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Greatest common divisors and Vojta's conjecture for blowups of algebraic tori

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Abstract We give results and inequalities bounding the greatest common divisor of multivariable polynomials evaluated at S-unit arguments, generalizing to an arbitrary number of variables results of Bugeaud–Corvaja–Zannier, Hernández–Luca, and Corvaja–Zannier. In closely related results, and in line with observations of Silverman, we prove special cases of Vojta's conjecture for blowups of toric varieties. As an application, we classify when terms from simple linear recurrence sequences can have a large greatest common divisor (in an appropriate sense). The primary tool used in the proofs is Schmidt's Subspace Theorem from Diophantine approximation.

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1 Introduction

We study two closely related problems: greatest common divisors of multivariable polynomials evaluated at S-unit arguments and certain cases of Vojta's conjecture involving blowups of the algebraic torus $\mathbb{G}_m^n$. The main result obtained towards the first problem is a generalization of results of Bugeaud et

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al. [4], Hernández and Luca [21], and Corvaja and Zannier [7,8] to polynomials in an arbitrary number of variables:

Theorem 1.1 *Let n be a positive integer, $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ a finitely generated group, and $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n) \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ nonconstant coprime polynomials such that not both of them vanish at $(0, 0, \ldots, 0)$. Let $h(\alpha)$ denote the (absolute logarithmic) height of an algebraic number α. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$\log \gcd(f(u_1, \ldots, u_n), g(u_1, \ldots, u_n)) < \epsilon \max\{h(u_1), \ldots, h(u_n)\}$$

for all $(u_1, \ldots, u_n) \in \Gamma \setminus Z$.

The greatest common divisor on the left-hand side of the inequality is a generalized notion of the usual quantity for integers, adapted to algebraic numbers, which also notably includes archimedean contributions (Definition 1.4). As an application of Theorem 1.1 and related results, we study when terms from two simple linear recurrence sequences can have a "large" greatest common divisor (Theorem 1.11).

In a related set of results, we prove a family of special cases of Vojta's conjecture. Towards this end, in Sect. 1.2 we formulate a version of Vojta's conjecture attached to pairs (X, V), where X is a nonsingular complete variety and $V \subset X$ is an open subvariety (the pair (X, X) recovers a standard version of Vojta's conjecture for X). With this formulation, we prove Vojta's conjecture for pairs (X, V), where X is a suitable blowup of an n-dimensional toric variety and V is the corresponding blowup of $\mathbb{G}_m^n \subset X$. The connection between Vojta's conjecture and Bugeaud–Corvaja–Zannier's result (and its generalizations) was originally observed by Silverman [39].

A special case of these results is the following inequality: let $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ be a finitely generated subgroup and let Y be a closed subscheme of $\mathbb{P}^n$ of codimension at least 2, appropriately in general position with the boundary of $\mathbb{G}_m^n$ in $\mathbb{P}^n$ (see Theorem 1.16 for the precise condition). Let h_Y be a height associated to Y and let h denote the standard height on $\mathbb{P}^n$. Then there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that

$$h_Y(P) < \epsilon h(P) \tag{1}$$

for all $P \in \Gamma \setminus Z$. This result may be viewed as a geometric or projective version of Theorem 1.1.

In the next two sections we describe and contextualize the above results in more detail. The remainder of the paper is organized as follows. In Sect. 2 we collect together and give the necessary background material. In Sect. 3 we prove Theorem 1.1 and some related results, and in the following Sect. 4 we derive consequences for Vojta's conjecture. In the final section, Sect. 5, we give an application to greatest common divisors of terms from simple linear recurrence sequences. As in nearly all previous work on these topics, the primary tool in our proofs is Schmidt's Subspace Theorem from Diophantine approximation.

1.1 Greatest common divisors

In 2003, Bugeaud et al. [4] initiated a new line of results by proving the following simply-stated theorem.

Theorem 1.2 (Bugeaud et al. [4]) *Let $a, b \in \mathbb{Z}$ be multiplicatively independent integers. Then for every $\epsilon > 0$,*

$$\log \gcd(a^n - 1, b^n - 1) \leq \epsilon n \tag{2}$$

for all but finitely many positive integers n.

Despite the simplicity of the statement, the proof required the powerful Schmidt Subspace Theorem from Diophantine approximation. It follows from a result of Adleman et al. [1, Prop. 10] (see also [4, Remark (2)]) that there is a constant $c > 0$ such that for all pairs of integers $a, b > 1$, there exist infinitely many positive integers n with

$$\log \gcd(a^n - 1, b^n - 1) > e^{c \log n / \log \log n}.$$

Thus, the inequality (2) is close to being optimal.

In proving a conjecture of Győry et al. [18], Corvaja and Zannier [7] and, independently, Hernández and Luca [21], gave an improvement to Theorem 1.2 where $\gcd(a^n - 1, b^n - 1)$ is replaced by $\gcd(u - 1, v - 1)$ for multiplicatively independent S-unit integers u and v.

Theorem 1.3 (Corvaja and Zannier [7] and Hernández and Luca [21]) *Let $p_1, \ldots, p_t \in \mathbb{Z}$ be prime numbers and let $S = \{\infty, p_1, \ldots, p_t\}$. Then for every $\epsilon > 0$,*

$$\log \gcd(u - 1, v - 1) \leq \epsilon \max\{\log |u|, \log |v|\} \tag{3}$$

for all but finitely many multiplicatively independent S-unit integers $u, v \in \mathbb{Z}_S^*$.

More generally, Corvaja and Zannier [8] replace a^n and b^n by elements of a finitely generated subgroup $\Gamma \subset \mathbb{G}_m^2(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^* \times \overline{\mathbb{Q}}^*$ and replace $u - 1$ and

$v-1$ by other pairs of polynomials. Before stating this generalization, we discuss extending the quantities in (3) to all algebraic numbers.

First, note that if u is an integer, then $\log|u|$ is the same as the absolute logarithmic height $h(u)$. Thus, in extending (3) to arbitrary algebraic numbers, the right-hand side readily generalizes. To make sense of greatest common divisors for arbitrary algebraic numbers, we note that if a and b are integers, not both zero, then

$$\begin{aligned}\log\gcd(a,b) &= \sum_{p \text{ prime}} \min\{\operatorname{ord}_p(a), \operatorname{ord}_p(b)\}\log p\\ &= -\sum_{v\in M_{\mathbb{Q}}^0} \log\max\{|a|_v, |b|_v\}\\ &= -\sum_{v\in M_{\mathbb{Q}}^0} \log^-\max\{|a|_v, |b|_v\},\end{aligned}$$

where $M_{\mathbb{Q}}^0$ denotes the set of nonarchimedean places of $\mathbb{Q}$ and $\log^- z = \min\{0, \log z\}$.

Extending this sum to archimedean places and to all algebraic numbers, following [8,39], we make the following definition (see Sect. 2 for our conventions on absolute values).

Definition 1.4 Let $\alpha, \beta \in \overline{\mathbb{Q}}$ be two algebraic numbers, not both zero. We define the generalized logarithmic greatest common divisor of α and β by

$$\log\gcd(\alpha,\beta) = -\sum_{v\in M_k} \log^-\max\{|\alpha|_v, |\beta|_v\},$$

where k is any number field containing both α and β.

Equivalently, it is easily seen that

$$\log\gcd(\alpha,\beta) = h([1:\alpha:\beta]) - h([\alpha:\beta]),$$

where h is the standard height (on the appropriate projective spaces), and in particular, Definition 1.4 is independent of the choice of number field k containing α and β. For completeness, if $\alpha=\beta=0$, then by convention we also define $\log\gcd(\alpha,\beta)=\infty$ and $-\log^-\max\{|\alpha|_v, |\beta|_v\}=\infty$ (for any place v). Since this situation will always correspond to a codimension two phenomenon, it will be easily avoided (sometimes implicitly, and without comment).

Finally, we note that the condition that u and v are multiplicatively independent can be rephrased as saying that (u,v) is not an element of a proper algebraic subgroup of $\mathbb{G}_m^2$. In fact, Corvaja and Zannier [8, Prop. 2] show that the multiplicative independence condition in Theorem 1.3 can be replaced by the assumption that (u, v) does not lie in one of finitely many proper algebraic subgroups of $\mathbb{G}_m^2$ (depending only on ϵ). Explicitly, one needs to exclude subgroups given by an equation $u^p = v^q$ with p and q coprime integers satisfying $|p|, |q| \leq 1/\epsilon$ (in fact, Corvaja and Zannier prove this in the more general setting discussed below, with u and v lying in a finitely generated subgroup of $\overline{\mathbb{Q}}^*$). When one replaces $u - 1$ and $v - 1$ in Theorem 1.3 by more general polynomials in u and v, proper algebraic subgroups must be replaced, in general, by translates of proper algebraic subgroups.

We now state Corvaja and Zannier's generalization of Theorem 1.3.

Theorem 1.5 (Corvaja and Zannier [8]) *Let $\Gamma \subset \mathbb{G}_m^2(\overline{\mathbb{Q}})$ be a finitely generated group. Let $f(x, y), g(x, y) \in \overline{\mathbb{Q}}[x, y]$ be nonconstant coprime polynomials such that not both of them vanish at* $(0, 0)$. *For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^2$ such that*

$$\log \gcd(f(u, v), g(u, v)) < \epsilon \max\{h(u), h(v)\}$$

for all $(u, v) \in \Gamma \setminus Z$.

Theorem 1.1 from the introduction is the same assertion, generalized in the obvious way from $\mathbb{G}_m^2$ to $\mathbb{G}_m^n$ for $n \geq 2$.

Corvaja and Zannier also show that the nonvanishing hypothesis on f and g can be dropped for $u, v \in \mathcal{O}_{k,S}^*$ if one removes the contribution to $\log \gcd(f(u, v), g(u, v))$ coming from places in S.

Theorem 1.6 (Corvaja and Zannier [8]) *Let k be a number field and S a finite set of places of k containing the archimedean places. Let $f(x, y), g(x, y) \in k[x, y]$ be nonconstant coprime polynomials. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^2$ such that*

$$-\sum_{w \in M_k \setminus S} \log^- \max\{|f(u, v)|_w, |g(u, v)|_w\} < \epsilon \max\{h(u), h(v)\}$$

for all $(u, v) \in \mathbb{G}_m^2(\mathcal{O}_{k,S}) \setminus Z$.

In the same way, we also generalize Theorem 1.6 to higher dimensions (Theorem 3.2).

In a different direction, Luca [28] proved a generalization of Theorem 1.3 where u and v are only assumed to be close (in an appropriate sense) to S-units. Analogues of Theorems 1.2, 1.3, and 1.5 have also been considered [2,8–10,13,38] in the context of function fields (both in characteristic 0 and positive characteristic).

Another line of related results comes into view via the deep analogies between Nevanlinna theory and Diophantine approximation. Using Vojta's dictionary [42, Ch. 3], statements in Diophantine approximation for integral points on $\mathbb{G}_m^n$ should correspond to certain statements in Nevanlinna theory for a holomorphic map $f : \mathbb{C} \to \mathbb{G}_m^n$. From this point of view, Noguchi et al. [31, Main Theorem and Th. 5.1] (see also [30, § 6.5]) proved a general inequality for holomorphic maps to semi-abelian varieties (and associated jet spaces), which contains as a special case an analogue of Theorem 1.1. The results of [31] are stated in a formulation analogous to inequality (1) (and results of Sect. 1.2), but one can also formulate statements directly analogous to Theorem 1.1, involving counting functions of common zeros of appropriate holomorphic functions (see [32] for this and some related problems).

As an application of our results, in Sect. 5 we study greatest common divisors of terms from simple linear recurrence sequences. Consider a sequence of power sums given by

$$F(n) = \sum_{i=1}^{r} c_i \alpha_i^n, \quad n \in \mathbb{N}, \tag{4}$$

where $\alpha_i, c_i \in \mathbb{C}^*$, $i = 1, \dots, r$. As is well-known, such sequences satisfy a linear recurrence relation of the form

$$F(n) = A_1 F(n-1) + \cdots + A_r F(n-r), \quad n = r, r+1, r+2, \dots, \tag{5}$$

for some constants $A_i \in \mathbb{C}$. To situate things in a larger context, we note that more generally, F satisfies a relation of the form (5) if and only if one may write

$$F(n) = \sum_{i=1}^{s} f_i(n) \alpha_i^n, \quad n \in \mathbb{N}, \tag{6}$$

for some nonzero polynomials $f_i \in \mathbb{C}[x]$ and distinct $\alpha_i \in \mathbb{C}^*$, classically called the *roots* of F. In fact, $\alpha_1, \dots, \alpha_s$ are precisely the distinct roots of the corresponding characteristic polynomial

$$X^r - A_1 X^{r-1} - \cdots - A_r,$$

and f_i is a polynomial of some degree strictly smaller than the multiplicity of α_i in the characteristic polynomial. In particular, F has the form (4) if and only if F satisfies a linear recurrence relation as above and every root of the associated characteristic polynomial is simple. In accordance with this, such a linear recurrence is called *simple*.

Before discussing greatest common divisors of linear recurrences, we discuss the related topic of divisibility relations between linear recurrences. The set of linear recurrences of the form (6) forms a ring under the usual addition and multiplication of sequences. Then, as in any ring, we obtain a notion of divisibility between the elements, and so a notion of divisibility between linear recurrences.

If F and G are linear recurrences and G divides F, then $F(n)/G(n)$ obviously lies in a finitely generated ring for all $n \in \mathbb{N}$ with $G(n) \neq 0$. The Hadamard quotient theorem, proven by van der Poorten (following an incomplete argument of Pourchet), gives a converse to this statement.

Theorem 1.7 (Hadamard quotient theorem, Pourchet-van der Poorten [33,34,40,41]) *Let F and G be linear recurrences and let R be a finitely generated subring of $\mathbb{C}$ with $F(n), G(n) \in R$ for all $n \in \mathbb{N}$. If $G(n)$ divides $F(n)$ in R for all $n \in \mathbb{N}$, then G divides F (in the ring of linear recurrences).*

Corvaja and Zannier [6] proved a similar result, but under the much weaker hypothesis that $G(n)$ divides $F(n)$ for infinitely many $n \in \mathbb{N}$ (as opposed to all n). We let $F(q \bullet +r)$ denote the sequence $n \mapsto F(qn+r)$.

Theorem 1.8 (Corvaja and Zannier [6]) *Let F and G be linear recurrences and let R be a finitely generated subring of $\mathbb{C}$ with $F(n), G(n) \in R$ for all $n \in \mathbb{N}$. If $G(n)$ divides $F(n)$ in R for infinitely many $n \in \mathbb{N}$, then there exist positive integers q and r such that $G(qn+r) = P(n)H(n)$, $n \in \mathbb{N}$, for some polynomial $P \in \mathbb{C}[x]$ and some linear recurrence H dividing the linear recurrence $F(q \bullet +r)$. In particular, if G is a simple linear recurrence, then there exist q and r such that $G(q \bullet +r)$ divides $F(q \bullet +r)$.*

We now turn to greatest common divisors among linear recurrences. From now on, we consider only *algebraic* linear recurrences, i.e., $f_i \in \overline{\mathbb{Q}}[x]$ and $\alpha_i \in \overline{\mathbb{Q}}^*$ in (6) [or $\alpha_i, c_i \in \overline{\mathbb{Q}}^*$ in (4)]. Theorem 1.2 provides a starting point and a prototype for such results. More generally, it is clear that Theorem 1.5 can be used to analyze $\log \gcd(F(n), G(n))$ (or $\log \gcd(F(m), G(n))$) for algebraic simple linear recurrences when the involved roots generate a small-rank group. In this vein, we have work of Luca [26–28] and Hernández and Luca [20], including results for nonsimple linear recurrences. For instance, in [28], Luca proved:

Theorem 1.9 (Luca [28]) *Let a and b be nonzero integers which are multiplicatively independent and let $f_1, f_2, g_1, g_2 \in \mathbb{Z}[x]$ be nonzero polynomials. Let*

$$F(n) = f_1(n)a^n + f_2(n),$$
$$G(n) = g_1(n)b^n + g_2(n).$$

Then for all $\epsilon > 0$,

$$\log \gcd(F(n), G(m)) < \epsilon \max\{m, n\}$$

for all but finitely many pairs of positive integers (m, n).

Fuchs [17] (see also [16]) proved a result for simple linear recurrences, including cases where the roots generate a group of arbitrarily large rank.

Theorem 1.10 (Fuchs [17]) *Let*

$$F(n) = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_s\alpha_s^n,$$
$$G(n) = d_1\beta_1^n + d_2\beta_2^n + \cdots + d_t\beta_t^n,$$

define two simple linear recurrence sequences of integers, where $c_1, \ldots, c_s$, $d_1, \ldots, d_t \in \mathbb{Q}^*$ *and* $\alpha_1 > \cdots > \alpha_s > 0$ *and* $\beta_1 > \cdots > \beta_t > 0$ *are integers with* α_1 *and* $\alpha_2 \ldots \alpha_s \beta_1 \ldots \beta_t$ *coprime. Let* $\epsilon > 0$. *Then*

$$\log \gcd(F(n), G(n)) < \epsilon n$$

for all but finitely many positive integers n.

Before stating our main application, we discuss greatest common divisors in certain rings of linear recurrences. Let $\Gamma \subset \overline{\mathbb{Q}}^*$ be a torsion-free multiplicative subgroup of rank t with generators $u_1, \ldots, u_t$. It is well-known (and easy to check) that the ring R_Γ of algebraic simple linear recurrences with roots in Γ is isomorphic to the ring of Laurent polynomials $\overline{\mathbb{Q}}[T_1, \ldots, T_t, T_1^{-1}, \ldots, T_t^{-1}]$, with the isomorphism induced by mapping T_i to the linear recurrence F_i defined by $F_i(n) = u_i^n$. In particular, the ring R_Γ is a unique factorization domain.

Now suppose that F and G are algebraic simple linear recurrences whose combined roots generate a torsion-free subgroup Γ of $\overline{\mathbb{Q}}^*$. Then we say that F and G are coprime if they are coprime as elements of R_Γ (i.e., 1 is a greatest common divisor of F and G in R_Γ). Otherwise, we say that F and G have a nontrivial common factor. Whenever we use this terminology, we tacitly assume that the roots of F and G generate a torsion-free group under multiplication. We also note that these properties are stable under enlarging Γ [6, p. 438], that is, F and G are coprime in R_Γ if and only if they are coprime in $R_{\Gamma'}$, where Γ' is any finitely generated torsion-free subgroup of $\overline{\mathbb{Q}}^*$ containing Γ.

With this terminology, we can state our application to greatest common divisors of terms from simple linear recurrence sequences.

Theorem 1.11 *Let*

$$F(n) = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_s\alpha_s^n,$$
$$G(n) = d_1\beta_1^n + d_2\beta_2^n + \cdots + d_t\beta_t^n,$$

define two algebraic simple linear recurrence sequences. Let k be a number field such that $c_i, \alpha_i, d_j, \beta_j \in k$ for $i = 1, \ldots, s$, $j = 1, \ldots, t$. Let

$$S_0 = \{v \in M_k : \max\{|\alpha_1|_v, \ldots, |\alpha_s|_v, |\beta_1|_v, \ldots, |\beta_t|_v\} < 1\}.$$

(a) There exists $\delta > 0$ and an integer N such that

$$-\log^- \max\{|F(m)|_v, |G(n)|_v\} > \delta \min\{m, n\}$$

for all $v \in S_0$ and all integers $m, n \geq N$.

(b) Let $\epsilon > 0$. If the inequality

$$\sum_{v \in M_k \setminus S_0} -\log^- \max\{|F(m)|_v, |G(n)|_v\} > \epsilon \max\{m, n\}$$

has infinitely many solutions (m, n), then all but finitely many of them satisfy one of finitely many linear relations

$$(m, n) = (a_i t + b_i, c_i t + d_i), \quad t \in \mathbb{Z}, i = 1, \ldots, r,$$

where $a_i, b_i, c_i, d_i \in \mathbb{Z}$, $a_i c_i \neq 0$, and the linear recurrences $F(a_i \bullet + b_i)$ and $G(c_i \bullet + d_i)$ have a nontrivial common factor for $i = 1, \ldots, r$.
In particular, if $S_0 = \emptyset$, then the same statement holds for the inequality

$$\log \gcd(F(m), G(n)) > \epsilon \max\{m, n\}.$$

Theorem 1.11 asserts, roughly speaking, that $F(m)$ and $G(n)$ can have a "large" greatest common divisor only for obvious algebraic reasons: the roots of F and G have a nontrivial common divisor, or m and n lie in arithmetic progressions such that F and G, restricted to the arithmetic progressions, have a nontrivial common divisor (as linear recurrences).

1.2 Vojta's conjecture

In this section, we develop and describe our results from the point of view of Vojta's conjecture. The perspective here was influenced by and borrows from Silverman's paper [39].

We begin by stating Vojta's conjecture (i.e., the "Main Conjecture" from [42, Conj. 3.4.3]; see also [5, Conj. 15.6]). We will have nothing to say about the more difficult "General Conjecture" of Vojta for algebraic points [5, Conj. 25.1].

Conjecture 1.12 (Vojta's Main Conjecture) *Let X be a nonsingular complete variety with canonical divisor K. Let D be a normal crossings divisor on X, and let A be a big divisor on X. Let k be a number field over which X and D are both defined, and let S be a finite set of places of k. Let $\epsilon > 0$. Then there exists a proper Zariski closed subset Z of X such that*

$$m_{D,S}(P) + h_K(P) \leq \epsilon h_A(P) + O(1)$$

for all points $P \in X(k)\backslash Z$.

We will also find it useful to formulate a version of Vojta's conjecture for nonsingular varieties V (not necessarily complete). More precisely, we consider V embedded in a nonsingular complete variety X such that $D_0 = X\backslash V$ is a normal crossings divisor (with both X and D defined over some number field). In this case, we call (X, V) an *admissible pair*. Note that by a theorem of Nagata, every variety can be embedded as an open subvariety of a complete variety, and then using Hironaka's theorem on resolution of singularities (we always assume characteristic 0), it follows that for any such nonsingular variety V we can always find a nonsingular complete variety X such that $X\backslash V$ is a normal crossings divisor. The variety X is, of course, not unique (except when V is a curve).

If (X, V) is an admissible pair and $D_0 = X\backslash V$, then we define $K_{(X,V)} = K_X + D_0$. We will say that D is a normal crossings divisor on (X, V) if D is an effective divisor on X such that $D + D_0$ is a normal crossings divisor on X. With this terminology, we state the following version of Vojta's conjecture for the pair (X, V), where h_{K_X} is replaced by $h_{K_{(X,V)}}$ and the inequality holds for a set of integral points on V (equivalently, a set of D_0-integral points) instead of the full set of all rational points:

Conjecture 1.13 *Let (X, V) be an admissible pair and let D be a normal crossings divisor on (X, V). Let A be a big divisor on X. Let k be a number field over which X, V, and D are all defined and let S be a finite set of places of k containing the archimedean places. Let $\epsilon > 0$. Then there exists a proper Zariski closed subset Z of X such that for any set R of S-integral points on V we have*

$$m_{D,S}(P) + h_{K_{(X,V)}}(P) \leq \epsilon h_A(P) + O(1)$$

for all points $P \in R\backslash Z$.

When $V = X$ is complete, Conjecture 1.13 is exactly Vojta's conjecture for X. Conversely, it is not hard to see (Lemma 2.3) that Conjecture 1.13 follows from Conjecture 1.12. Thus, the main use of Conjecture 1.13 is as an organizational tool for our results, and to highlight a certain natural class of cases of Vojta's conjecture.

From this point of view, we prove Vojta's conjecture for blowups of $\mathbb{G}_m^n$, with the compactification coming from any projective toric variety (e.g., $\mathbb{P}^n$ or $(\mathbb{P}^1)^n$). More precisely, if X is a nonsingular projective toric variety of dimension n (containing, by definition, $V = \mathbb{G}_m^n$ as a dense open subvariety) and $\pi : \tilde{X} \to X$ is a suitable birational morphism, then Vojta's conjecture holds for the pair $(\tilde{X}, \pi^{-1}(\mathbb{G}_m^n))$ (assuming it is an admissible pair).

Before stating the full result we make two further definitions. For a birational morphism $\pi : \tilde{X} \to X$, we let $\operatorname{Exc}(\pi)$ denote the exceptional locus of π, that is, the locus of points of $\tilde{X}$ where π is not a local isomorphism. We say that $Y \subset X$ is in general position with the boundary $X \backslash V$ if Y does not contain any point of intersection of n distinct irreducible components of $X \backslash V$ (where $n = \dim X$). The main result towards Vojta's conjecture is the following theorem.

Theorem 1.14 *Let X be a nonsingular projective toric variety of dimension n. Let $\tilde{X}$ be a nonsingular projective variety, $\pi : \tilde{X} \to X$ a birational morphism, $\tilde{V} = \pi^{-1}(\mathbb{G}_m^n)$, $D_0 = \tilde{X} \backslash \tilde{V}$ a (reduced) divisor, A a big divisor on $\tilde{X}$, and D an effective divisor on $\tilde{X}$. Additionally, suppose that $\pi(\operatorname{Exc}(\pi)) \cup \operatorname{Supp} \pi_* D$ is in general position with $X \backslash \mathbb{G}_m^n$. Let k be a number field over which $\tilde{X}$, X, D, D_0, and π are defined, S a finite set of places of k containing the archimedean places, and $\epsilon > 0$. Then there exists a proper Zariski closed subset Z of $\tilde{X}$ such that the inequality*

$$m_{D,S}(P) + h_{K_{\tilde{X}}+D_0}(P) \leq \epsilon h_A(P) + O(1)$$

holds for all points $P \in \pi^{-1}(\mathbb{G}_m^n(\mathcal{O}_{k,S})) \backslash Z$. In particular, if $(\tilde{X}, \tilde{V})$ is admissible then Vojta's conjecture holds for the pair $(\tilde{X}, \tilde{V})$.

The last statement of the theorem is clear once one makes the easy observation that if $\pi(\operatorname{Exc}(\pi))$ is in general position with $X \backslash \mathbb{G}_m^n$ and $D + D_0$ is a normal crossings divisor, then $\operatorname{Supp} \pi_* D$ is in general position with $X \backslash \mathbb{G}_m^n$. Thus, the general position condition on $\operatorname{Supp} \pi_* D$ is a weakening of the normal crossings condition present in Vojta's conjecture. We also note that $R \subset X(k)$ is a set of (D_0, S)-integral points for some S if and only if $R \subset \pi^{-1}(\mathbb{G}_m^n(\mathcal{O}_{k,S'}))$ for some S' (where S and S' are finite sets of places of k containing the archimedean places).

Remark 1.15 The inequality of Theorem 1.14 is true without the $O(1)$ term, but then the exceptional set Z must be allowed to depend on the choice of $m_{D,S}$,

$h_{K_X+D_0}$, and h_A (which are only determined up to $O(1)$), as well as on the choice of ϵ. Hence, we will typically include an $O(1)$ term in such inequalities involving height functions (and when we omit it, the exceptional set Z will depend on the choice of height functions in addition to other data). Moreover, the $O(1)$ term allows one to choose Z to consist only of positive-dimensional components. Since all of our results depend ultimately on applications of the Schmidt Subspace Theorem, it follows easily from our proofs and Vojta's refinement of the Subspace Theorem (Remark 2.5), that Z may be chosen independent of k and S (or Γ) here and elsewhere (up to finitely many points in inequalities without an $O(1)$ term, such as Theorem 1.1). However, since the constructions in the proofs depend crucially on ϵ, we do not obtain any independence of Z from ϵ.

From a slightly different perspective, recall that every birational morphism of projective varieties $\pi : \tilde{X} \to X$ is a blowup along some closed subscheme Y of X [19, Th. 7.17]. The inequality on $\tilde{X}$ of Theorem 1.14 is closely related to inequalities on X for heights associated to closed subschemes Y (in fact, this is the basis of the proof of Theorem 1.14). For simplicity, we assume now that $X = \mathbb{P}^n$ and we identify $\mathbb{G}_m^n$ with $\mathbb{P}^n \setminus \cup_{i=0}^n H_i$, where $H_0, \ldots, H_n$ are the $n+1$ coordinate hyperplanes of $\mathbb{P}^n$. Let Y be a nonsingular subvariety of $\mathbb{P}^n$ of codimension $r \geq 2$, such that Y intersects $\mathbb{P}^n \setminus \mathbb{G}_m^n = \cup_{i=0}^n H_i$ transversally. Let $\pi : \tilde{X} \to \mathbb{P}^n$ be the blowup along Y. Then from our assumptions (using the notation of Theorem 1.14), we have

$$K_{\tilde{X}} \sim \pi^* K_{\mathbb{P}^n} + (r-1)E \sim -\pi^* \left(\sum_{i=0}^n H_i \right) + (r-1)E \sim -D_0 + (r-1)E,$$

where $E = \pi^{-1}(Y)$ is the exceptional divisor. Thus, taking $D = 0$, the inequality in Theorem 1.14 becomes

$$h_E(P) \leq \epsilon h_A(P) + O(1).$$

Alternatively, to formulate the inequality on $\mathbb{P}^n$, we can take $h_A(P) = h(\pi(P))$ and use the equality

$$h_E(P) = h_Y(\pi(P)) + O(1), \quad \forall P \in X(k) \setminus \operatorname{Supp} E,$$

where h_Y is a height associated to the closed subscheme Y. In this case, this yields, together with Laurent's theorem (Theorem 2.1), the inequality (1) discussed in the beginning of the introduction. In fact, as an easy consequence of Theorem 1.1 and related results, we derive an inequality for h_Y under a general position assumption on Y in place of a transversality assumption. Let $P_0 = [1 : 0 : \cdots : 0], \ldots, P_n = [0 : 0 : \cdots : 0 : 1]$, or equivalently,

$$P_i = \bigcap_{\substack{j=0 \\ j \neq i}}^{n} H_j, \quad i = 0, \ldots, n.$$

Theorem 1.16 *Let Y be a closed subscheme of $\mathbb{P}^n$, defined over a number field, of codimension at least 2. Suppose that*

$$P_0, \ldots, P_n \notin Y.$$

Let $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ be a finitely generated subgroup. Then for all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that

$$h_Y(P) \leq \epsilon h(P) + O(1)$$

for all $P \in \Gamma \backslash Z \subset \mathbb{P}^n(\overline{\mathbb{Q}})$.

The $n = 2$ cases of Theorems 1.14 and 1.16 follow (essentially) from Corvaja and Zannier's Theorem 1.5. In this case, Yasufuku [45] has extended Theorem 1.14, showing for instance that one can eliminate the general position condition on $\pi(\text{Exc}(\pi))$. Moreover, for certain special blowups of $\mathbb{P}^n$ involving the boundary of $\mathbb{G}_m^n$, Yasufuku [44] proves an inequality as in Theorem 1.14 for *rational points*, and with the height h_{D_0} replaced by the sum of local heights $m_{D_0,S}$. Lastly, for completeness, we mention a result of McKinnon [29], fully proving Vojta's conjecture over a number field k for a blowup of a product $E \times E$, where E is an elliptic curve over k and $E(k)$ has rank one (or more generally, rank one over $\text{End}_k(E)$).

2 Notation and background material

We collect together some notation and background material that will be used throughout.

2.1 Algebra and algebraic geometry

Let k be a field and let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. For $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we define

$$\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$$

and write

$$|\mathbf{i}| = i_1 + \cdots + i_n = \deg \mathbf{x}^{\mathbf{i}}.$$

Let X be a projective variety of dimension n and let D be a Cartier divisor on X. We let $\operatorname{Supp} D$ denote the support of D and $h^0(D) = \dim H^0(X, \mathcal{O}(D))$. Recall that D is called a big divisor if

$$\limsup_{m\to\infty} \frac{h^0(mD)}{m^n} > 0.$$

If $D_1, \ldots, D_q$ are effective Cartier divisors on X, then we say that $D_1, \ldots, D_q$ are in general position if for any subset $I \subset \{1, \ldots, q\}$, $|I| \leq n+1$, we have $\operatorname{codim} \cap_{i\in I} \operatorname{Supp} D_i \geq |I|$, where we set $\dim \emptyset = -1$. In particular, the supports of any $n+1$ divisors in general position have empty intersection. If D and E are Cartier divisors on X, then we write $D \sim E$ if D and E are linearly equivalent, and define $D \geq E$ if $D - E$ is effective.

Let $\phi : Y \to X$ be a morphism of projective varieties. If the image of ϕ is not contained in the support of D, then we let $\phi^* D$ denote the pullback Cartier divisor. More generally, if Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}$, we let $\phi^* Z$ denote the closed subscheme of Y associated to the ideal sheaf $\phi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$.

Assume now that X is a nonsingular complex projective variety (in particular, Cartier and Weil divisors coincide on X). Let D be a divisor on X. We say that D has normal crossings if every point P in the support of D has an analytic open neighborhood in X with analytic local coordinates $z_1, \ldots, z_n$ such that D is locally defined by $z_1 \cdot z_2 \cdots z_i = 0$ for some i. If Y is a projective variety and $\phi : X \to Y$ is a morphism, we let $\phi_* D$ denote the pushforward of D.

2.2 Algebraic tori, toric varieties, and Laurent's theorem

We give a short naïve treatment of the algebraic torus $\mathbb{G}_m^n$, sufficient for our purposes. Let k be a field of characteristic 0. As an affine variety over k, we identify $\mathbb{G}_m^n$ with the Zariski open subset

$$x_1 \cdots x_n \neq 0$$

of affine space $\mathbb{A}^n$. Coordinate-wise multiplication

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$$

gives $\mathbb{G}_m^n(\overline{k})$ the structure of a group, so that $\mathbb{G}_m^n(\overline{k}) = (\overline{k}^*)^n$ as groups, where $\overline{k}^*$ is the multiplicative group of nonzero elements of $\overline{k}$. More generally, if R is a subring of $\overline{k}$, we let

$$\mathbb{G}_m^n(R) = (R^*)^n \subset (\overline{k}^*)^n,$$

where R^* is the group of units of R.

An *algebraic subgroup* of $\mathbb{G}_m^n$ is a Zariski closed subset $Z \subset \mathbb{G}_m^n$ such that $Z(\bar{k})$ is a subgroup of $\mathbb{G}_m^n(\bar{k})$. As is well-known, every algebraic subgroup of $\mathbb{G}_m^n$ is defined by a system of equations

$$x_1^{i_1} \cdots x_n^{i_n} = 1, \quad (i_1, \ldots, i_n) \in \Lambda,$$

for some subgroup Λ of $\mathbb{Z}^n$. A *translate* of an algebraic subgroup $H \subset \mathbb{G}_m^n$ is a coset of the form gH, where $g \in \mathbb{G}_m^n(\bar{k})$.

We will use throughout the following fundamental result of Laurent [24], describing the Zariski closure of subsets of finitely generated subgroups of $\mathbb{G}_m^n(\overline{\mathbb{Q}})$.

Theorem 2.1 (Laurent) *Let $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ be a finitely generated subgroup and let $R \subset \Gamma$ be a subset. Then the Zariski closure of R in $\mathbb{G}_m^n$ is a finite union of translates of algebraic subgroups of $\mathbb{G}_m^n$.*

In fact, Laurent proved a stronger statement, conjectured by Lang, where Γ is only assumed to be a subgroup of $\mathbb{G}_m^n(\mathbb{C})$ of finite $\mathbb{Q}$-rank.

We will also frequently want to work with compactifications of $\mathbb{G}_m^n$. A natural class of such compactifications arises from toric varieties. A *toric variety* X is a variety over k containing $\mathbb{G}_m^n$ as a dense open subvariety such that the action of $\mathbb{G}_m^n$ on itself extends to an algebraic action of $\mathbb{G}_m^n$ on X. For instance, $\mathbb{P}^n$ and $(\mathbb{P}^1)^n$ are projective toric varieties, with embeddings of $\mathbb{G}_m^n$ given, respectively, by

$$\begin{aligned} \mathbb{G}_m^n &\hookrightarrow \mathbb{P}^n \\ (x_1, \ldots, x_n) &\mapsto [1 : x_1 : \cdots : x_n] \end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_m^n &\hookrightarrow (\mathbb{P}^1)^n \\ (x_1, \ldots, x_n) &\mapsto ([x_1 : 1], \ldots, [x_n : 1]). \end{aligned}$$

Toric varieties admit a rich combinatorial description and theory, for which we refer to [11] for the general theory and for the few facts that we will need in Sect. 4.

2.3 Absolute values, heights, and Schmidt's Subspace Theorem

Let k be a number field. Recall that we have a canonical set M_k of places (or absolute values) of k consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \overline{\sigma} : k \to \mathbb{C}$. For $v \in M_k$, let k_v denote the completion of k with respect to v. We normalize our absolute values so that $|p|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p}$ lies above a rational prime p, and $|x|_v = |\sigma(x)|^{[k_v:\mathbb{R}]/[k:\mathbb{Q}]}$ if v corresponds to an embedding σ (in which case we say that v is archimedean).

Let S be a finite set of places of k containing the archimedean places. We use $\mathcal{O}_k$, $\mathcal{O}_{k,S}$, and $\mathcal{O}_{k,S}^*$ to denote the ring of integers of k, ring of S-integers of k, and group of S-units of k, respectively.

For $v \in M_k$ and $\alpha \in k$, we define the height

$$h(\alpha) = \sum_{v \in M_k} \log \max\{|\alpha|_v, 1\}.$$

More generally, for a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$, we have the standard (absolute logarithmic) height

$$h(P) = \sum_{v \in M_k} \log \max\{|x_0|_v, \ldots, |x_n|_v\}.$$

Note that this is independent of the choice of number field k (with $P \in \mathbb{P}^n(k)$), and it is independent of the choice of homogeneous coordinates $x_0, \ldots, x_n \in k$ by the product formula:

$$\prod_{v \in M_k} |x|_v = 1$$

for all $x \in k^*$.

Let X be a projective (or more generally, complete) variety defined over a number field k. The classical theory of heights [3,22,23] associates to every Cartier divisor D on X a height function $h_D : X(k) \to \mathbb{R}$ and local height functions, $\lambda_{D,v} : X(k) \setminus \operatorname{Supp} D \to \mathbb{R}$, $v \in M_k$, well-defined up to bounded functions, such that

$$\sum_{v \in M_k} \lambda_{D,v}(P) = h_D(P) + O(1)$$

for all $P \in X(k) \setminus \operatorname{Supp} D$. When D is effective, $\lambda_{D,v}(P)$ is essentially the negative of the logarithm of the v-adic distance between P and D, and in particular, if D is a hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial $F \in k[x_0, \ldots, x_n]$ of degree d, then a local height function for D is given by

$$\lambda_{D,v}(P) = \log \max_i \frac{|x_i|_v^d}{|F(P)|_v} = \log \frac{|P|_v^d}{|F(P)|_v}, \quad v \in M_k,$$

where $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k)\setminus \operatorname{Supp} D$ and $|P|_v = \max_i |x_i|_v$ (this last quantity, of course, depends on the choice of homogeneous coordinates for P, but we only ever use it in ratios which are well-defined).

Generalizing the theory of heights for Cartier divisors, Silverman [37] developed a theory of heights for arbitrary closed subschemes of projective varieties. We give here a quick summary of the relevant properties of such heights (see [37] for the general theory and details).

Let Y be a closed subscheme of a projective variety X, both defined over a number field k. One can associate to Y local height functions $\lambda_{Y,v} : X(k)\setminus Y \to \mathbb{R}$, $v \in M_k$, unique up to a bounded function (more precisely, unique up to an M_k-bounded function; see [37]). A global height function h_Y, unique up to a bounded function, can be constructed as the sum of local height functions. If $Y = D$ is an effective (Cartier) divisor (which we identify with the associated closed subscheme), these height functions agree with the previously discussed height functions associated to divisors. Local height functions satisfy the following properties: if Y and Z are two closed subschemes of X, defined over k, and $v \in M_k$, then up to $O(1)$,

$$\begin{aligned}
&\lambda_{Y\cap Z,v} = \min\{\lambda_{Y,v}, \lambda_{Z,v}\},\\
&\lambda_{Y+Z,v} = \lambda_{Y,v} + \lambda_{Z,v},\\
&\lambda_{Y,v} \le \lambda_{Z,v}, \quad \text{if} \quad Y \subset Z,\\
&\lambda_{Y,v} \le c\lambda_{Z,v}, \quad \text{if} \quad \operatorname{Supp} Y \subset \operatorname{Supp} Z,
\end{aligned}$$

for some constant $c > 0$, where $\operatorname{Supp} Y$ denotes the support of Y. If $\phi : W \to X$ is a morphism of projective varieties, then

$$\lambda_{Y,v}(\phi(P)) = \lambda_{\phi^*Y,v}(P) + O(1), \quad \forall P \in W(k)\setminus\phi^*Y.$$

Here, $Y \cap Z$, $Y + Z$, $Y \subset Z$, and ϕ^*Y are defined in terms of the associated ideal sheaves (see Sect. 2.1 and [37]). We will typically avoid points in $Y(k)$, but at times it will be convenient to define $\lambda_{Y,v}(P) = \infty$ for $P \in Y(k)$, with typical conventions for ∞ (e.g., so that the formula for $\lambda_{Y\cap Z,v}$ holds for all $P \in X(k)$).

Global height functions satisfy similar properties (except the first property above, which becomes $h_{Y\cap Z} \le \min\{h_Y, h_Z\} + O(1)$). Global height functions satisfy two additional properties that we will use. First, if D and E are linearly equivalent Cartier divisors on X, then

$$h_D(P) = h_E(P) + O(1)$$

for all $P \in X(\bar{k})$. Second, if D is any Cartier divisor on X and A is an ample divisor on X, then there is a constant $c > 0$ such that

$$h_D(P) < ch_A(P) + O(1)$$

for all $P \in X(\bar{k})$. More generally, if A is a big divisor, then there is a constant $c > 0$ and a proper Zariski closed subset $Z \subset X$ such that

$$h_D(P) < ch_A(P) + O(1)$$

for all $P \in X(\bar{k})\setminus Z$. We will use a slightly extended version of this fact.

Lemma 2.2 *Let $\phi : X \to Y$ be a birational map of nonsingular projective varieties over a number field k. Let A be a divisor on X and B be a big divisor on Y, both defined over k. Then there exists a constant c and a proper Zariski closed subset $Z \subset X$ such that*

$$h_A(P) \leq ch_B(\phi(P))$$

for all $P \in X(\bar{k})\setminus Z$.

Proof Let $U \subset X$ be an open subset such that ϕ restricts to a morphism on U. We can resolve ϕ to a morphism $\tilde{\phi} : \tilde{X} \to Y$, where $\phi \circ \pi = \tilde{\phi}$ on $\pi^{-1}(U)$ and $\pi : \tilde{X} \to X$ is a birational morphism. Then by functoriality,

$$h_A(\pi(P)) = h_{\pi^*A}(P) + O(1),$$
$$h_B(\tilde{\phi}(P)) = h_{\tilde{\phi}^*B}(P) + O(1),$$

for all $P \in \tilde{X}(\bar{k})$. Since $\tilde{\phi}^*B$ is again big on $\tilde{X}$, there exists a constant c and a proper Zariski closed subset $\tilde{Z} \subset \tilde{X}$ such that

$$h_{\pi^*A}(P) < ch_{\tilde{\phi}^*B}(P)$$

for all $P \in \tilde{X}(\bar{k})\setminus\tilde{Z}$. Combining the above, we find that

$$h_A(P) \leq ch_B(\phi(P)) + O(1)$$

for all $P \in X(\bar{k})\setminus Z$, where $Z = \pi(\tilde{Z}) \cup (X\setminus U)$. Finally, by an appropriate version of Northcott's theorem for big divisors [42, Prop. 1.2.9(h)], we may remove the $O(1)$ at the expense of possibly enlarging c and Z. □

We also define two functions related to height functions, depending on a finite set of places S of k. We define the *proximity function*

$$m_{Y,S}(P) = \sum_{v \in S} \lambda_{Y,v}(P)$$

and the *counting function*

$$N_{Y,S}(P) = h_Y(P) - m_{Y,S}(P) = \sum_{v \in M_k \setminus S} \lambda_{Y,v}(P)$$

for $P \in X(k) \setminus Y$. Both functions are well-defined up to a bounded function.

Lastly, we define the notion of a set of (D, S)-integral points. Let D be an effective Cartier divisor on X, $h_D = \sum_{v \in M_k} \lambda_{D,v}$ a height function associated to D, and S a finite set of places of k containing the archimedean places. A set of points $R \subset X(k) \setminus \operatorname{Supp} D$ is called a *set of* (D, S)*-integral points on* X if there exist constants c_v, $v \in M_k$, such that $c_v = 0$ for all but finitely many v, and for all $v \in M_k \setminus S$,

$$\lambda_{D,v}(P) \leq c_v$$

for all $P \in R$. In this case, clearly

$$m_{D,S}(P) = h_D(P) + O(1)$$

for all $P \in R$. We will also call a set R of (D, S)-integral points a set of S-integral points on $V = X \setminus \operatorname{Supp} D$ (indeed, this notion is independent of how we write $V = X \setminus \operatorname{Supp} D$ [42, Cor. 1.4.2, Th. 1.4.11]).

Sets of integral points on $\mathbb{G}_m^n$, subsets of finitely generated subgroups of $\mathbb{G}_m^n(\overline{\mathbb{Q}})$, and subsets of $\mathbb{G}_m^n(\mathcal{O}_{k,S})$ (varying k and S) are all essentially equivalent objects. In particular, if X is a nonsingular projective toric variety and $D = X \setminus \mathbb{G}_m^n$, then $R \subset \mathbb{G}_m^n(k)$ is a set of (D, S)-integral points on X for some S if and only if $R \subset \mathbb{G}_m^n(\mathcal{O}_{k,S'})$ for some S' (where S and S' are finite sets of places of k containing the archimedean places).

We note that Conjecture 1.13 is an easy consequence of Vojta's conjecture.

Lemma 2.3 *Conjecture* 1.12 *implies Conjecture* 1.13.

Proof Let $R \subset X(k)$ be a set of (D_0, S)-integral points on X. Then by definition,

$$m_{D_0,S}(P) = h_{D_0}(P) + O(1)$$

for all $P \in R$. By additivity of heights and proximity functions, we obtain

$$\begin{aligned} m_{D+D_0,S}(P) + h_K(P) &= m_{D,S}(P) + m_{D_0,S}(P) + h_K(P) + O(1) \\ &= m_{D,S}(P) + h_{K+D_0}(P) + O(1) \end{aligned}$$

for all $P \in R$. Thus, Conjecture 1.13 follows from Conjecture 1.12 applied to the normal crossings divisor $D + D_0$ on X and restricted to the set of points R. □

The essential tool in all of our proofs is Schmidt's Subspace Theorem [36]. We state a general formulation of the theorem, including subsequent improvements by Schlickewei [35] to allow for arbitrary number fields and finite sets of places.

Theorem 2.4 (Subspace Theorem) *Let S be a finite set of places of a number field k. For each $v \in S$, let $H_{0,v}, \dots, H_{n,v} \subset \mathbb{P}^n$ be hyperplanes over k in general position. Let $\epsilon > 0$. Then there exists a finite union of hyperplanes $Z \subset \mathbb{P}^n$ such that the inequality*

$$\sum_{v\in S}\sum_{i=0}^{n} \lambda_{H_{i,v},v}(P) < (n+1+\epsilon)h(P) + O(1)$$

holds for all points $P \in \mathbb{P}^n(k)\setminus Z$.

Remark 2.5 In [43], Vojta proved that the exceptional set Z may be chosen to depend only on $\cup_{v\in S}\cup_{0\le i\le n} H_{i,v}$, and not on ϵ, k, or S. Moreover, it follows from [43] that such a set Z is effectively computable (however, note that the implicit constant in the $O(1)$ in the inequality is ineffective, as already happens in Roth's theorem). In contrast to this, it is known [25] that the exceptional set Z in Vojta's conjecture (Conjecture 1.12) must, in general, be allowed to depend on ϵ.

It will also be convenient to use the Subspace Theorem in the equivalent affine form:

Theorem 2.6 *Let S be a finite set of places of a number field k. For each $v \in S$, let $L_{1,v}, \dots, L_{N,v}$ be linearly independent linear forms over k in $N \ge 2$ variables. Let $\epsilon > 0$. Then there exists a finite union of proper subspaces $Z \subset k^N$ such that the inequality*

$$\sum_{v\in S}\sum_{i=1}^{N} \log\frac{|P|_v}{|L_{i,v}(P)|} < (N+\epsilon)h(P) + O(1)$$

holds for all points $P \in k^N\setminus Z$.

Here, if $P = (x_1, \dots, x_N) \in k^N$, $N \geq 2$, then $h(P)$ denotes the height of the associated projective point $[x_1 : \cdots : x_N] \in \mathbb{P}^{N-1}(k)$.

3 Proof of Theorem 1.1

Given a finitely generated subgroup $\Gamma \subset \mathbb{G}_m^n(\overline{\mathbb{Q}})$ and polynomials $f, g \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$, there exists a number field k and a finite set of places S of k (containing the archimedean places) such that $\Gamma \subset \mathbb{G}_m^n(\mathcal{O}_{k,S})$ and $f, g \in k[x_1, \dots, x_n]$. Thus, Theorem 1.1 is equivalent to the following statement.

Theorem 3.1 *Let k be a number field and S a finite set of places of k containing the archimedean places. Let $f, g \in k[x_1, \dots, x_n]$ be coprime polynomials that do not both vanish at the origin. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$-\sum_{v \in M_k} \log^- \max\{|f(u_1, \dots, u_n)|_v, |g(u_1, \dots, u_n)|_v\} < \epsilon \max\{h(u_1), h(u_2), \dots, h(u_n)\}$$

for all $(u_1, \dots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z$.

We prove the theorem by breaking up the sum on the left-hand side in to a sum over places $v \notin S$ and a sum over places $v \in S$. The inequality for the latter sum (Theorem 3.3) follows from work of Evertse [15], but we give a self-contained proof here for completeness (see also [8, Prop. 1]).

We first consider the sum over places not in S, in which case one may drop the vanishing hypothesis on f and g. As mentioned in the introduction, in the case $n = 2$ this result is due to Corvaja and Zannier [8].

Theorem 3.2 *Let k be a number field and let S be a finite set of places of k containing the archimedean places. Let $f, g \in k[x_1, \dots, x_n]$ be coprime polynomials. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$-\sum_{v \in M_k \setminus S} \log^- \max\{|f(u_1, \dots, u_n)|_v, |g(u_1, \dots, u_n)|_v\} < \epsilon \max\{h(u_1), h(u_2), \dots, h(u_n)\}$$

for all $(u_1, \dots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z$.

Proof Let m be a positive integer. For a subset $T \subset k[x_1, \dots, x_n]$, we let

$$T_m = \{p \in T \mid \deg p \le m\},$$

where $\deg p$ denotes the (total) degree of the polynomial p.

Consider the ideal $(f, g) \subset k[x_1, \dots, x_n]$. If $(f, g) = (1)$ is the entire polynomial ring, then it is elementary that the left-hand side of the inequality of the theorem is bounded by a constant, and the result holds (with Z a finite set). Suppose now that (f, g) is a proper ideal. Let $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S})$. For $v \in S$, we construct a basis B_v for $V_m = k[x_1, \dots, x_n]_m/(f, g)_m$ as follows. Choose a monomial $\mathbf{x}^{\mathbf{i}_1} \in k[x_1, \dots, x_n]_m$ so that $|\mathbf{u}^{\mathbf{i}_1}|_v$ is minimal subject to the condition $\mathbf{x}^{\mathbf{i}_1} \notin (f, g)$. Suppose now that $\mathbf{x}^{\mathbf{i}_1}, \dots, \mathbf{x}^{\mathbf{i}_j}$ have been constructed and are linearly independent modulo $(f, g)_m$, but don't span $k[x_1, \dots, x_n]_m$ modulo $(f, g)_m$. Then we let $\mathbf{x}^{\mathbf{i}_{j+1}} \in k[x_1, \dots, x_n]_m$ be a monomial such that $|\mathbf{u}^{\mathbf{i}_{j+1}}|_v$ is minimal subject to the condition that $\mathbf{x}^{\mathbf{i}_1}, \dots, \mathbf{x}^{\mathbf{i}_{j+1}}$ are linearly independent modulo $(f, g)_m$. In this way, we construct a basis of V_m with monomial representatives $\mathbf{x}^{\mathbf{i}_1}, \dots, \mathbf{x}^{\mathbf{i}_{N'}}$, where $N' = N'_m = \dim V_m$. Let $I_v = \{\mathbf{i}_1, \dots, \mathbf{i}_{N'}\}$. We also choose a basis $\phi_1, \dots, \phi_N$ of the k-vector space $(f, g)_m$, where $N = N_m = \dim(f, g)_m$. Now for each $\mathbf{i}$, $|\mathbf{i}| \le m$, we have that

$$\mathbf{x}^{\mathbf{i}} + \sum_{j=1}^{N'} c_{\mathbf{i},j}\mathbf{x}^{\mathbf{i}_j} \in (f, g)_m$$

for some choice of coefficients $c_{\mathbf{i},j} \in k$. Then for each such $\mathbf{i}$ there is a linear form $L_{\mathbf{i}}^v$ over k such that

$$L_{\mathbf{i}}^v(\phi_1, \dots, \phi_N) = \mathbf{x}^{\mathbf{i}} + \sum_{j=1}^{N'} c_{\mathbf{i},j}\mathbf{x}^{\mathbf{i}_j}.$$

Note that $\{L_{\mathbf{i}}^v(\phi_1, \dots, \phi_N) \mid |\mathbf{i}| \le m, \mathbf{i} \notin I_v\}$ is a basis for $(f, g)_m$, and $\{L_{\mathbf{i}}^v \mid |\mathbf{i}| \le m, \mathbf{i} \notin I_v\}$ is a set of N linearly independent linear forms in N variables. Let

$$P = \phi(\mathbf{u}) := (\phi_1(\mathbf{u}), \dots, \phi_N(\mathbf{u})) \in k^N.$$

From the triangle inequality and the definition of $\mathbf{x}^{\mathbf{i}_1}, \dots, \mathbf{x}^{\mathbf{i}_{N'}}$, for any $\mathbf{i}$ with $|\mathbf{i}| \le m, \mathbf{i} \notin I_v$, we have the key inequality

$$\log |L_{\mathbf{i}}^v(P)|_v \le \log |\mathbf{u}^{\mathbf{i}}|_v + C_v,$$

where the constant C_v depends only on $v \in S$ and the set $\{\mathbf{i}_1, \dots, \mathbf{i}_{N'}\}$ (and not on $\mathbf{u}$).

We will apply the Subspace Theorem with the choice of linear forms $L_{\mathbf{i}}^v$, $|\mathbf{i}| \le m$, $\mathbf{i} \notin I_v$, for each $v \in S$. We want to estimate the sum

$$\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log \frac{|P|_v}{|L_{\mathbf{i}}^v(P)|_v}.$$

Towards this end, we estimate the sums

$$-\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log |L_{\mathbf{i}}^v(P)|_v$$

and

$$\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log |P|_v$$

separately.

We have

$$-\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log |L_{\mathbf{i}}^v(P)|_v \ge -\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log |\mathbf{u}^{\mathbf{i}}|_v - CN,$$

where $C = \sum_{v \in S} C_v$. Since $\mathbf{u}^{\mathbf{i}}$ is an S-unit, by the product formula,

$$\sum_{v \in S} \log |\mathbf{u}^{\mathbf{i}}|_v = \sum_{v \in M_k} \log |\mathbf{u}^{\mathbf{i}}|_v = 0.$$

It follows that

$$\begin{aligned} -\sum_{v \in S} \sum_{\substack{|\mathbf{i}| \le m \\ \mathbf{i} \notin I_v}} \log |\mathbf{u}^{\mathbf{i}}|_v &= -\sum_{v \in S} \sum_{|\mathbf{i}| \le m} \log |\mathbf{u}^{\mathbf{i}}|_v + \sum_{v \in S} \sum_{\mathbf{i} \in I_v} \log |\mathbf{u}^{\mathbf{i}}|_v \\ &= \sum_{v \in S} \sum_{\mathbf{i} \in I_v} \log |\mathbf{u}^{\mathbf{i}}|_v. \end{aligned}$$

Using the easy inequality

$$-\sum_{v \in S} \sum_{\mathbf{i} \in I_v} \log |\mathbf{u}^{\mathbf{i}}|_v \le N'm(h(u_1) + \cdots + h(u_n)) \le N'mn \max_i h(u_i),$$

we find that

$$-\sum_{v\in S}\sum_{\substack{|\mathbf{i}|\le m\\ \mathbf{i}\notin I_v}} \log|L_{\mathbf{i}}^v(P)|_v \ge -N'mn\max_i h(u_i) - CN.$$

On the other hand,

$$\sum_{v\in S}\sum_{\substack{|\mathbf{i}|\le m\\ \mathbf{i}\notin I_v}} \log|P|_v = N\sum_{v\in S}\log|P|_v = N\left(h(P) - \sum_{v\in M_k\backslash S}\log|P|_v\right).$$

Since $\phi_i \in (f,g)$, we can write $\phi_i = fp_i + gq_i$ for some $p_i, q_i \in k[x_1,\ldots,x_n]$. Then for $v\in M_k\backslash S$,

$$\log|\phi_i(\mathbf{u})|_v \le \log^-\max\{|f(\mathbf{u})|_v, |g(\mathbf{u})|_v\} + C'_{i,v},$$

where $C'_{i,v}$ depends only on v and the coefficients of f, g, p_i, q_i, and $C'_{i,v} = 0$ for all but finitely many $v\in M_k\backslash S$.

Then

$$\sum_{v\in S}\sum_{\substack{|\mathbf{i}|\le m\\ \mathbf{i}\notin I_v}} \log|P|_v \ge N\left(h(P) - \sum_{v\in M_k\backslash S}\log^-\max\{|f(\mathbf{u})|_v, |g(\mathbf{u})|_v\} + C'\right)$$

for some constant C' depending only on f, g, and the basis $\phi_1,\ldots,\phi_N$.

We will also make use of the estimate

$$h(P) \le mn\max_i h(u_i) + O(1).$$

Schmidt's Subspace Theorem (with, say, $\epsilon = 1$) implies that there exists a finite union Z of proper subspaces of k^N such that

$$\sum_{v\in S}\sum_{\substack{|\mathbf{i}|\le m\\ \mathbf{i}\notin I_v}} \log\frac{|Q|_v}{|L_{\mathbf{i}}^v(Q)|_v} \le (N+1)h(Q)$$

for all $Q\in k^N\backslash Z$.

Using the above estimates, if $P = \phi(\mathbf{u}) \notin Z$, we find that

$$N\left(h(P) - \sum_{v \in M_k \setminus S} \log^- \max\{|f(\mathbf{u})|_v, |g(\mathbf{u})|_v\} - C''\right) - N'mn \max_i h(u_i) \leq (N+1)h(P)$$

or

$$N\left(-\sum_{v \in M_k \setminus S} \log^- \max\{|f(\mathbf{u})|_v, |g(\mathbf{u})|_v\} - C'''\right) \leq (N'+1)mn \max_i h(u_i) \tag{7}$$

for some constants C'' and C'''.

Since f and g are coprime, the ideal (f, g) defines a closed subset of $\mathbb{A}^n$ of codimension at least 2. As is well-known from the theory of Hilbert functions and Hilbert polynomials, this easily implies that $N' = O(m^{n-2})$ and $N = \frac{m^n}{n!} + O(m^{n-1})$. Let $\epsilon > 0$. Then choosing m large enough, depending on ϵ, (7) implies that

$$-\sum_{v \in M_k \setminus S} \log^- \max\{|f(\mathbf{u})|_v, |g(\mathbf{u})|_v\} \leq \epsilon \max_i h(u_i) + O(1), \tag{8}$$

as long as $\mathbf{u}$ does not lie in the proper closed subset of $\mathbb{G}_m^n$ coming from the exceptional set in the application of the Subspace Theorem.

Finally, we note that the choice of linear forms in the application of Schmidt's Subspace Theorem depends not on $\mathbf{u}$, but on the choice of the monomial bases B_v, $v \in S$. Since for fixed m there are only finitely many monomials of degree at most m, and hence only finitely man choices for these bases, we see that for fixed m the given argument leads to only finitely many applications of Schmidt's Subspace Theorem (over all choices of $\mathbf{u}$). Therefore the inequality (8) is valid for all $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S})$, outside of some proper Zariski closed subset Z of $\mathbb{G}_m^n$. We may also eliminate the $O(1)$ term in (8) by enlarging Z. By Laurent's theorem, after replacing Z by the Zariski closure of $\mathbb{G}_m^n(\mathcal{O}_{k,S}) \cap Z$ in $\mathbb{G}_m^n$, we may assume that Z is a finite union of translates of proper algebraic subgroups of $\mathbb{G}_m^n$. □

We now consider the sum over places in S. In fact, in this setting, we prove a stronger statement for a single polynomial that doesn't vanish at the origin. As mentioned, this inequality follows from work of Evertse [15]. We give a proof that is a slight variation of the proof for $n = 2$ in [8, Prop. 1].

Theorem 3.3 *Let k be a number field and S a finite set of places of k containing the archimedean places. Let* $f \in k[x_1, \ldots, x_n]$ *be a polynomial that doesn't vanish at the origin* $(0, \ldots, 0)$. *For all* $\epsilon > 0$, *there exists a finite union Z of translates of proper algebraic subgroups of* $\mathbb{G}_m^n$ *such that*

$$-\sum_{v \in S} \log^- |f(u_1, \ldots, u_n)|_v < \epsilon \max\{h(u_1), \ldots, h(u_n)\}$$

for all $(u_1, \ldots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z$.

Proof By Laurent's theorem, the set of points $\{(u_1, \ldots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \mid f(u_1, \ldots, u_n) = 0\}$ lies in a finite union of translates of proper algebraic subgroups of $\mathbb{G}_m^n$. Thus, we may always choose Z to contain all such points and we will (implicitly) ignore such points in the remainder of the proof.

We first prove a Diophantine approximation statement depending on the choice of a subset $S' \subset S$. Having chosen $S' \subset S$, let R consist of the set of points $(u_1, \ldots, u_n) \in \mathbb{G}_m^n(\mathcal{O}_{k,S})$ such that

$$S' = \{v \in S \mid \log |f(u_1, \ldots, u_n)|_v < 0\}.$$

Then for $(u_1, \ldots, u_n) \in R$,

$$\log^- |f(u_1, \ldots, u_n)|_v = \begin{cases} \log |f(u_1, \ldots, u_n)|_v & \text{if } v \in S', \\ 0 & \text{if } v \in S \setminus S'. \end{cases}$$

Let $d = \deg f$ and let $\phi : \mathbb{P}^n \to \mathbb{P}^N$, $\phi = (\phi_0, \ldots, \phi_N)$, $N = \binom{n+d}{n} - 1$, be the d-uple embedding of $\mathbb{P}^n$ given by the set of monomials of degree d in $k[x_0, \ldots, x_n]$ (in some order). Let $F = x_0^d f(x_1/x_0, \ldots, x_n/x_0)$ be the homogenization of f in $k[x_0, \ldots, x_n]$. Let V_d be the vector space of homogeneous polynomials of degree d, and let Mon_d consist of the set of all monomials in $k[x_0, \ldots, x_n]$ of degree d.

If $v \in S'$, we let

$$B_v = B = \{F(x_0, \ldots, x_n)\} \cup \text{Mon}_d \setminus \{x_0^d\},$$

so that B is obtained by replacing x_0^d in Mon_d by F. Since f doesn't vanish at the origin, x_0^d appears with a nonzero coefficient in F, and thus it's clear that B is a basis for V_d.

If $v \in S \setminus S'$, then we let $B_v = \text{Mon}_d$. Let $\epsilon > 0$. Then applying the Subspace Theorem on $\mathbb{P}^N$ with appropriate linear forms, we find that

$$\sum_{v \in S} \sum_{Q \in B_v} \log \frac{|\phi(P)|_v}{|Q(P)|_v} < (N + 1 + \epsilon) h(\phi(P)) \tag{9}$$

for all $P \in \mathbb{P}^n(k)\backslash Z$, where $Z = \phi^{-1}(Z')$ and Z' is a finite union of hyperplanes in $\mathbb{P}^N$. From the definition of B_v, we can rewrite the left-hand side of (9) as

$$\sum_{v\in S}\sum_{Q\in\mathrm{Mon}_d}\log\frac{|\phi(P)|_v}{|Q(P)|_v}-\sum_{v\in S'}\log\frac{|F(P)|_v}{|x_0^d(P)|_v}<(N+1+\epsilon)h(\phi(P)).$$

Suppose now that $(u_1,\ldots,u_n)\in R$ and let $P=[1:u_1:\cdots:u_n]\in\mathbb{P}^n(k)$. It follows immediately from the definitions that

$$\begin{aligned}-\sum_{v\in S'}\log\frac{|F(P)|_v}{|x_0^d(P)|_v}&=-\sum_{v\in S'}\log|f(u_1,\ldots,u_n)|_v\\&=-\sum_{v\in S}\log^-|f(u_1,\ldots,u_n)|_v.\end{aligned}$$

Since the u_i are S-units,

$$\begin{aligned}\sum_{v\in S}\sum_{Q\in\mathrm{Mon}_d}\log\frac{|\phi(P)|_v}{|Q(P)|_v}&=\sum_{v\in M_k}\sum_{Q\in\mathrm{Mon}_d}\log\frac{|\phi(P)|_v}{|Q(P)|_v}\\&=\sum_{v\in M_k}\sum_{Q\in\mathrm{Mon}_d}\log|\phi(P)|_v\\&=(N+1)\sum_{v\in M_k}\log|\phi(P)|_v\\&=(N+1)h(\phi(P)),\end{aligned}$$

where the second equality follows from the product formula. We also have

$$h(\phi(P))=dh(P)\le dn\max_i h(u_i).$$

Then combining everything, (9) implies that

$$-\sum_{v\in S}\log^-|f(u_1,\ldots,u_n)|_v<\epsilon dn\max_i h(u_i)$$

for all $(u_1,\ldots,u_n)\in R$ outside of some proper Zariski closed subset Z of $\mathbb{G}_m^n$. In fact, since there are only finitely many choices of the subset $S'\subset S$ (and so $\mathbb{G}_m^n(\mathcal{O}_{k,S})$ is partitioned into finitely many sets R as above), we find that the inequality holds for all $P\in\mathbb{G}_m^n(\mathcal{O}_{k,S})\backslash Z$, for some (possibly larger) proper closed subset $Z\subset\mathbb{G}_m^n$. Finally, by Laurent's theorem, after replacing Z by the Zariski closure of $\mathbb{G}_m^n(\mathcal{O}_{k,S})\cap Z$ in $\mathbb{G}_m^n$, we may assume that Z is a finite union of translates of proper algebraic subgroups of $\mathbb{G}_m^n$. □

Theorem 3.1 now follows immediately from combining Theorems 3.2 and 3.3.

4 Vojta's conjecture for blowups of $\mathbb{G}_m^n$

The goal of this section is to prove (Theorem 1.14) a general form of Vojta's Conjecture for blowups of $\mathbb{G}_m^n$ (in the sense of Conjecture 1.13), with the compactification coming from an embedding of $\mathbb{G}_m^n$ in a nonsingular projective toric variety. To illustrate elements of the general proof, in the next section we consider the simple case when $\mathbb{G}_m^n$ is embedded in projective space $\mathbb{P}^n$. By an easy direct explicit computation, we transform Theorems 3.2 and 3.3 into related Diophantine approximation statements on projective space. In the subsequent section, we extend these results to nonsingular projective toric varieties and derive Theorem 1.14.

4.1 Two inequalities on projective space

We state some easy consequences of Theorems 3.2 and 3.3 in terms of Diophantine approximation on projective space.

The first result we derive, Theorem 4.1, is not new (it traces back to Evertse's paper [15]), but the derivation given here will be generalized to projective toric varieties in the next section. Let k be a number field and let D be an effective divisor on $\mathbb{P}^n$ defined by a homogeneous polynomial $F \in k[x_0, \ldots, x_n]$ of degree $d > 0$. Recall that a local height associated to D and $v \in M_k$ is given by

$$\lambda_{D,v}(P) = \log \frac{|P|_v^d}{|F(P)|_v} = \log \frac{\max_i |x_i|_v^d}{|F(x_0, \ldots, x_n)|_v},$$

for $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \setminus \operatorname{Supp} D$.

Consider the $n + 1$ isomorphisms

$$\begin{aligned} \psi_i : \mathbb{P}^n \setminus \cup_{j=0}^n H_j &\to \mathbb{G}_m^n, \\ [x_0 : \cdots : x_n] &\mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right), \end{aligned}$$

for $i = 0, \ldots, n$, where $H_0, \ldots, H_n$ are the coordinate hyperplanes in $\mathbb{P}^n$. Let $f_i \in k[x_1, \ldots, x_n]$, $i = 0, \ldots, n$, be the ith dehomogenized polynomial such that

$$x_i^d f_i \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right) = F(x_0, \ldots, x_n).$$

Let $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(k) \setminus \cup_{j=0}^n H_j$ and let $v \in M_k$. Let i be such that

$$|x_i|_v = \max_j |x_j|_v.$$

Then using the representation $P = [x_0/x_i : \cdots : x_n/x_i]$, we have $|P|_v = 1$ and

$$\lambda_{D,v}(P) = \log \frac{|P|_v^d}{|F(P)|_v} = -\log |f_i(\psi_i(P))|_v \leq -\log^- |f_i(\psi_i(P))|_v.$$

Therefore,

$$\lambda_{D,v}(P) \leq \sum_{i=0}^{n} -\log^- |f_i(\psi_i(P))|_v + O(1). \tag{10}$$

Note that all of the $n+1$ polynomials f_i, $i = 0, \ldots, n+1$, are nonvanishing at the origin if and only if the support of the divisor D doesn't contain any of the $n+1$ points $P_0 = [1 : 0 : \cdots : 0], \ldots, P_n = [0 : 0 : \cdots : 0 : 1]$. Equivalently, this occurs if and only if D is in general position with the components of the boundary of $\mathbb{G}_m^n$ in $\mathbb{P}^n$ (i.e., the coordinate hyperplanes $H_0, \ldots, H_n$). Since the heights of the coordinates of $\psi_i(P)$ are trivially bounded by $h(P)$, Theorem 3.3 and (10) immediately imply the following (known) result.

Theorem 4.1 *Let D be an effective divisor on $\mathbb{P}^n$, defined over a number field k, such that*

$$P_0, \ldots, P_n \notin \operatorname{Supp} D.$$

Let S be a finite set of places of k containing the archimedean places. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that

$$m_{D,S}(P) \leq \epsilon h(P) + O(1)$$

for all points $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z \subset \mathbb{P}^n(k)$.

Remark 4.2 Let Y be any closed subscheme of $\mathbb{P}^n$ defined by an ideal $I \subset k[x_0, \ldots, x_n]$. If D is any divisor defined by a nonconstant polynomial $f \in I$, then $\lambda_{Y,v}(P) \leq \lambda_{D,v}(P) + O(1)$ for all $P \in \mathbb{P}^n(k) \setminus \operatorname{Supp} D$. Thus, Theorem 4.1 trivially extends to closed subschemes Y of $\mathbb{P}^n$ with $P_0, \ldots, P_n \notin Y$ (with $m_{D,S}(P)$ replaced by $m_{Y,S}(P)$ in the inequality). By a similar argument, to prove upper bounds for sums of local heights associated to closed subschemes

Y of codimension at least 2 in $\mathbb{P}^n$, it suffices to consider the case where Y has exactly codimension 2, defined by the vanishing of two homogeneous polynomials.

We now reformulate Theorem 3.2 in terms of heights on $\mathbb{P}^n$. Let $F, G \in k[x_0, \ldots, x_n]$ be coprime homogeneous polynomials of degrees d and e, respectively. Let

$$f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n)$$
$$g(x_1, \ldots, x_n) = G(1, x_1, \ldots, x_n)$$

be dehomogenizations of F and G, respectively.

Let Y be the closed subscheme of $\mathbb{P}^n$ defined by $F = G = 0$. Then from the previous local height formulas, for $v \in M_k$, a local height associated to Y can be taken to be

$$\lambda_{Y,v}(P) = \min \left\{ \log \frac{|P|_v^d}{|F(P)|_v}, \log \frac{|P|_v^e}{|G(P)|_v} \right\},$$

where $P \in \mathbb{P}^n(k) \setminus Y$. It is immediate from the local height formula that if $f, g \in \mathcal{O}_{k,S}[x_1, \ldots, x_n]$ and $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \subset \mathbb{P}^n(k)$, then

$$\lambda_{Y,v}(P) = -\log \max\{|f(P)|_v, |g(P)|_v\} = -\log^- \max\{|f(P)|_v, |g(P)|_v\}$$

for all $v \in M_k \setminus S$. In fact, this holds up to $O_v(1)$ (and identically for all but finitely many v) even without the S-integrality assumption on the coefficients of f and g, since after multiplying by a nonzero constant, f and g will have S-integral coefficients.

Then translating Theorem 3.2 into the language of heights yields the following theorem.

Theorem 4.3 *Let Y be a closed subscheme of $\mathbb{P}^n$ of codimension at least 2, defined over a number field k. Let S be a finite set of places of k containing the archimedean places. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$N_{Y,S}(P) \leq \epsilon h(P) + O(1)$$

for all $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z \subset \mathbb{P}^n(k)$.

Combining Theorems 4.1 and 4.3 (and Remark 4.2) gives Theorem 1.16 from the Introduction. These results are generalized to nonsingular projective toric varieties in the next section, where they are used in deriving the consequences towards Vojta's conjecture.

4.2 Toric varieties and their blowups

We first generalize Theorem 4.1 to an arbitrary nonsingular projective toric variety X. The proof of Theorem 4.1 proceeded via (10), which (implicitly) used the natural affine covering of $\mathbb{P}^n$ by the $n+1$ affine spaces $\mathbb{A}^n = \mathbb{P}^n \setminus H_i$, $i = 0, \ldots, n$. The general proof is similar and uses a natural covering of X by copies of $\mathbb{A}^n$ coming from the theory of toric varieties. We only use a few basic facts about toric varieties available in any standard reference (e.g., [11]).

Theorem 4.4 *Let X be a nonsingular projective toric variety of dimension n defined over a number field k. Let A be a big divisor on X. Let D be an effective divisor on X, defined over k, that is in general position with the boundary of $\mathbb{G}_m^n$ in X. Let S be a finite set of places of k and let $\epsilon > 0$. Then there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$m_{D,S}(P) \le \epsilon h_A(P) + O(1)$$

for all points $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z \subset X(k)$.

Note that in the case $X = \mathbb{P}^n$, the general position condition in the theorem is precisely the condition $P_0, \ldots, P_n \notin \operatorname{Supp} D$ of Theorem 4.1.

Proof It suffices to prove the inequality of the theorem with $m_{D,S}(P)$ replaced by $\lambda_{D,v}(P)$ for a single place $v \in S$.

Let Σ be the fan corresponding to the toric variety X. Then an affine covering of X is given by the affine varieties X_σ, where σ ranges over the maximal cones $\sigma \in \Sigma$. Let $\sigma \in \Sigma$ be a maximal cone. Since X is smooth and complete, the fan Σ is smooth and complete [11, Th. 3.1.19], and it follows that σ is an n-dimensional smooth cone. Thus, there is an isomorphism $i_\sigma : X_\sigma \to \mathbb{A}^n$ [11, Ex. 1.2.21] . This isomorphism restricts to an automorphism of $\mathbb{G}_m^n$, where we identify $\mathbb{G}_m^n \subset X_\sigma$ naturally as a subset of X and $\mathbb{G}_m^n \subset \mathbb{A}^n$ in the standard way, so that $\mathbb{A}^n \setminus \mathbb{G}_m^n$ consists of the affine coordinate hyperplanes defined by $x_i = 0$, $i = 1, \ldots, n$. In the latter case, it will be convenient to use the height $h(u_1, \ldots, u_n) = h(u_1) + \cdots + h(u_n)$ for $(u_1, \ldots, u_n) \in \mathbb{G}_m^n(k)$. Since i_σ yields an automorphism of $\mathbb{G}_m^n$, it follows from Lemma 2.2 that for some constant $C_{\sigma,A}$ and some closed subset $Z_{\sigma,A} \subset \mathbb{G}_m^n$, depending on σ and A,

$$h(i_\sigma(P)) \le C_{\sigma,A} h_A(P)$$

for all $P \in \mathbb{G}_m^n(k) \setminus Z_{\sigma,A} \subset X(k)$.

The pullback $(i_\sigma^{-1})^*(D|_{X_\sigma})$ of D to $\mathbb{A}^n$ is defined by some nonzero polynomial $f \in k[x_1, \ldots, x_n]$. Since D is in general position with the boundary of $\mathbb{G}_m^n$ in X, it follows that f does not vanish at the origin. From properties of local Weil functions, for $P \in X_\sigma(k_v)$,

$$\lambda_{D,v}(P) = -\log|f(i_\sigma(P))|_v + \alpha_v(i_\sigma(P))$$

for some continuous function $\alpha_v(P)$ on $\mathbb{A}^n(k_v)$. Let $U \subset \mathbb{A}^n(k_v)$ be an open subset in the v-topology with compact closure. It follows that

$$\lambda_{D,v}(P) = -\log|f(i_\sigma(P))|_v + O(1)$$

for all $P \in i_\sigma^{-1}(U)$. By Theorem 3.3, there exists a finite union $Z_{\sigma,A,U}$ of translates of algebraic subgroups such that for all $P \in i_\sigma^{-1}(U) \cap (\mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z_{\sigma,A,U})$,

$$\lambda_{D,v}(P) < \frac{\epsilon}{C_{\sigma,A}} h(i_\sigma(P)) + O(1) < \epsilon h_A(P) + O(1).$$

Since X is projective, $X(k_v)$ is compact (in the v-topology) and $X(k_v)$ is covered by finitely many such open sets $i_\sigma^{-1}(U)$ (for varying σ). Therefore, we find that there exists a finite union of translates of algebraic subgroups Z such that for all $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z$,

$$\lambda_{D,v}(P) < \epsilon h_A(P) + O(1).$$

□

Next, we extend Theorem 4.3 to nonsingular projective toric varieties.

Theorem 4.5 *Let X be a nonsingular projective toric variety of dimension n, and let Y be a closed subscheme of X of codimension at least 2, both defined over a number field k. Let A be a big divisor on X. Let S be a finite set of places of k containing the archimedean places. For all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that*

$$N_{Y,S}(P) \le \epsilon h_A(P) + O(1)$$

for all $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S}) \setminus Z \subset X(k)$.

Proof Let $\mathcal{I}_Y$ be the ideal sheaf of Y. We identify the coordinate ring $k[\mathbb{G}_m^n]$ with the ring $k[x_1, 1/x_1, \ldots, x_n, 1/x_n]$. Since Y has codimension at least 2 in X, we can find $f, g \in \Gamma(\mathbb{G}_m^n, \mathcal{I}_Y|_{\mathbb{G}_m^n})$ such that $f, g \in k[x_1, \ldots, x_n]$ are coprime polynomials. The set $\mathbb{G}_m^n(\mathcal{O}_{k,S}) \times (M_k \setminus S)$ is trivially M_k-bounded inside $\mathbb{G}_m^n$, and then it is immediate from properties of Weil functions [37, Prop. 2.4] that for $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S})$ and $v \in M_k \setminus S$,

$$\begin{aligned}\lambda_{Y,v}(P) &\leq -\log\max\{|f(P)|_v, |g(P)|_v\} + O_v(1),\\ &\leq -\log^- \max\{|f(P)|_v, |g(P)|_v\} + O_v(1),\end{aligned}$$

where $O_v(1)$ may be taken to be 0 for all but finitely many v. Then by Theorem 3.2 (and Lemma 2.2), for all $\epsilon > 0$ there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^n$ such that

$$\begin{aligned}N_{Y,S}(P) &\leq -\sum_{v\in M_k\setminus S} \log^- \max\{|f(P)|_v, |g(P)|_v\} + O(1)\\ &\leq \epsilon h_A(P) + O(1)\end{aligned}$$

for all $P \in \mathbb{G}_m^n(\mathcal{O}_{k,S})\setminus Z \subset X(k)$. □

Finally, we combine the previous two results with some geometry to deduce Theorem 1.14.

Proof of Theorem 1.14 From [12, p. 29],

$$K_{\tilde{X}} \sim \pi^* K_X + R,$$

for some effective divisor R supported on $\operatorname{Exc}(\pi)$. Since a canonical divisor K_X can naturally be taken [11, Th. 8.2.3] to be the negative of the boundary divisor $X\setminus\mathbb{G}_m^n$ and π is a birational morphism, $D_0 \sim -\pi^* K_X - E'$ for some effective divisor E' whose support is contained in $\operatorname{Exc}(\pi)$. Thus,

$$K_{\tilde{X}} + D_0 \sim R - E' = E'',$$

where E'' is a divisor whose support is contained in $\operatorname{Exc}(\pi)$.

Then up to $O(1)$,

$$m_{D,S}(P) + h_{K_{\tilde{X}}+D_0}(P) = m_{D,S}(P) + h_{E''}(P) = m_{D+E'',S}(P) + N_{E'',S}(P)$$

for all $P \in \tilde{X}(k)\setminus \operatorname{Supp}(D + E'')$. Since $\pi(\operatorname{Exc}(\pi)) \cup \operatorname{Supp}\pi_* D$ is in general position with $X\setminus\mathbb{G}_m^n$, we can find an effective divisor D' on X, again in general position with $X\setminus\mathbb{G}_m^n$, such that $D + E'' \leq \pi^* D'$ (i.e., $\pi^* D' - (D + E'')$ is effective). Then for $P \in \tilde{X}(k)\setminus \operatorname{Supp}\pi^* D'$,

$$m_{D+E'',S}(P) \leq m_{\pi^* D',S}(P) + O(1) = m_{D',S}(\pi(P)) + O(1).$$

Similarly, since E'' is supported on $\operatorname{Exc}(\pi)$ and $\pi(\operatorname{Exc}(\pi))$ has codimension at least 2 in X [12, p. 28], we can find a closed subscheme Y of X of codimension at least 2 such that

$$N_{E'',S}(P) \leq N_{\pi^*Y,S}(P) + O(1) = N_{Y,S}(\pi(P)) + O(1).$$

Let $\epsilon > 0$. Then it follows from combining the above with Lemma 2.2, Theorems 4.4, and 4.5 that there exists a proper Zariski closed subset Z of $\tilde{X}$ such that the inequality

$$m_{D,S}(P) + h_{K_{\tilde{X}}+D_0}(P) \leq \epsilon h_A(P) + O(1)$$

holds for all points $P \in \pi^{-1}(\mathbb{G}_m^n(\mathcal{O}_{k,S}))\backslash Z$. □

5 Greatest common divisors and simple linear recurrence sequences

In this section, we give an application of our results to greatest common divisors of terms from simple linear recurrence sequences. We begin by noting the following lemma, which is a straightforward consequence of elementary estimates, and is equivalent to part (a) of Theorem 1.11. This yields a trivial situation where terms from two simple linear recurrence sequences may have a "large" greatest common divisor.

Lemma 5.1 *Let*

$$F(n) = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_s\alpha_s^n,$$
$$G(n) = d_1\beta_1^n + d_2\beta_2^n + \cdots + d_t\beta_t^n,$$

define two algebraic simple linear recurrence sequences. Let $|\cdot|$ *be an absolute value on* $\overline{\mathbb{Q}}$ *such that* $|\alpha_i| < 1$ *and* $|\beta_j| < 1$ *for all* i *and* j. *Then there exists* $\delta > 0$ *and a positive integer* N *such that*

$$-\log^- \max\{|F(m)|, |G(n)|\} \geq \delta \min\{m, n\}$$

for all $m, n \geq N$.

The next result gives a counterpart to Lemma 5.1.

Lemma 5.2 *Let*

$$F(n) = c_0\alpha_0^n + c_1\alpha_1^n + \cdots + c_s\alpha_s^n$$

define a nondegenerate algebraic simple linear recurrence sequence. Let $|\cdot|$ *be an absolute value on* $\overline{\mathbb{Q}}$ *such that* $|\alpha_i| \geq 1$ *for some* i. *Let* $\epsilon > 0$. *Then*

$$-\log|F(n)| < \epsilon n$$

for all but finitely many $n \in \mathbb{N}$.

Recall that a linear recurrence is called *nondegenerate* if the ratios of distinct roots $\frac{\alpha_i}{\alpha_j}$ is not a root of unity for every $i \neq j$.

Proof Let k be a number field and S a finite set of places of k such that $c_i, \alpha_i \in \mathcal{O}_{k,S}^*$, $i = 0, \ldots, s$, and $|\cdot|$ restricted to k is equivalent to $|\cdot|_v$ for some $v \in S$ (note that if $|\cdot|$ is trivial, the lemma is obvious). It suffices to prove that

$$-\log |F(n)|_v < \epsilon n$$

for all but finitely many $n \in \mathbb{N}$.

Let H_i be the coordinate hyperplane in $\mathbb{P}^s$ defined by $x_i = 0, i = 0, \ldots, s$. Let H_{s+1} be the hyperplane in $\mathbb{P}^s$ defined by $c_0x_0 + c_1x_1 + \cdots + c_sx_s = 0$. Note that the $s+2$ hyperplanes $H_0, \ldots, H_{s+1}$ are in general position. Let

$$P_n = [\alpha_0^n : \cdots : \alpha_s^n] \in \mathbb{P}^s(k), \quad n \in \mathbb{N}.$$

Then the Schmidt Subspace Theorem gives that for some finite union of hyperplanes Z in $\mathbb{P}^s$,

$$\sum_{i=0}^{s+1} m_{H_i,S}(P_n) < (s+1+\epsilon)h(P_n) \tag{11}$$

for all points $P_n \in \mathbb{P}^s(k)\backslash Z$. In fact, since F is nondegenerate, by the Skolem–Mahler–Lech theorem [14, Theorem 2.1], only finitely many points P_n can belong to any given hyperplane in $\mathbb{P}^s$, and thus the inequality holds for all but finitely many n. Since $\alpha_i \in \mathcal{O}_{k,S}^*$ for all i, $m_{H_i,S}(P_n) = h(P_n), i = 0, \ldots, s$. Note also that

$$h(P_n) = nh(P_1)$$

for all $n \in \mathbb{N}$. Substituting into (11), we find that for all $\epsilon > 0$,

$$m_{H_{s+1},S}(P_n) < \epsilon n$$

for all but finitely many $n \in \mathbb{N}$. Now

$$m_{H_{s+1},S}(P_n) \geq \lambda_{H_{s+1},v}(P_n) + O(1)$$

and

$$\lambda_{H_{s+1},v}(P_n) = \log \frac{\max_i |\alpha_i^n|_v}{|c_0\alpha_0^n + c_1\alpha_1^n + \cdots + c_s\alpha_s^n|_v} \geq -\log|F(n)|_v$$

for all n, since by our hypotheses, $\max_i |\alpha_i^n|_v \geq 1$. Thus, for all $\epsilon > 0$,

$$-\log|F(n)|_v < \epsilon n$$

for all but finitely many $n \in \mathbb{N}$. □

We now prove the main result of this section, Theorem 1.11(b). For convenience, we restate the result.

Theorem 5.3 *Let*

$$F(n) = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_s\alpha_s^n,$$
$$G(n) = d_1\beta_1^n + d_2\beta_2^n + \cdots + d_t\beta_t^n,$$

define two algebraic simple linear recurrence sequences. Let k be a number field such that $c_i, \alpha_i, d_j, \beta_j \in k$ for $i = 1, \ldots, s$, $j = 1, \ldots, t$. Let

$$S_0 = \{v \in M_k : \max\{|\alpha_1|_v, \ldots, |\alpha_s|_v, |\beta_1|_v, \ldots, |\beta_t|_v\} < 1\}.$$

Let $\epsilon > 0$. If the inequality

$$\sum_{v \in M_k \setminus S_0} -\log^- \max\{|F(m)|_v, |G(n)|_v\} > \epsilon \max\{m, n\} \tag{12}$$

has infinitely many solutions (m, n), then all but finitely many of them satisfy one of finitely many linear relations

$$(m, n) = (a_i t + b_i, c_i t + d_i), \quad t \in \mathbb{Z}, i = 1, \ldots, r,$$

where $a_i, b_i, c_i, d_i \in \mathbb{Z}$, $a_i c_i \neq 0$, and the linear recurrences $F(a_i \bullet + b_i)$ and $G(c_i \bullet + d_i)$ have a nontrivial common factor for $i = 1, \ldots, r$.

In particular, if $S_0 = \emptyset$, then the same statement holds for the inequality

$$\log \gcd(F(m), G(n)) > \epsilon \max\{m, n\}.$$

Proof We begin with a couple of convenient reductions. First, by considering finitely many arithmetic progressions in m and n, we may reduce to the case where the combined roots of F and G generate a torsion-free group Γ of rank r (in particular, both F and G are nondegenerate). Let $S \supset S_0$ be a finite set of places of k, containing the archimedean places, such that $c_i, \alpha_i, d_j, \beta_j \in \mathcal{O}_{k,S}^*$ for all i and j. By Lemma 5.2,

$$\sum_{v \in M_k \setminus S_0} -\log^- \max\{|F(m)|_v, |G(n)|_v\} \leq \sum_{v \in M_k \setminus S} -\log^- \max\{|F(m)|_v, |G(n)|_v\} + \frac{\epsilon}{2} \max\{m, n\}$$

for all but finitely many $m, n \in \mathbb{N}$. Thus, it suffices to prove the statement of the theorem with the inequality (12) replaced by

$$\sum_{v \in M_k \setminus S} -\log^- \max\{|F(m)|_v, |G(n)|_v\} > \epsilon \max\{m, n\}. \tag{13}$$

Let $u_1, \ldots, u_r$ be generators for Γ. Let $f, g \in k[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$ be the Laurent polynomials corresponding to F and G. We may write

$$f(x_1, \ldots, x_r) = x_1^{i_1} \cdots x_r^{i_r} f_0(x_1, \ldots, x_r),$$
$$g(x_1, \ldots, x_r) = x_1^{j_1} \cdots x_r^{j_r} g_0(x_1, \ldots, x_r),$$

where $i_1, \ldots, i_r, j_1, \ldots, j_r \in \mathbb{Z}$ and $f_0, g_0 \in k[x_1, \ldots, x_r]$ with $x_i \nmid f_0 g_0$, $i = 1, \ldots, r$. Let F_0 and G_0 be the corresponding simple linear recurrence sequences. Since $u_1, \ldots, u_r \in \mathcal{O}_{k,S}^*$, it's trivial that

$$\sum_{v \in M_k \setminus S} -\log^- \max\{|F(m)|_v, |G(n)|_v\} = \sum_{v \in M_k \setminus S} -\log^- \max\{|F_0(m)|_v, |G_0(n)|_v\}.$$

Then it suffices to prove the statement of the theorem with (12) replaced by (13), and with F and G replaced by F_0 and G_0, respectively. Note that since $x_1, \ldots, x_r$ are units in $k[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$, replacing F and G by F_0 and G_0 has no effect on coprimality statements. Thus, we now assume that F and G correspond to polynomials f and g, respectively, in $k[x_1, \ldots, x_r, x_1^{-1}, \ldots, x_r^{-1}]$, and that $x_i \nmid fg$, $i = 1, \ldots, r$, viewed as polynomials in $k[x_1, \ldots, x_r]$.

We further reduce the problem to consideration of the case $m = n$ as follows. Let $\tilde{f}, \tilde{g} \in k[x_1, \ldots, x_{2r}]$ be the polynomials

$$\tilde{f}(x_1, \ldots, x_{2r}) = f(x_1, \ldots, x_r),$$
$$\tilde{g}(x_1, \ldots, x_{2r}) = g(x_{r+1}, \ldots, x_{2r}).$$

Then $\tilde{f}$ and $\tilde{g}$ are coprime in $k[x_1, \ldots, x_{2r}]$ (the polynomials involve different sets of variables). Let

$$\begin{aligned} P_{m,n} &= (u_1^m, \ldots, u_r^m, u_1^n, \ldots, u_r^n) \in \mathbb{G}_m^{2r}(\mathcal{O}_{k,S}), \quad m, n \in \mathbb{N}, \\ R &= \{P_{m,n} : m, n \in \mathbb{N}\}. \end{aligned}$$

By Theorem 3.2, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^{2r}$ such that

$$\begin{aligned} &\sum_{v \in M_k \setminus S} -\log^- \max\{|\tilde{f}(P_{m,n})|_v, |\tilde{g}(P_{m,n})|_v\} \\ &\quad < \epsilon \max\{h(u_1^m), \ldots, h(u_r^m), h(u_1^n), \ldots, h(u_r^n)\} \end{aligned}$$

for all points $P_{m,n} \in R \setminus Z$. Now we note that

$$\begin{aligned} \tilde{f}(P_{m,n}) &= F(m), \\ \tilde{g}(P_{m,n}) &= G(n), \end{aligned}$$

and

$$\begin{aligned} &\max\{h(u_1^m), \ldots, h(u_r^m), h(u_1^n), \ldots, h(u_r^n)\} \\ &\quad = \max\{h(u_1), \ldots, h(u_r)\} \max\{m, n\}. \end{aligned}$$

It follows that for all $\epsilon > 0$, there exists a finite union Z of translates of proper algebraic subgroups of $\mathbb{G}_m^{2r}$ such that

$$\sum_{v \in M_k \setminus S} -\log^- \max\{|F(m)|_v, |G(n)|_v\} < \epsilon \max\{m, n\}$$

for all pairs (m, n) such that $P_{m,n} \in R \setminus Z$. Since $u_1, \ldots, u_r$ are multiplicatively independent, every translate of a proper algebraic subgroup that contains infinitely many points $P_{m,n}$ lies in a translate of a proper algebraic subgroup of the form $x_i^a x_{i+r}^b = u_i^c$, where $a, b, c \in \mathbb{Z}$, and a and b are not both 0. If $P_{m,n}$ lies on such a translate, then $am + bn = c$. Therefore, by restricting m and n to finitely many arithmetic progressions (with a common variable), it suffices to show that if F and G are coprime and $\epsilon > 0$, then

$$\sum_{v \in M_k \setminus S} -\log^- \max\{|F(n)|_v, |G(n)|_v\} < \epsilon n$$

for all but finitely many $n \in \mathbb{N}$. Suppose now that F and G are coprime. The argument is similar to before. Let

$$P_n = (u_1^n, \ldots, u_r^n) \in \mathbb{G}_m^r(\mathcal{O}_{k,S}), \quad n \in \mathbb{N}.$$

Note that since $u_1, \ldots, u_r$ are multiplicatively independent, it is immediate that any translate of a proper algebraic subgroup of $\mathbb{G}_m^r$ contains only finitely many points P_n. Since the polynomials f and g corresponding to F and G, respectively, are coprime in $k[x_1, \ldots, x_r]$ by assumption and by our reductions, by Theorem 3.2,

$$\sum_{v \in M_k \setminus S} -\log^- \max\{|f(P_n)|_v, |g(P_n)|_v\} < \epsilon \max\{h(u_1^n), \ldots, h(u_r^n)\}$$

for all but finitely many $n \in \mathbb{N}$. Since $f(P_n) = F(n)$, $g(P_n) = G(n)$, and

$$\max\{h(u_1^n), \ldots, h(u_r^n)\} = n \max\{h(u_1), \ldots, h(u_r)\},$$

we obtain the desired inequality.

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