Invent. math. (2019) 217:77–94
https://doi.org/10.1007/s00222-019-00863-8

The Ax–Schanuel conjecture for variations of Hodge structures

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Received: 31 August 2018 / Accepted: 24 January 2019 / Published online: 8 February 2019

Abstract We extend the Ax–Schanuel theorem recently proven for Shimura varieties by Mok–Pila–Tsimerman to all varieties supporting a pure polarizable integral variation of Hodge structures. In fact, Hodge theory provides a number of conceptual simplifications to the argument. The essential new ingredient is a volume bound for Griffiths transverse subvarieties of period domains.

1 Introduction

1.1 History

Motivated by arithmetic considerations, much recent work has focused on functional transcendence, specifically on generalizations of the famous Ax–Schanuel theorem on the exponential function to the context of hyperbolic uniformizations. Indeed, the strategy of Pila and Zannier for proving the André–Oort conjecture is reliant on a functional transcendence result dubbed the 'Ax–Lindemann theorem' by Pila. The approach originates in the celebrated paper [13], where Pila used his counting theorem with Wilkie to establish the result in the case of the Shimura variety $X(1)^n$, for $n \geq 1$.

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The Ax–Lindemann theorem was finally established in full generality for Shimura varieties in [10] by Klingler, Ullmo, and Yafaev, and for mixed Shimura varieties by Gao [5]. Motivated by an analogous (though much more difficult to carry out) approach to the more general Zilber–Pink conjectures, Mok, Pila, and the second author recently proved the full Ax–Schanuel conjecture for general Shimura varieties [12]. In this paper we prove the Ax–Schanuel conjecture in the more general setting of variations of (pure) Hodge structures (formulated recently by Klingler [9, Conjecture 7.5]). This is motivated largely by a recent approach of Lawrence–Venkatesh [11] to proving analogs of the arithmetic Shafarevich conjecture for families of varieties with a generically immersive period map, which seems to require the theorem we prove to work in full generality.

1.2 Statement of results

Let $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ be the Deligne torus. Given a pure polarized Hodge structure $h : \mathbf{S} \to \mathbf{Aut}(H_\mathbb{Z}, Q_\mathbb{Z})$, the Mumford-Tate group $\mathbf{MT}_h \subset \mathbf{Aut}(H_\mathbb{Z}, Q_\mathbb{Z})$ is the $\mathbb{Q}$-Zariski closure of $h(\mathbf{S})$. The associated Mumford–Tate domain $D(\mathbf{MT}_h)$ is the $\mathbf{MT}_h(\mathbb{R})$-orbit of h in the full period domain of polarized Hodge structures on $(H_\mathbb{Z}, Q_\mathbb{Z})$. By a *weak Mumford–Tate domain* $D(\mathbf{M})$ we mean the $\mathbf{M}(\mathbb{R})$-orbit of h for some normal $\mathbb{Q}$-algebraic subgroup $\mathbf{M}$ of $\mathbf{MT}_h$.

Let X be a smooth algebraic variety over $\mathbb{C}$ supporting a pure polarized integral variation of Hodge structures $\mathscr{H}_\mathbb{Z}$. Let $\mathbf{MT}_{\mathscr{H}_\mathbb{Z}}$ be the generic Mumford–Tate group, and let $\Gamma \subset \mathbf{MT}_{\mathscr{H}_\mathbb{Z}}(\mathbb{Q})$ be the image of the monodromy representation $\pi_1(X) \to \mathbf{MT}_{\mathscr{H}_\mathbb{Z}}(\mathbb{Q})$ after possibly passing to a finite cover. Let $\mathbf{G}$ be the identity component of the $\mathbb{Q}$-Zariski closure of Γ. Let $D = D(\mathbf{G})$ be the associated weak Mumford–Tate domain and $\varphi : X \to \Gamma\backslash D$ the period map of $\mathscr{H}_\mathbb{Z}$. The compact dual $\check{D}$ of D is a projective variety containing D as an open set in the Archimedean topology.

Consider the fiber product

$$\begin{array}{ccccc} X \times D & \supset & W & \xrightarrow{\tilde{\varphi}} & D \\ & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & X & \xrightarrow[\varphi]{} & \Gamma\backslash D. \end{array}$$

In this situation, for any weak Mumford–Tate subdomain $D' = D(\mathbf{M}') \subset D$ such that $\Gamma \cap \mathbf{M}'(\mathbb{Q})$ is $\mathbb{Q}$-Zariski dense, $\varphi^{-1}\pi(D')$ is an algebraic subvariety of X by a result of Cattani–Deligne–Kaplan [2], and we refer to such subvarieties as *weak Mumford–Tate subvarieties* of X.

Theorem 1.1 (Ax–Schanuel for variations of Hodge structures) *In the above setup, let $V \subset X \times \check{D}$ be an algebraic subvariety, and let U be an irreducible analytic component of $V \cap W$ such that*

$$\operatorname{codim}_{X\times\check{D}}(U) < \operatorname{codim}_{X\times\check{D}}(V) + \operatorname{codim}_{X\times\check{D}}(W).$$

Then the projection of U to X is contained in a proper weak Mumford–Tate subvariety.

Theorem 1.1 for example implies that the (analytic) locus in X where the periods satisfy a given set of algebraic relations must be of the expected codimension unless there is a reduction in the generic Mumford–Tate group. See [9] for some related discussions.

1.3 Outline of the proof

We follow closely the strategy of proof in [12]. There are two serious complications that have to be addressed, which are as follows:

First, we need to find a suitable fundamental domain in D for the image of X in $\Gamma\backslash D$. This domain has to be definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, and have certain growth properties. In the Shimura case, this is done by using a Siegel set. In our current setup this seems more difficult, due to the absence of toroidal co-ordinates. Instead, we use Schmid's theory of degenerations of Hodge structures to define our fundamental domain, which also provides a new approach in the setting of Shimura varieties. For more details on this, see Sect. 3.

Second, the proof of Theorem 1.1 requires a volume bound on Griffiths transverse[1] subvarieties $X \subset D$ analogous to those proven by Hwang–To for Hermitian symmetric domains [8]. We prove this in Sect. 2 and the result is as follows:

Theorem 1.2 *There are constants $\beta, \rho > 0$ (only depending on D) such that for any $R > \rho$, any $x \in D$, and any positive-dimensional Griffiths transverse closed analytic subvariety $Z \subset B_x(R) \subset D$, we have*

$$\operatorname{vol}(Z) \geq e^{\beta R} \operatorname{mult}_x Z$$

where $B_x(R)$ is the radius R ball centered at x and $\operatorname{vol}(Z)$ *the volume with respect to the natural left-invariant metric on D.*

In Sect. 4 we establish all the required comparisons between the various height and distance functions that show up, and Sect. 5 completes the proof.

[1] It is essential to restrict to Griffiths transverse subvarieties, as the general statement is false since, for example, D contains compact subvarieties.

2 Volume estimates

In this section we prove Theorem 1.2; we begin with some general remarks. Without loss of generality, we may clearly assume D is a full period domain. Further, letting $\mathbb{H}$ be the upper half-plane, $D \times \mathbb{H}$ embeds isometrically into a period domain D' of weight one larger by tensoring with the weight one Hodge structure of an elliptic curve, and it therefore suffices to consider D of odd weight. We make both of these assumptions for the remainder of this section. For general background on period domains and Hodge structures, see for example [4].

2.1 Hodge norms

A point $x \in D$ yields a Hodge structure H_x on $H_{\mathbb{Z}}$ polarized by Q_Z. Recall that the Hodge metric $h_x(v, w) = Q_{\mathbb{Z}}(v, C_x\overline{w})$ is positive-definite, where C_x is the Weil operator of H_x. For any $w \in H_{\mathbb{C}}$ we can define the norm-squared function $h(w) : x \mapsto h_x(w) := h_x(w, w)$ on D. Note that $g^*h(w) = h(g^{-1}w)$ for $g \in \mathbf{G}(\mathbb{R})$. Recall also that a choice of point $x \in D$ naturally endows the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\mathbf{G}(\mathbb{R})$ with a weight zero Hodge structure $\mathfrak{g}_x$ polarized by the Killing form, and that the holomorphic tangent space at x is naturally identified with $\mathfrak{g}_x^-$, where as usual we give $\mathfrak{g}_x^{p,-p}$ grading p. We refer to the odd part of $\mathfrak{g}_x^-$ as the horizontal directions, and to $\mathfrak{g}_x^{-1,1}$ as the Griffiths transverse directions. We will use the same notation $h(X) : x \mapsto h_x(X)$ for $X \in \mathfrak{g}_{\mathbb{C}}$ for norms with respect to the induced Hodge metric on $\mathfrak{g}_{\mathbb{C}}$, as well as for the induced Hodge metrics on all tensor, wedge, symmetric powers etc. of $\mathfrak{g}_{\mathbb{C}}$.

The following lemma calculates the derivatives of the Hodge norm function $h(w)$. The computation can be expressed more compactly in terms of the connection operators on D, but we prefer a more elementary approach for ease of exposition.

Lemma 2.1 *For Hodge-pure horizontal (in particular Griffiths transverse) directions $X \in \mathfrak{g}_x^-$, we have*

$$\partial h(w)(X) = -2h_x(Xw, w)$$
$$\partial\overline{\partial} h(w)(X, \overline{X}) = 2h_x(Xw) + 2h_x(\overline{X}w)$$

Proof Note that in $\mathbb{C}[z, \overline{z}]/(z^2, \overline{z}^2)$ we have

$$\exp(-zX)\exp\left(zX + \overline{z}\overline{X} + \frac{|z|^2}{2}\left([X, \overline{X}]^{<0} + [\overline{X}, X]^{>0}\right)\right)$$
$$= (1 - zX)\left(1 + zX + \overline{z}\overline{X} + \frac{|z|^2}{2}\left([X, \overline{X}]^{<0} + [\overline{X}, X]^{>0}\right)\right.$$

$$
\begin{aligned}
&+\frac{|z|^2}{2}\left(X\overline{X}+\overline{X}X\right)\Big) \\
&= 1+\overline{z}\overline{X}+|z|^2\left(-X\overline{X}+\frac{1}{2}\left([X,\overline{X}]^{<0}+[\overline{X},X]^{>0}\right)+\frac{1}{2}\left(X\overline{X}+\overline{X}X\right)\right) \\
&= 1+\overline{z}\overline{X}+\frac{|z|^2}{2}\left(-[X,\overline{X}]+[X,\overline{X}]^{<0}+[\overline{X},X]^{>0}\right) \\
&= 1+\overline{z}\overline{X}+\frac{|z|^2}{2}\left(-[X,\overline{X}]^{\geq 0}+[\overline{X},X]^{>0}\right)
\end{aligned}
$$

which is in the parabolic stabilizing the Hodge flag at x. Thus, modulo $(z^2, \overline{z}^2)$ we have

$$
\exp(zX).x = \exp\left(M(zX,\overline{z}\overline{X})\right).x
$$

where $M(zX,\overline{z}\overline{X}) = zX+\overline{z}\overline{X}+\frac{|z|^2}{2}\left([X,\overline{X}]^{<0}+[\overline{X},X]^{>0}\right) \in \mathfrak{g}$. Hence,

$$
\begin{aligned}
\partial h(w)(X) &= \frac{\partial}{\partial z}\exp(zX)^* h(w)|_{z=0} \\
&= \frac{\partial}{\partial z} h_x\left(\exp\left(-M(zX,\overline{z}\overline{X})\right).w\right)|_{z=0} \\
&= h_x(-Xw,w)+h_x(w,-\overline{X}w) \\
&= -2h_x(Xw,w)
\end{aligned}
$$

where we have used that X is horizontal and thus conjugate self-adjoint with respect to h_x. Likewise,

$$
\begin{aligned}
\partial\overline{\partial}h(w)(X,\overline{X}) =& \frac{\partial^2}{\partial z\partial\overline{z}}\exp(zX)^* h(w)|_{z=0} \\
=& \frac{\partial^2}{\partial z\partial\overline{z}} h_x\left(\exp\left(-M(zX,\overline{z}\overline{X})\right).w\right)|_{z=0} \\
=& h_x(-Xw,-Xw)+h_x(-\overline{X}w,-\overline{X}w) \\
&+\operatorname{Re} h_x(-[X,\overline{X}]^{<0}w,w)+\operatorname{Re} h_x(-[\overline{X},X]^{>0}w,w) \\
&+\operatorname{Re} h_x((X\overline{X}+\overline{X}X)w,w) \\
&= 2h_x(Xw)+2h_x(\overline{X}w)
\end{aligned}
$$

where we have used that $[X,\overline{X}]^{<0} = [\overline{X},X]^{>0} = 0$ since X is Hodge pure, as well as the conjugate self-adjointness of X. □

2.2 Distance functions

Let $\pi : D \to D_W$ be the projection to the associated symmetric space by taking the Weil Jacobian Hodge structure. We briefly recall the basic definitions, and refer to [4, §3.5] for details.

For $x \in D$ and the associated Hodge structure H_x on $H_{\mathbb{Z}}$, the Weil Jacobian Hodge structure $H_{\pi(x)}$ is the (pure) weight one Hodge structure on $H_{\mathbb{Z}}$ given by $H^{1,0}_{\pi(x)} = H^{\mathrm{odd}}_x$ and $H^{0,1}_{\pi(x)} = H^{\mathrm{even}}_x$. For each $x \in D$, we denote by $\mathfrak{h}_x$ the Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ induced by the Weil Jacobian Hodge structure $H_{\pi(x)}$. Note that both Hodge structures $\mathfrak{g}_x$ and $\mathfrak{h}_x$ induce the same Hodge metric on $\mathfrak{g}_{\mathbb{C}}$. Further, $\mathfrak{h}_x$ only has $(-1, 1)$, $(0, 0)$, and $(1, -1)$ parts, so that in particular $\mathfrak{h}^+_x = \mathfrak{h}^{1,-1}_x$. Given a basepoint $x_0 \in D$, π is identified with $\mathbf{G}(\mathbb{R})/V \to \mathbf{G}(\mathbb{R})/K$, where V is the stabilizer of x_0 under $\mathbf{G}(\mathbb{R})$ and K is the unitary subgroup of $\mathbf{G}(\mathbb{R})$ with respect to h_{x_0}. Note that K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$.

Let v_0 be a unit-length generator of $\det \mathfrak{h}^+_{x_0}$ in $\bigwedge^{\dim D_W} \mathfrak{h}_{x_0}$, and define a function $\varphi_0 : D \to \mathbb{R}$ by

$$\varphi_0(x) := \log h_x(v_0).$$

Evidently, φ_0 factors through the projection π since $h_x = h_{\pi(x)}$. Moreover, if F_0 is the fiber of π containing x_0, then by the KAK decomposition of $\mathbf{G}(\mathbb{R})$, φ_0 in fact only depends on F_0 (and not on x_0) since K fixes v_0 up to a phase.

Lemma 2.2 *$i\partial\overline{\partial}\varphi_0$ is strictly positive on Griffiths transverse tangent directions at x_0.*

Proof Let $X \in \mathfrak{g}^{-1,1}_{x_0}$, and note that $X \in \mathfrak{h}^{-1,1}_{x_0} \oplus \mathfrak{h}^{1,-1}_{x_0}$ since the parity operators on $\mathfrak{g}_x$ and $\mathfrak{h}_x$ are the same. Let $X^{-1,1}$, $X^{1,-1}$ be the graded pieces of X with respect to the Weil Hodge structure. Fixing a basis Y_i of $\mathfrak{h}^+_{x_0}$, we see that

$$\mathrm{ad}(X)\,(Y_1 \wedge \cdots \wedge Y_k) = \sum_i (-1)^{i-1} Y_1 \wedge \cdots \wedge \mathrm{ad}(X^{-1,1})Y_i \wedge \cdots \wedge Y_k.$$

Since the Y_i are a basis for $\mathfrak{h}^+_{x_0}$and $\mathrm{ad}(X^{-1,1})Y_i \in \mathfrak{h}^-_{x_0}$ it follows that the vectors on the right-hand side are all linearly independent, so if $\mathrm{ad}(X)v_0 = 0$ then $\mathrm{ad}(X)\mathfrak{h}^+_{x_0} = 0$. Likewise, if $\mathrm{ad}(\overline{X})v_0 = 0$, then $\mathrm{ad}(X)\mathfrak{h}^-_{x_0} = 0$. Thus, if $i\partial\overline{\partial}\varphi_0(X, \overline{X}) = 0$ then by Lemma 2.1 $\mathrm{ad}(X)$ kills $\mathfrak{h}^{\mathrm{odd}}_{x_0}$ and in particular $\overline{X}$, but this implies $X = 0$ [4, Corollary 12.6.3(iii)]. □

Define the horizontal distance from x to x_0, denoted $d^{\mathrm{horiz}}_0(x)$, to be the geodesic distance $d^{D_W}_0(y)$ between $y := \pi(x)$ and $y_0 := \pi(x_0)$ with respect to the natural $\mathbf{G}(\mathbb{R})$-invariant metric on the symmetric space D_W. Let A be an $\mathbb{R}$-split torus of $\mathbf{G}(\mathbb{R})$ that is Killing-orthogonal to K. By the KAK decomposition of $\mathbf{G}(\mathbb{R})$, the distance $d_0^{D_W}(y)$ and $\varphi_0(x)$ are both determined by $d_0^{D_W}(ay_0)$ and $\varphi_0(ax_0)$, respectively, for $a \in A$. Since Ay_0 is a flat totally geodesic submanifold of D_W, and the restriction of the invariant metric is a Euclidean metric in exponential coordinates, we have

$$d_0^{D_W}(ay_0)^2 \sim \sum_i t_i^2 \tag{1}$$

where $a = \exp(\sum_i t_i T_i)$ for some chosen basis T_i of the Lie algebra $\mathfrak{a}$ of A.

The main result of this subsection is the following comparison. Note that both d_0^{horiz} and φ_0 vanish exactly on F_0.

Proposition 2.3 $d_0^{\text{horiz}}(x) \ll \varphi_0(x) + O(1)$ *and* $\varphi_0(x) \ll d_0^{\text{horiz}}(x) + O(1)$.

Proof Griffiths–Schmid [7, Theorem 8.1] show that a function closely related to our φ_0 is an exhaustion function of D. For D_W, their function is given by

$$\varphi_0'(gy_0) := \log h_{x_0}(gv_0)$$

whereas our function is $\varphi_0(gx_0) = \log h_{gx_0}(v_0) = \log h_{x_0}(g^{-1}v_0)$. Their result implies $\varphi_0' \to \infty$ at the boundary of D_W, which is equivalent to saying that $h_{x_0}(gv_0)$ goes to ∞ as $g \to \infty$ (in the sense of escaping any compact subset of $\mathbf{G}(\mathbb{R})$). Since $g \to \infty$ is equivalent to $g^{-1} \to \infty$, it follows that $\varphi_0 \to \infty$ at the boundary of D.

Now, consider the decomposition

$$v_0 = \sum_\alpha v_\alpha$$

by $\mathfrak{a}$-weights. Note that as A is Killing-orthogonal to K, $\mathfrak{a}$ is odd and therefore self adjoint with respect to h_{x_0}. It follows then that the decomposition of $\bigwedge^{\dim D_W} \mathfrak{g}_{\mathbb{C}}$ into $\mathfrak{a}$-weight spaces is orthogonal with respect to h_{x_0}, and thus for $T \in \mathfrak{a}$,

$$\varphi_0(\exp(T)v_0) = \log h_{x_0}(\exp(-T)v_0) = \log \sum_\alpha e^{-2\alpha(T)} h_{x_0}(v_\alpha).$$

Since $\varphi_0 \to \infty$ at the boundary, it follows that there can be no $T \in \mathfrak{a}\backslash\{0\}$ such that $\alpha(T) \geq 0$ for all α with $v_\alpha \neq 0$. Thus, if $\Xi \subset \mathfrak{a}^\vee$ is the convex hull of the α for which $v_\alpha \neq 0$, we must have $0 \in \Xi$.

For $\alpha \in \mathfrak{a}^\vee$ denote by $e^\alpha : A \to \mathbb{R}$ the function mapping $\exp(T)$ to $e^{\alpha(T)}$, for $T \in \mathfrak{a}$. Choosing a basis T_i of $\mathfrak{a}$, it then follows from the above that

$$\log \sum_i \left(e^{T_i^\vee}(a) + e^{-T_i^\vee}(a) \right) \ll \varphi_0\,(a x_0) + O(1)$$

and

$$\varphi_0\,(a x_0) \ll \log \sum_i \left(e^{T_i^\vee}(a) + e^{-T_i^\vee}(a) \right) + O(1)$$

which imply the claim by (1). □

2.3 Multiplicity bounds

For any $r > 0$ and $x_0 \in D$, denote by

$$B^{\varphi_0}(r) := \{x \in D \mid \varphi_0(x) < r\}$$

and for any Griffiths transverse analytic subvariety $Z \subset D$ of dimension d,

$$\mathrm{vol}^{\varphi_0}(Z) := \frac{1}{d!} \int_Z (i\partial\overline{\partial}\varphi_0)^d.$$

Proposition 2.4 *Let ω be the positive (1,1) form associated to the natural left-invariant Hermitian metric on D.*

(1) $i\partial\overline{\partial}\varphi_0 \geq_{\mathrm{trans}} 0$ and $i\partial\overline{\partial}\varphi_0 = O_{\mathrm{trans}}(\omega)$;
(2) $|\partial\varphi_0|^2 = O_{\mathrm{trans}}(i\partial\overline{\partial}\varphi_0)$.

In the statement of the proposition, the notations $O_{\mathrm{trans}}(\cdot)$ and $\geq_{\mathrm{trans}}$ mean the bound holds in Griffiths transverse tangent directions.

Proof By definition, $\omega_x(X, \overline{X}) \sim h_x(X)$. For horizontal X, $\mathrm{tr}(X\overline{X}) \sim h_x(X)$ is larger (up to a fixed positive constant) than the maximum eigenvalue of $X^* h_x$ with respect to h_x. For $X \in \mathfrak{g}_x^-$ Hodge-pure and horizontal, by Lemma 2.1 and the fact that for any function f we have

$$\partial\overline{\partial} f(h(w)) = f' \partial\overline{\partial} h(w) + f'' |\partial h(w)|^2$$

it follows that

$$\partial\overline{\partial}\varphi_0(X, \overline{X}) = 2\left(\frac{h_x(X v_0)}{h_x(v_0)} + \frac{h_x(\overline{X} v_0)}{h_x(v_0)} \right) - 4 \left| \frac{h_x(X v_0, v_0)}{h_x(v_0)} \right|^2$$

which is nonnegative by Cauchy–Schwarz (using that X is conjugate-adjoint) and bounded (up to a fixed positive constant) by the maximal eigenvalue of X^*h_x with respect to h_x, so (1) follows.

The second claim follows by Lemma 2.1 and the following lemma:

Lemma 2.5 *There is a $\beta > 0$ (only depending on D) such that for any $x \in D$, $w \in H_{\mathbb{C}}$, and $X \in \mathfrak{g}_x^{-1,1}$,*

$$h_x(w) \cdot \frac{h_x(Xw) + h_x(\overline{X}w)}{2} \geq (1+\beta)\,|h_x(Xw, w)|^2 .$$

Proof Let $w = \sum_i w^{i,n-i}$ be the decomposition into Hodge components at x, so that we have Hodge decompositions $Xw = \sum_i Xw^{i,n-i}$, $\overline{X}w = \sum_i \overline{X}w^{i,n-i}$.

Now let

$$a_i^2 = h_x(w^{i,n-i}), \quad b_{i-1}^2 = h_x(Xw^{i,n-i}), \quad c_{i+1}^2 = h_x(\overline{X}w^{i,n-i}),$$

and we'll also set $b_n = c_0 = 0$. Note that since X and $\overline{X}$ are adjoint we have

$$h_x(Xw, w) = \sum_i h_x(Xw^{i+1,n-i-1}, w^{i,n-i})$$

and

$$\begin{aligned}|h_x(Xw^{i+1,n-i-1}, w^{i,n-i})| &= |h_x(w^{i+1,n-i-1}, \overline{X}w^{i,n-i})| \\ &\leq \min(a_i b_i, a_{i+1}c_{i+1}).\end{aligned}$$

Thus it is sufficient to show that

$$\left(\sum_{i=0}^{n} a_i^2\right)\left(\sum_{i=0}^{n-1} b_i^2 + \sum_{i=1}^{n} c_i^2\right) \geq (2+\delta)\left(\sum_{i=0}^{n} a_i(r_i b_i + s_i c_i)\right)^2$$

for some choice of nonnegative r_i, s_i with $r_i + s_{i+1} = 1$ for $0 \leq i \leq n-1$. By the Cauchy–Schwartz inequality, the left-hand side is greater than or equal to $\left(\sum_{i=0}^{n} a_i \sqrt{b_i^2 + c_i^2}\right)^2$. Thus, it suffices to show for each i,

$$b_i^2 + c_i^2 \geq (2+\delta)\,(r_i b_i + s_i c_i)^2 .$$

Note that $x^2 + y^2 - 2(rx + sy)^2$ is positive definite if $(1-2r^2)(1-2s^2) > 4r^2s^2$.

Lemma 2.6 *There exist non-negative real numbers* $r_0, s_1, r_1, s_2, \dots, s_{n-1}, r_{n-1}, s_n$, *with* $r_i + s_{i+1} = 1$ *for* $0 \leq i \leq n-1$, $\max(r_0, s_n) < \frac{1}{\sqrt{2}}$, *and* $(1-2r_i^2)(1-2s_i^2) > 4r_i^2 s_i^2$ *for all* $1 \leq i \leq n-1$.

Proof Note that at $r_i = s_i = \frac{1}{2}$ we get exact equality, in that $(1-2r_i^2)(1-2s_i^2) = 4r_i^2 s_i^2$. Thus, we set $r_j = \frac{1}{2} + \delta_j$, where $\delta_0 = \frac{1}{9}$ and δ_{j+1} is sufficiently small in terms of δ_j to ensure $(1-2r_{j+1}^2)(1-2s_{j+1}^2) > 4r_j^2 s_j^2$. □

The statement now follows by picking the r_i, s_i from the previous lemma, and setting $(2+\delta)$ to be the largest number such that $x^2+y^2-(2+\delta)(r_i x+s_i y)^2$ is positive semi-definite for $1 \leq i \leq n-$ and $1-(2+\delta)s_0^2$ is nonnegative. □

□

The previous proposition implies that the exponential growth of the φ_0-volume of a Griffiths transverse subvariety of D:

Proposition 2.7 *There is a constant* $\beta > 0$ *such that for any* $R > 0$ *and any positive-dimensional Griffiths transverse closed analytic subvariety* $Z \subset B^{\varphi_0}(R)$,

$$e^{-\beta r} \operatorname{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(r))$$

is a nondecreasing function in $r \in [0, R]$.

Proof Let $d = \dim Z$. Let $\psi_0 = -e^{-\beta\varphi_0}$ for β the constant from Lemma 2.5. We have

$$i\partial\overline{\partial}\psi_0 = \beta e^{-\beta\varphi_0}\left(i\partial\overline{\partial}\varphi_0 - \beta|\partial\varphi_0|^2\right)$$

which is nonnegative in Griffiths transverse directions by the proof of Proposition 2.4(ii). By Stokes' theorem we have

$$\begin{aligned}
\operatorname{vol}^{\varphi}(Z \cap B^{\varphi_0}(r)) &= \int_{Z\cap B^{\varphi_0}(r)} (i\partial\overline{\partial}\varphi_0)^d \\
&= \int_{Z\cap\partial B^{\varphi_0}(r)} d^c\varphi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-1}e^{\beta r}\int_{Z\cap\partial B^{\varphi_0}(r)} d^c\psi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-1}e^{\beta r}\int_{Z\cap B^{\varphi_0}(r)} i\partial\overline{\partial}\psi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-d}e^{\beta d r}\int_{Z\cap B^{\varphi_0}(r)} (i\partial\overline{\partial}\psi_0)^d
\end{aligned}$$

which implies the claim, as $\psi_0|_Z$ is plurisubharmonic. □

Combining Proposition 2.7 with the comparison in Proposition 2.3, we are now ready to prove Theorem 1.2:

Proof of Theorem 1.2 Choose a fixed euclidean ball B centered around x_0 with respect to some coordinate system. By a classical result Federer (see for example [16]), we have an inequality of the form $\mathrm{vol}^{\mathrm{eucl}}(Z \cap B) \gg \mathrm{mult}_{x_0} Z$. Choose a fixed radius r_0 such that $B \subset B^{\varphi_0}(r_0)$. After possibly shrinking B, $i\partial\overline{\partial}\varphi_0$ is comparable to the euclidean Kähler form on B in Griffiths transverse directions by Lemma 2.2, and combining this with the above proposition we have

$$\mathrm{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(r)) \gg e^{\beta r}\, \mathrm{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(r_0)) \gg e^{\beta r}\, \mathrm{mult}_{x_0} Z \qquad (2)$$

for all $r > r_0$.

Now, as the fibers of $\pi : D \to D_W$ have fixed diameter with respect to the natural left-invariant metric on D, there is $\rho > 0$ such that for any $R > \rho$ we have $B_{x_0}(R) \supset B^{\mathrm{horiz}}_{x_0}(R)$, where $B_{x_0}(R)$ (resp. $B^{\mathrm{horiz}}_{x_0}(R)$) is the radius R ball centered at x_0 with respect to the metric on D (resp. the distance function d^{horiz}_0). By Proposition 2.3, after possible increasing ρ, there is a constant $C > 0$ such that

$$B_{x_0}(R) \supset B^{\mathrm{horiz}}_{x_0}(R) \supset B^{\varphi_0}(CR) \qquad (3)$$

for all $R > \rho$. Combining (3) and (2) with Proposition 2.4(1) yields the bound in Theorem 1.2. □

3 Definable fundamental sets

Throughout the following, by definable we mean definable with respect to the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. Let X be a smooth algebraic variety supporting a pure polarizable integral variation of Hodge structures $\mathscr{H}_{\mathbb{Z}}$, and let $(\overline{X}, E)$ be a log-smooth compactification of X. For simplicity we may assume that $\mathscr{H}_{\mathbb{Z}}$ has unipotent local monodromy and that the associated period map $\varphi : X \to \Gamma\backslash D$ is proper, although the argument carries through without making these assumptions. We may also assume that the monodromy Γ is torsion-free.

The structure of X as an algebraic variety canonically endows it with the structure of a definable complex analytic manifold, and the choice of compactification $(\overline{X}, E)$ allows us to choose a definable atlas of X of finitely many polydisks $\Delta^k \times (\Delta^*)^\ell$. Note that any polydisk chart P in such an atlas $\{P_i\}$ can be shrunk to yield a new such atlas, as the complement of $\bigcup_{P_i \neq P} P_i$ is contained in P and has compact closure in the interior closure of P in $\overline{X}$. Let

$$\exp : \Delta^k \times \mathbb{H}^\ell \to \Delta^k \times (\Delta^*)^\ell$$

be the standard universal cover, and choose a bounded vertical strip $\Sigma \subset \mathbb{H}$ such that $\Delta^k \times \Sigma^\ell$ is a fundamental set for the action of covering transformations. By the above remark, after shrinking a polydisk we may always restrict to a region in $\Delta^k \times \Sigma^\ell$ where $|z_i|$ is bounded away from 1 on the Δ factors and $\operatorname{Im} z_i$ is bounded away from 0 on the Σ factors.

Choose lifts $\tilde{\varphi} : \Delta^k \times \mathbb{H}^\ell \to D$ of the period map restricted to each chart, and let $\mathcal{F}$ be the disjoint union of $\Delta^k \times \Sigma^\ell$ over all charts. We then have a diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\tilde{\varphi}} & D \\ {\scriptstyle \exp}\downarrow & & \\ X & & \end{array} \tag{4}$$

and $\mathcal{F}$ has a natural definable structure.

Note that the embedding $D \subset \check{D}$ as a semialgebraic set gives D a canonical definable structure.

Lemma 3.1 *Both maps in* (4) *are definable.*

Proof The claim for the vertical map is obvious. By the nilpotent orbit theorem, for each polydisk $\tilde{\varphi} = e^{zN}\tilde{\psi}$ where $\tilde{\psi} = \psi \circ \exp$ for some extendable holomorphic function $\psi : \Delta^n \to D$ (after shrinking the polydisks). The action of $\mathbf{G}(\mathbb{R})$ on D is definable, and $e^{z \cdot N}$ is polynomial in z, so $\tilde{\varphi} : \Delta^k \times \Sigma^\ell \to D$ is definable. □

Fix a left-invariant metric h_D on D and let $\Phi = \tilde{\varphi}(\mathcal{F})$.

Proposition 3.2 *Let $Z \subset \check{D}$ be a closed algebraic subvariety. For all $\gamma \in$ $\mathbf{G}(\mathbb{Z})$,* $\operatorname{vol}(Z \cap \gamma\Phi) = O(1)$.

Proof Evidently it is enough to show $\operatorname{vol}(Z' \cap \Phi) = O(1)$ for all Z' in the same connected component of the Hilbert scheme of $\check{D}$ as Z. Further, it suffices to show $\operatorname{vol}(\tilde{\varphi}^{-1}(Z') \cap \Delta^k \times \Sigma^\ell) = O(1)$ for each lifted polydisk chart $\tilde{\varphi} : \Delta^k \times \mathbb{H}^\ell \to D$, where the volume is computed with respect to $\tilde{\varphi}^* h_D$.

For any holomorphic horizontal map $f : M \to \Gamma\backslash D$ we have $f^* h_D \ll \kappa_M$ where κ_M is the Kobayashi metric of M. In particular, for $M = \Delta^k \times \mathbb{H}^\ell$ the metric κ_M is the maximum over the coordinate-wise Poincaré metrics. After shrinking the polydisk, the factors in $\Delta^k \times \Sigma^\ell$ have finite volume with respect to the Kobayashi metric of the larger polydisk, and thus it is enough to uniformly bound the degree of the projection of $\tilde{\varphi}^{-1}(Z')$ to any subset of coordinates.

By definable cell decomposition, for any definable subset $L \subset \mathbb{R}^N$ and any coordinate projection $\mathbb{R}^N \to \mathbb{R}^M$, the number of connected components in the fibers of L is bounded. Applying this to the universal family of $\tilde{\varphi}^{-1}(Z') \subset \Delta^k \times \Sigma^\ell$, the claim follows.

□

4 Heights

Fix a basepoint $x_0 \in \Phi$ so that we have an identification $D \cong \mathbf{G}(\mathbb{R})/V$ for a compact subgroup $V \subset \mathbf{G}(\mathbb{R})$. Thinking of D as a space of Hodge structures on the fixed integral lattice $(H_{\mathbb{Z}}, Q_{\mathbb{Z}})$, as before we denote by h_x the induced Hodge metric on $H_{\mathbb{C}}$ corresponding to $x \in D$.

Definition 4.1 For $\gamma \in \mathbf{G}(\mathbb{Z})$ let $H(\gamma)$ be the height of γ with respect to the representation $\rho_{\mathbb{Z}} : \mathbf{G}(\mathbb{Z}) \to \mathrm{GL}(H_{\mathbb{Z}})$. For $g \in \mathbf{G}(\mathbb{R})$, we denote by $||\rho_{\mathbb{R}}(g)||$ the maximum Archimedean size of the entries of $\rho_{\mathbb{R}}(g)$, so that if $\gamma \in \mathbf{G}(\mathbb{Z})$ we have $H(\gamma) = ||\rho_{\mathbb{R}}(\gamma)||$.

For any $R > 0$ let $B_{x_0}(R) \subset D$ be the ball of radius R centered at x_0. The main goal of this section is to establish the following:

Theorem 4.2 *For any $R > 0$, every element γ of*

$$\{\gamma \in \mathbf{G}(\mathbb{Z}) \mid B_0(R) \cap \gamma^{-1}\Phi \neq \varnothing\}$$

has height $H(\gamma) = e^{O(R)}$.

Define $d_0(x) = d(x, x_0)$. We write $f \preceq g$ if $|f| \ll |g|^{O(1)} + O(1)$, and $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

Lemma 4.3 *Let $\lambda(x, x')$ be the maximal eigenvalue of h_x with respect to $h_{x'}$. Then*

(1) For all $g \in \mathbf{G}(\mathbb{R})$ we have $||\rho_{\mathbb{R}}(g)|| \asymp e^{d_0(gx_0)}$;
(2) $\lambda(x, x') \asymp e^{d(x,x')}$.

Proof Choose a maximal compact subgroup $K \subset \mathbf{G}(\mathbb{R})$ containing V and a left-invariant metric on the associated symmetric space $\mathbf{G}(\mathbb{R})/K$. Note that the diameters of the fibers of $\mathbf{G}(\mathbb{R})/V \to \mathbf{G}(\mathbb{R})/K$ are bounded. Choosing a K-orthogonal split maximal torus $A \subset \mathbf{G}(\mathbb{R})$ and a basis A_i of the Lie algebra $\mathfrak{a}$ of A, we have for any $g \in \mathbf{G}(\mathbb{R})$ with KAK decomposition $g = k_1 a k_2$

$$\sqrt{\sum_i t_i^2} \ll d_0(gx_0) = d_0(ax_0) + O(1) \ll \sqrt{\sum_i t_i^2} + O(1)$$

where $a = \exp(\sum_i t_i A_i)$. As

$$\max_i \exp(|t_i|) \preceq \rho_{\mathbb{R}}(g) \preceq \max_i \exp(|t_i|)$$

part (1) follows.

For part (2), note that by $\mathbf{G}(\mathbb{R})$-invariance we may restrict to the case $x' = x_0$. Setting $\rho = \rho_{\mathbb{R}}$ for convenience, note that $\mathrm{tr}(\rho(g)^* \rho(g))$ is a sum of the eigenvalues of h_{gx_0} with respect to h_{x_0}, where $\rho(g)^*$ is the adjoint of $\rho(g)$ with respect to h_{x_0}. Thus $\operatorname{tr}(\rho(g)^*\rho(g)) \asymp \lambda(gx_0, x_0)$. As $\operatorname{tr}(\rho(g)^*\rho(g))$ is the sum of the squares of the entries of $\rho(g)$, part (2) follows from part (1). □

We define a proximity function of the boundary by the minimal period length:

$$\mu(x) = \min_{v \in H_{\mathbb{Z}} \setminus \{0\}} h_x(v).$$

For any $v \in H_{\mathbb{C}}$ we have $\log \frac{h_{x_0}(v)}{h_x(v)} \ll d_0(x) + O(1)$ by part (2) of Lemma 4.3, and so we deduce the following:

Corollary 4.4 $-\log \mu \ll d_0 + O(1)$.

Proof There is some $v \in H_{\mathbb{Z}} \setminus \{0\}$ with $\log \mu(x) = \log h_x(v)$ and thus

$$-\log \mu = -\log h_x(v) \ll \log \frac{h_{x_0}(v)}{h_x(v)} + O(1) \ll d_0(x) + O(1)$$

where we have used that h_{x_0} is comparable to a standard Hermitian metric on $H_{\mathbb{C}}$, so that $h_{x_0}(v) \gg 1$ for any $v \in H_{\mathbb{Z}} \setminus \{0\}$. □

When restricted to the fundamental set Φ, we in fact have a comparison in the other direction:

Lemma 4.5 *For $x \in \Phi$ we have $d_0(x) \ll -\log \mu(x) + O(1)$.*

Proof We may assume $\mathcal{F}$ is a single $\Delta^k \times \Sigma^\ell$. After choosing logarithms $N_1, \ldots, N_\ell$ of the local monodromy operators of the variation over $\Delta^k \times (\Delta^*)^\ell$, let v_i be a fixed basis of $H_{\mathbb{Z}}$ descending to a basis of the multi-graded module associated to the ℓ weight filtrations, where we take each grading centered at 0. Let $w_i^{(j)}$ for $j = 1, \ldots, \ell$ be the weights of v_i w.r.t. N_j. By Cattani et al. [3], for every permutation π and on each region $S_\pi \subset \Delta^k \times \mathbb{H}^\ell$ of the form $\operatorname{Im} z_{\pi(1)} \gg \cdots \operatorname{Im} z_{\pi(\ell)} \gg 1$ we have

$$h_{\tilde{\varphi}(z)}(v_i) \sim \left(\frac{\operatorname{Im} z_{\pi(1)}}{\operatorname{Im} z_2}\right)^{w_i^{(1)}} \cdots \left(\frac{\operatorname{Im} z_{\pi(\ell-1)}}{\operatorname{Im} z_{\pi(\ell)}}\right)^{w_i^{(\ell-1)}} \cdot \left(\operatorname{Im} z_{\pi(\ell)}\right)^{w_i^{(\ell)}}.$$

where "$\sim$" means "within a bounded function of." As the set of weights is preserved under negation, it follows that $\max_i h_{\tilde{\varphi}(z)}(v_i) \sim (\min_i h_{\tilde{\varphi}(z)}(v_i))^{-1}$, and so by Lemma 4.3,

$$d_0(\tilde{\varphi}(z)) \ll \max_i \log h_{\tilde{\varphi}(z)}(v_i) \ll -\log \mu(\tilde{\varphi}(z)) + O(1)$$

uniformly on every such region. The S_π can be made to cover the region $\Delta^k \times \Sigma^\ell$ after shrinking Σ, and the result follows. □

Proof of Theorem 4.2 Suppose $x \in B_0(R) \cap \gamma^{-1}\Phi$ for $\gamma \in \mathbf{G}(\mathbb{Z})$. Putting together Lemma 4.5 and Corollary 4.4 we have

$$d_0(\gamma x) \ll -\log \mu(\gamma x) + O(1) = -\log \mu(x) + O(1) \ll d_0(x) + O(1)$$

and since

$$d_0(\gamma x_0) \leq d(\gamma x, \gamma x_0) + d(\gamma x, x_0) \leq d_0(x) + d_0(\gamma x)$$

we are finished by part (1) of Lemma 4.3. □

5 The proof of Theorem 1.1

The remainder of the proof follows the same general strategy as [12]. There are sufficiently many differences, however, that we include the necessary modifications.

Recall that D sits naturally as an open subset in its compact dual $\check{D}$ which has the structure of a projective variety. Let M be the Hilbert scheme[2] of all subvarieties of $X \times \check{D}$ with the same Hilbert polynomial as V. Moreover let $\mathcal{V} \to M$ be the universal family over M, with a natural embedding $\mathcal{V} \hookrightarrow (X \times \check{D}) \times M$.

Let $\mathcal{V}_W$ be the base-change to $W \times M$. The action of Γ on $X \times D$ lifts to $\mathcal{V}_W$, and we define $\mathcal{V}_X := \Gamma \backslash \mathcal{V}_W$, which is naturally an analytic variety. Note that as M is proper, $\mathcal{V}_W$ is proper over W, and likewise $\mathcal{V}_X$ is proper over X.

We endow $\mathcal{V}_X$ with a definable structure as follows. Since $\mathcal{V}$ is algebraic it has an induced definable structure. By Lemma 3.1, pulling back to $\mathcal{F} \times M$ and quotienting out by the definable equivalence relation $\mathcal{F} \to X$ we obtain the desired definable structure on $\mathcal{V}_X$.

Suppose now for the sake of contradiction that the conclusion of Theorem 1.1 is false in the above setup. Moreover, suppose that among all counterexamples, $\dim X$ is minimal, and subject to that assumption, $\operatorname{codim} V + \operatorname{codim} W - \operatorname{codim} U$ is as large as possible, and subject to that assumption, that $\dim U$ is maximal.

Define a closed analytic subvariety $T \subset \mathcal{V}_W$ consisting of all pairs (p, V') such that $V' \cap W$ has dimension at least $\dim U$ around p, and let T_0 be the irreducible component containing (p, V) for some (hence any) point $p \in U$. Let $Y := \Gamma \backslash T_0 \subset \mathcal{V}_X$, which is a closed definable analytic subvariety. Now, the projection $q : Y \to X$ is defineable and proper, so the image Z is a definable

[2] Strictly speaking, a compactification of X should be chosen.

closed complex analytic subvariety of X by Remmert's theorem, and therefore it is also algebraic by definable Chow [14] (see also [12]). Moreover, it contains $\mathrm{pr}_X(U)$, and thus it contains the smallest algebraic variety containing $\mathrm{pr}_X(U)$, so we may assume $Z = X$.

Consider the family $\mathscr{F}$ of algebraic varieties parametrized by T_0. Let $\Gamma_{\mathscr{F}} \subset \Gamma$ be the subgroup of elements γ such that a very general[3] fiber of $\mathscr{F}$ is stable under γ. The stabilizer of a very general fiber of $\mathscr{F}$ in Γ is then exactly $\Gamma_{\mathscr{F}}$. Let Θ be the identity component of the $\mathbb{Q}$-Zariski closure of $\Gamma_{\mathscr{F}}$ in $\mathbf{G}$.

Lemma 5.1 *Θ is a normal subgroup of* $\mathbf{G}$.

Proof Let W' be a connected component of W which intersects $X \times \Phi$. Note that W' is stable under the monodromy group Γ of X. Clearly $\mathscr{F}$ is stable under the image Γ_Y of $\pi_1(Y) \to \pi_1(X) \to \mathbf{G}(\mathbb{Z})$ which is finite index in Γ, and therefore Γ_Y is Zariski-dense in $\mathbf{G}$ by André [1].

Each element of Γ_Y sends a very general fiber of $\mathscr{F}$ to a very general fiber, so by the above remark $\Gamma_{\mathscr{F}} = \gamma \cdot \Gamma_{\mathscr{F}} \cdot \gamma^{-1}$ for all $\gamma \in \Gamma_Y$. It follows that Θ is invariant under conjugation by Γ_Y and hence by the Zariski closure of Γ_Y as well, which is all of $\mathbf{G}$. □

Proposition 5.2 *Θ is the identity subgroup.*

Proof Without loss of generality V is a very general fiber of F, and hence is invariant by exactly Θ. Since Θ is a $\mathbb{Q}$-group by construction, it follows that $\mathbf{G}$ is isogenous to $\Theta_1 \times \Theta_2$ with $\Theta_2 = \Theta$ and we have a splitting of weak Mumford–Tate domains $D = D_1 \times D_2$ with $D_i = D(\Theta_i)$. Replacing X by a finite cover we also have a splitting of the period map [6, Theorem III.A.1]

$$\varphi = \varphi_1 \times \varphi_2 : X \to \Gamma_1 \backslash D_1 \times \Gamma_2 \backslash D_2.$$

Moreover, φ_1, φ_2 satisfy Griffiths transversality (see the proof of [6, Theorem III.A.1]). Note that $V \subset X \times D$ by assumption, and as V is invariant under Θ_2 it is of the form $V_1 \times D_2$ where $V_1 \subset X \times D_1$.

Consider the period map $X \to \Gamma_1 \backslash D_1$, the resulting $W_1 \subset X \times D_1$, and the subvariety $V_1 \subset X \times D_1$. Let U_1 be the component of $V_1 \cap W_1$ onto which U projects. By assumption the theorem applies in this situation, and as U_1 cannot be contained in a proper weak Mumford–Tate subdomain (for then U would as well), we must have

$$\mathrm{codim}_{X \times D_1}(U_1) = \mathrm{codim}_{X \times D_1}(V_1) + \mathrm{codim}_{X \times D_1}(W_1).$$

[3] Recall that very general means in the complement of countably many proper closed subvarieties.

Note that the projection $W \to W_1$ has discrete fibers, so $\dim W = \dim W_1$ and $\dim U = \dim U_1$, whereas $\operatorname{codim} V_1 = \operatorname{codim} V$, which is a contradiction if φ_2 is non-constant. □

It follows that V is not invariant by any infinite subgroup of Γ. The proof of Theorem 1.1 is then completed by the following lemma, which produces a contradiction:

Lemma 5.3 *V is invariant by an infinite subgroup of* Γ.

Proof Consider the definable set

$$I := \{g \in \mathbf{G}(\mathbb{R}) \mid \dim (gV \cap W \cap (X \times \Phi)) = \dim U\}.$$

Clearly, I contains $\gamma \in \Gamma$ whenever U intersects $X \times \gamma^{-1}\Phi$. We may assume $1 \in I$, and take $x_0 \in \Phi$ the second coordinate of a point of intersection of U and $X \times \Phi$.

For any sufficiently large $R > 0$, consider the ball $B_{x_0}(R)$ centered at x_0. On the one hand, by Theorem 1.2 we have

$$\operatorname{vol}\left(U \cap \left(X \times B_{x_0}(R)\right)\right) \gg e^{\beta R}.$$

U is covered with bounded overlaps by $U \cap (X \times \gamma^{-1}\Phi)$ for $\gamma \in \mathbf{G}(\mathbb{Z})$, so by Proposition 3.2 it follows that I has $e^{\omega(R)}$ integer points.[4] On the other hand, by Theorem 4.2 each of these points has height $e^{O(R)}$, and it follows by the Pila–Wilkie theorem [15, Theorem 1.8] that I contains a real algebraic curve C containing arbitrarily many integer points, in particular at least 2 integer points.

If cV is constant in $c \in C$, then V is stable under $C \cdot C^{-1}$. Since C contains at least 2 integer points, it follows that V is stabilized by a non-identity integer point, completing the proof (since Γ is torsion free). So we assume that cV varies with $c \in C$. Note that since C contains an integer point that $\tilde{\varphi}(cV \cap W)$ is not contained in a weak Mumford-Tate subdomain for at least one $c \in C$, and thus for all but a countable subset of C (since there are only countably many families of weak Mumford–Tate subdomains).

We now have two cases to consider. First, suppose that $U \subset cV$ for all $c \in C$. Then we may replace V by $cV \cap c'V$ for a generic $c, c' \in C$ and lower $\dim V$, contradicting our induction hypothesis on $\dim V - \dim U$.

On the other hand, if it is not true that $U \subset cV$ for all $c \in C$ then $cV \cap W$ varies with C, and so we may set V' to be the Zariski closure of $C \cdot V$. This increases the dimension of V by 1, but then $\dim V' \cap W = \dim U + 1$ as well,

[4] Recall that for a function $f(R)$, saying $f(R) = \omega(R)$ means that for some positive constant $\delta > 0$ we have $f(R) \geq \delta R$.

and thus we again contradict our induction hypothesis, this time on $\dim U$. This completes the proof.

□

Acknowledgements The first author was partially supported by NSF grant DMS-1702149. We would like to thank the referee for helpful comments and for suggestions which improved the readability of the paper.

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