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Equidistribution of minimal hypersurfaces for generic metrics

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Abstract For almost all Riemannian metrics (in the C^∞ Baire sense) on a closed manifold M^{n+1}, $3 \le (n+1) \le 7$, we prove that there is a sequence of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M. This gives a quantitative version of the main result of Irie et al. (Ann Math 187(3):963–972, 2018), that established density of minimal hypersurfaces for generic metrics. As in Irie et al. (2018), the main tool is the Weyl Law for the Volume Spectrum proven by Liokumovich et al. (Ann Math 187(3):933–961, 2018).

1 Introduction

In 1982, Yau [25] conjectured that every closed Riemannian three-manifold contains infinitely many smooth, closed, immersed minimal surfaces. In [8],

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Irie and the first two authors settled Yau's conjecture in the generic case by proving a much stronger property holds true:

Theorem (Irie, Marques, and Neves [8]) *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq (n+1) \leq 7$. Then for a C^∞-generic Riemannian metric g on M, the union of all closed, smooth, embedded minimal hypersurfaces is dense.*

In our paper, we use the methods of [8] in a more quantitative way and prove an even stronger property: there is a sequence of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M.

Main Theorem *Let M^{n+1} be a closed manifold of dimension $n + 1$, with $3 \leq (n+1) \leq 7$. Then for a C^∞-generic Riemannian metric g on M, there exists a sequence $\{\Sigma_j\}_{j \in \mathbb{N}}$ of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M: for any $f \in C^\infty(M)$ we have*

$$\lim_{q \to \infty} \frac{1}{\sum_{j=1}^{q} \text{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} f \, d\Sigma_j = \frac{1}{\text{vol}_g M} \int_M f \, dM. \qquad (1)$$

Even more, for any symmetric $(0, 2)$-tensor h on M we have:

$$\lim_{q \to \infty} \frac{1}{\sum_{j=1}^{q} \text{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} \text{Tr}_{\Sigma_j}(h) \, d\Sigma_j = \frac{1}{\text{vol}_g M} \int_M \frac{n \, \text{Tr}_M \, h}{n + 1} dM. \qquad (2)$$

Equidistribution theorems have an old history in fields like number theory, ergodic theory and harmonic analysis. Equidistribution of closed geodesics is known in some cases, like for compact hyperbolic manifolds (Bowen' 72 [2] or Margulis [10], see also [26]). Equidistribution of totally geodesic surfaces is a well-studied problem for hyperbolic 3-manifolds [3,14,15,17,19]. Our theorem is the first of its kind for the higher-dimensional setting of minimal surfaces in general manifolds.

Remark Yau's Conjecture has been fully resolved by the third author [22]. He was able to localize the methods initially developed by the first two authors in [13], and proved that any compact (M^{n+1}, g), $3 \leq (n+1) \leq 7$, contains infinitely many smooth, embedded, closed minimal hypersurfaces. It would be interesting to know whether density and equidistribution of minimal hypersurfaces hold for all Riemannian metrics.

As in the Irie–Marques–Neves paper [8], the crucial tool in our proof is the Weyl law for the Volume Spectrum conjectured by Gromov [5] and recently proven by the first two authors jointly with Liokumovich in [9]:

Weyl law for the volume spectrum (Liokumovich, Marques, and Neves [9]) *There exists a universal constant $a(n) > 0$ such that for any compact Riemannian manifold (M^{n+1}, g) we have:*

$$\lim_{p \to \infty} \omega_p(M, g) p^{-\frac{1}{n+1}} = a(n) \mathrm{vol}(M, g)^{\frac{n}{n+1}}.$$

The volume spectrum of a compact Riemannian manifold (M^{n+1}, g) is a nondecreasing sequence of numbers $\{\omega_p(M, g) : p \in \mathbb{N}\}$ defined variationally by performing a min-max procedure for the area functional over multiparameter sweepouts. The first estimates for these numbers were proven in fundamental papers by Gromov in the late 1980s [4] and by Guth [6] more recently.

Our proof also uses a transversality argument, based on the Structure Theorem of White ([23], Theorem 2.1), that allows one to compute the derivative of the p-width as the derivative of the area of some minimal hypersurface. We combine this information with appropriately chosen N-parameter deformations of the metric, for N large, that generalize the one-parameter deformations of [7,8]. A key idea in the paper (that can be be deduced from Lemma (2)) is that metrics which are critical points of the functional $g \mapsto \omega_p(M, g) p^{-\frac{1}{n+1}} - a(n) \mathrm{vol}(M, g)^{\frac{n}{n+1}}$ when restricted to the N-parameter family of deformations have minimal hypersurfaces that obey some form of equidistribution. The fact that this functional is only Lipschitz continuous and thus not differentiable everywhere is a serious technical issue that the authors had to overcome.

We note that Property (1) is equivalent to saying that

$$\frac{\sum_{j=1}^{q} \mu_{\Sigma_j}}{\sum_{j=1}^{q} \mu_{\Sigma_j}(M)} \to \frac{\mu}{\mu(M)}$$

as measures, where $\mu_{\Sigma_j} = ||\Sigma_j||$ is the Radon measure $\mu_{\Sigma_j}(U) = \mathrm{vol}_g(\Sigma_j \cap U)$, $U \subset M$, and $\mu = dv_g$ is the Riemannian volume measure of (M, g). Property (1) follows from Property (2) by choosing $h = f \cdot g$.

The dimensional restriction in the Main Theorem is due to the fact that in higher dimensions min–max (even area-minimizing) minimal hypersurfaces can have singular sets. We use Almgren–Pitts theory [1,16], which together with Schoen–Simon regularity [18] produces smooth minimal hypersurfaces when $3 \leq (n + 1) \leq 7$. We expect that the methods of this paper can be generalized to handle the higher-dimensional singular case.

The Main Theorem raises many interesting and exciting new questions: namely whether equidistribution holds in the Grassmanian bundle, whether the minimal hypersurfaces realizing the width are the ones that become equidistributed, or whether there are conditions which ensure that the sequence of

unit measures $\mu_{\Sigma_j}/\mu_{\Sigma_j}(M)$ converges to the normalized volume measure. Any progress related with these directions would be highly desirable.

2 Preliminaries

We suppose that M is a closed manifold of dimension $3 \leq (n + 1) \leq 7$. For each $2 \leq q \leq \infty$, we denote by Γ_q the space of all C^q Riemannian metrics on M, endowed with the C^q topology. Given $g \in \Gamma_q$, we let $\mathcal{V}(g)$ be the set of stationary integral varifolds in (M, g) whose support is a closed, C^2, embedded, minimal hypersurface. Hence $V \in \mathcal{V}(g)$ if and only if there exist a disjoint collection $\{\Sigma_1, \ldots, \Sigma_s\}$ of closed, C^2, embedded, connected minimal hypersurfaces in (M, g) and integers $\{m_1, \ldots, m_s\} \subset \mathbb{N}$ such that $V = m_1 \Sigma_1 + \cdots + m_s \Sigma_s$. By elliptic regularity, each Σ_i is in fact of class C^q. The support of V is denoted by $\mathrm{spt}(V)$ and is equal to $\cup_{i=1}^s \Sigma_i$, while $||V||$ denotes the Radon measure induced by V on M.

We denote by $\mathcal{Z}_n(M; \mathbb{Z}_2)$ the space of modulo two n-dimensional flat chains T in M with $T = \partial U$ for some $(n + 1)$-dimensional modulo two flat chain U in M, endowed with the flat topology. This space is weakly homotopically equivalent to $\mathbb{RP}^\infty$ (see Section 4 of [12]). We denote by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_n(M; \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$. The mass ($n$-dimensional volume) of T is denoted by $M(T)$.

Let X be a finite dimensional simplicial complex. A continuous map $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is called a *p-sweepout* if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

We say X is *p-admissible* if there exists a p-sweepout $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ that has no concentration of mass, meaning

$$\lim_{r \to 0} \sup\{M(\Phi(x) \cap B_r(p)) : x \in X, p \in M\} = 0.$$

The set of all p-sweepouts Φ that have no concentration of mass is denoted by $\mathcal{P}_p$. Note that two maps in $\mathcal{P}_p$ can have different domains.

In [13], the first two authors defined

Definition The *p-width of* (M, g) is the number

$$\omega_p(M, g) = \inf_{\Phi \in \mathcal{P}_p} \sup\{M(\Phi(x)) : x \in \mathrm{dmn}(\Phi)\},$$

where $\mathrm{dmn}(\Phi)$ is the domain of Φ.

The next lemma gives that the normalized p-width $p^{-\frac{1}{(n+1)}}\omega_p(M, g)$ is a Lipschitz function of the metric on sets of uniformly equivalent metrics, with a Lipschitz constant that does not depend on p.

Lemma 1 *Let $\tilde{g}$ be a C^2 Riemannian metric on M, and let $C_1 < C_2$ be positive constants. Then there exists $K = K(\tilde{g}, C_1, C_2) > 0$ such that*

$$|p^{-\frac{1}{(n+1)}}\omega_p(M, g) - p^{-\frac{1}{(n+1)}}\omega_p(M, g')| \le K \cdot |g - g'|_{\tilde{g}}$$

for any $g, g' \in \{h \in \Gamma_2; C_1\tilde{g} \le h \le C_2\tilde{g}\}$ and any $p \in \mathbb{N}$.

Proof It follows from the Gromov–Guth bound ([4,6], see Theorem 5.1 of [13]) that there exists $C = C(\tilde{g})$ such that $\omega_p(M, \tilde{g}) \le Cp^{\frac{1}{(n+1)}}$ for every $p \in \mathbb{N}$.

Given $g, g' \in \{h \in \Gamma_2; C_1\tilde{g} \le h \le C_2\tilde{g}\}$, one can check (see Lemma 2.1 of [8]) that

$$
\begin{aligned}
\omega_p(M, g') - \omega_p(M, g) &\le \left(\left(\sup_{v \neq 0}\frac{g'(v, v)}{g(v, v)}\right)^{\frac{n}{2}} - 1\right)\omega_p(M, g) \\
&\le \left(\left(1 + \sup_{v \neq 0}\frac{|g(v, v) - g'(v, v)|}{g(v, v)}\right)^{\frac{n}{2}} - 1\right)\omega_p(M, g) \\
&\le \left(\left(1 + C_1^{-1}|g - g'|_{\tilde{g}}\right)^{\frac{n}{2}} - 1\right)\omega_p(M, g) \\
&\le \left(\left(1 + C_1^{-1}|g - g'|_{\tilde{g}}\right)^{\frac{n}{2}} - 1\right)C_2^{\frac{n}{2}}\omega_p(M, \tilde{g}) \\
&\le \left(\left(1 + C_1^{-1}|g - g'|_{\tilde{g}}\right)^{\frac{n}{2}} - 1\right)C_2^{\frac{n}{2}}Cp^{\frac{1}{n+1}},
\end{aligned}
$$

from which the result follows. $\square$

The next lemma concerns the differentiability properties of the p-width restricted to a generic finite-dimensional family of metrics. Let $I^N = [0, 1]^N$.

Lemma 2 *Let $g : I^N \to \Gamma_q$ be a smooth embedding, $N \in \mathbb{N}$. If $q \ge N + 3$, then there exists an arbitrarily small perturbation in the C^∞ topology $g' : I^N \to \Gamma_q$ of g such that there is a subset $\mathcal{A} \subset I^N$ of full N-dimensional Lebesgue measure with the following property: for any $p \in \mathbb{N}$ and any point t of $\mathcal{A}$, the function $s \mapsto \omega_p(g'(s))$ is differentiable at t and there exists a disjoint*

collection $\{\Sigma_1, \ldots, \Sigma_Q\}$ *of closed,* C^q, *embedded, minimal hypersurfaces of* $(M, g'(t))$ *together with integers* $\{m_1, \ldots, m_Q\} \subset \mathbb{N}$ *so that*

$$\omega_p(g'(t)) = \sum_{j=1}^{Q} m_j \text{vol}_{g'(t)}(\Sigma_j), \quad \sum_{j=1}^{Q} \text{index}(\Sigma_j) \leq p,$$

and

$$\frac{\partial}{\partial v}(\omega_p \circ g')_{|s=t} = \frac{\partial}{\partial v}\left(\sum_{j=1}^{Q} m_j \text{vol}_{g'(s)}(\Sigma_j)\right)_{|s=t}$$

$$= \sum_{j=1}^{Q} m_j \int_{\Sigma_j} \frac{1}{2} \text{Tr}_{\Sigma_j, g'(t)}\left(\frac{\partial g'}{\partial v}_{|s=t}\right) d\Sigma_j$$

for every $v \in \mathbb{R}^N$.

Proof Let $g : I^N \to \Gamma_q$ be a smooth embedding. Consider a sequence $\{S_i\}_i$ that enumerates all the diffeomorphism types of n-dimensional closed manifolds, and let $\mathfrak{M}(S_i)$ be the Banach manifold of pairs $(\gamma, [u])$ as in the Structure Theorem of White [23] (Theorem 2.1), where γ is a C^q Riemannian metric and $u : S_i \to M$ is a $C^{2,\alpha}$ embedding that is minimal with respect to γ. Define $\mathfrak{M} := \bigcup_i \mathfrak{M}(S_i)$ and the projection $\Pi : \mathfrak{M} \to \Gamma_q$ which sends $(\gamma, [u])$ to γ. Theorem 2.1 of [23] (see also [24]) gives that $\mathfrak{M}$ is a separable C^{q-2} Banach manifold and that Π is a C^{q-2} Fredholm map with Fredholm index zero. The pair $(\gamma, [u])$ is a critical point of Π if and only if u admits a nontrivial Jacobi field with respect to the metric γ.

We can perturb $g : I^N \to \Gamma_q$ slightly in the C^∞ topology to a C^∞ embedding $g' : I^N \to \Gamma_q$ that is transversal to $\Pi : \mathfrak{M} \to \Gamma_q$ by Smale's Transversality Theorem (Theorem 3.1 of [21]). Transversality implies $\tilde{I}^N = \Pi^{-1}(g'(I^N))$ is an N-dimensional submanifold of $\mathfrak{M}$ (Theorem 3.3 of [21]). Let $\pi = (g')^{-1} \circ \Pi_{|\tilde{I}^N}$, so $\pi : \tilde{I}^N \to I^N$. Let $\mathcal{A}'$ be the subset of points $t' \in I^N$ which are regular values of π and such that the Lipschitz function $t \mapsto \omega_p(t) := \omega_p(M, g'(t))$ is differentiable at t' for all p. This subset is of full Lebesgue measure in I^N by Rademacher's Theorem and Sard's Theorem. Note that if $t' \in \mathcal{A}'$, then $g'(t')$ is a regular value of Π.

Consider

$$\mathcal{S}'_{\kappa,c}(p) := \Big\{ t \in I^N; \exists V \in \mathcal{V}(g'(t)), \|V\|_{g'(t)}(M) = \omega_p(t),$$

$$\text{index}(\text{spt}(V)) \leq p, \max_{\text{spt}(V)} |A| \leq \kappa,$$

$$\text{and } V \text{ satisfies } (\star)_{c,\kappa,\sup_{s \in I^N} \omega_p(s)} \Big\}$$

for any $\kappa > 0$ and $c > 0$, where

$(\star_{c,\kappa,a})$: every two-sided, connected component of spt V, has varifold distance (in the metric $g'(0)$) at least c of any varifold 2Σ, where Σ is a one-sided, embedded, connected minimal hypersurface in $g'(s)$ with $|A| \leq \kappa$ and $\mathrm{vol}_{g'(s)}(\Sigma) \leq a$ for some $s \in I^N$.

Each $\mathcal{S}'_{\kappa,c}(p)$ is a closed set, by convergence properties of minimal hypersurfaces. Proposition 2.2 of [8] (which uses the index estimates of [11]) also holds for C^q metrics if we allow the minimal hypersurfaces to be C^2. This follows, for instance, by approximating the C^q metric by C^∞ metrics, applying Proposition 2.2 of [8] to these metrics and using Sharp's Compactness Theorem [20]. It implies

$$\cup_{\kappa,c\in\mathbb{Q}_+} \mathcal{S}'_{\kappa,c}(p) = I^N$$

for every $p \in \mathbb{N}$.

For $\kappa > 0$ and $c > 0$, define $\mathcal{S}_{\kappa,c}(p)$ to be the set of points where the Lebesgue density of $\mathcal{S}'_{\kappa,c}(p)$ is one. By the Lebesgue density theorem, $\mathcal{S}'_{\kappa,c}(p)\backslash\mathcal{S}_{\kappa,c}(p)$ has measure zero. Finally, define the full Lebesgue measure set

$$\mathcal{A} := \mathcal{A}' \cap \bigcap_p \bigcup_{\kappa,c} \mathcal{S}_{\kappa,c}(p).$$

Fix $p \in \mathbb{N}$, and let $t \in \mathcal{A}$. There exist $\kappa > 0$ and $c > 0$ such that $t \in \mathcal{S}_{\kappa,c}(p)$. Since the Lebesgue density of $\mathcal{S}'_{\kappa,c}(p)$ at t is one, we have that for any unit direction v, there is a sequence $\{t_m(v)\}_m \subset \mathcal{S}'_{\kappa,c}(p)$ converging to t with $\frac{t_m(v)-t}{|t_m(v)-t|}$ converging to v, so that

$$\lim_{m\to\infty} \frac{\omega_p(t_m(v)) - \omega_p(t)}{|t_m(v) - t|} = \frac{\partial}{\partial v}\omega_p(t). \tag{3}$$

Fix v and a corresponding sequence $\{t_m(v)\}_m$. By construction, for each m there is a $V_m \in \mathcal{V}(g'(t_m(v)))$ with mass $\omega_p(t_m(v))$, with $\mathrm{index}(\mathrm{spt}(V_m)) \leq p$, whose support has second fundamental form bounded by κ (which is independent of m) and such that every two-sided, connected component of spt V_m has varifold distance (in the metric $g'(0)$) at least c (also independent of m) from any varifold 2Σ, where Σ is a one-sided, connected minimal hypersurface in $g'(s)$ with $|A| \leq \kappa$ and $\mathrm{vol}_{g'(s)}(\Sigma) \leq \sup_{s\in I^N} \omega_p(s)$ for some $s \in I^N$. This implies no two-sided component of spt V_m can collapse, after maybe passing to a subsequence, to a one-sided component with multiplicity two. Choosing a subsequence and renumbering if necessary, V_m converges to a varifold

$V \in \mathcal{V}(g'(t))$ and the supports $\mathrm{spt}(V_m)$ converge in C^2 to $\mathrm{spt}(V)$. This convergence is with multiplicity one, because if not one could construct by a standard argument a nontrivial Jacobi field on one of the components of $\mathrm{spt}(V)$. This is not possible, since $g'(t)$ is a regular value of Π.

Consider a sequence $\{\Sigma_m\}$ of connected components of $\mathrm{spt}(V_m)$ that converges in C^2 to Σ. By elliptic regularity, the convergence is also in $C^{2,\alpha}$. The corresponding points

$$\tilde{z}_m = (g'(t_m(v)), [\Sigma_m]) \in \tilde{I}^N \subset \mathfrak{M}$$

converge to a point $z \in \Pi^{-1}(g'(t)) \subset \tilde{I}^N$, $z = (g'(t), [\Sigma])$. Note that since $t \in \mathcal{A}$, π is a local diffeomorphism from a neighborhood of z in $\tilde{I}^N$ to a neighborhood of t in I^N. We write $\tilde{z} = (g'(\pi(\tilde{z})), [\Sigma(\pi(\tilde{z}))])$ for any $\tilde{z}$ in this neighborhood of z. For sufficiently large m, $[\Sigma_m] = [\Sigma(t_m(v))]$. But then

$$\lim_{m \to \infty} \frac{\mathrm{vol}_{g'(t_m(v))}(\Sigma(t_m(v))) - \mathrm{vol}_{g'(t)}(\Sigma(t))}{|t_m(v) - t|} = \frac{\partial}{\partial v} \mathrm{vol}_{g'(s)}(\Sigma(s))_{|s=t}$$
$$= \frac{1}{2} \int_\Sigma \mathrm{Tr}_{\Sigma, g'(t)} \left(\frac{\partial g'}{\partial v}(t) \right) d\Sigma.$$

Taking into account the multiplicity of each connected component of $\mathrm{spt}(V_m)$, the limit in (3) becomes

$$\frac{\partial}{\partial v} \omega_p(t) = \int_V \frac{1}{2} \mathrm{Tr}_{V, g'(t)} \left(\frac{\partial g'}{\partial v}_{|s=t} \right) d\|V\|(M),$$

where V is of the form $\sum_{i=1}^Q m_i \tilde{\Sigma}_i$, with $\{\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_Q\}$ a disjoint collection of closed, $C^{2,\alpha}$, embedded, minimal hypersurfaces in $(M, g'(t))$ and $\{m_1, \ldots, m_Q\} \subset \mathbb{N}$, $\|V\|(M) = \omega_p(t)$, $\sum_{i=1}^Q \mathrm{index}(\tilde{\Sigma}_i) \leq p$, $\max_{\mathrm{spt}(V)} |A| \leq \kappa$ and V satisfies $(\star_{\kappa,c,\sup_{s \in I^N} \omega_p(s)})$. By elliptic regularity, each $\tilde{\Sigma}_i$ is of class C^q. Since t is a regular value of π, every embedded minimal hypersurface of $(M, g'(t))$ is non-degenerate. Because convergence of the supports can only happen with multiplicity one, there are only finitely many V's as above, say $\{V^{(1)}, \ldots, V^{(P)}\}$. For any unit direction $v \in \mathbb{R}^N$, one has

$$\frac{\partial}{\partial v} \omega_p(t) = \int_{V^{(l)}} \frac{1}{2} \mathrm{Tr}_{V^{(l)}, g'(t)} \left(\frac{\partial g'}{\partial v}_{|s=t} \right) d\|V^{(l)}\|(M)$$

for some $1 \leq l \leq P$. This means that there will be a single $1 \leq l \leq P$ such that the above formula is true for a linearly independent set $\{v_1, \ldots, v_N\}$, and hence for every v by linearity. This finishes the proof. $\square$

The next lemma concerns the gradient of Lipschitz functions that are almost constant. The convex hull of a set $K \subset \mathbb{R}^N$ is denoted by $\mathrm{Conv}(K)$.

Lemma 3 *Given $\delta > 0$ and $N \in \mathbb{N}$, there exists $\varepsilon > 0$ depending on δ and N such that the following is true: for any Lipschitz function $f : I^N \to \mathbb{R}$ satisfying*

$$|f(x) - f(y)| \leq 2\varepsilon$$

for every $x, y \in I^N$, and for any subset $\mathcal{A}$ of I^N of full measure, there exist $N+1$ sequences of points $\{y_{1,m}\}_m, \ldots, \{y_{N+1,m}\}_m$ contained in $\mathcal{A}$ and converging to a common limit $y \in (0, 1)^N$ such that:

- *f is differentiable at each $y_{i,m}$,*
- *the gradients $\nabla f(y_{i,m})$ converge to $N + 1$ vectors $v_1, \ldots, v_{N+1}$ with*

$$d_{\mathbb{R}^N}(0, \mathrm{Conv}(v_1, \ldots, v_{N+1})) < \delta,$$

Proof Suppose, by contradiction, that the lemma is false. Then there exists a sequence of Lipschitz functions $f_k : I^N \to \mathbb{R}$ satisfying

$$|f_k(x) - f_k(y)| \leq 1/k$$

for every $x, y \in I^N$, and a sequence of sets $\mathcal{A}_k \subset I^N$ of full measure, such that these sequences of points do not exist. Since f_k is Lipschitz, the set $\mathcal{D}_k \subset I^N$ of points where f_k is differentiable has full measure by Rademacher's Theorem. Hence the set $\mathcal{A}'_k = \mathcal{A}_k \cap \mathcal{D}_k$ has full measure also.

Choose a smooth function $g : I^N \to \mathbb{R}$ such that g is equal to 1 on the boundary of I^N and equal to 0 at $(1/2, \ldots, 1/2) \in I^N$. Then the Lipschitz function $h_k = f_k - \frac{2}{k}g$ achieves its maximum at an interior point $y_k \in (0, 1)^N$. Consider the set $V_k \subset \mathbb{R}^N$ of vectors v such that there exists a sequence $z_m \in \mathcal{A}'_k$ with $z_m \to y_k$ and $\nabla h_k(z_m) \to v$ as $m \to \infty$. The set V_k is bounded and closed. For almost all directions w in the unit sphere S^{N-1}, the set

$$\{t \in [0, d_{\mathbb{R}^N}(y_k, \partial I^N)] : y_k + tw \in \mathcal{A}'_k\}$$

has full measure in $[0, d_{\mathbb{R}^N}(y_k, \partial I^N)]$. For any such w, because h_k has a maximum point at y_k, there exists $v \in V_k$ with $\langle v, w \rangle \leq 0$. This implies that for any $w \in \mathbb{R}^N$, there exists $v \in V_k$ with $\langle v, w \rangle \leq 0$. By the Hahn-Banach Theorem, $0 \in \mathrm{Conv}(V_k)$. Caratheodory's Theorem gives vectors $\{\tilde{v}_1, \ldots, \tilde{v}_{N+1}\} \subset V_k$ such that $0 \in \mathrm{Conv}(\{\tilde{v}_1, \ldots, \tilde{v}_{N+1}\})$. Hence there exist $N + 1$ sequences of points $\{y_{1,m}^{(k)}\}_m, \ldots, \{y_{N+1,m}^{(k)}\}_m$ contained in $\mathcal{A}'_k$ and converging to $y_k \in (0, 1)^N$ such that:

- f_k is differentiable at each $y_{i,m}^{(k)}$,
- the gradients $\nabla f_k(y_{i,m}^{(k)})$ converge to $N+1$ vectors $v_1, \ldots, v_{N+1}$ with

$$d_{\mathbb{R}^N}(0, \text{Conv}(v_1, \ldots, v_{N+1})) \leq \frac{2}{k} \sup_{I^N} |\nabla g|.$$

If k is sufficiently large, $\frac{2}{k} \sup_{I^N} |\nabla g| < \delta$. Contradiction. □

The last lemma shows that one can make finitely many closed, embedded, minimal hypersurfaces nondegenerate by an arbitrarily small conformal change of the metric. This generalizes Proposition 2.3 of [8].

Lemma 4 *Suppose $g \in \Gamma_q$, $q \geq 2$. Let $\{\Sigma_1, \ldots, \Sigma_L\}$ be a finite collection of closed, embedded, connected, C^2 minimal hypersurfaces in (M, g). Then there exists a sequence of metrics $g_i \in \Gamma_q$, $i \in \mathbb{N}$, converging to g in the C^q topology such that Σ_j is a nondegenerate minimal hypersurface in (M, g_i) for all $j = 1, \ldots, L$ and $i \in \mathbb{N}$.*

Proof Each Σ_i is C^q by elliptic regularity. We can suppose $\Sigma_j \neq \Sigma_k$ when $j \neq k$. Choose $\delta > 0$ such that $B_r(q) \cap \Sigma_j$ is connected for every $j = 1, \ldots, L$, $0 < r \leq \delta$ and $q \in \Sigma_j$. We claim that there exists a point $p \in \Sigma_1 \setminus (\Sigma_2 \cup \cdots \cup \Sigma_L)$.

Pick $x_1 \in \Sigma_1$ arbitrary. If $x_1 \notin \Sigma_2$, set $x_2 = x_1$. Suppose $x_1 \in \Sigma_2$. If $B_\delta(x_1) \cap \Sigma_1 \subset \Sigma_2$, then $\Sigma_1 = \Sigma_2$ by unique continuation. This is not possible, hence there exists $x_2 \in B_\delta(x_1) \cap \Sigma_1$ but $x_2 \notin \Sigma_2$. In any case, we have found $x_2 \in \Sigma_1 \setminus \Sigma_2$. Suppose we have $x_j \in \Sigma_1 \setminus (\Sigma_2 \cup \cdots \cup \Sigma_j)$, $2 \leq j \leq L-1$. If $x_j \notin \Sigma_{j+1}$, set $x_{j+1} = x_j$. Assume $x_j \in \Sigma_{j+1}$, and define $\delta_j = \min\{\delta, \frac{1}{2}d(x_j, \Sigma_2 \cup \cdots \cup \Sigma_j)\} > 0$. If $B_{\delta_j}(x_j) \cap \Sigma_1 \subset \Sigma_{j+1}$, then $\Sigma_1 = \Sigma_{j+1}$ by unique continuation. This is impossible, hence there exists $x_{j+1} \in B_{\delta_j}(x_j) \cap \Sigma_1$ but $x_{j+1} \notin \Sigma_{j+1}$. In any case, we have found $x_{j+1} \in \Sigma_1 \setminus (\Sigma_2 \cup \cdots \cup \Sigma_{j+1})$. By induction, we find $x_L \in \Sigma_1 \setminus (\Sigma_2 \cup \cdots \cup \Sigma_L)$.

For similar reasons, there exist $p_l \in \Sigma_l \setminus (\cup_{k \neq l} \Sigma_k)$ for every $l = 1, \ldots, L$. Choose $\eta > 0$ sufficiently small so that η is smaller than the injectivity radius of the manifold and such that $\eta < \frac{1}{4} d_g(p_l, \cup_{k \neq l} \Sigma_k)$ for every l. By decreasing η if necessary, we can choose for each $l = 1, \ldots, L$, a C^q function $f_l : B_\eta(p_l) \to \mathbb{R}$ such that $f_l = 0$ and $\langle \nabla f_l, N_l \rangle > 0$ on $\Sigma_l \cap B_\eta(p_l)$, where N_l is a local choice of unit normal to Σ_l. We also choose, for each $l = 1, \ldots, L$, a smooth nonnegative function $\varphi_l : M \to \mathbb{R}$ such that $\varphi_l = 1$ on $B_{\eta/2}(p_l)$ and $\varphi_l = 0$ outside $B_{2\eta/3}(p_l)$.

Let $g_i = \exp(2\phi_i)g$, where $\phi_i = -\frac{1}{i}(\varphi_1 f_1^2 + \cdots + \varphi_L f_L^2)$. By the arguments of [8, Proposition 2.3], one can check that at any point y on Σ_l, $\phi_i = 0$, $\nabla \phi_i = 0$ and $\text{Hess}_g \phi_i(N, N) = -\frac{2}{i} \varphi_l(y) \langle \nabla f_l, N \rangle^2(y)$, where N is a unit normal to Σ_l at y with respect to the metric g (or g_i). This implies Σ_l remains minimal with respect to g_i for every l, and at points of Σ_l we have:

$$\text{Ric}_{g_i}(N, N) + |A_{\Sigma_l, g_i}|^2_{g_i} = \text{Ric}_g(N, N) + |A_{\Sigma_l, g}|^2_g + \frac{2n}{i}\varphi_l\langle\nabla f_l, N\rangle^2,$$

where $|A_{\Sigma_l, g}|$ is the norm of the second fundamental form of Σ_l with respect to g.

The Jacobi operator of Σ_l acting on normal vector fields is given by

$$L_{\Sigma_l, g}(X) = \Delta^{\perp}_{\Sigma_l, g}X + (\text{Ric}_g(N, N) + |A_{\Sigma_l, g}|^2_g)X.$$

Since g_i and g coincide on Σ, $\Delta^{\perp}_{\Sigma_l, g} = \Delta^{\perp}_{\Sigma_l, g_i}$ and hence

$$L_{\Sigma_l, g_i}(X) = L_{\Sigma_l, g}(X) + \frac{2n}{i}\varphi_l\langle\nabla f_l, N\rangle^2 X. \tag{4}$$

Fix l, and define $\tilde{L}_t(X) = L_{\Sigma_l, g}(X) + t\varphi_l\langle\nabla f_l, N\rangle^2 X$ on Σ_l, for $t \in \mathbb{R}$. It is known that the eigenvalues of $\tilde{L}_t$ depend continuously on the parameter t. Suppose that Σ_l is a degenerate minimal hypersurface in (M, g), and let Q be the unique integer such that $0 = \lambda_Q(\tilde{L}_0) < \lambda_{Q+1}(\tilde{L}_0)$. If t is sufficiently small, then $\lambda_{Q+1}(\tilde{L}_t) > 0$.

Let X be in the zero eigenspace E of $\tilde{L}_0$, $X \neq 0$. Then

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} &\left(-\frac{\int_{\Sigma_l}\langle\tilde{L}_t(X), X\rangle}{\int_{\Sigma_l}|X|^2}\right) \\
&= \frac{d}{dt}\bigg|_{t=0}\left(\frac{-\int_{\Sigma_l}\langle\tilde{L}_0(X), X\rangle - t\int_{\Sigma_l}\varphi_l\langle\nabla f_l, N\rangle^2|X|^2}{\int_{\Sigma_l}|X|^2}\right) \\
&= -\frac{\int_{\Sigma_l}\varphi_l\langle\nabla f_l, N\rangle^2|X|^2}{\int_{\Sigma_l}|X|^2} \\
&\leq -\frac{\int_{B_{\eta/2}(p_l)\cap\Sigma_l}\langle\nabla f_l, N\rangle^2|X|^2}{\int_{\Sigma_l}|X|^2}.
\end{aligned}$$

By unique continuation of solutions of linear elliptic equations and the finite-dimensionality of E, we can find a constant $c > 0$ such that

$$\frac{d}{dt}\bigg|_{t=0}\left(-\frac{\int_{\Sigma_l}\langle\tilde{L}_t(X), X\rangle}{\int_{\Sigma_l}|X|^2}\right) \leq -c \tag{5}$$

for every $X \in E \setminus \{0\}$.

Recall the min-max characterization of the eigenvalue $\lambda_Q(\tilde{L}_t)$:

$$\lambda_Q(\tilde{L}_t) = \inf_W \max_{X \in W \setminus \{0\}} \frac{-\int_{\Sigma_l} \langle \tilde{L}_t(X), X \rangle}{\int_{\Sigma_l} |X|^2}, \tag{6}$$

where the infimum is taken over all the Q-dimensional subspaces W of the space of smooth, normal vector fields on Σ_l. If $\tilde{W}$ is the subspace spanned by the eigensections of $\tilde{L}_0$ corresponding to eigenvalues $\lambda \leq 0$, then $\dim(W) = Q$. By combining (5) and (6), we have

$$\lambda_Q(\tilde{L}_t) \leq \max_{X \in \tilde{W} \setminus \{0\}} \frac{-\int_{\Sigma_l} \langle \tilde{L}_t(X), X \rangle}{\int_{\Sigma_l} |X|^2} \leq -\frac{c}{2} t$$

for sufficiently small $t \geq 0$. Therefore for sufficiently large i we have both $\lambda_Q(L_{\Sigma_l, g_i}) < 0$ and $\lambda_{Q+1}(L_{\Sigma_l, g_i}) > 0$. This implies Σ_l is nondegenerate with respect to (M, g_i) for sufficiently large i. Since this is true for every $l = 1, \ldots, L$, the Lemma is proved. $\qquad\square$

3 Proof of the main theorem

Let g be a smooth Riemannian metric on M, K be an integer and $\epsilon_1 > 0$ be a positive constant smaller than the injectivity radius of g. Let $\hat{B}_1, \ldots, \hat{B}_K$ be disjoint domains in M, with piecewise smooth boundary, such that the union of their closures covers M.

Let B_k be some neighborhood of $\hat{B}_k$. We suppose that each B_k is contained in a geodesic ball of radius ϵ_1. Choose also a smooth function $0 \leq \phi_k \leq 1$ that is equal to 1 on $\hat{B}_k$ and with $\mathrm{spt}(\phi_k) \subset B_k$, and a point $q_k \in \hat{B}_k$ for each k. We can also suppose that $q_k \notin B_l$ if $l \neq k$. Define the partition of unity $\psi_k = \frac{\phi_k}{\sum_q \phi_q}$. Hence $\psi_k(q_k) = 1$ and $\psi_k(q_l) = 0$ for $l \neq k$.

For a fixed k, let e be a unit vector in the tangent space of M at q_k. It determines by parallel transport along geodesics starting at q_k a unit vector field in B_k still denoted by e. We define a nonnegative symmetric $(0, 2)$-tensor $h(e)$ on B_k as follows: $h(e)(v, w) = \langle v, e \rangle_g \langle w, e \rangle_g$.

Now consider the space $\mathcal{B}_k$ of orthonormal bases at q_k; these $\mathcal{B}_k$ are endowed with a natural metric determined by g and of course are isometric to each other. For each k, pick L points $x_1^k, \ldots, x_L^k \in \mathcal{B}_k$ such that any point in $\mathcal{B}_k$ is at distance less than ϵ_1 to one of the x_l^k. Each x_l^k is an orthonormal basis $(x_{l,1}^k, \ldots, x_{l,n+1}^k)$ at q_k and so we can consider the family of symmetric $(0, 2)$-tensors $h_{l,j}^k = h(x_{l,j}^k)$. Note that by construction, in B_k, for any l the sum $\sum_{j=1}^{n+1} h_{l,j}^k$ is the metric g.

We denote by $\mathcal{C}_{g,\tilde{K},\epsilon_1}$ the set of all possible choices

$$(K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\})$$

as above, with $K \geq \tilde{K}$. The set $\mathcal{C}_{g,\tilde{K},\epsilon_1}$ is non-empty, as can be seen by taking a sufficiently fine triangulation of M.

Recall that $\mathcal{V}(g)$ denotes the set of stationary integral varifolds in (M, g) whose support is an embedded minimal hypersurface. We claim that in order to show the main theorem, it suffices to prove the following property.

(**P**): for any metric g, for every $\epsilon_1 > 0$, $\tilde{K} > 0$ and any choice of

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g,\tilde{K},\epsilon_1},$$

there is a metric $\tilde{g}$ arbitrarily close to g in the C^∞ topology such that there are varifolds $V_1, \ldots, V_J$ of $\mathcal{V}(\tilde{g})$ whose support $\mathrm{spt}(V_j)$ are nondegenerate, and coefficients $\alpha_1, \ldots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$ satisfying

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{\|V_i\|(M)} - \frac{1}{(n+1)} \frac{1}{\mathrm{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < \epsilon_1/K, \quad (7)$$

where the terms of the sum are computed for the metric $\tilde{g}$. Here

$$V(h) = \int_{G_n(M)} h(\nu, \nu) dV(p, \pi),$$

where $G_n(M)$ denotes the Grassmannian of n-dimensional planes of M and ν is a unit normal to the n-plane $\pi \subset T_p M$.

Indeed, let us explain why Property (**P**) implies the main theorem. We denote by $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$, with

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g,\tilde{K},\epsilon_1},$$

the family of metrics $\tilde{g} \in \Gamma_\infty$ at distance less than ϵ_1/K to g (computed with respect to g) in the C^K topology such that there are $\{V_1, \ldots, V_J\} \subset \mathcal{V}(\tilde{g})$ whose supports are nondegenerate, $\alpha_1, \ldots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$, which satisfy (7) for all k, l, j. If $g' \in \mathcal{M}(g, \epsilon_1, \tilde{K}, S)$, and $\{\Sigma_1', \ldots, \Sigma_Q'\}$ is any finite collection of nondegenerate minimal hypersurfaces in (M, g'), then for every metric $\tilde{g}$ that is sufficiently close to g', there is a unique collection $\{\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_Q\}$ of nondegenerate minimal hypersurfaces in $(M, \tilde{g})$ such that $\tilde{\Sigma}_i$ is close to Σ_i', $i = 1, \ldots, Q$. Moreover, $\tilde{\Sigma}_i$ converges smoothly to Σ_i' as $\tilde{g}$ converges to g'. This implies that $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$ is open in the C^∞ topology.

Define

$$\mathcal{M}(\epsilon_1, \tilde{K}) := \bigcup_{g \in \Gamma_\infty} \bigcup_{S \in \mathcal{C}_{g, \tilde{K}, \epsilon_1}} \mathcal{M}(g, \epsilon_1, \tilde{K}, S).$$

It is clearly open. Given an arbitrary metric $g \in \Gamma_\infty$, we can choose $S \in \mathcal{C}_{g, \tilde{K}, \epsilon_1}$. Property (**P**) implies that the metric g is a limit of metrics in $\mathcal{M}(g, \epsilon_1, \tilde{K}, S)$. This shows that $\mathcal{M}(\epsilon_1, \tilde{K})$ is also dense.

Define

$$\mathcal{M} := \bigcap_{m \in \mathbb{N}} \mathcal{M}(1/m, m).$$

Since each $\mathcal{M}(1/m, m)$ is open and dense, the intersection $\mathcal{M}$ is a residual subset (in the Baire sense) of the set of metrics. We will show that for any metric in $\mathcal{M}$, one can find sequences of minimal hypersurfaces like in the Main Theorem. For any metric, a symmetric $(0, 2)$-tensor h is diagonalizable at every point. The idea is to find a fine subdivision of M in domains B_k where h is approximately diagonal when expressed in the basis $x_{l(k)}^k$ for a certain $l(k) \in \{1, \ldots, L\}$.

Let $\tilde{g} \in \mathcal{M}$. Then $\tilde{g} \in \mathcal{M}(1/m, m)$ for every $m \in \mathbb{N}$. Fix m. Then by construction there exists a metric g such that $\tilde{g} \in \mathcal{M}(g, 1/m, m, S)$ for some choice of

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g, m, 1/m}.$$

In particular, g belongs to a $1/(mK)$-neighborhood of $\tilde{g}$ in the C^K topology. We also have $\{V_1, \ldots, V_J\} \subset \mathcal{V}(\tilde{g})$, $\alpha_1, \ldots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$, which satisfy

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < 1/(mK). \quad (8)$$

Note that $g, S, J, \{V_j\}, \{\alpha_j\}$ all depend on m.

Let h be a symmetric $(0, 2)$-tensor on M. The following computations are done with respect to the metric $\tilde{g}$, unless otherwise specified. We start by writing

$$\int_M \text{Tr}(h) = \sum_k \int_{B_k} \psi_k \text{Tr}(h),$$

and

$$\sum_i \alpha_i \frac{V_i(h)}{||V_i||(M)} = \sum_k \sum_i \alpha_i \frac{V_i(\psi_k h)}{||V_i||(M)}.$$

At each $q_k \in \hat{B}_k$, h is diagonalizable for the metric g in a g-orthonormal basis

$$u^k = (u_1^k, \ldots, u_{n+1}^k) \in \mathcal{B}_k$$

with eigenvalues $\lambda_1(k), \ldots, \lambda_{n+1}(k)$ and we note that $\sum_j \lambda_j(k)$ is the trace of h at q_k for the metric g. Let $l(k)$ be such that $x_{l(k)}^k$ is at distance less than $1/m$ from u^k in $\mathcal{B}_k$. We get on B_k (which are contained in balls of radius $1/m$) the following estimates with the metric $\tilde{g}$:

$$\left| h - \sum_{j=1}^{n+1} \lambda_j(k) h(u_j^k) \right|_{\tilde{g}} < \frac{C}{m}, \tag{9}$$

$$\left| \sum_{j=1}^{n+1} \lambda_j(k) h(u_j^k) - \sum_{j=1}^{n+1} \lambda_j(k) h_{l(k),j}^k \right|_{\tilde{g}} < \frac{C}{m}. \tag{10}$$

Here C depends only on $\tilde{g}$ and h, and might be different from line to line.

We have, by (8), that

$$\forall k \quad \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M \left(\sum_j \lambda_j(k) \psi_k \right) \right|$$
$$< C/(mK).$$

Hence

$$\sum_k \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M \left(\sum_j \lambda_j(k) \psi_k \right) \right|$$
$$< C/m,$$

and since $|\operatorname{Tr}_g h - \operatorname{Tr}_{\tilde{g}} h| < C/m$, we obtain readily

$$\sum_k \left| \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\text{vol}(M)} \int_M \psi_k \operatorname{Tr} h \right| < C/m.$$

Therefore

$$\left| \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{||V_i||(M)} - \frac{\int_M \operatorname{Tr} h}{(n+1)\text{vol}(M)} \right| < C/m.$$

But we also have by (9) and (10) that

$$\left|\sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h_{l(k),j}^k)}{||V_i||(M)} - \sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h(u_j^k))}{||V_i||(M)}\right| < C/m,$$

and

$$\left|\sum_k \sum_i \sum_j \alpha_i \frac{V_i(\psi_k \lambda_j(k) h(u_j^k))}{||V_i||(M)} - \sum_i \alpha_i \frac{V_i(h)}{||V_i||(M)}\right| < C/m,$$

so we conclude

$$\left|\sum_i \alpha_i \frac{V_i(h)}{||V_i||(M)} - \frac{\int_M \mathrm{Tr}\, h}{(n+1)\mathrm{vol}(M)}\right| < C/m.$$

In the paragraph that follows, all the integrals, traces and varifold values are computed with $\tilde{g}$. Each $V_i = V_{m,i}$, $i = 1, \ldots, J_m = J$, is of the form

$$V_i = \sum_{q=1}^{R_{m,i}} \Sigma_{m,i,q},$$

with $R_{m,i} \in \mathbb{N}$, $\Sigma_{m,i,q}$ a connected, closed, smooth, embedded, minimal hypersurface of $(M, \tilde{g})$. Choose integers $c_{m,i}, d_m \in \mathbb{N}$ such that $\alpha_i = \alpha_{m,i}$ satisfies

$$\left|\frac{\alpha_{m,i}}{||V_{m,i}||(M)} - \frac{c_{m,i}}{d_m}\right| < \frac{1}{m J_m ||V_{m,i}||(M)}.$$

In particular, $|1 - \sum_{i=1}^{J_m} \frac{c_{m,i}||V_{m,i}||(M)}{d_m}| < 1/m$ and

$$\left|\sum_i \frac{c_{m,i}}{d_m} V_{m,i}(h) - \frac{\int_M \mathrm{Tr}\, h}{(n+1)\mathrm{vol}(M)}\right| < C/m.$$

Hence

$$\lim_{m \to \infty} \frac{\sum_{i=1}^{J_m} c_{m,i} V_{m,i}(h)}{\sum_{i=1}^{J_m} c_{m,i}||V_{m,i}||(M)} = \frac{\int_M \mathrm{Tr}\, h}{(n+1)\mathrm{vol}(M)} \tag{11}$$

for any symmetric $(0, 2)$-tensor h. If we choose $h = f \cdot \tilde{g}$, with $f \in C^\infty(M)$, we get

$$\lim_{m \to \infty} \frac{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \int_{\Sigma_{m,i,q}} f}{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \mathrm{vol}(\Sigma_{m,i,q})} = \frac{\int_M f}{\mathrm{vol}(M)}. \tag{12}$$

Because

$$V_{m,i}(h) = \sum_{q=1}^{R_{m,i}} \int_{\Sigma_{m,i,q}} h(v,v) = \sum_{q=1}^{R_{m,i}} \int_{\Sigma_{m,i,q}} (\text{Tr}\, h - \text{Tr}_{\Sigma_{m,i,q}}\, h),$$

we can combine (11) with (12) to conclude

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i} \int_{\Sigma_{m,i,q}} \text{Tr}_{\Sigma_{m,i,q}}\, h}{\sum_{i=1}^{J_m} \sum_{q=1}^{R_{m,i}} c_{m,i}\,\text{vol}(\Sigma_{m,i,q})} = \frac{n \int_M \text{Tr}\, h}{(n+1)\text{vol}(M)}.$$

In other words, we just proved that, assuming Property (**P**), one can find for a generic metric a sequence of finite lists of closed, embedded, connected minimal hypersurfaces $\{\Sigma_{N,1}, \ldots, \Sigma_{N,P_N}\}_{N\in\mathbb{N}}$ such that the following is true: if we denote $\int_{\Sigma_{N,i}} \text{Tr}_{\Sigma_{N,i}}(h)\, d\Sigma_{N,i}$ (resp. $\text{vol}(\Sigma_{N,i})$) by $X_{N,i}$ (resp. $\bar{X}_{N,i}$), then

$$\left| \frac{\sum_{i=1}^{P_N} X_{N,i}}{\sum_{i=1}^{P_N} \bar{X}_{N,i}} - \alpha \right| \leq \varepsilon_N, \tag{13}$$

where $\alpha = \frac{1}{\text{vol}(M)} \int_M \frac{n\,\text{Tr}_M\, h}{n+1}\, dM$ and $\lim_{N\to\infty} \varepsilon_N = 0$. From the numbers $X_{N,i}$, $\bar{X}_{N,i}$, we want to construct two sequences $\{Y_j\}_{j\in\mathbb{N}}$ and $\{\bar{Y}_j\}_{j\in\mathbb{N}}$ such that

- for all j, there exist integers $N(j)$, $i(j)$ (chosen independently of h) with $Y_j = X_{N(j),i(j)}$ and $\bar{Y}_j = \bar{X}_{N(j),i(j)}$,
- moreover

$$\lim_{q\to\infty} \frac{\sum_{j=1}^q Y_j}{\sum_{j=1}^q \bar{Y}_j} = \alpha.$$

Note first that all the $\bar{X}_{N,i}$ are bounded below by a uniform positive constant v, according to the monotonicity formula, and that $|X_{N,i}| \leq C(h)\bar{X}_{N,i}$ where $C(h)$ is the maximum value that the absolute value of the trace of h can take over the Grassmannian $G_n(M)$.

Let $\{Q_N\}_{N\in\mathbb{N}}$ be a sequence of positive integers that will be chosen in the following order: Q_1 is chosen depending on $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$, Q_2 is chosen depending on Q_1, $\{\Sigma_{1,i}\}$, $\{\Sigma_{2,i}\}$, $\{\Sigma_{3,i}\}$, and similarly Q_{N_0} is chosen depending on $\{Q_1, \ldots, Q_{N_0-1}\}$, $\{\Sigma_{1,i}\}$, $\{\Sigma_{2,i}\}$, $\ldots \{\Sigma_{N_0+1,i}\}$.

If $1 \leq j \leq Q_1 P_1$, write $j = kP_1 + l$ where $k \in \{0, \ldots, Q_1 - 1\}$ and $l \in \{1, \ldots, P_1\}$. Then define $Y_j = X_{1,l}$ and $\bar{Y}_j = \bar{X}_{1,l}$ accordingly. Notice that

$$\left| \frac{\sum_{j=1}^{kP_1+l} Y_j}{\sum_{j=1}^{kP_1+l} \overline{Y}_j} - \alpha \right|$$

$$\leq \left| \frac{k(\sum_{i=1}^{P_1} X_{1,i} - \alpha \sum_{i=1}^{P_1} \overline{X}_{1,i}) + (\sum_{i=1}^{l} X_{1,i} - \alpha \sum_{i=1}^{l} \overline{X}_{1,i})}{k \sum_{i=1}^{P_1} \overline{X}_{1,i} + \sum_{i=1}^{l} \overline{X}_{1,i}} \right|$$

$$\leq \varepsilon_1 + C(h) + |\alpha|,$$

while

$$\left| \frac{\sum_{j=1}^{Q_1 P_1} Y_j}{\sum_{j=1}^{Q_1 P_1} \overline{Y}_j} - \alpha \right| \leq \varepsilon_1.$$

If $Q_1 P_1 + 1 \leq j \leq Q_1 P_1 + Q_2 P_2$, we write $j = Q_1 P_1 + kP_2 + l$ where $k \in \{0, \dots, Q_2-1\}$ and $l \in \{1, \dots, P_2\}$. Then define $Y_j = X_{2,l}$ and $\bar{Y}_j = \bar{X}_{2,l}$ accordingly. Now

$$\left| \frac{\sum_{j=1}^{Q_1 P_1+kP_2+l} Y_j}{\sum_{j=1}^{Q_1 P_1+kP_2+l} \overline{Y}_j} - \alpha \right|$$

$$\leq \left| \left(Q_1 \left(\sum_{i=1}^{P_1} X_{1,i} - \alpha \sum_{i=1}^{P_1} \overline{X}_{1,i} \right) + k \left(\sum_{i=1}^{P_2} X_{2,i} - \alpha \sum_{i=1}^{P_2} \overline{X}_{2,i} \right) \right. \right.$$

$$\left. \left. + \left(\sum_{i=1}^{l} X_{2,i} - \alpha \sum_{i=1}^{l} \overline{X}_{2,i} \right) \right) \right|$$

$$\cdot \frac{1}{\left(Q_1 \sum_{i=1}^{P_1} \overline{X}_{1,i} + k \sum_{i=1}^{P_2} \overline{X}_{2,i} + \sum_{i=1}^{l} \overline{X}_{2,i} \right)}$$

$$\leq \varepsilon_1 + \varepsilon_2 + \frac{C(h) + |\alpha|}{Q_1 P_1 v} \sum_{i=1}^{P_2} \overline{X}_{2,i}$$

$$\leq \varepsilon_1 + \varepsilon_2 + (C(h) + |\alpha|)\varepsilon_2,$$

if Q_1 is sufficiently large depending on $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$, while

$$\left| \frac{\sum_{j=1}^{Q_1 P_1+Q_2 P_2} Y_j}{\sum_{j=1}^{Q_1 P_1+Q_2 P_2} \overline{Y}_j} - \alpha \right| \leq 2\varepsilon_2,$$

if Q_2 is sufficiently large depending on Q_1, $\{\Sigma_{1,i}\}$ and $\{\Sigma_{2,i}\}$.

Proceeding this way we get a sequence $\{Q_N\}$ and a sequence $\{Y_j\}$ defined so that if $1 + \sum_{N=1}^{N=N_0} Q_N P_N \leq j \leq \sum_{N=1}^{N=N_0+1} Q_N P_N$, we write

$j = \sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1} + l$, where $k \in \{0, \dots, Q_{N_0+1} - 1\}$ and $l \in \{1, \dots, P_{N_0+1}\}$, and set $Y_j = \Sigma_{N_0+1,l}$, $\bar{Y}_j = \bar{X}_{N_0+1,l}$. We will have

$$\left| \frac{\sum_{j=1}^{\sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1}+l} Y_j}{\sum_{j=1}^{\sum_{N=1}^{N=N_0} Q_N P_N + k P_{N_0+1}+l} \bar{Y}_j} - \alpha \right| \le 2\varepsilon_{N_0} + \varepsilon_{N_0+1} + (C(h) + |\alpha|)\varepsilon_{N_0+1},$$

and

$$\left| \frac{\sum_{j=1}^{\sum_{N=1}^{N=N_0+1} Q_N P_N} Y_j}{\sum_{j=1}^{\sum_{N=1}^{N=N_0+1} Q_N P_N} \bar{Y}_j} - \alpha \right| \le 2\varepsilon_{N_0+1}.$$

This implies

$$\lim_{q \to \infty} \frac{\sum_{j=1}^{q} Y_j}{\sum_{j=1}^{q} \bar{Y}_j} = \alpha$$

for any h, and we are done.

Proof of the Property (**P**): Let g be a smooth Riemannian metric, $\epsilon_1 > 0$ and $\tilde{K} > 0$ be constants, and choose

$$S = (K, \{\hat{B}_k\}, \{B_k\}, \{\phi_k\}, \{q_k\}, \{x_l^k\}) \in \mathcal{C}_{g,\tilde{K},\epsilon_1}.$$

Let $\mathcal{U}$ be a C^∞ neighborhood of g. Let $N = KL(n+1)$. Choose $\epsilon' > 0$ sufficiently small and $q \ge N + 3$ sufficiently large so that if $g' \in \Gamma_\infty$ satisfies $\|g - g'\|_{C^q} < \epsilon'$, then $g' \in \mathcal{U}$. For each k, l, j we associate a variable $t_{k,l,j} \in [0, 1]$ and we order them by lexicographical order on the indices. We can find a smooth $(0, 2)$-tensor $\bar{h}_{l,j}^k$ so that $\|\bar{h}_{l,j}^k - h_{l,j}^k\|_{C^q} < \epsilon'$ and such that $\{\bar{h}_{l,j}^k\}_{l,j}$ is linearly independent in a neighborhood of q_k where ϕ_k is equal to 1 and $\phi_{k'}$ is zero for $k' \ne k$.

Consider the following N-parameter family of metrics. For a $t = (t_{k,l,j}) \in [0, 1]^N$, we define

$$\hat{g}(t) = g + 2 \sum_{k,l,j} \psi_k t_{k,l,j} \bar{h}_{l,j}^k.$$

As t goes to zero, we have the following expansion

$$\text{vol}(M, \hat{g}(t))^{\frac{n}{n+1}} = \text{vol}_g(M)^{\frac{n}{n+1}}$$
$$+ \frac{n}{(n+1)} \text{vol}_g(M)^{-\frac{1}{n+1}} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g + o(\|t\|_1) + O(\epsilon' \|t\|_1), \quad (14)$$

where $||t||_1 = \sum_{k,l,j} |t_{k,l,j}|$. Also

$$\frac{\partial}{\partial t_{k,l,j}} \text{vol}(M, \hat{g}(t)) = \int_M \psi_k \, \text{Tr}_{\hat{g}(t)}(\bar{h}^k_{l,j}) dv_{\hat{g}(t)} = \int_M \psi_k dv_g + o(1) + O(\epsilon').$$

We will say that a function $f : [0, \delta]^N \to \mathbb{R}$ is ϵ'-close to another function $g : [0, \delta]^N \to \mathbb{R}$ if, when appropriately rescaled to be functions defined on $[0, 1]^N$, they are at distance less than ϵ' in the L^∞ norm, i.e.

$$||\frac{1}{\delta} f_\delta - \frac{1}{\delta} g_\delta||_\infty < \epsilon'$$

with $f_\delta(s) = f(\delta s)$ and $g_\delta(s) = g(\delta s)$. By (14), the function

$$f_0(t) := \frac{\text{vol}(M, \hat{g}(t))^{n/(n+1)}}{\text{vol}_g(M)^{n/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$-close to the constant function equal to 1 on $[0, \delta]^N$, where $C = C(g)$ depends only on g and might differ from line to line, if δ is sufficiently small.

If $\delta > 0$ is sufficiently small, we also have that $\hat{g} : [0, \delta]^N \to \Gamma_q$ is an embedding and $||\hat{g}(t) - g||_{C^q} < \epsilon'/2$ for every $t \in [0, \delta]^N$. We can slightly perturb $\hat{g}$ in the C^∞ topology into a C^∞ map $g' : [0, \delta]^N \to \Gamma_q$ so that the conclusion of Lemma 2 is satisfied. In particular, we can assume $||g'(t) - \hat{g}(t)||_{C^q} < \epsilon'/4$ and $||\frac{\partial g'}{\partial v}(t) - \frac{\partial \hat{g}}{\partial v}(t)||_{C^q} < \epsilon'/4$ for any $t \in [0, \delta]^N$ and $v \in \mathbb{R}^N$, $|v| = 1$, and the function

$$f_1(t) := \frac{\text{vol}(M, g'(t))^{n/(n+1)}}{\text{vol}_g(M)^{n/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$-close to the constant function equal to 1 on $[0, \delta]^N$.

The normalized widths $p^{-\frac{1}{n+1}} \omega_p(g'(t))$ of $g'(t)$ ($t \in [0, \delta]^N$) are uniformly Lipschitz continuous on $[0, \delta]^N$ by Lemma 1. Hence, by the Weyl Law for the Volume Spectrum [9], the functions $t \mapsto p^{-\frac{1}{n+1}} \omega_p(g'(t))$ converge uniformly to the function $t \mapsto a(n) \text{Vol}(M, g'(t))^{\frac{n}{n+1}}$. Hence if p is sufficiently large, $|p^{-\frac{1}{n+1}} \omega_p(g'(t)) - a(n) \text{Vol}(M, g'(t))^{\frac{n}{n+1}}| < \delta\epsilon'$ and the function

$$f_2(t) := \frac{\omega_p(g'(t))}{a(n) \text{vol}_g(M)^{n/(n+1)} p^{1/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\text{vol}_g(M)} \sum_{k,l,j} t_{k,l,j} \int_M \psi_k dv_g$$

is $C\epsilon'$-close to the constant function equal to 1 on $[0, \delta]^N$.

Then at each $t \in \mathcal{A}$ (where $\mathcal{A}$ is given by Lemma 2), there is a varifold $V \in \mathcal{V}(g'(t))$ with support a minimal hypersurface Σ such that

$$
\begin{aligned}
\frac{\partial}{\partial t_{k,l,j}} p^{-\frac{1}{n+1}} \omega_p(g'(t)) &= p^{-\frac{1}{n+1}} \frac{\partial}{\partial t_{k,l,j}} ||V||(M, g'(t)) \\
&= p^{-\frac{1}{n+1}} ||V||(\psi_k \operatorname{Tr}_\Sigma(\bar{h}_{l,j}^k)) + O(\epsilon') \\
&= p^{-\frac{1}{n+1}} \left(||V||(\psi_k \operatorname{Tr}_{M,g'(t)} \bar{h}_{l,j}^k) - V(\psi_k \bar{h}_{l,j}^k) \right) + O(\epsilon') \\
&= p^{-\frac{1}{n+1}} \left(||V||(\psi_k) - V(\psi_k h_{l,j}^k) \right) + O(\epsilon'),
\end{aligned}
\tag{15}
$$

where $||V||(.)$, $V(.)$ and the traces are computed with respect to $g'(t)$.

Given $\eta > 0$, we can choose $0 < \epsilon' < \eta$ sufficiently small compared to $C = C(g)$ so that we can apply Lemma 3 to f_2 and to $\mathcal{A}$. We get sequences of points $\{y_{1,m}\}_m, \ldots, \{y_{N+1,m}\}$ in $\mathcal{A}$ converging to a common limit $y \in (0, \delta)^N$ such that the gradients $\nabla f_2(y_{i,m})$ converge to $N + 1$ vectors $v_1, \ldots, v_{N+1}$ with

$$
d_{\mathbb{R}^N} (0, \operatorname{Conv}(v_1, \ldots, v_{N+1})) < \eta.
$$

Let $\{\alpha_1, \ldots, \alpha_{N+1}\} \subset [0, 1]$ with $\sum_{i=1}^{N+1} \alpha_i = 1$ such that $|\alpha_1 v_1 + \cdots + \alpha_{N+1} v_{N+1}| < \eta$. Then for sufficiently large m, we have

$$
|\alpha_1 \nabla f_2(y_{1,m}) + \cdots + \alpha_{N+1} \nabla f_2(y_{N+1,m})| < \eta,
$$

and hence

$$
|\alpha_1 \frac{\partial f_2}{\partial t_{k,l,j}}(y_{1,m}) + \cdots + \alpha_{N+1} \frac{\partial f_2}{\partial t_{k,l,j}}(y_{N+1,m})| < \eta
\tag{16}
$$

for all k, l, j.

According to Lemma 2, each gradient based at $y_{i,m}$ corresponds to a varifold of mass $\omega_p(g'(y_{i,m}))$ whose support is a minimal hypersurface in $(M, g'(y_{i,m}))$ of index bounded by p. Hence for all i, by Sharp's Compactness Theorem [20] a subsequence in m of these varifolds converges to a varifold of $\mathcal{V}(g'(y))$ whose mass is $\omega_p(g'(y))$. By (15) and (16), we have $N + 1$ varifolds V_i in $\mathcal{V}(g'(y))$ such that

$$
\left| \sum_i \alpha_i \frac{||V_i||(\psi_k) - V_i(\psi_k h_{l,j}^k)}{a(n) \operatorname{vol}_g(M)^{n/(n+1)} p^{1/(n+1)}} - \frac{n}{(n+1)} \frac{1}{\operatorname{vol}_g(M)} \int_M \psi_k dv_g \right| < C\eta
$$

for all k, l, j, where $||V_i||(.)$ and $V_i(.)$ are computed with respect to $g'(y)$. This implies

$$
\forall k, l, j \quad \left| \sum_i \alpha_i \frac{||V_i||(\psi_k) - V_i(\psi_k h_{l,j}^k)}{||V_i||(M)} - \frac{n}{(n+1)} \frac{1}{\operatorname{vol}_g(M)} \int_M \psi_k dv_g \right| < C\eta.
$$

We also have that for all i, k, l, one has

$$\left| \sum_{j=1}^{n+1} V_i(\psi_k h_{l,j}^k) - ||V_i||(\psi_k) \right| = |V_i(\psi_k g) - ||V_i||(\psi_k)| < C\eta ||V_i||(M),$$

and $\left| \dfrac{1}{(n+1)} \dfrac{1}{\mathrm{vol}_g(M)} \int_M \psi_k dv_g - \dfrac{1}{(n+1)} \dfrac{1}{\mathrm{vol}_{g'(y)}(M)} \int_M \psi_k dv_{g'(y)} \right| < C\eta.$

We deduce the following:

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\mathrm{vol}_{g'(y)}(M)} \int_M \psi_k dv_{g'(y)} \right| < C\eta. \tag{17}$$

The metric $g'(y) \in \Gamma_q$ satisfies $||g'(y) - g||_{C^q} < 3\epsilon'/4$. We apply Lemma 4 to $\bigcup_{i=1}^{N+1} \mathrm{spt}(V_i)$ and find a C^q metric $\overline{g}$ such that $||\overline{g} - g||_{C^q} < 4\epsilon'/5$, each $\mathrm{spt}(V_i)$ is nondegenerate minimal with respect to $\overline{g}$ and (17) is still valid with $g'(y)$ replaced by $\overline{g}$. If $\tilde{g}$ is a C^∞ metric that is sufficiently close to $\overline{g}$ in the C^q topology, then $\tilde{g} \in \mathcal{U}$. Because of the nondegeneracy of $\mathrm{spt}\, V_i$ with respect to $\overline{g}$ and the Implicit Function Theorem, we can also assure that there are varifolds $V_1, \ldots, V_J$ of $\mathcal{V}(\tilde{g})$ whose support $\mathrm{spt}(V_j)$ are nondegenerate, and coefficients $\alpha_1, \ldots, \alpha_J \in [0, 1]$ with $\sum_i \alpha_i = 1$ satisfying

$$\forall k, l, j \quad \left| \sum_i \alpha_i \frac{V_i(\psi_k h_{l,j}^k)}{||V_i||(M)} - \frac{1}{(n+1)} \frac{1}{\mathrm{vol}_{\tilde{g}}(M)} \int_M \psi_k dv_{\tilde{g}} \right| < C\eta, \tag{18}$$

where the terms of the sum are computed for the metric $\tilde{g}$. Since η is arbitrarily small, we have proved Property (**P**).

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