Invent. math. (2020) 219:831–871
https://doi.org/10.1007/s00222-019-00918-w

Jacquet modules and local Langlands correspondence

Hiraku Atobe[1]

Received: 24 December 2018 / Accepted: 28 August 2019 /
Published online: 5 September 2019

Abstract In this paper, we explicitly compute the semisimplifications of all Jacquet modules of irreducible representations with generic L-parameters of p-adic split odd special orthogonal groups or symplectic groups. Our computation represents them in terms of linear combinations of standard modules with rational coefficients. The main ingredient of this computation is to apply Mœglin's explicit construction of local A-packets to tempered L-packets. Finally, we study the derivatives introduced by Mínguez.

Mathematics Subject Classification Primary 22E50; Secondary 11S37

1 Introduction

When G is a p-adic reductive group and $P = MN$ is a parabolic subgroup, there is the normalized parabolic induction functor

$$\mathrm{Ind}_P^G \colon \mathrm{Rep}(M) \to \mathrm{Rep}(G),$$

✉ Hiraku Atobe
atobe@math.sci.hokudai.ac.jp

[1] Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido 060-0810, Japan

where Rep(G) is the category of smooth admissible representations of G. The (normalized) Jacquet functor

$$\mathrm{Jac}_P\colon \mathrm{Rep}(G) \to \mathrm{Rep}(M)$$

is the left adjoint functor of Ind_P^G. For $\pi \in \mathrm{Rep}(G)$, the object $\mathrm{Jac}_P(\pi) \in \mathrm{Rep}(M)$ is called the **Jacquet module** of π with respect to P. In the representation theory of p-adic reductive groups, the parabolic induction functors and the Jacquet functors are ones of the most basic and important terminologies. One of the reasons why they are so important is that they are both exact functors.

The Jacquet modules have many applications. For example:

- Looking at the Jacquet modules of irreducible representation π of G, one can take a parabolic subgroup $P = MN$ and an irreducible supercuspidal representation ρ_M of M such that $\pi \hookrightarrow \mathrm{Ind}_P^G(\rho_M)$. Such a ρ_M is called the **cuspidal support** of π.
- **Casselman's criterion** says that the growth of matrix coefficients of an irreducible representation π is determined by exponents of the Jacquet modules of π.
- Mœglin explicitly constructed the **local** A**-packets**, which are the "local factors of Arthur's global classification", by taking Jacquet functors intelligently.

In this paper, we shall give an explicit description of the semisimplifications of Jacquet modules of tempered representations of split odd special orthogonal groups $\mathrm{SO}_{2n+1}(F)$ or symplectic groups $\mathrm{Sp}_{2n}(F)$, where F is a non-archimedean local field of characteristic zero. To do this, it is necessary to have some sort of classification of irreducible representations of these groups. We use the local Langlands correspondence established by Arthur [1] for such a classification.

The local Langlands correspondence attaches each irreducible representation π of $G = \mathrm{SO}_{2n+1}(F)$ (resp. $G = \mathrm{Sp}_{2n}(F)$) to its L-parameter (ϕ, η), where

$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$$

is a homomorphism from the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ of F to the Langlands dual group $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$) of G, and

$$\eta \in \mathrm{Irr}(A_\phi)$$

is an irreducible character of the component group $A_\phi = \pi_0(\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi)))$ associated to ϕ which is trivial on the image of the center $Z(\widehat{G})$ of $\widehat{G}$.

The Jacquet modules will be computed by two main theorems (Theorems 4.2 and 4.3) and Tadić's formula (Theorem 2.8) together with Lemma 2.7. Fix an irreducible unitary supercuspidal representation ρ of $\mathrm{GL}_d(F)$. By abuse of notation, we denote by the same notation ρ the irreducible representation of W_F corresponding to ρ by the local Langlands correspondence. Let $P_k = M_k N_k$ be the standard parabolic subgroup of G with Levi subgroup $M_k \cong \mathrm{GL}_k(F) \times G_0$ for some classical group G_0 of the same type as G. When $k = d$, for an irreducible representation π of G, if the semisimplification of $\mathrm{Jac}_{P_d}(\pi)$ is of the form

$$\mathrm{s.s.Jac}_{P_d}(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i$$

with τ_i (resp. π_i) being an irreducible representation of $\mathrm{GL}_d(F)$ (resp. G_0), we define a **partial Jacquet module** $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$ for $x \in \mathbb{R}$ by

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \pi_i.$$

Here, $|\cdot| \colon W_F \to \mathbb{R}^\times$ is the norm map normalized so that $|\mathrm{Frob}| = q^{-1}$, where $\mathrm{Frob} \in W_F$ is a fixed (geometric) Frobenius element, and q is the cardinality of the residual field of F. The first main theorem is the description of the partial Jacquet module $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$ for tempered π (Theorem 4.2).

For discrete series π, Theorem 4.2 has been proven by Xu [16, Lemma 7.3] to describe the cuspidal support of π in terms of its L-parameter. As related works, Aubert–Moussaoui–Solleveld [2–4] defined the "cuspidality" of L-parameters (ϕ, η) by a geometric way, and compared this notion with the cuspidal supports or the Bernstein components of corresponding π. Theorem 4.2 gives us more information for π than its cuspidal support. The main ingredient for the proof of Theorem 4.2 is Mœglin's explicit construction of tempered L-packets (Theorem 3.5).

The second main theorem (Theorem 4.3) is a reduction of the computation of the Jacquet module $\mathrm{s.s.Jac}_{P_k}(\pi)$ with respect to any maximal parabolic subgroup P_k to the ones of partial Jacquet modules $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$. Using Theorems 4.2 and 4.3 (together with Lemma 2.7), we can explicitly compute the semisimplifications of all Jacquet modules of irreducible tempered representations π. In fact, using a generalization of the standard module conjecture by Mœglin–Waldspurger (Theorem 3.4) and Tadić's formula (Theorem 2.8), we can apply this explicit computation to any irreducible representation π with generic L-parameter (ϕ, η).

This paper is organized as follows. In Sect. 2, we review some basic results on parabolically induced representations and Jacquet modules for classical groups. In particular, Tadić's formula, which computes the Jacquet modules of parabolically induced representations, is stated in Sect. 2.2. In Sect. 3, we explain the local Langlands correspondence and Mœglin's explicit construction of tempered L-packets. In Sect. 4, we state the main theorems (Theorems 4.2 and 4.3) and give some examples. Finally, we prove the main theorems in Sect. 5. Some complements are in Sect. 6. In Sect. 6.1, we give an irreducibility condition for standard modules. In Sect. 6.2, we study derivatives of irreducible tempered representations introduced by Mínguez. It was a key notion for the proof of the Howe duality conjecture by Gan–Takeda [7].

Notation

Let F be a non-archimedean local field of characteristic zero. We denote by W_F the Weil group of F. The norm map $|\cdot|\colon W_F \to \mathbb{R}^\times$ is normalized so that $|\mathrm{Frob}| = q^{-1}$, where $\mathrm{Frob} \in W_F$ is a fixed (geometric) Frobenius element, and $q = q_F$ is the cardinality of the residual field of F.

Each irreducible supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$ is identified with the irreducible bounded representation of W_F of dimension d via the local Langlands correspondence for $\mathrm{GL}_d(F)$. Through this paper, we fix such a ρ. For each positive integer a, the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a is denoted by S_a.

For a p-adic group G, we denote by $\mathrm{Rep}(G)$ (resp. $\mathrm{Irr}(G)$) the set of equivalence classes of smooth admissible (resp. irreducible) representations of G. For $\Pi \in \mathrm{Rep}(G)$, we write $\mathrm{s.s.}(\Pi)$ for the semisimplification of Π.

2 Induced representations and Jacquet modules

In this section, we recall some results on parabolically induced representations and Jacquet modules.

2.1 Representations of $\mathrm{GL}_k(F)$

Let $P = MN$ be a standard parabolic subgroup of $\mathrm{GL}_k(F)$, i.e., P contains the Borel subgroup consisting of upper half triangular matrices. Then the Levi subgroup M is isomorphic to $\mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F)$ with $k_1 + \cdots + k_r = k$. For smooth representations $\tau_1, \ldots, \tau_r$ of $\mathrm{GL}_{k_1}(F), \ldots, \mathrm{GL}_{k_r}(F)$, respectively, we denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r := \mathrm{Ind}_P^{\mathrm{GL}_k(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

A **segment** is a symbol $[x, y]$, where $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}$ and $x \geq y$. We identify $[x, y]$ with the set $\{x, x-1, \ldots, y\}$ so that $\#[x, y] = x - y + 1$. Then the normalized parabolically induced representation

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y$$

of $\mathrm{GL}_{d(x-y+1)}(F)$ has a unique irreducible subrepresentation, which is denoted by

$$\langle \rho; x, \ldots, y \rangle .$$

If $y = -x \leq 0$, this is called a **Steinberg representation** and is denoted by

$$\mathrm{St}(\rho, 2x+1) = \langle \rho; x, \ldots, -x \rangle ,$$

which is a discrete series representation of $\mathrm{GL}_{d(2x+1)}(F)$. In general, $\langle \rho; x, \ldots, y \rangle$ is the twist $|\cdot|^{\frac{x+y}{2}} \mathrm{St}(\rho, x - y + 1)$. We say that two segments $[x, y]$ and $[x', y']$ are **linked** if $[x, y] \not\subset [x', y']$, $[x', y'] \not\subset [x, y]$ as sets, and $[x, y] \cup [x, y']$ is also a segment. The linked-ness gives an irreducibility criterion for parabolically induced representations.

Theorem 2.1 (Zelevinsky [18, Theorem 9.7]) *Let* $[x, y]$ *and* $[x', y']$ *be segments, and let* ρ *and* ρ' *be irreducible unitary supercuspidal representations of* $\mathrm{GL}_d(F)$ *and* $\mathrm{GL}_{d'}(F)$*, respectively. Then the parabolically induced representation*

$$\langle \rho; x, \ldots, y \rangle \times \left\langle \rho'; x', \ldots, y' \right\rangle$$

is irreducible unless $[x, y]$ *are* $[x', y']$ *are linked, and* $\rho \cong \rho'$.

Let $\mathrm{Irr}_\rho(\mathrm{GL}_{dm}(F))$ be the subset of $\mathrm{Irr}(\mathrm{GL}_{dm}(F))$ consisting of τ with cuspidal support of the form $\rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_m}$, i.e.,

$$\tau \hookrightarrow \rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_m}$$

for some $x_1, \ldots, x_m \in \mathbb{R}$. We understand that $\mathbf{1}_{\mathrm{GL}_0(F)} \in \mathrm{Irr}_\rho(\mathrm{GL}_0(F))$. It is easy to see that

- for pairwise distinct irreducible unitary supercuspidal representations $\rho_1, \ldots, \rho_r$, if $\tau_i \in \mathrm{Irr}_{\rho_i}(\mathrm{GL}_{d_i m_i}(F))$ for $i = 1, \ldots, r$, then the parabolically induced representation $\tau_1 \times \cdots \times \tau_r$ is irreducible;

- any irreducible representation of $\mathrm{GL}_k(F)$ is of the above form for some $\tau_i \in \mathrm{Irr}_{\rho_i}(\mathrm{GL}_{d_i m_i}(F))$.

Lemma 2.2 *Let Ω_m be the subset of $\mathbb{R}^m$ consisting of elements*

$$\underline{x} = (\underbrace{x_1, x_1 - 1, \dots, y_1}_{x_1 - y_1 + 1}, \underbrace{x_2, x_2 - 1, \dots, y_2}_{x_2 - y_2 + 1}, \dots, \underbrace{x_t, x_t - 1, \dots, y_t}_{x_t - y_t + 1})$$

such that

- $x_i \geq y_i$ *and* $x_i - y_i \in \mathbb{Z}$ *for* $1 \leq i \leq t$;
- $x_1 \leq x_2 \leq \dots \leq x_t$;
- $y_{i-1} \leq y_i$ *if* $x_{i-1} = x_i$.

Let $\Delta_i = \langle \rho; x_i, x_i - 1, \dots, y_i \rangle$ be the essentially discrete series representation of $\mathrm{GL}_{d(x_i - y_i + 1)}(F)$ corresponding to the segment $[x_i, y_i]$. Then the parabolically induced representation $\Delta_{\underline{x}} := \Delta_1 \times \dots \times \Delta_t$ has a unique irreducible subrepresentation $\tau_{\underline{x}}$. The map $\underline{x} \mapsto \tau_{\underline{x}}$ gives a bijection

$$\Omega_m \to \mathrm{Irr}_\rho(\mathrm{GL}_{dm}(F)).$$

Proof This follows from the Langlands classification and Theorem 2.1, See also [18, Proposition 9.6]. □

For a partition $(k_1, \dots, k_r)$ of k, we denote by $\mathrm{Jac}_{(k_1, \dots, k_r)}$ the normalized Jacquet functor on $\mathrm{Rep}(\mathrm{GL}_k(F))$ with respect to the standard parabolic subgroup $P = MN$ with $M \cong \mathrm{GL}_{k_1}(F) \times \dots \times \mathrm{GL}_{k_r}(F)$. The Jacquet module of $\langle \rho; x, \dots, y \rangle$ with respect to a maximal parabolic subgroup is computed by Zelevinsky.

Proposition 2.3 ([18, Proposition 9.5]) *Suppose that $x \neq y$ and set $k = d(x - y + 1)$. Then $\mathrm{Jac}_{(k_1, k_2)}(\langle \rho; x, \dots, y \rangle) = 0$ unless $k_1 \equiv 0 \bmod d$. If $k_1 = dm$ with $1 \leq m \leq x - y$, we have*

$$\mathrm{Jac}_{(k_1, k_2)}(\langle \rho; x, \dots, y \rangle) = \langle \rho; x, \dots, x - (m-1) \rangle \boxtimes \langle \rho; x - m, \dots, y \rangle .$$

If

$$\mathrm{s.s.Jac}_{(d, k-d)}(\tau) = \bigoplus_{i \in I} \tau_i \boxtimes \tau_i',$$

for $x \in \mathbb{R}$, we define a **partial Jacquet module** $\mathrm{Jac}_{\rho|\cdot|^x}(\tau)$ by

$$\mathrm{Jac}_{\rho|\cdot|^x}(\tau) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \tau_i'.$$

For $\underline{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$, we also set

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}}} = \mathrm{Jac}_{\rho|\cdot|^{x_r}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{x_1}}.$$

This is a functor

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}}} \colon \mathrm{Rep}(\mathrm{GL}_k(F)) \to \mathrm{Rep}(\mathrm{GL}_{k-dr}(F)).$$

In particular, when $\tau \in \mathrm{Rep}(\mathrm{GL}_{dm}(F))$ is of finite length, for $\underline{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$, the partial Jacquet module $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}}}(\tau)$ is a representation of the trivial group $\mathrm{GL}_0(F)$ of finite length so that it is a finite dimensional $\mathbb{C}$-vector space.

Lemma 2.4 *Let $\underline{x} = (x_1, \dots, y_1, \dots, x_t, \dots, y_t) \in \Omega_m$ such that $x_{i-1} \leq x_i$ for $1 < i \leq t$, and $y_{i-1} \leq y_i$ if $x_{i-1} = x_i$ as in Lemma 2.2. For $(x, y) \in \{(x_i, y_i)\}_i$, if we set $m_{(x,y)} = \#\{i \mid (x_i, y_i) = (x, y)\}$, then for $\underline{y} \in \Omega_m$, we have*

$$\dim_{\mathbb{C}} \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\Delta_{\underline{x}}) = \begin{cases} \prod_{(x,y)\in\{(x_i,y_i)\}_i} m_{(x,y)}! & \text{if } \underline{y} = \underline{x}, \\ 0 & \text{if } \underline{y} < \underline{x}. \end{cases}$$

Here, we regard $\mathbb{R}^m$ as a totally ordered set with respect to the lexicographical order.

Proof Fix $z \in \mathbb{R}$. We note that $\mathrm{Jac}_{\rho|\cdot|^z}(\Delta_{\underline{x}}) \neq 0$ if and only if $z \in \{x_1, \dots, x_t\}$. Let $\underline{x'_1}, \dots, \underline{x'_l} \in \Omega_{m-1}$ be the elements obtained by removing z from a component of $\underline{x}$, and rearranging it (so that $l = \#\{i \mid x_i = z\}$). Then $\mathrm{Jac}_{\rho|\cdot|^z}(\Delta_{\underline{x}}) = \sum_{i=1}^{l} \Delta_{\underline{x'_i}}$. Using this, we obtain the lemma by induction on m. □

When $\underline{y} > \underline{x}$, one can also compute $\dim_{\mathbb{C}} \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\Delta_{\underline{x}})$ inductively.

Let $\mathcal{R}_k$ be the Grothendieck group of the category of smooth representations of $\mathrm{GL}_k(F)$ of finite length. By the semisimplification, we identify the objects in this category with elements in $\mathcal{R}_k$. Elements in $\mathrm{Irr}(\mathrm{GL}_k(F))$ form a $\mathbb{Z}$-basis of $\mathcal{R}_k$. Set $\mathcal{R} = \oplus_{k \geq 0} \mathcal{R}_k$. The parabolic induction functor gives a product

$$m \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \ \tau_1 \otimes \tau_2 \mapsto \mathrm{s.s.}(\tau_1 \times \tau_2).$$

This product makes $\mathcal{R}$ an associative commutative ring. On the other hand, the Jacquet functor gives a coproduct

$$m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$$

which is defined by the $\mathbb{Z}$-linear extension of

$$\mathrm{Irr}(\mathrm{GL}_k(F)) \ni \tau \mapsto \sum_{k_1=0}^{k} \mathrm{s.s.Jac}_{(k_1,k-k_1)}(\tau).$$

Then m and m^* make $\mathcal{R}$ a graded Hopf algebra, i.e., $m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ is a ring homomorphism.

2.2 Representations of $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{Sp}_{2n}(F)$

We set G to be split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$, i.e., G is the group of F-points of the split algebraic group of type B_n or C_n. Fix a Borel subgroup of G, and let $P = MN$ be a standard parabolic subgroup of G. Then the Levi part M is of the form $\mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F) \times G_0$ with $G_0 = \mathrm{SO}_{2n_0+1}(F)$ or $G_0 = \mathrm{Sp}_{2n_0}(F)$ such that $k_1 + \cdots + k_r + n_0 = n$. For a smooth representation $\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0$ of M, we denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 = \mathrm{Ind}_P^G(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0).$$

The functor $\mathrm{Ind}_P^G \colon \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ is exact.

On the other hand, for a smooth representation π of G, we denote the normalized Jacquet module with respect to P by

$$\mathrm{Jac}_P(\pi),$$

and its semisimplification by $\mathrm{s.s.Jac}_P(\pi)$. The functor $\mathrm{Jac}_P \colon \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ is exact. The Frobenius reciprocity asserts that

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \cong \mathrm{Hom}_M(\mathrm{Jac}_P(\pi), \sigma)$$

for $\pi \in \mathrm{Rep}(G)$ and $\sigma \in \mathrm{Rep}(M)$.

The maximal standard parabolic subgroup with Levi $\mathrm{GL}_k(F) \times G_0$ is denoted by $P_k = M_k N_k$ for $0 \le k \le n$.

Definition 2.5 Let π be a smooth representation of G. Consider $\mathrm{s.s.Jac}_{P_d}(\pi)$ (and a fixed irreducible supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$). If

$$\mathrm{s.s.Jac}_{P_d}(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i$$

with τ_i (resp. π_i) being an irreducible representation of $\mathrm{GL}_d(F)$ (resp. G_0), for $x \in \mathbb{R}$, we define a **partial Jacquet module** $\mathrm{Jac}_{\rho|\cdot|^x}(\pi)$ by

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) = \bigoplus_{\substack{i \in I \\ \tau_i \cong \rho|\cdot|^x}} \pi_i.$$

This is a representation of G_0, which is $\mathrm{SO}_{2(n-d)+1}(F)$ or $\mathrm{Sp}_{2(n-d)}(F)$.

Also, for $\underline{x} = (x_1, \dots, x_r) \in \mathbb{R}^r$, we set

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}}}(\pi) = \mathrm{Jac}_{\rho|\cdot|^{x_1},\dots,\rho|\cdot|^{x_r}}(\pi) = \mathrm{Jac}_{\rho|\cdot|^{x_r}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{x_1}}(\pi).$$

We recall some properties of $\mathrm{Jac}_{\rho|\cdot|^{x_1},\dots,\rho|\cdot|^{x_r}}$.

Lemma 2.6 ([16, Lemmas 5.3, 5.6]) *Let π be an irreducible representation of G.*

(1) *Suppose that $\mathrm{Jac}_{\rho|\cdot|^{x_1},\dots,\rho|\cdot|^{x_r}}(\pi)$ is nonzero. Then there exists an irreducible constituent σ of $\mathrm{Jac}_{\rho|\cdot|^{x_1},\dots,\rho|\cdot|^{x_r}}(\pi)$ such that we have an inclusion*

$$\pi \hookrightarrow \rho|\cdot|^{x_1} \times \cdots \times \rho|\cdot|^{x_r} \rtimes \sigma.$$

(2) *If $|x - y| \neq 1$, then $\mathrm{Jac}_{\rho|\cdot|^x,\rho|\cdot|^y}(\pi) = \mathrm{Jac}_{\rho|\cdot|^y,\rho|\cdot|^x}(\pi)$.*
(3) *If $\rho \not\cong \rho'$, then $\mathrm{Jac}_{\rho|\cdot|^x} \circ \mathrm{Jac}_{\rho'|\cdot|^{x'}} = \mathrm{Jac}_{\rho'|\cdot|^{x'}} \circ \mathrm{Jac}_{\rho|\cdot|^x}$ for any $x, x' \in \mathbb{R}$.*

Let $\mathcal{R}_n(G)$ be the Grothendieck group of the category of smooth representations of G of finite length, where $n = \mathrm{rank}(G)$, i.e., $G = \mathrm{SO}_{2n+1}(F)$ or $G = \mathrm{Sp}_{2n}(F)$. Set $\mathcal{R}(G) = \oplus_{n\geq 0}\mathcal{R}_n(G)$. The parabolic induction defines a module structure

$$\rtimes\colon \mathcal{R} \otimes \mathcal{R}(G) \to \mathcal{R}(G),\ \tau \otimes \pi \mapsto \mathrm{s.s.}(\tau \rtimes \pi),$$

and the Jacquet functor defines a comodule structure

$$\mu^*\colon \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)$$

by

$$\mathrm{Irr}(G) \ni \pi \mapsto \sum_{k=0}^{\mathrm{rank}(G)} \mathrm{s.s.Jac}_{P_k}(\pi).$$

When

$$\mu^*(\pi) = \sum_{i\in I} \tau_i \otimes \pi_i,$$

we define $\mu_\rho^*(\pi)$ by

$$\mu_\rho^*(\pi) = \sum_{\substack{i \in I \\ \tau_i \in \mathrm{Irr}_\rho(\mathrm{GL}_{dm}(F))}} \tau_i \otimes \pi_i.$$

Lemma 2.7 *If we define $\iota \colon \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)$ by $\pi \mapsto \mathbf{1}_{\mathrm{GL}_0(F)} \otimes \pi$, we have*

$$\mu^* = \circ_\rho \left((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_\rho^*) \right) \circ \iota,$$

where ρ runs over all irreducible unitary supercuspidal representations of $\mathrm{GL}_d(F)$ *for $d > 0$. Namely, for $\pi \in \mathrm{Irr}(G)$, if $\{\rho_1, \ldots, \rho_t\}$ is the finite set of irreducible unitary supercuspidal representations ρ of some* $\mathrm{GL}_d(F)$ *such that $\mu_\rho^*(\pi) \neq 0$, then*

$$\mu^*(\pi) = \left((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_{\rho_t}^*) \right) \circ \cdots \circ \left((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_{\rho_1}^*) \right) (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes \pi).$$

Proof Fix an irreducible representation π of G. First, we note that there are only finitely many ρ such that $\mu_\rho^*(\pi) \neq 0$. Second, we claim that

$$(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_{\rho'}^*) \circ \mu_\rho^*(\pi) = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_\rho^*) \circ \mu_{\rho'}^*(\pi)$$

for distinct ρ and ρ'. In fact, this is the sum of subrepresentations appearing $\mu^*(\pi)$ of the form

$$(\tau \times \tau') \otimes \pi_0 = (\tau' \times \tau) \otimes \pi_0,$$

where $\tau \in \mathrm{Irr}_\rho(\mathrm{GL}_{dm}(F))$ and $\tau' \in \mathrm{Irr}_{\rho'}(\mathrm{GL}_{d'm'}(F))$ for various m and m'. By the same argument, we have

$$\mu^*(\pi) = \circ_\rho \left((m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_\rho^*) \right) \circ \iota(\pi),$$

as desired. □

Tadić established a formula to compute μ^* for parabolically induced representations. The contragredient functor $\tau \mapsto \tau^\vee$ defines an automorphism $\vee \colon \mathcal{R} \to \mathcal{R}$ in a natural way. Let $s \colon \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ be the homomorphism defined by $\sum_i \tau_i \otimes \tau_i' \mapsto \sum_i \tau_i' \otimes \tau_i$.

Theorem 2.8 (Tadić [15]) *Consider the composition*

$$M^* = (m \otimes \mathrm{id}) \circ (\vee \otimes m^*) \circ s \circ m^* \colon \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}.$$

Then for the maximal parabolic subgroup $P_k = M_k N_k$ of G and for an admissible representation $\tau \boxtimes \pi$ of M_k, we have

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi).$$

In particular, we have the following.

Corollary 2.9 *For a segment* $[x, y]$, *we have*

$$\mu^*(\langle \rho; x, \dots, y \rangle \rtimes \pi) = \sum_{\substack{k,l \geq 0 \\ k+l \leq x-y+1}} \left(\left\langle \rho^\vee; \underbrace{-y, \dots, -y-k+1}_{k} \right\rangle \times \left\langle \rho; \underbrace{x, \dots, x-l+1}_{l} \right\rangle \right) \otimes \langle \rho; x-l, \dots, y+k \rangle \rtimes \mu^*(\pi).$$

3 Local Langlands correspondence

In this section, we review the local Langlands correspondence for split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$.

3.1 L-parameters

A homomorphism

$$\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})$$

is called a **representation** of the Weil–Deligne group $W_F \times \mathrm{SL}_2(\mathbb{C})$ if

- $\phi(\mathrm{Frob}) \in \mathrm{GL}_k(\mathbb{C})$ is semisimple;
- $\phi|W_F$ is smooth, i.e., has an open kernel;
- $\phi|\mathrm{SL}_2(\mathbb{C})$ is algebraic.

We say that a representation ϕ of $W_F \times \mathrm{SL}_2(\mathbb{C})$ is **tempered** if $\phi(W_F)$ is bounded. The local Langlands correspondence for $\mathrm{GL}_k(F)$ asserts that there is a canonical bijection

$$\mathrm{Irr}(\mathrm{GL}_k(F)) \longleftrightarrow \{\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})\}/\cong,$$

which preserves the tempered-ness.

We say that a representation $\phi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_k(\mathbb{C})$ is **symplectic** or **of symplectic type** (resp. **orthogonal** or **of orthogonal type**) if the image of ϕ is in $\mathrm{Sp}_k(\mathbb{C})$ (so that k is even) (resp. in $\mathrm{O}_k(\mathbb{C})$). An L-**parameter** for $\mathrm{SO}_{2n+1}(F)$ is a $2n$-dimensional symplectic representation

$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C}).$$

Similarly, an L-**parameter** for $\mathrm{Sp}_{2n}(F)$ is a $(2n+1)$-dimensional orthogonal representation

$$\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2n+1}(\mathbb{C})$$

with trivial determinant. For $G = \mathrm{SO}_{2n+1}(F)$ or $G = \mathrm{Sp}_{2n}(F)$, we let $\Phi(G)$ be the set of equivalence classes of L-parameters for G. We say that

- $\phi \in \Phi(G)$ is **discrete** if ϕ is a multiplicity-free sum of irreducible self-dual representations of the same type as ϕ;
- $\phi \in \Phi(G)$ is **of good parity** if ϕ is a sum of irreducible self-dual representations of the same type as ϕ;
- $\phi \in \Phi(G)$ is **tempered** if $\phi(W_F)$ is bounded;
- $\phi \in \Phi(G)$ is **generic** if the adjoint L-function $L(s, \phi, \mathrm{Ad})$ is regular at $s = 1$. Here, $L(s, \phi, \mathrm{Ad}) = L(s, \mathrm{Ad} \circ \phi)$ is the L-function associated to the composition of ϕ and the adjoint representation $\mathrm{Ad}\colon \mathrm{Sp}_{2n}(\mathbb{C}) \to \mathrm{GL}(\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{C})))$ or $\mathrm{Ad}\colon \mathrm{SO}_{2n+1}(\mathbb{C}) \to \mathrm{GL}(\mathrm{Lie}(\mathrm{SO}_{2n+1}(\mathbb{C})))$.

We denote by $\Phi_{\mathrm{disc}}(G)$ (resp. $\Phi_{\mathrm{gp}}(G)$, $\Phi_{\mathrm{temp}}(G)$, and $\Phi_{\mathrm{gen}}(G)$) the subset of $\Phi(G)$ consisting of discrete L-parameters (resp. L-parameters of good parity, tempered L-parameters, and generic L-parameters). Then we have inclusions

$$\Phi_{\mathrm{disc}}(G) \subset \Phi_{\mathrm{gp}}(G) \subset \Phi_{\mathrm{temp}}(G) \subset \Phi_{\mathrm{gen}}(G) \subset \Phi(G).$$

For $\phi \in \Phi(G)$, we can decompose

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_r\phi_r \oplus (\phi' \oplus \phi'^\vee),$$

where $\phi_1, \dots, \phi_r$ are distinct irreducible self-dual representations of the same type as ϕ, $m_i \geq 1$ is the multiplicity of ϕ_i in ϕ, and ϕ' is a sum of irreducible representations which are not self-dual or self-dual of the opposite type to ϕ. We define the **component group** A_ϕ of ϕ by

$$A_\phi = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})\alpha_{\phi_i} \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

Namely, A_ϕ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank r and $\{\alpha_{\phi_1}, \ldots, \alpha_{\phi_r}\}$ is a basis of A_ϕ with α_{ϕ_i} associated to ϕ_i. We set

$$z_\phi = \sum_{i=1}^{r} m_i \alpha_{\phi_i} \in A_\phi$$

and we call z_ϕ the **central element** in A_ϕ.

We shall introduce an **enhanced component group** $\mathcal{A}_\phi$ associated to $\phi \in \Phi(G)$. Write $\phi = \phi_{\mathrm{gp}} \oplus (\phi' \oplus \phi'^\vee)$, where ϕ_{gp} is the sum of irreducible self-dual subrepresentations of the same type as ϕ, and ϕ' is a sum of irreducible representations which are not of the same type as ϕ. We decompose

$$\phi_{\mathrm{gp}} = \bigoplus_{i \in I} \phi_i$$

into the sum of irreducible representations.

Definition 3.1 With the notation above, we define the **enhanced component group** $\mathcal{A}_\phi$ associated to ϕ by

$$\mathcal{A}_\phi = \bigoplus_{i \in I} (\mathbb{Z}/2\mathbb{Z})\alpha_i.$$

Namely, $\mathcal{A}_\phi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module whose rank is equal to the length of ϕ_{gp}.

By abuse of notation, we put $z_\phi = \sum_{i \in I} \alpha_i \in \mathcal{A}_\phi$ and call it the **central element** in $\mathcal{A}_\phi$. There is a canonical surjection

$$\mathcal{A}_\phi \twoheadrightarrow A_\phi, \ \alpha_i \mapsto \alpha_{\phi_i}.$$

This map preserves the central elements. The kernel of this map is generated by $\alpha_i + \alpha_j$ such that $\phi_i \cong \phi_j$. In particular, if ϕ is discrete, then this map is an isomorphism.

Remark 3.2 For the definition of this enhanced component group, we rely on the shape of ϕ. A similar definition will work for $\mathrm{SO}_{2n}(F)$ and unitary groups as well, but we do not know a uniform definition for general (connected) reductive groups, nor whether this notion makes sense.

3.2 Local Langlands correspondence

We denote by $\mathrm{Irr}_{\mathrm{disc}}(G)$ (resp. $\mathrm{Irr}_{\mathrm{temp}}(G)$) the set of equivalence classes of irreducible discrete series (resp. tempered) representations of G. The local Langlands correspondence established by Arthur is as follows:

Theorem 3.3 ([1, Theorem 2.2.1]) *Let G be a split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$.*

(1) *There exists a canonical surjection*

$$\mathrm{Irr}(G) \to \Phi(G).$$

*For $\phi \in \Phi(G)$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-**packet associated to** ϕ.*

(2) *There exists an injection*

$$\Pi_\phi \to \widehat{A_\phi},$$

which satisfies certain endoscopic character identities. Here, $\widehat{A_\phi}$ is the Pontryagin dual of A_ϕ. The image of this map is

$$\{\eta \in \widehat{A_\phi} \mid \eta(z_\phi) = 1\}.$$

When $\pi \in \Pi_\phi$ corresponds to $\eta \in \widehat{A_\phi}$, we write $\pi = \pi(\phi, \eta)$.

(3) *For $* \in \{\mathrm{disc}, \mathrm{temp}\}$,*

$$\mathrm{Irr}_*(G) = \bigsqcup_{\phi \in \Phi_*(G)} \Pi_\phi.$$

(4) *Assume that $\phi = \phi_\tau \oplus \phi_0 \oplus \phi_\tau^\vee \in \Phi_{\mathrm{temp}}(G)$, where*
- *$\phi_0 \in \Phi_{\mathrm{temp}}(G_0)$ with a classical group G_0 of the same type as G;*
- *ϕ_τ is a tempered representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ of dimension k.*

Let τ be the irreducible tempered representation of $\mathrm{GL}_k(F)$ corresponding to ϕ_τ. Then for $\pi_0 \in \Pi_{\phi_0}$, the induced representation $\tau \rtimes \pi_0$ decomposes into a direct sum of irreducible tempered representations of G. The L-packet Π_ϕ is given by

$$\Pi_\phi = \left\{\pi \;\middle|\; \pi \subset \tau \rtimes \pi_0,\ \pi_0 \in \Pi_{\phi_0}\right\}.$$

Moreover there is a canonical inclusion $A_{\phi_0} \hookrightarrow A_\phi$. If $\pi(\phi, \eta)$ is a direct summand of $\tau \rtimes \pi_0$ with $\pi_0 = \pi(\phi_0, \eta_0)$, then $\eta_0 = \eta | A_{\phi_0}$.

(5) *Assume that*

$$\phi = \phi_1|\cdot|^{s_1} \oplus \cdots \oplus \phi_r|\cdot|^{s_r} \oplus \phi_0 \oplus \phi_r^\vee|\cdot|^{-s_r} \oplus \cdots \oplus \phi_1^\vee|\cdot|^{-s_1},$$

where
- *$\phi_0 \in \Phi_{\mathrm{temp}}(G_0)$ with a classical group G_0 of the same type as G;*
- *ϕ_i is a tempered representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ of dimension k_i for $1 \le i \le r$;*
- *s_i is a real number such that $s_1 \ge \cdots \ge s_r > 0$.*

Let τ_i be the irreducible tempered representation of $\mathrm{GL}_{k_i}(F)$ corresponding to ϕ_i. Then the L-packet Π_ϕ consists of the unique irreducible quotients π of the standard modules

$$\tau_1|\cdot|^{s_1} \times \cdots \times \tau_r|\cdot|^{s_r} \rtimes \pi_0,$$

where π_0 runs over Π_{ϕ_0}. Moreover, there is a canonical inclusion $A_{\phi_0} \hookrightarrow A_\phi$, which is in fact bijective. If $\pi(\phi, \eta)$ is the unique irreducible quotient of the above standard module with $\pi_0 = \pi(\phi_0, \eta_0)$, then $\eta_0 = \eta|A_{\phi_0}$. In this case, we denote this standard module by $I(\phi, \eta)$.

The injection $\Pi_\phi \to \widehat{A_\phi}$ is not canonical when $G = \mathrm{Sp}_{2n}(F)$. To specify this, we implicitly fix an $F^{\times 2}$-orbit of non-trivial additive characters of F through this paper. Remark that our main results (Theorems 4.2 and 4.3 below) are independent of such a choice.

We have the following irreducibility criterion for standard modules.

Theorem 3.4 (Generalized standard module conjecture) *For $\phi \in \Phi(G)$, the standard module $I(\phi, \mathbf{1})$ attached to $\pi(\phi, \mathbf{1})$ is irreducible if and only if ϕ is generic. Moreover, if ϕ is generic, then all standard modules $I(\phi, \eta)$, where $\eta \in \widehat{A_\phi}$ with $\eta(z_\phi) = 1$, are irreducible.*

Proof The first assertion is the usual standard module conjecture proven in [5,8,9,14]. The second assertion was proven by Mœglin–Waldspurger [13, Corollaire 2.14] for special orthogonal groups and symplectic groups. Heiermann [10] also proved the second assertion in a more general setting. Note that their definition of generic L-parameters might look different from ours. The equivalence of two definitions is called a conjecture of Gross–Prasad and Rallis, which was proven by Gan–Ichino [6, Proposition B.1]. □

However, even if ϕ is not generic, there might exist an irreducible standard module $I(\phi, \eta)$. An example of such standard modules will be given by Corollary 6.1 and Example 6.2 below.

3.3 Extension to enhanced component groups

To describe Jacquet modules of $\pi(\phi, \eta)$ for $\phi \in \Phi_{\mathrm{gp}}(G)$, it is useful to extend $\pi(\phi, \eta)$ to the case where η is a character of the enhanced component group $\mathcal{A}_\phi$ as follows. Recall that there exists a canonical surjection $\mathcal{A}_\phi \twoheadrightarrow A_\phi$ so that we have an injection $\widehat{A_\phi} \hookrightarrow \widehat{\mathcal{A}_\phi}$. For $\eta \in \widehat{\mathcal{A}_\phi}$, set

$$\pi(\phi, \eta) = \begin{cases} \pi(\phi, \eta) & \text{if } \eta \in \widehat{A_\phi},\ \eta(z_\phi) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\pi(\phi, \eta)$ is irreducible or zero for any $\eta \in \widehat{\mathcal{A}_\phi}$.

More precisely, if $\phi = \oplus_{i \in I} \phi_i$ is the irreducible decomposition so that $\mathcal{A}_\phi = \oplus_{i \in I} (\mathbb{Z}/2\mathbb{Z})\alpha_i$, then $\pi(\phi, \eta)$ is nonzero if and only if $\eta(z_\phi) = 1$ and $\eta(\alpha_i) = \eta(\alpha_j)$ whenever $\phi_i \cong \phi_j$.

3.4 Mœglin's construction of tempered L-packets

The L-packets are used for a local classification. On the other hand, Arthur [1, Theorem 2.2.1] introduced the notion of A-**packets** for a global classification. Mœglin constructed the local A-packets in her consecutive works (e.g., [11, 12], etc.). For a detailed why Mœglin's local A-packets agree with Arthur's, one can see Xu's paper [17] in addition to the original papers of Mœglin. Since the **tempered** A-**packets** are the same notion as the tempered L-packets, Mœglin's construction can be applied to the tempered L-packets.

We explain Mœglin's construction of Π_ϕ for $\phi \in \Phi_{\mathrm{gp}}(G)$. Assume that

$$\phi = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i} \right) \oplus \phi_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_a \not\subset \phi_e$ for any $a > 0$. (In particular, ρ must be self-dual if $t \geq 1$.) Take a new L-parameter

$$\phi_\gg = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a'_i} \right) \oplus \phi_e$$

for a bigger group G' of the same type as G such that

- $a'_1 < \cdots < a'_t$;
- $a'_i \geq a_i$ and $a'_i \equiv a_i \bmod 2$ for any i;

Then we can identify $\mathcal{A}_{\phi_\gg}$ with $\mathcal{A}_\phi$ canonically, i.e., $\mathcal{A}_{\phi_\gg} = \mathcal{A}_\phi$ and if $\mathcal{A}_{\phi_\gg} \ni \alpha \mapsto \alpha_{\rho \boxtimes S_{a'_i}} \in A_{\phi_\gg}$, then $\mathcal{A}_\phi \ni \alpha \mapsto \alpha_{\rho \boxtimes S_{a_i}} \in A_\phi$. Let $\eta_\gg \in \widehat{\mathcal{A}_{\phi_\gg}}$ be the character corresponding to $\eta \in \widehat{\mathcal{A}_\phi}$, i.e., $\eta_\gg = \eta$ via $\mathcal{A}_{\phi_\gg} = \mathcal{A}_\phi$.

Theorem 3.5 (Mœglin) *With the notation above, we have*

$$\pi(\phi, \eta) = \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_t - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_t + 1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_1 - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_1 + 1}{2}}} (\pi(\phi_\gg, \eta_\gg)).$$

For a proof, one can specialize [17, Corollary 8.10], which treats all local A-packets. Using this theorem repeatedly, we can construct the L-packets Π_ϕ for $\phi \in \Phi_{\mathrm{gp}}(G)$ from the L-packets associated to discrete L-parameters for bigger groups.

Example 3.6 We construct Π_ϕ for $\phi = S_2 \oplus S_4 \oplus S_4 \oplus S_6 \oplus S_6 \in \Phi_{\mathrm{gp}}(\mathrm{SO}_{23}(F))$. Note that $A_\phi = (\mathbb{Z}/2\mathbb{Z})\alpha_{S_2} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{S_4} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{S_6}$ with $z_\phi = \alpha_{S_2}$. Let $\eta \in \widehat{A_\phi}$. We write $\eta(\alpha_{S_{2a}}) = \eta_a \in \{\pm 1\}$. If $\eta(z_\phi) = 1$, then $\eta_1 = +1$.

Now we take new L-parameters

$$\begin{aligned} \phi_{\gg} &= S_2 \oplus S_4 \oplus S_6 \oplus S_8 \oplus S_{10} \in \Phi_{\mathrm{disc}}(\mathrm{SO}_{31}(F)), \\ \phi' &= S_2 \oplus S_4 \oplus S_4 \oplus S_8 \oplus S_{10} \in \Phi_{\mathrm{disc}}(\mathrm{SO}_{29}(F)), \\ \phi'' &= S_2 \oplus S_4 \oplus S_4 \oplus S_6 \oplus S_{10} \in \Phi_{\mathrm{disc}}(\mathrm{SO}_{27}(F)), \end{aligned}$$

and we consider $\eta_{\gg} \in \widehat{A_{\phi_{\gg}}}$, $\eta' \in \widehat{A_{\phi'}}$ and $\eta'' \in \widehat{A_{\phi''}}$ given by

- $\eta_{\gg}(\alpha_{S_2}) = \eta'(\alpha_{S_2}) = \eta''(\alpha_{S_2}) = \eta_1 = +1$;
- $\eta_{\gg}(\alpha_{S_4}) = \eta_{\gg}(\alpha_{S_6}) = \eta'(\alpha_{S_4}) = \eta''(\alpha_{S_4}) = \eta_2$;
- $\eta_{\gg}(\alpha_{S_8}) = \eta_{\gg}(\alpha_{S_{10}}) = \eta'(\alpha_{S_8}) = \eta'(\alpha_{S_{10}}) = \eta''(\alpha_{S_6}) = \eta''(\alpha_{S_{10}}) = \eta_3$.

Then Theorem 3.5 says that

$$\begin{aligned} \mathrm{Jac}_{|\cdot|^{\frac{5}{2}}}(\pi(\phi_{\gg}, \eta_{\gg})) &= \pi(\phi', \eta'), \\ \mathrm{Jac}_{|\cdot|^{\frac{7}{2}}}(\pi(\phi', \eta')) &= \pi(\phi'', \eta''), \\ \mathrm{Jac}_{|\cdot|^{\frac{9}{2}}, |\cdot|^{\frac{7}{2}}}(\pi(\phi'', \eta'')) &= \pi(\phi, \eta). \end{aligned}$$

4 Description of Jacquet modules

In this section, we state the main theorems, which compute the semisimplifications of the Jacquet modules of $\pi(\phi, \eta)$ for $\phi \in \Phi_{\mathrm{gp}}(G)$.

4.1 Statements

Note that:

Lemma 4.1 *For $\phi \in \Phi_{\mathrm{gp}}(G)$ and $x \in \mathbb{R}$, if $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0$ for some $\pi \in \Pi_\phi$, then x is a non-negative half-integer and $\rho \boxtimes S_{2x+1} \subset \phi$.*

Proof When $\phi \in \Phi_{\mathrm{disc}}(G)$, it follows from [16, Lemma 7.3]. We may assume that $\phi \in \Phi_{\mathrm{gp}}(G) \setminus \Phi_{\mathrm{disc}}(G)$. Then there exists an irreducible representation $\rho' \boxtimes S_a$ which ϕ contains at least multiplicity two. By Theorem 3.3 (4), we have

$$\pi \hookrightarrow \mathrm{St}(\rho', a) \rtimes \pi_0$$

for some $\pi_0 \in \Pi_{\phi_0}$ with $\phi_0 = \phi - (\rho' \boxtimes S_a)^{\oplus 2}$. Then by Corollary 2.9, we have

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \hookrightarrow (1 \otimes \mathrm{St}(\rho', a)) \rtimes \mathrm{Jac}_{\rho|\cdot|^x}(\pi_0) + \begin{cases} 2\,\langle \rho; x, \dots, -(x-1)\rangle \rtimes \pi_0 & \text{if } \rho' \cong \rho, a = 2x+1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0$, then $\mathrm{Jac}_{\rho|\cdot|^x}(\pi_0) \neq 0$ or $\rho \boxtimes S_{2x+1} \cong \rho' \boxtimes S_a \subset \phi$. By induction, we conclude that $\rho \boxtimes S_{2x+1} \subset \phi$ as desired. □

The following is the first main theorem, which is a description of $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta))$.

Theorem 4.2 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta) \neq 0$. Fix a non-negative half-integer $x \in (1/2)\mathbb{Z}$. Write*

$$\phi = \phi_0 \oplus (\rho \boxtimes S_{2x+1})^{\oplus m}$$

with $\rho \boxtimes S_{2x+1} \not\subset \phi_0$ and $m > 0$.

(1) *Assume that $m \geq 3$. Take $\delta \in \{1, 2\}$ such that $\delta \equiv m \bmod 2$. Then*

$$\begin{aligned} &\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta)) \\ &= (m - \delta) \cdot \Big\langle \rho; \underbrace{x, x-1, \dots, -(x-1)}_{2x} \Big\rangle \rtimes \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta\right) \\ &+ \underbrace{\mathrm{St}(\rho, 2x+1) \times \cdots \times \mathrm{St}(\rho, 2x+1)}_{(m-\delta)/2} \rtimes \mathrm{Jac}_{\rho|\cdot|^x}\left(\pi\left(\phi_0 \oplus (\rho \boxtimes S_{2x+1})^{\oplus \delta}, \eta\right)\right). \end{aligned}$$

Here, we canonically identify the (usual) component groups of $\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}$ and $\phi_0 \oplus (\rho \boxtimes S_{2x+1})^{\oplus \delta}$ with A_ϕ, so that we regard η as a character of these groups.

(2) *Assume that $x > 0$ and $m = 1$. Set*

$$\phi' = \phi - (\rho \boxtimes S_{2x+1}) \oplus (\rho \boxtimes S_{2x-1}).$$

There is a canonical inclusion $\mathcal{A}_{\phi'} \hookrightarrow \mathcal{A}_\phi$, which is in fact bijective if $x > 1/2$. Let $\eta' \in \widehat{\mathcal{A}_{\phi'}}$ be the character corresponding to $\eta \in \widehat{\mathcal{A}_\phi}$, i.e., $\eta' = \eta|\mathcal{A}_{\phi'}$. Then

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta)) = \pi(\phi', \eta').$$

In particular, $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta))$ is irreducible or zero.

(3) *Assume that* $x > 0$ *and* $m = 2$*. Set* $\eta_+ = \eta$*, and take the unique character* $\eta_- \in \widehat{\mathcal{A}_\phi}$ *so that* $\eta_-|\mathcal{A}_{\phi_0} = \eta_+|\mathcal{A}_{\phi_0}$*,* $\eta_-(z_\phi) = \eta_+(z_\phi) = 1$ *but* $\eta_- \neq \eta_+$*. For* ϕ' *as in (2) and for* $\epsilon \in \{\pm\}$*, let* $\eta'_\epsilon \in \widehat{\mathcal{A}_{\phi'}}$ *be the character corresponding to* $\eta_\epsilon \in \widehat{\mathcal{A}_\phi}$ *via the canonical inclusion* $\mathcal{A}_{\phi'} \hookrightarrow \mathcal{A}_\phi$*. Then*

$$\begin{aligned}\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta)) = \langle \rho; x, x-1, \dots, -(x-1)\rangle \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0}) \\ + \pi(\phi', \eta'_+) - \pi(\phi', \eta'_-).\end{aligned}$$

(4) *Assume that* $x = 0$*. If* $m = 1$*, then* $\mathrm{Jac}_\rho(\pi(\phi,\eta)) = 0$*. If* $m = 2$*, then* $\mathrm{Jac}_\rho(\pi(\phi,\eta)) = \pi(\phi_0, \eta|\mathcal{A}_{\phi_0})$.

When $\phi \in \Phi_{\mathrm{disc}}(G)$, Theorem 4.2 has been already proven by Xu ([16, Lemma 7.3]). In (2) (resp. (3)), we note that $\pi(\phi', \eta')$ (resp. $\pi(\phi', \eta'_\epsilon)$) can be zero even if $\pi(\phi, \eta) \neq 0$. In (3), the character η_- is characterized so that

$$\pi(\phi, \eta_+) \oplus \pi(\phi, \eta_-) = \mathrm{St}(\rho, 2x+1) \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0}).$$

The second main theorem concerns $\mu^*_\rho(\pi)$.

Theorem 4.3 *Let* $\phi \in \Phi_{\mathrm{gp}}(G)$*, and write*

$$\phi = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i}\right) \oplus \phi_e$$

with $a_1 \leq \cdots \leq a_t$ *and* $\rho \boxtimes S_a \not\subset \phi_e$ *for any* $a > 0$*. Set* $x_i = \frac{a_i - 1}{2}$*. For* $0 \leq m \leq (2d)^{-1} \cdot \dim(\phi)$*, we denote by* $K^{(m)}_\phi$ *the set of tuples of integers* $\underline{k} = (k_1, \dots, k_t)$ *such that*

- $0 \leq k_i \leq a_i$ *for any* i*;*
- $k_{i-1} \geq k_i$ *if* $a_{i-1} = a_i$*;*
- $k_1 + \cdots + k_t = m$*.*

For $\underline{k} \in K^{(m)}_\phi$*, set*

$$\underline{x}(\underline{k}) = (\underbrace{x_1, \dots, x_1 - k_1 + 1}_{k_1}, \dots, \underbrace{x_t, \dots, x_t - k_t + 1}_{k_t}) \in \Omega_m.$$

For $\underline{k}, \underline{l} \in K^{(m)}_\phi$*, we set*

$$m_{\underline{k},\underline{l}} = \dim_{\mathbb{C}} \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{k})}}(\Delta_{\underline{x}(\underline{l})}),$$

and define $(m'_{\underline{k},\underline{l}})_{\underline{k},\underline{l}\in K_\phi^{(m)}}$ *to be the inverse matrix of* $(m_{\underline{k},\underline{l}})_{\underline{k},\underline{l}\in K_\phi^{(m)}}$*, i.e.,*

$$\sum_{\underline{k'}\in K_\phi^{(m)}} m'_{\underline{k},\underline{k'}} m_{\underline{k'},\underline{l}} = \begin{cases} 1 & \text{if } \underline{k} = \underline{l}, \\ 0 & \text{if } \underline{k} \neq \underline{l}. \end{cases}$$

Then for $\pi \in \Pi_\phi$*, we have*

$$\mu^*_\rho(\pi) = \sum_{m=0}^{(2d)^{-1}\dim(\phi)} \sum_{\underline{k},\underline{l}\in K_\phi^{(m)}} m'_{\underline{k},\underline{l}} \cdot \Delta_{\underline{x}(\underline{k})} \otimes \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{l})}}(\pi).$$

When we formally regard $(\Delta_{\underline{x}(\underline{k})})_{\underline{k}\in K_\phi^{(m)}}$ and $(\otimes\mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{l})}}(\pi))_{\underline{l}\in K_\phi^{(m)}}$ as column vectors, we have

$$\sum_{\underline{k},\underline{l}\in K_\phi^{(m)}} m'_{\underline{k},\underline{l}} \cdot \Delta_{\underline{x}(\underline{k})} \otimes \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{l})}}(\pi) = {}^t(\Delta_{\underline{x}(\underline{k})}) \cdot (m_{\underline{k},\underline{l}})^{-1} \cdot (\otimes\mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{l})}}(\pi)).$$

By Lemma 2.4, $(m_{\underline{k},\underline{l}})_{\underline{k},\underline{l}\in K_\phi^{(m)}}$ is a “triangular matrix”, which can be computed inductively. Here, we regard $K_\phi^{(m)}$ as a totally ordered set with respect to the lexicographical order. The diagonal entries $m_{\underline{k},\underline{k}}$ are given in Lemma 2.4 explicitly.

By Tadić’s formula (Theorem 2.8), Lemma 2.7, and Theorems 4.2, 4.3, we can deduce the following corollary.

Corollary 4.4 *We can compute* $\mu^*(\pi)$ *explicitly for any* $\pi \in \Pi_\phi$ *with* $\phi \in \Phi_{\mathrm{gen}}(G)$.

4.2 Examples

We shall give some examples.

Example 4.5 Fix two positive integers a, b such that $a \equiv b \mod 2$ and $a < b$, and consider

$$\phi = \rho \boxtimes (S_a \oplus S_b) \in \Phi_{\mathrm{disc}}(\mathrm{SO}_{d(a+b)+1}(F)),$$

where ρ is symplectic (resp. orthogonal) if $a \equiv 1 \bmod 2$ (resp. if $a \equiv 0 \bmod 2$). Then $\Pi_\phi = \{\pi_+(a,b), \pi_-(a,b)\}$ with generic $\pi_+(a,b)$ and non-generic $\pi_-(a,b)$. Note that both $\pi_+(a,b)$ and $\pi_-(a,b)$ are discrete series. We compute $\mu^*(\pi_\epsilon(a,b))$ for $\epsilon \in \{\pm\}$.

Note that $K_\phi^{(m)} = \{(k_1, k_2) \in \mathbb{Z}^2 \mid 0 \le k_1 \le a,\ 0 \le k_2 \le b,\ k_1 + k_2 = m\}$. For $(k_1, k_2) \in K_\phi^{(m)}$, since $x_1 = \frac{a-1}{2}$ and $x_2 = \frac{b-1}{2}$,

$$\Delta_{\underline{x}(k_1,k_2)} = \left\langle \rho; \frac{a-1}{2}, \dots, \frac{a+1}{2} - k_1 \right\rangle \times \left\langle \rho; \frac{b-1}{2}, \dots, \frac{b+1}{2} - k_2 \right\rangle.$$

This induced representation is irreducible unless $(a+3)/2 - k_1 \le (b+1)/2 - k_2 \le (a+1)/2$, i.e., $(b-a)/2 \le k_2 \le (b-a)/2 + k_1 - 1$. Moreover, one can easy to see that for $(l_1, l_2) \in K_\phi^{(m)}$, the virtual representation

$$\sum_{(k_1,k_2) \in K_\phi^{(m)}} m'_{(k_1,k_2),(l_1,l_2)} \cdot \Delta_{\underline{x}(k_1,k_2)}$$

is the unique irreducible subrepresentation $\tau_{\underline{x}(l_1,l_2)}$ of $\Delta_{\underline{x}(l_1,l_2)}$ (cf. see [18, Proposition 4.6]).

Note that

$$\mathrm{Jac}_{\rho|\cdot|^{\frac{a-1}{2}}, \dots, \rho|\cdot|^{\frac{a+1}{2}-k_1}} (\pi_\epsilon(a, b)) = \begin{cases} \pi_\epsilon(a - 2k_1, b) & \text{if } k_1 \le a/2, \\ 0 & \text{otherwise.} \end{cases}$$

Here, when $k_1 = a/2$, we understand that $\pi_+(0, b)$ is the unique element in $\Pi_{\rho \boxtimes S_b}$, and $\pi_-(0, b) = 0$. Moreover, for $(k_1, k_2) \in K_\phi^{(m)}$, when $k_1 \le a/2$ and $k_2 \le (b-a)/2 + k_1$, we have

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}(k_1,k_2)}} (\pi_\epsilon(a, b)) = \pi_\epsilon(a - 2k_1, b - 2k_2).$$

When $k_1 < a/2$ and $(b-a)/2 + k_1 + 1 \le k_2 \le b/2$, we have

$$\begin{aligned} \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(k_1,k_2)}} (\pi_\epsilon(a, b)) = &\left\langle \rho; \frac{a-1}{2} - k_1, \dots, -\frac{b-1}{2} + k_2 \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ &+ \pi_\epsilon(b - 2k_2, a - 2k_1) - \pi_{-\epsilon}(b - 2k_2, a - 2k_1). \end{aligned}$$

When $k_1 < a/2$ and $b/2 < k_2 \le (a+b)/2 - k_1$, we have

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}(k_1,k_2)}} (\pi_\epsilon(a, b)) = \left\langle \rho; \frac{a-1}{2} - k_1, \dots, -\frac{b-1}{2} + k_2 \right\rangle \rtimes \mathbf{1}_{\mathrm{SO}_1(F)}.$$

In particular, if $k_1 + k_2 = (a+b)/2$, then $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}(k_1,k_2)}} (\pi_\epsilon(a, b)) = \mathbf{1}_{\mathrm{SO}_1(F)}$. Hence s.s.$\mathrm{Jac}_{P_{d(a+b)/2}} (\pi_\epsilon(a, b))$ contains the irreducible representation

$$\left\langle \rho; \frac{a-1}{2}, \dots, \frac{a+1}{2} - k_1 \right\rangle \times \left\langle \rho; \frac{b-1}{2}, \dots, -\left(\frac{a-1}{2} - k_1 \right) \right\rangle \otimes \mathbf{1}_{\mathrm{SO}_1(F)}$$

with multiplicity one if $k_1 < a/2$, or if $k_1 = a/2$ and $\epsilon = +1$.

Example 4.6 Consider the L-parameter $\phi = S_2 \oplus S_4 \oplus S_4 \in \Phi_{\mathrm{gp}}(\mathrm{SO}_{11}(F))$. Then Π_ϕ has two elements $\pi_+(2,4,4)$ and $\pi_-(2,4,4)$ with generic $\pi_+(2,4,4)$ and non-generic $\pi_-(2,4,4)$. Then

$$K_\phi^{(m)}=\{(k_1,k_2,k_3) \in \mathbb{Z}^3 \mid 0 \leq k_1 \leq 2,\ 0\leq k_3 \leq k_2 \leq 4, k_1+k_2+k_3 = m\}$$

for $0 \leq m \leq 5$. Write $\Pi^\epsilon_{\underline{x}(\underline{k})} = \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{k})}}(\pi_\epsilon(2,4,4))$ for $\epsilon \in \{\pm\}$, and $\mathrm{St}_a = \mathrm{St}(\mathbf{1}_{\mathrm{GL}_1(F)}, a)$. We denote by $\det_a$ the determinant character of $\mathrm{GL}_a(F)$.

(1) When $m = 1$, we have $K_\phi^{(1)} = \{(1,0,0) > (0,1,0)\}$. Since

$$\left(\Delta_{\underline{x}(1,0,0)}\ \Delta_{\underline{x}(0,1,0)}\right) = \left(|\cdot|^{\frac{1}{2}}\ |\cdot|^{\frac{3}{2}}\right),$$

we have

$$(m_{\underline{k},\underline{k}'})^{-1}_{\underline{k},\underline{k}'} = \begin{pmatrix}1&0\\0&1\end{pmatrix}^{-1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Moreover

$$\begin{pmatrix}\Pi^\epsilon_{\underline{x}(1,0,0)}\\ \Pi^\epsilon_{\underline{x}(0,1,0)}\end{pmatrix} = \begin{pmatrix}\pi_\epsilon(4,4)\\ |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \pi_+(2) + \epsilon\cdot\pi_+(2,2,4)\end{pmatrix}.$$

Hence

$$\begin{aligned}\mathrm{s.s.Jac}_{P_1}(\pi_\epsilon(2,4,4)) &= |\cdot|^{\frac{1}{2}} \otimes \pi_\epsilon(4,4)\\ &\quad + |\cdot|^{\frac{3}{2}} \otimes \left(|\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \pi_+(2) + \epsilon\cdot\pi_+(2,2,4)\right).\end{aligned}$$

(2) When $m = 2$, we have $K_\phi^{(2)} = \{(2,0,0) > (1,1,0) > (0,2,0) > (0,1,1)\}$. Since

$$\begin{aligned}&\left(\Delta_{\underline{x}(2,0,0)}\ \Delta_{\underline{x}(1,1,0)}\ \Delta_{\underline{x}(0,2,0)}\ \Delta_{\underline{x}(0,1,1)}\right)\\ &= \left(\mathrm{St}_2|\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{3}{2}}\ |\cdot|^{1}\mathrm{St}_2\ |\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{3}{2}}\right),\end{aligned}$$

we have

$$(m_{\underline{k},\underline{k}'})^{-1}_{\underline{k},\underline{k}'} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix}^{-1} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&0&\frac{1}{2}\end{pmatrix}.$$

Moreover

$$\begin{pmatrix} \Pi^{\epsilon}_{\underline{x}(2,0,0)} \\ \Pi^{\epsilon}_{\underline{x}(1,1,0)} \\ \Pi^{\epsilon}_{\underline{x}(0,2,0)} \\ \\ \Pi^{\epsilon}_{\underline{x}(0,1,1)} \end{pmatrix} = \begin{pmatrix} 0 \\ |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \pi_{\epsilon}(2,4) - \pi_{-\epsilon}(2,4) \\ |\cdot|^{1}\mathrm{St}_2 \rtimes \pi_+(2) + |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ +\epsilon\left(|\cdot|^{\frac{1}{2}} \rtimes \pi_+(4) + \pi_+(2,4)\right) \\ (1+\epsilon)\cdot \pi_+(2,2,2) \end{pmatrix}.$$

Hence

$$\begin{aligned} \text{s.s.Jac}_{P_2}(\pi_{\epsilon}(2,4,4)) &= |\det{}_2|^1 \otimes \left(|\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \pi_{\epsilon}(2,4) - \pi_{-\epsilon}(2,4)\right) \\ &\quad + |\cdot|^1\mathrm{St}_2 \otimes \Big(|\cdot|^1\mathrm{St}_2 \rtimes \pi_+(2) + |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ &\qquad + \epsilon\left(|\cdot|^{\frac{1}{2}} \rtimes \pi_+(4) + \pi_+(2,4)\right)\Big) \\ &\quad + \left(|\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{3}{2}}\right) \otimes \frac{1+\epsilon}{2}\cdot \pi_+(2,2,2). \end{aligned}$$

(3) When $m = 3$, we have $K_{\phi}^{(3)} = \{(2,1,0) > (1,2,0) > (1,1,1) > (0,3,0) > (0,2,1)\}$. Since

$$\begin{aligned} &\left(\Delta_{\underline{x}(2,1,0)}\ \Delta_{\underline{x}(1,2,0)}\ \Delta_{\underline{x}(1,1,1)}\ \Delta_{\underline{x}(0,3,0)}\ \Delta_{\underline{x}(0,2,1)}\right) \\ &= \left(\mathrm{St}_2 \times |\cdot|^{\frac{3}{2}}\ \ |\cdot|^{\frac{1}{2}} \times |\cdot|^1\mathrm{St}_2\ \ |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{3}{2}}\ \ |\cdot|^{\frac{1}{2}}\mathrm{St}_3\ \ |\cdot|^1\mathrm{St}_2 \times |\cdot|^{\frac{3}{2}}\right), \end{aligned}$$

we have

$$(m_{\underline{k},\underline{k}'})^{-1}_{\underline{k},\underline{k}'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} \Pi^{\epsilon}_{\underline{x}(2,1,0)} \\ \Pi^{\epsilon}_{\underline{x}(1,2,0)} \\ \Pi^{\epsilon}_{\underline{x}(1,1,1)} \\ \Pi^{\epsilon}_{\underline{x}(0,3,0)} \\ \Pi^{\epsilon}_{\underline{x}(0,2,1)} \end{pmatrix} = \begin{pmatrix} 0 \\ |\cdot|^1\mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \epsilon\cdot\pi_+(4) \\ 2\cdot\pi_{\epsilon}(2,2) \\ |\cdot|^{\frac{3}{2}} \rtimes \pi_+(2) + \epsilon\cdot\pi_+(4) \\ (1+\epsilon)\left(|\cdot|^{\frac{1}{2}} \rtimes \pi_+(2) + \pi_+(2,2)\right) + \pi_-(2,2) \end{pmatrix}.$$

Hence

$$\begin{aligned}
&\text{s.s.Jac}_{P_3}(\pi_\epsilon(2,4,4)) \\
&\quad = \left(|\cdot|^{\frac{1}{2}} \times |\cdot|^1 \mathrm{St}_2\right) \otimes \left(|\cdot|^1 \mathrm{St}_2 \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \epsilon \cdot \pi_+(4)\right) \\
&\quad + \left(|\det{}_2|^1 \times |\cdot|^{\frac{3}{2}}\right) \otimes \pi_\epsilon(2,2) \\
&\quad + |\cdot|^{\frac{1}{2}} \mathrm{St}_3 \otimes \left(|\cdot|^{\frac{3}{2}} \rtimes \pi_+(2) + \epsilon \cdot \pi_+(4)\right) \\
&\quad + \left(|\cdot|^1 \mathrm{St}_2 \times |\cdot|^{\frac{3}{2}}\right) \otimes \left((1+\epsilon)|\cdot|^{\frac{1}{2}} \rtimes \pi_+(2) + \pi_+(2,2) + \epsilon \cdot \pi_{-\epsilon}(2,2)\right).
\end{aligned}$$

(4) When $m = 4$, we have $K_\phi^{(4)} = \{(2,2,0) > (2,1,1) > (1,3,0) > (1,2,1) > (0,4,0) > (0,3,1) > (0,2,2)\}$. Since

$$\begin{pmatrix} \Delta_{\underline{x}(2,2,0)} \\ \Delta_{\underline{x}(2,1,1)} \\ \Delta_{\underline{x}(1,3,0)} \\ \Delta_{\underline{x}(1,2,1)} \\ \Delta_{\underline{x}(0,4,0)} \\ \Delta_{\underline{x}(0,3,1)} \\ \Delta_{\underline{x}(0,2,2)} \end{pmatrix} = \begin{pmatrix} \mathrm{St}_2 \times |\cdot|^1 \mathrm{St}_2 \\ \mathrm{St}_2 \times |\cdot|^{\frac{3}{2}} \times |\cdot|^{\frac{3}{2}} \\ |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{1}{2}} \mathrm{St}_3 \\ |\cdot|^{\frac{1}{2}} \times |\cdot|^1 \mathrm{St}_2 \times |\cdot|^{\frac{3}{2}} \\ \mathrm{St}_4 \\ |\cdot|^{\frac{1}{2}} \mathrm{St}_3 \times |\cdot|^{\frac{3}{2}} \\ |\cdot|^1 \mathrm{St}_2 \times |\cdot|^1 \mathrm{St}_2 \end{pmatrix},$$

we have

$$\begin{aligned}
(m_{\underline{k},\underline{k}'})^{-1}_{\underline{k},\underline{k}'} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

Moreover

$$\begin{pmatrix} \Pi^{\epsilon}_{\underline{x}(2,2,0)} \\ \Pi^{\epsilon}_{\underline{x}(2,1,1)} \\ \Pi^{\epsilon}_{\underline{x}(1,3,0)} \\ \Pi^{\epsilon}_{\underline{x}(1,2,1)} \\ \Pi^{\epsilon}_{\underline{x}(0,4,0)} \\ \Pi^{\epsilon}_{\underline{x}(0,3,1)} \\ \Pi^{\epsilon}_{\underline{x}(0,2,2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ |\cdot|^{\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ |\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \epsilon\cdot\pi_+(2) \\ \pi_+(2) \\ (1+\epsilon)\cdot\pi_+(2) \\ (3+2\epsilon)|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + (1+2\epsilon)\cdot\pi_+(2) \end{pmatrix}.$$

Hence

$$\begin{aligned} &\mathrm{s.s.Jac}_{P_4}(\pi_\epsilon(2,4,4)) \\ &\quad = \left(|\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{1}{2}}\mathrm{St}_3\right) \otimes |\cdot|^{\frac{3}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ &\quad + \left(|\cdot|^{1}\mathrm{St}_2 \times |\det{}_2|^{1}\right) \otimes \left(|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \epsilon\cdot\pi_+(2)\right) \\ &\quad + \mathrm{St}_4 \otimes \pi_+(2) \\ &\quad + \left(|\cdot|^{\frac{1}{2}}\mathrm{St}_3 \times |\cdot|^{\frac{3}{2}}\right) \otimes (1+\epsilon)\cdot\pi_+(2) \\ &\quad + \left(|\cdot|^{1}\mathrm{St}_2 \times |\cdot|^{1}\mathrm{St}_2\right) \otimes \left((1+\epsilon)|\cdot|^{\frac{1}{2}} \rtimes \mathbf{1}_{\mathrm{SO}_1(F)} + \frac{1+\epsilon}{2}\cdot\pi_+(2)\right). \end{aligned}$$

(5) When $m = 5$, we have $K_\phi^{(5)} = \{(2,3,0) > (2,2,1) > (1,4,0) > (1,3,1) > (1,2,2) > (0,4,1) > (0,3,2)\}$. Since

$$\begin{pmatrix} \Delta_{\underline{x}(2,3,0)} \\ \Delta_{\underline{x}(2,2,1)} \\ \Delta_{\underline{x}(1,4,0)} \\ \Delta_{\underline{x}(1,3,1)} \\ \Delta_{\underline{x}(1,2,2)} \\ \Delta_{\underline{x}(0,4,1)} \\ \Delta_{\underline{x}(0,3,2)} \end{pmatrix} = \begin{pmatrix} \mathrm{St}_2 \times |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \\ \mathrm{St}_2 \times |\cdot|^{1}\mathrm{St}_2 \times |\cdot|^{\frac{3}{2}} \\ |\cdot|^{\frac{1}{2}} \times \mathrm{St}_4 \\ |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \times |\cdot|^{\frac{3}{2}} \\ |\cdot|^{\frac{1}{2}} \times |\cdot|^{1}\mathrm{St}_2 \times |\cdot|^{1}\mathrm{St}_2 \\ \mathrm{St}_4 \times |\cdot|^{\frac{3}{2}} \\ |\cdot|^{\frac{1}{2}}\mathrm{St}_3 \times |\cdot|^{1}\mathrm{St}_2 \end{pmatrix},$$

we have

$$(m_{\underline{k},\underline{k}'})^{-1}_{\underline{k},\underline{k}'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} \Pi^{\epsilon}_{\underline{x}(2,3,0)} \\ \Pi^{\epsilon}_{\underline{x}(2,2,1)} \\ \Pi^{\epsilon}_{\underline{x}(1,4,0)} \\ \Pi^{\epsilon}_{\underline{x}(1,3,1)} \\ \Pi^{\epsilon}_{\underline{x}(1,2,2)} \\ \Pi^{\epsilon}_{\underline{x}(0,4,1)} \\ \Pi^{\epsilon}_{\underline{x}(0,3,2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_{\mathrm{SO}_1(F)} \\ \mathbf{1}_{\mathrm{SO}_1(F)} \\ (1+\epsilon)\cdot \mathbf{1}_{\mathrm{SO}_1(F)} \\ 0 \\ (1+\epsilon)\cdot \mathbf{1}_{\mathrm{SO}_1(F)} \end{pmatrix}.$$

Hence

$$\begin{aligned} \mathrm{s.s.Jac}_{P_5}(\pi_\epsilon(2,4,4)) &= \left(|\cdot|^{\frac{1}{2}} \times \mathrm{St}_4\right) \otimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ &+ \left(|\cdot|^{\frac{1}{2}}\mathrm{St}_3 \times |\det_2|^1\right) \otimes \mathbf{1}_{\mathrm{SO}_1(F)} \\ &+ \left(|\cdot|^{\frac{1}{2}} \times |\cdot|^1\mathrm{St}_2 \times |\cdot|^1\mathrm{St}_2\right) \otimes \frac{1+\epsilon}{2}\cdot \mathbf{1}_{\mathrm{SO}_1(F)} \\ &+ \left(|\cdot|^{\frac{1}{2}}\mathrm{St}_3 \times |\cdot|^1\mathrm{St}_2\right) \otimes (1+\epsilon)\cdot \mathbf{1}_{\mathrm{SO}_1(F)}. \end{aligned}$$

Remark 4.7 In Theorem 4.3, one can replace $\Delta_{\underline{x}(\underline{k})}$ with its unique irreducible subrepresentation $\tau_{\underline{x}(\underline{k})}$. Then one should consider the matrix $(M_{\underline{k},\underline{l}}) = (\dim_{\mathbb{C}} \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{k})}}(\tau_{\underline{x}(\underline{l})}))$. One might seem that $(M_{\underline{k},\underline{l}})$ is easier than $(m_{\underline{k},\underline{l}})$. For instance, if ϕ is in Example 4.5, all $(M_{\underline{k},\underline{l}})$ are the identity matrix, but not so is some $(m_{\underline{k},\underline{l}})$. However, in general, $(M_{\underline{k},\underline{l}})$ is not always diagonal. In Example 4.6 (3) and (4), such non-diagonal $(M_{\underline{k},\underline{l}})$ would appear.

5 Proof of the main theorems

In this section, we prove the main theorems (Theorems 4.2 and 4.3).

5.1 The case of higher multiplicity

We prove Theorem 4.2 (1) in this subsection. It immediately follows from Tadić's formula (Corollary 2.9).

Proof of Theorem 4.2 (1) We prove the assertion by induction on m. Since

$$\pi(\phi, \eta) = \mathrm{St}(\rho, 2x+1) \rtimes \pi(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta)$$

with $\mathrm{St}(\rho, 2x+1) = \langle \rho; x, \ldots, -x \rangle$, by Corollary 2.9, $\text{s.s.Jac}_{P_d}(\pi(\phi, \eta))$ is equal to

$$\begin{aligned}&(\rho^\vee|\cdot|^x \otimes \langle \rho; x, \ldots, -(x-1)\rangle) \rtimes (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes \pi(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta))\\ &+ (\rho|\cdot|^x \otimes \langle \rho; x-1, \ldots, -x\rangle) \rtimes (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes \pi(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta))\\ &+ (\mathbf{1}_{\mathrm{GL}_0(F)} \otimes \mathrm{St}(\rho, 2x+1)) \rtimes \text{s.s.Jac}_{P_d}(\pi(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta)).\end{aligned}$$

Note that $\text{s.s.}(\langle \rho; x-1, \ldots, -x\rangle \rtimes \pi_0) \cong \text{s.s.}(\langle \rho^\vee; x, \ldots, -(x-1)\rangle \rtimes \pi_0)$ for any representation π_0. Since $\rho^\vee \cong \rho$, we have

$$\begin{aligned}\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta)) = 2 \cdot \langle \rho; x, \ldots, -(x-1)\rangle \rtimes \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta\right)\\ + \mathrm{St}(\rho, 2x+1) \rtimes \mathrm{Jac}_{\rho|\cdot|^x}\left(\pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta\right)\right).\end{aligned}$$

This proves the assertion when $m = 3$ or $m = 4$. When $m \geq 5$, since

$$\begin{aligned}&\mathrm{St}(\rho, 2x+1) \times \langle \rho; x, \ldots, -(x-1)\rangle \rtimes \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 4}, \eta\right)\\ &\cong \langle \rho; x, \ldots, -(x-1)\rangle \times \mathrm{St}(\rho, 2x+1) \rtimes \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 4}, \eta\right)\\ &\cong \langle \rho; x, \ldots, -(x-1)\rangle \rtimes \pi\left(\phi - (\rho \boxtimes S_{2x+1})^{\oplus 2}, \eta\right),\end{aligned}$$

we obtain the assertion by the induction hypothesis. □

5.2 The case of multiplicity one

Next, we prove Theorem 4.2 (2). Let $\phi = \phi_0 \oplus (\rho \boxtimes S_{2x+1})$ with $\rho \boxtimes S_{2x+1} \not\subset \phi_0$, and $\eta \in \widehat{\mathcal{A}_\phi}$. Set

$$\phi' = \phi - (\rho \boxtimes S_{2x+1}) \oplus (\rho \boxtimes S_{2x-1}).$$

Proof of Theorem 4.2 (2) First, we assume that $x > 1/2$ and $\pi(\phi', \eta') \neq 0$. We apply Mœglin's construction to $\Pi_{\phi'}$. Write

$$\phi' = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i}\right) \oplus \phi'_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_a \not\subset \phi'_e$ for any $a > 0$. Set

$$t_0 = \max\{i \in \{1, \dots, t\} \mid a_i = 2x - 1\}.$$

Take a new L-parameter

$$\phi'_{\gg} = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a'_i} \right) \oplus \phi'_e$$

such that

- $a'_1 < \cdots < a'_t$;
- $a'_i \geq a_i$ and $a'_i \equiv a_i \bmod 2$ for any i;
- $a'_{t_0} \geq 2x + 1$.

We can identify $\mathcal{A}_{\phi'_{\gg}}$ with $\mathcal{A}_{\phi'}$ canonically. Let $\eta'_{\gg} \in \widehat{\mathcal{A}_{\phi'_{\gg}}}$ be the character corresponding to $\eta' \in \widehat{\mathcal{A}_{\phi'}}$. Then Theorem 3.5 says that

$$\pi(\phi', \eta') = \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_t - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_t + 1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_1 - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_1 + 1}{2}}} (\pi(\phi'_{\gg}, \eta'_{\gg})).$$

When $i = t_0$, we note that

$$\mathrm{Jac}_{\rho|\cdot|^{\frac{a'_{t_0} - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_{t_0} + 1}{2}}} = \mathrm{Jac}_{\rho|\cdot|^x} \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_{t_0} - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_{t_0} + 3}{2}}}.$$

Since ϕ' does not contain $\rho \boxtimes S_{2x+1}$, for $i > t_0$ and $a_i < 2x' + 1 \leq a'_i$ with $2x' + 1 \equiv a_i \bmod 2$, we have $x' - x > 1$. By Lemma 2.6 (2), we see that $\pi(\phi', \eta')$ is the image of

$$\mathrm{Jac}_{\rho|\cdot|^{\frac{a'_t - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_t + 1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_{t_0} - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_{t_0} + 3}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a'_1 - 1}{2}}, \dots, \rho|\cdot|^{\frac{a_1 + 1}{2}}} (\pi(\phi'_{\gg}, \eta'_{\gg}))$$

under $\mathrm{Jac}_{\rho|\cdot|^x}$. However, by applying Theorem 3.5 again, we see that this representation is isomorphic to $\pi(\phi, \eta)$. Therefore $\pi(\phi', \eta') = \mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta))$, as desired.

Next, we consider the case where $x = 1/2$ and $\pi'(\phi', \eta') \neq 0$. We reduce this case to the case where ϕ is discrete ([16, Lemma 7.3]) using Theorem 3.5 repeatedly. The argument is similar to the first case so that we omit the detail.

Finally, we assume that $\pi(\phi', \eta') = 0$. We claim that $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta)) = 0$. When $\phi \in \Phi_{\mathrm{disc}}(G)$, this was proven in [16, Lemma 7.3]. When $\phi \in \Phi_{\mathrm{gp}}(G) \setminus \Phi_{\mathrm{disc}}(G)$, there exists an irreducible representation ϕ_1 which is contained in ϕ with multiplicity at least two. Then $\pi(\phi, \eta)$ is a subrepresentation of $\tau_1 \rtimes \pi(\phi - \phi_1^{\oplus 2}, \eta)$, where τ_1 is the irreducible tempered representation of $\mathrm{GL}_k(F)$

corresponding to ϕ_1. Since $\rho \boxtimes S_{2x+1}$ is contained in ϕ with multiplicity one, we have $\phi_1 \not\cong \rho \boxtimes S_{2x+1}$. This implies that

$$\mathrm{Jac}_{\rho|\cdot|^x}(\tau_1 \rtimes \pi(\phi - \phi_1^{\oplus 2}, \eta)) = \tau_1 \rtimes \mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi - \phi_1^{\oplus 2}, \eta)).$$

By the induction hypothesis, $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi - \phi_1^{\oplus 2}, \eta)) = 0$ unless $\phi_1 = \rho \boxtimes S_{2x-1}$ and ϕ contains it with multiplicity exactly two. In this case, one can take $\eta_- \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta_-) \neq 0$ and

$$\pi(\phi, \eta) \oplus \pi(\phi, \eta_-) = \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi - \phi_1^{\oplus 2}, \eta).$$

Then by the first case, we see that $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta_-)) \neq 0$ and $\mathrm{St}(\rho, 2x-1) \rtimes \mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi - \phi_1^{\oplus 2}, \eta))$ is irreducible. Hence $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta))$ must be zero. This completes the proof of Theorem 4.2 (2). □

By the same argument as the last part, one can prove that $\mathrm{Jac}_\rho(\pi(\phi, \eta)) = 0$ when $x = 0$ and $m = 1$.

5.3 Description of small standard modules

Before proving Theorem 4.2 (3), we describe the structures of some standard modules.

Lemma 5.1 *Let $\phi \in \Phi_{\mathrm{disc}}(G)$. Suppose that $x > 0$, and $\phi \supset \rho \boxtimes S_{2x-1}$ but $\phi \not\supset \rho \boxtimes S_{2x+1}$. Let $\eta \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta) \neq 0$. We set*

- *$\Pi = \rho|\cdot|^x \rtimes \pi(\phi, \eta)$ to be a standard module;*
- *σ to be the unique irreducible quotient of Π;*
- *$\phi' = \phi - (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$ so that there is a canonical injection $\mathcal{A}_\phi \hookrightarrow \mathcal{A}_{\phi'}$, which is bijection unless $x = 1/2$;*
- *$\eta' \in \widehat{\mathcal{A}_{\phi'}}$ to be the character satisfying $\eta'|\mathcal{A}_\phi = \eta$ (and $\pi(\phi', \eta') \neq 0$ if $x = 1/2$).*

Then there exists an exact sequence

$$0 \longrightarrow \pi(\phi', \eta') \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0.$$

In particular, Π has length two.

Proof We note that $\pi(\phi', \eta')$ is an irreducible subrepresentation of Π by Theorem 4.2 (2) and Lemma 2.6 (1).

If σ' is an irreducible subquotient of Π which is non-tempered, by Tadić's formula and Casselman's criterion, there exists a maximal parabolic subgroup P_k of G such that $\mathrm{s.s.Jac}_{P_k}(\sigma')$ contains an irreducible representation of the form $(\rho|\cdot|^{-x} \times \tau) \boxtimes \sigma_0$. In particular, we have $\mathrm{Jac}_{\rho|\cdot|^{-x}}(\sigma') \neq 0$. However, since $\mathrm{Jac}_{\rho|\cdot|^{-x}}(\Pi) = \pi(\phi, \eta)$ is irreducible, we see that $\sigma' = \sigma$, i.e., Π has only one irreducible non-tempered subquotient.

Let Π^{sub} be the maximal proper subrepresentation of Π, i.e., $\Pi/\Pi^{\mathrm{sub}} \cong \sigma$. By the above argument, all irreducible subquotients of Π^{sub} must be tempered. Since they have the same cuspidal support, they share the same Plancherel measure. This implies that all irreducible subquotients of Π^{sub} belong to the same L-packet $\Pi_{\phi'}$ (see [6, Lemma A.6]), so that they are all discrete series. Hence Π^{sub} is semisimple. In particular, any irreducible subquotient π' of Π^{sub} is a subrepresentation of Π, so that $\mathrm{Jac}_{\rho|\cdot|^{x}}(\pi') \neq 0$. However, since $\mathrm{Jac}_{\rho|\cdot|^{x}}(\Pi) = \pi(\phi, \eta)$ is irreducible, Π has only one irreducible subrepresentation. Therefore $\Pi^{\mathrm{sub}} = \pi(\phi', \eta')$. This completes the proof. □

We describe the standard module appearing in Theorem 4.2 (3). When $x = 1/2$, the standard module was described in Lemma 5.1. Hence we assume $x > 1/2$.

Proposition 5.2 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$. Suppose that $x > 1/2$ and $\phi \not\supset \rho \boxtimes S_{2x+1}$. Let $\eta \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta) \neq 0$. We set*

- *$\Pi = \langle \rho; x, x-1, \dots, -(x-1) \rangle \rtimes \pi(\phi, \eta)$ to be a standard module;*
- *σ to be the unique irreducible quotient of Π;*
- *$\phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$;*
- *η'_+ and η'_- to be the two distinct characters of $\widehat{\mathcal{A}_{\phi'}}$ such that $\eta'_\pm | \mathcal{A}_\phi = \eta$ and $\eta'_\pm(z_{\phi'}) = 1$.*

Then there exists an exact sequence

$$0 \longrightarrow \pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-) \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0.$$

In particular, Π has length 2 or 3 according to $\phi \supset \rho \boxtimes S_{2x-1}$ or not.

Proof First, we show that there is an inclusion $\pi(\phi', \eta'_\epsilon) \hookrightarrow \Pi$ for each $\epsilon \in \{\pm\}$. To do this, we may assume that $\pi(\phi', \eta'_\epsilon) \neq 0$. Note that ϕ' contains $\rho \boxtimes S_{2x+1}$ with multiplicity one. By Theorem 4.2 (2), we see that $\mathrm{Jac}_{\rho|\cdot|^{x}}(\pi(\phi', \eta'_\epsilon))$ is nonzero and is an irreducible subrepresentation of $\mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi, \eta)$. By Lemma 2.6 (1), we have an inclusion

$$\pi(\phi', \eta'_\epsilon) \hookrightarrow \rho|\cdot|^{x} \times \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi, \eta).$$

Since Π is a subrepresentation of $\rho|\cdot|^{x} \times \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi, \eta)$ such that

$$\mathrm{Jac}_{\rho|\cdot|^{x}} \left(\rho|\cdot|^{x} \times \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi, \eta) \right) = \mathrm{Jac}_{\rho|\cdot|^{x}}(\Pi),$$

the above inclusion factors through $\pi(\phi', \eta'_\epsilon) \hookrightarrow \Pi$.

If $\text{s.s.Jac}_{P_k}(\Pi)$ contains an irreducible representation $\tau \boxtimes \pi_0$ such that the central character of τ is of the form $\chi|\cdot|^s$ with χ unitary and $s < 0$, by Tadić's formula (Theorem 2.8) and Casselman's criterion, $\tau = |\cdot|^{-\frac{1}{2}}\text{St}(\rho, 2x) \times (\times_{i=1}^r \tau_i)$, where τ_i is a discrete series representation of $\text{GL}_{k_i}(F)$ such that the corresponding irreducible representation ϕ_i of $W_F \times \text{SL}_2(\mathbb{C})$ is contained in ϕ with multiplicity at least two, and $\pi_0 = \pi(\phi_0, \eta_0)$ with $\phi_0 = \phi - (\oplus_{i=1}^r \phi_i)^{\oplus 2}$ and $\eta_0 = \eta|\mathcal{A}_{\phi_0}$. Since such an irreducible representation $\tau \boxtimes \pi_0$ is also contained in $\text{s.s.Jac}_{P_k}(\sigma)$, we see that σ is the unique irreducible non-tempered subquotient of Π. Namely, if we let Π^{sub} be the maximal proper subrepresentation of Π, i.e., $\Pi/\Pi^{\text{sub}} \cong \sigma$, then all irreducible subquotients of Π^{sub} must be tempered. Moreover since these irreducible subquotients have the same cuspidal support so that they share the same Plancherel measure, they belong to the same L-packet $\Pi_{\phi'}$ (see [6, Lemma A.6]).

Now we show that Π^{sub} is isomorphic to $\pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-)$. We separate the cases as follows:

- ϕ is discrete and $\rho \boxtimes S_{2x-1} \not\subset \phi$;
- ϕ is discrete and $\rho \boxtimes S_{2x-1} \subset \phi$;
- ϕ is in general.

When ϕ is discrete and $\rho \boxtimes S_{2x-1} \not\subset \phi$, we note that $\phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$ is also discrete. Then since all irreducible subquotients of Π^{sub} are discrete series, Π^{sub} is semisimple. In particular, any irreducible subquotient π' of Π^{sub} is a subrepresentation of Π so that $\text{Jac}_{\rho|\cdot|^x}(\pi') \neq 0$. However, since $\text{Jac}_{\rho|\cdot|^x}(\Pi) = \text{Jac}_{\rho|\cdot|^x}(\pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-))$, we have $\Pi^{\text{sub}} \cong \pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-)$.

When ϕ is discrete and $\rho \boxtimes S_{2x-1} \subset \phi$, any irreducible subquotient π' of Π^{sub} belongs to $\Pi_{\phi'}$ with $\phi' = \phi'_0 \oplus (\rho \boxtimes S_{2x-1})^{\oplus 2}$, where $\phi'_0 = \phi - (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$ is discrete such that $\rho \boxtimes S_{2x-1} \not\subset \phi'_0$. Hence the Jacquet module $\text{s.s.Jac}_{P_{d(2x-1)}}(\pi')$ contains an irreducible representation of the form $\text{St}(\rho, 2x-1) \otimes \pi'_0$. By Tadić's formula (Corollary 2.9), the sum of irreducible representations of this form appearing in $\text{s.s.Jac}_{P_{d(2x-1)}}(\Pi)$ is

$$\text{St}(\rho, 2x-1) \otimes \text{s.s.}(\rho|\cdot|^x \rtimes \pi(\phi, \eta)).$$

By Lemma 5.1, we have an exact sequence

$$0 \longrightarrow \pi(\phi'_0, \eta'_0) \longrightarrow \rho|\cdot|^x \rtimes \pi(\phi, \eta) \longrightarrow \sigma' \longrightarrow 0,$$

where σ' is the unique irreducible quotient of $\rho|\cdot|^x \rtimes \pi(\phi, \eta)$, and $\eta'_0 \in \widehat{\mathcal{A}_{\phi'_0}}$ is the character corresponding to $\eta \in \widehat{\mathcal{A}_\phi}$ via the identification $\mathcal{A}_\phi = \mathcal{A}_{\phi'_0}$. Now there exists $\epsilon \in \{\pm\}$ such that $\pi(\phi', \eta'_{-\epsilon}) = 0$. Moreover, $\text{s.s.Jac}_{P_{d(2x-1)}}(\pi(\phi', \eta'_\epsilon)) \supset \text{St}(\rho, 2x-1) \otimes \pi(\phi'_0, \eta'_0)$ since

$$\eta'_\epsilon(\alpha_{\rho\boxtimes S_{2x+1}}) = \eta'_\epsilon(\alpha_{\rho\boxtimes S_{2x-1}}) = \eta(\alpha_{\rho\boxtimes S_{2x-1}}) = \eta'_0(\alpha_{\rho\boxtimes S_{2x+1}}).$$

On the other hand, since $\sigma \hookrightarrow \mathrm{St}(\rho, 2x-1) \times \rho|\cdot|^{-x} \rtimes \pi(\phi, \eta)$, we see that $\mathrm{s.s.Jac}_{P_{d(2x-1)}}(\sigma)$ is nonzero and contains $\mathrm{St}(\rho, 2x-1) \otimes \sigma'$. Hence

$$\mathrm{s.s.Jac}_{P_{d(2x-1)}}(\Pi) - \mathrm{s.s.Jac}_{P_{d(2x-1)}}(\pi(\phi', \eta'_\epsilon)) - \mathrm{s.s.Jac}_{P_{d(2x-1)}}(\sigma)$$

has no irreducible representation of the form $\mathrm{St}(\rho, 2x-1) \otimes \pi'_0$. This shows that $\Pi^{\mathrm{sub}} = \pi(\phi', \eta'_\epsilon)$.

In general, we prove the claim by induction on the dimension of ϕ. When ϕ is not discrete, there exists an irreducible representation ϕ_1 of $W_F \times \mathrm{SL}_2(\mathbb{C})$ which ϕ contains with multiplicity at least two. Note that $\phi_1 \not\cong \rho \boxtimes S_{2x+1}$. Set $\phi_0 = \phi - \phi_1^{\oplus 2}$, and $\eta_0 = \eta|\mathcal{A}_{\phi_0}$. Take Π_0, σ_0, ϕ'_0 and $\eta'_{0,\epsilon} \in \widehat{\mathcal{A}_{\phi'_0}}$ as in the statement of the proposition. By induction hypothesis, we have an exact sequence

$$0 \longrightarrow \pi(\phi'_0, \eta'_{0,+}) \oplus \pi(\phi'_0, \eta'_{0,-}) \longrightarrow \Pi_0 \longrightarrow \sigma_0 \longrightarrow 0.$$

Let τ be the irreducible discrete series representation of $\mathrm{GL}_k(F)$ corresponding to ϕ_1. The above exact sequence remains exact after taking the parabolic induction functor $\pi_0 \mapsto \tau \rtimes \pi_0$. Note that $\tau \times \langle \rho; x, x-1, \dots, -(x-1)\rangle \cong \langle \rho; x, x-1, \dots, -(x-1)\rangle \times \tau$ by Theorem 2.1. Since σ_0 is unitary, the parabolic induction $\tau \rtimes \sigma_0$ is semisimple. In particular, any irreducible subquotient of $\tau \rtimes \sigma_0$ is non-tempered. Considering the cases where

- ϕ contains ϕ_1 with multiplicity more than two;
- ϕ contains ϕ_1 with multiplicity exactly two and $\phi_1 \not\cong \rho \boxtimes S_{2x-1}$;
- ϕ contains ϕ_1 with multiplicity exactly two and $\phi_1 \cong \rho \boxtimes S_{2x-1}$

separately, we see that $\Pi^{\mathrm{sub}} \cong \pi(\phi', \eta'_+) \oplus \pi(\phi', \eta'_-)$ in all cases. This completes the proof. □

5.4 The case of multiplicity two

Finally, we prove Theorem 4.2 (3).

Lemma 5.3 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$, $\eta \in \widehat{\mathcal{A}_\phi}$ and $x > 0$. Suppose that ϕ contains both $\rho \boxtimes S_{2x+1}$ and $\rho \boxtimes S_{2x+3}$ with multiplicity one. Then we have*

$$\mathrm{Jac}_{\rho|\cdot|^{x+1},\rho|\cdot|^{x},\rho|\cdot|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \mathrm{Jac}_{\rho|\cdot|^{x},\rho|\cdot|^{x+1},\rho|\cdot|^{x}}(\pi(\phi, \eta)).$$

Proof We may assume that $\mathrm{Jac}_{\rho|\cdot|^{x+1},\rho|\cdot|^{x},\rho|\cdot|^{x}}(\pi(\phi, \eta)) \neq 0$. By Lemma 2.6 (1), there exists an irreducible subquotient σ of this Jacquet module such that

$$\pi(\phi, \eta) \hookrightarrow \rho|\cdot|^{x+1} \times \rho|\cdot|^{x} \times \rho|\cdot|^{x} \rtimes \sigma.$$

Since there exists an exact sequence

$$0 \longrightarrow \langle \rho; x+1, x\rangle \longrightarrow \rho|\cdot|^{x+1} \times \rho|\cdot|^{x} \longrightarrow \langle \rho; x, x+1\rangle \longrightarrow 0,$$

where $\langle \rho; x, x+1\rangle$ is the unique irreducible subrepresentation of $\rho|\cdot|^{x} \times \rho|\cdot|^{x+1}$, we see that $\pi(\phi, \eta)$ is a subrepresentation of $\langle \rho; x+1, x\rangle \times \rho|\cdot|^{x} \rtimes \sigma$ or $\langle \rho; x, x+1\rangle \times \rho|\cdot|^{x} \rtimes \sigma$. Since $\langle \rho; x+1, x\rangle \times \rho|\cdot|^{x} \cong \rho|\cdot|^{x} \times \langle \rho; x+1, x\rangle$ and $\langle \rho; x, x+1\rangle \times \rho|\cdot|^{x} \cong \rho|\cdot|^{x} \times \langle \rho; x, x+1\rangle$, we have $\mathrm{Jac}_{\rho|\cdot|^{x}}(\pi(\phi, \eta)) \neq 0$.

By Theorem 4.2 (2), $\mathrm{Jac}_{\rho|\cdot|^{x+1}}(\pi(\phi, \eta)) \neq 0$ and $\mathrm{Jac}_{\rho|\cdot|^{x}}(\pi(\phi, \eta)) \neq 0$ imply that $\sigma' = \mathrm{Jac}_{\rho|\cdot|^{x}, \rho|\cdot|^{x+1}, \rho|\cdot|^{x}}(\pi(\phi, \eta))$ is nonzero and irreducible. Moreover, we have $\mathrm{Jac}_{\rho|\cdot|^{x}}(\sigma') = 0$ and $\mathrm{Jac}_{\rho|\cdot|^{x+1}}(\sigma') = 0$. By Lemma 2.6 (1), we have an inclusion

$$\pi(\phi, \eta) \hookrightarrow \rho|\cdot|^{x} \times \rho|\cdot|^{x+1} \times \rho|\cdot|^{x} \rtimes \sigma'.$$

Since

$$\mathrm{Jac}_{\rho|\cdot|^{x+1}, \rho|\cdot|^{x}, \rho|\cdot|^{x}}(\rho|\cdot|^{x} \times \rho|\cdot|^{x+1} \times \rho|\cdot|^{x} \rtimes \sigma') = 2 \cdot \sigma',$$

we have $\mathrm{Jac}_{\rho|\cdot|^{x+1}, \rho|\cdot|^{x}, \rho|\cdot|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \sigma'$, as desired. □

Suppose that $x > 0$. Let $\phi = \phi_0 \oplus (\rho \boxtimes S_{2x+1})^{\oplus 2}$ with $\rho \boxtimes S_{2x+1} \not\subset \phi_0$, and $\eta \in \widehat{\mathcal{A}_\phi}$.

Lemma 5.4 *Set $\phi_1 = \phi - (\rho \boxtimes S_{2x+1})^{\oplus 2} \oplus (\rho \boxtimes S_{2x-1})^{\oplus 2}$ so that there is a canonical injection $\mathcal{A}_{\phi_1} \hookrightarrow \mathcal{A}_\phi$, which is bijective unless $x = 1/2$. Define $\eta_1 \in \widehat{\mathcal{A}_{\phi_1}}$ by $\eta_1 = \eta|\mathcal{A}_{\phi_1}$. Then we have*

$$\mathrm{Jac}_{\rho|\cdot|^{x}, \rho|\cdot|^{x}}(\pi(\phi, \eta)) = 2 \cdot \pi(\phi_1, \eta_1).$$

Proof Let $\eta_+ = \eta$ and $\eta_- \in \widehat{\mathcal{A}_\phi}$ be as in the statement of Theorem 4.2 (3). Define $\eta_{1,\pm} \in \widehat{\mathcal{A}_{\phi_1}}$ by $\eta_{1,\pm} = \eta_\pm|\mathcal{A}_\phi$ if $x > 1/2$. When $x = 1/2$, we set $\eta_{1,+} = \eta|\mathcal{A}_{\phi_1}$ and $\pi(\phi_1, \eta_{1,-}) = 0$ formally. Then we have

- $\pi(\phi, \eta_+) \oplus \pi(\phi, \eta_-) = \mathrm{St}(\rho, 2x+1) \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0})$;
- $\pi(\phi_1, \eta_{1,+}) \oplus \pi(\phi_1, \eta_{1,-}) = \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0})$;
- $\mathrm{Jac}_{\rho|\cdot|^{x}, \rho|\cdot|^{x}}(\mathrm{St}(\rho, 2x+1) \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0})) \cong 2 \cdot \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0})$.

Therefore, it is enough to show that $\mathrm{Jac}_{\rho|\cdot|^{x}, \rho|\cdot|^{x}}(\pi(\phi, \eta)) \subset 2 \cdot \pi(\phi_1, \eta_1)$.

We apply Mœglin's construction to Π_ϕ. Write

$$\phi = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i} \right) \oplus \phi_e$$

with $a_1 \leq \cdots \leq a_t$ and $\rho \boxtimes S_a \not\subset \phi_e$ for any $a > 0$. There exists $t_0 > 1$ such that $a_{t_0-1} = a_{t_0} = 2x+1$. Take a new L-parameter

$$\phi_{\gg} = \left(\bigoplus_{i=1}^{t} \rho \boxtimes S_{a_i'}\right) \oplus \phi_e$$

such that

- $a_1' < \cdots < a_t'$;
- $a_i' \geq a_i$ and $a_i' \equiv a_i \bmod 2$ for any i.

In particular, $a_{t_0}' \geq 2x+3$. We can identify $\mathcal{A}_{\phi_{\gg}}$ with $\mathcal{A}_{\phi}$ canonically. Let $\eta_{\gg} \in \widehat{\mathcal{A}_{\phi_{\gg}}}$ be the character corresponding to $\eta \in \widehat{\mathcal{A}_{\phi}}$. Then Theorem 3.5 says that

$$\pi(\phi,\eta) = \mathrm{Jac}_{\rho|\cdot|^{\frac{a_t'-1}{2}},\dots,\rho|\cdot|^{\frac{a_t+1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_1'-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}(\pi(\phi_{\gg},\eta_{\gg})).$$

Note that $(a_{t_0}+1)/2 = x+1$. By Lemma 2.6 (2), we see that

$$\begin{aligned}
&\mathrm{Jac}_{\rho|\cdot|^{x},\rho|\cdot|^{x}}(\pi(\phi,\eta))\\
&= \mathrm{Jac}_{\rho|\cdot|^{\frac{a_t'-1}{2}},\dots,\rho|\cdot|^{\frac{a_t+1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0+1}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0+1}+1}{2}}}\\
&\circ \mathrm{Jac}_{\rho|\cdot|^{x+1},\rho|\cdot|^{x},\rho|\cdot|^{x}}\left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0}+3}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_1'-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}(\pi(\phi_{\gg},\eta_{\gg}))\right).
\end{aligned}$$

By Lemma 5.3, we have

$$\begin{aligned}
&\mathrm{Jac}_{\rho|\cdot|^{x+1},\rho|\cdot|^{x},\rho|\cdot|^{x}}\left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0}+3}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_1'-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}(\pi(\phi_{\gg},\eta_{\gg}))\right)\\
&\subset 2\cdot \mathrm{Jac}_{\rho|\cdot|^{x},\rho|\cdot|^{x+1},\rho|\cdot|^{x}}\left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0}+3}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_1'-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}(\pi(\phi_{\gg},\eta_{\gg}))\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{Jac}_{\rho|\cdot|^{\frac{a_t'-1}{2}},\dots,\rho|\cdot|^{\frac{a_t+1}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0+1}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0+1}+1}{2}}}\\
&\circ \mathrm{Jac}_{\rho|\cdot|^{x},\rho|\cdot|^{x+1},\rho|\cdot|^{x}}\left(\mathrm{Jac}_{\rho|\cdot|^{\frac{a_{t_0}'-1}{2}},\dots,\rho|\cdot|^{\frac{a_{t_0}+3}{2}}} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^{\frac{a_1'-1}{2}},\dots,\rho|\cdot|^{\frac{a_1+1}{2}}}(\pi(\phi_{\gg},\eta_{\gg}))\right)\\
&= \pi(\phi_1,\eta_1)
\end{aligned}$$

by Theorem 3.5, we have $\mathrm{Jac}_{\rho|\cdot|^{x},\rho|\cdot|^{x}}(\pi(\phi,\eta)) \subset 2\cdot\pi(\phi_1,\eta_1)$, as desired. □

Now we can prove Theorem 4.2 (3).

Proof of Theorem 4.2 (3) By Corollary 2.9, we have

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_+)\oplus\pi(\phi,\eta_-)) = 2\cdot\langle\rho; x, x-1,\dots,-(x-1)\rangle \rtimes \pi(\phi_0, \eta|\mathcal{A}_{\phi_0}).$$

By Proposition 5.2, we have an exact sequence

$$0 \longrightarrow \pi(\phi',\eta'_+)\oplus\pi(\phi',\eta'_-) \longrightarrow \Pi \longrightarrow \sigma \longrightarrow 0,$$

where $\Pi = \langle\rho; x, x-1,\dots,-(x-1)\rangle \rtimes \pi(\phi_0,\eta|\mathcal{A}_{\phi_0})$, and σ is the unique irreducible quotient of Π. Fix $\epsilon \in \{\pm\}$. By Lemma 5.4, we see that $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_\epsilon)) \supset 2\cdot\pi(\phi',\eta'_\epsilon)$. On the other hand, since $\mathrm{s.s.Jac}_{P_{d(2x+1)}}(\pi(\phi,\eta_\epsilon)) \supset \mathrm{St}(\rho, 2x+1)\otimes\pi(\phi_0,\eta|\mathcal{A}_{\phi_0})$, we have

$$\mathrm{s.s.Jac}_{P_{2dx}}(\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_\epsilon))) \supset |\cdot|^{-\frac{1}{2}}\mathrm{St}(\rho,2x)\otimes\pi(\phi_0,\eta|\mathcal{A}_{\phi_0}).$$

This implies that $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_\epsilon))$ contains an irreducible non-tempered representation, which must be σ. Hence

$$\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_\epsilon)) \supset 2\cdot\pi(\phi',\eta_\epsilon)+\sigma = \Pi + \pi(\phi',\eta_\epsilon) - \pi(\phi',\eta_{-\epsilon}).$$

Considering $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi,\eta_+)\oplus\pi(\phi,\eta_-))$, we see that this inclusion must be an equality. □

If $x = 0$ and $m = 2$, we see that $\mathrm{Jac}_\rho(\pi(\phi,\eta)) \supset \pi(\phi_0,\eta|\mathcal{A}_{\phi_0})$. By the same argument, this inclusion must be an equality. This completes the proof of Theorem 4.2 (4), so that the ones of all statements of Theorem 4.2.

5.5 Description of μ^*_ρ

We prove Theorem 4.3 in this subsection. To do this, we need the following specious lemma.

Lemma 5.5 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\underline{x} = (x_1,\dots,x_m) \in \mathbb{R}^m$. Suppose that $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}}}(\pi) \neq 0$ for some $\pi \in \Pi_\phi$.*

(1) *If $x_m < 0$, then $x_i = -x_m$ for some i.*
(2) *Suppose that $\underline{x}$ is of the form*

$$\underline{x} = (\underbrace{x_1^{(1)},\dots,x_{m_1}^{(1)}}_{m_1},\dots,\underbrace{x_1^{(k)},\dots,x_{m_k}^{(k)}}_{m_k})$$

with $x^{(j)}_{i-1} > x^{(j)}_i$ *for* $1 \leq j \leq k$, $1 < i \leq m_j$, *and* $x^{(1)}_1 \leq \cdots \leq x^{(k)}_1$. *Then* $x^{(j)}_1 \geq 0$ *for* $j = 1, \dots, k$, *and*

$$\phi \supset \bigoplus_{j=1}^{k} \rho \boxtimes S_{2x^{(j)}_1+1}.$$

Proof We prove the lemma by induction on m. By Lemma 4.1, we see that $2x_1+1$ is a positive integer, and ϕ contains $\rho \boxtimes S_{2x_1+1}$. In particular, we obtain the lemma for $m = 1$.

Suppose that $m \geq 2$ and put $\underline{x}' = (x_2, \dots, x_m) \in \mathbb{R}^{m-1}$. By Theorem 4.2, one of the following holds.

- $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}'}}(\pi') \neq 0$ for some $\pi' \in \Pi_{\phi'}$ with $\phi' = \phi - (\rho \boxtimes S_{2x_1+1}) \oplus (\rho \boxtimes S_{2x_1-1})$;
- $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}'}}(\langle \rho; x_1, x_1-1, \dots, -(x_1-1)\rangle \rtimes \pi_0) \neq 0$ for some $\pi_0 \in \Pi_{\phi_0}$ with $\phi_0 = \phi - (\rho \boxtimes S_{2x_1+1})^{\oplus 2}$.

The former case can occur only if $x_1 > 0$, and the latter case can occur only if $\phi \supset (\rho \boxtimes S_{2x_1+1})^{\oplus 2}$.

We consider the former case. Assume that $x_1 > 0$ and $\mathrm{Jac}_{\rho|\cdot|^{\underline{x}'}}(\pi') \neq 0$ for some $\pi' \in \Pi_{\phi'}$ with $\phi' = \phi - (\rho \boxtimes S_{2x_1+1}) \oplus (\rho \boxtimes S_{2x_1-1})$. By the induction hypothesis, we have $x_i = -x_m$ for some $i \geq 2$ when $x_m < 0$, and

$$\phi' \supset (\rho \boxtimes S_{2x^{(1)}_1-1}) \oplus \left(\bigoplus_{j=2}^{k} \rho \boxtimes S_{2x^{(j)}_1+1} \right)$$

when $\underline{x}$ is of the form in (2) since $\underline{x}'$ is also of the form. This implies the assertion for ϕ.

We consider the latter case. Assume that $\phi \supset (\rho \boxtimes S_{2x_1+1})^{\oplus 2}$, and that

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{x}'}}(\langle \rho; x_1, x_1-1, \dots, -(x_1-1)\rangle \rtimes \pi_0) \neq 0$$

for some $\pi_0 \in \Pi_{\phi_0}$ with $\phi_0 = \phi - (\rho \boxtimes S_{2x_1+1})^{\oplus 2}$. By Corollary 2.9, we can divide

$$\{2, \dots, m\} = \{i_1, \dots, i_{m_1}\} \sqcup \{j_1, \dots, j_{m_2}\} \sqcup \{k_1, \dots, k_{m_3}\}$$

with $i_1 < \cdots < i_{m_1}$, $j_1 < \cdots < j_{m_2}$, $k_1 < \cdots < k_{m_3}$ and $m_2 + m_3 \leq 2x_1$ such that

- $\mathrm{Jac}_{\rho|\cdot|^{y_1}, \dots, \rho|\cdot|^{y_{m_1}}}(\pi_0) \neq 0$ with $y_t = x_{i_t}$;
- $x_{j_t} = x_1 + 1 - t$ for $t = 1, \dots, m_2$;

- $x_{k_t} = x_1 - t$ for $t = 1, \ldots, m_3$.

Considering the following four cases, we can prove the existence x_i satisfying $x_i = -x_m$ when $x_m < 0$.

- When $m = i_{m_1}$, by the induction hypothesis, we have $x_{i_t} = -x_m$ for some t.
- When $m = j_{m_2}$, we have $x_m = x_1 + 1 - m_2 < 0$ so that $x_{j_t} = -x_m$ with $t = 2x_1 + 2 - m_2$.
- When $m = k_{m_3}$ and $m_3 < 2x_1$, we have $x_m = x_1 - m_3 < 0$ so that $x_{k_t} = -x_m$ with $t = 2x_1 - m_3$.
- When $m = k_{m_3}$ and $m_3 = 2x_1$, we have $x_1 = -x_m$.

On the other hand, when $\underline{x}$ is of the form in (2), since $x_1^{(j)} \geq x_1^{(1)} = x_1$, there is at most one $j_0 \geq 2$ such that

$$x_1^{(j_0)} \in \{x_{j_t} \mid t = 1, \ldots, m_2\} \cup \{x_{k_t} \mid t = 1, \ldots, m_3\},$$

in which case, $x_1^{(j_0)} = x_1$. By the induction hypothesis, we have

$$\phi_0 \supset \bigoplus_{\substack{2 \leq j \leq k \\ j \neq j_0}} \rho \boxtimes S_{2x_1^{(j)}+1}.$$

This implies the assertion for ϕ. This completes the proof. □

Now we can prove Theorem 4.3.

Proof of Theorem 4.3 Since the subgroup of $\mathcal{R}_{dm}$ spanned by $\mathrm{Irr}_\rho(\mathrm{GL}_{dm}(F)) = \{\tau_{\underline{x}} \mid \underline{x} \in \Omega_m\}$ has another basis $\{\Delta_{\underline{x}} \mid \underline{x} \in \Omega_m\}$, we can write

$$\mu_\rho^*(\pi) = \sum_{m \geq 0} \sum_{\underline{x} \in \Omega_m} \Delta_{\underline{x}} \otimes \Pi_{\underline{x}}(\pi)$$

for some virtual representation $\Pi_{\underline{x}}(\pi)$. For $\underline{y} \in \Omega_m$, applying $\mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}$ to $\mathrm{s.s.Jac}_{P_{dm}}(\pi)$, we have

$$\mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\pi) = \sum_{\underline{x} \in \Omega_m} \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\Delta_{\underline{x}}) \otimes \Pi_{\underline{x}}(\pi) = \sum_{\underline{x} \in \Omega_m} m(\underline{y}, \underline{x}) \cdot \Pi_{\underline{x}}(\pi),$$

where $m(\underline{y}, \underline{x}) = \dim_{\mathbb{C}} \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\Delta_{\underline{x}})$. If $m'(\underline{x}', \underline{y}) \in \mathbb{Q}$ satisfies that

$$\sum_{\underline{y} \in \Omega_m} m'(\underline{x}', \underline{y}) m(\underline{y}, \underline{x}) = \delta_{\underline{x}', \underline{x}},$$

we have

$$\Pi_{\underline{x}}(\pi) = \sum_{\underline{y} \in \Omega_m} m'(\underline{x}, \underline{y}) \cdot \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\pi).$$

Hence we have

$$\mu_\rho^*(\pi) = \sum_{m \geq 0} \sum_{\underline{x}, \underline{y} \in \Omega_m} m'(\underline{x}, \underline{y}) \cdot \Delta_{\underline{x}} \otimes \mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\pi).$$

By Lemma 5.5, if $\mathrm{Jac}_{\rho|\cdot|^{\underline{y}}}(\pi) \neq 0$ for $\underline{y} \in \Omega_m$, then $\underline{y} = \underline{x}(\underline{k})$ for some $\underline{k} \in K_\phi^{(m)}$. If $m'(\underline{x}, \underline{y}) \neq 0$, then the image of $\underline{x}$ under the canonical map $\mathbb{R}^m \rightarrow \mathbb{R}^m/\mathfrak{S}_m$ coincides with the one of $\underline{y}$ since the same property holds for $m(\underline{x}, \underline{y})$. In particular, for fixed $\underline{k} \in K_\phi^{(m)}$, if $m'(\underline{x}, \underline{x}(\underline{k})) \neq 0$, then $\underline{x} = \underline{x}(\underline{k}')$ for some $\underline{k}' \in K_\phi^{(m)}$. Therefore, we have

$$\mu_\rho^*(\pi) = \sum_{m \geq 0} \sum_{\underline{k}, \underline{k}' \in K_\phi^{(m)}} m'(\underline{x}(\underline{k}'), \underline{x}(\underline{k})) \cdot \Delta_{\underline{x}(\underline{k}')} \otimes \mathrm{Jac}_{\rho|\cdot|^{\underline{x}(\underline{k})}}(\pi).$$

This completes the proof of Theorem 4.3. □

6 Complements

6.1 A remark on standard modules

As a consequence of Theorem 4.2, we can prove the irreducibility of certain standard modules.

Corollary 6.1 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta) \neq 0$. Suppose that $\phi \supset \rho \boxtimes S_{2x+1}$ for $x > 0$ but $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi, \eta)) = 0$. Then the standard module*

$$\Pi = \langle \rho; x, x-1, \ldots, -(x-1) \rangle \rtimes \pi(\phi, \eta)$$

is irreducible.

Proof Let σ be the unique irreducible quotient of Π, which is non-tempered. By the same argument as the proof of Proposition 5.2, we see that σ is the unique irreducible non-tempered subquotient of Π. Suppose that Π is reducible. If π' is another irreducible subquotient of Π, by considering its cuspidal support or its Plancherel measure, we see that $\pi' \in \Pi_{\phi'}$ with $\phi' = \phi \oplus (\rho \boxtimes S_{2x-1}) \oplus (\rho \boxtimes S_{2x+1})$. Since $\phi \supset \rho \boxtimes S_{2x+1}$, we see that

$\phi' \supset (\rho \boxtimes S_{2x+1})^{\oplus 2}$. By Theorem 4.2, $\mathrm{Jac}_{\rho|\cdot|^x}(\pi')$ contains an irreducible non-tempered representation. However, $\mathrm{Jac}_{\rho|\cdot|^x}(\Pi) = \mathrm{St}(\rho, 2x-1) \rtimes \pi(\phi, \eta)$ consists of tempered representations. This is a contradiction. □

Example 6.2 Consider $\phi = S_2 \oplus S_4 \oplus S_6 \in \Phi_{\mathrm{disc}}(\mathrm{SO}_{13}(F))$ and $\eta \in \widehat{A_\phi}$ given by $\eta(\alpha_{S_{2a}}) = (-1)^a$ for $a = 1, 2, 3$. Then $\pi(\phi, \eta)$ is an irreducible supercuspidal representation. Moreover, the standard module

$$\Pi = \left\langle \mathbf{1}_{\mathrm{GL}_1(F)}; \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \pi(\phi, \eta)$$

of $\mathrm{SO}_{23}(F)$ is irreducible. Note that the L-parameter $\phi' = \phi \oplus |\cdot|^{\frac{1}{2}} S_5 \oplus |\cdot|^{-\frac{1}{2}} S_5$ of Π is non-generic since

$$\begin{aligned} L(s, \phi', \mathrm{Ad}) = &\zeta_F(s-1)\zeta_F(s)^3\zeta_F(s+1)^{13}\zeta_F(s+2)^{10} \\ &\times \zeta_F(s+3)^{12}\zeta_F(s+4)^5\zeta_F(s+5)^3 \end{aligned}$$

has a pole at $s = 1$, where ζ_F is the local zeta function associated to F. In particular, the standard module

$$\Pi_0 = \left\langle \mathbf{1}_{\mathrm{GL}_1(F)}; \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\rangle \rtimes \pi(\phi, \mathbf{1})$$

is reducible by Theorem 3.4.

6.2 Mínguez's derivatives

The following notion is introduced by Mínguez. Fix $x > 0$ such that $\rho \boxtimes S_{2x+1}$ is good with respect to G, i.e., we assume that there exists $\phi \in \Phi_{\mathrm{gp}}(G)$ such that $\phi \supset \rho \boxtimes S_{2x+1}$. For $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G)$, we set

$$\mathrm{Jac}^{\rho|\cdot|^x}(\pi) = \underbrace{\mathrm{Jac}_{\rho|\cdot|^x} \circ \cdots \circ \mathrm{Jac}_{\rho|\cdot|^x}}_{k}(\pi),$$

where $k \geq 0$ is the maximal integer such that $\mathrm{Jac}^{\rho|\cdot|^x}(\pi) \neq 0$. we call $\mathrm{Jac}^{\rho|\cdot|^x}(\pi)$ the **$\rho|\cdot|^x$-derivative of π**.

Proposition 6.3 *Let $\phi \in \Phi_{\mathrm{gp}}(G)$ and $\eta \in \widehat{\mathcal{A}_\phi}$ such that $\pi(\phi, \eta) \neq 0$. Suppose that $x > 0$ and $\mathrm{Jac}_{\rho|\cdot|^x}(\pi) \neq 0$, so that the multiplicity m of $\rho \boxtimes S_{2x+1}$ in ϕ is positive. For $1 \leq k \leq m$, set*

$$\phi^{(k)} = \phi - (\rho \boxtimes S_{2x+1})^{\oplus k} \oplus (\rho \boxtimes S_{2x-1})^{\oplus k}$$

and $\eta^{(k)} \in \widehat{\mathcal{A}_{\phi^{(k)}}}$ *to be* $\eta|\mathcal{A}_{\phi^{(k)}}$ *via the canonical inclusion* $\mathcal{A}_{\phi^{(k)}} \hookrightarrow \mathcal{A}_\phi$. *Then*

$$\mathrm{Jac}^{\rho|\cdot|^x}(\pi(\phi, \eta)) = k! \cdot \pi(\phi^{(k)}, \eta^{(k)})$$

with

$$k = \begin{cases} m & \text{if } \pi(\phi^{(m)}, \eta^{(m)}) \neq 0, \\ m-1 & \text{if } \pi(\phi^{(m)}, \eta^{(m)}) = 0. \end{cases}$$

In particular, for the above k, we have an inclusion

$$\pi(\phi, \eta) \hookrightarrow \underbrace{\rho|\cdot|^x \times \cdots \times \rho|\cdot|^x}_{k} \rtimes \pi(\phi^{(k)}, \eta^{(k)}),$$

and $\mathrm{Jac}_{\rho|\cdot|^x}(\pi(\phi^{(k)}, \eta^{(k)})) = 0$.

Proof When $m = 1$ (resp. $m = 2$), it is Theorem 4.2 (2) (resp. Lemma 5.4). In general, by Theorem 4.2 (1), we can reduce the assertion to the case of $m \in \{1, 2\}$. □

Remark 6.4 (1) Embedding an irreducible representation π into $(\rho|\cdot|^x)^{\times k} \rtimes \pi_0$ with k maximal is the key idea of the proof of the Howe duality conjecture by Gan–Takeda [7].
(2) If ρ is not self-dual, we should define $\mathrm{Jac}^{\rho|\cdot|^x}$ as the maximal nonzero iterated composition of $\mathrm{Jac}_{\rho^\vee|\cdot|^x} \circ \mathrm{Jac}_{\rho|\cdot|^x}$. Then Proposition 6.3 can be extended to any $\phi \in \Phi_{\mathrm{temp}}(G)$ and any supercuspidal unitary representation ρ of $\mathrm{GL}_d(F)$. We leave the detail for readers.
(3) In particular, one can show that for any $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G)$, the $\rho|\cdot|^x$-derivative $\mathrm{Jac}^{\rho|\cdot|^x}(\pi)$ is an isotypic of an irreducible tempered representation.

Acknowledgements The author expresses gratitude to Alberto Mínguez for telling the notion of his derivatives and pointing out Proposition 6.3. Thanks are also due to the referee for the careful readings and the helpful comments.

References

1. Arthur, J.: The endoscopic classification of representations. Orthogonal and symplectic groups. American Mathematical Society Colloquium Publications, vol. 61. American Mathematical Society, Providence (2013)
2. Aubert, A.-M., Moussaoui, A., Solleveld, M.: Generalizations of the Springer correspondence and cuspidal Langlands parameters. Manuscripta Math. **157**(1–2), 121–192 (2018)
3. Aubert, A.-M., Moussaoui, A., Solleveld, M.: Graded Hecke Algebras for Disconnected Reductive Group, Geometric Aspects of the Trace Formula, Simons Symposia, pp. 23–84. Springer, New York (2018)

4. Aubert, A.-M., Moussaoui, A., Solleveld, M.: Affine Hecke algebras for Langlands parameters. arXiv:1701.03593v2
5. Casselman, W., Shahidi, F.: On irreducibility of standard modules for generic representations. Ann. Sci. École Norm. Sup. (4) **31**(4), 561–589 (1998)
6. Gan, W.T., Ichino, A.: The Gross-Prasad conjecture and local theta correspondence. Invent. Math. **206**(3), 705–799 (2016)
7. Gan, W.T., Takeda, S.: A proof of the Howe duality conjecture. J. Am. Math. Soc. **29**(2), 473–493 (2016)
8. Heiermann, V., Muić, G.: On the standard modules conjecture. Math. Z. **255**(4), 847–853 (2007)
9. Heiermann, V., Opdam, E.: On the tempered L-functions conjecture. Am. J. Math. **135**(3), 777–799 (2013)
10. Heiermann, V.: A note on standard modules and Vogan L-packets. Manuscripta Math. **150**(3–4), 571–583 (2016)
11. Mœglin, C.: Sur certains paquets d'Arthur et involution d'Aubert–Schneider–Stuhler généralisée. Represent. Theory **10**, 86–129 (2006)
12. Mœglin, C.: Paquets d'Arthur discrets pour un groupe classique p-adique. Automorphic forms and L-functions II. Local aspects. Contemp. Math. **489**, 179–257. Israel Mathematics Conference, Proceedings of the American Mathematical Society, Providence (2009)
13. Mœglin, C., Waldspurger, J.-L.: La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général. Sur les conjectures de Gross et Prasad. II. Astérisque **347**, 167–216 (2012)
14. Muić, G.: A proof of Casselman–Shahidi's conjecture for quasi-split classical groups. Can. Math. Bull. **44**(3), 298–312 (2001)
15. Tadić, M.: Structure arising from induction and Jacquet modules of representations of classical p-adic groups. J. Algebra **177**(1), 1–33 (1995)
16. Xu, B.: On the cuspidal support of discrete series for p-adic quasisplit $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$. Manuscripta Math. **154**(3–4), 441–502 (2017)
17. Xu, B.: On Mœglin's parametrization of Arthur packets for p-adic quasisplit $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$. Can. J. Math. **69**(4), 890–960 (2017)
18. Zelevinsky, A.V.: Induced representations of reductivep-adic groups. II. On irreducible representations ofGL(n). Ann. Sci. École Norm. Sup. (4) **13**(2), 165–210 (1980)