$, and $r \in (0, \delta_\Omega(p))$. Then there exists $q \in B(x_0, r)$ such that, for some constant $\kappa \in (0, 1/10)$,*

(a) $\delta_\Omega(q) \geq \kappa r$, *and*

(b)

$$\kappa \frac{\omega^p(B(x_0, r))}{r^{n-1}} \leq g(p, q) \leq \kappa^{-1} \frac{\omega^p(B(x_0, r))}{r^{n-1}}.$$

The constant κ depends only on Λ, n, and C_0, the AD-regularity constant of $\partial\Omega$.

Proof This follows easily from Lemmas 9.4 and 9.7. $\qquad\square$

Lemma 11.6 (Short paths). *Let $p \in \Omega$, $x_0 \in \mathsf{WA}(p, \Lambda)$, and for $0 < r_0 \leq \delta_\Omega(p)/4$, $0 < \tau_0, \lambda_0 \leq 1$, let $q \in \Omega$ be such that*

$$q \in B(x_0, r_0), \quad \delta_\Omega(q) \geq \tau_0 r_0, \quad g(p, q) \geq \lambda_0 \frac{\delta_\Omega(q)}{\delta_\Omega(p)^n}. \tag{11.6}$$

Then there exist constants $A_1 > 1$ and $0 < a_1, \lambda_1 < 1$ such that for every $r \in (r_0, \delta_\Omega(p)/2)$, there exists some point $q' \in \Omega$ such that

$$q' \in B(x_0, A_1 r), \quad \delta_\Omega(q') \geq \kappa \, |x_0 - q'| \geq \kappa \, r, \quad g(p, q') \geq \lambda_1 \frac{\delta_\Omega(q')}{\delta_\Omega(p)^n}, \tag{11.7}$$

(with κ as in Lemma 11.5) and such that q and q' can be joined by a curve γ such that

$$\gamma \subset \{y \in B(x_0, A_1 r) : \mathrm{dist}(y, \partial\Omega) > a_1 \, r_0\}.$$

The parameters λ_1, A_1, a_1 depend only on $C_0, \Lambda, \lambda_0, \tau_0$ and the ratio r/r_0.

Proof All the parameters in the lemma will be fixed along the proof. We assume that $A_1 \gg \kappa^{-1} > 1$. First note that we may assume that $r < 2A_1^{-1}|x_0 - p|$. Otherwise, we just take a point $q' \in \Omega$ such that $|p - q'| = \delta_\Omega(p)/2$, which clearly satisfies the properties in (11.7). Further, both q and q' belong to the open connected set

$$U := \{x \in \Omega : g(p, x) > c_2 \, r_0 \, \delta_\Omega(p)^{-n}\}$$

for a sufficiently small $c_2 > 0$. The fact that U is connected is well known. This follows from the fact that, for any $\lambda > 0$, any connected component of $\{g(p, \cdot) > \lambda\}$ should contain p. Otherwise there would be a connected component where $g(p, \cdot) - \lambda$ is positive and harmonic with zero boundary values. So, by maximum principle, $g(p, \cdot) - \lambda$ should equal λ in the whole component, which is a contradiction. So there is only one connected component.

We just let γ be a curve contained in U. Note that

$$\mathrm{dist}(U, \partial\Omega) \geq c \, r_0^{\frac{1}{\alpha}} \, \delta_\Omega(p)^{1 - \frac{1}{\alpha}} \geq a \, r_0,$$

for a sufficiently small $a > 0$ because, by boundary Hölder continuity,

$$g(p, x) \lesssim \left(\frac{\delta_\Omega(x)}{\delta_\Omega(p)}\right)^\alpha \frac{1}{\delta_\Omega(p)^{n-1}}$$

if $\mathrm{dist}(x, \partial\Omega) \leq \delta_\Omega(p)/2$. Further, the fact that $g(p, x) \leq c|x - p|^{1-n}$ ensures that $U \subset B(p, C\delta_\Omega(p))$, for a sufficiently big constant C depending on r/r_0.

So from now on we assume that $r < 2A_1^{-1}|x_0 - p|$. By Lemma 11.5 we know there exists some point $\widetilde{q} \in \Omega$ such that

$$\widetilde{q} \in B(x_0, \kappa^{-1}r), \quad \delta_\Omega(\widetilde{q}) \geq r \geq \kappa \, |x_0 - \widetilde{q}| \geq \kappa \, \delta_\Omega(\widetilde{q}) \geq \kappa \, r,$$
$$g(p, \widetilde{q}) \geq c \, \frac{\delta_\Omega(\widetilde{q})}{\delta_\Omega(p)^n}, \tag{11.8}$$

with c depending on κ and Λ.

Assume that q and $\widetilde{q}$ cannot be joined by a curve γ as in the statement of the lemma. Otherwise, we choose $q' = \widetilde{q}$ and we are done. For $t > 0$, consider the open set

$$V^t = \left\{ x \in B(x_0, \tfrac{1}{4}A_1 r) : g(p, x) > t\, r_0\, \delta_\Omega(p)^{-n} \right\}.$$

We fix $t > 0$ small enough such that $q, \widetilde{q} \in V^{2t} \subset V^t$. Such t exists by (11.6) and (11.8), and it may depend on $\Lambda, \lambda, r/r_0$.

Let V_1 and V_2 be the respective components of V^t to which q and $\widetilde{q}$ belong. We have

$$V_1 \cap V_2 = \varnothing,$$

because otherwise there is a curve contained in $V^t \subset B(x_0, \tfrac{1}{4}A_1 r)$ which connects q and $\widetilde{q}$, and further this is far away from $\partial\Omega$. Indeed, we claim that

$$\operatorname{dist}(V^t, \partial\Omega) \gtrsim_{A_1, \Lambda, t, r/r_0} r_0. \tag{11.9}$$

To see this, note that by the Hölder continuity of $g(p, \cdot)$ in $B(x_0, \tfrac{1}{2}A_1 r)$, for all $x \in V^t$, we have

$$
\begin{aligned}
t\, \frac{r_0}{\delta_\Omega(p)^n} \le g(p, x) &\lesssim \sup_{y \in B(x_0, \frac{1}{2}A_1 r)} g(p, y) \left(\frac{\delta_\Omega(x)}{A_1 r} \right)^\alpha \\
&\le \fint_{B(x_0, \frac{3}{4}A_1 r)} g(p, y)\, dy \left(\frac{\delta_\Omega(x)}{A_1 r} \right)^\alpha \\
&\lesssim_{A_1, \Lambda} \frac{A_1 r}{\delta_\Omega(p)^n} \left(\frac{\delta_\Omega(x)}{A_1 r} \right)^\alpha,
\end{aligned}
$$

where in the last inequality we used Lemma 9.5 and that $x_0 \in \mathsf{WA}(p, \Lambda)$. This yields our claim.

Next we wish to apply the Alt–Caffarelli–Friedman formula with

$$
\begin{aligned}
u_1(x) &= \chi_{V_1} \left(\delta_\Omega(p)^n\, g(p, x) - t\, r_0 \right)^+, \\
u_2(x) &= \chi_{V_2} \left(\delta_\Omega(p)^n\, g(p, x) - t\, r_0 \right)^+.
\end{aligned}
$$

It is clear that both satisfy the hypotheses of Theorem 11.1. For $i = 1, 2$ and $0 < s < A_1 r$, we denote

$$J_i(x_0, s) = \frac{1}{s^2} \int_{B(x_0, s)} \frac{|\nabla u_i(y)|^2}{|y - x_0|^{n-1}}\, dy,$$

so that $J(x_0, s) = J_1(x_0, s) J_2(x_0, s)$. We claim that:

(i) $J_i(x_0, s) \lesssim_\Lambda 1$ for $i = 1, 2$ and $0 < s < \frac{1}{4}A_1 r$.

(ii) $J_i(x_0, 2r) \gtrsim_{\Lambda, \lambda, r/r_0} 1$ for $i = 1, 2$.

The condition (i) follows from (11.3) and the fact that

$$g(p, y) \lesssim \frac{s}{\delta_\Omega(p)^n} \quad \text{for all } y \in B(x_0, s), \tag{11.10}$$

which holds by Lemma 9.5 and subharmonicity, since $x_0 \in \mathsf{WA}(p, \Lambda)$. Concerning (ii), note first that

$$|\nabla u_1(y)| \lesssim \delta_\Omega(p)^n \frac{g(p, y)}{\delta_\Omega(y)} \lesssim_{\tau_0} \delta_\Omega(p)^n \frac{r_0}{\delta_\Omega(p)^n} = 1 \quad \text{for all } y \in B(q, \tau_0 r_0/2),$$

where we first used Cauchy estimates and then the pointwise bounds of $g(\cdot, \cdot)$ in (11.10) with $s \approx \delta_\Omega(y)$. Thus, using also that $q \in V^{2t}$, we infer that $u_1(y) > t\, r_0/2$ in some ball $B(q, ctr_0)$ with c possibly depending on $\Lambda, \lambda, r/r_0$. Analogously, we deduce that $u_2(y) > t\, r_0/2$ in some ball $B(\widetilde{q}, ctr_0)$. Let B be the largest open ball centered at q not intersecting ∂V_1 and let $y_0 \in \partial V_1 \cap \partial B$. Then, by considering the convex hull $H \subset B$ of $B(q, ctr_0)$ and y_0 and integrating in spherical coordinates (with the origin in y_0), one can check that

$$\int_H |\nabla u_1|\, dy \gtrsim_t r_0^{n+1}.$$

An analogous estimate holds for u_2, and then it easily follows that

$$J_i(x_0, 2r_0) \gtrsim_t 1,$$

which implies (ii). We leave the details for the reader.

From the conditions (i) and (ii) and the fact that $J(x, r)$ is non-decreasing we infer that

$$J(x_0, s) \approx_{\Lambda, \lambda, r/r_0} 1 \quad \text{for } 2r < s < \frac{1}{4}A_1 r.$$

and also

$$J_i(x_0, s) \approx_{\Lambda, \lambda, r/r_0} 1 \quad \text{for } i = 1, 2 \text{ and } 2r < s < \frac{1}{4}A_1 r. \tag{11.11}$$

Assume that $\frac{1}{4}A_1 = 2^m$ for some big $m > 1$. Since $J(x_0, s)$ is non-decreasing we infer that there exists some $h \in [1, m-1]$ such that

$$J(x_0, 2^{h+1}r) \le C(\Lambda, \lambda, r/r_0)^{1/m} J(x_0, 2^h r),$$

because otherwise, by iterating the reverse inequality, we get a contradiction. Now from Lemma 11.4 we deduce that, given any $\varepsilon > 0$, for m big enough, there are constant $k_i \approx_{\Lambda,\lambda,r/r_0} 1$ and a unit vector e such that

$$\|u_1 - k_1\,((\cdot - x_0)\cdot e)^+\|_{\infty, B(x_0, 2^h r)} + \|u_2 - k_2\,((\cdot - x_0)\cdot e)^-\|_{\infty, B(x_0, 2^h r)} \le \varepsilon\, 2^h\, r. \tag{11.12}$$

As a matter of fact, $\|u_i\|_{\infty, B(x_0, 2^{h+1} r)} \approx_{\Lambda,\lambda,r/r_0} 2^h r$ by (11.3), (11.11), and (11.10); $\|u_i\|_{\mathrm{Lip}^\alpha, B(x_0, 2^{h+1} r)} \lesssim_{\Lambda,\lambda,r/r_0} (2^h r)^{1-\alpha}$ by Lemma 9.6; and the option (a) in Lemma 11.4 cannot hold (since we have $\|u_i\|_{\infty, B(x_0, 2^h r)} \approx_{\Lambda,\lambda,r/r_0} 2^h r$).

In particular, for ε small, (11.12) implies that if $q' := x_0 + 2^{h-1} r e$, then one has $u_1(q') \approx_{\Lambda,\lambda,r/r_0} 2^{h-1} r$, and also that

$$u_1(y) \approx_{\Lambda,\lambda,r/r_0} 2^{h-1} r > 0 \quad \text{for all } y \in B(q', 2^{h-2} r).$$

Thus $B(q', 2^{h-2} r) \subset \Omega$ and so q' is at a distance at least $2^{h-2} r$ from $\partial\Omega$, and also

$$g(p, q') \ge \frac{u_1(q')}{\delta_\Omega(p)^n} \approx_{\Lambda,\lambda,r/r_0} \frac{2^h\, r}{\delta_\Omega(p)^n}.$$

Further, since q and q' are both in V_1 by definition, there is a curve γ which joins q and q' contained in V_1 satisfying

$$\operatorname{dist}(\gamma, \partial\Omega) \gtrsim_{A_1,\Lambda,t,r/r_0} r_0,$$

by (11.9). So q' satisfies all the required properties in the lemma and we are done. $\qquad\square$

12 Types of cubes

From now on we fix $R_0 \in \mathcal{D}$ and $p \in \Omega$ and we assume that we are under the assumptions of the Main Lemma 10.2.

We need now to define two families HD and LD of high density and low density cubes, respectively. Let $A \gg 1$ be some fixed constant. We denote by HD (high density) the family of maximal cubes $Q \in \mathcal{D}$ which are contained in R_0 and satisfy

$$\frac{\omega^p(2Q)}{\sigma(2Q)} \ge A\, \frac{\omega^p(2R_0)}{\sigma(2R_0)}.$$

We also denote by LD (low density) the family of maximal cubes $Q \in \mathcal{D}$ which are contained in R_0 and satisfy

$$\frac{\omega^p(Q)}{\sigma(Q)} \le A^{-1} \frac{\omega^p(R_0)}{\sigma(R_0)}$$

(notice that $\omega^p(R_0) \approx \omega^p(2R_0) \approx 1$ by assumption). Observe that the definition of the family HD involves the density of $2Q$, while the one of LD involves the density of Q.

We denote

$$B_H = \bigcup_{Q \in \mathsf{HD}} Q \quad \text{and} \quad B_L = \bigcup_{Q \in \mathsf{LD}} Q.$$

Lemma 12.1 *We have*

$$\sigma(B_H) \lesssim \frac{1}{A} \sigma(R_0) \quad and \quad \omega^p(B_L) \le \frac{1}{A} \omega^p(R_0).$$

Proof By Vitali's covering theorem, there exists a subfamily $I \subset \mathsf{HD}$ so that the cubes $2Q$, $Q \in I$, are pairwise disjoint and

$$\bigcup_{Q \in \mathsf{HD}} 2Q \subset \bigcup_{Q \in I} 6Q.$$

Then, since σ is doubling, we obtain

$$\sigma(B_H) \lesssim \sum_{Q \in I} \sigma(2Q) \le \frac{1}{A} \sum_{Q \in I} \frac{\omega^p(2Q)}{\omega^p(2R_0)} \sigma(2R_0) \lesssim \frac{1}{A} \sigma(R_0).$$

Next we turn our attention to the low density cubes. Since the cubes from LD are pairwise disjoint, we have

$$\omega^p(B_L) = \sum_{Q \in \mathsf{LD}} \omega^p(Q) \le \frac{1}{A} \sum_{Q \in \mathsf{LD}} \frac{\sigma(Q)}{\sigma(R_0)} \omega^p(R_0) \le \frac{1}{A} \omega^p(R_0).$$

$\square$

From the above estimates and the fact that the harmonic measure belongs to weak-A_∞ (cf. (10.1)), we infer that if A is chosen big enough, then

$$\omega^p(B_H) \le \varepsilon_0 \, \omega^p(2B_{R_0}) \le \frac{1}{4} \omega^p(R_0)$$

and thus

$$\omega^p(B_H \cup B_L) \le \frac{1}{4}\,\omega^p(R_0) + \frac{1}{A}\,\omega^p(R_0) \le \frac{1}{2}\,\omega^p(R_0).$$

As a consequence, denoting $G_0 = R_0 \backslash (B_H \cup B_L))$, we deduce that

$$\omega^p(G_0) \ge \frac{1}{2}\,\omega^p(R_0) \approx \omega^p(2B_{R_0}),$$

which implies that

$$\sigma(G_0) \gtrsim \sigma(2B_{R_0}) \approx \sigma(R_0),$$

again using the fact that ω^p belongs to weak-A_∞ in B_{R_0}. So we have:

Lemma 12.2 *Assuming A big enough, the set $G_0 := R_0 \backslash (B_H \cup B_L))$ satisfies*

$$\omega^p(G_0) \approx 1 \quad and \quad \sigma(G_0) \approx \sigma(R_0),$$

with the implicit constants depending on C_0 and the weak-A_∞ condition in B_{R_0}.

We denote by G the family of those cubes $Q \in \mathcal{D}(R_0)$ which are not contained in $\bigcup_{P \in \mathsf{HD} \cup \mathsf{LD}} P$. In particular, such cubes $Q \in \mathsf{G}$ do not belong to $\mathsf{HD} \cup \mathsf{LD}$ and

$$A^{-1}\frac{\omega^p(R_0)}{\sigma(R_0)} \le \frac{\omega^p(Q)}{\sigma(Q)} \lesssim \frac{\omega^p(2Q)}{\sigma(2Q)} \le A\,\frac{\omega^p(2R_0)}{\sigma(2R_0)}. \tag{12.1}$$

From this fact, it follows easily that G_0 is contained in the set $\mathsf{WA}(p, \Lambda)$ defined in Sect. 11.2, assuming Λ big enough, and so Lemma 12.2 ensures that (11.5) holds.

The following lemma is an immediate consequence of Lemma 11.5.

Lemma 12.3 *For every cube $Q \in \mathsf{G}$ there exists some point $z_Q \in 2B_Q \cap \Omega$ such that $\delta_\Omega(z_Q) \ge \kappa_0 \ell(Q)$ and*

$$g(p, z_Q) > c_3\,\frac{\ell(Q)}{\sigma(R_0)}, \tag{12.2}$$

for some $\kappa_0, c_3 > 0$, which depend on A and on the weak-A_∞ constants in B_{R_0}.

If $z_Q \in 2B_Q \cap \Omega$ and $\delta_\Omega(z_Q) \geq \kappa_0 \ell(Q)$, we say that z_Q is κ_0-*corkscrew* for Q. If (12.2) holds, we say that z_Q is a c_3-*good corkscrew* for Q. Abusing notation, quite often we will not write "for Q".

We will need the following auxiliary result:

Lemma 12.4 *Let $Q \in \mathcal{D}$ and let z_Q be a λ-good c_4-corkscrew, for some $\lambda, c_4 > 0$. Suppose that $\ell(Q) \geq c_5 \ell(R_0)$. Then there exists some C-good Harnack chain that joins z_Q and p, with C depending on λ, c_5.*

Proof Consider the open set $U = \{x \in \Omega : g(p, x) > \lambda \ell(Q)/\sigma(R_0)\}$. This is connected and thus there exists a curve $\gamma \subset U$ that connects z_Q and p. By Hölder continuity, any point $x \in \Omega$ such that $\delta_\Omega(x) \leq \delta_\Omega(p)/2$, satisfies

$$g(p, x) \leq c \left(\frac{\delta_\Omega(x)}{\ell(R_0)} \right)^\alpha \frac{1}{\ell(R_0)^{n-1}}.$$

Since $g(p, x) > \lambda \ell(Q)/\sigma(R_0) \gtrsim_{c_5, \lambda} \ell(R_0)^{1-n}$ for all $x \in U$, we then deduce that $\text{dist}(U, \partial\Omega) \geq c_6 \ell(R_0)$ for some $c_6 > 0$ depending on λ and c_5. Thus,

$$\text{dist}(\gamma, \partial\Omega) \geq c_6 \ell(R_0).$$

From the fact that $g(p, x) \leq |p - x|^{1-n}$ for all $x \in \Omega$, we infer that any $x \in U$ satisfies

$$\lambda \frac{\ell(Q)}{\sigma(R_0)} < g(p, x) \leq \frac{1}{|p - x|^{n-1}}.$$

Therefore,

$$|p - x| < \left(\frac{\sigma(R_0)}{\lambda \ell(Q)} \right)^{1/(n-1)} \lesssim_{c_5, \lambda} \ell(R_0).$$

So $U \subset B(p, C_2 \ell(R_0))$ for some C_2 depending on λ and c_5. Next we consider a Besicovitch covering of γ with balls B_i of radius $c_6 \ell(R_0)/2$. By volume considerations, it easily follows that the number of balls B_i is bounded above by some constant C_3 depending on λ and c_5, and thus this is a C-good Harnack chain, with $C = C(\lambda, c_5)$. □

Lemma 12.5 *There exists some constant κ_1 with $0 < \kappa_1 \leq \kappa_0$ such that the following holds for all $\lambda > 0$. Let $Q \in \mathsf{G}$, $Q \neq R_0$, and let z_Q be a λ-good κ_1-corkscrew. Then there exists some cube $R \in \mathsf{G}$ with $Q \subsetneq R \subset R_0$ and $\ell(R) \leq C \ell(Q)$ and a λ'-good κ_1-corkscrew z_R such that z_Q and z_R can be joined by a $C'(\lambda)$-good Harnack chain, with $\lambda' > 0$ and C depending on λ.*

The proof below yields a constant $\lambda' < \lambda$. On the other hand, the lemma ensures that z_R is still a κ_1-corkscrew, which will be important for the arguments to come.

Proof This follows easily from Lemma 11.6. For completeness we will show the details.

By choosing $\Lambda = \Lambda(A) > 0$ big enough, $G_0 \cap Q \subset \mathsf{WA}(p, \Lambda)$ and thus there exists some $x_0 \in Q \cap \mathsf{WA}(p, \Lambda)$. We let

$$\kappa_1 = \min\left(\kappa_0, \kappa\right),$$

where κ_0 is defined in Lemma 12.3 and κ in Lemma 11.5 (and thus it depends only on A and C_0). We apply Lemma 11.6 to x_0, $q = z_Q$, with $r_0 = 3r(B_Q)$, $\lambda_0 \approx \lambda$, and $r = 4r(B_Q)$. To this end, note that

$$\delta_\Omega(q) \geq \kappa_1 \ell(Q) = \kappa_1 \frac{1}{4} \ell(r(B_Q)) = \kappa_1 \frac{1}{12} r_0.$$

Hence there exists $q' \in B(x_0, A_1 r)$ such that

$$\delta_\Omega(q') \geq \kappa |x_0 - q'| \geq \kappa r, \qquad g(p, q') \geq \lambda_1 \frac{\delta_\Omega(q')}{\delta_\Omega(p)^n}, \qquad (12.3)$$

and such that q and q' can be joined by a curve γ such that

$$\gamma \subset \{y \in B(x_0, A_1 r) : \mathrm{dist}(y, \partial\Omega) > a_1 r_0\}, \qquad (12.4)$$

with λ_1, A_1, a_1 depending on $C_0, A, \lambda, \kappa_1$. Now let $R \in \mathcal{D}$ be the cube containing x_0 such that

$$\frac{1}{2} r(B_R) < |x_0 - q'| \leq r(B_R).$$

Observe that

$$r(B_R) \geq |x_0 - q'| \geq r = 4r(B_Q) \quad \text{and} \quad r(B_R) < 2|x_0 - q'| \leq 2A_1 r \lesssim_\lambda \ell(Q).$$

Also, we may assume that $\ell(R) \leq \ell(R_0)$ because otherwise we have $\ell(Q) \gtrsim A_1 \delta_\Omega(p)$ and then the statement in the lemma follows from Lemma 12.4. So we have $Q \subsetneq R \subset R_0$.

From (12.3) we get

$$\delta_\Omega(q') \geq \kappa |x_0 - q'| \geq \frac{1}{2} \kappa r(B_R) = 2\kappa \ell(R) > \kappa_1 \ell(R)$$

and

$$g(p, q') \geq c \lambda_1 \frac{2\kappa \, \ell(R)}{\sigma(R_0)}.$$

Hence, q' is a λ'-good κ_1-corkscrew, for $\lambda' = c\lambda_1 2\kappa$.

From (12.4) and arguing as in the end of the proof of Lemma 12.4 we infer that $z_Q = q$ and $z_R = q'$ can be joined by a $C(\lambda)$-good Harnack chain. $\qquad \square$

From now on we will assume that all corkscrew points for cubes $Q \in \mathsf{G}$ are κ_1-corkscrews, unless otherwise stated.

13 The corona decomposition and the Key Lemma

13.1 The corona decomposition

Recall that the $b\beta$ coefficient of a ball was defined in (9.1). For each $Q \in \mathcal{D}$, we denote

$$b\beta(Q) = b\beta_{\partial\Omega}(100B_Q).$$

Now we fix a constant $0 < \varepsilon \ll \min(1, \kappa_1)$. Given $R \in \mathcal{D}(R_0)$, we denote by $\mathsf{Stop}(R)$ the maximal family of cubes $Q \in \mathcal{D}(R)\backslash\{R\}$ satisfying that either $Q \notin \mathsf{G}$ or $b\beta(\widehat{Q}) > \varepsilon$, where $\widehat{Q}$ is the parent of Q. Recall that the family G was defined in (12.1). Note that, by maximality, $\mathsf{Stop}(R)$ is a family of pairwise disjoint cubes.

We define

$$\mathsf{T}(R) := \{Q \in \mathcal{D}(R) : \nexists \ S \in \mathsf{Stop}(R) \ \text{such that} \ Q \subset S\}.$$

In particular, note that $\mathsf{Stop}(R) \not\subset \mathsf{T}(R)$.

We now define the family of the top cubes with respect to R_0 as follows: first we define the families Top_k for $k \geq 1$ inductively. We set

$$\mathsf{Top}_1 = \{R \in \mathcal{D}(R_0) \cap \mathsf{G} : \ell(R) = 2^{-10}\ell(R_0)\}.$$

Assuming that Top_k has been defined, we set

$$\mathsf{Top}_{k+1} = \bigcup_{R \in \mathsf{Top}_k} (\mathsf{Stop}(R) \cap \mathsf{G}),$$

and then we define

$$\mathsf{Top} = \bigcup_{k \geq 1} \mathsf{Top}_k.$$

Notice that the family of cubes $Q \in \mathcal{D}(R_0)$ with $\ell(Q) \leq 2^{-10}\ell(R_0)$ which are not contained in any cube $P \in \mathsf{HD} \cup \mathsf{LD}$ is contained in $\bigcup_{R \in \mathsf{Top}} \mathsf{T}(R)$, and this union is disjoint. Also, all the cubes in that union belong to G.

The following lemma is an easy consequence of our construction. Its proof is left for the reader.

Lemma 13.1 *We have*

$$\mathsf{Top} \subset \mathsf{G}.$$

Also, for each $R \in \mathsf{Top}$,

$$\mathsf{T}(R) \subset \mathsf{G}.$$

Further, for all $Q \in \mathsf{T}(R) \cup \mathsf{Stop}(R)$,

$$\omega^p(2Q) \leq C\,A\,\frac{\sigma(Q)}{\sigma(R_0)}.$$

Remark that the last inequality holds for any cube $Q \in \mathsf{Stop}(R)$ because its parent $\widehat{Q}$ belongs to $\mathsf{T}(R)$ and so $\widehat{Q}$ is not contained in any cube from HD, which implies that $\omega^p(2Q) \leq \omega^p(2\widehat{Q}) \lesssim A\,\frac{\sigma(\widehat{Q})}{\sigma(R_0)} \approx A\,\frac{\sigma(Q)}{\sigma(R_0)}$.

Using that $\partial\Omega$ is n-UR (by the assumption in the Main Lemma 10.2), it is easy to prove that the cubes from Top satisfy a Carleson packing condition. This is shown in the next lemma.

Lemma 13.2 *We have*

$$\sum_{R \in \mathsf{Top}} \sigma(R) \leq M(\varepsilon)\,\sigma(R_0).$$

Proof For each $Q \in \mathsf{Top}$ we have

$$\sigma(Q) = \sum_{P \in \mathsf{Stop}(Q) \cap \mathsf{G}} \sigma(P) + \sum_{P \in \mathsf{Stop}(Q) \backslash \mathsf{G}} \sigma(P) + \sigma\left(Q \backslash \bigcup_{P \in \mathsf{Stop}(Q)} P\right).$$

Then we get

$$\sum_{Q \in \mathsf{Top}} \sigma(Q) \leq \sum_{Q \in \mathsf{Top}} \sum_{P \in \mathsf{Stop}(Q) \cap \mathsf{G}} \sigma(P)$$

$$+ \sum_{Q \in \mathsf{Top}} \sum_{P \in \mathsf{Stop}(Q) \backslash \mathsf{G}} \sigma(P) + \sum_{Q \in \mathsf{Top}} \sigma\left(Q \backslash \bigcup_{P \in \mathsf{Stop}(Q)} P\right).$$

$$(13.1)$$

Note now that, because of the stopping conditions, for all $Q \in \mathsf{Top}$, if $P \in \mathsf{Stop}(Q) \cap \mathsf{G}$, then the parent $\widehat{P}$ of P satisfies $b\beta_{\partial\Omega}(100B_{\widehat{P}}) > \varepsilon$. Hence, by Theorems 9.1 and 9.2,

$$\sum_{Q \in \mathsf{Top}} \sum_{P \in \mathsf{Stop}(Q) \cap \mathsf{G}} \sigma(P) \leq \sum_{P \in \mathcal{D}(R_0): b\beta_{\partial\Omega}(100B_{\widehat{P}}) > \varepsilon} \sigma(P) \leq C(\varepsilon)\, \sigma(R_0).$$

On the other hand, the cubes $P \in \mathsf{Stop}(Q) \backslash \mathsf{G}$ with $Q \in \mathsf{Top}$ do not contain any cube from Top, by construction. Hence, they are disjoint and thus

$$\sum_{Q \in \mathsf{Top}} \sum_{P \in \mathsf{Stop}(Q) \backslash \mathsf{G}} \sigma(P) \leq \sigma(R_0).$$

By an analogous reason,

$$\sum_{Q \in \mathsf{Top}} \sigma\left(Q \backslash \bigcup_{P \in \mathsf{Stop}(Q)} P \right) \leq \sigma(R_0).$$

Using (13.1) and the estimates above, the lemma follows. $\square$

Given a constant $K \gg 1$, next we define

$$G_0^K = \left\{ x \in G_0 : \sum_{R \in \mathsf{Top}} \chi_R(x) \leq K \right\}, \tag{13.2}$$

By Chebyshev and the preceding lemma, we have

$$\sigma(G_0 \backslash G_0^K) \leq \sigma(R_0 \backslash G_0^K) \leq \frac{1}{K} \int_{R_0} \sum_{R \in \mathsf{Top}} \chi_R \, d\sigma \leq \frac{M(\varepsilon)}{K} \sigma(R_0).$$

Therefore, if K is chosen big enough (depending on $M(\varepsilon)$ and the constants on the weak-A_∞ condition), by Lemma 12.2 we get

$$\sigma(G_0 \backslash G_0^K) \leq \frac{1}{2} \sigma(G_0),$$

and thus

$$\sigma(G_0^K) \geq \frac{1}{2} \sigma(G_0) \gtrsim \sigma(R_0).$$

We distinguish now two types of cubes from Top. We denote by Top_a the family of cubes $R \in \mathsf{Top}$ such that $\mathsf{T}(R) = \{R\}$, and we set $\mathsf{Top}_b = \mathsf{Top} \backslash \mathsf{Top}_a$.

Notice that, by construction, if $R \in \mathsf{Top}_b$, then $b\beta(R) \leq \varepsilon$. On the other hand, this estimate may fail if $R \in \mathsf{Top}_a$.

13.2 The truncated corona decomposition

For technical reasons, we need now to define a truncated version of the previous corona decomposition. We fix a big natural number $N \gg 1$. Then we let $\mathsf{Top}^{(N)}$ be the family of the cubes from Top with side length larger than $2^{-N}\ell(R_0)$. Given $R \in \mathsf{Top}^{(N)}$ we let $\mathsf{T}^{(N)}(R)$ be the subfamily of the cubes from $\mathsf{T}(R)$ with side length larger than $2^{-N}\ell(R_0)$, and we let $\mathsf{Stop}^{(N)}(R)$ be a maximal subfamily from $\mathsf{Stop}(R) \cup \mathcal{D}_N(R_0)$, where $\mathcal{D}_N(R_0)$ is the subfamily of the cubes from $\mathcal{D}(R_0)$ with side length $2^{-N}\ell(R_0)$. We also denote $\mathsf{Top}_a^{(N)} = \mathsf{Top}^{(N)} \cap \mathsf{Top}_a$ and $\mathsf{Top}_b^{(N)} = \mathsf{Top}^{(N)} \cap \mathsf{Top}_b$.

Observe that, since $\mathsf{Top}^{(N)} \subset \mathsf{Top}$, we also have

$$\sum_{R \in \mathsf{Top}^{(N)}} \chi_R(x) \leq \sum_{R \in \mathsf{Top}} \chi_R(x) \leq K \quad \text{for all } x \in G_0^K.$$

13.3 The key lemma

The main ingredient for the proof of the Main Lemma 10.2 is the following result.

Lemma 13.3 (Key Lemma). *Given $\eta \in (0, 1)$ and $\lambda \in (0, c_3]$ (with c_3 as in (12.2)), there exists an exceptional family $\mathsf{Ex}(R) \subset \mathsf{Stop}(R) \cap \mathsf{G}$ satisfying*

$$\sum_{P \in \mathsf{Ex}(R)} \sigma(P) \leq \eta \, \sigma(R)$$

such that, for every $Q \in \mathsf{Stop}(R) \cap \mathsf{G}\backslash\mathsf{Ex}(R)$, any λ-good corkscrew for Q can be joined to some λ'-good corkscrew for R by a $C(\lambda, \eta)$-good Harnack chain, with λ' depending on λ, η.

This lemma will be proved in the next Sects. 14 and 15. Using this result, in Sect. 16 we will build the required carrot curves for the Main Lemma 10.2, which join the pole p to points from a suitable big piece of R_0. If the reader prefers to see how this is applied before its long proof, they may go directly to Sect. 16. A crucial point in the Key Lemma is that the constant ε in the definition of the stopping cubes of the corona decomposition does not depend on the constants λ or η above.

To prove the Key Lemma 13.3 we will need first to introduce the notion of "cubes with well separated big corkscrews" and we will split $\mathsf{T}^{(N)}(R)$ into

subtrees by introducing an additional stopping condition involving this type of cubes. Later on, in Sect. 14 we will prove the "Geometric Lemma", which relies on a geometric construction which plays a fundamental role in the proof of the Key Lemma.

13.4 The cubes with well separated big corkscrews

Let $Q \in \mathcal{D}$ be a cube such that $b\beta(Q) \leq C_4\varepsilon$. For example, Q might be a cube from $Q \in \mathsf{T}^{(N)}(R) \cup \mathsf{Stop}^{(N)}(R)$, with $R \in \mathsf{Top}_b^{(N)}$ (which in particular implies that $b\beta(R) \leq \varepsilon$). We denote by L_Q a best approximating n-plane for $b\beta(Q)$, and we choose z_Q^1 and z_Q^2 to be two fixed points in B_Q such that $\mathrm{dist}(z_Q^i, L_Q) = r(B_Q)/2$ and lie in different components of $\mathbb{R}^{n+1} \setminus L_Q$. So z_Q^1 and z_Q^2 are corkscrews for Q. We will call them "big corkscrews".

Since any corkscrew x for Q satisfies $\delta_\Omega(x) \geq \kappa_1 \ell(Q)$ and we have chosen $\varepsilon \ll \kappa_1$, it turns out that

$$\mathrm{dist}(x, L_Q) \geq \frac{1}{2} \kappa_1 \ell(Q) \gg \varepsilon \ell(Q).$$

As a consequence, x can be joined either to z_Q^1 or to z_Q^2 by a C-good Harnack chain, with C depending only on n, C_0, κ_1, and thus only on n, C_0 and the weak-A_∞ constants in B_{R_0}. The following lemma follows by the same reasoning:

Lemma 13.4 *Let $Q, Q' \in \mathcal{D}$ be cubes such that $b\beta(Q), b\beta(Q') \leq C_4\varepsilon$ and Q' is the parent of Q. Let $z_Q^i, z_{Q'}^i$, for $i = 1, 2$, be big corkscrews for Q and Q' respectively. Then, after relabeling the corkscrews if necessary, z_Q^i can be joined to $z_{Q'}^i$ by a C-good Harnack chain, with C depending only on n, C_0, κ_1.*

Given $\Gamma > 0$, we will write $Q \in \mathsf{WSBC}(\Gamma)$ (or just $Q \in \mathsf{WSBC}$, which stands for "well separated big corkscrews") if $b\beta(Q) \leq C_4\varepsilon$ and the big corkscrews z_Q^1, z_Q^2 can *not* be joined by any Γ-good Harnack chain. The parameter Γ will be chosen below. For the moment, let us say that $\Gamma^{-1} \ll \varepsilon$. The reader should think that in spite of $b\beta(Q) \leq C_4\varepsilon$, the possible existence of "holes of size $C \varepsilon\ell(Q)$ in $\partial\Omega$" makes possible the connection of the big corkscrews by means of Γ-Harnack chains passing through these holes. Note that if $b\beta(Q) \leq C_4\varepsilon$ and $Q \notin \mathsf{WSBC}(\Gamma)$, then any pair of corkscrews for Q can be connected by a $C(\Gamma)$-good Harnack chain, since any of these corkscrews can be joined by a good chain to one of the big corkscrews for Q, as mentioned above.

13.5 The tree of cubes of type **WSBC** and the subtrees

Given $R \in \mathsf{Top}_b^{(N)}$, denote by $\mathsf{Stop}_{\mathsf{WSBC}}(R)$ the maximal subfamily of cubes $Q \in \mathcal{D}(R)$ which satisfy that either

- $Q \notin \mathsf{WSBC}(\Gamma)$, or
- $Q \notin \mathsf{T}^{(N)}(R)$.

Also, denote by $\mathsf{T}_{\mathsf{WSBC}}(R)$ the cubes from $\mathcal{D}(R)$ which are not contained in any cube from $\mathsf{Stop}_{\mathsf{WSBC}}(R)$. So this tree is empty if $R \notin \mathsf{WSBC}(\Gamma)$. Notice also that $\mathsf{Stop}_{\mathsf{WSBC}}(R) \not\subset \mathsf{T}_{\mathsf{WSBC}}(R)$.

Observe that if $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R)$, it may happen that $Q \notin \mathsf{WSBC}(\Gamma)$. However, unless $Q = R$, it holds that $Q \in \mathsf{WSBC}(\Gamma')$, with $\Gamma' > \Gamma$ depending only on Γ and C_0 (because the parent of Q belongs to $\mathsf{WSBC}(\Gamma)$).

For each $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \backslash \mathsf{Stop}(R)$, we denote

$$\mathsf{SubTree}(Q) = \mathcal{D}(Q) \cap \mathsf{T}^{(N)}(R), \qquad \mathsf{SubStop}(Q) = \mathsf{Stop}(R) \cap \mathcal{D}(Q).$$

So we have

$$\mathsf{T}^{(N)}(R) = \mathsf{T}_{\mathsf{WSBC}}(R) \cup \bigcup_{Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R)} \mathsf{SubTree}(Q),$$

and the union is disjoint. Observe also that we have the partition

$$\mathsf{Stop}(R) = \big(\mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{Stop}(R)\big) \cup \bigcup_{Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \backslash \mathsf{Stop}(R)} \mathsf{SubStop}(Q).$$

$$(13.3)$$

14 The geometric lemma

14.1 The geometric lemma for the tree of cubes of type **WSBC**

Let $R \in \mathsf{Top}_b^{(N)}$ and suppose that $\mathsf{T}_{\mathsf{WSBC}}(R) \neq \varnothing$. We need now to define a family $\mathsf{End}(R)$ of cubes from $\mathcal{D}$, which in a sense can be considered as a regularized version of $\mathsf{Stop}_{\mathsf{WSBC}}(R)$. The first step consists of introducing the following auxiliary function:

$$d_R(x) := \inf_{Q \in \mathsf{T}_{\mathsf{WSBC}}(R)} (\ell(Q) + \mathrm{dist}(x, Q)), \quad \text{for } x \in \mathbb{R}^{n+1}.$$

Observe that d_R is 1-Lipschitz.

For each $x \in \partial\Omega$ we take the largest cube $Q_x \in \mathcal{D}$ such that $x \in Q_x$ and

$$\ell(Q_x) \le \frac{1}{300} \inf_{y \in Q_x} d_R(y). \tag{14.1}$$

We consider the collection of the different cubes Q_x, $x \in \partial\Omega$, and we denote it by $\mathsf{End}(R)$.

Lemma 14.1 *Given* $R \in \mathsf{Top}_b^{(N)}$, *the cubes from* $\mathsf{End}(R)$ *are pairwise disjoint and satisfy the following properties:*

(a) *If* $P \in \mathsf{End}(R)$ *and* $x \in 50B_P$, *then* $100\,\ell(P) \le d_R(x) \le 900\,\ell(P)$.
(b) *There exists some absolute constant* C *such that if* $P, P' \in \mathsf{End}(R)$ *and* $50B_P \cap 50B_{P'} \ne \varnothing$, *then* $C^{-1}\ell(P) \le \ell(P') \le C\,\ell(P)$.
(c) *For each* $P \in \mathsf{End}(R)$, *there at most* N *cubes* $P' \in \mathsf{End}(R)$ *such that* $50B_P \cap 50B_{P'} \ne \varnothing$, *where* N *is some absolute constant.*
(d) *If* $P \in \mathsf{End}(R)$ *and* $\mathrm{dist}(P, R) \le 20\,\ell(R)$, *then there exists some* $Q \in \mathsf{T_{WSBC}}(R)$ *such that* $P \subset 22Q$ *and* $\ell(Q) \le 2000\,\ell(P)$.

Proof The proof is a routine task. For the reader's convenience we show the details. To show (a), consider $x \in 50B_P$. Since $d_R(\cdot)$ is 1-Lipschitz and, by definition, $d_R(x_P) \ge 300\,\ell(P)$, we have

$$d_R(x) \ge d_R(x_P) - |x - x_P| \ge d_R(x_P) - 50\,r(B_P)$$
$$\ge 300\,\ell(P) - 200\,\ell(P) = 100\,\ell(P).$$

To prove the converse inequality, by the definition of $\mathsf{End}(R)$, there exists some $z' \in \widehat{P}$, the parent of P, such that

$$d_R(z') \le 300\,\ell(\widehat{P}) = 600\,\ell(P).$$

Also, we have

$$|x - z'| \le |x - x_P| + |x_P - z'| \le 50\,r(B_P) + 2\ell(P) \le 300\,\ell(P).$$

Thus,

$$d_R(x) \le d_R(z') + |x - z'| \le (600 + 300)\,\ell(P).$$

The statement (b) is an immediate consequence of (a), and (c) follows easily from (b). To show (d), observe that, for any $S \in \mathsf{T_{WSBC}}(R)$,

$$\ell(P) \le \frac{d_R(x_P)}{300} \le \frac{\ell(S) + \mathrm{dist}(x_P, S)}{300} \le \frac{\ell(P) + \ell(S) + \mathrm{dist}(P, S)}{300}.$$

Thus,

$$\ell(P) \leq \frac{\ell(S) + \mathrm{dist}(P, S)}{299}.$$

In particular, choosing $S = R$, we deduce

$$\ell(P) \leq \frac{\ell(R) + \mathrm{dist}(P, R)}{299} \leq \frac{21}{299} \ell(R) \leq \ell(R),$$

and thus, using again that $\mathrm{dist}(P, R) \leq 20\ell(R)$, it follows that $P \subset 22R$. Let $S_0 \in \mathsf{T_{WSBC}}(R)$ be such that $d_R(x_P) = \ell(S_0) + \mathrm{dist}(x_P, S_0)$, and let $Q \in \mathcal{D}$ be the smallest cube such that $S_0 \subset Q$ and $P \subset 22Q$. Since $S_0 \subset R$ and $P \subset 22R$, we deduce that $S_0 \subset Q \subset R$, implying that $Q \in \mathsf{T_{WSBC}}(R)$.

So it just remains to check that $\ell(Q) \leq 2000\,\ell(P)$. To this end, consider a cube $\widetilde{Q} \supset S_0$ such that

$$\ell(P) + \ell(S_0) + \mathrm{dist}(P, S_0) \leq \ell(\widetilde{Q}) \leq 2\big(\ell(P) + \ell(S_0) + \mathrm{dist}(P, S_0)\big).$$

From the first inequality, it is clear that $P \subset 2\widetilde{Q}$ and then, by the definition of Q, we infer that $Q \subset \widetilde{Q}$. This inclusion and the second inequality above imply that

$$\ell(Q) \leq \ell(\widetilde{Q}) \leq 2\big(\ell(P) + \ell(S_0) + \mathrm{dist}(x_P, S_0)\big) = 4\ell(P) + 2\,d_R(x_P).$$

By (a) we know that $d_R(x_P) \leq 900\,\ell(P)$, and so we derive $\ell(Q) \leq 2000\,\ell(P)$. □

Lemma 14.2 *Given $R \in \mathsf{Top}_b^{(N)}$, if $Q \in \mathsf{End}(R)$ and $\mathrm{dist}(P, R) \leq 20\,\ell(R)$, then $b\beta(Q) \leq C\,\varepsilon$ and $Q \in \mathsf{WSBC}(\Gamma')$, with $\Gamma' = c_6\,\Gamma$, for some absolute constants $C, c_6 > 0$.*

Proof This immediate from the fact that, by (d) in the previous lemma, there exists some cube $Q' \in \mathsf{T_{WSBC}}(R)$ such that $Q \subset 22Q'$ and $\ell(Q') \leq 2000\,\ell(Q)$, so that $b\beta(Q') \leq \varepsilon$ and $Q' \in \mathsf{WSBC}(\Gamma)$. □

As in Sect. 3, we make a standard Whitney decomposition of the open set Ω. With a harmless abuse of notation we let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω, which satisfy for some $M_0 > 20$ and $D_0 \geq 1$

(i) $10I \subset \Omega$;

(ii) $M_0 I \cap \partial\Omega \neq \varnothing$;

(iii) there are at most D_0 cubes $I' \in \mathcal{W}$ such that $10I \cap 10I' \neq \varnothing$. Further, for such cubes I', we have $\ell(I') \approx \ell(I)$, where $\ell(I')$ stands for the side length of I'.

From the properties (i) and (ii) it is clear that $\mathrm{dist}(I, \partial\Omega) \approx \ell(I)$. We assume that the Whitney cubes are small enough so that

$$\mathrm{diam}(I) < \frac{1}{100}\,\mathrm{dist}(I, \partial\Omega). \tag{14.2}$$

To construct this Whitney decomposition one just needs to replace each cube $I \in \mathcal{W}$, as in [49, Chapter VI], by its descendants $I' \in \mathcal{D}_k(I)$, for some fixed $k \geq 1$.

For each $I \in \mathcal{W}$, as much as in Lemma 9.4, we denote by B^I a ball concentric with I and radius $C_5\ell(I)$, where C_5 is a universal constant big enough so that

$$g(p, x) \lesssim \frac{\omega^p(B^I)}{\ell(I)^{n-1}} \quad \text{for all } x \in 4I,$$

and whenever $p \notin 5I$. Obviously, the ball B^I intersects $\partial\Omega$, and the family $\{B^I\}_{I\in\mathcal{W}}$ does not have finite overlapping.

Given a bounded measurable set $F \subset \mathbb{R}^{n+1}$ with $|F| > 0$, and a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, we denote by $m_F f$ the mean of f in F with respect to Lebesgue measure. That is,

$$m_F f = \fint_F f\,dx.$$

To state the Geometric Lemma we need some additional notation. Given a cube $R' \in \mathsf{T}_{\mathsf{WSBC}}(R)$, we denote by $\widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$ the family of cubes from $\mathcal{D}$ with side length at most $\ell(R')$ which are contained in $100B_{R'}$ and are not contained in any cube from $\mathsf{End}(R)$. We also denote by $\widetilde{\mathsf{End}}(R')$ the subfamily of the cubes from $\mathsf{End}(R)$ which are contained in some cube from $\widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$. Note that $\widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$ is not a tree, in general, but a union of trees. Further, from Lemma 14.1(a), it follows easily that

$$\mathsf{T}_{\mathsf{WSBC}}(R) \cup \mathsf{Stop}_{\mathsf{WSBC}}(R) \subset \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R) \cap \mathcal{D}(R).$$

Lemma 14.3 (Geometric Lemma). *Let $0 < \gamma < 1$, and assume that the constant $\Gamma = \Gamma(\gamma)$ in the definition of WSBC is big enough. Let $R \in \mathsf{Top}_b^{(N)} \cap \mathsf{WSBC}(\Gamma)$ and let $R' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ be such that $\ell(R') = 2^{-k_0}\ell(R)$, with $k_0 = k_0(\gamma) \geq 1$ big enough. Then there are two open sets $V_1, V_2 \subset CB_{R'} \cap \Omega$ with disjoint closures which satisfy the following properties:*

(a) *There are subfamilies $\mathcal{W}_i \subset \mathcal{W}$ such that $V_i = \bigcup_{I\in\mathcal{W}_i} 1.1\,\mathrm{int}(I)$.*
(b) *Each V_i contains a ball B_i with $r(B_i) \approx \ell(R')$, and each corkscrew point for R' contained in $2B_{R'} \cap V_i$ can be joined to the center z_i of B_i by a good*

Harnack chain contained in V_i. Further, any point $x \in V_i$ can be joined to z_i by a good Harnack chain (not necessarily contained in V_i).

(c) *For each $Q \in \left(\mathsf{T}_{\mathsf{WSBC}}(R) \cup \mathsf{Stop}_{\mathsf{WSBC}}(R)\right) \cap \mathcal{D}(R')$ there are big corkscrews $z_Q^1 \in V_1 \cap 2B_Q$ and $z_Q^2 \in V_2 \cap 2B_Q$, and if $\widehat{Q}$ is an ancestor of Q which also belongs to $\mathsf{T}_{\mathsf{WSBC}}(R) \cap \mathcal{D}(R')$, then z_Q^i can be joined to $z_{\widehat{Q}}^i$ by a good Harnack chain, for each $i = 1, 2$.*

(d) *$(\partial V_1 \cup \partial V_2) \cap 10B_{R'} \subset \bigcup_{P \in \widetilde{\mathsf{End}}(R')} 2B_P$.*

(e) *For each $P \in \widetilde{\mathsf{End}}(R')$ such that $2B_P \cap 10B_{R'} \neq \varnothing$, let $\mathcal{W}_P$ be the family of Whitney cubes $I \subset V_1 \cup V_2$ such that $1.1\overline{I} \cap \partial(V_1 \cup V_2) \cap 2B_P \neq \varnothing$, so that*

$$\partial V_i \cap 2B_P \subset \bigcup_{I \in \mathcal{W}_P} 1.1\overline{I}.$$

Then

(i)

$$m_{4I} g(p, \cdot) \leq \gamma \, \frac{\ell(P)}{\sigma(R_0)} \quad \text{for each } I \in \mathcal{W}_P,$$

and

(ii)

$$\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n \quad \text{and} \quad \sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P),$$

for some universal constant $C > 1$.

The constants involved in the Harnack chain and corkscrew conditions may depend on ε, Γ, and γ.[5]

14.2 Proof of the geometric Lemma 14.3

In this whole subsection we fix $R \in \mathsf{Top}_b^{(N)}$ and we assume $\mathsf{T}_{\mathsf{WSBC}}(R) \neq \varnothing$, as in Lemma 14.3. We let $R' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ be such that $\ell(R') = 2^{-k_0}\ell(R)$, with $k_0 = k_0(\gamma) \geq 1$ big enough, as in Lemma 14.3, and we consider the associated families $\widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$ and $\widetilde{\mathsf{End}}(R')$.

Remark 14.4 By arguments analogous to the ones in Lemma 14.2, it follows easily that if $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$, for $R' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ such that $\ell(R') =$

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[5] To guarantee the existence of the sets V_i and the fact that they are contained in Ω we use the assumption that $\Omega = \mathbb{R}^{n+1} \setminus \partial\Omega$.

$2^{-k_0}\ell(R)$, then there exists some cube $S \in \mathsf{T}_{\mathsf{WSBC}}(R)$ such that $Q \subset 22S$ and $\ell(S) \leq 2000\ell(Q)$. This implies that $b\beta(Q) \leq C\varepsilon$ and $Q \in \mathsf{WSBC}(c_6\Gamma)$ too.

In order to define the open sets V_1, V_2 described in the lemma, first we need to associate some open sets $U_1(Q), U_2(Q)$ to each $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')$. We distinguish two cases:

- For $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$, we let $\mathcal{J}_i(Q)$ be the family of Whitney cubes $I \in \mathcal{W}$ which intersect

$$\{y \in 20B_Q : \mathrm{dist}(y, L_Q) > \varepsilon^{1/4}\ell(Q)\}$$

  and are contained in the same connected component of $\mathbb{R}^{n+1} \setminus L_Q$ as z_Q^i, and then we set

$$U_i(Q) = \bigcup_{I \in \mathcal{J}_i(Q)} 1.1\,\mathrm{int}(I).$$

- For $Q \in \widetilde{\mathsf{End}}(R')$ the definition of $U_i(Q)$ is more elaborated. First we consider an auxiliary ball $\widetilde{B}_Q$, concentric with B_Q, such that $19B_Q \subset \widetilde{B}_Q \subset 20B_Q$ and having thin boundaries for ω^p. This means that, for some absolute constant C,

$$\omega^p\left(\{x \in 2\widetilde{B}_Q : \mathrm{dist}(x, \partial\widetilde{B}_Q) \leq t\,r(\widetilde{B}_Q)\}\right) \leq C\,t\,\omega^p(2\widetilde{B}_Q) \quad \text{for all } t > 0.$$
  (14.3)

  The existence of such ball $\widetilde{B}_Q$ follows by well known arguments (see for example [50, p.370]).

  Next we denote by $\mathcal{J}(Q)$ the family of Whitney cubes $I \in \mathcal{W}$ which intersect $\widetilde{B}_Q$ and satisfy $\ell(I) \geq \theta\,\ell(Q)$ for $\theta \in (0,1)$ depending on γ (the reader should think that $\theta \ll \varepsilon$ and that $\theta = 2^{-j_1}$ for some $j_1 \gg 1$), and we set

$$U(Q) = \bigcup_{I \in \mathcal{J}(Q)} 1.1\,\mathrm{int}(I). \tag{14.4}$$

  For a fixed $i = 1$ or 2, let $\{D_j^i(Q)\}_{j \geq 0}$ be the connected components of $U(Q)$ which satisfy one of the following properties:
  - either $z_Q^i \in D_j^i(Q)$ (recall that z_Q^i is a big corkscrew for Q), or
  - there exists some $y \in D_j^i(Q)$ such that $g(p, y) > \gamma\,\ell(Q)\,\sigma(R_0)^{-1}$ and there is a $C_6(\gamma, \theta)$-good Harnack chain that joins y to z_Q^i, for some constant $C_6(\gamma, \theta)$ to be chosen below.

  Then we let $U_i(Q) = \bigcup_j D_j^i(Q)$. After reordering the sequence, we assume that $z_Q^i \in D_0^i(Q)$. We let $\mathcal{J}_i(Q)$ be the subfamily of cubes from $\mathcal{J}(Q)$ contained in $U_i(Q)$.

In the case $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$, from the definitions, it is clear that the sets $U_i(Q)$ are open and connected and

$$\overline{U_1(Q)} \cap \overline{U_2(Q)} = \varnothing. \tag{14.5}$$

In the case $Q \in \widetilde{\mathsf{End}}(R')$, the sets $U_i(Q)$ may fail to be connected. However, (14.5) still holds if Γ is chosen big enough (which will be the case). Indeed, if some component D_j^i can be joined by $C_6(\gamma, \theta)$-good Harnack chains both to z_Q^1 and z_Q^2, then there is a $C(\gamma, \theta)$-good Harnack chain that joins z_Q^1 to z_Q^2, and thus Q does not belong to $\mathsf{WSBC}(c_6\Gamma)$ if Γ is taken big enough, which cannot happen by Lemma 14.2. Note also that the two components of

$$\{y \in \widetilde{B}_Q : \operatorname{dist}(y, L_Q) > \varepsilon^{1/2} \ell(Q)\}$$

are contained in $D_0^1(Q) \cup D_0^2(Q)$, because $b\beta(Q) \leq C\varepsilon$ and we assume $\theta \ll \varepsilon$.

The following is immediate:

Lemma 14.5 *Assume that we relabel appropriately the sets $U_i(P)$ and corkscrews z_P^i for $P \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')$. Then for all $Q, \widehat{Q} \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')$ such that $\widehat{Q}$ is the parent of Q we have*

$$\left[z_Q^1, z_{\widehat{Q}}^1\right] \subset U_1(Q) \cap U_1(\widehat{Q}) \quad \text{and} \quad \left[z_Q^2, z_{\widehat{Q}}^2\right] \subset U_2(Q) \cap U_2(\widehat{Q}). \tag{14.6}$$

Further,

$$\operatorname{dist}\left([z_Q^i, z_{\widehat{Q}}^i], \partial\Omega\right) \geq c\,\ell(Q) \quad \text{for } i = 1, 2,$$

where c depends at most on n and C_0.

The labeling above can be chosen inductively. First we fix the sets $U_i(T)$ and corkscrews x_T^i for every maximal cube T from $\widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$ (contained in $100B_{R'}$ and with side length equal to $\ell(R')$). Further we assume that, for any maximal cube T, the corkscrew x_T^i is at the same side of $L_{R'}$ as $z_{R'}^i$, for each $i = 1, 2$ (this property will be used below). Later we label the sons of each T so that (14.6) holds for any son Q of T. Then we proceed with the grandsons of T, and so on. We leave the details for the reader.

The following result will be used later to prove the property (e)(i).

Lemma 14.6 *Suppose that the constant $k_0(\gamma)$ in Lemma 14.3 is big enough. Let $Q \in \widetilde{\mathsf{End}}(R')$ and assume θ small enough and $C_6(\gamma, \theta)$ big enough in the definition of $U_i(Q)$. If $y \in \widetilde{B}_Q$ satisfies $g(p, y) > \gamma\,\ell(Q)\,\sigma(R_0)^{-1}$, then $y \in U_1(Q) \cup U_2(Q)$.*

Recall $\widetilde{B}_Q$ is the ball with thin boundary appearing in (14.3).

Proof By the definition of $U_i(Q)$, it suffices to show that y belongs to some component $D^i_j(Q)$ and that there is a $C_6(\gamma, \theta)$-good Harnack chain that joins y to z^i_Q. To this end, observe that by the boundary Hölder continuity of $g(p, \cdot)$,

$$\gamma \frac{\ell(Q)}{\sigma(R_0)} \leq g(p, y) \leq C \left(\frac{\delta_\Omega(y)}{\ell(Q)} \right)^\alpha m_{30B_Q} g(p, \cdot) \leq C \left(\frac{\delta_\Omega(y)}{\ell(Q)} \right)^\alpha \frac{\ell(Q)}{\sigma(R_0)},$$

where in the last inequality we used Lemma 9.5. Thus,

$$\delta_\Omega(y) \geq c \, \gamma^{1/\alpha} \, \ell(Q),$$

and if θ is small enough, then y belongs to some connected component of the set $U(Q)$ in (14.4). By Lemma 14.1(d) there is a cube $Q' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ such that $Q \subset 22Q'$ and $\ell(Q') \approx \ell(Q)$. In particular, $\mathsf{WA}(p, \Lambda) \cap Q' \supset G_0 \cap Q' \neq \varnothing$ and thus, by applying Lemma 11.6 with $q = y$ and $r_0 = Cr(B_Q)$ (for a suitable $C > 1$), it follows that there exists a κ_1-corkscrew $y' \in C(\gamma) B_Q$, with $C(\gamma) > 20$ say, such that y can be joined to y' by a $C'(\gamma)$-good Harnack chain. Assuming that the constant $k_0(\gamma)$ in Lemma 14.3 is big enough, it turns out that $y' \in 2B_{Q''}$ for some $Q'' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ such that $22Q'' \supset Q$. Since all the cubes S such that $Q \subset S \subset 22Q''$ satisfy $b\beta(S) \leq C\varepsilon$, by applying Lemma 13.4 repeatedly, it follows that y' can be joined either to z^1_Q or z^2_Q by a $C''(\gamma)$-good Harnack chain. Then, joining both Harnack chains, it follows that y can be joined either to z^1_Q or z^2_Q by a $C'''(\gamma)$-good Harnack chain. So y belongs to one of the components D^i_j, assuming $C_6(\gamma, \theta)$ big enough. $\qquad\square$

From now on we assume θ small enough and $C_6(\gamma, \theta)$ big enough so that the preceding lemma holds. Also, we assume $\theta \ll \varepsilon^4$. We define

$$V_1 = \bigcup_{Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')} U_1(Q), \qquad V_2 = \bigcup_{Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')} U_2(Q).$$

Next we will show that

$$\overline{V_1} \cap \overline{V_2} = \varnothing.$$

Since the number of cubes $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')$ is finite (because of the truncation in the corona decomposition), this is a consequence of the following:

Lemma 14.7 *Suppose Γ is big enough in the definition of* WSBC *(depending on θ). For all $P, Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R') \cup \widetilde{\mathsf{End}}(R')$, we have*

$$\overline{U_1(P)} \cap \overline{U_2(Q)} = \varnothing.$$

Proof We suppose that $\ell(Q) \geq \ell(P)$ We also assume that $\overline{U_1(P)} \cap \overline{U_2(Q)} \neq \varnothing$ and then we will get a contradiction. Notice first that if $\ell(P) = \ell(Q) = 2^{-j}\ell(R')$ for some $j \geq 0$, then the corkscrews z_P^i and z_Q^i are at the same side of L_Q for each $i = 1, 2$. This follows easily by induction on j.

Case 1. Suppose first that $P, Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$. Since the cubes from $\mathcal{J}_2(Q)$ have side length at least $c\,\varepsilon^{1/4}\,\ell(Q)$, it follows that at least one of the cubes from $\mathcal{J}_1(P)$ has side length at least $c'\,\varepsilon^{1/4}\,\ell(Q)$, which implies that $\ell(P) \geq c''\,\varepsilon^{1/4}\,\ell(Q)$, by the construction of $U_1(P)$.

Since $U_1(P) \cap U_2(Q) \neq \varnothing$, there exists some curve $\gamma = \gamma(z_P^1, z_Q^2)$ that joins z_P^1 and z_Q^2 such that $\mathrm{dist}(\gamma, \partial\Omega) \geq c\,\varepsilon^{1/2}\,\ell(Q)$ because all the cubes from $\mathcal{J}_2(Q)$ have side length at least $c\,\varepsilon^{1/4}\,\ell(Q)$, and the ones from $\mathcal{J}_1(P)$ have side length $\geq c\,\varepsilon^{1/4}\,\ell(P) \geq c\,\varepsilon^{1/2}\,\ell(Q)$.

Let $\widehat{P}$ be the ancestor of P such that $\ell(\widehat{P}) = \ell(Q)$. From the fact that $\overline{U_1(P)} \cap \overline{U_2(Q)} \neq \varnothing$, we deduce that $20B_P \cap 20B_Q \neq \varnothing$ and thus $20B_{\widehat{P}} \cap 20B_Q \neq \varnothing$, and so $20B_{\widehat{P}} \subset 60B_Q$. This implies that $z_{\widehat{P}}^1$ is in the same connected component as z_Q^1 and also that $\mathrm{dist}([z_Q^1, z_{\widehat{P}}^1], \partial\Omega) \gtrsim \ell(Q)$, because $b\beta(100B_Q) \leq \varepsilon \ll 1$ and they are at the same side of L_Q.

Consider now the chain $P = P_1 \subset P_2 \subset \cdots \subset P_m = \widehat{P}$, so that P_{i+1} is the parent of P_i. Form the curve $\gamma' = \gamma'(z_{\widehat{P}}^1, z_P^1)$ with endpoints $z_{\widehat{P}}^1$ and z_P^1 by joining the segments $[z_{P_i}^1, z_{P_{i+1}}^1]$. Since these segments satisfy

$$\mathrm{dist}\big([z_{P_i}^1, z_{P_{i+1}}^1], \partial\Omega\big) \geq c\,\ell(P_i) \geq c\,\ell(P) \geq c\,\varepsilon^{1/4}\,\ell(Q),$$

it is clear that $\mathrm{dist}(\gamma', \partial\Omega) \geq c\,\varepsilon^{1/4}\,\ell(Q)$.

Next we form a curve $\gamma'' = \gamma''(z_Q^1, z_Q^2)$ which joins z_Q^1 to z_Q^2 by joining $[z_Q^1, z_{\widehat{P}}^1]$, $\gamma'(z_{\widehat{P}}^1, z_P^1)$, and $\gamma(z_P^1, z_Q^2)$. It follows easily that this is contained in $90B_Q$ and that $\mathrm{dist}(\gamma'', \partial\Omega) \geq c\,\varepsilon^{1/2}\,\ell(Q)$. However, this is not possible because z_Q^1 and z_Q^2 are in different connected components of $\mathbb{R}^{n+1} \setminus L_Q$ and $b\beta(Q) \leq \varepsilon \ll \varepsilon^{1/2}$ (since we assume $\varepsilon \ll 1$).

Case 2. Suppose now that $Q \in \widetilde{\mathsf{End}}(R')$. The arguments are quite similar to the ones above. In this case, the cubes from $\mathcal{J}_2(Q)$ have side length at least $\theta\,\ell(Q)$ and thus at least one of the cubes from $\mathcal{J}_1(P)$ has side length at least $c\,\theta\,\ell(Q)$, which implies that $\ell(P) \geq c'\,\theta\,\ell(Q)$.

Now there exists a curve $\gamma = \gamma(z_P^1, z_Q^2)$ that joints z_P^1 and z_Q^2 such that $\mathrm{dist}(\gamma, \partial\Omega) \geq c\,\theta^2\,\ell(Q)$ because all the cubes from $\mathcal{J}_2(Q)$ have side length at least $\theta\,\ell(Q)$, and the ones from $\mathcal{J}_1(P)$ have side length $\theta\,\ell(P) \geq c\,\theta^2\,\ell(Q)$.

We consider again cubes $\widehat{P}$ and $P_1, \ldots, P_m$ defined exactly as above. By the same reasoning as above, $\mathrm{dist}([z_Q^1, z_{\widehat{P}}^1], \partial\Omega) \gtrsim \ell(Q)$. We also define the curve $\gamma' = \gamma'(z_{\widehat{P}}^1, z_P^1)$ which joins $z_{\widehat{P}}^1$ to z_P^1 in the same way. In the present case we have

$$\mathrm{dist}(\gamma', \partial\Omega) \gtrsim \ell(P) \geq c\,\theta\,\ell(Q).$$

Again construct a curve $\gamma'' = \gamma''(z_Q^1, z_Q^2)$ which joins z_Q^1 to z_Q^2 by gathering $[z_Q^1, z_{\widehat{P}}^1]$, $\gamma'(z_{\widehat{P}}^1, z_P^1)$, and $\gamma(z_P^1, z_Q^2)$. This is contained in CB_Q (for some $C > 1$ possibly depending on γ) and satisfies $\mathrm{dist}(\gamma'', \partial\Omega) \geq c\,\theta^2\,\ell(Q)$. From this fact we deduce that z_Q^1 and z_Q^2 can be joined by $C(\theta)$-good Harnack chain. Taking Γ big enough (depending on $C(\theta)$), this implies that the big corkscrews for Q can be joined by a $(c_6\Gamma)$-good Harnack chain, which contradicts Lemma 14.2.

Case 3. Finally suppose that $P \in \widetilde{\mathsf{End}}(R')$. We consider the same auxiliary cube $\widehat{P}$ and the same curve $\gamma = \gamma(z_P^1, z_Q^2)$ satisfying $\mathrm{dist}(\gamma, \partial\Omega) \geq c\,\theta\,\ell(P)$. By joining the segments $[z_{P_i}^2, z_{P_{i+1}}^2]$, we construct a curve $\gamma_2' = \gamma_2'(z_{\widehat{P}}^2, z_P^2)$ analogous to $\gamma' = \gamma'(z_{\widehat{P}}^1, z_P^1)$ from the case 2, so that this joins $z_{\widehat{P}}^2$ to z_P^2 and satisfies $\mathrm{dist}(\gamma_2', \partial\Omega) \gtrsim \ell(P)$.

We construct a curve γ''' that joins z_P^1 to z_P^2 by joining $\gamma(z_P^1, z_Q^2), [z_Q^2, z_{\widehat{P}}^2]$, and $\gamma_2'(z_{\widehat{P}}^2, z_P^2)$. Again this is contained in CB_Q and it holds $\mathrm{dist}(\gamma''', \partial\Omega) \geq c\,\theta\,\ell(P)$. This implies that z_P^1 and z_P^2 can be joined by $C(\theta)$-good Harnack chain. Taking Γ big enough, we deduce the big corkscrews for P can be joined by a $(c_6\Gamma)$-good Harnack chain, which is a contradiction. $\square$

By the definition of V_1 and V_2 it is clear that the properties (a), (b) and (c) in Lemma 14.3 hold. So to complete the proof of the lemma it just remains to prove (d) and (e).

Proof of Lemma 14.3(d) Let $x \in (\partial V_1 \cup \partial V_2) \cap 10B_{R'}$. We have to show that there exists some $S \in \widetilde{\mathsf{End}}(R')$ such that $x \in 2B_S$. To this end we consider $y \in \partial\Omega$ such that $|x - y| = \delta_\Omega(x)$. Since $x_{R'} \in \partial\Omega$, it follows that $y \in 20B_{R'}$. Let $S \in \widetilde{\mathsf{End}}(R')$ be such that $y \in S$. Observe that

$$\ell(S) \leq \frac{1}{300}\,d_R(y) \leq \frac{1}{300}\left(\ell(R') + 20\,r(B_{R'})\right) = \frac{81}{300}\,\ell(R') \leq \frac{1}{3}\,\ell(R').$$

$$(14.7)$$

We claim that $x \in 2B_S$. Indeed, if $x \notin 2B_S$, taking also into account (14.7), there exists some ancestor Q of S contained in $100B_{R'}$ such that $x \in 2B_Q$ and

$\delta_\Omega(x) = |x - y| \approx \ell(Q)$. From the fact that $S \subsetneq Q \subset 100B_{R'}$ we deduce that $Q \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$. By the construction of the sets $U_i(Q)$, it is immediate to check that the condition that $\delta_\Omega(x) \approx \ell(Q)$ implies that $x \in U_1(Q) \cup U_2(Q)$. Thus $x \in V_1 \cup V_2$ and so $x \notin \partial(V_1 \cup V_2) = \partial V_1 \cup \partial V_2$ (for this identity we use that $\mathrm{dist}(V_1, V_2) > 0$), which is a contradiction. $\square$

To show (e), first we need to prove the next result:

Lemma 14.8 *For each $i = 1, 2$, we have*

$$\partial V_i \cap 10B_{R'} \subset \bigcup_{Q \in \widetilde{\mathsf{End}}(R')} \partial U_i(Q).$$

Proof Clearly, we have

$$\partial V_i \cap 10B_{R'} \subset \bigcup_{\substack{P \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R'): \\ P \cap 10B_{R'} \neq \varnothing}} \partial U_i(P) \cup \bigcup_{\substack{Q \in \widetilde{\mathsf{End}}(R'): \\ Q \cap 10B_{R'} \neq \varnothing}} \partial U_i(Q).$$

So it suffices to show that

$$\bigcup_{\substack{P \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R'): \\ P \cap 10B_{R'} \neq \varnothing}} \partial U_i(P) \cap \partial V_i \cap 10B_{R'} = \varnothing. \qquad (14.8)$$

Let $x \in \partial U_i(P) \cap \partial V_i \cap 10B_{R'}$, with $P \in \widetilde{\mathsf{T}}_{\mathsf{WSBC}}(R')$, $P \cap 10B_{R'} \neq \varnothing$. From the definition of $U_i(P)$, it follows easily that

$$\delta_\Omega(x) \gtrsim \varepsilon^{1/4} \ell(P). \qquad (14.9)$$

On the other hand, by Lemma 14.3(d), there exists some $Q \in \widetilde{\mathsf{End}}(R')$ such that $x \in 2B_Q$. By the definition of $U_i(Q)$, since $\theta \ll \varepsilon$, it also follows easily that

$$\left\{ y \in 2B_Q : \delta_\Omega(y) > \varepsilon^{1/2} \ell(Q) \right\} \subset V_1 \cup V_2.$$

Hence, $\mathrm{dist}(\partial V_i \cap 2B_Q, \partial\Omega) \leq \varepsilon^{1/2} \ell(Q)$, and so

$$\delta_\Omega(x) \leq \varepsilon^{1/2} \ell(Q). \qquad (14.10)$$

We claim that $\ell(Q) \lesssim \ell(P)$. Indeed, from the fact that $x \in \partial U_i(P) \subset 30B_P$, we infer that

$$30B_P \cap 2B_Q \neq \varnothing.$$

Suppose that $\ell(Q) \geq \ell(P)$. This implies that $B_P \subset 33B_Q$. Consider now a cube $S \subset P$ belonging to $\widetilde{\mathsf{End}}(R')$. Since $B_S \cap 33B_Q \neq \varnothing$, by Lemma 14.1 (b) we have

$$\ell(Q) \approx \ell(S) \leq \ell(P),$$

which proves our claim. Together with (14.9) and (14.10), this yields

$$\varepsilon^{1/4}\ell(P) \lesssim \delta_\Omega(x) \lesssim \varepsilon^{1/2} \ell(Q) \lesssim \varepsilon^{1/2} \ell(P),$$

which is a contradiction for ε small enough. So there does not exist any $x \in \partial U_i(P) \cap \partial V_i \cap 10B_{R'}$, which proves (14.8). □

Proof of Lemma 14.3(e) Let $P \in \widetilde{\mathsf{End}}(R')$ be such that $2B_P \cap 10B_{R'} \neq \varnothing$. The statement (i) is an immediate consequence of Lemma 14.6. In fact, this lemma implies that any $y \in 2B_P$ such that $g(p, y) > \gamma \ell(P)\sigma(R_0)^{-1}$ is contained in $U_1(P) \cup U_2(P)$ and thus in $V_1 \cup V_2$. In particular, $y \notin \partial(V_1 \cup V_2) = \partial V_1 \cup \partial V_2$. Thus, if $y \in 2B_P \cap \partial V_i$, then

$$g(p, y) \leq \gamma \frac{\ell(P)}{\sigma(R_0)}.$$

It is easy to check that this implies the statement (i) in Lemma 14.3(e) (possibly after replacing γ by $C\gamma$).

Next we turn our attention to (ii). To this end, denote by J_P the subfamily of the cubes $Q \in \widetilde{\mathsf{End}}(R')$ such that $30B_Q \cap 2B_P \neq \varnothing$. By Lemma 14.8,

$$\partial V_i \cap 2B_P \subset \bigcup_{Q \in J_P} \partial U_i(Q) \cap 2B_P. \tag{14.11}$$

We will show that

$$\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n \quad \text{and} \quad \sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P), \tag{14.12}$$

where $\mathcal{W}_P$ is the family of Whitney cubes $I \subset V_1 \cup V_2$ such that $1.1\overline{I} \cap \partial(V_1 \cup V_2) \cap 2B_P \neq \varnothing$. To this end, observe that, by (14.11) and the construction of $U_i(Q)$, for each $I \in \mathcal{W}_P$ there exists some $Q \in J_P$ such that $I \subset 30B_Q$ and either $\ell(I) \approx \theta\ell(Q)$ or $1.1\overline{I} \cap \partial\widetilde{B}_Q \neq \varnothing$. Using the n-ADRity of σ, it is immediate to check that for each $Q \in J_P$,

$$\sum_{\substack{I \subset 30B_Q: \\ \ell(I) = \theta\ell(Q)}} \ell(I)^n \lesssim \ell(Q)^n.$$

Also,

$$\sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \ell(I)^n \lesssim \sum_{\substack{I \in \mathcal{W} \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} H^n(2I \cap \partial \widetilde{B}_Q) \lesssim H^n(\partial \widetilde{B}_Q) \lesssim \ell(Q)^n.$$

Since the number of cubes $Q \in J_P$ is uniformly bounded (by Lemma 14.1(b)) and $\ell(Q) \approx \ell(P)$, the above inequalities yield the first estimate in (14.12).

To prove the second one we also distinguish among the two types of cubes $I \in J_P$ above. First, by the bounded overlap of the balls B^I such that $\ell(I) = \theta \, \ell(Q)$, we get

$$\sum_{\substack{I \subset 30B_Q \\ \ell(I) \approx \theta \ell(Q)}} \omega^p(B^I) \lesssim \omega^p(CB_P), \tag{14.13}$$

since the balls B^I in the sum are contained CB_P for a suitable universal constant $C > 1$. To deal with the cubes $I \in \mathcal{W}$ such that $1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing$ we intend to use the thin boundary property of $\widetilde{B}_Q$ in (14.3). To this end, we write

$$\sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \omega^p(B^I) = \sum_{k \geq 0} \sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing \\ \ell(I) = 2^{-k}\ell(Q)}} \omega^p(B^I) \lesssim \sum_{k \geq 0} \omega^p(\mathcal{U}_{2^{-k+1} \operatorname{diam}(Q)}(\partial \widetilde{B}_Q)),$$

where $\mathcal{U}_d(A)$ stands for the d-neighborhood of A. By (14.3) it follows that

$$\omega^p(\mathcal{U}_{2^{-k}\ell(Q)}(\partial \widetilde{B}_Q)) \lesssim 2^{-k}\omega^p(C'B_Q),$$

and thus

$$\sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \omega^p(B^I) \lesssim \omega^p(C'B_Q) \lesssim \omega^p(CB_P),$$

for a suitable $C > 1$. Together with (14.13), this yields the second inequality in (14.12), which completes the proof of Lemma 14.3(e). □

15 Proof of the key lemma

We fix $R_0 \in \mathcal{D}$ and a corkscrew point $p \in \Omega$ as in the preceding sections. We consider $R \in \mathsf{Top}_b^{(N)}$ and we assume $\mathsf{T}_{\mathsf{WSBC}}(R) \neq \varnothing$, as in Lemma 14.3. We let $R' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ be such that $\ell(R') = 2^{-k_0}\ell(R)$, with $k_0 = k_0(\gamma) \geq 1$ big

enough. Given $\lambda > 0$ and $i = 1, 2$, we set

$$\mathsf{H}_i(R') = \left\{Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathcal{D}(R') \cap \mathsf{G} : g(p, z_Q^i) > \lambda \, \ell(Q) \, \sigma(R_0)^{-1}\right\}, \tag{15.1}$$

so that by Lemma 12.3, $\mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathcal{D}(R') \cap \mathsf{G} = \mathsf{H}_1(R') \cup \mathsf{H}_2(R')$. Here we are assuming that the corkscrews z_Q^i belong to the set V_i from Lemma 14.3, that λ is small enough, and we are taking into account that, by the arguments in Sect. 13.4, any corkscrew for Q can be joined to one of the big corkscrews z_Q^1 by some C-good Harnack chain.

Lemma 15.1 (Baby Key Lemma). *Let p, R_0, R, R' be as above. Given $\lambda > 0$, define also $\mathsf{H}_i(R')$ as above. For a given $\tau > 0$, suppose that*

$$\sigma\left(\bigcup_{Q \in \mathsf{H}_i(R')} Q\right) \geq \tau \, \sigma(R').$$

If γ is small enough in the definition of V_i in Lemma 14.3 (depending on τ and λ), then

$$g(p, z_{R'}^i) \geq c(\lambda, \tau) \, \frac{\ell(R')}{\sigma(R_0)}.$$

Remark that Γ depends on γ (see Lemma 14.3), and thus the families $\mathsf{WSBC}(\Gamma)$, $\mathsf{Stop}_{\mathsf{WSBC}}(R)$, $\mathsf{H}_i(R')$ also depend on γ. The reader should thing that $\Gamma \to \infty$ as $\gamma \to 0$.

A key fact in this lemma is that the constants λ, τ can be taken arbitrarily small, without requiring $\varepsilon \to 0$ as $\lambda \tau \to 0$. Instead, the lemma requires $\gamma \to 0$, which does not affect the packing condition in Lemma 13.2.

We denote

$$\mathsf{Bdy}(R') = \bigcup_{P \in \widetilde{\mathsf{End}}(R') : 2B_P \cap 10B_{R'} \neq \varnothing} \mathcal{W}_P,$$

with $\mathcal{W}_P$ as in the Lemma 14.3. That is, $\mathcal{W}_P$ is the family of Whitney cubes $I \subset V_1 \cup V_2$ such that $1.1\overline{I} \cap \partial(V_1 \cup V_2) \cap 2B_P \neq \varnothing$. So the family $\mathsf{Bdy}(R')$ contains Whitney cubes which intersect the boundaries of V_1 or V_2 and are close to $10B_{R'}$.

Let us introduce some extra piece of notation. Given $q \in \mathbb{R}^{n+1}$ and $0 < r < s$ we let

$$A(q, r, s) = B(q, s) \setminus \overline{B(q, r)}.$$

To prove Lemma 15.1, first we need the following auxiliary result.

Lemma 15.2 *Let p, R_0, R, R' be as above and, for $i = 1$ or 2, let $Q \in \mathsf{H}_i(R')$. Let V_i be as in Lemma 14.3 and let $q \in \Omega$ be a corkscrew point for Q which belongs to V_i. Denote $r = 2\ell(R')$ and for $\delta \in (0, 1/100)$ set*

$$A_r^\delta = \left\{ x \in A(q, r, 2r) \cap \Omega : \delta_\Omega(x) > \delta\,r \right\}.$$

Then we have

$$
g(p, q) \lesssim \frac{1}{r} \sup_{y \in A_r^\delta \cap V_i} \frac{g(p, y)}{\delta_\Omega(y)} \int_{A_r^\delta} g(q, x)\,dx
$$
$$
+ \frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(q,r,2r)} g(p, x)\,dx \int_{A(q,r,2r)} g(q, x)\,dx
$$
$$
+ \sum_{I \in \mathsf{Bdy}(R')} \frac{1}{\ell(I)} \int_{2I} \big| g(p, x)\,\nabla g(q, x) - \nabla g(p, x)\,g(q, x) \big|\,dx.
$$

Let us note that the fact that q is a corkscrew for Q contained in V_i implies that $\mathrm{dist}(q, \partial V_i) \approx \ell(Q)$, by the construction of the sets V_i in Lemma 14.3.

Proof We fix $i = 1$, for definiteness. Recall that $V_1 = \bigcup_{I \in \mathcal{W}_1} 1.1\,\mathrm{int}(I)$. For each $I \in \mathcal{W}_1$, consider a smooth function η_I such that $\chi_{0.9I} \leq \eta_I \leq \chi_{1.09I}$ with $\|\nabla \eta_I\|_\infty \lesssim \ell(I)^{-1}$ and

$$
\eta := \sum_{I \in \mathcal{W}_1} \eta_I \equiv 1 \quad \text{on } V_1 \cap 10 B_{R'} \setminus \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

It follows that $\mathrm{supp}\,\eta \subset V_1$ and so $\mathrm{supp}\,\eta \cap V_2 = \varnothing$, and also

$$
\mathrm{supp}(\nabla \eta) \cap 10 B_{R'} \subset \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

Let φ_0 be a smooth function such that $\chi_{B(q,1.2r)} \leq \varphi_0 \leq \chi_{B(q,1.8r)}$, with $\|\nabla \varphi_0\|_\infty \lesssim 1/r$. Then we set

$$
\varphi = \eta\,\varphi_0.
$$

So φ is smooth, and it satisfies

$$
\mathrm{supp}\,\nabla \varphi \subset \big(A(q, r, 2r) \cap V_1 \big) \cup \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

Observe that, in a sense, φ is a smooth version of the function $\chi_{B(q,r) \cap V_1}$.

Since $g(p, q) = g(p, q) \varphi(q)$ and $g(p, \cdot) \varphi$ is a continuous function from $W_0^{1,2}(\Omega)$, we have

$$
\begin{aligned}
g(p, q) &= \int_\Omega \nabla(g(p, \cdot) \varphi)(x) \nabla g(q, x) \, dx \\
&= \int_\Omega g(p, x) \nabla \varphi(x) \nabla g(q, x) \, dx + \int_\Omega \varphi(x) \nabla g(p, x) \nabla g(q, x) \, dx \\
&=: I_1 + I_2.
\end{aligned}
$$

First we estimate I_2. For ε with $0 < \varepsilon < 1/10$, we consider a smooth function φ_ε such that $\chi_{B(q, \varepsilon \delta_\Omega(q))} \le \varphi_\varepsilon \le \chi_{B(q, 2\varepsilon \delta_\Omega(q))}$, with $\|\nabla \varphi_\varepsilon\|_\infty \lesssim 1/(\varepsilon \delta_\Omega(q))$. Since $\varphi_\varepsilon \varphi = \varphi_\varepsilon$, we have

$$
\begin{aligned}
I_2 &= \int_\Omega \varphi_\varepsilon(x) \nabla g(p, x) \nabla g(q, x) \, dx \\
&\quad + \int_\Omega \varphi(x)(1 - \varphi_\varepsilon(x)) \nabla g(p, x) \nabla g(q, x) \, dx \\
&=: I_{2,a} + I_{2,b}.
\end{aligned}
$$

To deal with $I_{2,a}$ we use the fact that for $x \in B(q, 2\varepsilon \delta_\Omega(q))$ we have

$$
|\nabla g(q, x)| \lesssim \frac{1}{|x - q|^n} \quad \text{and} \quad |\nabla g(p, x)| \lesssim \frac{g(p, q)}{\delta_\Omega(q)}.
$$

Then we get

$$
|I_{2,a}| \lesssim \frac{g(p, q)}{\delta_\Omega(q)} \int_{B(q, 2\varepsilon \delta_\Omega(q))} \frac{1}{|x - q|^n} \, dx \lesssim \frac{g(p, q)}{\delta_\Omega(q)} \varepsilon \delta_\Omega(q) = \varepsilon \, g(p, q).
$$

Let us turn our attention to $I_{2,b}$. We denote $\psi = \varphi(1 - \varphi_\varepsilon)$. Integrating by parts, we get

$$
I_{2,b} = \int \nabla g(p, x) \nabla(\psi \, g(q, \cdot))(x) \, dx - \int \nabla g(p, x) \nabla \psi(x) \, g(q, x) \, dx.
$$

Observe now that the first integral vanishes because $\psi \, g(q, \cdot) \in W_0^{1,2}(\Omega) \cap C(\overline{\Omega})$ and vanishes at $\partial \Omega$ and at p. Hence, since $\nabla \psi = \nabla \varphi - \nabla \varphi_\varepsilon$, we derive

$$
I_{2,b} = -\int \nabla g(p, x) \nabla \varphi(x) \, g(q, x) \, dx + \int \nabla g(p, x) \nabla \varphi_\varepsilon(x) \, g(q, x) \, dx = I_3 + I_4.
$$

To estimate I_4 we take into account that $|\nabla\varphi_\varepsilon| \lesssim \chi_{A(q,\varepsilon\delta_\Omega(q),2\varepsilon\delta_\Omega(q))}/(\varepsilon\delta_\Omega(q))$, and then we derive

$$|I_4| \lesssim \frac{1}{\varepsilon\,\delta_\Omega(q)} \int_{A(q,\varepsilon\delta_\Omega(q),2\varepsilon\delta_\Omega(q))} |\nabla g(p,x)|\,g(q,x)\,dx.$$

Using now that, for x in the domain of integration,

$$g(q,x) \lesssim \frac{1}{(\varepsilon\,\delta_\Omega(q))^{n-1}} \quad \text{and} \quad |\nabla g(p,x)| \lesssim \frac{g(p,q)}{\delta_\Omega(q)},$$

we obtain

$$|I_4| \lesssim \frac{1}{\varepsilon\,\delta_\Omega(q)} \frac{1}{(\varepsilon\,\delta_\Omega(q))^{n-1}} \frac{g(p,q)}{\delta_\Omega(q)} (\varepsilon\,\delta_\Omega(q))^{n+1} \lesssim \varepsilon\,g(p,q).$$

From the above estimates we infer that

$$g(p,q) \leq |I_1 + I_3| + c\,\varepsilon\,g(p,q).$$

Since neither I_1 nor I_3 depend on ε, letting $\varepsilon \to 0$ we get

$$g(p,q) \leq |I_1 + I_3|$$
$$\leq \left| \int g(p,x)\,\nabla\varphi(x)\,\nabla g(q,x)\,dx - \int \nabla g(p,x)\,\nabla\varphi(x)\,g(q,x)\,dx \right|$$
$$\leq \int |\nabla\varphi(x)| \big| g(p,x)\,\nabla g(q,x) - \nabla g(p,x)\,g(q,x) \big|\,dx.$$

We denote

$$\widetilde{F} = \bigcup_{I \in \mathrm{Bdy}(R')} 2I,$$
$$\widetilde{A}_r^\delta = \big\{ x \in A(q,1.2r,1.8r) \cap V_1 \setminus \widetilde{F} : \delta_\Omega(x) > \delta\,r \big\},$$

and

$$\widetilde{A}_{r,\delta} = \big\{ x \in A(q,1.2,1.8r) \cap V_1 \setminus \widetilde{F} : \delta_\Omega(x) \leq \delta\,r \big\}.$$

Next we split the last integral as follows:

$$
\begin{aligned}
g(p, q) \leq & \int_{\widetilde{A}_r^\delta} |\nabla\varphi(x)| \left| g(p, x)\,\nabla g(q, x) - \nabla g(p, x)\,g(q, x) \right| dx \\
& + \int_{\widetilde{A}_{r,\delta}} |\nabla\varphi(x)| \left| g(p, x)\,\nabla g(q, x) - \nabla g(p, x)\,g(q, x) \right| dx \\
& + \int_{\widetilde{F}} |\nabla\varphi(x)| \left| g(p, x)\,\nabla g(q, x) - \nabla g(p, x)\,g(q, x) \right| dx \\
=: & \ J_1 + J_2 + J_3.
\end{aligned} \tag{15.2}
$$

Concerning J_1, we have

$$
|\nabla g(p, x)| \lesssim \frac{g(p, x)}{\delta_\Omega(x)} \quad \text{and} \quad |\nabla g(q, x)| \lesssim \frac{g(q, x)}{\delta_\Omega(x)} \quad \text{for all } x \in \widetilde{A}_r^\delta.
$$

Thus, using also that $|\nabla\varphi| \lesssim 1/r$ outside $\widetilde{F}$,

$$
J_1 \lesssim \frac{1}{r} \sup_{x \in A_r^\delta \cap V_1} \frac{g(p, x)}{\delta_\Omega(x)} \int_{A_r^\delta} g(q, x)\, dx. \tag{15.3}
$$

Regarding J_2, using Cauchy–Schwarz, we get

$$
\begin{aligned}
J_2 \lesssim & \ \frac{1}{r} \int_{\widetilde{A}_{r,\delta}} \left| g(p, x)\,\nabla g(q, x) - \nabla g(p, x)\,g(q, x) \right| dx \\
\leq & \ \frac{1}{r} \left(\int_{\widetilde{A}_{r,\delta}} g(p, x)^2\, dx \right)^{1/2} \left(\int_{\widetilde{A}_{r,\delta}} |\nabla g(q, x)|^2\, dx \right)^{1/2} \\
& + \frac{1}{r} \left(\int_{\widetilde{A}_{r,\delta}} |\nabla g(p, x)|^2\, dx \right)^{1/2} \left(\int_{\widetilde{A}_{r,\delta}} g(q, x)^2\, dx \right)^{1/2}. \tag{15.4}
\end{aligned}
$$

To estimate the integral $\int_{\widetilde{A}_{r,\delta}} g(p, x)^2\, dx$, we take into account that, for all $x \in \widetilde{A}_{r,\delta}$,

$$
g(p, x) \lesssim \delta^\alpha \fint_{A(q, r, 2r)} g(p, y)\, dy.
$$

Then we deduce

$$
\int_{\widetilde{A}_{r,\delta}} g(p, x)^2\, dx \lesssim \frac{\delta^\alpha}{r^{n+1}} \left(\int_{A(q, r, 2r)} g(p, x)\, dx \right)^2.
$$

Next we estimate the integral $\int_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 \, dx$. By covering $\widetilde{A}_{r,\delta}$ by a finite family of balls of radius $r/100$ and applying Caccioppoli's inequality to each one, it follows that

$$\fint_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 \, dx \lesssim \frac{1}{r^2} \fint_{A(q,1.1r,1.9r)} g(q,x)^2 \, dx.$$

Since

$$g(q,x) \lesssim \fint_{A(q,r,2r)} g(q,y) \, dy \quad \text{for all } x \in A(q,1.1r,1.9r),$$

we get

$$\fint_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 \, dx \lesssim \frac{1}{r^2} \fint_{A(q,1.1r,1.9r)} g(q,x)^2 \, dx$$

$$\lesssim \frac{1}{r^{n+3}} \left(\int_{A(q,r,2r)} g(q,x) \, dx \right)^2.$$

So we obtain

$$\left(\fint_{\widetilde{A}_{r,\delta}} g(p,x)^2 \, dx \right)^{1/2} \left(\fint_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 \, dx \right)^{1/2}$$

$$\lesssim \frac{\delta^{\alpha/2}}{r^{n+2}} \int_{A(q,r,2r)} g(p,x) \, dx \int_{A(q,r,2r)} g(q,x) \, dx.$$

By interchanging, p and q, it is immediate to check that an analogous estimate holds for the second summand on the right hand side of (15.4). Thus we get

$$J_2 \lesssim \frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(q,r,2r)} g(p,x) \, dx \int_{A(q,r,2r)} g(q,x) \, dx. \tag{15.5}$$

Concerning J_3, we just take into account that $|\nabla \varphi| \lesssim 1/\ell(I)$ in $2I$, and then we obtain

$$J_3 \lesssim \sum_{I \in \mathsf{Bdy}(R')} \frac{1}{\ell(I)} \int_{2I} \left| g(p,x) \, \nabla g(q,x) - \nabla g(p,x) \, g(q,x) \right| dx.$$

Together with (15.2), (15.3), and (15.5), this yields the lemma. $\square$

Proof of Lemma 15.1 We fix $i = 1$, for definiteness. By a Vitali type covering theorem, there exists a subfamily $\widetilde{\mathsf{H}}_1(R') \subset \mathsf{H}_1(R')$ such that the balls $\{8B_Q\}_{Q \in \widetilde{\mathsf{H}}_1(R')}$ are disjoint and

$$\sum_{Q \in \mathsf{H}_1(R')} \sigma(Q) \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sigma(Q).$$

By Lemma 15.2, for each $Q \in \widetilde{\mathsf{H}}_1(R')$ we have

$$g(p, z_Q^1) \lesssim \frac{1}{r} \sup_{y \in 2B_{R'} \cap V_1 : \delta_\Omega(y) \geq \delta \ell(R')} \frac{g(p, y)}{\delta_\Omega(y)} \int_{A(z_Q^1, r, 2r)} g(z_Q^1, x) \, dx$$

$$+ \frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(z_Q^1, r, 2r)} g(p, x) \, dx \int_{A(z_Q^1, r, 2r)} g(z_Q^1, x) \, dx$$

$$+ \sum_{I \in \mathsf{Bdy}(R')} \frac{1}{\ell(I)} \int_{2I} \left| g(p, x) \, \nabla g(z_Q^1, x) - \nabla g(p, x) \, g(z_Q^1, x) \right| dx$$

$$=: I_1(Q) + I_2(Q) + I_3(Q),$$

with $r = 2\ell(R')$. Since $g(p, z_Q^1) > \lambda \, \ell(Q)/\sigma(R_0)$, we derive

$$\lambda \tau \sigma(R') \lesssim \lambda \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sigma(Q) \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} g(p, z_Q^1) \, \ell(Q)^{n-1} \, \sigma(R_0)$$

$$\lesssim \sum_{j=1}^{3} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_j(Q) \, \ell(Q)^{n-1} \, \sigma(R_0). \tag{15.6}$$

Estimate of $\sum_{Q \in \widetilde{\mathcal{W}} \mathsf{H}_1(R')} I_1(Q) \, \ell(Q)^{n-1}$

We have

$$\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_1(Q) \, \ell(Q)^{n-1}$$

$$\leq \frac{1}{r} \sup_{y \in 2B_{R'} \cap V_1 : \delta_\Omega(y) \geq \delta \ell(R')} \frac{g(p, y)}{\delta_\Omega(y)} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \int_{A(z_Q^1, r, 2r)} g(z_Q^1, x) \, dx \, \ell(Q)^{n-1}.$$

Note now that

$$\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \int_{A(z_Q^1, r, 2r)} g(z_Q^1, x) \, dx \, \ell(Q)^{n-1} \lesssim \int_{2B_{R'}} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^x(4Q) \, dx$$

$$\leq \int_{2B_{R'}} 1 \, dx \lesssim \ell(R')^{n+1},$$

where we used the fact that the cubes $4Q$, with $Q \in \widetilde{\mathsf{H}}_1(R')$, are pairwise disjoint. Since $r \approx \ell(R')$, we derive

$$\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_1(Q) \, \ell(Q)^{n-1} \lesssim \sup_{y \in 2B_{R'} \cap V_1 : \delta_\Omega(y) \geq \delta \ell(R')} \frac{g(p, y)}{\delta_\Omega(y)} \sigma(R').$$

Estimate of $\sum_{Q \in \tilde{\mathcal{W}}\mathsf{H}_1(R')} I_2(Q) \, \ell(Q)^{n-1}$

First we estimate $\int_{A(z_Q^1, r, 2r)} g(p, x) \, dx$ by applying Lemma 9.5:

$$\int_{A(z_Q^1, r, 2r)} g(p, x) \, dx \leq \int_{2B_{R'}} g(p, x) \, dx \lesssim \ell(R')^{n+1} \frac{\omega^p(8B_{R'})}{\ell(R')^{n-1}}$$

$$\lesssim \ell(R')^2 \frac{\sigma(R')}{\sigma(R_0)} \approx \frac{r^{n+2}}{\sigma(R_0)}.$$

So we have

$$\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_2(Q) \, \ell(Q)^{n-1} \lesssim \frac{\delta^{\alpha/2}}{r \, \sigma(R_0)} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \int_{A(z_Q^1, r, 2r)} g(z_Q^1, x) \, dx \, \ell(Q)^{n-1}$$

$$\lesssim \frac{\delta^{\alpha/2}}{r \, \sigma(R_0)} \int_{2B_{R'}} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^x(4Q) \, dx$$

$$\lesssim \frac{\delta^{\alpha/2}}{r \, \sigma(R_0)} \int_{2B_{R'}} 1 \, dx \lesssim \frac{\delta^{\alpha/2} \sigma(R')}{\sigma(R_0)}.$$

Estimate of $\sum_{Q \in \tilde{\mathcal{W}} \mathsf{H}_1(R')} I_3(Q)\, \ell(Q)^{n-1}$

Note first that, for each $I \in \mathsf{Bdy}(R')$, since $z_Q^1 \notin 4I$, using the subharmonicity of $g(p, \cdot)$ and $g(z_Q^1, \cdot)$ in $4I$, and Caccioppoli's inequality,

$$
\frac{1}{\ell(I)} \int_{2I} \left| g(p, x)\, \nabla g(z_Q^1, x) \right| dx \lesssim \frac{1}{\ell(I)} \sup_{x \in 2I} g(p, x) \int_{2I} |\nabla g(z_Q^1, x)|\, dx
$$
$$
\lesssim \ell(I)^{n-1} m_{4I} g(p, \cdot) \quad m_{4I} g(z_Q^1, \cdot).
$$

By very similar estimates, we also get

$$
\frac{1}{\ell(I)} \int_{2I} \left| \nabla g(p, x)\, g(z_Q^1, x) \right| dx \lesssim \ell(I)^{n-1} m_{4I} g(p, \cdot) \quad m_{4I} g(z_Q^1, \cdot).
$$

Recall now that, by Lemma 14.3(e)(i),

$$
m_{4I} g(p, \cdot) \le \gamma\, \frac{\ell(P)}{\sigma(R_0)}
$$

for each $I \in \mathcal{W}_P$, with $P \in \widetilde{\mathsf{End}}(R')$ such that $2B_P \cap 10B_{R'} \neq \varnothing$.

We distinguish two types of Whitney cubes $I \in \mathsf{Bdy}(R')$. We write $I \in T_1$ if $\ell(I) \ge \gamma^{1/2}\ell(P)$ for some P such that $I \in \mathcal{W}_P$ and $2B_P \cap 10B_{R'} \neq \varnothing$, and we write $I \in T_2$ otherwise (there may exist more than one P such that $I \in \mathcal{W}_P$, but if $\mathcal{W}_P \cap \mathcal{W}_{P'} \neq \varnothing$, then $\ell(P) \approx \ell(P')$). So we split

$$
\sum_{Q \in \tilde{\mathsf{H}}_1(R')} I_3(Q)\, \ell(Q)^{n-1} \le \sum_{Q \in \tilde{\mathsf{H}}_1(R')} \sum_{I \in \mathsf{Bdy}(R')} \ell(I)^{n-1} m_{4I} g(p, \cdot) \quad m_{4I} g(z_Q^1, \cdot)\, \ell(Q)^{n-1}
$$
$$
= \sum_{Q \in \tilde{\mathsf{H}}_1(R')} \sum_{I \in T_1} \cdots + \sum_{Q \in \tilde{\mathsf{H}}_1(R')} \sum_{I \in T_2} \cdots =: S_1 + S_2. \qquad (15.7)
$$

Concerning the sum S_1 we have

$$
S_1 \lesssim \gamma \sum_{Q \in \tilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_1} \frac{\ell(P)}{\sigma(R_0)}\, \ell(I)^{n-1} m_{4I} g(z_Q^1, \cdot)\, \ell(Q)^{n-1}
$$
$$
\lesssim \gamma^{1/2} \sum_{Q \in \tilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \frac{\ell(I)^n}{\sigma(R_0)}\, m_{4I} g(z_Q^1, \cdot)\, \ell(Q)^{n-1}
$$

Next we take into account that

$$\ell(Q)^{n-1} \, m_{4I} g(z_Q^1, \cdot) \lesssim \omega^{x_I}(4Q),$$

where x_I stands for the center of I and $C > 1$ is some absolute constant. This follows from Lemma 9.5 if x_I is far from Q, and it can be deduced from Lemma 9.3 when x_I is close to Q (in this case, $\omega^{x_I}(4Q) \approx 1$). Then we derive

$$S_1 \lesssim \gamma^{1/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \omega^{x_I}(4Q) \frac{\ell(I)^n}{\sigma(R_0)}.$$

Since $\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^{x_I}(4Q) \lesssim 1$ for each I, we get

$$S_1 \lesssim \gamma^{1/2} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \frac{\ell(I)^n}{\sigma(R_0)}.$$

By Lemma 14.3(e)(ii), we have $\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n$, and so we deduce

$$S_1 \lesssim \gamma^{1/2} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \frac{\sigma(P)}{\sigma(R_0)} \lesssim \gamma^{1/2} \frac{\sigma(R')}{\sigma(R_0)}.$$

Next we turn our attention to the sum S_2 in (15.7). Recall that

$$S_2 = \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2} \ell(I)^{n-1} \, m_{4I} g(p, \cdot) \; m_{4I} g(z_Q^1, \cdot) \, \ell(Q)^{n-1}.$$

Let us remark that we assume the condition that $I \in \mathcal{W}_P$ for some $2P \in \widetilde{\mathsf{End}}(R')$ such that $2B_P \cap 10B_{R'} \neq \varnothing$ to be part of the definition of $I \in T_2$. Using the estimate $m_{4I} g(p, \cdot) \lesssim \omega^p(B^I) \, \ell(I)^{1-n}$, we derive

$$S_2 \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2} \omega^p(B^I) \, m_{4I} g(z_Q^1, \cdot) \, \ell(Q)^{n-1}$$

$$= \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2 : 20I \cap 20B_Q \neq \varnothing} \cdots + \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2 : 20I \cap 20B_Q = \varnothing} \cdots =: A + B.$$

To estimate the term A we take into account that if $20I \cap 20B_Q \neq \varnothing$ and $I \in \mathcal{W}_P$, then $\ell(P) \lesssim \ell(Q)$ and thus $\ell(I) \lesssim \gamma^{1/2} \, \ell(Q)$ because $I \in T_2$. As a consequence, $I \subset 21B_Q$ and also, by the Hölder continuity of $g(z_Q^1, \cdot)$, if we

let B be a ball concentric with B^I with radius comparable to $\ell(Q)$ and such that $\text{dist}(z_Q^1, B) \approx \ell(Q)$, we obtain

$$m_{2B^I} g(z_Q^1, \cdot) \lesssim \left(\frac{r(B^I)}{r(B)}\right)^\alpha m_B g(z_Q^1, \cdot) \lesssim \gamma^{\alpha/2} \frac{1}{\ell(Q)^{n-1}},$$

where $\alpha > 0$ is the exponent of Hölder continuity. Hence,

$$A \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing \\ 20B_P \cap 20B_Q \neq \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_2} \omega^p(B^I).$$

By Lemma 14.3(e)(ii), we have $\sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P)$, and using also that, for P as above, $CB_P \subset C'B_Q$ for some absolute constant C', we obtain

$$A \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^p(C'B_Q) \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \frac{\sigma(Q)}{\sigma(R_0)} \lesssim \gamma^{\alpha/2} \frac{\sigma(R')}{\sigma(R_0)}.$$

Finally, we turn our attention to the term B. We have

$$B = \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2 : 20I \cap 20B_Q = \varnothing} \omega^p(B^I) m_{4I} g(z_Q^1, \cdot) \ell(Q)^{n-1}$$

$$= \sum_{I \in T_2} \omega^p(B^I) \fint_{4I} \sum_{Q \in \widetilde{\mathsf{H}}_1(R') : 20I \cap 20B_Q = \varnothing} g(z_Q^1, x) \ell(Q)^{n-1} \, dx$$

$$\lesssim \sum_{I \in T_2} \omega^p(B^I) \fint_{4I} \sum_{Q \in \widetilde{\mathsf{H}}_1(R') : 20I \cap 20B_Q = \varnothing} \omega^x(8B_Q) \, dx.$$

We claim now that, in the last sum, if one assumes that $20I \cap 20B_Q = \varnothing$, then $\text{dist}(I, 8B_Q) \geq c\, \gamma^{-1/2} \ell(I)$. To check this, take $P \in \widetilde{\mathsf{End}}(R')$ such that $I \in \mathcal{W}_P$. Then note that

$$\ell(P) \leq \frac{1}{300} d_R(x_P) \leq \frac{1}{300} \left(\text{dist}(x_P, Q) + \ell(Q)\right)$$

$$\leq \frac{1}{300} \left(\text{dist}(x_P, I) + \text{diam}(I) + \text{dist}(I, 8B_Q) + C\ell(Q)\right).$$

Using that $I \cap 2B_P \neq \varnothing$, $\text{diam}(I) \leq C\gamma^{1/2}\ell(P) \ll \ell(P)$, and $\ell(Q) \leq \text{dist}(I, 8B_Q)$, we get

$$\ell(P) \leq \frac{1}{300} \big(\operatorname{dist}(I, 8B_Q) + 3r(B_P) + C\,\ell(Q) \big) \leq C\,\operatorname{dist}(I, 8B_Q) + \frac{12}{300}\,\ell(P),$$

which implies that

$$\ell(I) \leq C\gamma^{1/2}\,\ell(P) \leq C\,\gamma^{1/2}\,\operatorname{dist}(I, 8B_Q),$$

and yields our claim.

Taking into account that the balls $\{8B_Q\}_{Q \in \widetilde{\mathsf{H}}_1(R')}$ are disjoint and the Hölder continuity of $\omega^{(\cdot)}(\partial\Omega \backslash c\gamma^{-1/2}I)$, for all $x \in 4I$ we get

$$\sum_{Q \in \widetilde{\mathsf{H}}_1(R'):20I \cap 20B_Q = \varnothing} \omega^x(8B_Q) \lesssim \omega^x(\partial\Omega \backslash c\gamma^{-1/2}I) \lesssim \gamma^{\alpha/2}.$$

Thus,

$$B \lesssim \gamma^{\alpha/2} \sum_{I \in T_2} \omega^p(B^I) \leq \gamma^{\alpha/2} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_2} \omega^p(B^I).$$

Recalling again that $\sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P)$, we deduce

$$B \lesssim \gamma^{\alpha/2} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \omega^p(CB_P) \lesssim \gamma^{\alpha/2} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'): \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \frac{\sigma(P)}{\sigma(R_0)} \lesssim \gamma^{\alpha/2}\,\frac{\sigma(R')}{\sigma(R_0)}.$$

Remark that for the second inequality we took into account that P is contained in a cube of the form $22P'$ with $P' \in \mathsf{T}_{\mathsf{WSBC}}(R)$ and $\ell(P') \approx \ell(P)$, by Lemma 14.1. This implies that $\omega^p(CB_P) \leq \omega^p(C'B_{P'}) \lesssim \sigma(P')\,\sigma(R_0)^{-1} \lesssim \sigma(P)\,\sigma(R_0)^{-1}$.

Gathering the estimates above and recalling (15.6), we deduce

$$\lambda\tau\,\sigma(R') \lesssim \sup_{y \in 2B_{R'} \cap V_1 : \delta_\Omega(y) \geq \delta\ell(R')} \frac{g(p, y)}{\delta_\Omega(y)}\,\sigma(R')\,\sigma(R_0) + \delta^{\alpha/2}\,\sigma(R') + \gamma^{\alpha/2}\,\sigma(R').$$

So, if δ and γ are small enough (depending on λ, τ), we infer that

$$\lambda\,\tau\,\sigma(R') \lesssim \sup_{y \in 2B_{R'} \cap V_1 : \delta_\Omega(y) \geq \delta\ell(R')} \frac{g(p, y)}{\delta_\Omega(y)}\,\sigma(R')\,\sigma(R_0).$$

That is, there exists some $y_0 \in 2B_{R'} \cap V_1$ with $\delta_\Omega(y_0) \geq \delta \ell(R')$ such that

$$\frac{g(p, y_0)}{\delta_\Omega(y)} \gtrsim \frac{\lambda \tau}{\sigma(R_0)},$$

with δ depending on λ, τ. Since $z_{R'}^1$ and y_0 can be joined by a C-good Harnack chain (for some C depending on δ and γ, and thus on λ, τ), we deduce that

$$\frac{g(p, z_{R'}^1)}{\ell(R')} \gtrsim \frac{c(\lambda, \tau)}{\sigma(R_0)},$$

as wished. $\square$

Lemma 15.3 *Let $\eta \in (0, 1)$ and $\lambda > 0$. Choose $\gamma = \gamma(\lambda, \tau)$ small enough as in Lemma* 15.1 *with $\tau = \eta/2$. Assume that the family* $\mathsf{WSBC}(\Gamma)$ *is defined by choosing Γ big enough depending on γ (and thus on λ and η) as in Lemma* 14.3. *Let $R \in \mathsf{Top}_b^{(N)}$ and suppose that $\mathsf{T_{WSBC}}(R) \neq \varnothing$. Then, there exists an exceptional family* $\mathsf{Ex_{WSBC}}(R) \subset \mathsf{Stop_{WSBC}}(R) \cap \mathsf{G}$ *satisfying*

$$\sum_{P \in \mathsf{Ex_{WSBC}}(R)} \sigma(P) \leq \eta \sigma(R)$$

such that, for every $Q \in \mathsf{Stop_{WSBC}}(R) \cap \mathsf{G} \backslash \mathsf{Ex_{WSBC}}(R)$, any λ-good corkscrew for Q can be joined to some λ'-good corkscrew for R by a $C(\lambda, \eta)$-good Harnack chain, with λ' depending on λ, η.

Proof For any $R' \in \mathcal{D}_{k_0}(R) \cap \mathsf{T_{WSBC}}(R)$, with $k_0 = k_0(\gamma)$, we define $\mathsf{H}_i(R')$ as in (15.1), so that

$$\mathsf{Stop_{WSBC}}(R) \cap \mathsf{G} \cap \mathcal{D}(R') = \mathsf{H}_1(R') \cup \mathsf{H}_2(R').$$

For each R', we set

$$\mathsf{Ex_{WSBC}}(R') = \bigcup_{i=1}^{2} \left\{ Q \in \mathsf{H}_i(R') : \sum_{P \in \mathsf{H}_i(R')} \sigma(P) \leq \tau \sigma(R') \right\}.$$

That is, for fixed $i = 1$ or 2, if $\sum_{P \in \mathsf{H}_i(R')} \sigma(P) \leq \tau \sigma(R')$, then all the cubes from $\mathsf{H}_i(R')$ belong to $\mathsf{Ex_{WSBC}}(R')$. In this way, it is clear that

$$\sum_{P \in \mathsf{Ex_{WSBC}}(R')} \sigma(P) \leq 2\tau \sigma(R) = \eta \sigma(R'). \tag{15.8}$$

We claim that the λ-good corkscrews of cubes from $\mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{G} \cap \mathcal{D}(R')\backslash\mathsf{Ex}_{\mathsf{WSBC}}(R')$ can be joined to some $\widetilde{\lambda}$-good corkscrew for R' by a $\widetilde{C}$-good Harnack chain, with $\widetilde{\lambda}$ depending on λ, η, and $\widetilde{C}$ depending on Γ and thus on λ, η too. Indeed, if $Q \in \mathsf{H}_i(R')\backslash\mathsf{Ex}_{\mathsf{WSBC}}(R')$ and z_Q^i is λ-good corkscrew belonging to V_i (we use the notation of Lemma 15.1 and 14.3), then $\sum_{P \in \mathsf{H}_i(R')} \sigma(P) > \tau \sigma(R')$ by the definition above and thus Lemma 15.1 ensures that $g(p, z_{R'}^i) \geq c(\lambda, \tau)\frac{\ell(R')}{\sigma(R_0)}$. So $z_{R'}^i$ is a $\widetilde{\lambda}$-good corkscrew, which by Lemma 14.3(c) can be joined to z_Q^i by a $\widetilde{C}$-good Harnack chain. In turn, this $\widetilde{\lambda}$-good corkscrew for R' can be joined to some λ'-good corkscrew for R by a C'-good Harnack chain, by applying Lemma 13.4 k_0 times, with C' depending on k_0 and thus on λ and η.

On the other hand, the cubes $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{G}$ which are not contained in any cube $R' \in \mathcal{D}_{k_0}(R) \cap \mathsf{T}_{\mathsf{WSBC}}(R)$ satisfy $\ell(Q) \geq 2^{-k_0}\ell(R)$, and then, arguing as above, their associated λ-good corkscrews can be joined to some λ'-good corkscrew for R by a C'-good Harnack chain, by applying Lemma 13.4 at most k_0 times. Hence, if we define

$$\mathsf{Ex}_{\mathsf{WSBC}}(R) = \bigcup_{R' \in \mathcal{D}_{k_0}(R)} \mathsf{Ex}_{\mathsf{WSBC}}(R'),$$

taking into account (15.8), the lemma follows. □

Proof of the Key Lemma 13.3 We choose $\Gamma = \Gamma(\lambda, \eta)$ as in Lemma 15.3 and we consider the associated family $\mathsf{WSBC}(\Gamma)$. In case that $\mathsf{T}_{\mathsf{WSBC}}(R) = \varnothing$, we set $\mathsf{Ex}(R) = \varnothing$. Otherwise, we consider the family $\mathsf{Ex}_{\mathsf{WSBC}}(R)$ from Lemma 15.3, and we define

$$\mathsf{Ex}(R) = \big(\mathsf{Ex}_{\mathsf{WSBC}}(R) \cap \mathsf{Stop}(R)\big) \cup \bigcup_{Q \in \mathsf{Ex}_{\mathsf{WSBC}}(R)\backslash\mathsf{Stop}(R)} \big(\mathsf{SubStop}(Q) \cap \mathsf{G}\big).$$

It may be useful for the reader to compare the definition above with the partition of $\mathsf{Stop}(R)$ in (13.3). By Lemma 15.3 we have

$$\sum_{P \in \mathsf{Ex}(R)} \sigma(P) \leq \sum_{Q \in \mathsf{Ex}_{\mathsf{WSBC}}(R)} \sigma(P) \leq \eta\,\sigma(R).$$

Next we show that for every $P \in \mathsf{Stop}(R) \cap \mathsf{G}\backslash\mathsf{Ex}(R)$, any λ-good corkscrew for P can be joined to some λ'-good corkscrew for R by a $C(\lambda, \eta)$-good Harnack chain. In fact, if $P \in \mathsf{Stop}_{\mathsf{WSBC}}(R)$, then $P \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{G}\backslash\mathsf{Ex}_{\mathsf{WSBC}}(R)$ since such cube P cannot belong to $\mathsf{SubStop}(Q)$ for any $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R)\backslash\mathsf{Stop}(R)$ (recall the partition (13.3)), and thus the existence of such Harnack chain is ensured by Lemma 15.3. On the other

hand, if $P \notin \mathsf{Stop}_{\mathsf{WSBC}}(R)$, then P is contained in some cube $Q(P) \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \backslash \mathsf{WSBC}(\Gamma)$. Consider the chain $P = S_1 \subset S_2 \subset \cdots \subset S_m = Q(P)$, so that each S_i is the parent of S_{i-1}. For $1 \leq i \leq m$, choose inductively a big corkscrew x_i for S_i in such a way that x_1 is at the same side of L_P as the good λ corkscrew z_P for P, and x_{i+1} is at the same side of L_{S_i} as x_i for each i. Using that $b\beta(S_i) \leq C\varepsilon \ll 1$ for all i, it easy to check that the line obtained by joining the segments $[z_P, x_1], [x_1, x_2], \ldots, [x_{m-1}, x_m]$ is a good carrot curve and so gives rise to a good Harnack chain that joins z_P to x_m. It may happen that x_m is not a λ-good corkscrew. However, since $Q(P) \notin \mathsf{WSBC}(\Gamma)$, it turns out that x_m can be joined to some c_3-good corkscrew $z_{Q(P)}$ for $Q(P)$ by some $C(\Gamma)$-good Harnack chain, with c_3 given by (12.2) (and thus independent of λ and η), because $Q(P) \in \mathsf{G}$. Note that since $\lambda \leq c_3$, $z_{Q(P)}$ is also a λ-good corkscrew. In turn, since $Q(P) \notin \mathsf{Ex}_{\mathsf{WSBC}}(R)$, $z_{Q(P)}$ can be joined to some λ'-good corkscrew for R by another $C'(\lambda, \eta)$-good Harnack chain. Altogether, this shows that z_P can be connected to some λ'-good corkscrew for R by a $C''(\lambda, \eta)$-good Harnack chain, which completes the proof of the lemma. $\square$

Below we will write $\mathsf{Ex}(R, \lambda, \eta)$ instead of $\mathsf{Ex}(R)$ to keep track of the dependence of this family on the parameters λ and η.

16 Proof of the main Lemma 10.2

16.1 Notation

Recall that by the definition of G_0^K in (13.2), $\sum_{R \in \mathsf{Top}} \chi_R(x) \leq K$ for all $x \in G_0^K$. For such x, let Q be the smallest cube from Top that contains x, and denote $n_0(x) = \log_2 \frac{\ell(R_0)}{\ell(Q)}$, so that $Q \in \mathcal{D}_{n_0(x)}(R_0)$. Next let $N_0 \in \mathbb{Z}$ be such that

$$\sigma\left(\{x \in G_0^K : n_0(x) \leq N_0 - 1\}\right) \geq \frac{1}{2}\sigma(G_0^K),$$

and denote

$$\widetilde{G}_0^K = \{x \in G_0^K : n_0(x) \leq N_0 - 1\}.$$

Fix

$$N = N_0 - 1,$$

and set

$$\mathsf{T}_a' = \mathcal{D}_N(R_0) \cup \mathsf{Top}_a^{(N)},$$

and also

$$\mathsf{T}_b' = \mathsf{Top}_b^{(N)} \backslash \mathcal{D}_N(R_0)$$

(recall that $\mathsf{Top}_a^{(N)}$ and $\mathsf{Top}_b^{(N)}$ were defined in Section 13.2). So if $R \in \mathsf{T}_a' \backslash \mathcal{D}_N(R_0)$, then $\mathsf{Stop}^N(R)$ coincides the family of sons of R, and it $R \in \mathsf{T}_b'$ this will not be the case, in general. Next we denote by T_a and T_b the respective subfamilies of cubes from T_a' and T_b' which intersect $\widetilde{G}_0^K$.

For $j \geq 0$, we set

$$\mathsf{T}_b^j = \left\{ R \in \mathsf{T}_b : \sum_{Q \in \mathsf{T}_b : Q \supset R} \chi_Q = j \text{ on } R \right\}.$$

We also denote

$$\mathsf{S}_b^j = \left\{ Q \in \mathcal{D} : Q \in \mathsf{Stop}^N(R) \text{ for some } R \in \mathsf{T}_b^j \right\}, \qquad \mathsf{S}_b = \bigcup_j \mathsf{S}_b^j,$$

and we let T_a^j be the subfamily of cubes $R \in \mathsf{T}_a$ such that there exists some $Q \in \mathsf{S}_b^{j-1}$ such that $Q \supset R$ and R is not contained in any cube from S_b^k with $k \geq j$.

16.2 Two auxiliary lemmas

Lemma 16.1 *The following properties hold for the family T_b^1:*

(a) *The cubes from T_b^1 are pairwise disjoint and cover $\widetilde{G}_0^K$, assuming N_0 big enough.*

(b) *If $R \in \mathsf{T}_b^1$, then $\ell(R) \approx_K \ell(R_0)$.*

(c) *Given $R \in \mathcal{D}(R_0)$ with $\ell(R) \geq c\,\ell(R_0)$ (for example, $R \in \mathsf{T}_b^1$) and $\lambda > 0$, if z_R is a λ-good corkscrew point for R, then there is a $C(\lambda, c)$-good Harnack chain that joins z_R to p.*

Proof Concerning the statement (a), the cubes from T_b^1 are pairwise disjoint by construction. Suppose that $x \in \widetilde{G}_0^K$ is not contained in any cube from T_b^1. By the definition of the family Top^N, this implies that all the cubes $Q \subset R_0$ with $2^{-N}\ell(R_0) \leq \ell(Q) \leq 2^{-10}\ell(R_0)$ containing x belong to T_a. However, there are at most K cubes Q of this type, which is not possible if N is taken big enough. So the cubes from T_b^1 cover $\widetilde{G}_0^K$.

The proof of (b) is analogous. Given $R \in \mathsf{T}_b^1$, all the cubes Q which contain R and have side length smaller or equal that $2^{-10}\ell(R_0)$ belong to T_a. Hence

there at most $K - 1$ cubes Q of this type, because $\widetilde{G}_0^K \cap R \neq \varnothing$. Thus, $\ell(R) \geq 2^{-K-10}\ell(R_0)$.

The statement (c) is an immediate consequence of (b) and Lemma 12.4. $\square$

Lemma 16.2 *Let $Q \in \mathsf{T}_a^j \cup \mathsf{T}_b^j$ for some $j \geq 2$ and let z_Q be a λ-good corkscrew for Q, with $\lambda > 0$. There exists some constant $\gamma_0(\lambda, K) > 0$ such if $\ell(Q) \leq \gamma_0(\lambda, K)\ell(R_0)$, then there exists some cube $R \in \mathsf{S}_b$ such that $R \supset Q$ with a λ'-good corkscrew z_R for R such that z_R can be joined to z_Q by a $C(\lambda, K)$-good Harnack chain, with λ' depending on λ and K.*

Proof We assume $\gamma_0(\lambda, K) > 0$ small enough. Then we can apply Lemma 12.5 $K + 1$ times to get cubes $R_1, \ldots, R_{K+1}$ satisfying:

- $Q \subsetneq R_1 \subsetneq R_2 \subsetneq \cdots \subsetneq R_{K+1}$ and $\ell(R_{K+1}) \leq 2^{-10}\ell(R_0)$,
- each R_j has an associated λ'-good corkscrew z_{R_i} (with λ' depending on λ, K) and there exists a $C(\lambda, K)$-good Harnack chain joining z_Q and $z_{R_1}, \ldots, z_{R_{K+1}}$.

Since $Q \cap \widetilde{G}_0^K \neq \varnothing$, at least one of the cubes $R_1, \ldots, R_{K+1}$, say R_j, does not belong to Top. This implies that $R_j \in \mathsf{T}^{(N)}(\widetilde{R})$ for some $\widetilde{R} \in \mathsf{T}_b$. Let $R \in \mathsf{Stop}^{(N)}(\widetilde{R})$ be the stopping cube that contains Q. Then Lemma 14.3 ensures that there is a good Harnack chain that connects z_{R_j} to some corkscrew z_R for R. Notice that $\ell(R_j) \approx_{\lambda, K} \ell(Q) \approx_{\lambda, K} \ell(R)$ because $Q \subset R \subset R_j$. This implies that $g(p, z_R) \approx_{K, \lambda} g(p, z_{R_j}) \approx_{K, \lambda} g(p, z_Q)$. Further, gathering the Harnack chain that joins z_Q to $z_{\widetilde{R}}$ and the one that joins z_{R_j} to z_R, we obtain the good Harnack chain required by the lemma. $\square$

16.3 The algorithm to construct good Harnack chains

We will construct good Harnack chains that join good corkscrews from "most" cubes from $\mathcal{D}_N(R_0)$ that intersect $\widetilde{G}_0^K$ to good corkscrews from cubes belonging to $R \in \mathsf{T}_b^1$, and then we will join these latter good corkscrews to p using the fact that $\ell(R) \approx \ell(R_0)$. To this end we choose $\eta > 0$ such that

$$\eta \leq \frac{1}{2K} \frac{\sigma(\widetilde{G}_0^K)}{\sigma(R_0)},$$

and we denote

$$m = \max_{x \in \widetilde{G}_0^K} \sum_{R \in \mathsf{T}_b} \chi_R(x)$$

(so that $m \leq K$) and we apply the following algorithm: we set $a_{m+1} = c_3$, so that (12.2) ensures that for each $Q \in \mathsf{T}_a \cup \mathsf{T}_b$ there exists some good a_{m+1}-good corkscrew z_Q. For $j = m, m - 1, \ldots, 1$, we perform the following procedure:

(1) Join a_{j+1}-good corkscrews of cubes Q from $\mathsf{T}_a^{j+1} \cup \mathsf{T}_b^{j+1}$ such that $\ell(Q) \leq c_j' \, \ell(R_0)$ to a_j'-good corkscrews of cubes $R(Q)$ from $\mathsf{S}_b^1 \cup \ldots \cup \mathsf{S}_b^j$ by C_j'-good Harnack chains, with $a_j' \leq a_{j+1}$, so that $R(Q)$ is an ancestor of Q. This step can be performed because of Lemma 16.2, with $c_j' = \gamma_0(a_{j+1}, K)$ in the lemma. The constants a_j', c_j', and C_j' depend on a_{j+1} and K.

(2) Set

$$\mathsf{NC}_j = \bigcup_{R \in \mathsf{T}_b^j} \mathsf{Ex}(R, a_j', \eta),$$

and join a_j'-good corkscrews for all cubes $Q \in \mathsf{S}_b^j \setminus \mathsf{NC}_j$ to a_j-good corkscrews for cubes $R(Q) \in \mathsf{T}_b^j$ by C_j-good Harnack chains, with $a_j \leq a_j'$, so that $R(Q)$ is an ancestor of Q. To this end, one applies Lemma 13.3, which ensures the existence of such Harnack chains connecting a_j'-good corkscrew points for cubes from $\mathsf{S}_b^j \setminus \mathsf{NC}_j$ to a_j-good corkscrew points for cubes from T_b^j. The constants a_j and C_j depend on a_j' and K.

After iterating the procedure above for $j = m, m-1 \ldots, 1$ and joining some Harnack chains arisen in the different iterations, we will have constructed C-good Harnack chains that join a_{m+1}-good corkscrew points for all cubes $Q \in \mathsf{T}_a$ not contained in $\bigcup_{j=1}^m \bigcup_{P \in \mathsf{NC}_j} P$ to a_1-good corkscrews of some ancestors $R(Q)$ belonging either T_b^1 or, more generally, such that $\ell(R(Q)) \gtrsim \ell(R_0)$. The constants c_j', a_j', a_j, C_j worsen at each step j. However, this is not harmful because the number of iterations of the procedure is at most m, and $m \leq K$.

Denote by I_N the cubes from $\mathcal{D}_N(R_0)$ which intersect $\widetilde{G}_0^K$ and are not contained in any cube from $\{P \in \mathsf{NC}_j : j = 1, \cdots m\}$. By the algorithm above we have constructed good Harnack chains that join a_{m+1}-good corkscrew points for all cubes $Q \in I_N$ to some a_1-good corkscrew for cubes $R(Q) \in \mathcal{D}(R_0)$ with $\ell(R(Q)) \approx \ell(R_0)$. Also, by applying Lemma 16.1 (c) we can connect the a_1-good corkscrew for $R(Q)$ to p by a good Harnack chain.

Consider now an arbitrary point $x \in \widetilde{G}_0^K \cap Q$, with $Q \in I_N$. By the definition of $\widetilde{G}_0^K$ and the choice $N = N_0$, all the cubes $P \in \mathcal{D}$ containing x with side length smaller or equal than $\ell(Q)$ satisfy $b\beta(P) \leq \varepsilon$. Then, by an easy geometric argument (see the proof of Lemma 13.3 for a related one) it is easy to check that there is a good Harnack chain joining any good corkscrew

for Q to x. Hence, for all the points $x \in \bigcup_{Q \in I_N} Q \cap \widetilde{G}_0^K$ there is a good Harnack chain that joins x to p.

Finally, observe that, for each j, by Lemma 13.3,

$$
\sum_{P \in \mathsf{NC}_j} \sigma(P) = \sum_{R \in \mathsf{T}_b^j} \sum_{P \in \mathsf{Ex}(R, a_j', \eta)} \sigma(P) \le \eta \sum_{R \in \mathsf{T}_b^j} \sigma(R) \le \eta \, \sigma(R_0)
$$
$$
\le \frac{1}{2K} \sigma(\widetilde{G}_0^K).
$$

Therefore,

$$
\sum_{j=1}^{m} \sum_{P \in \mathsf{NC}_j} \sigma(P) \le \frac{m}{2K} \sigma(\widetilde{G}_0^K) \le \frac{1}{2} \sigma(\widetilde{G}_0^K),
$$

and thus

$$
\sum_{Q \in I_N} \sigma(Q) \ge \sigma(\widetilde{G}_0^K) - \sum_{j=1}^{m} \sum_{P \in \mathsf{NC}_j} \sigma(P) \ge \frac{1}{2} \sigma(\widetilde{G}_0^K) \approx \sigma(R_0).
$$

This finishes the proof of the Main Lemma 10.2. □

Appendix A: Some counter-examples

We shall discuss some counter-examples which show that our background hypotheses in Theorem 1.1 (namely, n-ADR and interior corkscrew condition) are natural, and in some sense in the nature of best possible. In the first two examples, Ω is a domain satisfying an interior corkscrew condition, such that $\partial\Omega$ satisfies exactly one (but not both) of the upper or the lower n-ADR bounds, and for which harmonic measure ω fails to be weak-A_∞ with respect to surface measure σ on $\partial\Omega$. In this setting, in which full n-ADR fails, there is no established notion of uniform rectifiability, but in each case, the domain will enjoy some substitute property which would imply uniform rectifiability of the boundary in the presence of full n-ADR. Moreover, these examples may be constructed in such a way that the failure of the condition (either upper or lower n-ADR) can be expressed quantitatively, with a bound that may be taken arbitrarily close to a true n-ADR bound; see (A.3) and (A.6) below.

In the last example, we construct an open set Ω with n-ADR boundary, and for which $\omega \in$ weak-A_∞ with respect to surface measure, but for which the interior corkscrew condition fails, and $\partial\Omega$ is not n-UR.

Example 1 Failure of the upper n-ADR bound. In [8], the authors construct an example of a Reifenberg flat domain $\Omega \subset \mathbb{R}^{n+1}$ for which surface measure $\sigma = H^n \lfloor_{\partial\Omega}$ is locally finite on $\partial\Omega$, but for which the upper n-ADR bound

$$\sigma(\Delta(x, r)) \leq Cr^n \tag{A.1}$$

fails, and for which harmonic measure ω is not absolutely continuous with respect to σ. Note that the hypothesis of Reifenberg flatness implies in particular that Ω and $\Omega_{ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ are both NTA domains, hence both enjoy the corkscrew condition, so by the relative isoperimetric inequality, the lower n-ADR bound

$$\sigma(\Delta(x, r)) \geq cr^n \tag{A.2}$$

holds. Thus, it is the failure of (A.1) which causes the failure of absolute continuity: in the presence of (A.1), the results of [21] apply, and one has that $\omega \in A_\infty(\sigma)$, and that $\partial\Omega$ satisfies a "big pieces of Lipschitz graphs" condition (see [21] for a precise statement), and hence is n-UR. We note that by a result of Badger [10], a version of the Lipschitz approximation result of [21] still holds for NTA domains with locally finite surface measure, even in the absence of the upper n-ADR condition.

In addition, given any $\varepsilon > 0$, the construction in [8] can be made in such a way that (A.1) fails "within ε", i.e., so that

$$\sigma(\Delta(x, r)) \leq Cr^{n-\varepsilon}, \quad \forall x \in \partial\Omega,\, r < 1. \tag{A.3}$$

Let us sketch an argument to explain why this is so; we refer the interested reader to [8] for more details.

The domain Ω in [8] is obtained by enlarging a Wolff snowflake, that we will denote here by D. Both Ω and D are δ-Reifenberg flat, with δ as small as wished in the construction (recall that Wolff snowflakes can be taken δ-Reifenberg flat, with δ as small as wished).

It is shown in [8, Theorem 3.1] that for all $x \in \partial\Omega$ and $r < 1$,

$$H^n(B(x, r) \cap \partial\Omega) \lesssim \max(r^n, r^\alpha \mu(B(x, Cr))) \leq \max(r^n, \mu(B(x, Cr))) \tag{A.4}$$

where μ is some measure supported on ∂D satisfying $\mu(B(x, r)) \gtrsim r^{n-\alpha}$ for all x in some compact set $E \subset \partial\Omega \cap \partial D$, and some $\alpha > 0$. In the construction in [8], the authors take $\mu = \omega_D$, the harmonic measure for D. Further, from results of Kenig and Toro it follows that harmonic measure in a δ-Reifenberg flat domain D satisfies

$$\omega_D(B(x, r)) \lesssim r^{n-\varepsilon}\omega_D(B(x, 1)), \quad \forall x \in \partial D,\, r < 1,$$

with $\varepsilon \to 0$ as $\delta \to 0$ (see [40, Theorem 4.1]). As a consequence, the measure μ satisfies

$$\mu(B(x, r)) \lesssim r^{n-\varepsilon}, \quad \forall x \in \mathbb{R}^{n+1}, \, r < 1,$$

with ε as small as wished depending on δ. From (A.4), it follows that

$$H^n(B(x, r) \cap \partial\Omega) \lesssim \max(r^n, r^{n-\varepsilon}) \leq r^{n-\varepsilon}, \quad \forall x \in \partial\Omega, r < 1.$$

Example 2 Failure of the lower n-ADR bound. In [2, Example 5.5], the authors give an example of a domain satisfying the interior corkscrew condition, whose boundary is rectifiable (indeed, it is contained in a countable union of hyperplanes), and satisfies the upper n-ADR condition (A.1), but not the lower n-ADR condition (A.2), but for which surface measure σ fails to be absolutely continuous with respect to harmonic measure, and in fact, for which the non-degeneracy condition

$$A \subset \Delta_x := B(x, 10\delta_\Omega(x)) \cap \partial\Omega, \quad \sigma(A) \geq (1-\eta)\sigma(\Delta_x) \implies \omega^x(A) \geq c,$$
$$(A.5)$$

fails to hold uniformly for $x \in \Omega$, for any fixed positive η and c, and therefore ω cannot be weak-A_∞ with respect to σ. We note that in the presence of the full n-ADR condition, if $\partial\Omega$ were contained in a countable union of hyperplanes (as it is in the example), then in particular it would satisfy the "BAUP" condition of [23], and thus would be n-UR [23, Theorem I.2.18, p. 36].

Moreover, given any $\varepsilon > 0$, the parameters in the example of [2] can be chosen in such a way that the lower ADR bound fails "within ε", i.e., so that

$$H^n(\Delta(x, r)) \gtrsim \min(r^{n+\varepsilon}, r^n), \quad \forall x \in \partial\Omega. \tag{A.6}$$

To see this, we proceed as follows. We follow closely the construction in [2, Example 5.5], with some modification of the parameters. Fix $\varepsilon > 0$, and set

$$c_k := 2^{-k(n+\varepsilon)}.$$

For $k \geq 1$, and $n \geq 2$, set

$$\Sigma_k := \{(x, t) \in \mathbb{R}^{n+1}_+ : t = 2^{-k}, x \in \overline{\Delta(0, 2^{-\varepsilon k} c_k)} + c_k \mathbb{Z}^n\},$$

where for $x \in \mathbb{R}^n$, $\Delta(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ is the usual n-disk of radius r centered at x. Define

$$\Omega := \mathbb{R}^{n+1}_+ \setminus \left(\cup_{k=1}^\infty \Sigma_k\right), \qquad \Omega_k := \mathbb{R}^{n+1}_+ \setminus \Sigma_k,$$

each of which is clearly open and connected. Notice that Ω satisfies the interior Corkscrew condition (since the sets Σ_k are located at heights which are sufficently separated). Moreover, it is easy to see that $\partial\Omega$ satisfies the upper ADR condition and that $\mathbb{R}^n \times \{0\} \subset \partial\Omega$.

On the other hand, the lower ADR bound fails. To see this, let $X = (x, 0) \in \partial\Omega$, and choose $\mathbf{m}_{k,x} \in \mathbb{Z}^n$ and $X_k = (c_k \mathbf{m}_{k,x}, 2^{-k}) \in \Sigma_k \subset \partial\Omega$ such that $X_k \to X$. Set $B_k = B(X_k, 2^{-k-2})$, and observe that $H^n(B_k \cap \partial\Omega)/(2^{-kn}) \approx 2^{-kn\varepsilon} \to 0$ as $k \to \infty$, or equivalently

$$H^n(B_k \cap \partial\Omega) \approx r_k^{n+\varepsilon'},$$

where B_k has radius $r_k \approx 2^{-k}$, and $\varepsilon' = n\varepsilon$. We shall show that this behavior is in fact typical, and that (A.6) holds, with ε' in place of ε.

Let $\omega^{(\cdot)} := \omega_\Omega^{(\cdot)}$ and $\omega_k^{(\cdot)} := \omega_{\Omega_k}^{(\cdot)}$ denote harmonic measure for the domains Ω and Ω_k respectively.

Claim $\omega^{(\cdot)}(F) = 0$, with $F := \mathbb{R}^n \times \{0\}$. Thus, in particular (A.5) fails.

It remains to verify (A.6), and the claim. As regards the former, note that for $X = (x, 0) \in F$, we have the trivial standard lower n-ADR bound $H^n(\Delta(X, r)) \gtrsim r^n$, whereas for $X = (x, 2^{-k}) \in \Sigma_k$, we have

$$H^n \lfloor_{\partial\Omega}(B(X, r)) \geq H^n \lfloor_{\partial\Omega_k}(B(X, r)) \gtrsim \begin{cases} r^n, & r < 2^{-\varepsilon k} c_k, \\ 2^{-kn\varepsilon} c_k^n, & 2^{-\varepsilon k} c_k \leq r \leq c_k \\ 2^{-kn\varepsilon} r^n, & c_k < r \leq 2^{-k+1} \\ r^n, & r > 2^{-k+1}. \end{cases}$$
(A.7)

The first and fourth of these estimates are of course the standard lower n-ADR bound. For $r \leq c_k$, the second estimate is bounded below by $2^{-kn\varepsilon} r^n$, and in turn, with $r \lesssim 2^{-k}$, the second and third estimates are therefore bounded below by

$$2^{-kn\varepsilon} r^n \gtrsim r^{n+n\varepsilon} = r^{n+\varepsilon'},$$

which yields (A.6) with $\varepsilon' = n\varepsilon$ in place of ε.

Let us now prove the claim. We first recall some definitions. Given an open set $O \subset \mathbb{R}^{n+1}$, and a compact set $K \subset O$, we define the capacity of K relative to O as

$$\text{cap}(K, O) = \inf\left\{ \iint_O |\nabla\phi|^2 \, dY : \phi \in C_0^\infty(O), \ \phi \geq 1 \text{ in } K \right\}.$$

Also, the inhomogeneous capacity of K is defined as

$$\text{Cap}(K) = \inf \left\{ \iint_{\mathbb{R}^{n+1}} \left(|\phi|^2 + |\nabla\phi|^2 \right) dY : \phi \in C_0^\infty(\mathbb{R}), \ \phi \geq 1 \text{ in } K \right\}.$$

Combining [25, Theorem 2.38], [1, Theorem 2.2.7] and [1, Theorem 4.5.2] we have that if K is a compact subset of $\overline{B}$, where B is a ball with radius smaller than 1, then

$$\text{cap}(K, 2B) \gtrsim \text{Cap}(K) \gtrsim \sup_\mu \mu(K) \tag{A.8}$$

where the implicit constants depend only on n, the sup runs over all Radon positive measures μ supported on K, for which

$$W(\mu)(X) := \int_0^1 \frac{\mu(B(X,t))}{t^{n-1}} \frac{dt}{t} \leq 1, \qquad \forall X \in \text{supp}\,\mu.$$

Fix $k \geq 2$, and set

$$\beta = \beta_k := 2^{k(n-1)} c_k = 2^{k(n-1)} 2^{-k(n+\varepsilon)} = 2^{-k(1+\varepsilon)},$$

by definition of c_k. Our next goal is to show that

$$\text{cap}\left(\overline{B(X_0,s)} \cap \Sigma_k, B(X_0, 2s)\right) \gtrsim s^{n-1}, \quad X_0 := (x_0, 2^{-k}) \in \Sigma_k, \ \beta \leq s < 1. \tag{A.9}$$

For a fixed X_0 and s, write $K = \overline{B(X_0,s)} \cap \Sigma_k$, set $\mu = 2^{kn\varepsilon} s^{-1} H^n \lfloor_K$, and note that for $X \in K$, similarly to (A.7), we have

$$\mu(B(X,r)) \approx 2^{kn\varepsilon} s^{-1} \begin{cases} r^n, & r < 2^{-\varepsilon k} c_k, \\ 2^{-kn\varepsilon} c_k^n, & 2^{-\varepsilon k} c_k \leq r \leq c_k \\ 2^{-kn\varepsilon} r^n, & c_k < r \leq s \\ 2^{-kn\varepsilon} s^n, & r > s. \end{cases} \tag{A.10}$$

To compute $W(\mu)(X)$ for $X \in K$ write

$$W(\mu)(X) = \int_0^1 \frac{\mu(B(X,t))}{t^{n-1}} \frac{dt}{t} = \int_0^{2^{-\varepsilon k} c_k} + \int_{2^{-\varepsilon k} c_k}^{c_k} + \int_{c_k}^s + \int_s^1$$
$$=: I + II + III + IV.$$

Then, since $s \geq \beta = 2^{k(n-1)} c_k = 2^{-k(1+\varepsilon)}$,

$$I + II \lesssim 2^{kn\varepsilon} s^{-1} \left(2^{-\varepsilon k} c_k + 2^{-kn\varepsilon} c_k^n \int_{2^{-\varepsilon k} c_k}^\infty \frac{dt}{t^n} \right) \lesssim 2^{\varepsilon k(n-1)} c_k s^{-1} \lesssim 1.$$

Furthermore, the last two estimates in (A.10) easily imply that $III + IV \lesssim 1$ and hence $W(\mu)(X) \lesssim 1$ for every $X \in K$. This, (A.8), and (A.10) imply as desired (A.9):

$$\mathrm{cap}\big(\overline{B(X_0, s)} \cap \Sigma_k, B(X_0, 2s)\big) \gtrsim \mu(K) \gtrsim s^{n-1}.$$

Set

$$P_k := \left\{ \left(x, 2^{-k} - \beta\right) \in \mathbb{R}_+^{n+1} : x \in \mathbb{R}^n \right\},$$

and observe that for $X \in P_k$,

$$\beta \leq \delta_k(X) := \mathrm{dist}(X, \partial\Omega_k) = \mathrm{dist}(X, \Sigma_k) \leq 2\beta.$$

Recall that $F = \mathbb{R}^n \times \{0\}$, and define

$$u(X) := \omega_k^X(F), \qquad X \in \Omega_k.$$

Observe that $u \in W^{1,2}(\Omega_k) \cap C(\overline{\Omega_k})$ since $\partial\Omega_k$ is ADR (constants depend on k but we just use this qualitatively) and χ_F is a Lipschitz function on $\partial\Omega_k$. Fix $Z_0 \in P_k$ and let $Z_0' \in \Sigma_k$ be such that $|Z_0 - Z_0'| = \mathrm{dist}(Z_0, \partial\Omega_k) \leq 2\beta$. Let $\Omega_{Z_0} = \Omega_k \cap B(Z_0', \frac{3}{4}2^{-k})$, which is an open connected bounded set. We can now apply the usual capacitary estimates (see, e.g., [25, Theorem 6.18]) to find a constant $\alpha = \alpha(n) > 0$ such that

$$u(Z_0) \lesssim \exp\left(-\alpha \int_{3\beta}^{2^{-k-2}} \frac{ds}{s}\right) \approx \left(2^k \beta\right)^\alpha = 2^{-\alpha\varepsilon k}.$$

where we have used (A.9), the definition of β, and the fact that $u \equiv 0$ on $\partial\Omega_k \cap B(Z_0', 2^{-k-1})$. Note that the last estimate holds for any $Z_0 \in P_k$ and therefore, by the maximum principle,

$$u(x, t) \lesssim 2^{-\alpha\varepsilon k}, \qquad (x, t) \in \Omega_k, \quad t > 2^{-k} - \beta.$$

In particular, if we set $X_0 := (0, \ldots, 0, 1) \in \mathbb{R}_+^{n+1}$, then by another application of the maximum principle,

$$\omega^{X_0}(F) \leq \omega_k^{X_0}(F) = u(X_0) \lesssim 2^{-\alpha\varepsilon k} \to 0,$$

as $k \to \infty$, and the claim is established.

Example 3 Failure of the interior corkscrew condition. The example is based on the construction of Garnett's 4-corners Cantor set $\mathcal{C} \subset \mathbb{R}^2$ (see, e.g., [23,

Chapter 1]). Let I_0 be a unit square positioned with lower left corner at the origin in the plane, and in general for each $k = 0, 1, 2, \ldots$, we let I_k be the unit square positioned with lower left corner at the point $(2k, 0)$ on the x-axis. Set $\Omega_0 := I_0$. Let Ω_1 be the first stage of the 4-corners construction, i.e., a union of four squares of side length 1/4, positioned in the corners of the unit square I_1, and similarly, for each k, let Ω_k be the k-th stage of the 4-corners construction, positioned inside I_k. Note that $\text{dist}(\Omega_k, \Omega_{k+1}) = 1$ for every k. Set $\Omega := \cup_k \Omega_k$. It is easy to check that $\partial\Omega$ is n-ADR, and that the non-degeneracy condition (A.5) holds in Ω for some uniform positive η and c, and thus by the criterion of [11], $\omega \in \text{weak-}A_\infty(\sigma)$. On the other hand, the interior corkscrew condition clearly fails to hold in Ω (it holds only for decreasingly small scales as k increases), and certainly $\partial\Omega$ cannot be n-UR: indeed, if it were, then $\partial\Omega_k$ would be n-UR, with uniform constants, for each k, and this would imply that $\mathcal{C}$ itself was n-UR, whereas in fact, as is well known, it is totally non-rectifiable. One can produce a similar set in 3 dimensions by simply taking the cylinder $\Omega' = \Omega \times [0, 1]$. Details are left to the interested reader.

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