Invent. math. (2018) 213:759–809
https://doi.org/10.1007/s00222-018-0797-x

A C^0 counterexample to the Arnold conjecture

Lev Buhovsky[1] · **Vincent Humilière**[2] · **Sobhan Seyfaddini**[3]

Received: 19 October 2016 / Accepted: 15 March 2018 / Published online: 11 April 2018

Abstract The Arnold conjecture states that a Hamiltonian diffeomorphism of a closed and connected symplectic manifold (M, ω) must have at least as many fixed points as the minimal number of critical points of a smooth function on M. It is well known that the Arnold conjecture holds for Hamiltonian *homeomorphisms* of closed symplectic surfaces. The goal of this paper is to provide a counterexample to the Arnold conjecture for Hamiltonian homeomorphisms in dimensions four and higher. More precisely, we prove that every closed and connected symplectic manifold of dimension at least four admits a Hamiltonian homeomorphism with a single fixed point.

Lev Buhovsky: This author also uses the spelling "Buhovski" for his family name.

✉ Sobhan Seyfaddini
sobhan.seyfaddini@imj-prg.fr

Lev Buhovsky
levbuh@post.tau.ac.il

Vincent Humilière
vincent.humiliere@imj-prg.fr

[1] School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel

[2] Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, 75005 Paris, France

[3] CNRS, Sorbonne Université, Université Paris Diderot, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, 75005 Paris, France

Keywords C^0 Symplectic geometry · Symplectic and Hamiltonian homeomorphisms · Arnold conjecture · Hamiltonian dynamics

Contents

1 Introduction and main results

1.1 The Arnold conjecture

Let (M, ω) denote a closed and connected symplectic manifold. This paper is concerned with the celebrated conjecture of Arnold on fixed points of Hamiltonian diffeomorphisms.

Conjecture (Arnold). *A Hamiltonian diffeomorphism of M must have at least as many fixed points as the minimal number of critical points of a smooth function on M.*

What makes this conjecture so remarkable is the large number of fixed points predicted by it. This is often interpreted as a manifestation of symplectic rigidity. In contrast to Arnold's conjecture, the classical Lefschetz fixed-point theorem cannot predict the existence of more than one fixed point for a general diffeomorphism. Ever since its inception, this simple and beautiful conjecture has been a powerful driving force in the development of symplectic topology. The most important breakthrough towards a solution of this conjecture came with Floer's invention of what is now called *Hamiltonian Floer homology* which established a variant of the Arnold conjecture on a large class of symplectic manifolds [9,10,12]. The above version of the Arnold conjecture

has been established on symplectically aspherical[1] manifolds by Rudyak and Oprea [33] who built on earlier works of Floer [11] and Hofer [20]. We should mention that prior to the discovery of Floer homology, the Arnold conjecture was proven by Eliashberg [7] on closed surfaces (see also Sikorav [35]), by Conley and Zehnder [6] on higher dimensional tori, and by Fortune and Weinstein [13,14] on complex projective spaces.

1.2 The Arnold conjecture and Hamiltonian homeomorphisms

Throughout this paper we will denote by $\mathrm{Symp}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ the groups of symplectic and Hamiltonian diffeomorphisms of (M,ω), respectively. As is nowadays standard, we call *symplectic homeomorphism* any homeomorphism which can be written as a uniform limit of symplectic diffeomorphisms; the set of all symplectic homeomorphisms is denoted by $\mathrm{Sympeo}(M,\omega)$; see Sect. 2.1.

As a first attempt at defining *Hamiltonian homeomorphisms*, we will say that a homeomorphism ϕ of M is a *Hamiltonian homeomorphism* if it can be written as a uniform limit of Hamiltonian diffeomorphisms. This class of homeomorphisms has been studied very extensively, from a dynamical point of view, in the case of closed surfaces.[2] For example, Matsumoto [28], building on an earlier paper of Franks [15], has proven that Hamiltonian homeomorphisms of surfaces satisfy the Arnold conjecture. An important development in the study of Hamiltonian homeomorphisms of surfaces has been Le Calvez's theory of transverse foliations [25] which has not only proven the Arnold conjecture but also the Conley conjecture on periodic points of these homeomorphisms [26].

In striking contrast to the rich theory in dimension two, there are virtually no results on fixed point theory of Hamiltonian homeomorphisms in higher dimensions. Indeed, none of the powerful tools of surface dynamics seem to generalize in an obvious manner to dimensions higher than two. Our first theorem proves that in fact one can not hope to prove the Arnold conjecture in higher dimensions.

Theorem 1 *Every closed and connected symplectic manifold of dimension at least 4 admits a Hamiltonian homeomorphism with a single fixed point.*

This theorem might suggest that, in dimensions higher than two, one should search for a different notion of Hamiltonian homeomorphisms. Indeed, such notion does exist within the field of continuous, or C^0, symplectic topology.

[1] M is said to be symplectically aspherical if ω and c_1, the first Chern class of M, both vanish on $\pi_2(M)$.

[2] This is precisely the class of area preserving homeomorphisms with vanishing mean rotation vector.

Motivated in part by developing a continuous analogue of smooth Hamiltonian dynamics, Müller and Oh have suggested an alternative, more restrictive, definition for Hamiltonian homeomorphisms; see Sect. 2.1 for the precise definition. From this point onward by Hamiltonian homeomorphisms we will mean those homeomorphisms of M prescribed by Definition 7. We denote the set of all Hamiltonian homeomorphisms by $\mathrm{Hameo}(M,\omega)$.

The group $\mathrm{Hameo}(M,\omega)$ has met some success. Indeed, recent results in C^0-symplectic topology [22–24] have demonstrated that Hamiltonian homeomorphisms inherit some of the important dynamical properties of smooth Hamiltonian diffeomorphisms; see Theorem 8. Furthermore, they have played a key role in the development of C^0-symplectic topology over the past several years. However, our main theorem proves that the Arnold conjecture is not true for this notion of Hamiltonian homeomorphisms either. In fact, as we will explain below, it shows that there is no hope for proving the Arnold conjecture, as formulated above, for any alternate definition of Hamiltonian homeomorphisms which satisfies a minimal set of requirements.

Theorem 2 (Main theorem). *Let (M,ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists $f \in \mathrm{Hameo}(M,\omega)$ with a single fixed point. Furthermore, f can be chosen to satisfy either of the following additional properties.*

1. *Let $\mathcal{H}$ be a normal subgroup of $\mathrm{Sympeo}(M,\omega)$ which contains $\mathrm{Ham}(M,\omega)$ as a proper subset. Then, $f \in \mathcal{H}$.*
2. *Let p denote the unique fixed point of f. Then, f is a symplectic diffeomorphism of $M\backslash\{p\}$.*

A few remarks are in order. First, we should point out that every Hamiltonian homeomorphism possesses at least one fixed point. This is because a Hamiltonian homeomorphism is by definition a uniform limit of Hamiltonian diffeomorphisms and it is a non-trivial fact that a Hamiltonian diffeomorphism has at least one fixed point[3].

With regards to the second property, we point out that it is natural to expect f to have at least one non-smooth point. Indeed, since Hamiltonian Floer homology predicts that a Hamiltonian diffeomorphism can never have as few as one fixed point, our homeomorphism f must necessarily be non-smooth on any symplectic manifold (M,ω) with the property[4] that $\mathrm{Hameo}(M,\omega) \cap \mathrm{Diff}(M) = \mathrm{Ham}(M,\omega)$.

Lastly, we remark that it is well known that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$. Hence, it is reasonable to expect that any alternative

[3] This fact is an immediate consequence of Floer's proof of the Arnold conjecture; see also [17].

[4] It can be shown that this property holds for closed symplectic surfaces, as well as for the standard $\mathbb{C}P^2$ and monotone $S^2 \times S^2$.

candidate, say $\mathcal{H}$, for the group of Hamiltonian homeomorphisms should contain $\mathrm{Ham}(M, \omega)$ and be a normal subgroup of $\mathrm{Sympeo}(M, \omega)$. It is indeed the case that $\mathrm{Hameo}(M, \omega) \trianglelefteq \mathrm{Sympeo}(M, \omega)$. Therefore, the first property in the above theorem states that there is no hope of proving the Arnold conjecture for any alternate definition of Hamiltonian homeomorphisms.

1.3 Does there exist a fixed point theory for Hamiltonian homeomorphisms?

In Gromov's view [18], symplectic topology is enriched by a beautiful interplay between rigidity and flexibility. Recent results, such as [3,22,31], have demonstrated that this contrast between rigidity and flexibility permeates, in a surprising fashion, to C^0 symplectic topology as well. Symplectic rigidity manifests itself when symplectic phenomena survive under C^0 limits; see [2,5,8,22,31] for some examples. On the other hand, there exist instances where passage to C^0 limits results in spectacular loss of rigidity and prevalence of flexibility; see [3] for an example.

The main theorem of our paper tells us that fixed points of Hamiltonian diffeomorphisms become completely flexible under C^0 limits. It is interesting to contrast this prevalence of flexibility with the strong rigidity results of Franks [15], Matsumoto [28], and Le Calvez [25,26] in the two-dimensional setting. Given the main result of this article, one might conclude that there is no hope of developing a sensible fixed point theory for any notion of Hamiltonian homeomorphisms in dimensions greater than two. However, there exist some interesting open questions which remain unanswered.

The most prominent open question is that of the Conley conjecture which in its simplest form states that a Hamiltonian diffeomorphism on an aspherical symplectic manifold has infinitely many periodic points. This conjecture was proven by Hingston [19] on tori and Ginzburg [16] in the more general setting. As mentioned earlier, the Conley conjecture has been proven for Hamiltonian homeomorphisms of surfaces by Le Calvez [25,26]. We have not been able to construct a counterexample to the Conley conjecture in higher dimensions.

The second question relates to the theory of spectral invariants. For the sake of simplicity, we limit this discussion to the case of symplectically aspherical manifolds. In that case, the theory of spectral invariants, which was introduced by Viterbo, Oh and Schwarz [29,34,37], associates to each smooth Hamiltonian H, a collection of real numbers $\{c(a, H) \in \mathbb{R} : a \in H_*(M)\backslash\{0\}\}$, where $H_*(M)$ denotes the singular homology of M. These numbers are referred to as the spectral invariants of H and they correspond to critical values of the associated action functional. Hence, the number of distinct spectral invariants of a Hamiltonian H gives a lower bound for the number of fixed points of the time-1 map ϕ_H^1.

Recall that the cup length of M is defined by $cl(M) := \max\{k+1 : \exists a_1, \ldots, a_k \in H^*(M) : \forall i, \deg(a_i) \neq 0 \text{ and } a_1 \cup \cdots \cup a_k \neq 0\}$. Combining techniques from Hamiltonian Floer theory and Lusternik–Shnirelman theory, Floer [11] and Hofer [20] proved that if a Hamiltonian diffeomorphism, of an aspherical symplectic manifold M, has fewer spectral invariants than the cup length of M, then it must have infinitely many fixed points; see also [21].

It is well-known that one can associate spectral invariants to any continuous Hamiltonian function; see for example [30]. In an interesting twist, it turns out that the Hamiltonian homeomorphism that we construct in the proof of Theorem 2 is *generated by a continuous Hamiltonian which has at least as many distinct spectral invariants as* $cl(M)$. Hence, we see that the correspondence between spectral invariants and fixed points breaks down in the continuous setting. See Remark 20. This leads us to the following question:

Question 3 *Suppose that H is a continuous Hamiltonian with fewer spectral invariants than the cup length of M. Does ϕ_H^1, the time-1 map of the flow of H, have infinitely many fixed points?*

A positive answer to this question could be interpreted as a C^0 version of the Arnold conjecture.

We end this section with a brief discussion which will add to the importance of the above question. This concerns the theory of barcodes, or persistence modules. As pointed out in [32], Hamiltonian Floer theory allows one to associate a so-called barcode to any smooth Hamiltonian; see also [1,27,36]. Barcodes can be viewed as generalizations of spectral invariants. The barcode of a smooth Hamiltonian encodes all the information contained in the filtered Floer homology of that Hamiltonian. In the same way that one can associate spectral invariants to a continuous function, one can also associate a barcode to a continuous Hamiltonian function. In yet another interesting twist, it turns out that the Hamiltonian homeomorphism of Theorem 2 can be generated by a continuous Hamiltonian which has the same barcode as a C^2-small Morse function. See Remark 21.

1.4 A brief outline of the construction

Construction of the homeomorphism f, as prescribed in Theorem 2, takes place in two major steps. The first step, which is the more difficult of the two, can be summarized in the following theorem.

Theorem 4 *Let (M, ω) denote a closed and connected symplectic manifold of dimension at least 4. There exists $\psi \in \mathrm{Hameo}(M, \omega)$ and an embedded tree $T \subset M$ such that*

1. *T is invariant under ψ, i.e. $\psi(T) = T$,*
2. *All of the fixed points of ψ are contained in T,*
3. *ψ is smooth in the complement of T.*

For the proof of Theorem 2, we will in fact need a refined version of the above result; see Theorem 13. The proof of this theorem forms the technical heart of our paper. An important ingredient used in the construction of the invariant tree T is a quantitative h-principle for curves. Quantitative h-principles have recently been introduced to C^0 symplectic topology by Buhovsky and Opshtein and have had numerous fascinating applications; see [3]. We should point out that M having dimension at least four is used in a crucial way in the proof of this theorem.

The second major step of our construction consists of "collapsing" the invariant tree T to a single point which will be the fixed point of our homeomorphism f. Here is a brief outline of how this is done. Fix a point $p \in M$. We construct a sequence $\varphi_i \in \mathrm{Symp}(M, \omega)$ such that φ_i converges uniformly to a map $\varphi : M \to M$ with the following two properties:

1. $\varphi(T) = p$,
2. φ is a symplectic diffeomorphism from $M\backslash T$ to $M\backslash\{p\}$.

Note that the first property implies that φ is not a 1-1 map and hence, the sequence φ_i^{-1} is not convergent. Define $f : M \to M$ as follows: $f(p) = p$ and

$$\forall x \in M\backslash\{p\},\ \ f(x) = \varphi \circ \psi \circ \varphi^{-1}(x).$$

It is not difficult to see that p is the unique fixed point of f. Indeed, on $M\backslash\{p\}$, the map f is conjugate to $\psi : M\backslash T \to M\backslash T$ which is fixed point free by construction.

By picking the above sequence of symplectomorphisms φ_i carefully, it is possible to ensure that the sequence of conjugations $\varphi_i \psi \varphi_i^{-1}$ converges uniformly to f. The uniform convergence of $\varphi_i \psi \varphi_i^{-1}$ to f relies heavily on the invariance of the tree T and it occurs despite the fact that the sequence φ_i^{-1} diverges. The details of this are carried out in Sect. 3.1. It follows that f can be written as the uniform limit of a sequence of Hamiltonian diffeomorphisms.

It is not difficult to see that f is smooth on the complement of its unique fixed point. However, proving that f is a Hamiltonian homeomorphism and that it satisfies the first property listed in Theorem 2 requires some more work; see Sect. 3.1.

1.5 Organization of the paper

In Sect. 2, we recall some preliminary results from C^0 symplectic geometry. Symplectic and Hamiltonian homeomorphisms are introduced in Sect. 2.1. In Sect. 2.2, we introduce a quantitative h-principle for curves which plays an important role in our construction.

In Sect. 3.1, we prove that the existence of a Hamiltonian homeomorphism with an invariant tree, as described in Theorem 4, implies the main theorem of the paper. In Sect. 3.2, we prove the existence of a Hamiltonian homeomorphism as described in Theorem 4, assuming a technical and important result: Theorem 25. Section 3.3, which occupies the rest of the paper, is dedicated to the proof of Theorem 25. This section contains the technical heart of the paper.

2 Preliminaries from C^0-symplectic topology

In this section we introduce some of our notation and recall some of the basic notions of C^0-symplectic geometry. In Sect. 2.1 we give precise definitions for symplectic and Hamiltonian homeomorphisms. In Sect. 2.2 we state a *quantitative h-principle* for curves which will play a crucial role in the proof of Theorem 2.

2.1 Symplectic and Hamiltonian homeomorphisms

Throughout the rest of this paper, (M, ω) will denote a closed and connected symplectic manifold whose dimension is at least 4. We equip M with a Riemannian distance d. Given two maps $\phi, \psi : M \to M$, we denote

$$d_{C^0}(\phi, \psi) = \sup_{x \in M} d(\phi(x), \psi(x)).$$

We will say that a sequence of maps $\phi_i : M \to M$, converges uniformly, or C^0-converges, to ϕ, if $d_{C^0}(\phi_i, \phi) \to 0$ as $i \to \infty$. Of course, the notion of C^0-convergence does not depend on the choice of the Riemannian metric.

Recall that a symplectic diffeomorphism is a diffeomorphism $\theta : M \to M$ such that $\theta^*\omega = \omega$. The set of all symplectic diffeomorphisms of M is denoted by $\mathrm{Symp}(M, \omega)$.

Definition 5 A homeomorphism $\theta : M \to M$ is said to be symplectic if it is the C^0-limit of a sequence of symplectic diffeomorphisms. We will denote the set of all symplectic homeomorphisms by $\mathrm{Sympeo}(M, \omega)$.

The Eliashberg–Gromov theorem states that a symplectic homeomorphism which is smooth is itself a symplectic diffeomorphism. We remark that if θ

is a symplectic homeomorphism, then so is θ^{-1}. In fact, it is easy to see that Sympeo(M, ω) forms a group.

Remark 6 More generally, one can define a symplectic homeomorphism to be a homeomorphism which is locally a C^0-limit of symplectic diffeomorphisms; see [3] for further details.

Recall that a smooth Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ gives rise to a Hamiltonian flow ϕ_H^t. A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. The set of all Hamiltonian diffeomorphisms is denoted by Ham(M, ω); this is a normal subgroup of Symp(M, ω). We next define Hamiltonian homeomorphisms as introduced by Müller and Oh [30].

Definition 7 (Hamiltonian homeomorphisms). Denote by B an open (possibly not proper) subset of M. Let $(\phi^t)_{t\in[0,1]}$ be an isotopy of M which is compactly supported in B. We say that ϕ^t is a hameotopy, or a continuous Hamiltonian flow, of B if there exists a sequence of smooth and compactly supported Hamiltonians $H_i : [0, 1] \times B \to \mathbb{R}$ such that:

1. The sequence of flows $\phi_{H_i}^t$ C^0-converges to ϕ^t, uniformly in t, i.e. $\max_{t\in[0,1]} d_{C^0}(\phi_{H_i}^t, \phi^t) \to 0$ as $i \to \infty$.
2. The sequence of Hamiltonians H_i converges uniformly to a continuous function $H : [0, 1] \times M \to \mathbb{R}$, i.e. $\|H_i - H\|_\infty \to 0$ as $i \to \infty$, where $\|\cdot\|_\infty$ denotes the sup norm. Furthermore,

We say that H generates ϕ^t, denote $\phi^t = \phi_H^t$, and call H a continuous Hamiltonian.

A homeomorphism is called a Hamiltonian homeomorphism if it is the time-1 map of a continuous Hamiltonian flow. We will denote the set of all Hamiltonian homeomorphisms by Hameo(B, ω).

It is not difficult to check that Hameo(M, ω) is a normal subgroup of Sympeo(M, ω).

A continuous Hamiltonian H generates a unique continuous Hamiltonian flow; see [30]. Conversely, Viterbo [38] and Buhovsky–Seyfaddini [4] (see also [23]) proved that a continuous Hamiltonian flow has a unique (up to addition of a function of time) continuous generator.

One can easily check that generators of continuous Hamiltonian flows satisfy the same composition formulas as their smooth counterparts. Namely, if ϕ_H^t is a continuous Hamiltonian flow, then $(\phi_H^t)^{-1}$ is a continuous Hamiltonian flow generated by $\bar{H}(t, x) = -H(t, \phi_H^t(x))$; given another continuous Hamiltonian flow ϕ_K^t, the isotopy $\phi_H^t\phi_K^t$ is also a continuous Hamiltonian flow, generated by $H\#K(t, x) := H(t, x) + K(t, (\phi_H^t)^{-1}(x))$.

We will finish this section by recalling an important dynamical property of continuous Hamiltonian flows. Recall that a submanifold C of a symplectic manifold (M, ω) is called coisotropic if for all $p \in C$, $(T_pC)^\omega \subset T_pC$ where $(T_pC)^\omega$ denotes the symplectic orthogonal of T_pC. For instance, hypersurfaces and Lagrangians are coisotropic. A coisotropic submanifold carries a natural foliation $\mathcal{F}$ which integrates the distribution $(TC)^\omega$; $\mathcal{F}$ is called the characteristic foliation of C.

Assume that C is a closed and connected coisotropic submanifold of M and suppose that H is a smooth Hamiltonian. The following is a standard and important fact which relates Hamiltonian flows to coisotropic submanifolds: $H|_C$ is a function of time if and only if ϕ_H^t (preserves C and) flows along the characteristic foliation of C. By flowing along characteristics we mean that for any point $p \in C$ and any time $t \geqslant 0$, $\phi_H^t(p) \in \mathcal{F}(p)$, where $\mathcal{F}(p)$ stands for the characteristic leaf through p.

The following theorem, which was proven in [22], establishes the aforementioned property for continuous Hamiltonian flows.

Theorem 8 *Denote by C a closed and connected coisotropic submanifold and suppose that ϕ_H^t is a continuous Hamiltonian flow. The restriction of H to C is a function of time if and only if ϕ_H^t preserves C and flows along the leaves of its characteristic foliation.*

The above theorem indicates that continuous Hamiltonian flows inherit some of the fundamental dynamical properties of their smooth counterparts. In light of this, it would seem reasonable to expect the Arnold conjecture to hold for Hamiltonian homeomorphisms. But of course, Theorem 2 tells us that this is quite far from reality.

2.2 A quantitative *h*-principle for curves

Quantitative h-principles were introduced in [3], where they were used to construct interesting examples of symplectic homeomorphisms. We will need the following quantitative h-principle for curves in the construction of our counterexample to the Arnold conjecture.

Proposition 9 (Quantitative h-principle for curves). *Denote by (M, ω) a symplectic manifold of dimension at least 4. Let $\varepsilon > 0$. Suppose that $\gamma_0, \gamma_1 : [0, 1] \to M$ are two smooth embedded curves such that*

(i) *γ_0 and γ_1 coincide near $t = 0$ and $t = 1$,*
(ii) *there exists a homotopy, rel.end points, from γ_0 to γ_1 under which the trajectory of any point of γ_0 has diameter less than ε, and the symplectic area of the element of $\pi_2(M, \gamma_1 \sharp \overline{\gamma_0})$ defined by this homotopy has area 0.*

Then, for any $\rho > 0$, there exists a compactly supported Hamiltonian F, generating a Hamiltonian isotopy $\varphi^s : M \to M,\ s \in [0,1]$ such that

1. *F vanishes near $\gamma_0(0)$ and $\gamma_0(1)$ (in particular, φ^s fixes γ_0 and γ_1 near the extremities),*
2. *$\varphi^1 \circ \gamma_0 = \gamma_1$,*
3. *$d_{C^0}(\varphi^s, Id) < 2\varepsilon$ for each $s \in [0,1]$, and $\|F\|_\infty \leqslant \rho$,*
4. *F is supported in a 2ε-neighborhood of the image of γ_0.*

The existence of a Hamiltonian F satisfying only properties 1 and 2 is well known. The aspect of the above proposition which is non-standard is the fact that F can be picked such that properties 3 and 4 are satisfied as well. We should point out that the above proposition is a variation of a (considerably more difficult) quantitative h-principle for discs which appeared in Theorem 2 of [3]. The proof we will present is an adaptation of the arguments therein and thus, it will not be a detailed proof.

In the next section of the paper, we will need the following fact, which is standard, and follows from Proposition 9, and hence will not be proven here.

Lemma 10 *Denote by (M, ω) an exact symplectic manifold of dimension at least 4. Suppose that $\gamma_0, \gamma_1 : [0,1] \to M$ are two curves such that*

(i) *γ_0 and γ_1 coincide near $t = 0, 1$,*
(ii) *$\int_0^1 \gamma_0^* \lambda = \int_0^1 \gamma_1^* \lambda$ where λ is any 1-form such that $\omega = d\lambda$,*
(iii) *there exists a homotopy, rel. end points, from γ_0 to γ_1.*

Then, for any $\rho > 0$, there exists a compactly supported Hamiltonian F, generating a Hamiltonian isotopy $\varphi^s : M \to M,\ s \in [0,1]$ such that

1. *F vanishes near the extremities of γ_0 and γ_1,*
2. *$\varphi^1 \circ \gamma_0 = \gamma_1$,*
3. *$\|F\|_\infty < \rho$.*

Proof of Proposition 9 First, by a slight Hamiltonian perturbation of γ_0 via a Hamiltonian diffeomorphism generated by a C^∞-small Hamiltonian function which vanishes near $\{\gamma_0(0), \gamma_0(1)\}$, we can, without loss of generality, assume that $\gamma_0 = \gamma_1$ on $[0, \delta] \cup [1-\delta, 1]$, and that the images of $\gamma_0|_{(\delta, 1-\delta)}$ and $\gamma_1|_{(\delta, 1-\delta)}$ are disjoint, where $\delta > 0$ is a small positive real number. By assumption there exists a homotopy $h : [0,1] \times [0,1] \to M$ such that $h(0,t) = \gamma_0(t)$, $h(1,t) = \gamma_1(t)$ and for any fixed t, the path $s \mapsto h(s,t)$ is of diameter smaller than ε. Since the dimension of M is at least 4, by the weak Whitney immersion theorem, we can approximate h by a smooth map $h' : [0,1] \times [0,1] \to M$ such that $h'(0,t) = \gamma_0(t)$ and $h'(1,t) = \gamma_1(t)$ for $t \in [0,1]$, $h'(s,t) = \gamma_0(t) = \gamma_1(t)$ for $(s,t) \in [0,1] \times ([0,\delta] \cup [1-\delta, 1])$, and such that the restriction $h'|_{[0,1] \times (\delta, 1-\delta)}$ is a smooth immersion with a finite number of self-intersection points occuring inside the relative interior $h((0,1) \times (\delta, 1 -$

$\delta)$), and whose image $h'([0,1]\times(\delta,1-\delta))$ does not intersect $\gamma_0([0,\delta]\cup[1-\delta,1])$. Furthermore, similarly as was done in Lemma A.1 from [3], one can find a smooth map $h'' : [0,1]\times[0,1]\to M$ whose image lies in an arbitrarily small neighborhood of $h'([0,1]\times[0,1])$, such that as before we have $h''(0,t)=\gamma_0(t)$ and $h''(1,t)=\gamma_1(t)$ for $t\in[0,1]$, $h''(s,t)=\gamma_0(t)=\gamma_1(t)$ for $(s,t)\in[0,1]\times([0,\delta]\cup[1-\delta,1])$, and the image $h''([0,1]\times(\delta,1-\delta))$ does not intersect $\gamma_0([0,\delta]\cup[1-\delta,1])$, but moreover such that the restriction $h''|_{[0,1]\times(\delta,1-\delta)}$ is a smooth embedding, and such that for any fixed t the diameter of the curve $s\mapsto h(s,t)$ is less than 2ε. Note that by construction, h, h' and h'' give the same element of $\pi_2(M,\gamma_1\sharp\overline{\gamma_0})$. Abusing our notation, we will denote h'' by h again.

Let m be a sufficiently large positive integer. Then, for each $1\leqslant i<m-3$, the image $h([0,1]\times[\frac{i}{m},\frac{i+3}{m}])$ has diameter less than 2ε, for given $0\leqslant i,j\leqslant m-1$ we have $h([0,1]\times[\frac{i}{m},\frac{i+1}{m}])\cap h([0,1]\times[\frac{j}{m},\frac{j+1}{m}])\neq\emptyset$ only if $j\in\{i-1,i,i+1\}$, and moreover we can find a neighborhood U_i of each $h([0,1]\times[\frac{i}{m},\frac{i+1}{m}])$ such that U_i is diffeomorphic to a ball, such that we again have $U_i\cap U_j\neq\emptyset$ only if $j\in\{i-1,i,i+1\}$, and such that the diameter of $U_{i-1}\cup U_i\cup U_{i+1}$ is less than 2ε, for every $0<i<m-1$. Moreover, the union $U=U_0\cup\cdots\cup U_m$ can be assumed to be diffeomorphic to a ball, as well as $U_i\cap U_{i+1}$, for each $0\leqslant i<m-1$. Then in particular, ω is exact on U, i.e. $\omega=d\lambda$ on U, for some differential 1-form λ on U. By our assumptions, $\int_0^1\gamma_0^*\lambda=\int_0^1\gamma_1^*\lambda$.

Step 1: Mapping points to points For each $1\leqslant i\leqslant m-1$, we pick a Hamiltonian G_i which is supported in $U_{i-1}\cap U_i$

such that

$$\phi^1_{G_i}(\gamma_0(t))=\gamma_1(t),\quad\forall t\in\left[\tfrac{i}{m}-\kappa,\tfrac{i}{m}+\kappa\right],$$

where $\kappa>0$ is sufficiently small. In particular, the G_i's have mutually disjoint supports.

We let $G:=\sum_{i=1}^{m-1}G_i$ and let $\gamma_0'=\phi^1_G\circ\gamma_0$. We also remark that each G_i can be picked such that $\|G_i\|_\infty$ is as small as one wishes. Hence, we may assume that $\|G\|_\infty\leqslant\rho/3$.

Step 2: Adjusting the actions Note that the two curves $\gamma_1|_{[\frac{i}{m},\frac{i+1}{m}]}$ and $\gamma_0'|_{[\frac{i}{m},\frac{i+1}{m}]}$ coincide near their end-points and are both contained in U_i. We would like to find a Hamiltonian diffeomorphism which is supported in U_i and maps $\gamma_0'|_{[\frac{i}{m},\frac{i+1}{m}]}$ to $\gamma_1|_{[\frac{i}{m},\frac{i+1}{m}]}$. However, there is an obstruction to finding such a Hamiltonian diffeomorphism. These two curves do not necessarily have the same action. The goal of this step is to modify γ_0' to remove this obstruction. □

Claim 11 *There exists a Hamiltonian H with the following properties:*

1. *The support of H is contained in U, and we have $\phi_H^t(U_i) \subset U_i$ for every $t \in [0,1]$ and $0 \leqslant i \leqslant m$.*
2. *The curve $\gamma_0'' := (\phi_H^1) \circ \gamma_0'$ coincides with γ_1 near $t = \frac{i}{m}$, for $i = 0, 1, \ldots, m$,*
3. *$\gamma_0''|_{[\frac{i}{m}, \frac{i+1}{m}]}$ has the same action as $\gamma_1|_{[\frac{i}{m}, \frac{i+1}{m}]}$, for $i = 0, 1, \ldots, m-1$,*
4. *$\|H\|_\infty \leqslant \rho/3$.*
5. *For each $0 \leqslant i, j \leqslant m-1$, the images of $\gamma_0''|_{(\frac{i}{m}, \frac{i+1}{m})}$ and of $\gamma_1|_{(\frac{j}{m}, \frac{j+1}{m})}$ intersect only if $i = j$.*

Proof Denote $\gamma_{0,1}' = \gamma_0'$. We perform $(m-1)$ steps, where at step i $(1 \leqslant i \leqslant m-1)$ we construct a curve $\gamma_{0,i+1}'$, and find a Hamiltonian isotopy from $\gamma_{0,i}'$ to $\gamma_{0,i+1}'$.

Let us describe the ith step. Let $\gamma_{0,i}'$ be the curve provided by the previous step. First, perturb the curve $\gamma_{0,i}'$ in an arbitrarily small neighborhood of $t = \frac{i}{m} - \frac{\kappa}{2}$, so that the perturbed curve $\gamma_{0,i}''$ satisfies:

- $\gamma_{0,i}''$ coincides with $\gamma_{0,i}'$ on $[0, \frac{i}{m} - \frac{3\kappa}{4}] \cup [\frac{i}{m} - \frac{\kappa}{4}, 1]$,
- we have $\gamma_{0,i}''([\frac{i}{m} - \frac{3\kappa}{4}, \frac{i}{m} - \frac{\kappa}{4}]) \subset U_{i-1} \cap U_i$,
- the λ-actions of the restrictions of $\gamma_{0,i}''$ and γ_1 to $[\frac{i-1}{m}, \frac{i}{m}]$ coincide.

Such perturbation can be performed similarly as in the Remark A.13 from [3].

Now, we claim that there exists a smooth Hamiltonian function H_i supported in $U_{i-1} \cap U_i$, such that $\gamma_{0,i}'' = (\phi_{H_i}^1) \circ \gamma_{0,i}'$ on $[0, \frac{i}{m} + \kappa]$, and such that $\|H_i\|_\infty \leqslant \rho/6$. For doing this, roughly speaking, it is sufficient to isotope (via the Hamiltonian flow) a small segment of $\gamma_{0,i}'$ so that it coincides with $\gamma_{0,i}''$, near $t = \frac{i}{m}$, or more precisely, for $t \in [\frac{i}{m} - \frac{3\kappa}{4}, \frac{i}{m} + \kappa]$. And notice, that we are not restricted to keeping the "right end-point" $\gamma_{0,i}''(\frac{i}{m} + \kappa)$ fixed along the isotopy. Therefore, for keeping the Hofer norm of the isotopy small, we can just "shrink" the curve $\gamma_{0,i}'|_{[\frac{i}{m} - \frac{7\kappa}{8}, \frac{i}{m} + \kappa]}$ to the small segment $\gamma_{0,i}'|_{[\frac{i}{m} - \frac{7\kappa}{8}, \frac{i}{m} - \frac{3\kappa}{4}]} = \gamma_{0,i}''|_{[\frac{i}{m} - \frac{7\kappa}{8}, \frac{i}{m} - \frac{3\kappa}{4}]}$ near the left end-point, and then "expand this segment" to coincide with $\gamma_{0,i}''|_{[\frac{i}{m} - \frac{7\kappa}{8}, \frac{i}{m} + \kappa]}$.

For a more precise explanation, denote $a = \frac{i}{m} - \frac{7\kappa}{8}$, $b = \frac{i}{m} + \kappa$, choose a smooth function $c : [a, b] \to [a, a + \frac{\kappa}{8}]$ such that $c(t) = t$ for $t \in [a, a + \frac{\kappa}{16}]$ and $c'(t) > 0$ on $[a, b]$, and consider families of curves $\alpha_s, \beta_s : [a, b] \to M$, $s \in [0, 1]$, where $\alpha_s(t) = \gamma_{0,i}'(st + (1-s)c(t))$, $\beta_s(t) = \gamma_{0,i}''(st + (1-s)c(t))$. Note that $\alpha_0 \equiv \beta_0$. It is easy to see that one can find Hamiltonian functions H_i', H_i'', supported in arbitrarily small neighborhood of $\gamma_{0,i}'([a + \frac{\kappa}{8}, b])$ and $\gamma_{0,i}''([a + \frac{\kappa}{8}, b])$ respectively, such that $\alpha_s = \phi_{H_i'}^s \circ \alpha_0$ and $\beta_s = \phi_{H_i''}^s \circ \beta_0$ for each $s \in [0, 1]$, and such that $\|H_i'\|_\infty, \|H_i''\|_\infty \leqslant \rho/12$. Now let H_i

be the Hamiltonian function of the Hamiltonian flow $(\phi^s_{H_i''} \circ (\phi^s_{H_i'})^{-1})_{s\in[0,1]}$. The function H_i is supported in $U_{i-1} \cap U_i$, satisfies $\|H_i\|_\infty \leqslant \rho/6$, and $\gamma''_{0,i} = (\phi^1_{H_i}) \circ \gamma'_{0,i}$ on $[0, \frac{i}{m} + \kappa]$. Now define the curve $\gamma'_{0,i+1} : [0, 1] \to M$ by $\gamma'_{0,i+1} := \phi^1_{H_i} \circ \gamma'_{0,i}$.

After performing all the $(m-1)$ steps, the λ-actions of $\gamma'_{0,m-1}$ and γ_1 on $[\frac{i-1}{m}, \frac{i}{m}]$ coincide for any $1 \leqslant i \leqslant m-1$. But since the actions of $\gamma'_{0,m-1}$ and γ_1 coincide on the whole $[0, 1]$, it also follows that the actions of $\gamma'_{0,m-1}$ and γ_1 coincide on $[\frac{m-1}{m}, 1]$. Note that since all H_i have disjoint supports (since the support of H_i is contained in $U_{i-1} \cap U_i$), if we denote $H' = H_1 + \cdots + H_{m-1}$, then $\|H'\|_\infty \leqslant \rho/6$ and $\gamma'_{0,m-1} = (\phi^1_{H'}) \circ \gamma'_{0,1} = (\phi^1_{H'}) \circ \gamma'_0$.

It is possible that for different $0 \leqslant i, j \leqslant m-1$, the images of $\gamma'_{0,m-1}|_{(\frac{i}{m}, \frac{i+1}{m})}$ and of $\gamma_1|_{(\frac{j}{m}, \frac{j+1}{m})}$ intersect. But then, one can easily find a C^∞-small Hamiltonian function H'', supported inside an arbitrarily small neighborhood of $\bigcup_{i=0}^{m-1} \gamma_1([\frac{i}{m} + \frac{\kappa}{5}, \frac{i+1}{m} - \frac{\kappa}{5}])$, such that in particular we have $\|H''\|_\infty \leqslant \rho/6$, such that $\gamma''_0 := \phi^1_{H''} \circ \gamma'_{0,m-1}$ satisfies the property 5 from the statement of the Claim, and moreover such that the Hamiltonian function $H := H'' \sharp H'$ that generates the flow $(\phi^t_{H''} \circ \phi^t_{H'})$ satisfies the property 1 from the statement of the Claim. □

Step 3: Mapping $\gamma''_0|_{[\frac{i}{m}, \frac{i+1}{m}]}$ **to** $\gamma_1|_{[\frac{i}{m}, \frac{i+1}{m}]}$.

Claim 12 *There exist Hamiltonians* K_i *such that*

1. K_i *is supported in* U_i,
2. *the support of* K_i *intersects the images of* $\gamma''_0|_{[\frac{j}{m}, \frac{j+1}{m}]}$ *and of* $\gamma_1|_{[\frac{j}{m}, \frac{j+1}{m}]}$ *only for* $j = i$,
3. $\|K_i\|_\infty \leqslant \rho/6$,
4. $\phi^1_{K_i} \circ \gamma''_0|_{[\frac{i}{m}, \frac{i+1}{m}]} = \gamma_1|_{[\frac{i}{m}, \frac{i+1}{m}]}$.

Proof Before passing to the proof, let us remark that in general one can apply directly Lemma A.3 (a) of [3] to our situation, but we would not have obtained the estimate on $\|\cdot\|_\infty$. Therefore the proof is more subtle. Let us roughly explain the steps of the proof. The first step is to make a very small (C^∞) perturbation of the curve γ''_0 (via the Hamiltonian diffeomorphism $\phi^1_{K_i}$ below), in order to put the curves $\gamma''_0|_{[\frac{i}{m}, \frac{i+1}{m}]}$ and $\gamma_1|_{[\frac{i}{m}, \frac{i+1}{m}]}$ in general position. Then, we use the following idea. If two curves which coincide near the endpoints and with equal actions, are C^∞-close, then clearly one can find a very small (with respect to the Hofer norm) Hamiltonian function that moves the first curve to the second. However, if such curves are not C^∞-close, then we can use a "conjugation trick": Instead of moving the first curve to the second via a Hamiltonian diffeomorphism, we find a third curve which is C^∞-close to the

first curve, and such that the pair "(first curve, second curve)" could be mapped to the pair "(first curve, third curve)" via a symplectomorphism. After that, it is clearly enough to move the first curve to the third curve by a Hamiltonian flow with a very small Hofer's norm, and then conjugate the flow with the symplectomorphism. The details of this are carried out below.

The restrictions $\gamma_0''|_{[\frac{i}{m},\frac{i+1}{m}]}$ and $\gamma_1|_{[\frac{i}{m},\frac{i+1}{m}]}$ both lie in U_i, have the same λ-actions, and coincide near the endpoints. Let $\kappa > 0$ be such that $\gamma_0''(t) = \gamma_1(t)$ for $t \in [\frac{i}{m}, \frac{i}{m} + \kappa] \cup [\frac{i+1}{m} - \kappa, \frac{i+1}{m}]$ (we use the old notation κ in a new situation, and in fact it is enough to replace the old κ by $\frac{\kappa}{5}$). One can slightly perturb γ_0'' via a Hamiltonian diffeomorphism $\phi^1_{K_i'}$ generated by a C^∞-small Hamiltonian function K_i', so that $\gamma_{01}'' := \phi^1_{K_i'} \circ \gamma_0''$ satisfies $\gamma_{01}''(t) = \gamma_1(t)$ for $t \in [\frac{i}{m}, \frac{i}{m} + \kappa] \cup [\frac{i+1}{m} - \kappa, \frac{i+1}{m}]$, and moreover $\gamma_{01}''((\frac{i}{m} + \kappa, \frac{i+1}{m} - \kappa)) \cap \gamma_1((\frac{i}{m} + \kappa, \frac{i+1}{m} - \kappa)) = \emptyset$. We may assume that K_i' is supported in U_i, the support of K_i' intersects the image of $\gamma_0''|_{[\frac{j}{m},\frac{j+1}{m}]}$ and of $\gamma_1|_{[\frac{j}{m},\frac{j+1}{m}]}$ only for $j = i$, and that $\|K_i'\|_\infty \leqslant \rho/12$.

Now, one can clearly find a C^∞-small Hamiltonian function K_i'' supported in U_i, so that the support of K_i'' intersects the image of $\gamma_0''|_{[\frac{j}{m},\frac{j+1}{m}]}$ and of $\gamma_1|_{[\frac{j}{m},\frac{j+1}{m}]}$ only for $j = i$, and such that the curve $\gamma_1' := (\phi^1_{K_i''}) \circ \gamma_1$ coincides with γ_{01}'' on $[\frac{i}{m}, \frac{i}{m} + \kappa'] \cup [\frac{i+1}{m} - \kappa', \frac{i+1}{m}]$, and moreover $\gamma_1'((\frac{i}{m} + \kappa', \frac{i+1}{m} - \kappa')) \cap \gamma_1((\frac{i}{m} + \kappa', \frac{i+1}{m} - \kappa')) = \emptyset$, for some $\kappa' > \kappa$. We may assume that $\|K_i''\| \leqslant \rho/12$.

Denote $\kappa'' = (\kappa+\kappa')/2$. The curves $\gamma_{01}''|_{[\frac{i}{m}+\kappa'', \frac{i+1}{m}-\kappa'']}$ and $\gamma_1'|_{[\frac{i}{m}+\kappa'', \frac{i+1}{m}-\kappa'']}$ lie in U_i, coincide near the endpoints, have the same λ-action, and their images do not intersect $\gamma_1([0, 1])$. By Lemma A.3 (a) of [3], there exists a Hamiltonian function K_i''' supported in U_i and away from the endpoints $\gamma_{01}''(\frac{i}{m} + \kappa'')$, $\gamma_{01}''(\frac{i+1}{m} - \kappa'')$, such that $\gamma_1' = \phi^1_{K_i'''} \circ \gamma_{01}''$ on $[\frac{i}{m} + \kappa'', \frac{i+1}{m} - \kappa'']$. By a general position argument and a cut-off argument, we may further assume that the support of K_i''' does not intersect $\gamma_1([0, 1])$, as well as $\gamma_0''([\frac{j}{m}, \frac{j+1}{m}])$ for $j \neq i$. Denote $\psi := \phi^1_{K_i'''}$.

To finish the proof, let K_i be the Hamiltonian function that generates the flow $(\psi^{-1} \circ (\phi^s_{K_i''})^{-1} \circ \psi \circ \phi^s_{K_i'})_{s\in[0,1]}$. Then K_i has all the desired properties. For instance, for the property 4 of the statement of the Claim, we first of all have $\gamma_1 = \psi^{-1} \circ \gamma_1$ since the support of ψ does not intersect the curve γ_1, and then we get

$$\begin{aligned}\gamma_1 &= \psi^{-1} \circ \gamma_1 = \psi^{-1} \circ \left(\phi^1_{K_i''}\right)^{-1} \circ \gamma_1' = \psi^{-1} \circ \left(\phi^1_{K_i''}\right)^{-1} \circ \psi \circ \gamma_{01}'' \\ &= \psi^{-1} \circ \left(\phi^1_{K_i''}\right)^{-1} \circ \psi \circ \phi^1_{K_i'} \circ \gamma_0'' = \phi^1_{K_i} \circ \gamma_0''\end{aligned}$$

on $[\frac{i}{m}, \frac{i+1}{m}]$. □

Let $K_{\text{odd}} = K_1 + K_3 + \cdots$ and $K_{\text{even}} = K_2 + K_4 + \cdots$, where K_i's are provided to us by the above claim. We let K be a Hamiltonian such that $\phi_K^s = \phi_{K_{\text{odd}}}^s \circ \phi_{K_{\text{even}}}^s$.

One can deduce the following facts, without much difficulty, from the above claim:

1. $\phi_K^s(U_i) \subset U_{i-1} \cup U_i \cup U_{i+1}$ for each $0 \leqslant i \leqslant m$ and $s \in [0, 1]$ (where we put $U_{-1} = U_{m+1} = \emptyset$),
2. $\phi_K^1 \circ \gamma_0'' = \gamma_1$,
3. $\|K\|_\infty \leqslant \rho/3$.

We now let F be a Hamiltonian such that $\phi_F^s = \phi_K^s \circ \phi_H^s \circ \phi_G^s$. Examining the properties of K, H, and G we see that

1. F vanishes near the extremities of γ_0 (and hence γ_1), in particular ϕ_F^s fixes the extremities,
2. $\phi_F^1 \circ \gamma_0 = \gamma_1$,
3. $d_{C^0}(\phi_F^s, Id) < 2\varepsilon$ for each $s \in [0, 1]$, and $\|F\|_\infty \leqslant \rho$,
4. F is supported in a 2ε-neighborhood of γ_0.

□

3 Proof of Theorem 2

The most important step towards the proof of Theorem 2 will be to establish the following result which is a refined version of Theorem 4. Throughout this section (M, ω) will denote a closed and connected symplectic manifold whose dimension is at least four.

Theorem 13 *Let H be a Morse function on M. If H is sufficiently C^2-small, then for every $\varepsilon, \rho > 0$, there exists $\psi \in \text{Hameo}(M, \omega)$ and an embedded tree T (see Definition 14) such that:*

- *T is ψ-invariant, i.e. $\psi(T) = T$,*
- *T contains all the fixed points of ψ,*
- *$d_{C^0}(\phi_H^1, \psi) < \varepsilon$ and $\psi \circ \phi_H^{-1}$ is generated by a continuous Hamiltonian F such that $\|F\|_\infty < \rho$,*
- *ψ coincides with a Hamiltonian diffeomorphism in the complement of any neighborhood of T.*

The notion of an *embedded tree* which appears in the above theorem is defined as follows.

Definition 14 We will say that a compact subset T of a smooth manifold M is *an embedded tree* if there exists a finite tree T_0 and an injective continuous

map $\chi : T_0 \to M$, such that $T = \chi(T_0)$ and the map χ is a smooth embedding of the interior of each edge of T_0.

Note that we do not ask more than continuity at the vertices of T_0. Note also that the restriction of χ to any compact interval included in the interior of any edge is a smooth embedding.

In Sect. 3.1 below, we explain why Theorem 13 implies Theorem 2. The proof of Theorem 13 then occupies Sects. 3.2 and 3.3.

3.1 From an invariant tree to a single fixed point

In this section, we explain how one can build a Hamiltonian homeomorphism with a single fixed point from a Hamiltonian homeomorphism preserving an embedded tree (see Definition 14) that contains all of its fixed points; in other terms, we prove that Theorem 13 implies Theorem 2. This will rely on the following proposition.

Proposition 15 *Let ψ be a symplectic homeomorphism of (M, ω), and $T \subset M$ be an embedded tree which is invariant under ψ, that is $\psi(T) = T$. Assume that all the fixed points of ψ are contained in T. Then, there exists $f \in \mathrm{Sympeo}(M, \omega)$, with only one fixed point p, and such that $f \circ \psi^{-1} \in \mathrm{Hameo}(M, \omega)$.*

Moreover, if ψ is smooth on $M \backslash T$, then f can be chosen to be smooth on $M \backslash \{p\}$. If ψ coincides with a Hamiltonian diffeomorphism in the complement of any neighborhood of T, then f can be chosen in any normal subgroup of $\mathrm{Sympeo}(M, \omega)$ *which contains* $\mathrm{Ham}(M, \omega)$.

Note that in the last sentence of the above proposition, we do not claim that f can be chosen to be smooth on $M \backslash \{p\}$ and simultaneously be contained in any normal subgroup of $\mathrm{Sympeo}(M, \omega)$ which contains $\mathrm{Ham}(M, \omega)$. As will be clear from the proof, our method of building f in the normal closure of $\mathrm{Ham}(M, \omega)$ has the effect of creating a second non-smooth point. We do not know if both properties can be satisfied at the same time.

Proof of Theorem 2 Theorem 13 provides us with a Hamiltonian homeomorphism ψ which satisfies all the requirements of Proposition 15. Thus, there exists a symplectic homeomorphism f with only one fixed point p and such that $f \circ \psi^{-1} \in \mathrm{Hameo}(M, \omega)$. Since $\psi \in \mathrm{Hameo}(M, \omega)$ we deduce that $f \in \mathrm{Hameo}(M, \omega)$ as well. Moreover, ψ coincides with a Hamiltonian diffeomorphism in the complement of any neighborhood of T. This implies that f can be chosen in any normal subgroup of $\mathrm{Sympeo}(M, \omega)$ containing $\mathrm{Ham}(M, \omega)$. It also implies that ψ is smooth in the complement of T and hence that f can be chosen to be smooth in the complement of p. □

The following lemma will be useful for the proof Proposition 15.

Lemma 16 *Let (M, ω) be a symplectic manifold of dimension at least 4 and let $T \subset M$ be an embedded tree. Let d be a Riemannian distance on M. Then, for every $\varepsilon > 0$ and every open neigborhood U of T, there exists a Hamiltonian function H supported in U such that* $\operatorname{diam}(\phi_H^1(T)) \leqslant \varepsilon$ *and* $\|H\|_\infty \leqslant \varepsilon$.

Proof Let T_0 be a finite tree and $\chi : T_0 \to T$ a map as in Definition 14. Let U be an open set containing T. Without loss of generality, by replacing U by a smaller open set if needed, we may assume that U is simply connected. We first pick a Hamiltonian function H_0 supported in U, such that $\phi_{H_0}^1$ maps all the vertices of T inside a ball B included in U and of diameter less than ε. By continuity of χ, there exist an open neighborhood W of the set of vertices of T and a closed subinterval $J_i \subset \operatorname{Int}(I_i)$, for every edge I_i, $i = 1, \ldots, r$ of T_0, such that

- $T = (W \cap T) \cup \chi(J_1) \cup \cdots \cup \chi(J_r)$,
- $\phi_{H_0}^1(W) \subset B$.

To achieve the proof, we only need to find a Hamiltonian isotopy which moves the pieces of curves $\chi(J_1), \ldots, \chi(J_r)$ into B with the endpoints kept in B along the isotopy. This can be achieved by successively choosing Hamiltonian functions H_i's (for $i = 1, \ldots, r$) such that for all i, $\phi_{H_i}^1(\phi_{H_0}^1(\chi(J_i))) \subset B$ and the support of H_i meets neither $\phi_{H_0}^1(W \cap T)$ nor any of the curves $\phi_{H_k}^1(\chi(J_k))$, for $k = 1, \ldots, i-1$ and $\chi(J_k)$, for $k = i+1, \ldots, r$. Then, the Hamiltonian diffeomorphism $h = \phi_{H_r}^1 \circ \cdots \circ \phi_{H_1}^1 \circ \phi_{H_0}^1$ sends T into B, hence $\operatorname{diam}(h(T)) \leqslant \varepsilon$.

Moreover, it is a standard fact that the above Hamiltonian functions $H_0, \ldots, H_r$ can be chosen arbitrarily small in the $\|\cdot\|_\infty$ norm. Thus, h can be generated by a Hamiltonian H satisfying $\|H\|_\infty \leqslant \varepsilon$. □

Proof of Proposition 15 Let ψ and T be as in the statement of the proposition. We will build the symplectic homeomorphism f as a C^0-limit of conjugates of ψ.

For that purpose, let $W_0 \supset W_1 \supset \cdots \supset W_k \supset \cdots \supset T$ be a sequence of nested open neighborhoods of T such that $\bigcap_{k \geqslant 0} W_k = T$. □

Claim 17 *There exists a sequence of open sets $(U_i)_{i \in \mathbb{N}}$, a sequence of Hamiltonian diffeomorphisms $(h_i = \phi_{H_i}^1)_{i \in \mathbb{N}}$ and a subsequence $(W_{k_i})_{i \in \mathbb{N}}$ with the following properties: For all $i \in \mathbb{N}$,*

- $U_i = \varphi_i(W_{k_i})$, *where* $\varphi_i = h_i \circ \cdots \circ h_1 \circ h_0$,
- $U_{i+1} \subset U_i$,
- H_{i+1} *is supported in* U_i,
- $\operatorname{diam}(\overline{U_{i+1}}) \leqslant \frac{1}{3^i}$,

- $\|H_i\|_\infty \leqslant \frac{1}{3^i}$.

Proof We will construct these sequences by induction. First set $U_0 = W_0$, $H_0 = 0$ and $k_0 = 0$. Then, assume that we have constructed sequences $(U_i)_{i\in\{0,\dots,j\}}$, $(H_i)_{i\in\{0,\dots,j\}}$, $(W_{k_i})_{i\in\{0,\dots,j\}}$, with the desired properties.

According to Lemma 16, applied to the tree $\varphi_j(T)$, we can pick a Hamiltonian function H_{j+1} supported in U_j, such that $\|H_{j+1}\|_\infty \leqslant \frac{1}{3^{j+1}}$ and, denoting $h_{j+1} = \phi^1_{H_{j+1}}$, $h_{j+1}(\varphi_j(T))$ is included in a ball $B_j \subset U_j$, of diameter less than $\frac{1}{3^j}$. Then pick $k_{j+1} > k_j$ sufficiently large so that

$$h_{j+1}\left(\varphi_j(W_{k_{j+1}})\right) \subset B_j.$$

Then, $\operatorname{diam}(h_{j+1}(\varphi_j(W_{k_{j+1}}))) \leqslant \frac{1}{3^j}$ and we can set $U_{j+1} = h_{j+1}(\varphi_j(W_{k_{j+1}}))$. Since $B_j \subset U_j$, we have $U_{j+1} \subset U_j$ and the three sequences $(U_i)_{i\in\{0,\dots,j+1\}}$, $(H_i)_{i\in\{0,\dots,j+1\}}$, $(W_{k_i})_{i\in\{0,\dots,j+1\}}$ still have the required properties. By induction, we obtain the claimed infinite sequences. □

Since $\operatorname{diam}(\overline{U_i})$ converges to 0 when i goes to infinity and since the U_i's are nested, the intersection of the closures $\bigcap_{i\in\mathbb{N}} \overline{U_i}$ is a single point. Let p denote this point.

Consider the sequence of maps $\varphi_i = h_i \circ \cdots \circ h_1$. By construction, if $x \in T$ then $\varphi_i(x) \in U_i$, hence it converges to p. Moreover, for every neighborhood U of T, the restrictions $\varphi_i|_{M\setminus U}$ stabilize for i large. For every $x \notin T$, denote by $\varphi(x)$ the point $\varphi_i(x)$ for i large enough. The map φ is a diffeomorphism from $M\setminus T$ to $M\setminus\{p\}$.

We define for all $x \in M\setminus\{p\}$,

$$f(x) = \varphi \circ \psi \circ \varphi^{-1}(x),$$

and $f(p) = p$. We see that p is the unique fixed point of f. Indeed, if we assume that f admits another fixed point $q \neq p$ then, $\varphi(q)$ would be a fixed point of ψ which is not contained in T and this would be a contradiction.

The first part of Proposition 15 thus follows if we prove that f is a symplectic homeomorphism. This will be a consequence of the next claim, which requires us to introduce additional notations.

Without loss of generality, we may assume that for all i, the function $H_i(t,\cdot)$ vanishes for t in the complement of an open subinterval of $[0,1]$. Let $\tau_0 = 0$ and for every $i = 1, 2, \dots$, let $\tau_i = \sum_{k=1}^{i} \frac{1}{2^k}$. We consider the sequence of smooth Hamiltonian functions K_i defined as concatenations of time-reparametrizations of the H_i's as follows:

$$K_i(t,x) = \begin{cases} 2^{k+1} H_k\left(2^{k+1}(t-\tau_k), x\right), & \forall x \in M,\ k = 0, 1, \dots, i,\ t \in [\tau_k, \tau_{k+1}] \\ 0 & \forall x \in M,\ t \in [\tau_{i+1}, 1]. \end{cases}$$

Each Hamiltonian K_i generates the smooth isotopy φ_i^t given by:

$$\varphi_i^t = \begin{cases} \phi_{H_k}^{2^{k+1}(t-\tau_k)} \circ h_{k-1} \circ \cdots \circ h_0, & \forall k = 0, 1, \ldots, i,\ t \in [\tau_k, \tau_{k+1}] \\ h_i \circ \cdots \circ h_0, & \forall t \in [\tau_{i+1}, 1]. \end{cases}$$

Note that by construction, for all indices $i < j$, one has $K_i = K_j$ and $\varphi_i^t = \varphi_j^t$ on the time interval $[0, \tau_{i+1}]$. Also, note that for $t = 1$, we have $\varphi_i^1 = \varphi_i$.

Claim 18 *The isotopies $\varphi_i^t \circ \psi \circ (\varphi_i^t)^{-1}$ C^0-converges to an isotopy of homeomorphisms f^t as $i \to \infty$, satisfying $f^1 = f$. In particular, f is a symplectic homeomorphism.*

Proof of Claim 18 We denote $f_i^t = \varphi_i^t \circ \psi \circ (\varphi_i^t)^{-1}$ and let, for $t \in [0, 1)$ f^t be the homeomorphism defined by $f^t = f_i^t$ for any $i \in \mathbb{N}$ such that $\tau_{i+1} \geqslant t$. We set $f^1 = f$.

First, note that by construction f is a bijection. Note also that on any time interval, of the form $[0, 1 - \delta]$, with $\delta > 0$, f^t coincides with f_i^t for i large enough. We will prove that the sequences f_i^t and $(f_i^t)^{-1}$ respectively C^0-converge to f^t and $(f^t)^{-1}$. This will imply that f^t is a homeomorphism for all t (including $t = 1$) and prove the claim.

Let B be a ball around p. For i large enough, $U_i \subset B$, hence $\varphi_j^t(W_{k_i}) \subset B$ for all $j \geqslant i$ and $t \in [\tau_{j+1}, 1]$. The uniform continuity of ψ and the invariance of T implies that for $j \geqslant i$ large enough, $\psi(W_{k_j}) \subset W_{k_i}$. Thus,

$$f_j^t(U_j) = \varphi_j^t \circ \psi \circ \left(\varphi_j^t\right)^{-1}(U_j) \subset B \tag{1}$$

for j large enough and $t \in [\tau_{j+1}, 1]$. Let j_0 be such a large j.

For all j, it can be easily checked that

$$f_{j+1}^t = \rho_j^t \circ f_j^t \circ \left(\rho_j^t\right)^{-1},$$

where

$$\rho_j^t = \begin{cases} \mathrm{Id}, & \forall t \in [0, \tau_{j+1}] \\ \phi_{H_{j+1}}^{2^{j+2}(t-\tau_{j+1})}, & \forall t \in [\tau_{j+1}, \tau_{j+2}] \\ h_{j+1}, & \forall t \in [\tau_{j+2}, 1]. \end{cases}$$

It follows that if $t \in [0, \tau_{j+1}]$, then $(f_{j+1}^t)^{-1}(x) = (f_j^t)^{-1}(x)$ for all $x \in M$. If $t \in [\tau_{j+1}, 1]$, then for all $j \geqslant j_0$, ρ_j^t is supported in $U_j \subset U_{j_0} \subset B$, thus for all $x \notin B$, (1) yields $(f_j^t)^{-1} \circ (\rho_j^t)^{-1}(x) = (f_j^t)^{-1}(x) \notin U_j$. This implies that

$(f_{j+1}^t)^{-1}(x) = (f_j^t)^{-1}(x)$ for all $x \notin B$. Thus, the sequence $(f_i^t)^{-1}$ stabilizes in the complement of B, independently of t. Moreover, by construction the limit point of $(f_i^t)^{-1}(x)$ for $x \notin B$ is nothing but $(f^t)^{-1}(x)$.

It follows that for j large enough, $f_j^t \circ (f^t)^{-1}$ is supported in B, hence is C^0-close to Id. This shows that f_i^t C^0-converges to f^t. The same argument shows that $(f_j^t)^{-1}$ converges to $(f^t)^{-1}$. □

We will now prove that the homeomorphism $f \circ \psi^{-1}$ is a Hamiltonian homeomorphism.

The homeomorphism $f \circ \psi^{-1}$ is the time one map of the isotopy $f^t \circ \psi^{-1}$. According to Claim 18, this isotopy is the C^0-limit of the isotopies $\varphi_i^t \circ \psi \circ (\varphi_i^t)^{-1} \circ \psi^{-1}$. Since these isotopies are generated by the continuous Hamiltonians

$$F_i(t, x) = K_i(t, x) - K_i\left(t, \varphi_i^t \circ \psi^{-1} \circ \left(\varphi_i^t\right)^{-1}(x)\right),$$

the following claim implies that $f^t \circ \psi^{-1}$ is a hameotopy and hence that $f \circ \psi^{-1}$ is a Hamiltonian homeomorphism.

Claim 19 *The continuous Hamiltonians F_i converge uniformly as $i \to \infty$.*

Proof The condition $\|H_i\|_\infty \leqslant \frac{1}{3^i}$ implies that the sequence of Hamiltonians K_i converges uniformly to the continuous function K given for all $x \in M$ and all $t \in [0, 1)$ by $K(t, x) = 2^{k+1} H_k(2^{k+1}(t - \tau_k), x)$ with $k \in \mathbb{N}$ such that $t \in [\tau_k, \tau_{k+1}]$, and $K(1, x) = 0$. Now according to Claim 18, $\varphi_i^t \circ \psi^{-1} \circ (\varphi_i^t)^{-1}$ converges uniformly to $(f^t)^{-1}$. Therefore, $F_i(t, x)$ converges uniformly to $K(t, x) - K(t, (f^t)^{-1}(x))$.

We now pursue the proof of Proposition 15. Since φ is a smooth from $M \backslash T$ to $M \backslash \{p\}$, it is obvious that if ψ is smooth in the complement of T, then the above construction provides a symplectic homeomorphism f which is smooth in the complement of p.

Let us now assume that ψ coincides with a Hamiltonian diffeomorphism on the complement of any neighborhood of T. Let f be constructed as above. We will modify f to define a new symplectic homeomorphism $\tilde{f}$ which is in any normal subgroup of $\mathrm{Sympeo}(M, \omega)$ containing $\mathrm{Ham}(M, \omega)$.

Let K be a smooth Hamiltonian whose time-one map does not fix the point p. Then $\phi_K^1(p) \neq f \circ \phi_K^1(p)$ and we can find a small enough ball B around p, such that $\phi_K^1(B) \cap f \circ \phi_K^1(B) = \emptyset$.

For i large enough, f coincides with $\varphi_i \circ \psi \circ \varphi_i^{-1}$ on $M \backslash B$. Since by assumption ψ coincides with some Hamiltonian diffeomorphism on $M \backslash \varphi_i^{-1}(B)$, we deduce that we can write $f = h \circ g$, where h is a Hamiltonian diffeomorphism and g is a symplectic homeomorphism which is the identity on the complement of B. Now set

$$\tilde{f} = h \circ g \circ \phi_K^1 \circ g^{-1} \circ \left(\phi_K^1\right)^{-1}.$$

We see that $\tilde{f}$ belongs to any normal subgroup of $\mathrm{Sympeo}(M, \omega)$ containing $\mathrm{Ham}(M, \omega)$. We claim that p is the only fixed point of $\tilde{f}$. To see this, first note that $\phi_K^1 \circ g^{-1} \circ (\phi_K^1)^{-1}$ is the identity on the complement of $\phi_K^1(B)$. It follows in particular that $f = h \circ g$ and $\tilde{f}$ coincide on $M \backslash \phi_K^1(B)$, hence that p is the only fixed point of $\tilde{f}$ in $M \backslash \phi_K^1(B)$. But if $x \in \phi_K^1(B)$, then $\tilde{f}(x)$ belongs to $f \circ \phi_K^1(B)$, hence is distinct from x.

We point out that since g has only one non-smooth point, we see that $\tilde{f}$ has two non-smooth points, p and $\phi_K^1(p)$. □

Remark 20 We can now justify what we claimed in the discussion of Sect. 1.3, namely the fact that our Hamiltonian homeomorphism f with a unique fixed point can be generated by a continuous Hamiltonian admiting $cl(M)$ distinct spectral invariants. As in Sect. 1.3, we restrict our discussion to the case of an aspherical symplectic manifold.

We begin by recalling that spectral invariants of a smooth Hamiltonian depend on the Hamiltonian Lipschitz continuously; see [34]. It follows that one can define spectral invariants for any continuous function.

The argument showing that f can be generated by a continuous Hamiltonian with, at least, $cl(M)$ distinct spectral invariants requires four steps:

(i) The initial Hamiltonian H of the construction (see Theorem 13) is a C^2-small Morse function. For such a Hamiltonian, Floer theory is nothing but Morse theory and it follows from the classical Lusternik–Schnirelman theory that H must have at least $cl(M)$ distinct spectral invariants.

(ii) The Hamiltonian homeomorphism ψ obtained in Theorem 13 can be chosen so that $\psi \circ \phi_H^{-1}$ is generated by a Hamiltonian arbitrarily small in $\|\cdot\|_\infty$ norm. By continuity of spectral invariants, this implies that ψ can be generated by a (continuous) Hamiltonian, which will be denoted by G, with spectral invariants close to those of H.

(iii) In the above proof of Proposition 15, the Hamiltonian homeomorphisms $f_i = \varphi_i \circ \psi \circ \varphi_i^{-1}$ are conjugate to ψ. Hence, they can be generated by the Hamiltonians $G \circ \varphi_i^{-1}$ which have the same spectral invariants as G.

(iv) Our Hamiltonian homeomorphism f is constructed so that for i large, $f_i^{-1} \circ f$ is generated by a uniformly small Hamiltonian function $\widetilde{F}_i$. Thus, f is generated by the Hamiltonian $(G \circ \varphi_i^{-1}) \# \widetilde{F}_i$ whose spectral invariants are close to those of $G \circ \varphi_i^{-1}$ and hence to those of H. By choosing small enough perturbations, we can ensure that at least $cl(M)$ of these spectral invariants are distinct.

Remark 21 We should point out the argument presented in the above remark can be modified to prove the following (stronger) statement on closed sym-

plectic manifolds which are not necessarily aspherical: the Hamiltonian homeomorphism f can be generated by a continuous Hamiltonian, say G, whose spectral invariants are exactly the same as the spectral invariants of the initial C^2-small Morse function H.

In fact, one could go even further: it is possible to show that the continuous Hamiltonian G has the exact same barcode as the initial C^2-small Morse function H. Hence, despite the fact that the time-1 map of G has only one fixed point, from a Floer theoretic point of view G can not be distinguished from H. For further information on the theory of barcodes see [1,27,32,36].

Although these claims, and their proofs, are very interesting, in the interest of not lengthening the paper we do not present them here.

3.2 Building the tree from a Morse function

In this section we begin the proof of Theorem 13. Our first step will be to perturb our initial C^2-small Morse function H so that it satisfies a number of additional properties. This is the content of the following lemma. In order to simplify our presentation, throughout the rest of this section, we will refer to local maxima/minima of a function as maxima/minima. An extremum point of a function will be a point which is either a local maximum or a local minimum.

Lemma 22 *On every closed symplectic manifold (M, ω) and for every Morse function $\tilde{H}$ on M, there exists a Morse function H on M, arbitrarily C^1-close to $\tilde{H}$, with the following set of properties:*

1. *The function H takes distinct values at distinct critical points,*
2. *Every critical point p of H which is an extremum admits a neighborhood with Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in which H is of the form $H(p) + c\sum(x_i^2 + y_i^2)$, where $c \in \mathbb{R}\backslash\{0\}$ is a constant which can be chosen to have arbitrarily small magnitude,*
3. *For every critical point of H which is not extremal, there exist local Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in which $H = c(x_1^2 - y_1^2) + Q$, where c is some non-zero constant, and Q is a quadratic form in the variables $(x_2, \ldots, x_n, y_2, \ldots, y_n)$.*

The relevant consequence of the third condition is that locally near p, the Hamiltonian flow of H preserves the symplectic 2-plane $P = \{(x_1, 0, \ldots, 0, y_1, 0, \ldots, 0)\}$ and acts as a linear hyperbolic flow on it. Indeed, $X_H = X_{c(x_1^2 - y_1^2)} + X_Q$, where $X_H, X_{c(x_1^2-y_1^2)}, X_Q$ denote the associated Hamiltonian vector fields. It can easily be checked that $X_Q|_P = 0$. Hence, the restriction of the Hamiltonian flow of H to the plane P coincides with the Hamiltonian flow of $c(x_1^2 - y_1^2)$.

Proof We can assume without loss of generality that $\tilde{H}$ is a Morse function on M which has pairwise distinct critical values. We then modify $\tilde{H}$ near every critical point as follows. Let p be a critical point of $\tilde{H}$. Up to addition of a constant, we may assume that $\tilde{H}(p) = 0$.

Assume that p is a local minimum and let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be Darboux coordinates near p. We first make a C^2-small perturbation of $\tilde{H}$ so that in a small neighborhood of p, it coincides with its Hessian near p. Then, note that given two positive definite quadratic forms $Q_1 \leqslant Q_2$ and two open subsets U, V such that $0 \in U \subset \overline{U} \subset V$, there always exists a smooth function h which coincides with Q_1 on U and with Q_2 on the complement of V, and having 0 as its only critical point. Now define H by replacing $\tilde{H}$ by such a function h obtained in a neighborhood of p from the quadratic forms $Q_1 = c\sum x_i^2 + y_i^2$ for some small $c > 0$ and for Q_2 the Hessian of H. This perturbation can be made arbitrarily C^1-small by using a small neighborhood of p. Local maxima are worked out similarly.

Now assume that p is not an extremum. By the Morse lemma, and since p is not an extremum, there exists a local chart on a neighborhood of p, parametrised via coordinates $(v_1, \dots, v_n, w_1, \dots, w_n)$ by a small open ball W centered at the origin in the Euclidean space $\mathbb{R}^{2n}$, such that $p = (0, 0, \dots, 0)$ in these coordinates, and moreover $\tilde{H}$ has the form $\tilde{H} = \tilde{H}(p) + v_1^2 - w_1^2 + \sum_{i=2}^n \pm v_i^2 \pm w_i^2$. Now, let $(x_1, \dots, x_n, y_1, \dots, y_n)$ be Darboux coordinates in a neighborhood of p, such that $p = (0, 0, \dots, 0)$ in these coordinates as well. Then, choose a diffeomorphism ϕ of M, supported in a very small ball around p, such that near p, ϕ carries the coordinate system $(v_1, \dots, v_n, w_1, \dots, w_n)$ to the coordinate system $(x_1, \dots, x_n, y_1, \dots, y_n)$. Consider its "rescalings" $\phi_\lambda : M \to M$, for $\lambda \in (0, 1)$, defined by

$$\phi_\lambda(v_1, \dots, v_n, w_1, \dots, w_n) = \lambda\phi\left(\frac{v_1}{\lambda}, \dots, \frac{v_n}{\lambda}, \frac{w_1}{\lambda}, \dots, \frac{w_n}{\lambda}\right),$$

for $(v_1, \dots, v_n, w_1, \dots, w_n) \in \lambda W \subset \mathbb{R}^{2n}$, and by the identity on the complement of λW. Now, replacing $\tilde{H}$ by the pushforward $H := (\phi_\lambda)_* \tilde{H} = \tilde{H} \circ (\phi_\lambda)^{-1}$, we get that for small enough λ, H is a smooth function on M which is C^1-close to $\tilde{H}$, and which has the form $H = H(p) + x_1^2 - y_1^2 + \sum_{i=2}^n \pm x_i^2 \pm y_i^2$ in a small neighborhood of p. □

In the above lemma, the fact that H and $\tilde{H}$ are C^1-close implies that $d_{C^0}(\phi_{\tilde{H}}^1, \phi_H^1)$ is small and $\phi_H^1 \circ \phi_{\tilde{H}}^{-1}$ is generated by a uniformly small Hamiltonian. Hence, Theorem 13 now amounts to the following proposition.

Proposition 23 *Let H be a Morse function on M satisfying the properties of Lemma 22. If H is sufficiently C^2-small, then for every $\varepsilon, \rho > 0$, there exists $\psi \in \mathrm{Hameo}(M, \omega)$ and an embedded tree T (see Definition 14) such that:*

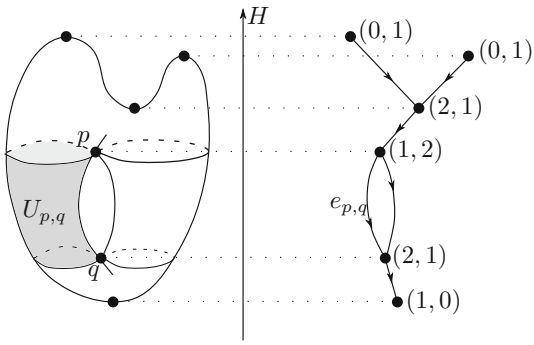

Fig. 1 The Reeb graph of a height function H on a 2-torus, the bi-degrees of the different vertices (Note that the bi-degree (1, 1) cannot happen on a surface) and an example of an edge $e_{p,q}$ with the corresponding open set $U_{p,q}$

- *T is ψ-invariant, i.e. $\psi(T) = T$,*
- *T contains all the fixed points of ψ,*
- *$d_{C^0}(\phi_H^1, \psi) < \varepsilon$ and $\psi \circ \phi_H^{-1}$ is generated by a continuous Hamiltonian F such that $\|F\|_\infty < \rho$,*
- *ψ coincides with a Hamiltonian diffeomorphism in the complement of any neighborhood of T.*

Our proof of Proposition 23 will make use of the notion of Reeb graph whose definition we now recall.

Definition 24 We assume that a function $H : M \to \mathbb{R}$ is given. For every point $x \in M$ we define $\mathcal{C}(x)$ as the connected component of x in the level set $H^{-1}(H(x))$. For a subset $X \subset M$, we define $\mathcal{C}(X) = \bigcup_{x \in X} \mathcal{C}(x)$.

The Reeb graph is the quotient space $\mathcal{R} = M/\sim$, where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $\mathcal{C}(x) = \mathcal{C}(y)$.

It follows from basic Morse theory that if H is a Morse function whose critical values are pairwise distinct, then the space $\mathcal{R}$ actually carries the structure of a graph whose vertices correspond to the critical points of H and such that an edge $e_{p,q}$ between two critical points p and q corresponds to a connected open subset $U_{p,q} \subset M$ such that:

- $U_{p,q}$ contains no critical point of H,
- The canonical projection $U_{p,q} \to U_{p,q}/\sim$ is a fibration onto an interval (this interval can be seen as parametrizing the edge),
- The closure $\overline{U_{p,q}}$ has two boundary components, one containing p and the other containing q.

By construction, H descends to a well defined function on $\mathcal{R}$, which we still denote H. This function is monotone on each edge. We assign orientations to the edges so that H is decreasing on each of them. As for any oriented graph, any vertex p admits a bi-degree which is a couple $(d_-(p), d_+(p))$, where $d_-(p)$ is the number of edges with head end at p and $d_+(p)$ is the number

of edges with tail end at p. This bi-degree is related to the Morse index of p. Indeed, local maxima have bi-degree $(0, 1)$, local minima have bi-degree $(1, 0)$, and all the other points have bi-degree $(1, 1)$, $(2, 1)$ or $(1, 2)$ (but $(2, 1)$ is only possible for points of Morse index $\dim(M) - 1$ and $(1, 2)$ is only possible for points of Morse index 1).

Proof of Proposition 23 Let $\varepsilon, \rho > 0$. Since M is connected, so is the Reeb graph $\mathcal{R}$ of H, and there exists a subgraph $\mathcal{T}$ of $\mathcal{R}$ which is a tree and contains all the vertices of $\mathcal{R}$. To each vertex of $\mathcal{T}$ we associate the corresponding critical point of H. This gives an embedding of the vertices of $\mathcal{T}$ into M. The complicated part of the proof will be to embed the edges.

By assumption, near every critical point p of H which is not extremal, there exist Darboux coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that $H = c(x_1^2 - y_1^2) + Q'$, where $c \neq 0$ and Q' is a quadratic form whose kernel is the plane $P = \{(x_1, 0, \dots, 0, y_1, 0, \dots, 0)\}$. As mentioned after the statement of Lemma 22, this implies in particular that locally near p, the flow of H preserves P and acts on it as the flow of the quadratic form $c(x_1^2 - y_1^2)$, which is linear hyperbolic. Thus the flow of H admits in P two orbits converging to p when time goes to $+\infty$ and two orbits converging to p when time goes to $-\infty$. Moreover, in the plane P and still locally near the point p, these four orbits are the frontiers of four regions; two of them correspond to $H < H(p)$ and the two others to $H > H(p)$.

To each edge $e_{p,q}$ with tail end at p, we associate one of the two orbits converging to p in the past. The local picture above shows that this orbit belongs to the closure $\overline{U_{p,q}}$. Whenever there are two such edges (i.e. $d_+(p) = 2$) we demand that the two associated orbits are distinct. Again, the local picture shows that this is possible. We let $x_{p,q}$ be a point located near p on the orbit associated to $e_{p,q}$. To summarize the situation, we have:

$$\overline{U_{p,q}} \ni \phi_H^t(x_{p,q}) \overset{t \to -\infty}{\longrightarrow} p.$$

We associate in a similar way to each edge $e_{q,p}$ with head end at p a point $y_{q,p}$ such that:

$$\overline{U_{q,p}} \ni \phi_H^t(y_{q,p}) \overset{t \to +\infty}{\longrightarrow} p.$$

For each oriented edge $e_{p,q}$ we now choose a path $\gamma_{p,q} : [0, 1] \to M$ satisfying the following properties (Fig. 2):

- For all $t \in [0, 1]$, $\frac{d}{dt}(H \circ \gamma_{p,q}(t)) < 0$,
- $\gamma_{p,q}(0) = p$ if p is a local maximum and $\gamma_{p,q}(0) = x_{p,q}$ otherwise,
- $\gamma_{p,q}(1) = q$ if q is a local minimum and $\gamma_{p,q}(1) = y_{p,q}$ otherwise.

Note that it follows from the above properties that $U_{p,q} = \mathcal{C}(\gamma((0, 1)))$.

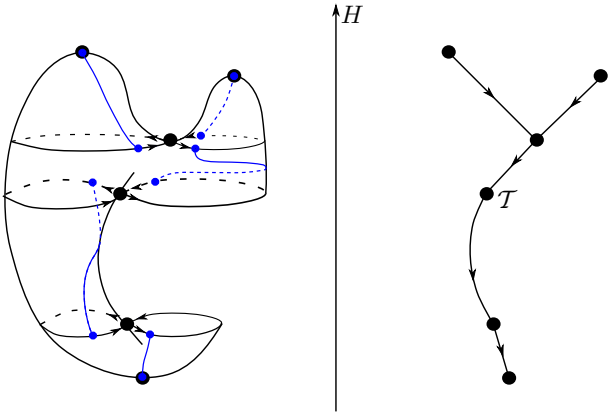

Fig. 2 The right-hand side represents the tree $\mathcal{T}$ obtained from the Reeb graph of Fig. 1. On the left hand side, the arrows correspond to the four orbits converging to non extremal points, and the blue curves represent possible choices for the curves $\gamma_{p,q}$

For each non extremal critical point p, and for each piece of orbit β which is either of the form $\{\phi_H^t(x_{p,q}) : t \in (-\infty, 1]\}$ or $\{\phi_H^t(y_{q,p}) : t \in [-1, +\infty)\}$, we choose some open sets $V_p(\beta)$, such that

$$\text{image}(\beta) \subset V_p(\beta) \subset M \backslash \{p\}.$$

We choose these open sets sufficiently small such that all the $V_p(\beta)$'s, for p ranging over all non-extremal critical points of H and β over all pieces of orbits as above, are pairwise disjoint.

We are now ready to construct the perturbation of H and the embedding of $\mathcal{T}$. We will proceed by induction on the edges of $\mathcal{T}$. For that purpose we number the edges $e_1, e_2, \ldots, e_N$, and for each index i, we let p_i and q_i be respectively the tail end and head end of e_i. With the previous notations, this means that $e_i = e_{p_i,q_i}$. We also rename the points $x_i = x_{p_i,q_i}$ and $y_i = y_{p_i,q_i}$, when they are defined and the paths $\gamma_i = \gamma_{p_i,q_i}$. For the sake of brevity, we omit the piece of orbit from the notation and denote V_{p_i} and V_{q_i} the corresponding open subsets of the form $V_p(\beta)$ when they are defined; it will always be clear from the context which orbit is considered. If p_i is a local maximum, we set $V_{p_i} = \emptyset$; if q_i is a local minimum, we set $V_{q_i} = \emptyset$.

Assume that for some $j \geqslant 0$, we have realized the edges $e_1, \ldots, e_j$ in M, i.e, we have built smooth embeddings $\alpha_1, \ldots, \alpha_j : \mathbb{R} \to M$ and Hamiltonian homeomorphisms $\theta_1, \ldots, \theta_j$, such that the following holds:

1. $\lim_{t \to -\infty} \alpha_i(t) = p_i$ and $\lim_{t \to +\infty} \alpha_i(t) = q_i$,
2. For all $i \in \{1, \ldots, j\}$ and all $t \in \mathbb{R}$, $\theta_i \circ \phi_H^1(\alpha_i(t)) = \alpha_i(t+1)$,
3. $\theta_i \circ \phi_H^1$ has the same fixed points as ϕ_H^1,

4. For all $i \in \{1, \dots, j\}$, α_i takes values in $V_{p_i} \cup \mathcal{C}(\gamma_i((0,1))) \cup V_{q_i}$, and the images of the α_i's are pairwise disjoint,
5. For all $i \in \{1, \dots, j\}$, θ_i is generated by a continuous Hamiltonian F_i supported in $\{p_i\} \cup V_{p_i} \cup \mathcal{C}(\gamma_i((0,1))) \cup V_{q_i} \cup \{q_i\}$, and the interiors of the supports of the θ_i's are pairwise disjoint,
6. $d_{C^0}(\theta_i, \mathrm{Id}) < \varepsilon$ and $\|F_i\|_\infty < \rho$.
7. For any neighborhoods of p_i, q_i, the homeomorphism θ_i coincides with a Hamiltonian diffeomorphism in the complement of the union of those neighborhoods.

We want to show that it is possible to build α_{j+1} and θ_{j+1} such that the above Properties 1–7 still hold. After shrinking the open sets $V_{p_{j+1}}$, $V_{q_{j+1}}$ if needed so that they intersect neither the supports of $F_1, \dots, F_j$ nor the images of the curves $\alpha_1, \dots, \alpha_j$, we construct α_{j+1} and θ_{j+1} by applying Theorem 25 below to the critical points p_{j+1}, q_{j+1}, to the curve γ_{j+1} and to the open subsets $V_{p_{j+1}}$, $V_{q_{j+1}}$.

Following this process by induction, we get curves $\alpha_1, \dots, \alpha_N$ and Hamiltonian homeomorphisms $\theta_1, \dots, \theta_N$ satisfying Properties 1–7. The union of the images α_i, denoted by T, is an embedding of the tree $\mathcal{T}$ in M in the sense of Definition 14. Moreover, it is easy to check that the Hamiltonian homeomorphism $\psi = \theta_N \circ \cdots \circ \theta_1 \circ \phi_H^1$ meets the requirements of Proposition 23. □

3.3 Connecting two critical points

The goal of this section is to explain the construction of the individual edges of the invariant tree of Proposition 23. This is achieved in Theorem 25.

3.3.1 Statement of the result

Theorem 25 *Let H be a Morse function as described in Lemma 22 and denote by p, q two critical points of H, at most one of them being an extremum. We assume that there exists a smooth embedded curve $\gamma : [0,1] \to M$ such that*

- *$\gamma(0) = p$ if p is a maximum of H, and $\phi_H^t(\gamma(0)) \to p$ when t goes to $-\infty$ if p is not a maximum,*
- *$\gamma(1) = q$ if q is a minimum of H, and $\phi_H^t(\gamma(1)) \to q$ when t goes to $+\infty$ if q is not a minimum,*
- *for all $t \in [0,1]$, $\frac{d}{dt}(H(\gamma(t))) < 0$.*

Let $\varepsilon, \rho > 0$ be positive real numbers, let V_p, V_q be open sets such that

$$\begin{cases} V_p = \emptyset, & \text{if } p \text{ is a maximum,} \\ V_p \supset \{\phi_H^t(\gamma(0)) : -\infty < t \leqslant 1\}, & \text{if } p \text{ is not a maximum,} \end{cases}$$

and

$$\begin{cases} V_q = \emptyset, & \text{if } q \text{ is a minimum,} \\ V_q \supset \left\{\phi_H^t(\gamma(1)) : -1 \leqslant t < +\infty\right\}, & \text{if } q \text{ is not a minimum.} \end{cases}$$

Then, there exist $\theta \in \mathrm{Hameo}(M, \omega)$ *and a smooth embedded curve* $\alpha : \mathbb{R} \to M$*, with the following properties:*

1. $\lim_{t\to-\infty} \alpha(t) = p$ *and* $\lim_{t\to+\infty} \alpha(t) = q$,
2. $\theta \circ \phi_H^1(\alpha(t)) = \alpha(t+1)$ *for all* $t \in \mathbb{R}$,
3. $\theta \circ \phi_H^1$ *has the same set of fixed points as* ϕ_H^1,
4. *Fot all* $t \in \mathbb{R}$, $\alpha(t)$ *belongs to the open set* $V_p \cup \mathcal{C}(\gamma((0,1))) \cup V_q$,
5. θ *is generated by a continuous Hamiltonian* F *which is supported in* $\{p\} \cup V_p \cup \mathcal{C}(\gamma((0,1))) \cup V_q \cup \{q\}$.
6. $d_{C^0}(\mathrm{Id}, \theta) < \varepsilon$ *and* $\|F\|_\infty < \rho$.
7. *For any neighborhoods of* p, q*, the homeomorphism* θ *coincides with a Hamiltonian diffeomorphism in the complement of the union of those neighborhoods.*

One of the main difficulties we will have to face in proving Theorem 25 is to perform perturbations of ϕ_H^1 without creating new fixed points. Away from the fixed points of ϕ_H^1, i.e. critical points of H, a C^0-small perturbation will not create such fixed points. However, a C^0-small perturbation near a critical point can create new fixed points, and for this reason neighborhoods of critical points will require special treatment. In order to surmount these difficulties, we will build the perturbation and the invariant curve of Theorem 25 in three steps: First, we build one end of the curve near one of the two critical points (this construction is achieved in Sect. 3.3.2 in the case of a local minimum/maximum). We then extend the invariant curve such that it reaches a sufficiently small neighborhood of the second critical point (Sect. 3.3.3). Finally, we finish the construction of the invariant curve in the neighborhood of the second critical point (Sect. 3.3.4).

3.3.2 Connecting a max/min to a nearby point

We denote by $\mathbb{R}^{2n}$ the Euclidean space of dimension $2n$ equipped with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. In the following theorem, $\|x\|$ denotes the Euclidean norm of a point $x \in \mathbb{R}^{2n}$.

Theorem 26 *Let* $H : \mathbb{R}^{2n} \to \mathbb{R}$ *be a Hamiltonian of the form* $H = c\sum_{i=1}^n (x_i^2 + y_i^2)$*, where* c *is a non-zero constant. There exists* $A > 0$ *such that if* $|c| < A$*, then the following statement holds for any point* $x \in \mathbb{R}^{2n}$ *and any* $\varepsilon, \delta > 0$.

Denote by $B \subset \mathbb{R}^{2n}$ an open ball which is centered at the origin and contains the point x. There exists a Hamiltonian homeomorphism θ, whose support is compactly contained in B, and a smooth injective immersion $\alpha : [-1, \infty) \to B \setminus \{0\}$ with the following properties:

1. *$\alpha(t) = \phi_H^t(x)$ for $t \in [-1, 0]$ and $\alpha(t) \to 0$ as $t \to \infty$, where 0 denotes the origin,*
2. *$\theta \circ \phi_H^1(\alpha(t)) = \alpha(t+1)$ for all $t \in [-1, \infty)$,*
3. *$\theta \circ \phi_H^1$ has only one fixed point and that is the origin,*
4. *The support of θ intersects the complement of the ball $B(0, \|x\|)$ of radius $\|x\|$ only in an ε-neighborhood of $\{\phi_H^t(x) : t \in [1, 2]\}$,*
5. *$d_{C^0}(Id, \theta) < \varepsilon$ and $\|F\|_\infty \leqslant \delta$, where F denotes a continuous Hamiltonian such that $\phi_F^1 = \theta$,*
6. *For any neighborhood of 0, there exists a compactly supported Hamiltonian diffeomorphism of B which coincides with θ in the complement of that neighborhood.*

Proof of Theorem 26 Without loss of generality we may assume that $\|x\| = 1$. We remark here that throughout the proof we will use the fact that ϕ_H^t, for each t, is a linear isometry of $\mathbb{R}^{2n}$, without explicitly mentioning it. We will assume that $\varepsilon > 0$ is very small in comparison to $\|\phi_H^1(x) - x\|$.

We will now pick a sequence of curves $\alpha_0, \alpha_1, \ldots$, which will be joined together, at a later stage, to form the invariant curve α. Let $x_0 = x$ and define $\alpha_0 : [0, 1] \to B$, $t \mapsto \phi_H^t(x_0)$. Let $0 < \rho < 1$ be a constant such that $1 - \rho$ is very small in comparison to ε. Now, let $x_1 = \rho\, \phi_H^1(x_0)$ and more generally, for each $i \geqslant 1$, $x_i = \rho^i \phi_H^i(x_0)$ and let $\alpha_i : [i, i+1] \to B$ denote the curve $\alpha_i(t) = \rho^i \phi_H^t(x_0)$ for each $t \in [i, i+1]$. Denote $y_i := \phi_H^1(x_i)$. Note that α_i satisfies the following identity:

$$\alpha_i(t) = \phi_H^{t-i}(x_i) = \phi_H^{t-i-1}(y_i), \quad \forall t \in [i, i+1]. \tag{2}$$

Step 1: Preliminary preparations for the construction of the invariant curve. For each non-negative integer i, let $a_i : [0, 1] \to B$ be the curve $a_i(t) := (1-t)y_i + t x_{i+1}$. Observe that, the length of a_i is $\rho^i - \rho^{i+1}$. Note that $\rho^i - \rho^{i+1} < \rho^i \varepsilon$. Furthermore, the image of a_i is disjoint from the remaining a_j's and is contained in the shell $\{z \in \mathbb{R}^{2n} : \rho^{i+1} \leqslant \|z\| \leqslant \rho^i\}$. See Fig. 3.

We will now introduce some of the notation which will be used throughout the proof. Let U_0 be a shell which is a slight enlargement of the shell $\{z \in \mathbb{R}^{2n} : \rho \leqslant \|z\| \leqslant 1\}$. For $i > 0$, we set $U_i := \rho^i U_0$. Note that U_i is a shell which is a slight enlargement of the shell $\{z \in \mathbb{R}^{2n} : \rho^{i+1} \leqslant \|z\| \leqslant \rho^i\}$ and furthermore, $U_i \cap U_j = \emptyset$ if $j \notin \{i-1, i, i+1\}$.

Next, let W_0 be a small neighborhood of the image of the curve a_0 which is contained in U_0 and has the following properties: the diameter of W_0 is less

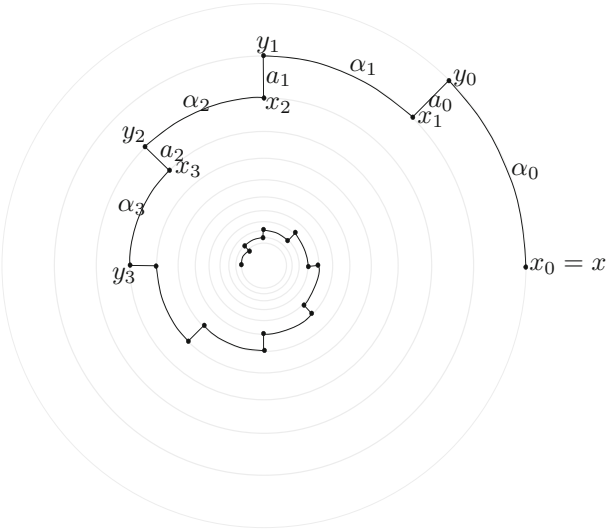

Fig. 3 Settings for the beginning of the proof of Theorem 26

than ε, W_0 is convex, $W_0 \cup \rho W_0$ is also convex, and there exists a small positive number κ such that $\phi_H^t(y_0), \phi_H^t(x_1) \in W_0$ if and only if $t \in (-2\kappa, 2\kappa)$. Lastly, we require that $W_0 \cap \rho\phi_H^1(W_0) = \emptyset$. Observe that, for this last requirement it is sufficient to take $A < \pi$. For $i > 0$, we define $W_i := \rho^i \phi_H^i(W_0)$, or equivalently, $W_i = \rho\phi_H^1(W_{i-1})$. What is important to observe about the sets W_i is that they have the following properties: W_i is a small neighborhood of the image of a_i which is contained in U_i and the diameter of W_i is less than $\rho^i\varepsilon$. The sets W_i are convex and so are $\phi_H^1(W_i) \cup W_{i+1}$. The W_i's are pairwise disjoint and furthermore $W_j \cap \phi_H^1(W_i) \neq \emptyset$ only if $j = i + 1$. Also, note that for $t \in [-1, 1]$ we have $\phi_H^t(y_i), \phi_H^t(x_{i+1}) \in W_i$ if and only if $t \in (-2\kappa, 2\kappa)$. Finally, we remark that among the sets W_j the only one which intersects the image of $\phi_H^1 \circ \alpha_i$ is W_i. Our set up is summarized in Fig. 3.

Next, we find Hamiltonians, say G_i, such that G_i is supported in W_i, and

$$\phi_{G_i}^1\left(\phi_H^t(y_i)\right) = \phi_H^t(x_{i+1}), \quad \forall\, t \in [-\kappa, \kappa]. \tag{3}$$

Let $G := \sum_{i=0}^{\infty} G_i$; we remark that by Lemma 10, we can pick G_i such that $\|G_i\|_\infty$ is arbitrarily small. Note that the G_i's are supported in W_i's which are pairwise disjoint. Therefore, G is well-defined and in fact $\|G\|_\infty$ can be made arbitrarily small. Furthermore, G generates a hameotopy whose flow is the (infinite) composition $\Pi_{i=1}^{\infty}\phi_{G_i}^t$: to see this observe that the flows $\phi_{G_i}^t$ are supported in W_i's which are disjoint and whose diameters are smaller than $\rho^i\varepsilon$. We point out that ϕ_G^1 coincides with a Hamiltonian diffeomorphism of B in the complement of any neighborhood of 0.

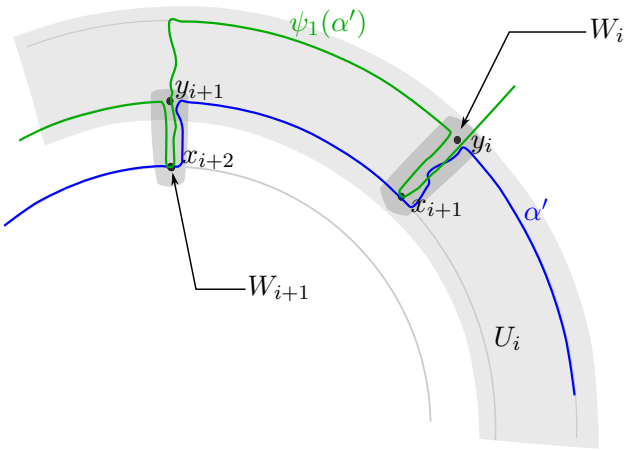

Fig. 4 The curve α' in blue and its image by ψ_1 in green

Let ψ_1 denote the Hamiltonian homeomorphism $\phi^1_G \circ \phi^1_H$.

Step 2: A first approximation to the invariant curve α. Consider the smooth curve $\alpha'(t) : [-1, \infty) \to B$ given by the following formula:

$$\alpha'(t) := \begin{cases} \phi^t_H(x) & t \in [-1, 0], \\ \phi^1_{G_i} \circ \alpha_i(t) & t \in [i, i+1], \quad \forall i \geqslant 0. \end{cases}$$

To see that the above formula defines a smooth curve one can check, using Eqs. (2) and (3), $\alpha'(t) = \phi^{t-i}_H(x_i)$ for $t \in [i-\kappa, i+\kappa]$ for each $i \geqslant 1$. One can ensure that α' is injective, by perturbing the Hamiltonians G_i, if necessary, and by picking the constant A, from the statement of the theorem, to be sufficiently small[5]; we leave it to the reader to check the details of this. The curve α' and its image under ψ_1 are drawn in Fig. 4.

It is evident that $\psi_1(\alpha'(t)) \neq \alpha'(t+1)$ for $t > 0$. In Steps 3 and 4, we will be modifying α' and ψ_1 to establish the invariance stated in the second property from the statement of the theorem. The next few claims record some properties of these two curves which will be used later in Steps 3 and 4. □

Claim 27 *For A sufficiently small, $\psi_1(\alpha'(t)) = \alpha'(t+1)$ for $t \in [-1, 0]$.*

Proof If $t \in [-1, 0]$, then one can easily conclude from the definition of α' that we have $\psi_1(\alpha'(t)) = \phi^1_{G_0}\phi^1_H\phi^t_H(x) = \phi^1_{G_0}\phi^{t+1}_H(x) = \alpha'(t+1)$. □

Claim 28 *For each $i \geqslant 0$, the two curves $\psi_1(\alpha'|_{[i,i+1]})$ and $\alpha'|_{[i+1,i+2]}$ coincide near their endpoints. More precisely, $\psi_1(\alpha'(t)) = \alpha'(t+1)$ for $t \in [i, i+\kappa] \cup [i+1-\kappa, i+1]$.*

[5] $A \leqslant \frac{\pi}{2}$ is sufficiently small for our purposes.

Proof We will check that if $t \in [i-\kappa, i+\kappa]$, then $\psi_1 \circ \alpha'(t) = \alpha'(t+1)$. To see this write $t = i+s$, where $s \in [-\kappa, \kappa]$, and note that $\psi_1 \circ \alpha'(t) = \phi_G^1 \phi_H^1 \phi_H^s(x_i) = \phi_G^1 \phi_H^s(y_i) = \phi_{G_i}^1 \phi_H^s(y_i) = \phi_H^s(x_{i+1}) = \alpha'(t+1)$. The last equality follows from the fact that for $i \geqslant 1$, we have $\alpha'(t) = \phi_H^{t-i}(x_i)$ when $t \in [i-\kappa, i+\kappa]$. □

For each $i \geqslant 0$, let V_i be the shell $U_i \cup U_{i+1}$. Note that $V_i \cap V_j = \emptyset$ if $j \notin \{i-2, i-1, i, i+1, i+2\}$.

Claim 29 *For each $i \geqslant 0$, there exists a homotopy, which is constant near endpoints, from $\psi_1 \circ \alpha'|_{[i,i+1]}$ to $\alpha'|_{[i+1,i+2]}$ which is supported in V_i and furthermore, under this homotopy the trajectory of any point of $\psi_1 \circ \alpha'|_{[i,i+1]}$ has diameter less than $2\rho^i\varepsilon$.*

Proof We point out that, as a consequence of Claim 28, the above two curves coincide near their endpoints. The idea of the proof of this claim is very simple: the straight-line homotopy $F(s,t) = (1-s)\psi_1 \circ \alpha'|_{[i,i+1]} + s\alpha'|_{[i+1,i+2]}$ satisfies all the required properties. The fact that $W_i \cup \rho W_i$ is convex, and of diameter less that $2\rho^i\varepsilon$, is used here to ensure that the trajectories of the points $\psi_1 \circ \alpha'|_{[i,i+1](t)}$ for t near i, $i+1$ are of diameter less than $2\rho^i\varepsilon$. Checking the proof of this claim in detail, although quite straight forward, is rather tedious. Hence, we will omit the proof. □

Step 3: Constructing the curve α Let λ be the 1-form $\sum_{i=1}^n x_i dy_i$. We define the action of any curve $a : [0,1] \to \mathbb{R}^{2n}$ to be the integral $\int_0^1 a^*\lambda$. We will now finish the construction of the curve α.

We begin by defining $\alpha|_{[-1,1]} = \alpha'|_{[-1,1]}$ and proceed to inductively construct the curve α such that it satisfies the following properties: for each $i \geqslant -1$, the two curves $\psi_1 \circ \alpha|_{[i,i+1]}$ and $\alpha|_{[i+1,i+2]}$ have the same action, and α coincides with α' for all values of t except near $t = i + \frac{1}{2}$, where $i \geqslant 1$. Suppose that, for some $k \geqslant 1$, we have constructed such $\alpha : [-1, k] \to \mathbb{R}^{2n}$. Next, we make a C^0 small perturbation of the curve $\alpha'|_{[k,k+1]}$, in an arbitrarily small neighborhood of the point $\alpha'(k+\frac{1}{2})$ to obtain a new curve, which we will call $\alpha|_{[k,k+1]}$, such that $\psi_1 \circ \alpha|_{[k-1,k]}$ and $\alpha|_{[k,k+1]}$ have the same action. It is evident from the construction that α coincides with α' for all t except for values of t near $i+\frac{1}{2}$, where $i \geqslant 1$. It is also clear that α be can be picked to be arbitrarily C^0 close to α'. We should add that it is a well-known fact that one can make an arbitrary adjustment to the action of a curve by performing a C^0-small perturbation; see for example Remark A.13 of [3].

We finish this step by pointing out that the curve α satisfies the first of the four properties listed in the statement of Theorem 26.

Step 4: Turning α into an invariant curve In this final step of the proof, we will perturb ψ_1 to a Hamiltonian diffeomorphism ψ such that $\psi \circ \alpha(t) = \alpha(t+1)$.

Claim 30 *For each $i \geqslant 0$, there exists a Hamiltonian K_i supported in $V_i = U_i \cup U_{i+1}$ such that*

1. $\phi^1_{K_i} \circ \psi_1 \circ \alpha(t) = \alpha(t+1)$ *for each* $t \in [i, i+1]$,
2. $d_{C^0}(Id, \phi^1_{K_i}) < 4\rho^i\varepsilon$ *and* K_i *can be picked such that* $\|K_i\|_\infty$ *is as small as one wishes,*
3. $\mathrm{supp}(K_i)$ *is contained within the* 4ε *neighborhood of* $\alpha([i+1, i+2])$,
4. $\alpha([j, j+1]) \cap \mathrm{supp}(K_i) = \emptyset$ *if* $j \neq i+1$. *Similarly,* $\psi_1 \circ \alpha([j, j+1]) \cap \mathrm{supp}(K_i) = \emptyset$ *for* $j \neq i$.

Proof In the previous step we constructed α such that $\psi_1 \circ \alpha|_{[i,i+1]}$ and $\alpha|_{[i+1,i+2]}$ have the same action. Furthermore, since we obtained $\alpha|_{[i,i+1]}$ from $\alpha'|_{[i,i+1]}$ by making a C^0-small perturbation of $\alpha'|_{[i,i+1]}$ near $t = i + \frac{1}{2}$ we conclude, by Claim 29, that there exists a homotopy, which is constant near endpoints, from $\psi_1 \circ \alpha|_{[i,i+1]}$ to $\alpha|_{[i+1,i+2]}$ which is supported in V_i and furthermore, under this homotopy the trajectory of any point of $\psi_1 \circ \alpha|_{[i,i+1]}$ has diameter less than $2\rho^i\varepsilon$. The first three properties of the claim then follow immediately from Proposition 9. As for the fourth property, since the dimension of M is at least four, by making a small perturbation of the homotopy from $\psi_1 \circ \alpha|_{[i,i+1]}$ to $\alpha|_{[i+1,i+2]}$, we may assume the image of the homotopy does not intersect $\alpha([j, j+1])$, for $j \neq i+1$, or $\psi_1 \circ \alpha([j, j+1])$, for $j \neq i$. Fix small $\kappa_i > 0$ such that the homotopy is constant on $\psi_1 \circ \alpha|_{[i,i+\kappa_i]\cup[i+1-\kappa_i]}$. Let W be a small neighborhood of the image of the restricted homotopy from $\psi_1 \circ \alpha|_{[i+\kappa_i, i+1-\kappa_i]}$ to $\alpha|_{[i+1+\kappa_i, i+2-\kappa_i]}$ such that $\alpha([j, j+1]) \cap W = \emptyset$, if $j \neq i+1$ and $\psi_1 \circ \alpha([j, j+1]) \cap W = \emptyset$ for $j \neq i$. Now, it is easy to see that K_i can be picked to have its support contained in W. This implies the fourth property of the claim. □

Note that for each i, $V_i \cap V_j \neq \emptyset$ only if $j \in \{i-2, i-1, i, i+1, i+2\}$. Hence, we see that the sets $V_0, V_3, V_6, \ldots$ are mutually disjoint and therefore the supports of $K_0, K_3, K_6, \ldots$ are mutually disjoint as well. This combined with the fact that $\|K_i\|_\infty$ can be picked to be as small as we wish, implies that the sum $F_0 := K_0 + K_3 + K_6 + \cdots$ defines a continuous function. Furthermore, as a consequence of the disjointness of the supports of these functions, we see that F_0 is a continuous Hamiltonian whose flow is $\phi^t_{F_0} = \phi^t_{K_0} \circ \phi^t_{K_3} \circ \phi^t_{K_6} \cdots$. Similarly, we define $F_1 = K_1 + K_4 + K_7 + \cdots$ and $F_2 = K_2 + K_5 + K_8 + \cdots$. These functions generate the hameotopies $\phi^t_{F_1} = \phi^t_{K_1} \circ \phi^t_{K_4} \circ \phi^t_{K_7} \cdots$ and $\phi^t_{F_2} = \phi^t_{K_2} \circ \phi^t_{K_5} \circ \phi^t_{K_8} \cdots$, respectively. Observe that, as a consequence of the second item in Claim 30, F_0, F_1, F_2 can be picked such that their norms are as small as one wishes.

We define $\theta := \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0} \circ \phi^1_G$ and $\psi := \theta \circ \phi^1_H = \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0} \circ \psi_1$. Clearly, θ is a Hamiltonian homeomorphism which is compactly supported in B.

We will now check that θ and the curve α satisfy the properties listed in the statement of Theorem 26. We have already checked the fact that the first property is satisfied.

First, we will show that $\psi(\alpha(t)) = \alpha(t+1)$, for all $t \in [-1, \infty)$. If $t \in [-1, 0]$, then this follows from Claim 27. Indeed, for $t \in [-1, 1]$ we have $\alpha(t) = \alpha'(t)$. Moreover, for $t \in [-1, 0]$ we have $\psi(\alpha'(t)) = \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0}(\psi_1(\alpha'(t))) = \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0}(\alpha'(t+1)) = \alpha'(t+1)$. The reason the last equality holds is that, by the 4th item of Claim 30, $\alpha'(t+1) = \alpha(t+1)$ is not contained in the support of any of F_0, F_1, F_2. Next, consider $t \in [0, \infty)$. Fix i and $t \in [i, i+1]$. According to Claim 30, $\alpha(t+1), \psi(\alpha(t)) \notin \text{supp}(K_j)$ for any j other than $j = i$. Hence, $\psi(\alpha(t)) = \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0} \circ \psi_1(\alpha(t)) = \phi^1_{K_i} \circ \psi_1(\alpha(t)) = \alpha(t+1)$.

To establish the remaining properties we will need the following claim. We define a nested sequence of balls $B_i \supset B_{i+1}$ as follows: For $i \geqslant 0$ we set $B_i = \{0\} \cup (\cup_{k \geqslant i} V_k)$. Note that B_0 is a slight enlargement of the unit ball and furthermore B_0 contains the supports of all F_0, F_1, F_2, and G.

Claim 31 *Suppose that $p \in B_0 - \{0\}$. Let i denote the smallest integer such that $p \in U_i$. Then, for each $j \in \{0, 1, 2\}$,*

- *if $i > 0$, then $\phi^1_{F_j}(p) \in B_{i-1}$ and $\|\phi^1_{F_j}(p) - p\| \leqslant 4\rho^{i-1}\varepsilon$.*
- *if $i = 0$, then $\phi^1_{F_j}(p) \in B_0$ and $\|\phi^1_{F_j}(p) - p\| \leqslant 4\varepsilon$.*
- *$\phi^1_G(p) \in B_{i-1}$ and $\|\phi^1_G(p) - p\| \leqslant \rho^{i-1}\varepsilon$.*

Proof We begin with the statement about $\phi^1_{F_j}(p)$. We will only prove this for $j = 0, i > 0$ and leave the remaining cases to the reader. Recall that $F_0 = K_0 + K_3 + K_6 + \cdots$ and that $\text{supp}(K_m) \subset V_m = U_m \cup U_{m+1}$ for all m. Hence, the point p can only be in the support of K_{i-1}, K_i, K_{i+1}. Now, only one of these three Hamiltonians enters the definition of F_0 and so we see that $\phi^1_{F_0}(p) \in \{\phi^1_{K_{i-1}}(p), \phi^1_{K_i}(p), \phi^1_{K_{i+1}}(p)\}$. The result then follows from Claim 30. The statement about $\phi^1_G(p)$ is proven similarly. □

We will now prove that ψ has no fixed points other than 0. Recall that $\psi = \theta \circ \phi^1_H$. Using the above claim, and the fact that $\theta = \phi^1_{F_2} \circ \phi^1_{F_1} \circ \phi^1_{F_0} \circ \phi^1_G$, we see that

1. $\|\theta \circ \phi^1_H(p) - \phi^1_H(p)\| \leqslant 13\rho^{i-4}\varepsilon$, when $p \in B_i$ and $i \geqslant 4$
2. $\|\theta \circ \phi^1_H(p) - \phi^1_H(p)\| \leqslant 13\varepsilon$, when $p \in B_i$ and $i \leqslant 3$.

Suppose that $p \in U_i$ where $i \geqslant 4$. We will show that p can not be a fixed point of $\theta \circ \phi^1_H$. First, note that since U_i is a slight enlargement of the shell $\{z : \rho^{i+1} \leqslant \|z\| \leqslant \rho^i\}$, we can assume that $|p| \geqslant \rho^{i+2}$. Recall that, we picked ε to be very small in comparison to the number $C = \|\phi^1_H(z) - z\|$, where z is any point such that $\|z\| = 1$. It follows that $\|\phi^1_H(p) - p\| \geqslant C\rho^{i+2}$. Hence, we

see that if $C\rho^{i+2} > 13\rho^{i-4}\varepsilon$, then $\theta \circ \phi_H^1(p) \neq p$. Of course, by picking ε to be sufficiently small we can make sure that the inequality $C\rho^{i+2} > 13\rho^{i-4}\varepsilon$ holds. (Recall that we picked ρ such that $1-\rho$ is small in comparison to ε.) We leave the case where $p \in U_i$ and $i \leqslant 3$ to the reader.

Next, we will show that the support of θ satisfies the fourth property from the statement of the theorem. Indeed, examining the proof we can see that the Hamiltonians G_i, K_i were all picked to be supported within at most a 4ε neighborhood of $\alpha([i+1, i+2])$. Of these Hamiltonians it is only G_0, K_0 whose supports intersect the complement of $B(0, \|x\|)$. This implies that the support of θ intersects the complement of $B(0, \|x\|)$ only in a 4ε-neighborhood of $\alpha([1, 2])$. Now, we leave it to the reader to check that $\alpha([1, 2])$ is contained within an ε-neighborhood of $\{\phi_H^t(x) : t \in [1, 2]\}$. Replacing ε by $\frac{\varepsilon}{5}$ throughout the proof yields the result.

We now check the 5th property from the statement of the theorem. Since $\theta = \phi_{F_2}^1 \circ \phi_{F_1}^1 \circ \phi_{F_0}^1 \circ \phi_G^1$, it is an immediate consequence of Claim 31 that $d_{C^0}(\theta, Id) \leqslant 13\varepsilon$. Hence, if we replace ε by $\frac{\varepsilon}{13}$ then we obtain $d_{C^0}(Id, \theta) \leqslant \varepsilon$. Next, let F denote the generating Hamiltonian of the continuous Hamiltonian flow $\phi_{F_2}^t \circ \phi_{F_1}^t \circ \phi_{F_0}^t \circ \phi_G^t$. It follows from the composition formulas mentioned in Sect. 2.1 that $\|F\|_\infty \leqslant \|F_0\|_\infty + \|F_1\|_\infty + \|F_2\|_\infty + \|G\|_\infty$. As we have already mentioned, F_0, F_1, F_2, G can be picked to have norms as small one wishes. Hence, the same is true for $\|F\|_\infty$.

Finally, we check the 6th and final property. Recall the nested sequence of balls $B_i \supset B_{i+1}$ introduced before Claim 31. Let B_i^c denote the complement of B_i. Observe that in B_i^c, for any i, each of $\phi_{F_0}^1, \phi_{F_1}^1, \phi_{F_2}^1$, and ϕ_G^1 coincides with a Hamiltonian diffeomorphism. Furthermore, using Claim 31, it can easily be checked that each of these homeomorphisms maps B_i^c into B_{i+1}^c. Combining these facts together we see that θ coincides with a Hamiltonian diffeomorphism in the complement of each B_i. □

3.3.3 Connecting two non critical points by an invariant curve

The main goal of this section is to prove the following technical result which is needed for the construction of the invariant curve of Theorem 25.

Theorem 32 *Denote by $H : M \to \mathbb{R}$ a smooth autonomous Hamiltonian on a closed symplectic manifold (M, ω) of dimension at least 4. Let z, w be two points in M. Suppose that there exists a smooth embedded curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = z, \gamma(1) = w$, such that H is not constant on γ, and the map $\Gamma : [0, 1] \times \mathbb{R} \to M$ defined by $\Gamma(s, t) = \phi_H^t(\gamma(s))$ is a smooth embedding on $(\{0\} \times [-2, 1]) \cup ((0, 1) \times [0, 1]) \cup (\{1\} \times [-1, 2])$.*

Then, for any constants $\varepsilon, \rho > 0$ and any neighborhood U of $\Gamma([0, 1] \times [0, 1])$, there exist

- *a Hamiltonian diffeomorphism* $\psi : M \to M$,
- *and a smooth embedded curve* $\alpha : [0, k+1] \to M$, *where* $k \in \mathbb{N}$,

such that:

1. $\psi(\alpha(s)) = \alpha(s+1)$ *for any* $s \in [0, k]$,
2. *There exists* $t_0 \in (1,2)$ *such that for all* $s \in [0, t_0]$, $\alpha(s) = \phi_H^{s-2}(z)$, *and* $\alpha([t_0, k+1]) \subset U \cup \mathcal{C}(\gamma((0,1)))$,
3. *Let* V *be any neighborhood of the point* $\phi_H^{\frac{1}{2}}(w)$. *Then,* α *can be chosen to satisfy the following property:*

$$\alpha(k+s) \begin{cases} = \phi_H^s(w), & s \in \left[0, \frac{1}{2} - b\right] \cup \left[\frac{1}{2} + b, 1\right] \\ \in V, & s \in \left[\frac{1}{2} - b, \frac{1}{2} + b\right], \end{cases}$$

   for some small $b > 0$. *Furthermore,* $\psi \circ \alpha(k+s) = \phi_H^{s+1}(w)$ *for* $s \in [0, \frac{1}{2} - b] \cup [\frac{1}{2} + b, 1]$,
4. $d_{C^0}(\psi, \phi_H^1) < \varepsilon$ *and the Hamiltonian diffeomorphism* $\theta := \psi \circ \phi_H^{-1}$ *is generated by a Hamiltonian, say* F, *such that* $\|F\|_\infty < \rho$. *Furthermore,* F *is supported inside* $U \cup \mathcal{C}(\gamma((0,1)))$,
5. ψ *has the same set of fixed points as* ϕ_H^1.

Proof of Theorem 32 By replacing U with a smaller open subset we can assume that there exists a diffeomorphism $\Phi : U \to (-c, 1+c) \times (-c, 1+c) \times (-c, c) \times \cdots \times (-c, c) \subset \mathbb{R}^{2n}$, where $c > 0$ is sufficiently small. By slightly decreasing c, if necessary, we may assume that Φ is a bi-Lipschitz map, where we consider the metric g on M and the standard euclidean metric on $(-c, 1+c) \times (-c, 1+c) \times (-c, c) \times \cdots \times (-c, c)$. We can further suppose that this diffeomorphism identifies $\Gamma(s,t)$ with $(s, t, 0, \ldots, 0)$ for all $s, t \in [0,1]$. Note that here we are relying on the fact that Γ is an embedding on $[0,1] \times [0,1]$.

Step 1: Preliminary preparations for the construction of the invariant curve We begin by picking $\delta > 0$ such that $\delta < \frac{\varepsilon}{2}$ and $\forall x, y \in M$

$$\text{if } d(x,y) < \delta, \text{ then } \quad \forall t \in [0,1],\ d\left(\phi_H^t(x), \phi_H^t(y)\right) < \frac{\varepsilon}{2}. \tag{4}$$

Pick $m \in \mathbb{N}$ large enough such that $d(\gamma(\frac{i}{m}), \gamma(\frac{i+1}{m})) < \frac{\delta}{2}$. For each $0 \leqslant i < m$, we define small neighborhoods U_i of $\Gamma([\frac{i}{m}, \frac{i+1}{m}] \times [0,1])$ by

$$U_i := \Phi^{-1}\left(\left(\tfrac{i}{m} - a, \tfrac{i+1}{m} + a\right) \times (-c, 1+c) \times (-c, c) \times \cdots \times (-c, c)\right),$$

where $a > 0$ is taken to be so small that $U_i \cap U_j \neq \emptyset$ only if $j \in \{i-1, i, i+1\}$. Clearly, $U_i \subset U$ for all i.

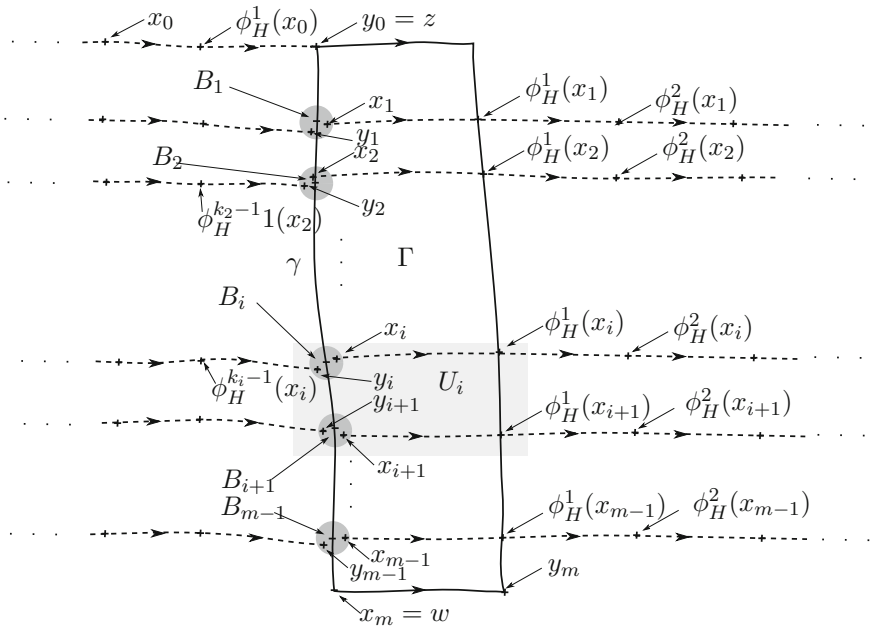

Fig. 5 Settings from Step 1

Take $r > 0$ satisfying the following two criteria: r is small in comparison to δ and the r-neighborhood of $\gamma(\frac{i}{m})$ is contained in U_i. For each $0 < i < m$ choose a ball B_i in the r-neighborhood of $\gamma(\frac{i}{m})$ such that $B_i \subset U_i \cap \mathcal{C}(\gamma((0,1)))$, such that $H(B_i) \cap H(B_j) = \emptyset$ for any $0 < i < j < m$, and such that moreover $H(x_0), H(x_m) \notin H(B_i)$ for each $0 < i < m$. It is not hard to show that such a choice of B_i is always possible; here it is important to remember that, because Γ is an embedding, γ does not pass through any of the critical points of H. Note that we may also assume that $B_{i+1} \subset U_i$ for $0 \leqslant i < m-1$.

We can ensure that for each $x \in B_i \cup B_{i+1}$ the image of the curve $t \mapsto \phi_H^t(x)$, $t \in [0,1]$ is contained in U_i; this will be used in Step 4.

By the Poincaré recurrence theorem, we can find points $x_i \in B_i$, $0 < i < m$, and integers $k_i \geqslant 2$ with the property that $\phi_H^{k_i}(x_i) \in B_i$. Let $y_i := \phi_H^{k_i}(x_i)$. By an arbitrarily C^2-small perturbation of H inside the open set U away from the curves $(\phi_H^t(x_0))_{t\in[-2,0]}$ and $(\phi_H^t(x_m))_{t\in[0,2]}$, if needed, we may assume that the curve $(\phi_H^t(x_i))_{t\in[-1,k_i+1]}$ is embedded, for each i. If i is 0 or m, we set $x_0 = \phi_H^{-2}(z)$, $y_0 = z$, $k_0 = 2$ and $x_m = w$, $y_m = \phi_H^1(x_m)$, $k_m = 1$. We remark that for each $0 \leqslant i < m$, the points x_i and x_{i+1} do not belong to the same level set of H; this will be used in the next step of the proof. Figure 5, below, describes the settings from Step 1.

Step 2: Turning the sequence $x_0, \dots, x_m$ into a genuine orbit For each $0 \leqslant i \leqslant m-1$ pick smooth embedded curves $a_i : [0,1] \to M$ such that

- $a_i(0) = y_i$ and $a_i(1) = x_{i+1}$,
- the diameter of a_i is smaller than δ [note that $d(y_i, x_{i+1}) < \delta$],
- the image of a_i is disjoint from the image of a_j, for $j \neq i$, and it is contained in the open set U_i,
- for each i the image of a_i does not intersect any of the curves $t \mapsto \phi_H^t(x_j)$, $t \in [-1, k_j + 1]$, for any value of j, except at the two points $y_i = \phi_H^{k_i}(x_i)$ and $x_{i+1} = \phi_H^0(x_{i+1})$. Moreover, for each i the image of a_i does not intersect the curves $(\phi_H^t(x_0))_{t\in[-2,0]}$ and $(\phi_H^t(x_m))_{t\in[0,2]}$, except at the point x_0 and x_m, respectively.

The above conditions, combined with the fact that the points x_i were picked such that y_i and x_{i+1} are not on the same trajectory of ϕ_H^t, guarantee that for each $0 \leqslant i \leqslant m-1$ we can find $W(a_i)$, a small neighborhood of the image of a_i, with the following properties: $W(a_i) \subset U_i$, the diameter of $W(a_i)$ is less than δ, these neighborhoods are disjoint, and each of the curves $(\phi_H^t(y_i))_{t\in[-1,1]}$ and $(\phi_H^t(x_{i+1}))_{t\in[-1,1]}$ takes values in $W(a_i)$ if and only if $t \in (-2\kappa_i, 2\kappa_i)$ where $\kappa_i > 0$ is small. Now, let $\kappa = \min\{\kappa_1, \dots, \kappa_{m-1}\}$. By shrinking the neighborhoods $W(a_i)$ we may assume the following:
If $t \in [-1, 1]$, then for $0 \leqslant i \leqslant m-1$ we have

$$\phi_H^t(y_i) \in W(a_i) \iff \phi_H^t(x_{i+1}) \in W(a_i) \iff t \in (-2\kappa, 2\kappa). \tag{5}$$

If $t \in [-2, 1]$, then

$$\phi_H^t(y_0) \in W(a_0) \iff t \in (-2\kappa, 2\kappa). \tag{6}$$

If $t \in [-1, 2]$, then

$$\phi_H^t(x_m) \in W(a_m) \iff t \in (-2\kappa, 2\kappa). \tag{7}$$

Next, we find a Hamiltonian, say G_1, the time-1 map of whose flow we will denote by τ, such that G_1 is supported in the union of $W(a_i)$'s, and

$$\tau\left(\phi_H^t(y_i)\right) = \phi_H^t(x_{i+1}), \quad \forall\, t \in [-\kappa, \kappa]. \tag{8}$$

Let $\psi_1 := \tau\phi_H^1$; observe that $\psi_1^{k_i}(x_i) = x_{i+1}$ for each $0 \leqslant i \leqslant m-1$. Also, note that $d_{C^0}(\tau, Id) < \delta$ and thus $d_{C^0}(\psi_1, \phi_H^1) < \delta < \varepsilon$. Lastly, we remark that G_1 can be picked such that $\|G_1\|_\infty$ is as small as desired.

Step 3: A first approximation to the invariant curve α Let $k = k_0 + k_1 + \cdots + k_{m-1}$ and consider the smooth curve $\alpha_1 : [0, k+1] \to M$ given by the following formula: For $0 \leqslant i \leqslant m$, let $K_i = \sum_{j=0}^{i} k_j$ and define

$$\alpha_1(t) := \begin{cases} \phi_H^t(x_0) & t \in [0,1] \\ \tau \circ \phi_H^t(x_0) & t \in [1, K_0] \\ \phi_H^{t-K_i}(x_{i+1}) & t \in \left[K_i, K_{i+1}-1\right], 0 \leqslant i \leqslant m-2 \\ \tau \circ \phi_H^{t-K_i}(x_{i+1}) & t \in \left[K_{i+1}-1, K_{i+1}\right], 0 \leqslant i \leqslant m-2 \\ \phi_H^{t-k}(x_m) & t \in [k, k+1] \end{cases}$$

The fact that the above formula yields a smooth curve is an immediate consequence of Eq. (8). Since the x_i's belong to different level sets of H, by slightly perturbing Hamiltonian diffeomorphism τ if needed we can guarantee that this curve is embedded. □

Remark 33 In this remark, we will prove that the curve α_1 satisfies the second property from the statement of the theorem.

We take $t_0 = K_0 - 2\kappa = 2 - 2\kappa$. Then, it follows from the definitions of α_1 and τ, and the fact that $x_0 = \phi_H^{-2}(z)$, that $\alpha_1(t) = \phi_H^{t-2}(z)$ for $t \in [0, t_0]$. We remark that $\alpha_1([t_0, k+1]) \subset U \cup \mathcal{C}(\gamma((0,1)))$. This can easily be checked from the definition of α_1 and noting the following facts: τ is supported in U, and $\{x_i : 1 \leqslant i \leqslant m\} \subset U \cap \mathcal{C}(\gamma((0,1))$. In Step 4 we will have to modify α_1 to obtain α. However, we will never change α_1 on $[0, t_0]$ and all of the modifications will take place inside U. Hence, the second property from the statement of the theorem will continue to hold for α as well.

Remark 34 Note that $\alpha_1(k+s) = \phi_H^s(w)$, for $s \in [0,1]$, and $\psi_1 \circ \alpha_1(k+s) = \phi_H^{s+1}(w)$ for $s \in [0,1]$. So at this stage the third property is also satisfied for any choice of V and any sufficiently small value of b. Throughout the remainder of the proof we will have to modify the curve α_1. However, the modifications on $[k, k+1]$ will be such that the the formula given in the third property will remain true. Furthermore, we will also modify ψ_1 to a different map ψ. Now, ψ will be constructed such that the support of $\theta := \psi \circ \phi_H^{-1}$ will not intersect the curve $\phi_H^1 \circ \alpha([k, k+1])$. It will then follow that $\psi \circ \alpha(k+s) = \phi_H^{s+1}(w)$ for $s \in [0, \frac{1}{2} - b] \cup [\frac{1}{2} + b, 1]$. This establishes the third property from the statement of the theorem.

Remark 35 For $0 \leqslant l \leqslant k$, we let $\alpha_1|_{[l,l+1]} : [0,1] \to M$ denote the curve $\alpha_1|_{[l,l+1]}(t) := \alpha_1(l+t),\ \forall t \in [0,1]$. Note that $\psi_1 \circ \alpha_1|_{[l-1,l]} = \alpha_1|_{[l,l+1]}$ for $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$. For the remaining values of l the two curves $\psi_1 \circ \alpha_1|_{[l-1,l]}$ and $\alpha_1|_{[l,l+1]}$ only coincide near their endpoints. Indeed, it can easily be checked, using Eqs. (5) and (8), that both of the above curves coincide with the curve $t \mapsto \phi_H^t(x_{i+1})$ on $[0, \kappa] \cup [1-\kappa, 1]$.

Remark 36 Let $l = K_i, 0 \leqslant i \leqslant m-1$. The two curves $\psi_1 \circ \alpha_1|_{[l-1,l]}$ and $\alpha_1|_{[l,l+1]}$ are both contained in U_i. This follows from the construction of τ

and the fact that the curves $t \mapsto \phi_H^t(x_i)$ and $t \mapsto \phi_H^t(y_i)$, $t \in [0, 1]$, are both contained in U_i. We leave the details of this to the reader.

We will next check that for each $l \in \{K_i : 0 \leqslant i \leqslant m-1\}$ the C^0 distance between the two curves $\psi_1 \circ \alpha_1|_{[l-1,l]}$ and $\alpha_1|_{[l,l+1]}$ is small. More precisely, we will prove:

Claim 37 *$d(\psi_1 \circ \alpha_1(l-1+t), \alpha_1(l+t)) < \varepsilon$, for each $t \in [0, 1]$, and for each $l \in \{K_i : 0 \leqslant i \leqslant m-1\}$.*

Proof of Claim Using the definition of α_1, it can easily be checked that $\alpha_1(K_i + t) = \phi_H^t(x_{i+1})$ and $\alpha_1(K_i - 1 + t) = \tau\phi_H^{k_i-1+t}(x_i) = \tau\phi_H^{-1+t}(y_i)$. It follows that $\psi_1 \circ \alpha_1(K_i - 1 + t) = \tau\phi_H^1\tau\phi_H^{-1+t}(y_i)$.

Now, recall that the curves $t \mapsto \phi_H^t(y_i)$ and $t \mapsto \phi_H^t(x_i)$ intersect the support of τ only for $t \in (-2\kappa, 2\kappa)$. Using this fact it can be checked that

$$\tau\phi_H^1\tau\phi_H^{-1+t}(y_i) = \begin{cases} \tau\phi_H^t(y_i), & t \in [0, 2\kappa], \\ \phi_H^t(y_i), & t \in [2\kappa, 1-2\kappa], \\ \tau\phi_H^1\left(\tau\phi_H^{-1+t}(y_i)\right), & t \in [1-2\kappa, 1]. \end{cases}$$

If $t \in [0, 2\kappa]$ then $\tau\phi_H^t(y_i), \phi_H^t(x_{i+1}) \in W(a_i)$ and the set $W(a_i)$ has diameter less than δ, thus $d(\tau\phi_H^t(y_i), \phi_H^t(x_{i+1})) < \delta < \varepsilon$.

If $t \in [2\kappa, 1-2\kappa]$, then we must prove that $d(\phi_H^t(y_i), \phi_H^t(x_{i+1})) < \varepsilon$. This follows immediately from Eq. (4) and the fact that $d(y_i, x_{i+1}) < \delta$.

Finally, we consider the case $t \in [1-2\kappa, 1]$: Write $\phi_H^t(x_i) = \phi_H^1(\phi_H^{-1+t}(x_i))$. Note that in this case $\tau\phi_H^{-1+t}(y_i)$ and $\phi_H^{-1+t}(x_{i+1})$ are both contained in the set $W(a_i)$ which has diameter less than δ. We see, via Eq. (4), that

$$\begin{aligned} d\left(\tau\phi_H^1\left(\tau\phi_H^{-1+t}(y_i)\right), \phi_H^t(x_{i+1})\right) &\leqslant d_{C^0}(\tau, Id) \\ &+ d\left(\phi_H^1\left(\tau\phi_H^{-1+t}(y_i)\right), \phi_H^1\left(\phi_H^{-1+t}(x_{i+1})\right)\right) < \delta + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

which proves our claim. □

Step 4: Constructing the curve α Let $\mathcal{V}$ denote a contractible open subset of M. Since $\mathcal{V}$ is contractible there exists a 1-form λ such that $\omega = d\lambda$ inside $\mathcal{V}$. Using the 1-form λ we can define the action of any curve $a : [0, 1] \to \mathcal{V}$ to be the integral $\int_0^1 a^*\lambda$. Of course, the action of a curve depends on λ, however the difference of action between two curves with the same endpoints does not depend on the choice of λ or $\mathcal{V}$.

For any $0 \leqslant l \leqslant k$, the two curves $\alpha_1|_{[l,l+1]}$ and $\psi_1 \circ \alpha_1|_{[l-1,l]}$ coincide near their endpoints. Furthermore, one can find a contractible neighborhood of

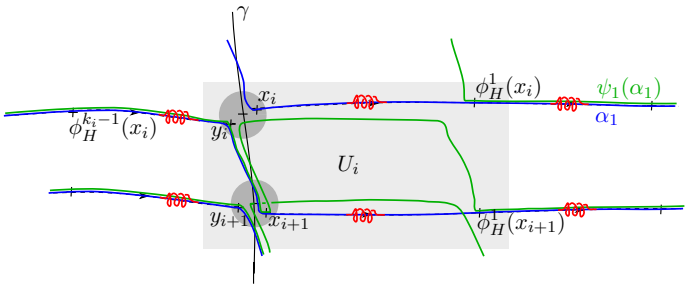

Fig. 6 A portion of the curve α_1 in blue, its image by ψ_1 in green, and the curve α obtained from α_1 by local perturbations which are represented in red

$\alpha_1|_{[l,l+1]}$ which contains $\psi_1 \circ \alpha_1|_{[l-1,l]}$: Indeed, as mentioned in Remark 35, if $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$, then $\psi_1 \circ \alpha_1|_{[l-1,l]} = \alpha_1|_{[l,l+1]}$. Hence, we can take any sufficiently small neighborhood of $\alpha_1|_{[l,l+1]}$. If $l = K_i$, $0 \leqslant i \leqslant m-1$, then by Remark 36 both curves are contained in U_i which is contractible.

If $l = 1$ the two curves $\alpha_1|_{[l,l+1]}$ and $\psi_1 \circ \alpha_1|_{[l-1,l]}$ coincide. Beginning with $l = 2$, for each $2 \leqslant l \leqslant k$ we successively make C^0-small perturbations of $\alpha_1|_{[l,l+1]}$ near $t = l + \frac{1}{2}$ to obtain an embedded curve α with the property that the two curves $\alpha|_{[l,l+1]}$ and $\psi_1 \circ \alpha|_{[l-1,l]}$ have the same action and are still contained in the same contractible open set mentioned in the previous paragraph. Here, we are again using the fact that one can make an arbitrary adjustment to the action of a curve by performing a C^0-small perturbation; see Remark A.13 of [3]. See Fig. 6.

We will ensure that the perturbations made in the previous paragraph satisfy the following properties. In the case where $l \in \{K_i : 0 \leqslant i \leqslant m-1\}$, we require the C^0 distance between $\alpha|_{[l,l+1]}$ and $\psi_1 \circ \alpha|_{[l-1,l]}$ to be less than ε. The fact that this can be achieved follows from Claim 37 and by taking the perturbations from the previous paragraph to be sufficiently small. Next consider the case where $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$. In the previous paragraph, we obtained α from α_1 by making a C^0 small perturbation of α_1 in the interior of a small closed interval which contains the point $t = l + \frac{1}{2}$; let $I_l \subset [l, l+1]$ denote this interval. Since the intervals I_l can be taken to be arbitrarily small and the action-adjusting perturbations from the previous step can be taken to have arbitrarily small support, we can pick contractible neighborhoods V_l of $\alpha|_{I_l}$ such that these neighborhoods are pairwise disjoint and the image of $\psi_1 \circ \alpha|_{I_{l-1}}$ is contained in V_l for each l. Furthermore, we require that $\phi_H^1(V_l) \cap V_l = \emptyset$ and that $\alpha(t) \in V_l$ only if $t \in (l, l+1)$. For $l = K_i$ we of course can assume that $\psi_1 \circ \alpha|_{[l-1,l]}$ and $\alpha|_{[l,l+1]}$ are contained in U_i. We remark that by taking V_k and I_k to be sufficiently small we can ensure that the formula for α from the third property of the statement of the theorem continues to hold.

Lastly, we point out that since $\alpha_1([2, k+1])$ is contained in $U \cup \mathcal{C}(\gamma((0,1)))$ (see Remark 33), by picking the sets V_l to be small enough, we may assume that each V_l is contained in $U \cup \mathcal{C}(\gamma((0,1)))$.

Step 5: Turning α into an invariant curve In this final step of the proof, we will perturb ψ_1 to a Hamiltonian diffeomorphism ψ such that $\psi \circ \alpha|_{[l-1,l]} = \alpha|_{[l,l+1]}$. Recall that in Step 4 we arranged that the two curves $\psi_1 \circ \alpha|_{[l-1,l]}$ and $\alpha|_{[l,l+1]}$ have the same action.

First assume that $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$. Then, we can find a Hamiltonian diffeomorphism η_l, generated by a Hamiltonian which is compactly supported in V_l, such that $\eta_l \circ \psi_1 \circ \alpha|_{I_{l-1}} = \alpha|_{I_l}$. Now, let η be the composition of the η_l's for $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$ and write $\psi_2 = \eta \circ \psi_1$ and observe that

$$\psi_2 \circ \alpha|_{[l-1,l]} = \alpha|_{[l,l+1]}. \tag{9}$$

Since the supports of η_l's are disjoint and can be taken to be as small as one wishes, we obtain the following inequalities

$$d_{C^0}\left(\psi_2, \phi_H^1\right) = d_{C^0}\left(\eta \circ \psi_1, \phi_H^1\right) \leqslant d_{C^0}(\eta, Id) + d_{C^0}\left(\psi_1, \phi_H^1\right) < \delta < \varepsilon.$$

We remark that we can assume that η is the time-1 map of a Hamiltonian, say G_2, whose norm $\|G_2\|_\infty$ can be made as small as one wishes. Existence of such Hamiltonian G_2 follows from Lemma 10.

We will now deal with the case $l \in \{K_i : 0 \leqslant i \leqslant m-1\}$. Fix i and to simplify our notation denote $\gamma_0(t) := \psi_2 \circ \alpha|_{[K_i-1,K_i]}(t) = \psi_2 \circ \alpha(K_i - 1 + t)$ and $\gamma_1(t) := \alpha|_{[K_i,K_i+1]}(t) = \alpha(K_i + t)$. The curves γ_0, γ_1 coincide for $t \in [0,\kappa] \cup [1-\kappa, 1]$: by Remark 35 this was true for α_1, and α differs from α_1 only for t near $K_i + \frac{1}{2}$. Recall from Step 4 that the C^0 distance between γ_0 and γ_1 is less than ε and that both of these curves are contained in U_i.

Claim 38 *There exists a homotopy $F_i(s,t) : [0,1] \times [\kappa, 1-\kappa] \to U_i$ rel. end points from $\gamma_0|_{[\kappa,1-\kappa]}$ to $\gamma_1|_{[\kappa,1-\kappa]}$ such that under this homotopy the trajectory of any point of $\gamma_0|_{[\kappa,1-\kappa]}$ has diameter less than $C\varepsilon$ for some constant $C > 0$ that only depends on U, the diffeomorphism Φ introduced before Step 1 and the Riemannian metric.*

Proof The existence of the homotopy F_i follows from the fact that the diffeomorphism Φ picked at the beginning of the proof identifies U_i with the box $(\frac{i}{m} - a, \frac{i+1}{m} + a) \times (-c, 1+c) \times (-c, c) \times \cdots \times (-c, c)$. We can take F_i to be the straight-line homotopy in these coordinates. The statement about diameter of trajectories of points on $\gamma_0|_{[\kappa,1-\kappa]}$ follows from the fact that the C^0 distance between γ_0 and γ_1 is less than ε. The constant C is given by the ratio of the pull back, via Φ, of the Riemannian metric d to $(-c, 1+c) \times (-c, 1+c) \times (-c, c) \times \cdots \times (-c, c)$ and the usual Euclidean metric. □

The dimension of M is at least 4 and thus we can perturb the homotopy F_i, provided to us by the above claim, to ensure that its image intersects the image of α only for $t \in (K_i, K_i+1)$ and the image of $\psi_2 \circ \alpha$ only for $t \in (K_i-1, K_i)$. Let $W_i \subset U_i$ be a small neighborhood of the image of the homotopy F_i which intersects the images of α and $\psi_2 \circ \alpha$ only for $t \in (K_i, K_i+1)$ and $t \in (K_i-1, K_i)$, respectively. Lastly, recall that we picked the sets U_i in Step 1 such that U_i can only intersect U_{i-1} and U_{i+1}. Hence, W_i can only intersect W_{i-1} and W_{i+1}.

Now, we apply the h-principle of Proposition 9 to obtain a Hamiltonian, say G_i', generating a Hamiltonian isotopy φ_i^t, $t \in [0,1]$, such that

1. G_i' is supported in W_i and $\|G_i'\|_\infty$ is as small as one wishes,
2. $\varphi_i^1(\psi_2 \circ \alpha|_{[K_i-1,K_i]}) = \alpha|_{[K_i,K_i+1]}$,
3. $d_{C^0}(Id, \varphi_i^1) < 2C\varepsilon$.

Let ϕ_{odd} denote the composition of the diffeomorphisms φ_i^1 for odd i. Note that ϕ_{odd} is the time-1 map of the Hamiltonian G_{odd} which is the sum of all G_i' for odd i. Define ϕ_{even} and G_{even} similarly. Let $\psi := \phi_{\text{even}}\phi_{\text{odd}}\psi_2$.

We remark that W_{m-1} is disjoint from $\phi_H^1 \circ \alpha([k, k+1])$. As pointed out in Remark 34, this ensures that the third property from the statement of the theorem holds.

We will now check that it is indeed true that $\psi \circ \alpha(t) = \alpha(t+1)$ for $t \in [0,k]$. If $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$, then $\psi \circ \alpha|_{[l-1,l]} = \psi_2 \circ \alpha|_{[l-1,l]}$: this is because we picked the sets W_i, whose union contains the supports of ϕ_{even} and ϕ_{odd}, such that they do not intersect the image of $\psi_2 \circ \alpha|_{[l-1,l]}$. We have already checked, in Eq. (9), that $\psi_2 \circ \alpha(t) = \alpha(t+1)$ for $t \in [l-1, l]$. We must next check that $\psi \circ \alpha|_{[l-1,l]} = \alpha|_{[l,l+1]}$ for $l \in \{K_i : 0 \leqslant i \leqslant m-1\}$. Fix i and let $l = K_i$. Then, $\phi_{\text{even}}\phi_{\text{odd}}\psi_2 \circ \alpha|_{[K_i-1,K_i]} = \varphi_i^1 \psi_2 \circ \alpha|_{[K_i-1,K_i]}$ because we picked the sets W_i such that the image of $\psi_2 \circ \alpha|_{[K_i-1,K_i]}$ only intersects W_i. Now, φ_i^1 was picked such that $\varphi_i^1(\psi_2 \circ \alpha|_{[K_i-1,K_i]}) = \alpha|_{[K_i,K_i+1]}$.

The rest of the proof is dedicated to verifying the fourth and the fifth properties from the statement of the theorem. We will first check that $d_{C^0}(\psi, \phi_H^1) < (4C+1)\varepsilon$. First, note that $d_{C^0}(\phi_{\text{odd}}, Id) < 2C\varepsilon$ because ϕ_{odd} is the composition of the diffeomorphisms φ_i^1, for odd i, which have disjoint supports and each of which satisfies $d_{C^0}(\varphi_i^1, Id) < 2C\varepsilon$. Similarly, $d_{C^0}(\phi_{\text{even}}, Id) < 2C\varepsilon$. It follows that $d_{C^0}(\psi, \phi_H^1) = d_{C^0}(\phi_{\text{even}}\phi_{\text{odd}}\psi_2, \phi_H^1) \leqslant d_{C^0}(\phi_{\text{even}}\phi_{\text{odd}}, Id) + d_{C^0}(\psi_2, \phi_H^1) < 4C\varepsilon + \varepsilon$; recall that we proved earlier that $d_{C^0}(\psi_2, \phi_H^1) < \varepsilon$. By going back to Step 1 and replacing ε with $\frac{\varepsilon}{4C+1}$ we obtain $d_{C^0}(\psi, \phi_H^1) < \varepsilon$.

It is fairly easy to see that we obtained ψ from ϕ_H^1 by composing it, on the left, with the Hamiltonian diffeomorphism $\theta := \phi_{even} \circ \phi_{odd} \circ \eta \circ \tau$; we denote by F the generating Hamiltonian of θ. Now, the supports of the diffeomorphisms $\tau, \eta, \phi_{odd}, \phi_{even}$ are all contained in the union of U and the open sets V_l, $l \notin \{K_i : 0 \leqslant i \leqslant m-1\}$. Recall that, as was mentioned at the

end of Step 4, each of the sets V_l is contained in $U \cup \mathcal{C}(\gamma((0,1)))$. This proves the claim about the support of F.

As was remarked throughout the proof, the generating Hamiltonians of $\tau, \eta, \phi_{odd}, \phi_{even}$, which we denoted by $G_1, G_2, G_{\text{odd}}, G_{\text{even}}$, respectively, were picked to have norms as small as one wishes. This proves the claim about $\|F\|_\infty$; we have established the fourth property in the statement of the theorem.

To finish the proof of the theorem, it remains to show that ψ has the same set of fixed points as ϕ_H^1. Now, $\psi = \theta \circ \phi_H^1$ and θ is supported in the union of U and the open sets V_l. Since U and the V_l's do not contain any of the fixed points of ϕ_H^1 we conclude that the fixed points of ϕ_H^1 are all fixed points of ψ as well. Now, to show that ψ has no additional fixed points we will check that any point x in the support of θ cannot be a fixed point of ψ. First, suppose that $x \in U$. Since U is a small neighborhood of the compact set $\Gamma([0,1] \times [0,1])$, which contains no fixed points of ϕ_H^1, there exists a positive number, say ϱ, such that $d(x, \phi_H^1(x)) > \varrho$ for all $x \in U$. Since $d_{C^0}(Id, \theta) < \varepsilon$, picking ε to be smaller than ϱ guarantees that $\theta\phi_H^1(x) \neq x$. Next, suppose that $x \notin U$. Hence, it must be the case that x is contained in one of the V_l's. We leave it to the reader to check, using the condition $\phi_H^1(V_l) \cap V_l = \emptyset$, that x can not be a fixed point of ψ. This completes the proof of Theorem 32. □

3.3.4 Proof of Theorem 25

In this section, we prove Theorem 25 with the help of Theorems 26 and 32.

Let H be a Morse function as described in Lemma 22, and let $\varepsilon > 0$. Let p, q be critical points of H, at least one of the two being non-extremal and let $\gamma : [0,1] \to M$ be a curve as in the statement of Theorem 25. We will give the proof under the assumption that q is not a minimum. The proof in the case where q is a minimum (hence p is not a maximum), can be easily adapted from this case.

Let $\varepsilon, \rho > 0$, let V_p be either the empty set if p is a maximum or an open set containing $\{\phi_H^t(\gamma(0)) : -\infty < t \leqslant 1\}$ is p is not a maximum, and let V_q be an open set containing $\{\phi_H^t(\gamma(1)) : -1 \leqslant t < +\infty\}$.

Our construction will be carried out in three steps: first in the neighborhood of p, then from the neighborhood of p to that of q, and finally in the neighborhood of q. Figure 7 below might help to grasp the construction.

Step 1: Construction in the neighborhood of p We need to consider two cases.

The simpler case is where p is not a maximum. In this case, we set $z = \gamma(0)$ and $\alpha_1(t) = \phi_H^{t-2}(z)$ for all $t \in (-\infty, 1]$. At this point there is no need for a perturbation of ϕ_H^1, thus we set $\theta_1 = \text{Id}$. It is clear that, α_1 and θ_1 satisfy the requirements of Theorem 25 for α and θ.

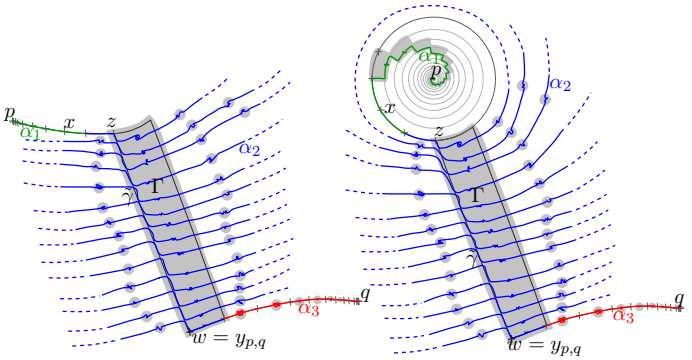

Fig. 7 The case "p not a maximum" is on the left, and the case "p a maximum" is on the right. The curves α_1, α_2 and α_3 are respectively in green, blue and red. The supports of the successive perturbations on ϕ_H^1 are included in the grey regions

Now consider the case when p is a maximum. In this case, $\gamma(0) = p$ and it is our assumption that in some Darboux coordinates around p, H is of the form $c\sum_{i=1}^{n}(x_i^2 + y_i^2)$ where c is a small negative constant. We denote by $\|\cdot\|$ the standard euclidean norm in these coordinates, and by $B(0, r)$ the euclidean ball centered at 0 and of radius r. Let z be a point on the image of γ, distinct from p, but close enough to p so that we may apply Theorem 26 to the Hamiltonian $H' = -H$ and to the point $x = \phi_H^{-2}(z) = \phi_{H'}^2(z)$. Given a ball B centered at p and containing x, and given $\varepsilon > 0$, Theorem 26 provides us with a curve $\alpha' : [-1, +\infty) \to B\backslash\{p\}$ and a Hamiltonian homeomorphism θ' satisfying :

- $\alpha'(t) = \phi_{H'}^t(x)$ for $t \in [-1, 0]$ and $\alpha'(t) \to p$ as $t \to +\infty$,
- $\theta' \circ \phi_{H'}^1(\alpha'(t)) = \alpha'(t+1)$ for all $t \in [-1, \infty)$,
- $\theta'(p) = p$ and $\theta' \circ \phi_{H'}^1$ has the same fixed points as $\phi_{H'}^1$,
- The support of θ' is contained in the union of the open ball $B(0, \|x\|)$ and the ε-neighborhood of $\{\phi_{H'}^t(x) : t \in [1, 2]\}$,
- $d_{C^0}(Id, \theta') < \varepsilon'$ and $\|F'\|_\infty \leqslant \rho$, where F' denotes a continuous Hamiltonian such that $\phi_{F'}^1 = \theta'$,
- For any neighborhood of p, there exists a compactly supported Hamiltonian diffeomorphism of B which coincides with θ' in the complement of that neighborhood.

Now define $\alpha_1(t) = \theta'^{-1}(\alpha'(-t))$ for all $t \in (-\infty, 1]$ and $\theta_1 = \theta'^{-1}$. We deduce the following from the above properties:

1. $\alpha_1(t) = \phi_H^{t-2}(z)$ for $t \in [0, 1]$ and $\alpha_1(t) \to p$ as $t \to -\infty$,
2. $\theta_1 \circ \phi_H^1(\alpha_1(t)) = \alpha_1(t+1)$ for all $t \in (-\infty, 0]$,
3. $\theta_1 \circ \phi_H^1$ has the same fixed points as ϕ_H^1,
4. α_1 takes values in the closed punctured ball centered at p and having z on its boundary, which is contained in $\mathcal{C}(\gamma((0, 1)))$.

5. The support of θ_1 is contained in the union of the open ball $B(0, \|x\|)$ and the ε-neighborhood of $\{\phi_H^t(x) : t \in [-2, -1]\}$,
6. $d_{C^0}(Id, \theta_1) < \varepsilon$ and $\|F_1\|_\infty < \rho$, where F_1 denotes a continuous Hamiltonian such that $\phi_{F_1}^1 = \theta_1$,
7. For any neighborhood of p, there exists a compactly supported Hamiltonian diffeomorphism of B which coincides with θ_1 in the complement of that neighborhood.

We see that α_1 and θ_1 are compatible with the requirements of Theorem 25.

Step 2: From a neighborhood of p to a neighborhood of q In both cases above we have set $z = \gamma(a)$ with $0 \leqslant a < 1$ ($a > 0$ if p is a maximum, $a = 0$ otherwise), and let us now set $w = \gamma(1)$ and $\tilde{\gamma}(t) = \gamma(a + (1 - a)t)$ for $t \in [0, 1]$. The curve $\tilde{\gamma}$ satisfies $\tilde{\gamma}(0) = z$ and $\tilde{\gamma}(1) = w$. It also satisfies that $\frac{d}{dt} H \circ \tilde{\gamma}(t) < 0$, which implies in particular that if H is C^2-small enough then the map $\Gamma : (s, t) \mapsto \phi_H^t(\tilde{\gamma}(s))$ is a smooth embedding of the set $(\{0\} \times [-2, 1]) \cup ((0, 1) \times [0, 1]) \cup (\{1\} \times [-1, 2])$. In the case where p is a maximum, note that since the flow of H preserves level sets, the compact set $\Gamma([0, 1] \times [0, 1])$ does not intersect the open ball $B(0, \|x\|) = B(0, \|z\|)$. Moreover, if ε and c are small enough, $\Gamma([0, 1] \times [0, 1])$ does not intersect the ε-neighborhood of $\{\phi_H^t(x) : t \in [-2, -1]\}$ either. Hence, $\Gamma([0, 1] \times [0, 1])$ does not intersect the support of θ_1.

We are now in the situation to apply Theorem 32: Let $U \supset \Gamma([0, 1] \times [0, 1])$ be an open set chosen so small that:

- If p is not a maximum,

$$U \cup \mathcal{C}(\tilde{\gamma}((0, 1))) = U \cup \mathcal{C}(\gamma((0, 1))) \subset V_p \cup \mathcal{C}(\gamma((0, 1))) \cup V_q, \quad (10)$$

- If p is a maximum, U does not intersect the support of θ_1 and

$$U \cup \mathcal{C}(\tilde{\gamma}((0, 1))) = U \cup \mathcal{C}(\gamma((a, 1))) \subset \mathcal{C}(\gamma((0, 1))) \cup V_q. \quad (11)$$

- U does not intersect the piece of orbit $\{\phi_H^t(w) : t \in [1 + \frac{1}{3}, +\infty)\}$.

Then, for every $\varepsilon, \rho > 0$, we can find a curve $\tilde{\alpha}_2 : [0, k + 1] \to M$, with $\alpha_2(0) = x = \phi_H^{-2}(z)$ and $\alpha_2(k) = w$, and a Hamiltonian diffeomorphism ψ which is C^0-close to ϕ_H^1 and satisfies a certain list of properties. We set $\theta_2 = \psi \circ \phi_H^{-1}$. The properties of α_2 and $\psi = \theta_2 \circ \phi_H^1$ listed in Theorem 32 have the following consequences:

1. $\alpha_2(t) = \alpha_1(t) = \phi_H^{t-2}(z)$ for all $t \in [0, 1]$,
2. $\theta_2 \circ \phi_H^1(\alpha_2(t)) = \alpha_2(t + 1)$ for all $t \in [0, k]$,
3. $\theta_2 \circ \phi_H^1$ has the same set of fixed points as ϕ_H^1,

4. There exists $t_0 \in (1,2)$ such that $\alpha_2(t) = \phi_H^t(x)$ for all $t \in [0,t_0]$ and $\alpha_2([t_0, k+1]) \subset U \cup \mathcal{C}(\tilde{\gamma}((0,1)))$. If p is not a maximum then $\{\phi_H^t(x) : t \in [0,t_0]\} \subset V_p$ by definition of V_p; if p is a maximum, then $\{\phi_H^t(x) : t \in [0,t_0]\}$ is contained in the boundary of the ball $B(0, \|x\|)$, hence in $\mathcal{C}(\gamma((0,1)))$. Using the inclusions (10) and (11), we thus get that α_2 takes values in $V_p \cup \mathcal{C}(\gamma((0,1))) \cup V_q$.
5. θ_2 is generated by a Hamiltonian F_2 which is supported in the open set $U \cup \mathcal{C}(\tilde{\gamma}((0,1)))$, which does not intersect the support of θ_1,
6. $d_{C^0}(\theta_2, \mathrm{Id}) < \varepsilon$ and $\|F_2\|_\infty < \rho$,
7. θ_2 is a smooth Hamiltonian diffeomorphism.

All the points above follow immediately from Theorem 32. Note that it also follows from Theorem 32 that for any neighborhood V of $\phi_H^{\frac{1}{2}}(w)$, the curve α_2 can be chosen so that for some $b > 0$, and all $t \in [k, k+1]$,

$$\alpha_2(t) \begin{cases} = \phi_H^{t-k}(w), & t \in \left[k, k+\frac{1}{2}-b\right] \cup \left[k+\frac{1}{2}+b, k+1\right] \\ \in V, & t \in \left[k+\frac{1}{2}-b, k+\frac{1}{2}+b\right]. \end{cases} \tag{12}$$

Step 3: Construction in the neighborhood of q By assumption, $\phi_H^t(w) = \phi_H^t(\gamma(1)) \to q$ when $t \to +\infty$. Let $\tilde{\alpha}_3 : [k, +\infty) \to M$ be the smooth curve defined by

$$\tilde{\alpha}_3(t) = \begin{cases} \alpha_2(t), & t \in [k, k+1) \\ \phi_H^{t-k}(w), & t \in [k+1, +\infty). \end{cases}$$

It has the following properties:

- $\tilde{\alpha}_3(t) \to q$ when $t \to +\infty$,
- For some small $\kappa > 0$, the image of $\tilde{\alpha}_3|_{[k+1+\kappa, +\infty)}$ does not intersect the supports of θ_1 nor θ_2,
- For all $t \in [k, +\infty) \setminus (k+\frac{1}{2}-b, k+\frac{1}{2}+b)$,

$$\theta_2 \circ \theta_1 \circ \phi_H^1(\tilde{\alpha}_3(t)) = \phi_H^1(\tilde{\alpha}_3(t)) = \tilde{\alpha}_3(t+1),$$

- According to (12), for $t \in [k+\frac{1}{2}-b, k+\frac{1}{2}+b]$, $\theta_2 \circ \theta_1 \circ \phi_H^1(\tilde{\alpha}_3(t))$ belongs to $\phi_H^1(V)$ which is an arbitrarily small neighborhood of $\tilde{\alpha}_3(k+1+\frac{1}{2})$.

For every integer $l \in \{k+1, k+2, \ldots\}$, we let B_l be a small ball centered at $\tilde{\alpha}_3(l+\frac{1}{2})$ and included in the open set V_q. We may assume V is small enough so that $V \subset U$ and $\phi_H^1(V) \subset B_{k+1}$. We also assume that the balls B_l are all disjoint, and that they do not intersect the supports of θ_1 nor θ_2. Similarly as in Step 4 of the proof of Theorem 32, we successively make C^0-small perturbations of $\tilde{\alpha}_3|_{[l,l+1]}$ in B_l, for $l \in \{k+1, k+2, \ldots\}$, to obtain an embedded curve

α_3, such that the curves $\phi_H^1(\alpha_3|_{[l-1,l]})$ and $\alpha_3|_{[l,l+1]}$ have the same action for all l. We can choose those perturbations so that, for each l, these two curves coincide in the complement of a very small interval J_l centered at $l+\frac{1}{2}$, and they both send J_l into B_l. Using Lemma 10 (as in Step 5 in the proof of Theorem 32), this property implies that we can find Hamiltonian diffeomorphisms η_l, generated by Hamiltonians which are compactly supported in the B_l's, such that $\eta_l(\phi_H^1(\alpha_3(t))) = \alpha_3(t+1)$ for all $t \in [l-1, l]$. Moreover, Lemma 10 tells us that these generating Hamiltonians can be chosen with arbitrary small $\|\cdot\|_\infty$ norm.

Let θ_3 be the composition of all the η_l's for $l \in \{k+1, k+2, \ldots\}$. By construction, θ_3 is generated by a Hamiltonian F_3 whose support does not intersect the support of θ_1 nor that of θ_2.

Moreover, the η_l's can be chosen C^0-close to Id by shrinking the balls B_{k+1}, $B_{k+2}, \ldots$. If we shrink each ball B_l so that $\operatorname{diam} B_l < \inf_{z\in B_l} d(z, \phi_H^1(z))$, then $\theta_3 \circ \phi_H^1$ has no fixed point in B_l. Indeed, the triangle inequality yields for all $z \in B_l$:

$$d\left(\theta_3 \circ \phi_H^1(z), z\right) \geqslant d\left(z, \phi_H^1(z)\right) - d\left(\theta_3 \circ \phi_H^1(z), \phi_H^1(z)\right) > 0.$$

Therefore, the following list of properties is satisfied:

1. $\alpha_3(t) \to q$ when $t \to +\infty$,
2. $\theta_3 \circ \phi_H^1(\alpha_3(t)) = \alpha_3(t+1)$ for all $t \in [k, +\infty)$,
3. $\theta_3 \circ \phi_H^1$ has the same set of fixed points as ϕ_H^1,
4. For all $t \in [k, +\infty)$, $\alpha_3(t) \in V_q$,
5. $\operatorname{supp}(F_3) \subset V_q \cup \{q\}$,
6. $d_{C^0}(\mathrm{Id}, \theta_3) < \varepsilon$ and $\|F_3\|_\infty < \rho$.
7. For any neighborhood of q, θ_3 coincides with a Hamiltonian diffeomorphism in the complement of that neighborhood.

End of the proof Let θ be the Hamiltonian homeomorphism $\theta = \theta_1 \circ \theta_2 \circ \theta_3$ and let α be the smooth curve which coincides with α_1 on $(-\infty, 1]$, with α_2 on $[0, k+1]$, and with α_3 on $[k, +\infty)$. The respective Properties 1–7 established for each θ_1, θ_2 and θ_3 in the previous steps, together with the fact that their supports are all disjoint, imply the corresponding Properties 1–7 in Theorem 25 whose proof in now achieved. □

Acknowledgements We would like to thank Yasha Eliashberg, Viktor Ginzburg, Helmut Hofer, Rémi Leclercq, Frédéric Le Roux, Patrice Le Calvez, Emmanuel Opshtein, Leonid Polterovich, and Claude Viterbo for fruitful discussions. LB: The research leading to this project began while I was a Professeur Invité at the Université Pierre et Marie Curie. I would like to express my deep gratitude to the members of the Institut Mathématique de Jussieu–Paris Rive Gauche, especially the team Analyse Algébrique, for their warm hospitality. SS: I would like to thank the School of Mathematics at the Institute for Advanced Study and the Department of Mathematics at MIT, where different parts of this project were carried out, for their hospitality. LB was partially supported by the Israel Science Foundation Grant 1380/13, by the Alon

Fellowship, and by the Raymond and Beverly Sackler Career Development Chair. VH was partially supported by the Agence Nationale de la Recherche, Projects ANR-11-JS01-010-01 and ANR-12-BS020-0020. SS was partially supported by the NSF Postdoctoral Fellowship Grant No. DMS-1401569.

References

1. Barannikov, S.A.: The framed Morse complex and its invariants. In: Arnold, V.I. (ed.) Singularities and Bifurcations. Transl. ed. by Sossinsky, A.B., vol. 21 of Advances in Soviet Mathematics, pp. 93–115. American Mathematical Society, Providence, RI (1994)
2. Buhovsky, L.: The 2/3-convergence rate for the Poisson bracket. Geom. Funct. Anal. **19**(6), 1620–1649 (2010)
3. Buhovsky, L., Opshtein, E.: Some quantitative results in C^0 symplectic geometry. Invent. Math. **205**(1), 1–56 (2016)
4. Buhovsky, L., Seyfaddini, S.: Uniqueness of generating Hamiltonians for topological Hamiltonian flows. J. Symplectic Geom. **11**(1), 37–52 (2013)
5. Cardin, F., Viterbo, C.: Commuting Hamiltonians and Hamilton–Jacobi multi-time equations. Duke Math. J. **144**(2), 235–284 (2008)
6. Conley, C.C., Zehnder, E.: The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnol′ d. Invent. Math. **73**(1), 33–49 (1983)
7. Eliashberg, Y.: Estimates on the Number of Fixed Points of Area Preserving Transformations. Syktyvkar University, Syktyvkar (1979). (preprint)
8. Entov, M., Polterovich, L.: C^0-rigidity of Poisson brackets. In: Symplectic Topology and Measure Preserving Dynamical Systems, vol. 512 of Contemporary Mathematics, pp. 25–32. American Mathematical Society, Providence, RI (2010)
9. Floer, A.: Proof of the Arnol′ d conjecture for surfaces and generalizations to certain Kähler manifolds. Duke Math. J. **53**(1), 1–32 (1986)
10. Floer, A.: The unregularized gradient flow of the symplectic action. Commun. Pure Appl. Math. **41**, 775–813 (1988)
11. Floer, A.: Cuplength estimates on Lagrangian intersections. Commun. Pure Appl. Math. **42**(4), 335–356 (1989)
12. Floer, A.: Symplectic fixed points and holomorphic spheres. Commun. Math. Phys. **120**(4), 575–611 (1989)
13. Fortune, B.: A symplectic fixed point theorem for $\mathbf{C}\mathrm{P}^n$. Invent. Math. **81**(1), 29–46 (1985)
14. Fortune, B., Weinstein, A.: A symplectic fixed point theorem for complex projective spaces. Bull. Am. Math. Soc. (N.S.) **12**(1), 128–130 (1985)
15. Franks, J.: Rotation vectors and fixed points of area preserving surface diffeomorphisms. Trans. Am. Math. Soc. **348**(7), 2637–2662 (1996)
16. Ginzburg, V.L.: The Conley conjecture. Ann. Math. (2) **172**(2), 1127–1180 (2010)
17. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. Invent. Math. **82**(2), 307–347 (1985)
18. Gromov, M.: Soft and hard symplectic geometry. In: Proceedings of the International Congress of Mathematicians, vols. 1, 2 (Berkeley, CA, 1986), pp. 81–98. American Mathematical Society, Providence, RI (1987)
19. Hingston, N.: Subharmonic solutions of Hamiltonian equations on tori. Ann. Math. (2) **170**(2), 529–560 (2009)
20. Hofer, H.: Lusternik–Schnirelman-theory for Lagrangian intersections. Ann. Inst. H. Poincaré Anal. Non Linéaire **5**(5), 465–499 (1988)
21. Howard, W.: Action selectors and the fixed point set of a Hamiltonian diffeomorphism. ArXiv:1211.0580

22. Humilière, V., Leclercq, R., Seyfaddini, S.: Coisotropic rigidity and C^0-symplectic geometry. Duke Math. J. **164**(4), 767–799 (2015)
23. Humilière, V., Leclercq, R., Seyfaddini, S.: New energy-capacity-type inequalities and uniqueness of continuous Hamiltonians. Comment. Math. Helv. **90**(1), 1–21 (2015)
24. Humilière, V., Leclercq, R., Seyfaddini, S.: Reduction of symplectic homeomorphisms. Ann. Sci. Éc. Norm. Supér. (4) **49**(3), 633–668 (2016)
25. Le Calvez, P.: Une version feuilletée équivariante du théorème de translation de Brouwer. Publ. Math. Inst. Hautes Études Sci. **102**, 1–98 (2005)
26. Le Calvez, P.: Periodic orbits of Hamiltonian homeomorphisms of surfaces. Duke Math. J. **133**(1), 125–184 (2006)
27. Le Peutrec, D., Nier, F., Viterbo, C.: Precise Arrhenius law for p-forms: the Witten Laplacian and Morse–Barannikov complex. Ann. Henri Poincaré **14**(3), 567–610 (2013)
28. Matsumoto, S.: Arnold conjecture for surface homeomorphisms. In: Proceedings of the French–Japanese Conference "Hyperspace Topologies and Applications" (La Bussière, 1997), vol. 104, pp. 191–214 (2000)
29. Oh, Y.-G.: Construction of Spectral Invariants of Hamiltonian Paths on Closed Symplectic Manifolds. The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 525–570. Birkhauser, Boston (2005)
30. Oh, Y.-G., Müller, S.: The group of Hamiltonian homeomorphisms and C^0-symplectic topology. J. Symplectic Geom. **5**(2), 167–219 (2007)
31. Opshtein, E.: C^0-rigidity of characteristics in symplectic geometry. Ann. Sci. Éc. Norm. Supér. (4) **42**(5), 857–864 (2009)
32. Polterovich, L., Shelukhin, E.: Autonomous Hamiltonian flows, Hofer's geometry and persistence modules. Selecta Math. (N.S.) **22**(1), 227–296 (2016)
33. Rudyak, Y.B., Oprea, J.: On the Lusternik–Schnirelmann category of symplectic manifolds and the Arnold conjecture. Math. Z. **230**(4), 673–678 (1999)
34. Schwarz, M.: On the action spectrum for closed symplectically aspherical manifolds. Pac. J. Math. **193**(2), 419–461 (2000)
35. Sikorav, J.-C.: Points fixes d'une application symplectique homologue à l'identité. J. Differ. Geom. **22**(1), 49–79 (1985)
36. Usher, M., Zhang, J.: Persistent homology and Floer–Novikov theory. ArXiv:1502.07928 (2015)
37. Viterbo, C.: Symplectic topology as the geometry of generating functions. Math. Ann. **292**, 685–710 (1992)
38. Viterbo, C.: On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows. Int. Math. Res. Not. **11**, 1–9 (2006)