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Local theta correspondence of tempered representations and Langlands parameters

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Abstract In this paper, we give an explicit determination of the theta lifting for symplectic-orthogonal and unitary dual pairs over a nonarchimedean field F of characteristic 0. We determine when theta lifts of tempered representations are nonzero, and determine the theta lifts in terms of the local Langlands correspondence.

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1 Introduction

The theory of local theta correspondence was initiated by Roger Howe almost 40 years ago and has since been a major theme in representation theory and the theory of automorphic forms. In this paper, we shall address some basic questions concerning the local theta correspondence. Let us briefly recall the setup in broad strokes, leaving the precise exposition to the main body of the paper.

Let F be a nonarchimedean local field of characteristic 0 and let E be F itself or a quadratic field extension of F. Fix $\epsilon = \pm 1$ and set $\epsilon_0 = \epsilon$ if $E = F$ and $\epsilon_0 = 0$ if E is a quadratic field. Consider a $-\epsilon$-Hermitian space W_n over E of dimension n with associated isometry group $\mathrm{U}(W_n)$. Likewise, let V_m be an ϵ-Hermitian space over E of dimension m with associated isometry group $\mathrm{U}(V_m)$. Then

$$\mathrm{U}(W_n) \times \mathrm{U}(V_m) \longrightarrow \mathrm{Sp}(\mathrm{Res}_{E/F}(W_n \otimes_E V_m))$$

forms a reductive dual pair in the above symplectic group.

After fixing some extra data, the dual pair $\mathrm{U}(W_n) \times \mathrm{U}(V_m)$ has a Weil representation ω_{W_n,V_m}. For an irreducible representation π of $\mathrm{U}(W_n)$, the maximal π-isotypic quotient of ω_{W_n,V_m} has the form

$$\pi \boxtimes \Theta_{W_n,V_m}(\pi)$$

for some smooth representation $\Theta_{W_n,V_m}(\pi)$ of $\mathrm{U}(V_m)$ (known as the big theta lift of π). It was shown by Kudla that $\Theta_{W_n,V_m}(\pi)$ has finite length (possibly zero). The following basic result is known as the Howe duality conjecture (see [14,15,20,45]):

Theorem 1.1 *If $\Theta_{W_n,V_m}(\pi)$ is nonzero, then it has a unique irreducible quotient $\theta_{W_n,V_m}(\pi)$.*

We call $\theta_{W_n,V_m}(\pi)$ the small theta lift of π to $\mathrm{U}(V_m)$ and shall interpret it to be 0 if $\Theta_{W_n,V_m}(\pi)$ is zero. After the above theorem, it is natural to consider the following two basic problems:

Problem A Determine precisely when $\theta_{W_n,V_m}(\pi)$ is nonzero.

Problem B Determine $\theta_{W_n,V_m}(\pi)$ precisely when it is nonzero.

In this paper, we shall address these two problems for tempered representations π.

To formulate answers to these two problems, especially Problem B, it is necessary to have some sort of classification of irreducible representations of the groups $\mathrm{U}(W_n)$ and $\mathrm{U}(V_m)$. Such a classification is provided by the local

Langlands correspondence (LLC). The recent results of Arthur [1], Mok [35], Kaletha–Mínguez–Shin–White [23] and Gan–Savin [13] meant that the LLC is almost completely known for the groups considered in this paper.

The LLC classifies the irreducible representations π of $\mathrm{U}(W_n)$ by their L-parameters (ϕ, η), where

$$\phi\colon WD_E \to \mathrm{GL}_N(\mathbb{C})$$

is a conjugate self-dual representation of the Weil–Deligne group $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$ of a certain sign, and

$$\eta \in \mathrm{Irr}(A_\phi)$$

is an irreducible character of the component group A_ϕ associated to ϕ. We may think of ϕ as the last name of the representation π and η its first name. Thus we shall address Problems A and B in terms of the last names and first names of tempered representations.

Before going on, let us give a reformulation of Problem A. Let $\mathcal{V} = (V_m)$ be a Witt tower of ϵ-Hermitian spaces over E so that $V_{m+2} = V_m + \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic plane. In particular, $m = \dim_E(V_m)$ is of a fixed parity. Then one has a Witt tower of local theta correspondence associated to the dual pair $\mathrm{U}(W_n) \times \mathrm{U}(V_m)$. It is known by Kudla that the number

$$m_{\mathcal{V}}(\pi) = \min\left\{m \;\middle|\; \Theta_{V_m,W_n}(\pi) \neq 0\right\}$$

is finite. Moreover, $\Theta_{V_m,W_n}(\pi) \neq 0$ for all $m \geq m_{\mathcal{V}}(\pi)$. The number $m_{\mathcal{V}}(\pi)$ is called the first occurrence index of π in the Witt tower $\mathcal{V}$. Addressing Problem A for π is equivalent to determining the first occurrence index $m_{\mathcal{V}}(\pi)$ of π in every Witt tower $\mathcal{V}$.

For this purpose, the so-called conservation relation reduces our workload by half. More precisely, given any Witt tower $\mathcal{V}$, there is a companion Witt tower $\mathcal{V}' = (V'_m)$. We shall denote the two Witt towers by (V_m^+) and (V_m^-) and denote the first occurrence indices of π by $m^{\pm}(\pi)$ accordingly. The conservation relation, shown by Kudla–Rallis [26] and Sun–Zhu [43], says that

$$m^+(\pi) + m^-(\pi) = 2 \cdot (n + \epsilon_0 + 1).$$

This shows that

$$m^{\mathrm{down}}(\pi) = \min\left\{m^+(\pi), m^-(\pi)\right\} \leq n + \epsilon_0 + 1$$

and

$$m^{\mathrm{up}}(\pi) = \max\left\{m^+(\pi), m^-(\pi)\right\} \geq n + \epsilon_0 + 1.$$

To address Problems A and B, we need to determine:

- the value of $m^{\mathrm{down}}(\pi)$ and which of $m^{\pm}(\pi)$ it is equal to;
- the L-parameter $(\theta_m^{\pm}(\phi), \theta_m^{\pm}(\eta))$ of $\theta_{V_m^{\pm}, W_n}(\pi)$ if it is nonzero;

in terms of the L-parameter (ϕ, η) of π.

Let us describe our results in the special case of discrete series representations when $\mathrm{U}(W) \times \mathrm{U}(V) = \mathrm{Mp}_{2n} \times \mathrm{O}_{2m+1}$. More precisely, let W_{2n} be the $2n$-dimensional symplectic space and $V_{2m+1}^{\pm}$ be the two $(2m+1)$-dimensional quadratic spaces of discriminant 1, with V_{2m+1}^{+} the split quadratic space. Let π be an irreducible (genuine) discrete series representation of $\mathrm{Mp}(W_{2n})$, with L-parameter (ϕ, η). Thus

$$\phi = \bigoplus_{i=1}^{r} \phi_i$$

is a direct sum of distinct irreducible symplectic representations of the Weil–Deligne group $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ of F and η is a character of the component group

$$A_{\phi} = \bigoplus_{i=1}^{r} \mathbb{Z}/2\mathbb{Z} a_i,$$

which is a $\mathbb{Z}/2\mathbb{Z}$-vector space with a canonical basis $\{a_i\}$ indexed by the summands ϕ_i of ϕ. Let z_{ϕ} denote the element $\sum_{i=1}^{r} a_i \in A_{\phi}$. On the other hand, since $\mathrm{O}(V_{2m+1}^{\pm}) \cong \mathrm{SO}(V_{2m+1}^{\pm}) \times \mathbb{Z}/2\mathbb{Z}$, an irreducible representation of $\mathrm{O}(V_{2m+1}^{\pm})$ is parametrized by (ϕ', η', ν') where

- ϕ' is a symplectic representation of WD_F;
- η' is an irreducible character of the component group $A_{\phi'}$;
- $\nu' = \pm 1$ is a sign, with $\nu' = 1$ corresponding to the trivial character of $\mathbb{Z}/2\mathbb{Z}$.

Now we consider the theta liftings of π to the two Witt towers $\mathcal{V}^{\pm}$. The conservation relation says that

$$m^{\mathrm{down}}(\pi) + m^{\mathrm{up}}(\pi) = 4n + 4,$$

so that

$$m^{\mathrm{down}}(\pi) \leq 2n + 1 \quad \text{and} \quad m^{\mathrm{up}}(\pi) \geq 2n + 3.$$

Our main results in this case are summarized in the following three theorems:

Theorem 1.2 (1) *$m^{\text{down}}(\pi) = m^{\epsilon}(\pi)$ if and only if $\epsilon = \eta(z_\phi)$. We call $\mathcal{V}^{\eta(z_\phi)}$ the going-down tower, and $\mathcal{V}^{-\eta(z_\phi)}$ the going-up tower.*

(2) *Consider the set $\mathcal{T}$ containing 0 and all even integers $l > 0$ satisfying the following conditions:*
- *(chain condition) ϕ contains $S_2 + S_4 + \cdots + S_l$, where S_k denotes the (unique) k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$;*
- *(initial condition) if e_k denotes the basis element of A_ϕ associated to S_k, then $\eta(e_2) = 1$;*
- *(alternating condition) $\eta(e_i) = -\eta(e_{i+2})$ for even $2 \le i \le l-2$.*

Let

$$l(\pi) = \max \mathcal{T}.$$

Then

$$m^{\text{down}}(\pi) = 2n + 1 - l(\pi) \quad \textit{and} \quad m^{\text{up}}(\pi) = 2n + 3 + l(\pi).$$

In particular, the above theorem addresses Problem A.

Theorem 1.3 *Consider the going-down tower $\mathcal{V}^{\eta(z_\phi)}$. For each V_{2m+1} in this Witt tower, with $2m + 1 \ge m^{\text{down}}(\pi) = 2n + 1 - l(\pi)$, consider the theta lift $\theta_{W_{2n}, V_{2m+1}}(\pi)$ and let its L-parameter be given by $(\theta_{2m+1}(\phi), \theta_{2m+1}(\eta), \nu_{2m+1}(\phi, \eta))$.*

(1) *One has:*

$$\nu_{2m+1}(\phi, \eta) = \eta(z_\phi) \cdot \epsilon(1/2, \phi).$$

(2) *If $m < n$, then*

$$\theta_{2m+1}(\phi) = \phi - S_{2n-2m}.$$

Hence $\theta_{2m+1}(\phi)$ is a discrete series parameter and there is a natural injection $A_{\theta_{2m+1}(\phi)} \hookrightarrow A_\phi$. For the basis element a_i of $A_{\theta_{2m+1}(\phi)}$ associated to an irreducible summand ϕ_i, one has

$$\theta_{2m+1}(\eta)(a_i)/\eta(a_i) = \epsilon\left(1/2, \phi_i \otimes S_{2(n-m)-1}\right) \cdot \epsilon(1/2, \phi_i)$$
$$= \begin{cases} -1 & \text{if } \phi_i = S_{2k} \text{ for some } 1 \le k \le n-m-1, \\ 1 & \text{otherwise.} \end{cases}$$

(3) *If $m = n$, then*

$$\theta_{2m+1}(\phi) = \phi \quad \textit{and} \quad \theta_{2m+1}(\eta) = \eta.$$

Hence $\theta_{2m+1}(\phi)$ is a discrete series parameter.

(4) *If $m > n$, then $\theta_{2m+1}(\pi)$ is non-tempered and is the unique Langlands quotient of the standard module*

$$\times_{i=1}^{m-n} |\cdot|^{m-n+\frac{1}{2}-i} \rtimes \theta_{2n+1}(\pi).$$

In particular,

$$\theta_{2m+1}(\phi) = \phi \oplus \left(\bigoplus_{i=1}^{m-n} |\cdot|^{m-n+\frac{1}{2}-i} \oplus |\cdot|^{-\left(m-n+\frac{1}{2}-i\right)} \right),$$

so that there is a natural identification $A_{\theta_{2m+1}(\phi)} \cong A_{\theta_{2n+1}(\phi)}$, and

$$\theta_{2m+1}(\eta) = \theta_{2n+1}(\eta).$$

Theorem 1.4 *Consider the going-up tower $\mathcal{V}^{-\eta(z_\phi)}$. For each V_{2m+1} in this Witt tower, consider the theta lift $\theta_{W_{2n},V_{2m+1}}(\pi)$ and let its L-parameter be given by $(\theta_{2m+1}(\phi), \theta_{2m+1}(\eta), \nu_{2m+1}(\phi, \eta))$.*

(1) *One has:*

$$\nu_{2m+1}(\phi, \eta) = \eta(z_\phi) \cdot \epsilon(1/2, \phi).$$

(2) *If $2m + 1 = m^{\mathrm{up}}(\pi)$, then $\theta_{2m+1}(\pi)$ is a tempered representation with*

$$\theta_{2m+1}(\phi) = \phi + S_{l(\pi)+2},$$

so that there is a natural inclusion

$$A_\phi \hookrightarrow A_{\theta_{2m+1}(\phi)}.$$

For the basis element a_i of $A_{\theta_{2m+1}(\phi)}$ associated to an irreducible summand ϕ_i, one has

$$\theta_{2m+1}(\eta)(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i \otimes S_{l(\pi)+1}) \cdot \epsilon(1/2, \phi_i)$$
$$= \begin{cases} -1 & \text{if } \phi_i = S_{2k} \text{ for some } 1 \le k \le l(\pi)/2, \\ 1 & \text{otherwise.} \end{cases}$$

(3) *If $2m + 1 > m^{\mathrm{up}}(\pi)$ (so that $m - n - 1 - l(\pi) > 0$), then $\theta_{2m+1}(\pi)$ is non-tempered and is the unique Langlands quotient of the standard module*

$$\times_{i=1}^{m-n-1-l(\pi)/2} |\cdot|^{m-n+\frac{1}{2}-i} \rtimes \theta_{m^{\mathrm{up}}(\pi)}(\pi).$$

In particular,

$$\theta_{2m+1}(\phi) = \phi \oplus S_{l(\pi)+2} \oplus \left(\bigoplus_{i=1}^{m-n-1-l(\pi)/2} |\cdot|^{m-n+\frac{1}{2}-i} \oplus |\cdot|^{-(m-n+\frac{1}{2}-i)} \right),$$

so that there is a natural identification $A_{\theta_{2m+1}(\phi)} \cong A_{\theta_{m^{\mathrm{up}}(\pi)}(\phi)}$ *and*

$$\theta_{2m+1}(\eta) = \theta_{m^{\mathrm{up}}(\pi)}(\eta).$$

Taken together, the above two theorems give precise determination of the theta lifts of any discrete series representation π of $\mathrm{Mp}(W_{2n})$. In the case of tempered π, the results are in the same spirit, though slightly more involved to state.

We note that Problems A and B have been extensively studied by Muić [36–39], Mœglin [31,32] and Matić [28–30], at least for the symplectic-orthogonal dual pairs and the metaplectic-orthogonal dual pairs. Their work uses the Mœglin–Tadić classification of discrete series representations of classical groups in terms of supercuspidal representations. At that point, the Mœglin–Tadić classification was conditional, and it may be viewed as a preliminary form of the LLC. As such, the formulation of the answers to Problems A and B in the various papers of Muić may seem somewhat complicated, as are the proofs. The formulation of our main results and their proofs are neater and more transparent. There are several reasons for this:

- the LLC affords a more efficient language to describe the answers;
- the theory of local theta correspondence is in a more mature state today than at the time of Muić's work. For example, the conservation relation is now known and we do exploit it to simplify life;
- we make use of a wider spectrum of tools than Muić. For example, we use results of Gan–Ichino [11] on the behaviour of the standard gamma factors and Plancherel measures in the local theta correspondence, as well as results of Gan–Takeda [14] and Gan–Savin [13]. In the proofs of some of these results, the doubling see-saw diagram plays a crucial role. In addition, Problems A and B in the almost equal rank case were resolved in [12] for the unitary case and [2] for symplectic-orthogonal case by the local intertwining relation given by Arthur [1]. Muić, on the other hand, mainly made use of the computation of Jacquet modules and Kudla's filtration.

However, the main innovation of this paper is the exploitation of the local Gross–Prasad conjecture (GP), which is now established, in addressing Problems A and B. Recall that the GP conjecture comes in two flavours: the Bessel case and the Fourier–Jacobi case. For tempered representations, the Bessel case was proved by Waldpsurger [47–50] for special orthogonal groups, and Beuzart-Plessis [4–6] for unitary groups. In [12] and [2], the Fourier–Jacobi case (for tempered representations) was deduced from the Bessel case by using the theta correspondence in the almost equal rank case. In particular, in the almost equal rank case, Problems A and B were fully addressed in [12] for unitary dual pairs, [2] and [3] for symplectic-orthogonal dual pairs, and [13] for metaplectic-orthogonal dual pairs, and these allow one to deduce the Fourier–Jacobi case of the GP conjecture from the Bessel case. In this paper, with the GP conjecture in hand, we turn the table around and use it to understand the theta correspondence for general dual pairs.

Let us give a brief summary of the contents of this paper. After describing some background material on theta correspondence and the LLC in Sects. 2 and 3, our main results are given in Sect. 4. In order not to overburden the reader with too much background material, we have placed the more precise description of LLC in Appendices A and B. The local Gross–Prasad conjecture and Prasad's conjectures (which resolve Problems A and B for almost equal rank dual pairs) are placed in Appendices C and D, respectively. Note that in a prequel to this paper [3], we have discussed the LLC for full orthogonal groups and established the GP conjecture for full orthogonal groups. Finally the proofs of the main results are given in Sects. 5 and 6.

2 Local theta correspondence

In this section, we fix some notations.

2.1 Fields

Let F be a nonarchimedean local field of characteristic 0 and residue characteristic p. Let $\mathfrak{o}_F$ be the ring of integers of F, $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$, ϖ_F be a uniformizer of $\mathfrak{o}_F$, and q_F be the cardinality of $\mathfrak{o}_F/\mathfrak{p}_F$. The absolute value $|\cdot|_F$ on F is normalized by $|\varpi_F|_F = q_F^{-1}$. We fix a non-trivial additive character ψ of F.

Let E be either F itself or a quadratic extension of F, and $\omega_{E/F}$ be the quadratic character of $F^\times$ corresponding to E via the local class field theory. We denote the generator of $\mathrm{Gal}(E/F)$ by c. We define a non-trivial additive character ψ_E of E by $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. If $E \neq F$, we fix an element $\delta \in E^\times$ such that $\mathrm{tr}_{E/F}(\delta) = 0$, and set

$$\psi_a^E(x) = \psi\left(\frac{a}{2}\mathrm{tr}_{E/F}(\delta x)\right)$$

for $x \in E$ and $a \in F^\times$. If $a = 1$, we simply write $\psi^E = \psi_1^E$. One should not confuse ψ_E with ψ^E. If $E = F$, we set

$$\psi_a(x) = \psi(ax)$$

for $x \in F$ and $a \in F^\times$.

2.2 Spaces

Fix $\epsilon = \pm 1$ in $E^\times$. Let

$$\begin{aligned} W_n &= \text{a } -\epsilon\text{-Hermitian space over } E \text{ of dimension } n \text{ over } E, \\ V_m &= \text{an } \epsilon\text{-Hermitian space over } E \text{ of dimension } m \text{ over } E. \end{aligned}$$

We set

$$l = n - m + \epsilon_0 \quad \text{with} \quad \epsilon_0 = \begin{cases} \epsilon & \text{if } E = F, \\ 0 & \text{if } E \neq F, \end{cases}$$

and

$$\kappa = \begin{cases} 1 & \text{if } l \text{ is odd }, \\ 2 & \text{if } l \text{ is even }. \end{cases}$$

We define the discriminant $\mathrm{disc}(V_m)$ and $\mathrm{disc}(W_n)$ as in [11, §2.2]. Note that

$$\mathrm{disc}(V_m) \in \begin{cases} F^\times/F^{\times 2} & \text{if } E = F, \\ F^\times/N_{E/F}(E^\times) & \text{if } E \neq F \text{ and } \epsilon = +1, \\ \delta^m \cdot F^\times/N_{E/F}(E^\times) & \text{if } E \neq F \text{ and } \epsilon = -1. \end{cases}$$

2.3 Groups

We will consider the isometry groups associated to the pair (V_m, W_n) of $\pm\epsilon$-Hermitian spaces. More precisely, we set:

$$G(W_n) = \begin{cases} \text{the metaplectic group } \mathrm{Mp}(W_n), & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd }, \\ \text{the isometry group of } W_n, & \text{otherwise.} \end{cases}$$

We define $H(V_m)$ similarly by switching the roles of W_n and V_m.

For a vector space X over E, we denote the general linear group of X by $\mathrm{GL}(X)$. Let $\det_X = \det_{\mathrm{GL}(X)}$ be the determinant on $\mathrm{GL}(X)$.

2.4 Representations

Let G be a p-adic group. We denote the category of smooth representations of G by $\mathrm{Rep}(G)$. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible smooth (genuine) representations of G. We also denote by $\mathrm{Irr}_{\mathrm{temp}}(G)$

(resp. $\mathrm{Irr}_{\mathrm{disc}}(G)$) the subset of $\mathrm{Irr}(G)$ of classes of irreducible tempered representations (resp. discrete series representations).

For a parabolic subgroup $P = MN$ of G, let δ_P be the modulus character of P. For $(\pi_0, \mathcal{V}_0) \in \mathrm{Rep}(M)$, we define the normalized induction $\mathrm{Ind}_P^G(\pi_0)$ by the space of smooth functions $f : G \to \mathcal{V}_0$ such that

$$f(mng) = \delta_P(m)^{\frac{1}{2}} \cdot \pi_0(m) f(g) \quad \text{for } m \in M, n \in N \text{ and } g \in G.$$

The group G acts on $\mathrm{Ind}_P^G(\pi_0)$ by right translation. For $(\pi, \mathcal{V}) \in \mathrm{Rep}(G)$, we define the normalized Jacquet module $R_P(\pi)$ by $R_P(\pi) = \mathcal{V}/\mathcal{V}(N)$, where $\mathcal{V}(N)$ is the subspace generated by $\pi(n)v - v$ for $n \in N$ and $v \in \mathcal{V}$. Note that $\mathcal{V}(N)$ is an M-subrepresentation of $\mathcal{V}$. The group M acts on $R_P(\pi)$ by

$$m \cdot (v \bmod \mathcal{V}(N)) = \delta_P(m)^{-\frac{1}{2}} \cdot \pi(m)v \bmod \mathcal{V}(N)$$

for $m \in M$ and $v \in \mathcal{V}$.

We have the normalized induction functor

$$\mathrm{Ind}_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(G)$$

and the normalized Jacquet functor

$$R_P : \mathrm{Rep}(G) \to \mathrm{Rep}(M).$$

Let $\overline{P} = M\overline{N}$ be the opposite parabolic subgroup to P. Then there exist two Frobenius reciprocity formulas:

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_P^G(\pi_0)\right) \cong \mathrm{Hom}_M(R_P(\pi), \pi_0) \quad \text{(standard Frobenius reciprocity)}$$

and

$$\mathrm{Hom}_G\left(\mathrm{Ind}_P^G(\pi_0), \pi\right) \cong \mathrm{Hom}_M\left(\pi_0, R_{\overline{P}}(\pi)\right) \quad \text{(Bernstein's Frobenius reciprocity).}$$

2.5 Parabolic inductions

We shall use Tadić's notation for induced representations. Let W_n be a $-\epsilon$-Hermitian space, and $G(W_n)$ as in Sect. 2.3. If X_t is a t-dimensional isotropic subspace of W_n, we decompose

$$W = X_t \oplus W_{n-2t} \oplus X_t^*,$$

where X_t^* is a t-dimensional isotropic subspace of W_n such that $X_t \oplus X_t^*$ is non-degenerate, and W_{n-2t} is the orthogonal complement of $X_t \oplus X_t^*$ in W_n. We denote by $P(X_t) = L(X_t) \cdot U(X_t)$ the maximal parabolic subgroup stabilizing X_t, where $L(X_t) = \mathrm{GL}(X_t) \times G(W_{n-2t})$ is the Levi subgroup of $P(X_t)$ stabilizing X_t^*. If $\tau \in \mathrm{Irr}(\mathrm{GL}(X_t))$ and $\pi_0 \in \mathrm{Irr}(G(W_{n-2t}))$, we write

$$\tau \rtimes \pi_0 := \mathrm{Ind}_{P(X_t)}^{G(W_n)} (\tau \otimes \pi_0) .$$

More generally, a standard parabolic subgroup P of $G(W)$ has the Levi factor of the form $\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_r}(E) \times G(W_{n_0})$, and we set

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 := \mathrm{Ind}_P^{G(W_n)} (\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0) ,$$

where τ_i is a representation of $\mathrm{GL}_{n_i}(E)$ and π_0 is a representation of $G(W_{n_0})$. When $G(W_n) = \mathrm{Mp}(W_n)$ is a metaplectic group, we will follow the convention of [13, § 2.2–2.5] for the normalized parabolic induction.

2.6 Galois conjugate

Recall that c denotes the generator of $\mathrm{Gal}(E/F)$. Let X be a vector space over E of dimension t. Choose a basis $\{x_j\}$ of X, and we set

$$i \colon \mathrm{GL}_t(E) \to \mathrm{GL}(X), \quad g \mapsto [(x_1, \ldots, x_t) \mapsto (x_1, \ldots, x_t) g] .$$

For a representation τ of $\mathrm{GL}(X)$, we define the c-conjugate ${}^c\tau$ of τ by

$${}^c\tau(h) := \tau \left(i \circ c \circ i^{-1}(h) \right)$$

for $h \in \mathrm{GL}(X)$. Let $\{x'_j\}$ be another basis of X and we denote by $i' \colon \mathrm{GL}_t(E) \to \mathrm{GL}(X)$ the corresponding map. If $A \in \mathrm{GL}_t(E)$ satisfies

$$(x'_1, \ldots, x'_t) = (x_1, \ldots, x_t) \cdot A,$$

then we have $i'(g) = i(A \cdot g \cdot A^{-1})$, and so

$$i' \circ c \circ i'^{-1}(h) = i \left(A \cdot {}^cA^{-1} \right) \cdot i \circ c \circ i^{-1}(h) \cdot i \left(A \cdot {}^cA^{-1} \right)^{-1}$$

for $h \in \mathrm{GL}(X)$. This shows that the equivalence class of ${}^c\tau$ is independent of the choice of a basis of X.

2.7 MVW functor

Let δ be an F-linear automorphism on W_n such that $\delta G(W_n)\delta^{-1} = G(W_n)$. For a representation π of $G(W_n)$, we denote by π^δ the representation of $G(W_n)$ defined by conjugation, i.e., $\pi^\delta(g) = \pi(\delta g \delta^{-1})$. The following proposition is in Chapter 4.II.1 in [33].

Proposition 2.1 *Let π be an irreducible admissible representation of $G(W_n)$ and $\pi^\vee$ be the contragredient of π. Let δ be an E-conjugate linear automorphism on W_n such that*

$$\langle \delta x, \delta y \rangle = \langle y, x \rangle$$

for $x, y \in W_n$. Here, $\langle -, - \rangle$ denotes the Hermitian pairing of W_n. Then, $\pi^\delta \cong \pi^\vee$.

Fix δ as in Proposition 2.1. We define a functor

$$\mathrm{MVW}\colon \mathrm{Rep}(G(W_n)) \to \mathrm{Rep}(G(W_n))$$

by $\pi^{\mathrm{MVW}} = \pi^\delta$. Note that MVW is independent of the choice of δ. By the definition and Proposition 2.1, we see that

- MVW is an involution, i.e., $(\pi^{\mathrm{MVW}})^{\mathrm{MVW}} \cong \pi$;
- MVW is a covariant functor;
- $\mathrm{Ind}_{P(X_t)}^{G(W_n)}(\tau \otimes \pi_0)^{\mathrm{MVW}} \cong \mathrm{Ind}_{P(X_t)}^{G(W_n)}({}^c\tau \otimes \pi_0^{\mathrm{MVW}})$ for $\tau \in \mathrm{Irr}(\mathrm{GL}(X_t))$ and $\pi_0 \in \mathrm{Rep}(G(W_{n-2t}))$;
- if π is irreducible, then $\pi^{\mathrm{MVW}} \cong \pi^\vee$.

We will use MVW in the following form.

Lemma 2.2 *Let P be a standard parabolic subgroup of $G(W_n)$ with the Levi factor of the form $\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_r}(E) \times G(W_{n_0})$. Then for $\tau_i \in \mathrm{Irr}(\mathrm{GL}_{n_i}(E))$, $\pi_0 \in \mathrm{Irr}(G(W_{n_0}))$ and $\pi \in \mathrm{Irr}(G(W_n))$, the following are equivalent:*

(1) *π is a subrepresentation of $\tau_1 \times \cdots \times \tau_r \rtimes \pi_0$;*
(2) *π is a quotient of ${}^c\tau_1^\vee \times \cdots \times {}^c\tau_r^\vee \rtimes \pi_0$.*

Proof Use both the contragredient functor and the MVW functor. □

2.8 Weil representations

Let $(V, W) = (V_m, W_n)$ be as in Sect. 2.2. We consider the Weil representation of the pair $G(W) \times H(V)$. We fix a pair of characters $\boldsymbol{\chi} = (\chi_{V_m}, \chi_{W_n})$ of $E^\times$ as in [11, §3.2]. When there is no fear of confusion, χ_{V_m} and χ_{W_n} are simply denoted by χ_V and χ_W, respectively. Note that ${}^c\chi_V^{-1} = \chi_V$ and ${}^c\chi_W^{-1} = \chi_W$. Moreover, if V_m (resp. W_n) is a symplectic space, then $\chi_V = \mathbf{1}$ (resp. $\chi_W = \mathbf{1}$). These data and ψ give a splitting $G(W) \times H(V) \to \mathrm{Mp}(W \otimes V)$ of the dual pair. More precisely, see [17,25] and [11, § 3.3]. Pulling back the Weil representation of $\mathrm{Mp}(W \otimes V)$ to $G(W) \times H(V)$ via this splitting, we obtain the associated Weil representation $\omega_{V,W,\boldsymbol{\chi},\psi}$ of $G(W) \times H(V)$. We simply write $\omega_{V,W}$ for the Weil representation.

2.9 Theta correspondence

Let $\omega_{V,W}$ be the Weil representation of $G(W) \times H(V)$. For $\pi \in \mathrm{Irr}(G(W))$, the maximal π-isotypic quotient of $\omega_{V,W}$ is of the form

$$\pi \boxtimes \Theta_{V,W}(\pi),$$

where $\Theta_{V,W}(\pi)$ is a smooth representation of $H(V)$. We emphasize that $\Theta_{V,W}(\pi)$ depends on $\boldsymbol{\chi}$ and ψ also. It was shown by Kudla [24] that $\Theta_{V,W}(\pi)$ is either zero or of finite length.

The following result is proven by Waldspurger [45] when $p \neq 2$ and by [14,15] in general.

Theorem 2.3 (Howe duality conjecture) *If $\Theta_{V,W}(\pi)$ is nonzero, then $\Theta_{V,W}(\pi)$ has a unique irreducible quotient $\theta_{V,W}(\pi)$.*

2.10 First occurrence and tower property

Fix $\epsilon = \pm 1$. Let W_n be a $-\epsilon$-Hermitian space as in Sect. 2.2. For an anisotropic ϵ-Hermitian space V_{m_0} and $r \geq 0$, we put

$$V_{m_0+2r} = V_{m_0} \oplus \mathbb{H}^r,$$

where $\mathbb{H}$ is the hyperbolic plane. The collection

$$\mathcal{V} = \left\{ V_{m_0+2r} \;\middle|\; r \geq 0 \right\}$$

is called a Witt tower of spaces. Note that $\mathrm{disc}(V_m)$ and the parity of $\dim(V_m)$ depend only on the Witt tower $\mathcal{V}$ to which V_m belongs. One can consider a tower of the theta correspondence associated to reductive dual pairs $\{(G(W_n), H(V_m)) \mid V_m \in \mathcal{V}\}$. For $\pi \in \mathrm{Irr}(G(W_n))$, we have the representation $\Theta_{V_m,W_n}(\pi)$ of $H(V_m)$. The number

$$m_{\mathcal{V}}(\pi) = \min\left\{ m \;\middle|\; \Theta_{V_m,W_n}(\pi) \neq 0 \right\}$$

is finite and is called the first occurrence index of π for the Witt tower $\mathcal{V}$, and the representation $\theta_{V_{m_{\mathcal{V}}(\pi)},W_n}(\pi)$ is called the first occurrence of π for this Witt tower.

The following proposition is often referred to as the tower property of theta correspondence (see [24]).

Proposition 2.4 *Let $m_{\mathcal{V}}(\pi)$ be the first occurrence index of π for the Witt tower $\mathcal{V} = \{V_m\}$. Then we have $\Theta_{V_m,W_n}(\pi) \neq 0$ for any $m \geq m_{\mathcal{V}}(\pi)$.*

If $E \neq F$ or $\epsilon = +1$, for a given Witt tower $\mathcal{V} = \{V_m\}$, there exists another Witt tower $\mathcal{V}' = \{V'_{m'}\}$ such that

- $V_m \not\cong V'_{m'}$;
- $\dim(V_m) \equiv \dim(V'_{m'}) \bmod 2$;
- $\mathrm{disc}(V_m) = \mathrm{disc}(V'_{m'})$ if $E = F$ and $\epsilon = +1$.

We call $\mathcal{V}'$ the companion Witt tower of $\mathcal{V}$. Also, by a companion space of V_m, we mean V_m or V'_m. For each $\pi \in \mathrm{Irr}(G(W_n))$, we may consider two first occurrence indices $m_{\mathcal{V}}(\pi)$ and $m_{\mathcal{V}'}(\pi)$. Let $\mathcal{V}^+ = \{V_m^+\}$ be the Witt tower whose anisotropic space is

$$\begin{cases} 0 & \text{if } E \neq F \text{ and } m \text{ is even,} \\ (E, 1) & \text{if } E \neq F, m \text{ is odd and } \epsilon = +1, \\ (E, \delta) & \text{if } E \neq F, m \text{ is odd and } \epsilon = -1, \\ 0 & \text{if } E = F, m \text{ is even and } \mathrm{disc}(V_m) = 1, \\ \left(F(\sqrt{d}), \mathrm{tr}_{F(\sqrt{d})/F}\right) & \text{if } E = F, m \text{ is even and } d := \mathrm{disc}(V_m) \neq 1 \text{ in } F^\times/F^{\times 2}, \\ (F, 2\mathrm{disc}(V_m)) & \text{if } E = F \text{ and } m \text{ is odd.} \end{cases}$$

Here, we consider V_m as a vector space equipped with a suitable Hermitian pairing. For example, by $(F, 2\mathrm{disc}(V_m))$, we mean the one dimensional space equipped with the bilinear form

$$(x, y) \mapsto 2dxy,$$

where $d \in F^\times$ satisfies $d \bmod F^{\times 2} = \mathrm{disc}(V_m)$ in $F^\times/F^{\times 2}$. Note that this space has discriminant $\mathrm{disc}(V_m)$. We denote the other Witt tower by $\{V_m^-\}$. Then for each $\pi \in \mathrm{Irr}(G(W_n))$, we have two first occurrence indices $m^\pm(\pi) := m_{\mathcal{V}^\pm}(\pi)$.

On the other hand, if $E = F$ and $\epsilon = -1$, then there is only a single tower of symplectic spaces $\mathcal{V} = \{V_m\}$. In this case, a companion space of V_m is just V_m. However, since π is a representation of the orthogonal group $G(W_n) = \mathrm{O}(W_n)$, we may consider its twist $\pi \otimes \det$. Thus we have the two towers of theta lifts

$$\Theta_{V_m,W_n}(\pi) \quad \text{and} \quad \Theta_{V_m,W_n}(\pi \otimes \det).$$

Hence we may define two first occurrence indices for each $\pi \in \mathrm{Irr}(G(W_n))$. When n is odd, we define $m^{\pm}(\pi)$ by

$$m^{\pm}(\pi) := \min \left\{ m \mid \Theta_{V_m,W_n}(\pi') \neq 0 \text{ with } \pi' \in \{\pi, \pi \otimes \det\} \right. \\ \left. \text{such that } \pi'(-\mathbf{1}_{W_n}) = \pm \mathrm{id} \right\}.$$

When n is even, we define $m^{\pm}(\pi)$ by

$$m^{+}(\pi) := \min \left\{ \min \left\{ m \mid \Theta_{V_m,W_n}(\pi) \neq 0 \right\},\ \min \left\{ m \mid \Theta_{V_m,W_n}(\pi \otimes \det) \neq 0 \right\} \right\},$$
$$m^{-}(\pi) := \max \left\{ \min \left\{ m \mid \Theta_{V_m,W_n}(\pi) \neq 0 \right\},\ \min \left\{ m \mid \Theta_{V_m,W_n}(\pi \otimes \det) \neq 0 \right\} \right\}.$$

Hence, in this case, $m^{+}(\pi) \leq m^{-}(\pi)$ by convention.

In any case, for each $\pi \in \mathrm{Irr}(G(W_n))$, we have two first occurrence indices $m^{\pm}(\pi)$. We put

$$m^{\mathrm{up}}(\pi) = \max \left\{ m^{+}(\pi), m^{-}(\pi) \right\} \quad \text{and} \quad m^{\mathrm{down}}(\pi) = \min \left\{ m^{+}(\pi), m^{-}(\pi) \right\}.$$

The following proposition is often referred to as the conservation relation (see [43]).

Proposition 2.5 *For any* $\pi \in \mathrm{Irr}(G(W_n))$, *we have*

$$m^{\mathrm{up}}(\pi) + m^{\mathrm{down}}(\pi) = 2n + 2 + 2\epsilon_0.$$

This relation shows that

$$m^{\mathrm{up}}(\pi) \geq n + 1 + \epsilon_0 \quad \text{and} \quad m^{\mathrm{down}}(\pi) \leq n + 1 + \epsilon_0.$$

If we put

$$l = n - m^{\mathrm{down}}(\pi) + \epsilon_0,$$

then we have $l \geq -1$.

3 Parametrization of irreducible representations

In this section, we explain the local Langlands correspondence (LLC) quickly. More precisely, see Appendix B.

Let $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of E. We define $\Phi(H(V_m))$, which is a set of equivalence classes of representations of WD_E, in the various cases as follows:

$$\begin{cases} \Phi(\mathrm{O}(V_m)) = \{\phi\colon WD_F \to \mathrm{Sp}(m-1, \mathbb{C})\} / \cong, & \text{if } m \text{ is odd,} \\ \Phi(\mathrm{Sp}(V_m)) = \{\phi\colon WD_F \to \mathrm{SO}(m+1, \mathbb{C})\} / \cong, & \\ \Phi(\mathrm{O}(V_m)) = \left\{\phi\colon WD_F \to \mathrm{O}(m, \mathbb{C}) \;\middle|\; \det(\phi) = \chi_V\right\} / \cong, & \text{if } m \text{ is even,} \\ \Phi(\mathrm{Mp}(V_m)) = \{\phi\colon WD_F \to \mathrm{Sp}(m, \mathbb{C})\} / \cong . & \end{cases}$$

For the unitary group $\mathrm{U}(m)$, we define $\Phi(\mathrm{U}(m))$ to be the set of equivalence classes of conjugate self-dual representations of WD_E of sign $(-1)^{m-1}$. For the notion of conjugate self-dual representations, see Appendix A.3.

We say that $\phi \in \Phi(H(V_m))$ is tempered if $\phi(W_E)$ is bounded. We denote by $\Phi_{\mathrm{temp}}(H(V_m))$ the subset of equivalence classes of tempered ϕ. For $\phi \in \Phi(H(V_m))$, we denote by $L(s, \phi)$, $\varepsilon(s, \phi, \psi')$, and $\gamma(s, \phi, \psi')$ the L-, ε-, and γ-factors associated to ϕ, respectively. Here, ψ' is a non-trivial additive character of E. The root number $\varepsilon(1/2, \phi, \psi')$ is also denoted by $\varepsilon(\phi)$ or $\varepsilon(\phi, \psi')$.

For an irreducible representation ϕ_0 of WD_E, we denote the multiplicity of ϕ_0 in ϕ by $m_\phi(\phi_0)$. We can decompose

$$\phi = m_1\phi_1 + \cdots + m_r\phi_r + \phi' + {}^c\phi'^\vee,$$

where $\phi_1, \ldots, \phi_r$ are distinct irreducible representations of WD_E of the same type as ϕ, $m_i = m_\phi(\phi_i)$, and ϕ' is a sum of irreducible representations of WD_E which are not of the same type as ϕ. We define the component group A_ϕ by

$$A_\phi = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

Namely, A_ϕ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank r and $\{a_1, \ldots, a_r\}$ is a basis of A_ϕ with a_i associated to ϕ_i. For $a = a_{i_1} + \cdots + a_{i_k} \in A_\phi$ with $1 \le i_1 < \cdots < i_k \le r$, we put

$$\phi^a = \phi_{i_1} \oplus \cdots \oplus \phi_{i_k}.$$

Also, we set

$$z_\phi := \sum_{i=1}^{r} m_\phi(\phi_i) \cdot a_i = \sum_{i=1}^{r} m_i \cdot a_i \in A_\phi.$$

We call z_ϕ the central element in A_ϕ. There is a homomorphism

$$\det\colon A_\phi \to \mathbb{Z}/2\mathbb{Z}, \quad \sum_{i=1}^{r} \varepsilon_i a_i \mapsto \sum_{i=1}^{r} \varepsilon_i \cdot \dim(\phi_i) \pmod 2,$$

where $\varepsilon_i \in \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$.

The LLC classifies $\mathrm{Irr}(H(V_m))$ as follows:

Desideratum 3.1 (1) *There exists a partition*

$$\bigsqcup_{V_m^\bullet} \mathrm{Irr}(H(V_m^\bullet)) = \bigsqcup_{\phi \in \Phi(H(V_m))} \Pi_\phi,$$

where $V_m^\bullet$ runs over all companion spaces of V_m. We call Π_ϕ the L-packet of ϕ.

(2) *$\pi \in \mathrm{Irr}(H(V_m^\bullet))$ is tempered if and only if π belongs to Π_ϕ for tempered ϕ.*

(3) *There exists a map*

$$\iota\colon \Pi_\phi \to \widehat{A_\phi},$$

which satisfies certain character identities. Here, we denote by $\widehat{A_\phi}$ the Pontryagin dual of A_ϕ.

(4) *The map ι is surjective unless $H(V_m) = \mathrm{Sp}(V_m)$ is a symplectic group. In this case, the image of ι is given by*

$$\left\{\eta \in \widehat{A_\phi} \;\middle|\; \eta(z_\phi) = 1\right\}.$$

(5) *The map ι is injective unless $H(V_m) = \mathrm{O}(V_m)$ is an odd orthogonal group (i.e., m is odd). In this case, each fiber of this map is of the form*

$$\{\pi,\ \pi \otimes \det\}.$$

Hence the map

$$\Pi_\phi \to \widehat{A_\phi} \times \{\pm 1\},\ \pi \mapsto (\iota(\pi), \omega_\pi(-1))$$

is bijective, where ω_π is the central character of π.

(6) *Suppose that V_m^- exists. Then $\pi \in \Pi_\phi$ is a representation of $H(V_m^-)$ if and only if $\iota(\pi)(z_\phi) = -1$.*

Therefore, unless $H(V_m) = \mathrm{O}(V_m)$ is an odd orthogonal group, $\pi \in \mathrm{Irr}(H(V_m))$ is parametrized by (ϕ, η), where $\phi \in \Phi(H(V_m))$ and $\eta \in \widehat{A_\phi}$. If

$H(V_m) = \mathrm{O}(V_m)$ is an odd orthogonal group, $\pi \in \mathrm{Irr}(H(V_m))$ is parametrized by the triple (ϕ, η, ν), where $\phi \in \Phi(H(V_m))$, $\eta \in \widehat{A_\phi}$ and $\nu \in \{\pm 1\}$. The pair (ϕ, η) is called the L-parameter for π. We also call ϕ and η the last name and the first name of π, respectively.

Remark 3.2 The map $\iota \colon \Pi_\phi \to \widehat{A_\phi}$ may not be canonical. To specify ι, we need to choose a Whittaker datum for $H(V_m)$. More precisely, see Remark B.2 below.

Suppose that $H(V_m) = \mathrm{O}(V_m)$ is an even orthogonal group (i.e., m is even). Then the following are equivalent:

- $\phi \in \Phi(\mathrm{O}(V_m))$ contains an irreducible orthogonal representation of WD_F of odd dimension;
- some $\pi \in \Pi_\phi$ satisfies $\pi \not\cong \pi \otimes \det$;
- any $\pi \in \Pi_\phi$ satisfies $\pi \not\cong \pi \otimes \det$.

4 Main results

The purpose of this paper is to describe theta lifts of tempered representations in terms of the local Langlands correspondence. In this section, we state the main results over 3 theorems. Though we formulate the main results as 3 theorems, these are proven together (in Sect. 6).

We denote by S_r the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension r. When χ is a character of $E^\times$, we regard χ as a character of W_E via the local class field theory $W_E^{\mathrm{ab}} \cong E^\times$, so that $\chi S_r = \chi \boxtimes S_r$ is a representation of $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$. The first main theorem gives an answer to Problem A in Sect. 1 for tempered representations.

Theorem 4.1 *Let (V_m, W_n) and $\kappa \in \{1, 2\}$ be as in Sect. 2.2, and $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η).*

(1) *Consider the set $\mathcal{T}$ containing $\kappa - 2$ and all integers $l > 0$ with $l \equiv \kappa \bmod 2$ satisfying the following conditions:*
- *(chain condition) ϕ contains $\chi_V S_r$ for $r = \kappa, \kappa + 2, \dots, l$;*
- *(odd-ness condition) the multiplicity $m_\phi(\chi_V S_r)$ is odd for $r = \kappa, \kappa + 2, \dots, l - 2$;*
- *(initial condition) if $\kappa = 2$, then*

$$\eta(e_2) = \begin{cases} \epsilon \cdot \delta(\chi_V = \mathbf{1}) & \text{if } E = F \text{ and } m \not\equiv n \bmod 2, \\ -1 & \text{if } E \neq F \text{ and } m \equiv n \bmod 2; \end{cases}$$

- *(alternating condition) $\eta(e_r) = -\eta(e_{r+2})$ for $r = \kappa, \kappa + 2, \dots, l - 2$.*

Here, e_r is the element in A_ϕ corresponding to $\chi_V S_r$, and for a character χ, we put

$$\delta(\chi = \mathbf{1}) = \begin{cases} +1 & \text{if } \chi = \mathbf{1}, \\ -1 & \text{otherwise.} \end{cases}$$

Let

$$l(\pi) = \max \mathcal{T}.$$

Then

$$m^{\text{down}}(\pi) = n + \epsilon_0 - l(\pi) \quad \text{and} \quad m^{\text{up}}(\pi) = n + 2 + \epsilon_0 + l(\pi).$$

(2) *If $l(\pi) = -1$, then $m^{\text{up}}(\pi) = m^{\text{down}}(\pi)$. Suppose that $l(\pi) \geq 0$. Then ϕ contains χ_V if $\kappa = 1$. Moreover, $m^{\text{down}}(\pi) = m^\alpha(\pi)$ if and only if*

$$\alpha = \begin{cases} \eta(z_\phi + e_1) & \text{if } \kappa = 1, \\ \eta(z_\phi) \cdot \varepsilon(\phi) \cdot \varepsilon(\phi \otimes \chi_V) \cdot \chi_V(-1)^{\frac{n}{2}} & \text{if } E = F, m \not\equiv n \bmod 2 \text{ and } \epsilon = +1, \\ \eta(z_\phi) \cdot \varepsilon(\phi) & \text{if } E = F, m \not\equiv n \bmod 2 \text{ and } \epsilon = -1, \\ \eta(z_\phi) \cdot \varepsilon\left(\phi \otimes \chi_V^{-1}, \psi_2^E\right) & \text{if } E \neq F \text{ and } m \equiv n \bmod 2. \end{cases}$$

We call $\mathcal{V}^{\text{down}} := \mathcal{V}^\alpha$ (resp. $\mathcal{V}^{\text{up}} := \mathcal{V}^{-\alpha}$) the going-down tower (resp. the going-up tower) with respect to π.

Remark 4.2 Recall that when $(G(W_n), H(V_m)) = (\mathrm{O}(W_n), \mathrm{Sp}(V_m))$ with even n, by the definition, $m^{\text{down}}(\pi) = m^+(\pi)$ for each $\pi \in \mathrm{Irr}(\mathrm{O}(W_n))$ (see Sect. 2.10). In this case, (2) asserts that if $\pi \in \mathrm{Irr}(\mathrm{O}(W_n))$ satisfies that $\Theta_{V_m,W_n}(\pi) \neq 0$ and $\Theta_{V_m,W_n}(\pi \otimes \det) = 0$ for some $m \leq n$, then the L-parameter (ϕ, η) of π satisfies that $\phi \supset \mathbf{1}$ and $\eta(z_\phi + e_1) = 1$. This follows from Prasad's conjecture (Theorem D.2 below).

The proof of Theorem 4.1 is given in Sect. 6. We give an indication for the relevant results. To prove (1), it is enough to show the following two statements:

- If $\Theta_{V_m^\bullet,W_n}(\pi) \neq 0$, then $l := n - m + \epsilon_0 \in \mathcal{T}$.
- $n - m^{\text{down}}(\pi) + \epsilon_0 + 2 \notin \mathcal{T}$.

For the first assertion, (chain condition) and (odd-ness condition) follow from Corollary 6.2, and (initial condition) and (alternating condition) follow from Proposition 6.8. The second assertion follows from Corollary 6.13, Proposition 6.10 and Prasad's conjecture (Theorem D.2). The assertion (2) follows from Prasad's conjecture (Theorem D.2) together with a comparison of central elements z_ϕ (Proposition 6.7) unless $E = F$, $m \not\equiv n \bmod 2$ and

$\epsilon = -1$. In this case, we compare the central character of $\pi \in \mathrm{Irr}(\mathrm{O}(W_n))$ with the central element z_ϕ (Proposition 6.20).

The second and third main theorems describe the L-parameter for $\theta_{V_m,W_n}(\pi)$.

Theorem 4.3 *Let (V_m, W_n) and $\kappa \in \{1,2\}$ be as in Sect. 2.2, and $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η). Assume that V_m belongs to the going-down tower $\mathcal{V}^{\mathrm{down}}$, $m \geq m^{\mathrm{down}}(\pi)$ and $m \equiv m^{\mathrm{down}}(\pi) \bmod 2$. Put $m_1 = n + \epsilon_0 + 2 - \kappa$. Let $(\theta_m(\phi), \theta_m(\eta))$ be the L-parameter for $\theta_{V_m,W_n}(\pi)$.*

(1) *If $m^{\mathrm{down}}(\pi) \leq m < m_1$, then*

$$\theta_m(\phi) = (\phi \otimes \chi_V^{-1}\chi_W) - \chi_W S_l,$$

where $l = n - m + \epsilon_0 > 0$. In particular, there is a canonical injection $A_{\theta_m(\phi)} \hookrightarrow A_\phi$. If $l = 1$, then we have $\eta | A_{\theta_m(\phi)} = \theta_m(\eta)$. If $l > 1$, then $\theta_m(\eta)(a)/\eta(a)$ is equal to

$$\begin{cases} \varepsilon(\phi^a\chi_V^{-1} \otimes S_{l-1}) \cdot \varepsilon(\phi^a) \cdot \chi_V(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd}, \\ \varepsilon\left(\phi^a\chi_V^{-1} \otimes S_{l-1}\right) \cdot \varepsilon\left(\phi^a\chi_W\right) \cdot \chi_W(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F, \epsilon = -1 \text{ and } n \text{ is odd}, \\ \varepsilon\left(\phi^a\chi_V^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a\chi_V^{-1}\right)(-1)^{\frac{l-1}{2}} & \text{if } E = F \text{ and } m, n \text{ are even}, \\ \varepsilon\left(\phi^a\chi_V^{-1} \otimes S_{l-1}, \psi_2^E\right) & \text{if } E \neq F, \end{cases}$$

for any $a \in A_{\theta_m(\phi)} \subset A_\phi$.

(2) *If $m = m_1$ and $\kappa = 1$, then*

$$\theta_{m_1}(\phi) = \left(\phi \otimes \chi_V^{-1}\chi_W\right) \oplus \chi_W.$$

In particular, there is a canonical injection $A_\phi \hookrightarrow A_{\theta_{m_1}(\phi)}$. Moreover, we have $\theta_{m_1}(\eta)|A_\phi = \eta$.

(3) *If $m = m_1$ and $\kappa = 2$, then*

$$\theta_{m_1}(\phi) = \phi \otimes \chi_V^{-1}\chi_W.$$

In particular, there is a canonical identification $A_\phi = A_{\theta_{m_1}(\phi)}$. Moreover, $\theta_m(\eta)(a)/\eta(a)$ is equal to

$$\begin{cases} \varepsilon(\phi^a) \cdot \varepsilon\left(\phi^a \otimes \chi_V^{-1}\chi_W\right) \cdot \left(\chi_V^{-1}\chi_W\right)(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F, \\ \varepsilon\left(\phi^a \otimes \chi_V^{-1}, \psi_2^E\right) & \text{if } E \neq F \end{cases}$$

for any $a \in A_{\theta_{m_1}(\phi)} = A_\phi$.

(4) *If* $m > m_1$*, then* $\theta_m(\phi)$ *is equal to*

$$\theta_{m_1}(\phi) \oplus \left(\bigoplus_{i=1}^{(m-m_1)/2} \left(\chi_W \, |\cdot|_E^{\frac{m-n-\epsilon_0+1}{2}-i} \oplus \chi_W \, |\cdot|_E^{-\frac{m-n-\epsilon_0+1}{2}+i} \right) \right).$$

In particular, there is a canonical identification $A_{\theta_m(\phi)} = A_{\theta_{m_1}(\phi)}$*. Moreover, we have* $\theta_m(\eta)|A_{\theta_{m_1}(\phi)} = \theta_{m_1}(\eta)$*.*

(5) *If* $(G(W_n), H(V_m)) = (\mathrm{Mp}(W_n), \mathrm{O}(V_m))$ *with odd* m*, then* $\theta_{V_m,W_n}(\pi)$ *is parametrized by* $(\theta_m(\phi), \theta_m(\eta), \nu_m(\phi, \eta))$ *with*

$$\nu_m(\phi, \eta) = \eta(z_\phi) \cdot \varepsilon(\phi) \cdot \chi_V(-1)^{\frac{n}{2}}.$$

Remark 4.4 In Theorem 4.1 (2), we note that

$$\left[A_{\theta_{m_1}(\phi)} : A_\phi\right] = \begin{cases} 1 & \text{if } \phi \text{ contains } \chi_V, \\ 2 & \text{if } \phi \text{ does not contain } \chi_V. \end{cases}$$

If ϕ does not contain χ_V, then $m^+(\pi) = m^-(\pi) = m_1$ for any $\pi \in \Pi_\phi$ by Theorem 4.1 (1), and $z_{\theta_{m_1}(\phi)}$ is not contained in A_ϕ. The value $\theta_{m_1}(\eta)(z_{\theta_{m_1}(\phi)})$ is determined by Desideratum 3.1 (4) or (6).

The assertion (1) will be shown in Sect. 6.2. The assertions (2) and (3) are the (almost) equal rank cases (Theorems B.8, D.1 and D.2). The assertion (4) follows from [14, Proposition 3.2] (see Proposition 5.6 below). The assertion (5) is Proposition 6.19.

Theorem 4.5 *Let* (V_m, W_n) *and* $\kappa \in \{1, 2\}$ *be as in Sect. 2.2, and* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ *with L-parameter* (ϕ, η)*. Assume that* V_m *belongs to the going-up tower* $\mathcal{V}^{\mathrm{up}}$ *and*

$$m \geq m^{\mathrm{up}}(\pi) \geq n + \epsilon_0 + 2 \quad \text{i.e.,} \quad l(\pi) \geq 0.$$

Let $(\theta_m(\phi), \theta_m(\eta))$ *be the L-parameter for* $\theta_{V_m,W_n}(\pi)$*. We put* $l = m - n - \epsilon_0 - 2 \geq 0$*.*

(1) *Suppose that* $m = m^{\mathrm{up}}(\pi)$ *so that* $l = l(\pi)$*. If* $l = 0$ *or* $m_\phi(\chi_V S_l)$ *is odd, then*

$$\theta_m(\phi) = \left(\phi \otimes \chi_V^{-1}\chi_W\right) \oplus \chi_W S_{l+2},$$

so that $\theta_{V_m,W_n}(\pi)$ is tempered. In particular, there is a canonical injection $A_\phi \hookrightarrow A_{\theta_m(\phi)}$. Moreover, $\theta_m(\eta)(a)/\eta(a)$ is equal to

$$\begin{cases} \varepsilon\left(\phi^a\chi_V^{-1}\otimes S_{l+1}\right)\cdot\varepsilon(\phi^a)\cdot\chi_V(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E=F, \epsilon=+1 \text{ and } m \text{ is odd},\\ \varepsilon\left(\phi^a\chi_V^{-1}\otimes S_{l+1}\right)\cdot\varepsilon\left(\phi^a\chi_W\right)\cdot\chi_W(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E=F, \epsilon=-1 \text{ and } n \text{ is odd},\\ \varepsilon\left(\phi^a\chi_V^{-1}\otimes S_{l+1}\right)\cdot\det\left(\phi^a\chi_V^{-1}\right)(-1)^{\frac{l+1}{2}} & \text{if } E=F \text{ and } m,n \text{ are even},\\ \varepsilon\left(\phi^a\chi_V^{-1}\otimes S_{l+1},\psi_2^E\right) & \text{if } E\neq F,\end{cases}$$

for $a \in A_\phi \subset A_{\theta_m(\phi)}$.

(2) *Suppose that $m = m^{\mathrm{up}}(\pi)$ so that $l = l(\pi)$. If $l > 0$ and $m_\phi(\chi_V S_l) = 2h > 0$, then*

$$\theta_{m^{\mathrm{up}}}(\phi) = \left(\left(\phi\otimes\chi_V^{-1}\chi_W\right) - \chi_W S_l\right) \oplus \left(\chi_W S_{l+1}\otimes\left(|\cdot|_E^{\frac{1}{2}} + |\cdot|_E^{-\frac{1}{2}}\right)\right),$$

so that $\theta_{V_m,W_n}(\pi)$ is not tempered. In this case,

$$\pi \subset \chi_V\mathrm{St}_l \times \cdots \times \chi_V\mathrm{St}_l \rtimes \pi_0,$$

where $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G(W_{n-2lh}))$ has the L-parameter (ϕ_0, η_0) given by $\phi_0 = \phi - (\chi_V\mathrm{St}_l)^{\oplus 2h}$ and $\eta_0 = \eta|A_{\phi_0}$. Then

$$m_0 := m^{\mathrm{up}}(\pi_0) = m - 2lh - 2 \quad \text{and}$$
$$\theta_{m_0}(\phi_0) = \left(\phi\otimes\chi_V^{-1}\chi_W\right) - (\chi_W S_l)^{\oplus(2h-1)}.$$

In particular, there is a canonical identification $A_{\theta_{m_0}(\phi_0)} = A_{\theta_m(\phi)}$. Moreover, we have $\theta_m(\eta)|A_{\theta_{m_0}(\phi_0)} = \theta_{m_0}(\eta_0)$.

(3) *Suppose that $m > m_1 := m^{\mathrm{up}}(\pi)$. Then $\theta_m(\phi)$ is equal to*

$$\theta_{m_1}(\phi) \oplus \left(\bigoplus_{i=1}^{(m-m_1)/2}\left(\chi_W\,|\cdot|_E^{\frac{m-n-\epsilon_0+1}{2}-i} \oplus \chi_W\,|\cdot|_E^{-\frac{m-n-\epsilon_0+1}{2}+i}\right)\right).$$

In particular, there is a canonical identification $A_{\theta_m(\phi)} = A_{\theta_{m_1}(\phi)}$. Moreover, we have $\theta_m(\eta)|A_{\theta_{m_1}(\phi)} = \theta_{m_1}(\eta)$.

(4) *If $(G(W_n), H(V_m)) = (\mathrm{Mp}(W_n), \mathrm{O}(V_m))$ with odd m, then $\theta_{V_m,W_n}(\pi)$ is parametrized by $(\theta_m(\phi), \theta_m(\eta), \nu_m(\phi,\eta))$ with*

$$\nu_m(\phi,\eta) = \eta(z_\phi)\cdot\varepsilon(\phi)\cdot\chi_V(-1)^{\frac{n}{2}}.$$

Remark 4.6 Note that in (1), A_ϕ can have index 2 in $A_{\theta_m(\phi)}$. In this case, we see that

$$A_{\theta_m(\phi)} = A_\phi \oplus (\mathbb{Z}/2\mathbb{Z})e_{l(\pi)+2}.$$

By Theorem 4.1 (1), we have $\theta_m(\eta)(e_{l(\pi)+2}) = -\theta_m(\eta)(e_{l(\pi)})$. Together with this equation, we see that (1) describes $\theta_m(\eta)$ completely.

Under the assumption of Theorem 4.5 (1), we will show that $\theta_{V_m,W_n}(\pi)$ is tempered (Corollary 6.13). If we knew the temperedness of $\theta_{V_m,W_n}(\pi)$, we obtain Theorem 4.5 (1) by applying Theorem 4.3 (1) to $\theta_{V_m,W_n}(\pi)$. The assertions (2) and (3) will be shown in Sects. 6.6 and 6.7, respectively. The assertion (4) is Propositions 6.19.

The twisted epsilon factors appearing in Theorems 4.3 and 4.5 can be computed by using the following lemma.

Lemma 4.7 *Let $l \geq 3$ be an integer, χ_V be a character of $E^\times$ and ϕ be a representation of WD_E such that $\phi\chi_V^{-1}$ is (conjugate) self-dual of sign $(-1)^{l-1}$.*

(1) *If $E = F$ and l is even, then*

$$\varepsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}\right) = (-1)^{m_\phi(\chi_V S_{l-2})+\cdots+m_\phi(\chi_V S_2)} \cdot \varepsilon\left(\phi\chi_V^{-1}\right).$$

(2) *If $E = F$ and l is odd, then*

$$\varepsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi\chi_V^{-1}\right)(-1)^{\frac{l-1}{2}} = (-1)^{m_\phi(\chi_V S_{l-2})+\cdots+m_\phi(\chi_V S_1)}.$$

(3) *If $E \neq F$ and l is even, then*

$$\varepsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}, \psi_2^E\right) = (-1)^{m_\phi(\chi_V S_{l-2})+\cdots+m_\phi(\chi_V S_2)} \cdot \varepsilon\left(\phi\chi_V^{-1}, \psi_2^E\right).$$

(4) *If $E \neq F$ and l is odd, then*

$$\varepsilon(\phi\chi_V^{-1} \otimes S_{l-1}, \psi_2^E) = (-1)^{m_\phi(\chi_V S_{l-2})+\cdots+m_\phi(\chi_V S_1)}.$$

Proof This follows from Lemma A.4. □

5 Irreducibility and temperedness of theta lifts

In this section, we recall Kudla's filtration of the normalized Jacquet module of Weil representations, and prove the irreducibility and temperedness of theta lifts.

5.1 Kudla's filtration and irreducibility of big theta lifts

Let (V_m, W_n) be a pair of spaces as in Sect. 2.2. We denote the anisotropic space in the Witt tower $\mathcal{V} = \{V_m\}$ by V_{an}. Decompose

$$W_n = X_k + W_{n-2k} + X_k^* \quad \text{and} \quad V_m = Y_a + V_{m-2a} + Y_a^*,$$

where X_k, X_k^* (resp. Y_a, Y_a^*) are k-dimensional (resp. a-dimensional) isotropic subspaces of W_n (resp. V_m) such that $X_k + X_k^*$ (resp. $Y_a + Y_a^*$) is non-degenerate, and W_{n-2k} (resp. V_{m-2a}) is the orthogonal complement of $X_k + X_k^*$ (resp. $Y_a + Y_a^*$) in W_n (resp. V_m). Let $P(X_k)$ (resp. $Q(Y_a)$) be the maximal parabolic subgroup of $G(W_n)$ (resp. $H(V_m)$) stabilizing X_k (resp. Y_a). We denote the normalized Jacquet functor with respect to a parabolic subgroup P by R_P.

The following lemma is called Kudla's filtration.

Lemma 5.1 ([24]) *The normalized Jacquet module $R_{P(X_k)}(\omega_{V_m,W_n})$ has an equivariant filtration*

$$R_{P(X_k)}(\omega_{V_m,W_n}) = R^0 \supset R^1 \supset \cdots \supset R^k \supset R^{k+1} = 0,$$

whose successive quotient $J^a = R^a/R^{a+1}$ is described as follows:

$$J^a = \mathrm{Ind}_{P(X_{k-a},X_k)\times G(W_{n-2k})\times Q(Y_a)}^{\mathrm{GL}(X_k)\times G(W_{n-2k})\times H(V_m)} \left(\chi_V \left|\det_{X_{k-a}}\right|_E^{\lambda_{k-a}} \otimes \mathcal{S}(\mathrm{Isom}(Y_a, X_a')) \otimes \omega_{V_{m-2a},W_{n-2k}}\right),$$

where

- $\lambda_{k-a} = (m - n + k - a - \epsilon_0)/2$;
- $X_k = X_{k-a} + X_a'$ *with* $\dim(X_{k-a}) = k - a$ *and* $\dim(X_a') = a$, *and* $P(X_{k-a}, X_k)$ *is the maximal parabolic subgroup of* $\mathrm{GL}(X_k)$ *stabilizing* X_{k-a};
- $\mathrm{Isom}(Y_a, X_a')$ *is the set of invertible E-conjugate linear maps from Y_a to X_a' and* $\mathcal{S}(\mathrm{Isom}(Y_a, X_a'))$ *is the space of locally constant compactly supported functions on* $\mathrm{Isom}(Y_a, X_a')$;
- $\mathrm{GL}(X_a') \times \mathrm{GL}(Y_a)$ *acts on* $\mathcal{S}(\mathrm{Isom}(Y_a, X_a'))$ *as*

$$((g,h)\cdot f)(x) = \chi_V(\det(g))\chi_W(\det(h)) f(g^{-1}\cdot x \cdot h)$$

  for $(g,h) \in \mathrm{GL}(X_a') \times \mathrm{GL}(Y_a)$, $f \in \mathcal{S}(\mathrm{Isom}(Y_a, X_a'))$ *and* $x \in \mathrm{Isom}(Y_a, X_a')$.

If $m - 2a < \dim(V_{\mathrm{an}})$, we interpret R^a and J^a to be 0.

For a representation $\mathcal{U}$ of a totally disconnected locally compact group G, we denote by $\mathcal{U}_\infty$ the smooth part of $\mathcal{U}$, i.e., the G-submodule of smooth vectors in $\mathcal{U}$. Note that for $\pi \in \mathrm{Irr}(G(W_n))$, we have an isomorphism

$$\mathrm{Hom}_{G(W_n)}\left(\omega_{V_m,W_n},\pi\right)_\infty \cong \Theta_{V_m,W_n}(\pi)^\vee$$

as representations of $H(V_m)$. In the following proposition, in order to simplify notation, we will also use $\Theta_{V_m,W_n}(\pi)^\vee$ to denote the left hand side of the above equation when π is admissible but possibly reducible.

The following proposition is useful.

Proposition 5.2 *Assume that $l = n - m + \epsilon_0 > 0$ and $k > 0$. Let π_0 be an admissible representation of $G(W_{n-2k})$, and τ be an irreducible essentially discrete series representation of $\mathrm{GL}(X_k)$. Then we have: the space $\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n-2k})}(J^a, \chi_V{}^c\tau^\vee \otimes \pi_0)_\infty$ is isomorphic to*

$$\begin{cases} \mathrm{Ind}_{Q(Y_k)}^{H(V_m)}\left(\chi_W^{-1}\tau^\vee \otimes \Theta_{V_{m-2k},W_{n-2k}}(\pi_0)^\vee\right) & \text{if } a = k, \\ \mathrm{Ind}_{Q(Y_{k-1})}^{H(V_m)}\left(\chi_W^{-1}\mathrm{St}_{k-1}\left|\det_{Y_{k-1}}\right|_E^{\frac{k-l+1}{2}} \otimes \Theta_{V_{m-2k+2},W_{n-2k}}(\pi_0)^\vee\right) & \text{if } a = k-1 \text{ and } \tau = \mathrm{St}_k\left|\det_{X_k}\right|_E^{\frac{l-k}{2}}, \\ 0 & \text{otherwise} \end{cases}$$

as representations of $H(V_m)$.

Proof We put $\tau' = {}^c\tau^\vee$. For $a = k$, it is easy to see that

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n-2k})}\left(J^k, \chi_V\tau'\otimes\pi_0\right)_\infty \\ &\quad\cong \mathrm{Ind}_{Q(Y_k)}^{H(V_m)}\left(\chi_W^{-1}\tau^\vee \otimes \mathrm{Hom}_{G(W_{n-2k})}\left(\omega_{V_{m-2k},W_{n-2k}},\pi_0\right)_\infty\right)\end{aligned}$$

(c.f., [13, p. 1674–1676]).

Next, we assume that $a < k$. By Bernstein's Frobenius reciprocity, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n-2k})}\left(J^a, \chi_V\tau'\otimes\pi_0\right) \\ &\quad\cong \mathrm{Hom}_{\mathrm{GL}(X_{k-a})\times\mathrm{GL}(X'_a)\times G(W_{n-2k})}\Big(\chi_V\left|\det_{X_{k-a}}\right|_E^{\lambda_{k-a}} \otimes \mathcal{S}\left(\mathrm{Isom}(Y_a,X'_a)\right) \\ &\quad\otimes\omega_{V_{m-2a},W_{n-2k}}, R_{\overline{P(X_{k-a},X_k)}}(\chi_V\tau')\otimes\pi_0\Big),\end{aligned}$$

where $\overline{P(X_{k-a},X_k)}$ is the parabolic subgroup of $\mathrm{GL}(X_k)$ opposite to $P(X_{k-a},X_k)$. By [53, Proposition 9.5], the normalized Jacquet module $R_{\overline{P(X_{k-a},X_k)}}(\chi_V\tau')$ is given by

$$R_{\overline{P(X_{k-a},X_k)}}(\chi_V\tau') \cong \chi_V\tau_1\left|\det_{X_{k-a}}\right|_E^{e_1} \otimes \chi_V\tau_2\left|\det_{X'_a}\right|_E^{e_2},$$

where τ_1 (resp. τ_2) is an irreducible (unitary) discrete series representation of $\mathrm{GL}(X_{k-a})$ (resp. $\mathrm{GL}(X'_a)$), and $e_1, e_2 \in \mathbb{R}$ such that

$$e_1 < e_2 \quad\text{and}\quad e_1\cdot(k-a) + e_2\cdot a = 0.$$

Since $\mathrm{GL}(X_{k-a})$ acts on $\chi_V|\det_{X_{k-a}}|_E^{\lambda_{k-a}} \otimes \mathcal{S}(\mathrm{Isom}(Y_a, X'_a)) \otimes \omega_{V_{m-2a}, W_{n-2k}}$ by the character $\chi_V|\det_{X_{k-a}}|_E^{\lambda_{k-a}}$, if $\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n-2k})}(J^a, \chi_V\tau'\otimes\pi_0) \neq 0$, then we must have $k-a=1$. Moreover, by results of Zelevinsky (see [38, p. 105]), we must have $\tau' = \mathrm{St}_k|\det_{X_k}|_E^e$ for some $e \in \mathbb{R}$. Then we have

$$e_1 = e - \frac{k-1}{2} \quad \text{and} \quad e_2 = e + \frac{1}{2}.$$

We must have $e_1 = \lambda_1$ so that $e = (k-l)/2$. In this case, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n-2k})}\left(J^{k-1}, \chi_V\tau'\otimes\pi_0\right)_\infty \\
&\quad\cong \mathrm{Hom}_{\mathrm{GL}(X'_{k-1})\times G(W_{n-2k})}\left(\mathcal{S}\left(\mathrm{Isom}\left(Y_{k-1}, X'_{k-1}\right)\right)\otimes\omega_{V_{m-2k+2},W_{n-2k}}, \chi_V\mathrm{St}_{k-1}\left|\det_{X'_{k-1}}\right|_E^{\frac{k-l+1}{2}}\otimes\pi_0\right)_\infty, \\
&\quad\cong \mathrm{Ind}_{Q(Y_{k-1})}^{H(V_m)}\left(\chi_W^{-1}\mathrm{St}_{k-1}\left|\det_{X'_{k-1}}\right|_E^{\frac{k-l+1}{2}}\otimes\mathrm{Hom}_{G(W_{n-2k})}\left(\omega_{V_{m-2k+2},W_{n-2k}}, \pi_0\right)_\infty\right)
\end{aligned}$$

(c.f., [13, p. 1674–1676]). Hence the proposition follows. □

Corollary 5.3 *We put $n_0 = n-2k$ and $m_0 = m-2k$. Let $\pi \in \mathrm{Irr}(G(W_n))$, $\pi_0 \in \mathrm{Irr}(G(W_{n_0}))$ and τ be an irreducible essentially discrete series representation of $\mathrm{GL}(X_k)$. Assume that*

- $l = n-m+\epsilon_0 > 0$;
- $\tau \not\cong \mathrm{St}_k|\det_{X_k}|_E^{\frac{l-k}{2}}$;
- $\mathrm{Ind}_{P(X_k)}^{G(W_n)}(\chi_V\tau\otimes\pi_0) \twoheadrightarrow \pi$.

Then we have

$$\mathrm{Ind}_{Q(Y_k)}^{H(V_m)}\left(\chi_W\tau\otimes\Theta_{V_{m_0},W_{n_0}}(\pi_0)\right) \twoheadrightarrow \Theta_{V_m,W_n}(\pi).$$

Proof By Lemma 2.2, we have $\pi \hookrightarrow \mathrm{Ind}_{P(X_k)}^{G(W_n)}(\chi_V{}^c\tau^\vee\otimes\pi_0)$. Hence we have

$$\begin{aligned}
\Theta_{V_m,W_n}(\pi)^\vee &\cong \mathrm{Hom}_{G(W_n)}\left(\omega_{V_m,W_n}, \pi\right)_\infty \\
&\hookrightarrow \mathrm{Hom}_{G(W_n)}\left(\omega_{V_m,W_n}, \mathrm{Ind}_{P(X_k)}^{G(W_n)}\left(\chi_V{}^c\tau^\vee\otimes\pi_0\right)\right)_\infty \\
&\cong \mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n_0})}\left(R_{P(X_k)}\left(\omega_{V_m,W_n}\right), \chi_V{}^c\tau^\vee\otimes\pi_0\right)_\infty.
\end{aligned}$$

Since $\tau \not\cong \mathrm{St}_k|\det_{X_k}|_E^{\frac{l-k}{2}}$, by Proposition 5.2, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n_0})}\left(R_{P(X_k)}\left(\omega_{V_m,W_n}\right), \chi_V{}^c\tau^\vee \otimes \pi_0\right)_\infty\\ &\hookrightarrow \mathrm{Hom}_{\mathrm{GL}(X_k)\times G(W_{n_0})}\left(J^k, \chi_V{}^c\tau^\vee \otimes \pi_0\right)_\infty\\ &\cong \left(\mathrm{Ind}_{Q(Y_k)}^{H(V_m)}\left(\chi_W\tau \otimes \Theta_{V_{m_0},W_{n_0}}(\pi_0)\right)\right)^\vee.\end{aligned}$$

Taking the contragredient functor, we get the corollary. □

Corollary 5.3 implies an irreducibility condition of big theta lifts.

Proposition 5.4 *Let* $\pi \in \mathrm{Irr}(G(W_n))$ *whose last name is* $\phi \in \Phi(G(W_n))$. *Assume that*

- π *is tempered;*
- $\Theta_{V_m,W_n}(\pi) \neq 0$ *for* $l = n - m + \epsilon_0 > 0$*;*
- ϕ *contains* $\chi_V S_l$ *with multiplicity one.*

Then $\Theta_{V_m,W_n}(\pi)$ *is irreducible and tempered.*

Proof We prove this corollary by induction on n. If π is a discrete series representation, then by a similar argument as that for [11, Proposition C.1], we see that all irreducible subquotients of $\Theta_{V_m,W_n}(\pi)$ are discrete series representations. Hence $\Theta_{V_m,W_n}(\pi)$ is a direct sum of irreducible discrete series representations, and so $\Theta_{V_m,W_n}(\pi)$ is irreducible by the Howe duality conjecture (Theorem 2.3).

Suppose that π is not a discrete series representation. Then there exist $\tau \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}(X_k))$ and $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G(W_{n_0}))$ with $n_0 = n - 2k$ such that $\mathrm{Ind}_{P(X_k)}^{G(W_n)}(\chi_V\tau \otimes \pi_0) \twoheadrightarrow \pi$. By our assumption, $\tau \not\cong \mathrm{St}_l$. Also, $\tau \not\cong \mathrm{St}_k|\det_{X_k}|_E^{\frac{l-k}{2}}$ since τ is discrete series. Hence we can apply Corollary 5.3 to π. We have

$$\mathrm{Ind}_{Q(Y_k)}^{H(V_m)}\left(\chi_W\tau \otimes \Theta_{V_{m_0},W_{n_0}}(\pi_0)\right) \twoheadrightarrow \Theta_{V_m,W_n}(\pi).$$

By the induction hypothesis, we see that $\Theta_{V_{m_0},W_{n_0}}(\pi_0)$ is irreducible and tempered. Hence so is $\Theta_{V_m,W_n}(\pi)$, by the Howe duality conjecture (Theorem 2.3) and the fact that the induced representation is semisimple and tempered. □

5.2 Temperedness of theta lifts 1

First, we prove the following proposition.

Proposition 5.5 *Let* $\pi \in \mathrm{Irr}(G(W_n))$ *be such that* $\Theta_{V_m,W_n}(\pi) \neq 0$. *Assume one of the following:*

(1) *π is tempered and $m \leq n + 1 + \epsilon_0$;*
(2) *π is a discrete series representation and $\Theta_{V_m,W_n}(\pi)$ is the first lift to the going-up tower $\mathcal{V}^{\mathrm{up}}$ so that $m = m^{\mathrm{up}}(\pi)$.*

Then all irreducible subquotients of $\Theta_{V_m,W_n}(\pi)$ are tempered.

Proof The first case is similar to [11, Proposition C.1]. Hence we consider the second case. So we assume that π is a discrete series representation and $m = m^{\mathrm{up}}(\pi)$.

Fix an $H(V_m)$-invariant filtration of $\Theta_{V_m,W_n}(\pi)$:

$$\Theta_{V_m,W_n}(\pi) = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_c \supset \Sigma_{c+1} = 0$$

such that

$$\Pi_i := \Sigma_i / \Sigma_{i+1}$$

is irreducible for any i. Suppose that Π_k is non-tempered. We may assume that Π_i is tempered for $i = 0, \ldots, k-1$. Then there exists a maximal parabolic subgroup Q of $H(V_m)$ with Levi component $L_Q = \mathrm{GL}_t(E) \times H(V_{m_0})$ such that

$$\Pi_k \hookrightarrow \mathrm{Ind}_Q^{H(V_m)}(\tau |\det|_E^{-s_0} \otimes \sigma_0),$$

where $\tau \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_t(E))$, $s_0 > 0$ and $\sigma_0 \in \mathrm{Irr}(H(V_{m_0}))$. By a similar argument as that for [11, Proposition C.1], we have a nonzero $H(V_m)$-map

$$\Theta_{V_m,W_n}(\pi) \to \mathrm{Ind}_Q^{H(V_m)}\left(\tau\, |\det|_E^{-s_0} \otimes \sigma_0\right).$$

Hence we have

$$\begin{aligned} \pi^\vee &\hookrightarrow \mathrm{Hom}_{H(V_m)}\left(\omega_{V_m,W_n}, \mathrm{Ind}_Q^{H(V_m)}\left(\tau\, |\det|_E^{-s_0} \otimes \sigma_0\right)\right) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_t(E)\times H(V_{m_0})}\left(R_Q\left(\omega_{V_m,W_n}\right), \tau\, |\det|_E^{-s_0} \otimes \sigma_0\right), \end{aligned}$$

where R_Q denotes the normalized Jacquet functor with respect to Q. The last Hom space has been studied precisely in the proof of [14, Proposition 3.1]. According to (the proof of) this proposition, one of the following must occur:

(a) $\Theta_{V_{m-2},W_n}(\pi) \neq 0$;
(b) $\mathrm{Ind}_{P(X_a)}^{G(W_n)}(\chi_V \mathrm{St}_a \otimes \pi_0) \twoheadrightarrow \pi$ for some a and $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G(W_{n_0}))$.

However, (a) can not occur since $\Theta_{V_m,W_n}(\pi)$ is the first occurrence. Also, (b) contradicts that π is a discrete series representation. This completes the proof. □

We also need the following proposition in [14]:

Proposition 5.6 ([14, Proposition 3.2]) *Let* $\pi \in \mathrm{Irr}(G(W_n))$. *Assume that* $l = n - m + \epsilon_0 \le 0$ *and* $\theta_{V_m,W_n}(\pi)$ *is nonzero and tempered. We put* $V_{m+2r} = V_m \oplus \mathbb{H}^r$ *for* $r \ge 0$. *Then* $\theta_{V_{m+2r},W_n}(\pi)$ *is the unique irreducible quotient of the standard module*

$$\chi_W\,|\cdot|_E^{\frac{2r-1-l}{2}} \times \chi_W\,|\cdot|_E^{\frac{2r-3-l}{2}} \times \cdots \times \chi_W\,|\cdot|_E^{\frac{1-l}{2}} \rtimes \theta_{V_m,W_n}(\pi).$$

This proposition implies Theorem 4.3 (4). In fact, [14, Proposition 3.2] can be applied to a more general situation as we shall show in Proposition 6.18 below. Theorem 4.5 (3) is proven by showing that we can apply [14, Proposition 3.2] to $\theta_{V_{m^{\mathrm{up}}(\pi)},W_n}(\pi)$, which may be non-tempered, for $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$.

Also, Proposition 5.6 implies the following.

Corollary 5.7 *Let* $\pi \in \mathrm{Irr}(G(W_n))$. *Assume that* $l = n - m + \epsilon_0 < -1$ *and* $\theta_{V_m,W_n}(\pi)$ *is nonzero and tempered. Let* V_{m_0} *be the space which belongs to the same Witt tower as* V_m, *and* $l_0 = n - m_0 + \epsilon_0 = 0$ *or* -1. *Then* $\Theta_{V_{m_0},W_n}(\pi) = 0$.

Proof If $\theta_{V_{m_0},W_n}(\pi)$ were nonzero, it must be tempered by Proposition 5.5, so that $\theta_{V_m,W_n}(\pi)$ is non-tempered by Proposition 5.6. This contradicts the temperedness of $\theta_{V_m,W_n}(\pi)$. □

6 Proof of main theorems

In this section, we prove Theorems 4.1, 4.3 and 4.5.

6.1 Correspondence of last names

First, we study the correspondence of last names.

Proposition 6.1 *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ *with L-parameter* (ϕ, η). *Assume that* $\Theta_{V_m,W_n}(\pi) \ne 0$ *with* $l = n - m + \epsilon_0 > 0$. *Then* ϕ *contains* $\chi_V S_l$.

Proof Consider the standard gamma factors (see Appendix A.1). By Proposition A.1 and Desideratum B.1 (7), the gamma factor

$$\gamma\left(s, \phi \otimes \chi_V^{-1}, \psi_E\right) = \varepsilon\left(s, \phi \otimes \chi_V^{-1}, \psi_E\right) \frac{L\left(1-s, \phi^\vee \otimes \chi_V^{-1}\right)}{L\left(s, \phi \otimes \chi_V^{-1}\right)}$$

has a pole at $s = \frac{l+1}{2}$. This implies that $L(1-s, \phi^\vee \otimes \chi_V^{-1})$ has a pole at $s = \frac{l+1}{2}$. We decompose

$$\phi = \bigoplus_{i \geq 1} \phi_i \otimes S_i,$$

where ϕ_i is a tempered representation of W_E. Then we have

$$L\left(1-s, \phi^\vee \otimes \chi_V^{-1}\right) = \prod_{i \geq 1} L\left(1-s+\frac{i-1}{2}, \phi_i^\vee \otimes \chi_V^{-1}\right).$$

Since ϕ_i is tempered, only $L(1-s+\frac{l-1}{2}, \phi_l \otimes \chi_V^{-1})$ can have a pole at $s = \frac{l+1}{2}$. Moreover, if it has a pole, then $\phi_l \otimes \chi_V^{-1}$ must contain the trivial representation. Hence the proposition follows. □

Corollary 6.2 *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ *with* L*-parameter* (ϕ, η)*. Assume that* $\Theta_{V_m,W_n}(\pi) \neq 0$ *with* $l = n - m + \epsilon_0 > 0$*. Define* $\kappa \in \{1, 2\}$ *by* $\kappa \equiv l \bmod 2$*. Then* ϕ *contains* $\chi_V S_r$ *for* $r = \kappa, \kappa+2, \ldots, l$*. Moreover, the multiplicity* $m_\phi(\chi_V S_r)$ *is odd for* $r = \kappa, \kappa+2, \ldots, l-2$*.*

Proof By Propositions 6.1 and 2.4, we see that ϕ contains $\chi_V S_r$ for $r = \kappa, \kappa+2, \ldots, l$.

By an induction on n, we prove that $m_\phi(\chi_V S_r)$ is odd for any $r = \kappa + 2i$ with $0 \leq i < (l-\kappa)/2$. We may assume that $m_\phi(\chi_V S_r) \geq 2$. Then we can write

$$\phi = \chi_V S_r \oplus \phi_0 \oplus \chi_V S_r$$

for some $\phi_0 \in \Phi_{\mathrm{temp}}(G(W_{n_0}))$ with $n_0 = n - 2r$. We can find $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G(W_{n_0}))$ such that there is a surjection $\mathrm{Ind}_{P(X_r)}^{G(W_n)}(\chi_V \mathrm{St}_r \otimes \pi_0) \twoheadrightarrow \pi$. Then the L-parameter of π_0 is given by $(\phi_0, \eta|A_{\phi_0})$. Since $r < l$, by Corollary 5.3, we have a surjection $\mathrm{Ind}_{Q(Y_r)}^{H(V_m)}(\chi_W \mathrm{St}_r \otimes \Theta_{V_{m_0},W_{n_0}}(\pi_0)) \twoheadrightarrow \Theta_{V_m,W_n}(\pi)$ with $m_0 = m - 2r$. In particular, $\Theta_{V_{m_0},W_{n_0}}(\pi_0)$ is nonzero. Since $n_0 - m_0 + \epsilon_0 = l$, by the induction hypothesis, we see that $m_{\phi_0}(\chi_V S_r)$ is odd. Therefore $m_\phi(\chi_V S_r) = m_{\phi_0}(\chi_V S_r) + 2$ is also odd. □

Corollary 6.2 gives the (chain condition) and the (odd-ness condition) in Theorem 4.1 (1). Note that it is possible that $m_\phi(\chi_V S_l)$ is even as we shall see later. The parity of $m_\phi(\chi_V S_l)$ determines the temperedness of the first occurrence $\theta_{V'_{m^{\mathrm{up}}(\pi)},W_n}(\pi)$ to the going-up tower (Corollary 6.13).

Next, we determine the last name of theta lifts in a special case for the going-down tower.

Theorem 6.3 *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ whose last name is $\phi \in \Phi_{\mathrm{temp}}(G(W_n))$. Assume that $\Theta_{V_m,W_n}(\pi) \neq 0$ with $l = n - m + \epsilon_0 > 0$. Put*

$$\theta_{V_m,W_n}(\phi) = \left(\phi \otimes \chi_V^{-1}\chi_W\right) - \chi_W S_l.$$

Then $\theta_{V_m,W_n}(\phi) \in \Phi(H(V_m))$ and it is the last name of $\theta_{V_m,W_n}(\pi)$.

Proof Let $\phi_{\theta(\pi)}$ be the last name of $\theta_{V_m,W_n}(\pi)$. Consider the Plancherel measure (see Appendix A.2). By Theorem A.2, we have

$$\begin{aligned}
&\mu_\psi\left(\tau_s\chi_W \otimes \theta_{V_m,W_n}(\pi)\right)\\
&\quad = \mu_\psi\left(\tau_s\chi_V \otimes \pi\right)\gamma\left(s - \frac{l-1}{2}, \tau, \psi_E\right)^{-1}\gamma\left(-s - \frac{l-1}{2}, \tau^\vee, \psi_E^{-1}\right)^{-1}
\end{aligned}$$

for any supercuspidal representation τ of $\mathrm{GL}_k(E)$. Using Desideratum B.1 (8) and Lemma A.3, for any irreducible representation ϕ_τ of W_E, we have

$$\begin{aligned}
&\gamma\left(s, \phi_\tau\chi_W \otimes \phi_{\theta(\pi)}^\vee, \psi_E\right)\gamma\left(-s, (\phi_\tau\chi_W)^\vee \otimes \phi_{\theta(\pi)}, \psi_E^{-1}\right)\\
&\quad = \frac{\gamma\left(s, \phi_\tau\chi_V \otimes \phi^\vee, \psi_E\right)\gamma\left(-s, (\phi_\tau\chi_V)^\vee \otimes \phi, \psi_E^{-1}\right)}{\gamma\left(s - \frac{l-1}{2}, \phi_\tau, \psi_E\right)\gamma\left(-s - \frac{l-1}{2}, \phi_\tau^\vee, \psi_E^{-1}\right)}\\
&\quad = \gamma\left(s, \phi_\tau\chi_W \otimes \theta_{V_m,W_n}(\phi)^\vee, \psi_E\right)\gamma\left(-s, (\phi_\tau\chi_W)^\vee \otimes \theta_{V_m,W_n}(\phi), \psi_E^{-1}\right).
\end{aligned}$$

By Proposition 5.5 (1), we see that $\phi_{\theta(\pi)}$ is tempered. Hence by Lemma A.6, we have

$$\phi_{\theta(\pi)} = \theta_{V_m,W_n}(\phi),$$

as desired. In particular, we have $\theta_{V_m,W_n}(\phi) \in \Phi(H(V_m))$. □

6.2 Correspondence of first names

In this subsection, we compare the first name of $\theta_{V_m,W_n}(\pi)$ with the one of π. To do this, we need the following lemma.

Lemma 6.4 *Let $\pi \in \mathrm{Irr}(G(W_n))$. Assume that $\Theta_{V_m,W_n}(\pi) \neq 0$ and all irreducible subquotients of $\Theta_{V_m,W_n}(\pi)$ are tempered. Then all irreducible subquotients of $\Theta_{V_m,W_n}(\pi)$ belong to the same L-packet.*

Proof This follows from [12, Lemma A.1], [11, Lemma B.2, Proposition B.3] and [12, Lemma A.6]. □

In the following theorem, to avoid a confusion, we denote the characters associated to V_m and W_n by χ_{V_m} and χ_{W_n}, respectively.

Theorem 6.5 *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ *with* L*-parameter* (ϕ, η)*. Assume that* $\Theta_{V_m,W_n}(\pi) \neq 0$ *with* $l = n-m+\epsilon_0 > 1$*. Let* $(\theta(\phi), \theta(\eta))$ *be the* L*-parameter for* $\theta_{V_m,W_n}(\pi) \in \mathrm{Irr}(H(V_m))$*. Then we have*

$$\theta(\eta)(a)/\eta(a) = \begin{cases} \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon(\phi^a) \cdot \chi_{V_m}(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd}, \\ \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon\left(\phi^a \chi_{W_n}\right) \cdot \chi_{W_n}(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F, \epsilon = -1 \text{ and } n \text{ is odd}, \\ \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a \chi_{V_m}^{-1}\right)(-1)^{\frac{l-1}{2}} \cdot \nu^{\det(a)} & \text{if } E = F \text{ and } m, n \text{ are even}, \\ \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}, \psi_2^E\right) & \text{if } E \neq F, \end{cases}$$

where the constant $\nu \in \{\pm 1\}$ *is given by*

$$\nu = (-1)^{\frac{l-1}{2}} \cdot \eta(e_1 + e_l).$$

Proof If $E \neq F$, we choose a character χ of $E^\times$ such that $\chi | F^\times = \omega_{E/F}$. We shall treat the cases $\epsilon = +1$ and $\epsilon = -1$ separately.

Suppose that $\epsilon = +1$. Put

$$\omega = \begin{cases} \omega_\psi & \text{if } E = F, \\ \omega_{\psi,\chi} & \text{if } E \neq F. \end{cases}$$

Let L be the Hermitian space of dimension 1 such that

$$\mathrm{disc}(L) = \begin{cases} (-1)^{m+1} & \text{if } E = F, \\ (-1)^m & \text{if } E \neq F. \end{cases}$$

Put $V_{m+1} = V_m \oplus L$. If $E \neq F$, we set $\chi_L = \chi^{(-1)^m}$ and $\chi_{V_{m+1}} = \chi_{V_m}\chi_L$. We denote by $(G'(W_n), H(V_{m+1}))$ the pair of groups associated to (V_{m+1}, W_n) defined in Sect. 2.3. By Lemma C.6, we can find $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G'(W_n))$ such that

$$\begin{cases} \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \pi', \omega\right) \neq 0 & \text{if } E{=}F \text{ and } m \equiv 0 \bmod 2, \text{ or } E \neq F \text{ and } m \equiv 1 \bmod 2, \\ \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \pi', \overline{\omega}\right) \neq 0 & \text{otherwise} \end{cases}$$

so that

$$\begin{cases} \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \overline{\omega}, \pi'^\vee\right) \neq 0 & \text{if } E = F \text{ and } m \equiv 0 \bmod 2, \text{ or } E \neq F \text{ and } m \equiv 1 \bmod 2, \\ \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \omega, \pi'^\vee\right) \neq 0 & \text{otherwise.} \end{cases}$$

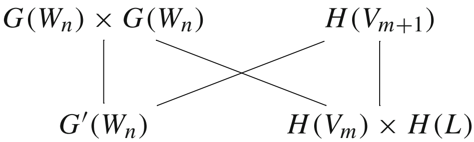
We put $\sigma = \theta_{V_m,W_n}(\pi) \in \mathrm{Irr}_{\mathrm{temp}}(H(V_m))$. Since $\pi \cong \theta_{W_n,V_m}(\sigma)$, we have

$$\mathrm{Hom}_{G'(W_n)}\left(\Theta_{W_n,V_m}(\sigma) \otimes \overline{\omega}, \pi'^{\vee}\right) \supset \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \overline{\omega}, \pi'^{\vee}\right) \neq 0$$

or

$$\mathrm{Hom}_{G'(W_n)}\left(\Theta_{W_n,V_m}(\sigma) \otimes \omega, \pi'^{\vee}\right) \supset \mathrm{Hom}_{G'(W_n)}\left(\pi \otimes \omega, \pi'^{\vee}\right) \neq 0.$$

The see-saw diagram



implies that

$$\mathrm{Hom}_{H(V_m)}\left(\Theta_{V_{m+1},W_n}(\pi'^{\vee}), \sigma\right) \neq 0.$$

Hence $\Theta_{V_{m+1},W_n}(\pi'^{\vee})$ has an irreducible subquotient σ' such that

$$\mathrm{Hom}_{H(V_m)}(\sigma', \sigma) \neq 0 \quad \text{so that} \quad \mathrm{Hom}_{H(V_m)}\left(\sigma^{\vee} \otimes \sigma', \mathbb{C}\right) \neq 0.$$

Since $\sigma^{\vee}$ and σ' are tempered, they are unitary, so that $\overline{\sigma^{\vee}} \cong \sigma$ and $\overline{\sigma'} \cong \sigma'^{\vee}$. Hence we have

$$\mathrm{Hom}_{H(V_m)}\left(\sigma \otimes \sigma'^{\vee}, \mathbb{C}\right) \neq 0.$$

By the GP conjectures (Theorems C.1 – C.4 and Corollary C.5), we have

$$\eta(a) = \begin{cases} \varepsilon\left(\phi^a \otimes \phi_{\pi'}\chi_{-1}\right) \cdot \varepsilon(\phi^a) \cdot \chi_{-1}(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F \text{ and } m \text{ is odd,} \\ \varepsilon\left(\phi^a \otimes \phi_{\pi'}\right) \cdot \varepsilon\left(\phi \otimes \phi_{\pi'}\right)^{\det(a)} \cdot \det(\phi^a)(-1)^{\frac{1}{2}\dim(\phi_{\pi'})} & \text{if } E = F \text{ and } m \text{ is even,} \\ \varepsilon\left(\phi^a \otimes \phi_{\pi'} \otimes \chi^{-1}, \psi_2^E\right) & \text{if } E \neq F \text{ and } m \text{ is odd,} \\ \omega_{E/F}(-1)^{\dim(\phi^a)} \cdot \varepsilon\left(\phi^a \otimes \phi_{\pi'} \otimes \chi, \psi_2^E\right) & \text{if } E \neq F \text{ and } m \text{ is even,} \end{cases}$$

$$\theta(\eta)(a) = \begin{cases} \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^{\vee}}\right) \cdot \det\left(\phi_{\sigma'^{\vee}}\right)(-1)^{\frac{1}{2}\dim(\theta(\phi)^a)} & \text{if } E = F \text{ and } m \text{ is odd,} \\ \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^{\vee}}\right) \cdot \det(\theta(\phi)^a)(-1)^{\frac{1}{2}\dim(\phi_{\sigma'^{\vee}})} \cdot \nu\left(\sigma'^{\vee}\right)^{\det(a)} & \text{if } E = F \text{ and } m \text{ is even,} \\ \omega_{E/F}(-1)^{(m+1)\dim(\theta(\phi)^a)} \cdot \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^{\vee}}, \psi_2^E\right) & \text{if } E \neq F \end{cases}$$

for $a \in A_{\theta(\phi)} \subset A_{\phi}$. Here,

- $\phi_{\pi'}$ and $\phi_{\sigma'^{\vee}}$ are the last names of π' and $\sigma'^{\vee}$, respectively;
- $\nu(\sigma'^{\vee}) \in \{\pm 1\}$ is the central value of $\sigma'^{\vee}$, i.e., $\sigma'^{\vee}(-\mathbf{1}_{V_{m+1}}) = \nu(\sigma'^{\vee}) \cdot \mathrm{id}$.

By Theorem 6.3, Lemma 6.4, Proposition B.4 and Theorem B.8, we have

$$\theta(\phi) = \left(\phi \otimes \chi_{V_m}^{-1} - S_l\right) \otimes \chi_{W_n} \quad \text{so that} \quad \theta(\phi)^a = \phi^a \otimes \chi_{V_m}^{-1} \chi_{W_n}$$

and

$$\phi_{\sigma'^\vee} = \begin{cases} \left(\phi_{\pi'} \otimes \chi_{V_{m+1}} - S_{l-1}\right) \otimes \chi_{W_n}^{-1} & \text{if } E \neq F \text{ or } m \text{ is odd,} \\ \left(\phi_{\pi'} \otimes \chi_{V_{m+1}} \chi_{-1} - S_{l-1}\right) \otimes \chi_{W_n}^{-1} & \text{if } E = F \text{ and } m \text{ is even.} \end{cases}$$

Therefore we have

$$\theta(\eta)(a)/\eta(a) = \begin{cases} \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon(\phi^a) \cdot \chi_{V_m}(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F \text{ and } m \text{ is odd,} \\ \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a \chi_{V_m}^{-1}\right) (-1)^{\frac{l-1}{2}} \cdot \nu^{\det(a)} & \text{if } E = F \text{ and } m \text{ is even,} \\ \varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}, \psi_2^E\right) & \text{if } E \neq F, \end{cases}$$

for some constant $\nu \in \{\pm 1\}$.

We shall determine this constant $\nu \in \{\pm 1\}$. So we assume that $E = F$ and m is even, hence $G(W_n) = \mathrm{Sp}(W_n)$ and $H(V_m) = \mathrm{O}(V_m)$. Since $\sigma = \theta_{V_m,W_n}(\pi) \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_m))$ satisfies that $\pi \cong \theta_{W_n,V_m}(\sigma)$ is nonzero and tempered, by Corollary 5.7, we have $\Theta_{W_m,V_m}(\sigma) = 0$. By Prasad conjecture (Theorem D.2), we have $\theta(\eta)(z_{\theta(\phi)} + e_1) = -1$. Since $z_{\theta(\phi)} + e_1 = z_\phi + e_1 + e_l$ in A_ϕ, we have $\eta(z_{\theta(\phi)} + e_1) = \eta(e_1 + e_l)$. On the other hand, if $a = z_\phi + e_1 + e_l$, we have

$$\begin{aligned} &\varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a \chi_{V_m}^{-1}\right) (-1)^{\frac{l-1}{2}} \cdot \nu^{\det(a)} \\ &\quad = \varepsilon\left(\phi \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon\left((S_1 \oplus S_l) \otimes S_{l-1}\right) \cdot \chi_{V_m}(-1)^{\frac{l-1}{2}} \cdot \nu. \end{aligned}$$

We have $\varepsilon((S_1 \oplus S_l) \otimes S_{l-1}) = -(-1)^{l-1} = -1$. Also, since $\det(\phi \chi_{V_m}^{-1}) = \chi_{V_m}$, by Lemma A.4 and the (odd-ness condition) proved in Corollary 6.2, we know

$$\begin{aligned} \varepsilon\left(\phi \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \chi_{V_m}(-1)^{\frac{l-1}{2}} &= (-1)^{m_\phi(\chi_{V_m} S_{l-2}) + m_\phi(\chi_{V_m} S_{l-4}) + \cdots + m_\phi(\chi_{V_m} S_1)} \\ &= (-1)^{\frac{l-1}{2}}. \end{aligned}$$

Hence we have $\nu = (-1)^{\frac{l-1}{2}} \cdot \eta(e_1 + e_l)$, as desired.

Now suppose that $\epsilon = -1$. Then $n \geq m - \epsilon_0 + 2$. If $n \leq 2 - \epsilon_0$, then $n = 2 - \epsilon_0$ and $m = 0$. In this case, the only representation of $G(W_n)$ which participates in the theta correspondence with $H(V_0)$ is the trivial representation, so that we have nothing to prove. In the other cases, there is a line L in W_n such that

$$\operatorname{disc}(L) = \begin{cases} (-1)^n & \text{if } E = F, \\ (-1)^{n-1} & \text{if } E \neq F. \end{cases}$$

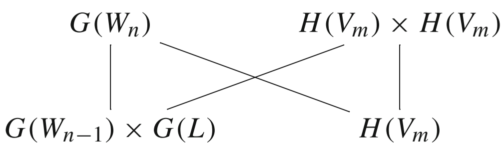
Let W_{n-1} be the orthogonal complement of L in W_n. If $E \neq F$, we set $\chi_L = \chi^{(-1)^{n-1}}$ and $\chi_{W_{n-1}} = \chi_{W_n} \chi^{(-1)^n}$. By Lemma C.6, we can find $\pi' \in \operatorname{Irr}_{\text{temp}}(G(W_{n-1}))$ such that

$$\operatorname{Hom}_{G(W_{n-1})}(\pi \otimes \pi', \mathbb{C}) \neq 0 \quad \text{so that} \quad \operatorname{Hom}_{G(W_{n-1})}(\pi, \pi'^{\vee}) \neq 0.$$

We put $\sigma = \theta_{V_m, W_n}(\pi) \in \operatorname{Irr}_{\text{temp}}(H(V_m))$. Since $\pi \cong \theta_{W_n, V_m}(\sigma)$, we have

$$\operatorname{Hom}_{G(W_{n-1})}\left(\Theta_{W_n, V_m}(\sigma), \pi'^{\vee}\right) \supset \operatorname{Hom}_{G(W_{n-1})}(\pi, \pi'^{\vee}) \neq 0.$$

The see-saw diagram



implies that

$$\begin{cases} \operatorname{Hom}_{H(V_m)}\left(\Theta_{V_m, W_{n-1}}(\pi'^{\vee}) \otimes \omega, \sigma\right) \neq 0 & \text{if } E = F \text{ and } n \equiv 0 \bmod 2, \\ & \text{or } E \neq F \text{ and } n \equiv 1 \bmod 2, \\ \operatorname{Hom}_{H(V_m)}\left(\Theta_{V_m, W_{n-1}}(\pi'^{\vee}) \otimes \overline{\omega}, \sigma\right) \neq 0 & \text{otherwise,} \end{cases}$$

where we put

$$\omega = \begin{cases} \omega_\psi & \text{if } E = F, \\ \omega_{\psi, \chi} & \text{if } E \neq F. \end{cases}$$

Hence $\Theta_{V_m, W_{n-1}}(\pi'^{\vee})$ has an irreducible subquotient σ' such that

$$\begin{cases} \operatorname{Hom}_{H(V_m)}\left(\sigma' \otimes \omega, \sigma\right) \neq 0 & \text{if } E = F \text{ and } n \equiv 0 \bmod 2, \\ & \text{or } E \neq F \text{ and } n \equiv 1 \bmod 2, \\ \operatorname{Hom}_{H(V_m)}\left(\sigma' \otimes \overline{\omega}, \sigma\right) \neq 0 & \text{otherwise,} \end{cases}$$

so that

$$\begin{cases} \mathrm{Hom}_{H(V_m)}\left(\sigma \otimes \sigma'^\vee, \omega\right) \neq 0 & \text{if } E = F \text{ and } n \equiv 0 \bmod 2, \\ & \text{or } E \neq F \text{ and } n \equiv 1 \bmod 2, \\ \mathrm{Hom}_{H(V_m)}\left(\sigma \otimes \sigma'^\vee, \overline{\omega}\right) \neq 0 & \text{otherwise.} \end{cases}$$

Here we use the fact that σ, σ' and ω are unitary. By the GP conjectures (Theorems C.1–C.4 and Corollary C.5), we have

$$\eta(a) = \begin{cases} \varepsilon(\phi^a \otimes \phi_{\pi'}) \cdot \det(\phi_{\pi'})(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F \text{ and } n \text{ is odd}, \\ \varepsilon(\phi^a \otimes \phi_{\pi'}) \cdot \det(\phi^a)(-1)^{\frac{1}{2}\dim(\phi_{\pi'})} \cdot \nu(\pi')^{\det(a)} & \text{if } E = F \text{ and } n \text{ is even}, \\ \omega_{E/F}(-1)^{(n-1)\dim(\phi^a)} \cdot \varepsilon\left(\phi^a \otimes \phi_{\pi'}, \psi_2^E\right) & \text{if } E \neq F, \end{cases}$$

$$\theta(\eta)(a) = \begin{cases} \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^\vee}\chi_{-1}\right) \cdot \varepsilon\left(\theta(\phi)^a\right) \cdot \chi_{-1}(-1)^{\frac{1}{2}\dim(\theta(\phi)^a)} & \text{if } E = F \text{ and } n \text{ is odd}, \\ \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^\vee}\right) \cdot \varepsilon\left(\theta(\phi) \otimes \phi_{\sigma'^\vee}\right)^{\det(a)} & \\ \quad \cdot \det\left(\theta(\phi)^a\right)(-1)^{\frac{1}{2}\dim(\phi_{\sigma'^\vee})} & \text{if } E = F \text{ and } n \text{ is even}, \\ \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^\vee} \otimes \chi^{-1}, \psi_2^E\right) & \text{if } E \neq F \text{ and } n \text{ is odd}, \\ \omega_{E/F}(-1)^{\dim(\theta(\phi)^a)} \cdot \varepsilon\left(\theta(\phi)^a \otimes \phi_{\sigma'^\vee} \otimes \chi, \psi_2^E\right) & \text{if } E \neq F \text{ and } n \text{ is even} \end{cases}$$

for $a \in A_{\theta(\phi)} \subset A_\phi$. Here,

- $\phi_{\pi'}$ and $\phi_{\sigma'^\vee}$ are the last names for π' and $\sigma'^\vee$, respectively;
- $\nu(\pi') \in \{\pm 1\}$ is the central value of π', i.e., $\pi'(-\mathbf{1}_{V_{m+1}}) = \nu(\pi') \cdot \mathrm{id}$.

By Theorem 6.3, Lemma 6.4, Proposition B.4 and Theorem B.8, we have

$$\theta(\phi) = \left(\phi \otimes \chi_{V_m}^{-1} - S_l\right) \otimes \chi_{W_n} \quad \text{so that} \quad \theta(\phi)^a = \phi^a \otimes \chi_{V_m}^{-1}\chi_{W_n}$$

and

$$\phi_{\sigma'^\vee} = \begin{cases} \left(\phi_{\pi'} \otimes \chi_{V_m} - S_{l-1}\right) \otimes \chi_{W_{n-1}}^{-1} & \text{if } E \neq F \text{ or } n \text{ is odd}, \\ \left(\phi_{\pi'} \otimes \chi_{V_m} - S_{l-1}\right) \otimes \chi_{W_{n-1}}^{-1}\chi_{-1} & \text{if } E = F \text{ and } n \text{ is even}. \end{cases}$$

Therefore we have

$$\theta(\eta)(a)/\eta(a) = \begin{cases} \varepsilon\left(\phi^a\chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon\left(\phi^a\chi_{V_m}^{-1}\chi_{W_n}\right) & \\ \quad \cdot \chi_{W_n}(-1)^{\frac{1}{2}\dim(\phi^a)} & \text{if } E = F \text{ and } n \text{ is odd}, \\ \varepsilon\left(\phi^a\chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a\chi_{V_m}^{-1}\right) & \\ \quad (-1)^{\frac{l-1}{2}} \cdot \nu^{\det(a)} & \text{if } E = F \text{ and } n \text{ is even}, \\ \varepsilon\left(\phi^a\chi_{V_m}^{-1} \otimes S_{l-1}, \psi_2^E\right) & \text{if } E \neq F, \end{cases}$$

for some constant $\nu \in \{\pm 1\}$.

We shall determine this constant $\nu \in \{\pm 1\}$. So we assume that $E = F$ and n is even, hence $G(W_n) = \mathrm{O}(W_n)$ and $H(V_m) = \mathrm{Sp}(V_m)$. Note that $\theta(\eta)(z_{\theta(\phi)}) = 1$. Also, by Prasad conjecture (Theorem D.2), we have $\eta(z_\phi + e_1) = 1$. Since $z_{\theta(\phi)} = z_\phi + e_l$ in A_ϕ, we have $\eta(z_{\theta(\phi)}) = \eta(e_1 + e_l)$. On the other hand, if $a = z_\phi + e_l$, we have

$$\begin{aligned} &\varepsilon\left(\phi^a \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \det\left(\phi^a \chi_{V_m}^{-1}\right)(-1)^{\frac{l-1}{2}} \cdot \nu^{\det(a)} \\ &\quad = \varepsilon\left(\phi \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \varepsilon\left(S_l \otimes S_{l-1}\right) \cdot \chi_{W_n}(-1)^{\frac{l-1}{2}} \cdot \nu. \end{aligned}$$

We have $\varepsilon(S_l \otimes S_{l-1}) = (-1)^{l-1} = 1$. Also, by Lemma A.4 and the (odd-ness condition) proved in Corollary 6.2, we have

$$\begin{aligned} \varepsilon\left(\phi \chi_{V_m}^{-1} \otimes S_{l-1}\right) \cdot \chi_{W_n}(-1)^{\frac{l-1}{2}} &= (-1)^{m_\phi(\chi_{V_m} S_{l-2}) + m_\phi(\chi_{V_m} S_{l-4}) + \cdots + m_\phi(\chi_{V_m} S_1)} \\ &= (-1)^{\frac{l-1}{2}}. \end{aligned}$$

Hence we have $\nu = (-1)^{\frac{l-1}{2}} \cdot \eta(e_1 + e_l)$, as desired. This completes the proof. □

Remark 6.6 Suppose that $E = F$ and m, n are even. After Proposition 6.8, which shows the (alternating condition), we will obtain $\eta(e_1 + e_l) = (-1)^{\frac{l-1}{2}}$ so that $\nu = 1$. By using Theorems 4.3 (5) and 4.5 (4), which are proven in Proposition 6.19, we can obtain $\nu = 1$ directly.

6.3 Comparison of central elements

Let (V_m, W_n) and $l = n - m + \epsilon_0$ be as in Sect. 2.2. Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$. Assume that $l \geq 2$ and $\sigma = \theta_{V_m, W_n}(\pi) \neq 0$ so that $\sigma \in \mathrm{Irr}(H(V_m))$. We denote the L-parameters for π and σ by (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$, respectively. In this subsection, we compare "$\eta_\pi(z_{\phi_\pi})$" with "$\eta_\sigma(z_{\phi_\sigma})$".

Let $\phi \in \Phi(G(W_n))$ (resp. $\phi' \in \Phi(H(V_m))$). If $l = n - m + \epsilon_0$ is odd and ϕ contains χ_V (resp. ϕ' contains χ_W), then we denote by e_1 the element in A_ϕ (resp. $A_{\phi'}$) corresponding to χ_V (resp. χ_W), i.e., $\phi^{e_1} = \chi_V$ (resp. $\phi'^{e_1} = \chi_W$) (for the definition of ϕ^a, see Sect. 3).

Proposition 6.7 *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ such that $\sigma = \theta_{V_m, W_n}(\pi) \in \mathrm{Irr}(H(V_m))$ is nonzero. Assume that $l = n - m + \epsilon_0 \geq 2$. We denote the L-parameters for π and σ by (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$, respectively. Then we have the following:*

(1) *If l is odd, then $\phi_\pi \supset \chi_V$ and $\phi_\sigma \supset \chi_W$.*

(2) *If* $E = F$, $m \not\equiv n \bmod 2$ *and* $\epsilon = +1$, *then*

$$\eta_\pi(z_{\phi_\pi}) = \eta_\sigma(z_{\phi_\sigma}) \cdot \varepsilon(\phi_\pi) \cdot \varepsilon(\phi_\pi \otimes \chi_V) \cdot \chi_V(-1)^{\frac{n}{2}}.$$

(3) *If* $E = F$, $m \not\equiv n \bmod 2$ *and* $\epsilon = -1$, *then*

$$\eta_\pi(z_{\phi_\pi}) = -\eta_\sigma(z_{\phi_\sigma}) \cdot \delta(\chi_W = \mathbf{1}) \cdot \varepsilon(\phi_\pi) \cdot \varepsilon(\phi_\pi \otimes \chi_W) \cdot \chi_W(-1)^{\frac{n-1}{2}}.$$

(4) *If* $E = F$, $m \equiv n \bmod 2$ *and* $\epsilon = +1$, *then* $\eta_\pi(z_{\phi_\pi} + e_1) = \eta_\sigma(z_{\phi_\sigma})$.
(5) *If* $E = F$, $m \equiv n \bmod 2$ *and* $\epsilon = -1$, *then* $\eta_\pi(z_{\phi_\pi}) = -\eta_\sigma(z_{\phi_\sigma} + e_1)$.
(6) *If* $E \neq F$ *and* l *is even, then*

$$\eta_\sigma(z_{\phi_\sigma}) = \varepsilon\left(\phi_\pi \otimes \chi_V^{-1}, \psi_2^E\right) \cdot \eta_\pi(z_{\phi_\pi}).$$

(7) *If* $E \neq F$ *and* l *is odd, then* $\eta_\pi(e_1) = -\eta_\sigma(e_1)$ *and* $\eta_\pi(z_{\phi_\pi} + e_1) = \eta_\sigma(z_{\phi_\sigma})$.

Proof (1) follows from Corollary 6.2 and Theorem 6.3.

The proofs of (2)–(7) are similar. So we prove (3) only. By the assumption, $G(W_n) = \mathrm{O}(W_n)$ is an odd orthogonal group and $H(V_m) = \mathrm{Mp}(V_m)$ is a metaplectic group. By Theorem B.6, there is unique $W_{m+1}^\bullet$ such that $\pi' = \theta_{W_{m+1}^\bullet, V_m}(\sigma)$ is nonzero. Let $(\phi_{\pi'}, \eta_{\pi'})$ be the L-parameter for π'. Note that $\theta_{W_n, V_m}(\sigma) = \pi$ is tempered and $m - n - \epsilon_0 < -1$. By applying Corollary 5.7 to $\sigma \in \mathrm{Irr}(\mathrm{Mp}(V_m))$ and $\mathrm{O}(W_n)$, we have $\Theta_{W_{m+1}, V_m}(\sigma) = 0$, where W_{m+1} is the space which belongs to the same Witt tower as W_n. This implies that $W_{m+1}^\bullet \neq W_{m+1}$. Since

$$\eta_\pi(z_{\phi_\pi}) = \begin{cases} +1 & \text{if } \mathrm{O}(W_n) \text{ is split,} \\ -1 & \text{if } \mathrm{O}(W_n) \text{ is not split,} \end{cases}$$
$$\eta_{\pi'}\left(z_{\phi_{\pi'}}\right) = \begin{cases} +1 & \text{if } \mathrm{O}\left(W_{m+1}^\bullet\right) \text{ is split,} \\ -1 & \text{if } \mathrm{O}\left(W_{m+1}^\bullet\right) \text{ is not split,} \end{cases}$$

we have

$$\eta_\pi(z_{\phi_\pi}) = -\eta_{\pi'}(z_{\phi_{\pi'}}).$$

On the other hand, by Theorem B.8, we have

$$\eta_{\pi'}\left(z_{\phi_{\pi'}}\right) = \eta_\sigma(z_{\phi_\sigma}) \cdot \varepsilon(\phi_\sigma) \cdot \varepsilon(\phi_\sigma \otimes \chi_W) \cdot \chi_W(-1)^{\frac{m}{2}}.$$

Since $\phi_\sigma = (\phi_\pi \otimes \chi_V^{-1}\chi_W) \oplus \chi_W S_l$ by Theorem 6.3, using Lemma A.5, we have

$$\eta_{\pi'}(z_{\phi_{\pi'}}) = \eta_\sigma(z_{\phi_\sigma}) \cdot \delta(\chi_W = \mathbf{1}) \cdot \varepsilon(\phi_\pi) \cdot \varepsilon(\phi_\pi \otimes \chi_W) \cdot \chi_W(-1)^{\frac{n-1}{2}}.$$

Hence we obtain (3).

Using Theorems B.8, D.1, D.2, Proposition 2.4, and Corollary 5.7, the proofs of the other cases are similar to that of the above case. □

If $l \in \{-1, 0, 1\}$, then we see that a similar assertion holds by using Theorem B.8 and Prasad's conjectures (Theorems D.1 and D.2). This implies Theorem 4.1 (2) unless $E = F$, $m \not\equiv n \bmod 2$ and $\epsilon = -1$. In this case, $G(W_n) = \mathrm{O}(W_n)$ is an odd orthogonal group, and the first occurrence indices $m^\pm(\pi)$ can be determined by the central character of $\pi \in \mathrm{Irr}(\mathrm{O}(W_n))$. Hence the remaining issue of Theorem 4.1 (2) is a relation between the central character of π and theta lifts $\Theta_{V_m,W_n}(\pi)$. It will be treated in Sect. 6.8 (Proposition 6.20).

6.4 Character conditions

In this subsection, we derive the (initial condition) and the (alternating condition) in Theorem 4.1 (1).

Proposition 6.8 *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η). Assume that $\Theta_{V_m,W_n}(\pi) \neq 0$ and $l = n - m + \epsilon_0 \geq 2$. Define $\kappa \in \{1, 2\}$ by $\kappa \equiv l \bmod 2$. Let e_i be the element in A_ϕ corresponding to $\chi_V S_i \subset \phi$. Then we have*

$$\eta(e_{\kappa+2i+2}) = -\eta(e_{\kappa+2i})$$

for $0 \leq i < (l - \kappa)/2$. Moreover, if $\kappa = 2$, then

$$\eta(e_2) = \begin{cases} \epsilon \cdot \delta(\chi_V = \mathbf{1}) & \text{if } E = F \text{ and } m \not\equiv n \bmod 2, \\ -1 & \text{if } E \neq F \text{ and } m \equiv n \bmod 2. \end{cases}$$

Proof Let $(\theta(\phi), \theta(\eta))$ be the L-parameter for $\theta_{V_m,W_n}(\pi)$. Note that

$$z_\phi = z_{\theta(\phi)} + e_l.$$

By applying Proposition 6.7 and Theorem 6.5 to $a = z_{\theta(\phi)} \in A_{\theta(\phi)} \subset A_\phi$, we have

$$\eta(e_l) = \frac{\eta(z_\phi)}{\theta(\eta)(z_{\theta(\phi)})} \cdot \frac{\theta(\eta)\left(z_{\theta(\phi)}\right)}{\eta\left(z_{\theta(\phi)}\right)}$$
$$= \begin{cases} \delta(\chi_V = \mathbf{1}) \cdot \varepsilon(\phi\chi_V^{-1} \otimes S_{l-1}) \cdot \varepsilon\left(\phi\chi_V^{-1}\right) & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd,} \\ -\varepsilon\left(\phi \otimes S_{l-1}\right) \cdot \varepsilon(\phi) & \text{if } E = F, \epsilon = -1 \text{ and } n \text{ is odd,} \\ \eta(e_1) \cdot \varepsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}\right) \cdot \chi_V(-1)^{\frac{l-1}{2}} & \text{if } E = F, \epsilon = +1 \text{ and } m \equiv n \equiv 0 \bmod 2, \\ -\varepsilon\left(\phi\chi_V^{-1}, \psi_2^E\right) \cdot \varepsilon(\phi\chi_V^{-1} \otimes S_{l-1}, \psi_2^E) & \text{if } E \neq F \text{ and } m \equiv n \bmod 2, \\ \eta(e_1) \cdot \varepsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}, \psi_2^E\right) & \text{if } E \neq F \text{ and } m \not\equiv n \bmod 2. \end{cases}$$

If $E = F$, $\epsilon = -1$ and $m \equiv n \equiv 0 \bmod 2$, applying Proposition 6.7 and Theorem 6.5 to $a = z_{\theta(\phi)} + e_1 = z_\phi + e_1 + e_l \in A_{\theta(\phi)} \subset A_\phi$. we obtain

$$\eta(e_l) = \frac{\eta(z_\phi + e_1)}{\theta(\eta)\left(z_{\theta(\phi)} + e_1\right)} \cdot \frac{\theta(\eta)\left(z_{\theta(\phi)} + e_1\right)}{\eta\left(z_{\theta(\phi)} + e_1\right)}$$
$$= \eta(e_1) \cdot \epsilon\left(\phi\chi_V^{-1} \otimes S_{l-1}\right) \cdot \chi_W(-1)^{\frac{l-1}{2}}.$$

By the tower property (Proposition 2.4), a similar equation for $\eta(e_{l-2i})$ holds for $i = 0, 1, \ldots, (l-\kappa)/2$. In particular, if $\kappa = 2$, then we have

$$\eta(e_2) = \begin{cases} \epsilon \cdot \delta(\chi_V = \mathbf{1}) & \text{if } E = F \text{ and } m \not\equiv n \bmod 2, \\ -1 & \text{if } E \neq F \text{ and } m \equiv n \bmod 2. \end{cases}$$

Moreover, we have

$$\frac{\eta\left(e_{\kappa+2i+2}\right)}{\eta\left(e_{\kappa+2i}\right)} = \frac{\varepsilon\left(\phi\chi_V^{-1} \otimes S_{\kappa+2i+1}, \psi_2^E\right)}{\varepsilon\left(\phi\chi_V^{-1} \otimes S_{\kappa+2i-1}, \psi_2^E\right)} \times \begin{cases} \det\left(\phi\chi_V^{-1}\right)(-1) & \text{if } E = F, \\ 1 & \text{if } E \neq F \end{cases}$$

for $0 \leq i < (l-\kappa)/2$. By Lemma A.4 and (odd-ness condition) in Theorem 4.1 (1), this is equal to $(-1)^{m_\phi(\chi_V S_{\kappa+2i})} = -1$. □

This is the (initial condition) and the (alternating condition) in Theorem 4.1 (1). In particular, we have

$$n - m^{\text{down}}(\pi) + \epsilon_0 \in \mathcal{T}$$

for any $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$.

Also, when $E = F$ and m, n are both even (so that l is odd), we have

$$\eta(e_1 + e_l) = (-1)^{\frac{l-1}{2}}.$$

Theorem 6.5 together with this equation implies Theorem 4.3 (1).

Remark 6.9 We may apply the result shown above (i.e., Theorem 4.3 (1)) to the going-up tower sometimes. Under the notation and assumption of Theorem 4.5 (1), we will show that $\theta_{V_m,W_n}(\pi)$ is tempered (Corollary 6.13). If we knew the temperedness of $\theta_{V_m,W_n}(\pi)$, Theorem 4.3 (1) implies Theorem 4.5 (1).

The following proposition says that $l(\pi) = \max \mathcal{T} = n - m^{\text{down}}(\pi) + \epsilon_0$ in a special case.

Proposition 6.10 *Let* $\pi \in \text{Irr}_{\text{temp}}(G(W_n))$ *with L-parameter* (ϕ, η)*. Assume that*

- $l = n - m^{\text{down}}(\pi) + \epsilon_0 \geq 0$*;*
- ϕ *contains* $\chi_V S_{l-2i}$ *for* $i = -1, 0, \ldots, (l-\kappa)/2$*;*
- *the first occurrence* $\sigma^{\text{up}} = \theta_{V'_{m^{\text{up}}(\pi)},W_n}(\pi)$ *to the going-up tower* $\mathcal{V}^{\text{up}}$ *is tempered.*

Then we have:

(1) *If* $l = 0$*, then the (initial condition) in Theorem 4.1 (1) does not hold. Namely,*

$$\eta(e_2) = \begin{cases} -\epsilon \cdot \delta(\chi_V = \mathbf{1}) & \text{if } E = F \text{ and } m \not\equiv n \bmod 2, \\ +1 & \text{if } E \neq F \text{ and } m \equiv n \bmod 2. \end{cases}$$

(2) *If* $l > 0$*, then* $m_\phi(\chi_V S_l)$ *is odd, and the (alternating condition) in Theorem 4.1 (1) does not hold. Namely,*

$$\eta(e_{l+2} + e_l) = -(-1)^{m_\phi(\chi_V S_l)} = +1.$$

Proof First, we prove (2). Let $(\phi_\sigma, \eta_\sigma)$ be the L-parameter for σ^{up}. Note that σ^{up} is tempered by the assumption, and $m^{\text{up}}(\pi) - n - \epsilon_0 = l + 2 \geq 2$ by the conservation relation (Proposition 2.5). By applying Theorem 6.3, Corollary 6.2 and Proposition 6.8 to σ^{up}, we have

$$\phi_\sigma = \left(\phi \otimes \chi_V^{-1}\chi_W\right) \oplus \chi_W S_{l+2},$$

and we see that $m_{\phi_\sigma}(\chi_W S_l) = m_\phi(\chi_V S_l)$ is odd, and $\eta_\sigma(e_{l+2} + e_l) = -1$. Therefore it is enough to show $\eta(e_{l+2} + e_l)/\eta_\sigma(e_{l+2} + e_l) = -1$. It follows from Theorem 4.3 (1). Hence we have (2). The proof of (1) is similar. □

By Proposition 5.5, if π is a discrete series representation, then the first occurrence $\sigma^{\text{up}} = \theta_{V'_{m^{\text{up}}(\pi)},W_n}(\pi)$ is tempered. Hence by Proposition 6.10, we see that

$$n - m^{\text{down}}(\pi) + \epsilon_0 + 2 = l + 2 \notin \mathcal{T}$$

if π is a discrete series representation. This completes the proof of Theorem 4.1 (1) for discrete series representations.

6.5 Temperedness of theta lifts 2

In this subsection, we discuss whether the first occurrence $\sigma = \theta_{V'_{m^{\mathrm{up}}(\pi)}, W_n}(\pi)$ for the going-up tower $\mathcal{V}^{\mathrm{up}}$ is tempered or not.

Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η). Assume that $l = n - m^{\mathrm{down}}(\pi) + \epsilon_0 \geq 0$. Define $\kappa \in \{1, 2\}$ by $\kappa \equiv l \bmod 2$. Then by Corollary 6.2, we know that ϕ contains $\chi_V S_{\kappa+2i}$ for $0 \leq i \leq (l-\kappa)/2$, and $m_\phi(\chi_V S_{\kappa+2i})$ is odd for $0 \leq i < (l-\kappa)/2$. Note that $m^{\mathrm{up}}(\pi) - n - \epsilon_0 = l + 2 \geq 2$.

Decompose $\phi = \phi' \oplus \phi_0 \oplus {}^c\phi'^\vee$ with $\phi_0 \in \Phi_{\mathrm{disc}}(G(W_{n_0}))$. Assume that

$$\chi_V \tau_1 \times \cdots \times \chi_V \tau_r \rtimes \pi_0 \twoheadrightarrow \pi$$

for some $\tau_i \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_{k_i}(E))$ and $\pi_0 \in \mathrm{Irr}_{\mathrm{disc}}(G(W_{n_0}))$ with $n_0 = n - 2\sum_{i=1}^r k_i$, so that the L-parameter of π_0 is given by $(\phi_0, \eta | A_{\phi_0})$. If $m \geq n + \epsilon_0$, then by a similar argument as that for [11, Proposition C.4], we have

$$\chi_W \tau_1 \times \cdots \times \chi_W \tau_r \rtimes \Theta_{V_{m_0}, W_{n_0}}(\pi_0) \twoheadrightarrow \Theta_{V_m, W_n}(\pi),$$

where $m_0 = m - 2\sum_{i=1}^r k_i$. In particular, if $\Theta_{V_m, W_n}(\pi)$ is nonzero, then $\Theta_{V_{m_0}, W_{n_0}}(\pi_0)$ is also nonzero.

Lemma 6.11 *Suppose that $m^{\mathrm{down}}(\pi) < m^{\mathrm{up}}(\pi)$. Then the going-down tower $\mathcal{V}^{\mathrm{down}}$ with respect to π is also the going-down tower $\mathcal{V}^{\mathrm{down}}$ with respect to π_0.*

Proof Set $m = n + \epsilon_0 + 2 - \kappa$. Then $l = n - m + \epsilon_0 = \kappa - 2 \in \{0, -1\}$. A tower $\mathcal{V}$ is the going-down tower with respect to π if and only if $\Theta_{V_m, W_n}(\pi)$ is nonzero for $V_m \in \mathcal{V}$. In this case, $\Theta_{V_{m_0}, W_{n_0}}(\pi_0)$ is also nonzero for $V_{m_0} \in \mathcal{V}$. This shows that $\mathcal{V}$ is also the going-down tower with respect to π_0. □

We determine the first occurrence index of π_0 in terms of the one of π.

Proposition 6.12 *Let notation be as above. If $m^{\mathrm{down}}(\pi) = n + \epsilon_0 - l$ with $l > 0$, then*

$$m^{\mathrm{down}}(\pi_0) = \begin{cases} n_0 + \epsilon_0 - l & \text{if } m_\phi(\chi_V S_l) \text{ is odd}, \\ n_0 + \epsilon_0 - l + 2 & \text{if } m_\phi(\chi_V S_l) \text{ is even}. \end{cases}$$

Proof Note that we have proven Theorem 4.1 (1) for the discrete series representation π_0. By Corollary 6.2, we see that $m_\phi(\chi_V S_{\kappa+2i})$ is odd for $0 \leq i <$

$(l-\kappa)/2$, where we define $\kappa \in \{1,2\}$ by $\kappa \equiv l \bmod 2$. If $m_\phi(\chi_V S_l)$ is even, then by applying Theorem 4.1 (1) to π_0, we have $m^{\text{down}}(\pi_0) = n_0+\epsilon_0-l+2$.

Suppose that $m_\phi(\chi_V S_l)$ is odd. Note that $m^{\text{up}}(\pi) = n+\epsilon_0+l+2$. By Lemma 6.11 and a remark before this lemma, we have $m^{\text{up}}(\pi_0) \le n_0+\epsilon_0+l+2$. Hence $m^{\text{down}}(\pi_0) \ge n_0+\epsilon_0-l$. On the other hand, by applying Theorem 4.1 (1) to π_0, we have $m^{\text{down}}(\pi_0) \le n_0+\epsilon_0-l$. Therefore we have $m^{\text{down}}(\pi_0) = n_0+\epsilon_0-l$. □

Corollary 6.13 *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ,η). Assume that $m^{\text{down}}(\pi) = n+\epsilon_0-l$ with $l \ge 0$, so that $m^{\text{up}}(\pi) = n+\epsilon_0+l+2$. Let $\sigma = \theta_{V'_{m^{\text{up}}(\pi)},W_n}(\pi)$ be the first occurrence for the going-up tower $\mathcal{V}^{\text{up}}$.*

(1) *If $l=0$, then σ is tempered.*

(2) *Suppose that $l>0$. Then σ is tempered if and only if $m_\phi(\chi_V S_l)$ is odd.*

Proof We prove (2). The proof of (1) is similar. So we assume that $l>0$.

If σ is tempered, then we have proven that $m_\phi(\chi_V S_l)$ is odd in Proposition 6.10.

Conversely, suppose that $m_\phi(\chi_V S_l)$ is odd. We may assume that

$$\chi_V\tau_1 \times \cdots \times \chi_V\tau_r \rtimes \pi_0 \twoheadrightarrow \pi$$

for some $\tau_i \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_{k_i}(E))$ and $\pi_0 \in \mathrm{Irr}_{\mathrm{disc}}(G(W_{n_0}))$ with $n_0 = n - 2\sum_{i=1}^r k_i$. As we have seen before Lemma 6.11, we have

$$\chi_W\tau_1 \times \cdots \times \chi_W\tau_r \rtimes \Theta_{V'_{m_0},W_{n_0}}(\pi_0) \twoheadrightarrow \Theta_{V'_m,W_n}(\pi),$$

where $m_0 = m - 2\sum_{i=1}^r k_i$ and $m = m^{\text{up}}(\pi)$. Hence there exists an irreducible subquotient σ_0 of $\Theta_{V'_{m_0},W_{n_0}}(\pi_0)$ such that

$$\chi_W\tau_1 \times \cdots \times \chi_W\tau_r \rtimes \sigma_0 \twoheadrightarrow \sigma.$$

Since $m_\phi(\chi_V S_l)$ is odd, by Proposition 6.12 together with the conservation relation (Proposition 2.5), we see that $\Theta_{V'_{m_0},W_{n_0}}(\pi_0)$ is the first lift of a discrete series representation π_0 to the going-up tower $\mathcal{V}^{\text{up}}$. By Proposition 5.5 (2), the irreducible subquotient σ_0 of $\Theta_{V'_{m_0},W_{n_0}}(\pi_0)$ is tempered. Therefore, σ is also tempered. □

Corollary 6.13 and Proposition 6.12 imply that

$$n - m^{\text{down}}(\pi) + \epsilon_0 + 2 = l + 2 \notin \mathcal{T}$$

for any tempered representation π. Hence we have $l(\pi) = \max \mathcal{T} = n - m^{\text{down}}(\pi) + \epsilon_0$. This completes the proof of Theorem 4.1 (1). Also, using Corollary 6.13, we obtain Theorem 4.5 (1) from Theorem 4.3 (1), as we noted in Remark 6.9.

6.6 Non-tempered first lifts

In this subsection, we prove Theorem 4.5 (2).

Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η). Assume that $l = l(\pi) = n - m^{\mathrm{down}}(\pi) + \epsilon_0 > 0$. Theorem 4.1 (1) implies that

- ϕ contains $\chi_V S_l, \chi_V S_{l-2}, \ldots, \chi_V S_\kappa$, where $\kappa \in \{1, 2\}$ is defined by $\kappa \equiv l \bmod 2$;
- $m_\phi(\chi_V S_{\kappa+2i})$ is odd for $0 \le i < (l - \kappa)/2$.

We put $m = m^{\mathrm{up}}(\pi)$. Note that $m - n - \epsilon_0 = l + 2$. Let $\sigma = \theta_{V_m, W_n}(\pi)$ be the first occurrence of π to the going-up tower $\mathcal{V}^{\mathrm{up}}$. By Corollary 6.13, we see that σ is non-tempered if and only if $m_\phi(\chi_V S_l)$ is even. In this subsection, we assume these conditions.

Suppose that σ is the Langlands quotient of the standard module

$$\tau_1 \, |\cdot|_E^{s_1} \times \cdots \times \tau_r \, |\cdot|_E^{s_r} \rtimes \sigma_0,$$

where $\tau_i \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_{k_i}(E)), \sigma_0 \in \mathrm{Irr}_{\mathrm{temp}}(H(V_{m_0})), 2k_1 + \cdots + 2k_r + m_0 = m$, and $s_1 \ge \cdots \ge s_r > 0$.

First, we have the following:

Proposition 6.14 *For any $i = 1, \ldots, r$, the exponent s_i is in $(1/2)\mathbb{Z}$.*

Proof Consider the Plancherel measure (see Appendix A.2). By Theorem A.2, we have

$$\begin{aligned}
&\mu\left(\chi_W \tau \, |\cdot|_E^s \otimes \sigma\right) \\
&\quad = \mu\left(\chi_V \tau \, |\cdot|_E^s \otimes \pi\right) \cdot \gamma\left(s - \frac{l-1}{2}, \tau, \psi_E\right)^{-1} \cdot \gamma\left(-s - \frac{l-1}{2}, \tau^\vee, \psi_E^{-1}\right)^{-1}
\end{aligned}$$

for any $\tau \in \mathrm{Irr}(\mathrm{GL}_k(E))$. In particular, by Desideratum B.1 (8), we have

$$\begin{aligned}
&\gamma\left(s, \chi_W \phi_\tau \otimes \phi_\sigma^\vee, \psi_E\right) \cdot \gamma\left(-s, \chi_W^{-1} \phi_\tau^\vee \otimes \phi_\sigma, \psi_E^{-1}\right) \\
&\quad = \gamma\left(s, \chi_V \phi_\tau \otimes \phi_\pi^\vee, \psi_E\right) \cdot \gamma\left(-s, \chi_V^{-1} \phi_\tau^\vee \otimes \phi_\pi, \psi_E^{-1}\right) \\
&\quad \cdot \gamma\left(s - \frac{l-1}{2}, \phi_\tau, \psi_E\right)^{-1} \cdot \gamma\left(-s - \frac{l-1}{2}, \phi_\tau^\vee, \psi_E^{-1}\right)^{-1}.
\end{aligned}$$

Let $\mathcal{A}$ be the set of $s_0 \in \mathbb{C}$ such that the left hand side of the above equation has a pole at $s = s_0$ for some unitary supercuspidal representation τ of $\mathrm{GL}_k(E)$. Looking at the right hand side, we see that

$$\left\{\mathrm{Re}(s_0) \;\middle|\; s_0 \in \mathcal{A}\right\} \subset \frac{1}{2}\mathbb{Z}.$$

Let ϕ_{τ_i} be the irreducible representation of WD_E corresponding to τ_i. We may decompose $\phi_{\tau_i} \cong \phi_i \boxtimes S_{d_i}$, where ϕ_i is an irreducible representation of W_E and d_i is a positive integer. Since

$$\phi_\sigma = \phi_{\tau_1} |\cdot|_E^{s_1} \oplus \cdots \oplus \phi_{\tau_r} |\cdot|_E^{s_r} \oplus \phi_{\sigma_0} \oplus {}^c\phi_{\tau_r}^\vee |\cdot|_E^{-s_r} \oplus \cdots \oplus {}^c\phi_{\tau_1}^\vee |\cdot|_E^{-s_1},$$

we have

$$\begin{aligned} &\gamma\left(s, \chi_W \phi_\tau \otimes \phi_\sigma^\vee, \psi_E\right) \cdot \gamma\left(-s, \chi_W^{-1} \phi_\tau^\vee \otimes \phi_\sigma, \psi_E^{-1}\right) \\ &= \Big[\prod_{i=1}^r \gamma\big(s - s_i, \chi_W \phi_\tau \otimes \phi_{\tau_i}^\vee, \psi_E\big)\gamma\big(s + s_i, \chi_W \phi_\tau \otimes {}^c\phi_{\tau_i}, \psi_E\big) \\ &\times \gamma\big(-s - s_i, \chi_W^{-1} \phi_\tau^\vee \otimes \phi_{\tau_i}^\vee, \psi_E^{-1}\big)\gamma\big(-s + s_i, \chi_W^{-1} \phi_\tau^\vee \otimes {}^c\phi_{\tau_i}, \psi_E^{-1}\big)\Big] \\ &\times \gamma\big(s, \chi_W \phi_\tau \otimes \phi_{\sigma_0}^\vee, \psi_E\big) \cdot \gamma\big(-s, \chi_W^{-1} \phi_\tau^\vee \otimes \phi_{\sigma_0}, \psi_E^{-1}\big). \end{aligned}$$

Now suppose that some s_j is not in $(1/2)\mathbb{Z}$. We may assume that $s_i \notin (1/2)\mathbb{Z}$ and s_i satisfies that

$$\max\left\{s_j + \frac{d_j - 1}{2} \mid s_j \notin \frac{1}{2}\mathbb{Z}\right\} = s_i + \frac{d_i - 1}{2}.$$

Taking $\phi_\tau = \chi_W^{-1}\phi_i$, in the above equation, we see that $\gamma(s, \chi_W \phi_\tau \otimes \phi_\sigma^\vee, \psi_E) \cdot \gamma(-s, \chi_W^{-1}\phi_\tau^\vee \otimes \phi_\sigma, \psi_E^{-1})$ has a pole at $s = 1 + s_i + (d_i - 1)/2$ since the local gamma factor $\gamma(s - s_i, \chi_W \phi_\tau \otimes \phi_{\tau_i}^\vee, \psi_E)$ has a pole at this point. Hence $1+s_i+(d_i-1)/2 \in \mathcal{A}$ but $1+s_i+(d_i-1)/2 \notin (1/2)\mathbb{Z}$. This is a contradiction. □

Corollary 6.15 *We have $s_i = 1/2$ and $\tau_i = \chi_W \mathrm{St}_{l+1}$ for any $i = 1, \ldots, r$.*

Proof By [14, Proposition 3.1], we know that $s_1 = 1/2$ and $\tau_1 = \chi_W \mathrm{St}_{l+1}$. Hence we have $s_i = 1/2$ for any $i = 1, \ldots, r$. Since each τ_i is a discrete series representation of a general linear group, we can interchange τ_i with τ_1 (see e.g., [53]). Hence we have $\tau_i = \chi_W \mathrm{St}_{l+1}$ for any $i = 1, \ldots, r$. □

The following is the key result.

Proposition 6.16 *We have $r = 1$.*

Proof By (the proof of) Proposition 3.1 in [14], we can find an irreducible representation σ_1 of $H(V_{m_1})$ such that

$$\mathrm{Ind}_{Q(Y_{l+1})}^{H(V_m)}\left(\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \otimes \sigma_1\right) \twoheadrightarrow \sigma,$$

and

$$\mathrm{Ind}_{P(X_l)}^{G(W_n)}\left(\chi_V \mathrm{St}_l \otimes \Theta_{W_{n_1},V_{m_1}}(\sigma_1)\right) \twoheadrightarrow \pi,$$

where we put $m_1 = m - 2(l+1)$ and $n_1 = n - 2l$. We have to show that σ_1 is tempered. Suppose for the sake of contradiction that σ_1 is not tempered. Then by Corollary 6.15, there exists $\sigma_2 \in \mathrm{Irr}(H(V_{m_2}))$ such that

$$\mathrm{Ind}_{Q(Y'_{l+1})}^{H(V_{m_1})}\left(\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \otimes \sigma_2\right) \twoheadrightarrow \sigma_1,$$

where $m_2 = m_1 - 2(l+1)$ and $V_{m_1} = Y'_{l+1} \oplus V_{m_2} \oplus (Y'_{l+1})^*$. Since $m_1 - n_1 - \epsilon_0 = l$, by Corollary 5.3, we have

$$\mathrm{Ind}_{P(X'_{l+1})}^{G(W_{n_1})}\left(\chi_V \mathrm{St}_{l+1} |\cdot|_E^{1/2} \otimes \Theta_{W_{n_2},V_{m_2}}(\sigma_2)\right) \twoheadrightarrow \Theta_{W_{n_1},V_{m_1}}(\sigma_1),$$

where $n_2 = n_1 - 2(l+1)$ and $W_{n_1} = X'_{l+1} \oplus W_{n_2} \oplus (X'_{l+1})^*$. Combining these maps, we have

$$\chi_V \mathrm{St}_l \times \chi_V \mathrm{St}_{l+1} |\cdot|_E^{1/2} \rtimes \Theta_{W_{n_2},V_{m_2}}(\sigma_2) \twoheadrightarrow \pi.$$

This contradicts the hypothesis that π is tempered by Casselman's criterion. □

Now we are ready to prove Theorem 4.5 (2). More precisely, we prove the following theorem:

Theorem 6.17 *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ *with* L*-parameter* (ϕ, η). *Assume that*

- $l = l(\pi) = n - m^{\mathrm{down}}(\pi) + \epsilon_0 > 0$;
- $m_\phi(\chi_V S_l) = 2h$ *for some* $h > 0$.

We write $\phi = \phi_0 \oplus (\chi_V S_l)^{\oplus 2h}$. *Put* $n_0 = n - 2hl$ *and* $m_0 = m - 2hl - 2$. *Let* $\pi_0 \in \mathrm{Irr}_{\mathrm{temp}}(G(W_{n_0}))$ *such that*

$$\chi_V \mathrm{St}_l \times \ldots \chi_V \mathrm{St}_l \rtimes \pi_0 \twoheadrightarrow \pi,$$

so that the L*-parameter of* π_0 *is* $(\phi_0, \eta|A_{\phi_0})$. *Here,* $\chi_V \mathrm{St}_l$ *appears* h*-times. We set* $m = m^{\mathrm{up}}(\pi)$ *and let* $\sigma = \theta_{V_m,W_n}(\pi)$ *be the first occurrence of* π *to the going-up tower* $\mathcal{V}^{\mathrm{up}}$. *Then we have*

$$\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \times \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \rtimes \sigma_0 \twoheadrightarrow \sigma,$$

where $\sigma_0 = \theta_{V_{m_0},W_{n_0}}(\pi_0)$, *and* $\chi_W \mathrm{St}_l$ *appears* $(h-1)$*-times. In particular, if we denote the L-parameter for* σ *(resp.* σ_0*) by* $(\phi_\sigma, \eta_\sigma)$ *(resp.* $(\phi_{\sigma_0}, \eta_{\sigma_0})$*), then we have*

$$\phi_{\sigma_0} = \phi\chi_V^{-1}\chi_W - (\chi_W S_l)^{\oplus(2h-1)} \quad \textit{and}$$
$$\phi_\sigma = \phi_{\sigma_0} + (\chi_W S_l)^{\oplus 2(h-1)} + \chi_W S_{l+1} \otimes \left(|\cdot|_E^{1/2} + |\cdot|_E^{-1/2}\right).$$

Moreover the canonical injection $A_{\phi_{\sigma_0}} \hookrightarrow A_{\phi_\sigma}$ *is in fact bijective, and we have* $\eta_\sigma | A_{\phi_{\sigma_0}} = \eta_{\sigma_0}$.

Proof By [14, Proposition 3.1], we can find $\sigma_1 \in \mathrm{Irr}(H(V_{m_1}))$ and $\pi_1 \in \mathrm{Irr}(G(W_{n_1}))$ with $m_1 = m - 2(l+1)$ and $n_1 = n - 2l$ such that

$$\mathrm{Ind}_{Q(Y_{l+1})}^{H(V_m)}\left(\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \otimes \sigma_1\right) \twoheadrightarrow \sigma, \quad \mathrm{Ind}_{P(X_l)}^{G(W_n)}(\chi_V \mathrm{St}_l \otimes \pi_1) \twoheadrightarrow \pi$$

and π_1 is a subquotient of $\Theta_{W_{n_1},V_{m_1}}(\sigma_1)$. Proposition 6.16 says that σ_1 is tempered. Hence $\theta_{W_{n_1},V_{m_1}}(\sigma_1)$ belongs to the same L-packet as π_1 by Proposition 5.5 and Lemma 6.4. Therefore we have

$$\begin{aligned}\phi_\sigma &= \phi_{\sigma_1} + \chi_W S_{l+1} \otimes \left(|\cdot|_E^{1/2} + |\cdot|_E^{-1/2}\right)\\ &= \left(\phi_{\pi_1}\chi_V^{-1}\chi_W + \chi_W S_l\right) + \chi_W S_{l+1} \otimes \left(|\cdot|_E^{1/2} + |\cdot|_E^{-1/2}\right)\\ &= \phi\chi_V^{-1}\chi_W - \chi_W S_l + \chi_W S_{l+1} \otimes \left(|\cdot|_E^{1/2} + |\cdot|_E^{-1/2}\right),\end{aligned}$$

where we denote by ϕ_{σ_1} and ϕ_{π_1} the last names for σ_1 and π_1, respectively.

In particular, there exists $\sigma_0 \in \mathrm{Irr}_{\mathrm{temp}}(H(V_{m_0}))$ whose L-parameter is $(\phi_{\sigma_0}, \eta_{\sigma_0})$ with

$$\phi_{\sigma_0} = \phi\chi_V^{-1}\chi_W - (\chi_W S_l)^{\oplus(2h-1)}, \quad \eta_{\sigma_0} = \eta_\sigma | A_{\phi_{\sigma_0}}$$

such that

$$\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \times \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \rtimes \sigma_0 \twoheadrightarrow \sigma,$$

with $\chi_W \mathrm{St}_l$ occuring $h-1$ times. Note that $\chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \rtimes \sigma_0$ is irreducible since ϕ_{σ_0} contains $\chi_W S_l$, so that

$$\chi_W \mathrm{St}_{l+1} |\cdot|_E^{1/2} \times \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \rtimes \sigma_0$$

is a standard module, which has a unique Langlands quotient. We have to show that $\sigma_0 = \theta_{V_{m_0},W_{n_0}}(\pi_0)$. Since $\chi_W \mathrm{St}_{l+1}|\cdot|_E^{1/2}$ and $\chi_W \mathrm{St}_l$ are not linked, we have

$$\chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{1/2} \times \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \rtimes \sigma_0$$
$$\cong \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \times \chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{1/2} \rtimes \sigma_0.$$

For the linked-ness and its properties, see [53] (in particular, see [53, Theorem 9.7]). By Lemma 2.2, we have

$$\sigma \hookrightarrow \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \times \chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{-1/2} \rtimes \sigma_0.$$

Since $m-n-\epsilon_0 = l+2$, by applying Corollary 5.3 to $\chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \times \chi_W \mathrm{St}_{l+1}|\cdot|_E^{-1/2} \rtimes \sigma_0$, we have

$$\begin{aligned}
\pi^\vee &\hookrightarrow \Theta_{W_n,V_m}(\sigma)^\vee \cong \mathrm{Hom}_{H(V_m)}\left(\omega_{V_m,W_n}, \sigma\right)_\infty \\
&\hookrightarrow \mathrm{Hom}_{H(V_m)}\left(\omega_{V_m,W_n}, \chi_W \mathrm{St}_l \times \cdots \times \chi_W \mathrm{St}_l \times \chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{-1/2} \rtimes \sigma_0\right)_\infty \\
&\cong \chi_V \mathrm{St}_l \times \cdots \times \chi_V \mathrm{St}_l \\
&\quad \rtimes \mathrm{Hom}_{H(V_{m-2(h-1)l})}\left(\omega_{V_{m-2(h-1)l},W_{n-2(h-1)l}}, \mathrm{Ind}_{Q(Y_{l+1})}^{H(V_{m-2(h-1)l})}\left(\chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{-1/2} \otimes \sigma_0\right)\right)_\infty.
\end{aligned}$$

We cannot apply Corollary 5.3 to

$$\mathrm{Hom}_{H(V_{m-2(h-1)l})}(\omega_{V_{m-2(h-1)l},W_{n-2(h-1)l}}, \mathrm{Ind}_{Q(Y_{l+1})}^{H(V_{m-2(h-1)l})}(\chi_W \mathrm{St}_{l+1}|\cdot|_E^{-1/2} \otimes \sigma_0))_\infty.$$

According to Proposition 5.2, J^l and J^{l+1} can contribute. However, since

$$\mathrm{Hom}_{\mathrm{GL}(Y_{l+1})\times H(V_{m_0})}\big(J^{l+1}, \chi_W \mathrm{St}_{l+1}\big|\cdot\big|_E^{-1/2} \otimes \sigma_0\big)_\infty$$
$$\cong \big(\mathrm{Ind}_{P(X_{l+1})}^{G(W_{n_0+2l})}\big(\chi_V \mathrm{St}_{l+1}\big|\cdot\big|_E^{1/2} \otimes \Theta_{W_{n_0-2},V_{m_0}}(\sigma_0)\big)\big)^\vee,$$

we have

$$\mathrm{Hom}_{G(W_n)}\left(\pi^\vee, \chi_V \mathrm{St}_l \times \cdots \times \chi_V \mathrm{St}_l \rtimes \mathrm{Hom}_{\mathrm{GL}(Y_{l+1})\times H(V_{m_0})}(J^{l+1}, \chi_W \mathrm{St}_{l+1}\,|\cdot|_E^{-1/2} \otimes \sigma_0)_\infty\right) = 0$$

by Casselman's temperedness criterion. Hence we have

$$\begin{aligned}
\pi^\vee &\hookrightarrow \chi_V \mathrm{St}_l \times \cdots \times \chi_V \mathrm{St}_l \rtimes \mathrm{Hom}_{\mathrm{GL}(Y_{l+1})\times H(V_{m_0})}\left(J^l, \chi_W \mathrm{St}_{l+1}|\cdot|_E^{-1/2} \otimes \sigma_0\right)_\infty \\
&\cong \chi_V \mathrm{St}_l \times \cdots \times \chi_V \mathrm{St}_l \times \left(\chi_V \mathrm{St}_l \rtimes \Theta_{W_{n-2hl},V_{m-2hl-2}}(\sigma_0)^\vee\right)
\end{aligned}$$

by Proposition 5.2. In particular, there exists an irreducible subquotient π_0' of $\Theta_{W_{n_0},V_{m_0}}(\sigma_0)$ such that

$$\chi_V \mathrm{St}_l \times \cdots \times \chi_V \mathrm{St}_l \rtimes \pi_0' \twoheadrightarrow \pi,$$

where $\chi_V \mathrm{St}_l$ appears h-times. This implies that the L-parameter for π_0' is given by $(\phi_0, \eta|A_{\phi_0})$, which is the same as the one for π_0. Also, if $G(W_n)$ is an odd orthogonal group, the central character of π_0' coincides with the one of π_0. Hence we have $\pi_0' \cong \pi_0$. Since ϕ_{σ_0} contains $\chi_W S_l$ with multiplicity one, by Proposition 5.4, we see that $\Theta_{W_{n_0},V_{m_0}}(\sigma_0)$ is irreducible, and so $\Theta_{W_{n_0},V_{m_0}}(\sigma_0) = \theta_{W_{n_0},V_{m_0}}(\sigma_0) = \pi_0$. In other words, we have $\sigma_0 = \theta_{V_{m_0},W_{n_0}}(\pi_0)$. This completes the proof. □

6.7 Higher lifts

In this subsection, we prove Theorem 4.5 (3).

Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$ with L-parameter (ϕ, η), and $\sigma = \theta_{V_m,W_n}(\pi) \in \mathrm{Irr}(H(V_m))$ be the first occurrence to the going-up tower i.e., $m = m^{\mathrm{up}}(\pi)$. Assume that σ is non-tempered. Then $l(\pi) + 2 = m - n - \epsilon_0 > 2$. Let $\sigma' = \theta_{V_{m'},W_n}(\pi)$ be a higher lift, i.e., $m' > m$. The assertion of Theorem 4.5 (3) follows from [14, Proposition 3.2] if we knew that this proposition can be applied to σ and σ'. So what we have to show is as follows:

Proposition 6.18 *We can apply [14, Proposition 3.2] to σ and σ'. Namely, the same assertion as Proposition 5.6 is true for $\sigma = \theta_{V_m,W_n}(\pi)$ and $\sigma' = \theta_{V_{m'},W_n}(\pi)$.*

Proof We freely use the notation of [14]. According to the proof of Proposition 3.2 in [14], it suffices show that only the 0-th piece R_0 of the filtration of Lemma 2.2 in [14] can contribute in the proof of Proposition 3.2 in [14] for σ and σ'.

Suppose that R_t contributes for some $t > 0$. Then we have a nonzero $\mathrm{GL}(Y_t)$-homomorphism

$$\chi_W \left|\det_{Y_t}\right|^s \rightarrow R_{\overline{Q(Y_t)}}(\sigma),$$

where

- $V_m = Y_t + V_{m_0} + Y_t^*$ with $m_0 = m - 2t$;
- $s = (m + r - n - \epsilon_0)/2 + t/2 > 0$ for some $r \geq 0$.

See also the argument after Lemma 2.2 in [14].

Put

$$\Sigma = \sum_{f \in \mathrm{Hom}_{\mathrm{GL}(Y_t)}\left(\chi_W \left|\det_{Y_t}\right|^s, R_{\overline{Q(Y_t)}}(\sigma)\right)} \mathrm{Im}(f).$$

This is a $\mathrm{GL}(Y_t) \times H(V_{m_0})$-subrepresentation of $R_{\overline{Q(Y_t)}}(\sigma)$ of the form

$$\Sigma = \chi_W \left|\det_{Y_t}\right|^s \boxtimes \Sigma_0,$$

where Σ_0 is a nonzero smooth representation of $H(V_{m_0})$. Since $R_{\overline{Q(Y_t)}}(\sigma)$ is finite length, so is Σ_0. Hence we can find an irreducible subrepresentation σ_0 of Σ_0. We obtain a nonzero $\mathrm{GL}(Y_t) \times H(V_{m_0})$-homomorphism

$$\chi_W \left|\det_{Y_t}\right|^s \boxtimes \sigma_0 \to R_{\overline{Q(Y_t)}}(\sigma).$$

By Bernstein's Frobenius reciprocity, we have a surjection

$$\mathrm{Ind}_{Q(Y_t)}^{H(V_m)} \left(\chi_W \left|\det_{Y_t}\right|^s \boxtimes \sigma_0\right) \twoheadrightarrow \sigma.$$

By Lemma 2.2, this surjection gives an injection

$$\sigma \hookrightarrow \mathrm{Ind}_{Q(Y_t)}^{H(V_m)} \left(\chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right).$$

Hence we have

$$\begin{aligned}\pi^* &\hookrightarrow \mathrm{Hom}_{H(V_m)}\left(\omega_{V_m,W_n}, \sigma\right)\\ &\hookrightarrow \mathrm{Hom}_{H(V_m)}\left(\omega_{V_m,W_n}, \mathrm{Ind}_{Q(Y_t)}^{H(V_m)}\left(\chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right)\right)\\ &\cong \mathrm{Hom}_{\mathrm{GL}(Y_t)\times H(V_{m_0})}\left(R_{Q(Y_t)}\left(\omega_{V_m,W_n}\right), \chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right).\end{aligned}$$

By Kudla's filtration (Lemma 5.1), we see that there is a nonzero homomorphism

$$\pi^\vee \to \mathrm{Hom}_{\mathrm{GL}(Y_t)\times H(V_{m_0})}\left(J^a, \chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right)_\infty$$

for some $0 \le a \le t$.

First, consider the case when $0 \le a < t$. By the definition of the normalized Jacquet module, we have

$$R_{\overline{Q(Y_{t-a},Y_t)}}\left(\chi_W \left|\det_{Y_t}\right|^{-s}\right) = \chi_W \left|\det_{Y_{t-a}}\right|^{-s+a/2} \boxtimes \chi_W \left|\det_{Y'_a}\right|^{-s-(t-a)/2}.$$

Note that $\mathrm{GL}(Y_{t-a})$ acts on J^a by the character

$$\chi_W \left|\det_{Y_{t-a}}\right|^{(n-m+\epsilon_0+t-a)/2}.$$

Since $t-a > 0 \ge -r/2$, we have

$$(n-m+\epsilon_0+t-a)/2 \ne -(m+r-n-\epsilon_0)/2 - t/2 + a/2.$$

Hence we have

$$\mathrm{Hom}_{G(W_n)}\left(\pi^\vee, \mathrm{Hom}_{\mathrm{GL}(Y_t)\times H(V_{m_0})}\left(J^a, \chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right)_\infty\right) = 0.$$

We conclude that there must be an injection

$$\pi^\vee \hookrightarrow \mathrm{Hom}_{\mathrm{GL}(Y_t)\times H(V_{m_0})}\left(J^t, \chi_W \left|\det_{Y_t}\right|^{-s} \boxtimes \sigma_0\right)_\infty.$$

However,

$$\mathrm{Hom}_{\mathrm{GL}(Y_t)\times H(V_{m_0})}(J^t, \chi_W|\det_{Y_t}|^{-s} \boxtimes \sigma_0)_\infty \cong (\mathrm{Ind}_{P_t}^{G(W_n)}(\chi_V|\det_{X_t}|^s \boxtimes \Theta_{W_{n_0},V_{m_0}}(\sigma_0)))^\vee.$$

Since $s > 0$, it has no irreducible tempered subrepresentations by Casselman's criterion.

We obtain a contradiction, so that R_t cannot contribute for any $t > 0$. □

6.8 Central characters of representations of odd orthogonal groups

Recall that for an odd orthogonal group $\mathrm{O}(V_m)$, our local Langlands correspondence described in Sect. 3 or Appendix B parametrizes $\mathrm{Irr}(\mathrm{O}(V_m))$ by the triples (ϕ, η, ν). More precisely, a pair (ϕ, η) corresponds to the set

$$\{\sigma,\ \sigma \otimes \det\}$$

for some $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$, and

$$\nu\colon \mathrm{Irr}(\mathrm{O}(V_m)) \to \{\pm 1\}$$

is given by the central character, i.e., $\sigma(-\mathbf{1}_{V_m}) = \nu(\sigma)\cdot\mathrm{id}$ for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$.

In this subsection, we consider the theta correspondence for $(\mathrm{Mp}(W_n), \mathrm{O}(V_m))$, i.e., $E = F$, $\epsilon = +1$, m is odd and n is even. We prove Theorems 4.3 (5), 4.5 (4) and complete the proof of Theorem 4.1 (2). Namely, we treat the following two issues:

(1) For $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{Mp}(W_n))$ with $\theta_{V_m,W_n}(\pi) \neq 0$, determine $\nu(\theta_{V_m,W_n}(\pi))$.
(2) For $\sigma \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_m))$, determine which tower $\{\Theta_{W_n,V_m}(\sigma)\}_n$ or $\{\Theta_{W_n,V_m}(\sigma\otimes\det)\}_n$ is the going-down tower.

First, we consider (1). Let $\pi \in \mathrm{Irr}(\mathrm{Mp}(W_n))$ and assume that $\sigma = \theta_{V_m,W_n}(\pi)$ is nonzero so that $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$. We define $\epsilon(V) \in \{\pm 1\}$ by

$$\epsilon(V) = \begin{cases} +1 & \text{if } \mathrm{O}(V_m) \text{ is split,} \\ -1 & \text{if } \mathrm{O}(V_m) \text{ is non-split.} \end{cases}$$

Note that $\epsilon(V) = \eta_\sigma(z_{\phi_\sigma})$ by Desideratum B.1 (3), where $(\phi_\sigma, \eta_\sigma)$ is the L-parameter for σ. The following proposition is Theorem 4.3 (5) and Theorem 4.5 (4).

Proposition 6.19 *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{Mp}(W_n))$ *with* L*-parameter* (ϕ_π, η_π). *Assume that* $\sigma = \theta_{V_m,W_n}(\pi)$ *is nonzero. Then we have*

$$\nu(\sigma) = \eta_\pi(z_{\phi_\pi}) \cdot \varepsilon(\phi_\pi) \cdot \chi_V(-1)^{\frac{n}{2}}.$$

Proof The Schrodinger model of the Weil representation allows one to relate the central characters of π and σ. In particular, if $z(\pi)$ denotes the central sign of π as defined in [13, Pg. 1658], we have:

$$z(\pi) = \nu(\sigma) \cdot \chi_V(-1)^{\frac{n}{2}}.$$

On the other hand, by the properties of the local Shimura correspondence [13, Theorem 1.4] and the definition of LLC for $\mathrm{Mp}(W_n)$, we see that

$$z(\pi) = \varepsilon(\phi_\pi) \cdot \eta_\pi(z_{\phi_\pi}).$$

Combining the two equations gives the desired result. □

Next, we consider (2). Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$. We compare the two towers $\{\Theta_{W_n,V_m}(\sigma)\}_n$ and $\{\Theta_{W_n,V_m}(\sigma \otimes \det)\}_n$.

Proposition 6.20 *Let* $\sigma \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{O}(V_m))$ *with* L*-parameter* $(\phi_\sigma, \eta_\sigma, \nu_\sigma)$. *Then* $\{\Theta_{W_n,V_m}(\sigma)\}_n$ *is the going-down tower with respect to* σ*, i.e.,*

$$\min\left\{n \;\middle|\; \Theta_{W_n,V_m}(\sigma) \neq 0\right\} \leq \min\left\{n \;\middle|\; \Theta_{W_n,V_m}(\sigma \otimes \det) \neq 0\right\}$$

if and only if

$$\nu_\sigma = \eta_\sigma(z_{\phi_\sigma}) \cdot \varepsilon(\phi_\sigma).$$

Proof Note that $\{\Theta_{W_n,V_m}(\sigma)\}_n$ is the going-down tower if and only if $\Theta_{W_{m-1},V_m}(\sigma)$ is nonzero. This is equivalent to $\nu_\sigma = \epsilon(V) \cdot \varepsilon(\phi_\sigma) = \eta_\sigma(z_{\phi_\sigma}) \cdot \varepsilon(\phi_\sigma)$ by [11, Theorem 11.1]. □

Together with Proposition 6.7, this completes the proof of Theorem 4.1 (2).

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Appendix A: Preparations for the local Langlands correspondence

In this appendix, we recall some basic results on standard gamma factors, Plancherel measures, and local factors associated to representations of Weil–Deligne groups.

A.1 Standard gamma factors

Fix a non-trivial additive character ψ of F. For $\pi \in \mathrm{Irr}(G(W_n))$ and a character χ of $E^\times$, let $\gamma(s, \pi, \chi, \psi)$ be the standard γ-factor defined by Lapid–Rallis [27] using the doubling method. For its properties, see [9,27] and [11, §10, §11]. The property which we need is as follows:

Proposition A.1 ([11, Theorem 11.2]) *Let* $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$. *Assume that* $\Theta_{V_m,W_n}(\pi) \neq 0$ *and* $l = n - m + \epsilon_0 > 0$. *Then* $\gamma(s, \pi, \chi_V^{-1}, \psi)$ *has a pole at* $s = \frac{l+1}{2}$.

A.2 Plancherel measures

Let G be a reductive group over F and $P = MU$ be a parabolic subgroup of G. For $\pi \in \mathrm{Irr}(M)$, consider the normalized induced representation

$$I_P^G(\pi) := \mathrm{Ind}_P^G(\pi).$$

We define an intertwining operator

$$J_{\overline{P}|P}(\pi)\colon I_P^G(\pi) \to I_{\overline{P}}^G(\pi)$$

by the integral

$$J_{\overline{P}|P}(\pi) f(g) = \int_{\overline{U}} f(\overline{u}g) d\overline{u} \quad \text{for } f \in I_P^G(\pi),$$

where $\overline{P} = M\overline{U}$ is the parabolic subgroup of G opposite to P. More precisely, the above integral converges if π belongs to a certain cone in its Bernstein component (which is a complex manifold), and admits a meromorphic continuation to the whole Bernstein component, being given by a rational function in π (see

[46, Théorème IV.1.1]). Then there exists a rational function μ of π such that

$$J_{P|\overline{P}}(\pi) \circ J_{\overline{P}|P}(\pi) = \mu(\pi)^{-1}.$$

The rational function μ is called the Plancherel measure associated to $I_P^G(\pi)$ (though some reader might find it more appropriate to use the term Plancherel measure or density for the product of the function μ with the formal degree of π if π is essentially square-integrable). The function μ is only well-defined up to a scalar since it depends on the choice of Haar measures on U and $\overline{U}$. We choose Haar measures as in [11, §B.2], which are determined by ψ. We denote the corresponding Plancherel measure by μ_ψ.

Let (V_m, W_n) be as in Sect. 2.2, and put $W_{n_1} = W_n + \mathbb{H}^k$ and $V_{m_1} = V_m + \mathbb{H}^k$ with $n_1 = n + 2k$ and $m_1 = m + 2k$. We consider the maximal parabolic subgroups $P = M_P U_P$ and $Q = M_Q U_Q$ of $G(W_{n_1})$ and $H(V_{m_1})$ with Levi components

$$M_P = \mathrm{GL}_k(E) \times G(W_n) \quad \text{and} \quad M_Q = \mathrm{GL}_k(E) \times H(V_m),$$

respectively.

Theorem A.2 ([11, Theorem 12.1]) *Let* $\pi \in \mathrm{Irr}(G(W_n))$ *and put* $\sigma = \theta_{V_m,W_n}(\pi)$. *Assume that* $\sigma \neq 0$, *so that* $\sigma \in \mathrm{Irr}(H(V_m))$. *For* $\tau \in \mathrm{Irr}(\mathrm{GL}_k(E))$ *and* $s \in \mathbb{C}$, *we put* $\tau_s = \tau|\det|_E^s$. *Then we have*

$$\frac{\mu_\psi(\tau_s\chi_V \otimes \pi)}{\mu_\psi(\tau_s\chi_W \otimes \sigma)} = \gamma\left(s - \frac{l-1}{2}, \tau, \psi_E\right) \cdot \gamma\left(-s - \frac{l-1}{2}, \tau^\vee, \psi_E^{-1}\right).$$

For metaplectic groups, we have to replace $\mathrm{GL}_k(E)$ with its double cover $\widetilde{\mathrm{GL}}_k(E)$. More precisely, see [13, § 2.2–§ 2.5] and [11, §2.5 and §2.6].

A.3 Representations of Weil–Deligne groups

We denote by W_E and $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$ the Weil group and Weil–Deligne group of E, respectively. Let I_E be the inertia subgroup of W_E. We fix a geometric Frobenius element Frob_E of W_E.

If $E \neq F$, we regard W_E as a subgroup W_F such that $W_F/W_E \cong \mathrm{Gal}(E/F)$ and fix $s \in W_F \setminus W_E$. If $E = F$, we put $s = 1$.

Let M be a finite dimensional vector space over $\mathbb{C}$. We say that a homomorphism $\phi\colon WD_E \to \mathrm{GL}(M)$ is a representation of WD_E if

- $\phi(\mathrm{Frob}_E)$ is semi-simple;
- the restriction of ϕ to W_E is smooth;
- the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic.

We call ϕ tempered if the image of W_E is bounded. Let $\phi^\vee$ be the contragredient representation of ϕ defined by $\phi^\vee(w) = {}^t\phi(w)^{-1}$. We define a representation ${}^c\phi$ of WD_E by ${}^c\phi(w) = \phi(sws^{-1})$. Then the equivalence class of ${}^c\phi$ is independent of the choice of s.

Fix $b \in \{\pm 1\}$. We say that ϕ is conjugate self-dual of sign b if there exists a non-degenerate bilinear form $B\colon M \times M \to \mathbb{C}$ such that

$$\begin{cases} B\left(\phi(w)x, \phi(sws^{-1})y\right) = B(x, y), \\ B(y, x) = b \cdot B(x, \phi(s^2)y) \end{cases}$$

for $x, y \in M$ and $w \in WD_E$. In this case, ϕ is equivalent to ${}^c\phi^\vee$. If $E = F$, then $s = 1$ and ${}^c\phi = \phi$. In this case, we call ϕ self-dual of sign b. We also say that ϕ is

$$\begin{cases} \text{orthogonal} & \text{if } \phi \text{ is self-dual of sign } +1, \\ \text{symplectic} & \text{if } \phi \text{ is self-dual of sign } -1, \\ \text{conjugate-orthogonal} & \text{if } \phi \text{ is conjugate self-dual of sign } +1, \\ \text{conjugate-symplectic} & \text{if } \phi \text{ is conjugate self-dual of sign } -1. \end{cases}$$

More precisely, see [10, §3].

For each positive integer k, there exists a unique irreducible algebraic representation S_k of $\mathrm{SL}_2(\mathbb{C})$ of dimension k. It is easy to see that S_k is (conjugate) self-dual of sign $(-1)^{k-1}$. Moreover we have

$$S_a \otimes S_b \cong \bigoplus_{k=1}^{\min\{a,b\}} S_{a+b+1-2k} = S_{a+b-1} \oplus S_{a+b-3} \oplus \cdots \oplus S_{|a-b|+1}$$

for positive integers a and b. We can prove this isomorphism by computing the character of $S_a \otimes S_b$ using the highest weight theory for $\mathrm{SL}_2(\mathbb{C})$.

A.4 Local factors

We define local factors associated to representations of WD_E. Fix a non-trivial additive character ψ' of E. A representation ϕ of WD_E is written by

$$\phi = \bigoplus_{n \geq 1} \phi_n \boxtimes S_n,$$

where (ϕ_n, M_n) is a representation of W_E. Let $M_n^{I_E}$ be the subspace of M_n consisting of I_E-fixed vectors. Note that $M_n^{I_E}$ is a subrepresentation of M_n and

$\phi_n(\mathrm{Frob}_E) \in \mathrm{GL}(M_n^{I_E})$ is independent of the choice of Frob_E. We define the local factors associated ϕ by

$$\begin{aligned} L(s,\phi) &= \prod_{n\geq 1} \det\left(\mathbf{1} - q_E^{-\left(s+\frac{n-1}{2}\right)} \phi_n(\mathrm{Frob}_E)\middle| M_n^{I_E}\right)^{-1} \\ &= \prod_{n\geq 1} L\left(s+\frac{n-1}{2}, \phi_n\right), \\ \varepsilon\left(s,\phi,\psi'\right) &= \prod_{n\geq 1} \varepsilon\left(s,\phi_n,\psi'\right)^n \det\left(-q_E^{\frac{1}{2}-s}\phi_n(\mathrm{Frob}_E)\middle| M_n^{I_E}\right)^{n-1}, \\ \gamma\left(s,\phi,\psi'\right) &= \varepsilon\left(s,\phi,\psi'\right)\frac{L\left(1-s,\phi^\vee\right)}{L(s,\phi)}. \end{aligned}$$

For the definition of $\varepsilon(s,\phi_n,\psi')$, see [44, §3]. For $c \in E^\times$, we define the non-trivial additive character ψ'_c of E by $\psi'_c(x) = \psi'(cx)$. It is known that

$$\varepsilon\left(s,\phi,\psi'_c\right) = \det(\phi)(c)\cdot|c|_E^{\dim(\phi)\left(s-\frac{1}{2}\right)}\cdot\varepsilon(s,\phi,\psi').$$

The local functional equation asserts that

$$\begin{aligned} \gamma(s,\phi,\psi')\cdot\gamma\left(1-s,\phi^\vee,\psi'^{-1}\right) = 1 \quad \text{or} \\ \varepsilon\left(s,\phi,\psi'\right)\cdot\varepsilon\left(1-s,\phi^\vee,\psi'\right) = \det(\phi)(-1). \end{aligned}$$

In particular, if ϕ is self-dual with $\det(\phi) = \mathbf{1}$, then $\varepsilon(1/2,\phi,\psi')$ is in $\{\pm 1\}$ and independent of ψ'. In this case, we write $\varepsilon(\phi) := \varepsilon(1/2,\phi,\psi')$. For $a \not\equiv b \bmod 2$, we have

$$\varepsilon(S_a \otimes S_b) = (-1)^{\min\{a,b\}}.$$

If $E \neq F$ and ϕ is conjugate self-dual, then we write $\varepsilon(\phi,\psi') := \varepsilon(1/2,\phi,\psi')$. By [10, Propostition 5.1], if $E \neq F$ and ${}^c\psi' = \psi'^{-1}$, then $\varepsilon(\phi,\psi') \in \{\pm 1\}$. Here, ${}^c\psi'(x) = \psi'({}^cx)$ for $x \in E$, where ${}^cx$ is the conjugate of x.

We need some lemmas for local factors.

Lemma A.3 *Let ϕ be an irreducible representation of W_E and l be a positive integer. Then we have*

$$\begin{aligned}&\varepsilon(s, \phi, \psi')^l \varepsilon\left(-s, \phi^\vee, \psi'^{-1}\right)^l \\ &\quad = \varepsilon\left(s - \frac{l-1}{2}, \phi, \psi'\right) \varepsilon\left(-s - \frac{l-1}{2}, \phi^\vee, \psi'^{-1}\right),\end{aligned}$$

and

$$\begin{aligned}&\gamma\left(s, \phi \otimes S_l, \psi'\right) \gamma\left(-s, \phi^\vee \otimes S_l, \psi'^{-1}\right) \\ &\quad = \gamma\left(s - \frac{l-1}{2}, \phi, \psi'\right) \gamma\left(-s - \frac{l-1}{2}, \phi^\vee, \psi'^{-1}\right).\end{aligned}$$

Proof Straightforward. □

Lemma A.4 *Let ψ' be a non-trivial additive character of E, ϕ be a representation of WD_E, and l be a positive integer. Assume that*

- *$\psi'|F = \mathbf{1}$, i.e., ${}^c\psi' = \psi'^{-1}$ if $E \neq F$;*
- *ϕ is conjugate self-dual of sign $(-1)^{l-1}$ if $E \neq F$;*
- *ϕ is self-dual of sign $(-1)^{l-1}$ if $E = F$.*

We define $\alpha_l(\phi) \in \{\pm 1\}$ by

$$\alpha_l(\phi) = \frac{\varepsilon\left(\phi \otimes S_{l+1}, \psi'\right)}{\varepsilon\left(\phi \otimes S_{l-1}, \psi'\right)} \times \begin{cases} \det(\phi)(-1) & \text{if } E = F, \\ 1 & \text{if } E \neq F. \end{cases}$$

Here, if $l = 1$, then we interpret $\varepsilon(\phi \otimes S_{l-1}, \psi') := 1$.

1. *Suppose that ϕ is irreducible. Then $\alpha_l(\phi) = -1$ if and only if $\phi = S_l$.*
2. *If $\phi = \phi_0 \oplus {}^c\phi_0^\vee$, then $\alpha_l(\phi) = 1$.*
3. *In general, $\alpha_l(\phi) = (-1)^{m_\phi(S_l)}$, where $m_\phi(S_l)$ is the multiplicity of S_l in ϕ.*

Proof Straightforward. □

For a character χ of $E^\times$, we put

$$\delta(\chi = \mathbf{1}) = \begin{cases} 1 & \text{if } \chi = \mathbf{1}, \\ -1 & \text{if } \chi \neq \mathbf{1}. \end{cases}$$

Lemma A.5 *Let χ be a quadratic character of $E^\times$, and k be a positive integer. Then $\chi \otimes S_{2k}$ is a symplectic representation of WD_E, and satisfies*

$$\varepsilon(\chi \otimes S_{2k}) = -\delta(\chi = \mathbf{1}) \cdot \chi(-1)^k.$$

Proof Since χ and S_{2k} is self-dual representations of sign $+1$ and -1, respectively, we see that $\chi \otimes S_{2k}$ has sign -1. By the definition of the ε-factor, we have

$$\begin{aligned}\varepsilon(\chi \otimes S_{2k}) &= \varepsilon(\chi, \psi)^{2k} \cdot \det\left(-\chi(\mathrm{Frob}_E)\middle|\mathbb{C}(\chi)^{I_E}\right)^{2k-1}\\ &= \chi(-1)^k \cdot \det\left(-\chi(\mathrm{Frob}_E)\middle|\mathbb{C}(\chi)^{I_E}\right)^{2k-1},\end{aligned}$$

where $\mathbb{C}(\chi)$ denotes the space of χ. If χ is ramified, then $\mathbb{C}(\chi)^{I_E} = 0$ so that $\det(-\chi(\mathrm{Frob}_E)|\mathbb{C}(\chi)^{I_E}) = 1$. If χ is unramified, then we have

$$\begin{aligned}&\det\left(-\chi(\mathrm{Frob}_E)\middle|\mathbb{C}(\chi)^{I_E}\right)\\ &= \begin{cases} -1 & \text{if } \chi = \mathbf{1},\\ 1 & \text{if } \chi \text{ is the unique non-trivial unramified quadratic character.}\end{cases}\end{aligned}$$

Hence for any quadratic character χ, we have $\det(-\chi(\mathrm{Frob}_E)|\mathbb{C}(\chi)^{I_E}) = -\delta(\chi = \mathbf{1})$. □

The following lemma is [13, Lemma 12.3] and [12, Lemma A.6].

Lemma A.6 *Let ϕ_1, ϕ_2 be a tempered representations of WD_E of the same dimension n. Assume that*

$$\begin{aligned}&\gamma\left(s, \phi_1^\vee \otimes \phi_\rho, \psi'\right) \cdot \gamma\left(-s, \phi_1 \otimes \phi_\rho^\vee, \psi'^{-1}\right)\\ &= \gamma\left(s, \phi_2^\vee \otimes \phi_\rho, \psi'\right) \cdot \gamma\left(-s, \phi_2 \otimes \phi_\rho^\vee, \psi'^{-1}\right)\end{aligned}$$

for every irreducible representation ϕ_ρ of W_E. Then

$$\phi_1 \cong \phi_2$$

as representations of WD_E.

Appendix B: Local Langlands correspondence

In this paper, we assume the local Langlands correspondence for classical groups, which parametrizes irreducible representations. For general linear groups, it was established by Harris–Taylor [18], Henniart [19], and Scholze [41]. For other classical groups, it is known by Arthur [1], Mok [35], and Kaletha–Mínguez–Shin–White [23], under some assumption on the stabilization of twisted trace formulas. For this assumption, see also the two books [34] of Mœglin–Waldspurger, and papers of Chaudouard–Laumon [7,8]. For metaplectic groups, it was established by the second author and Savin [13]. In this appendix, we summarize some of its properties which are used in this paper.

B.1 Parameters and its component groups

In this subsection, we define parameters and its component groups for (possibly disconnected) classical groups. More precisely, see [1] and [10].

Fix $\epsilon \in \{\pm 1\}$. Let V_m be an ϵ-Hermitian space of dimension m and $G = H(V_m)$ be the isometry group of V_m. Let $\Phi(H(V_m))$ be the set of equivalence classes of representations ϕ of WD_E of dimension m if V_m is an even-dimensional orthogonal space, or of dimension $m - \epsilon_0$ otherwise, which are

$$\begin{cases} \text{conjugate self-dual of sign } (-1)^{m-1}, & \text{if } E \neq F, \\ \text{self-dual of sign } +1 \text{ such that } \det(\phi) = \chi_V, & \text{if } E = F,\ \epsilon = +1 \text{ and } m \in 2\mathbb{Z}, \\ \text{self-dual of sign } -\epsilon \text{ such that } \det(\phi) = \mathbf{1}, & \text{otherwise.} \end{cases}$$

In particular, if $E = F$, $\epsilon = +1$ and $m = 1$, then $\Phi(H(V_1)) = \{$ the zero representation of $WD_E\}$. We call an element in $\Phi(H(V_m))$ a parameter for $H(V_m)$. We denote by $\Phi_{\text{temp}}(H(V_m))$ the subset of equivalence classes of tempered parameters, i.e., the subset of $\phi \in \Phi(H(V_m))$ such that $\phi(W_E)$ is bounded.

If $E = F$ and $G = H(V_m)$, we denote by $\widehat{G}$ the Langlands dual group of G. It is given by

$$\widehat{G} = \begin{cases} \mathrm{Sp}_{m-1}(\mathbb{C}) & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd,} \\ \mathrm{SO}_{m+1}(\mathbb{C}) & \text{if } E = F, \epsilon = -1, \\ \mathrm{SO}_m(\mathbb{C}) & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is even.} \end{cases}$$

Let $\phi \in \Phi(H(V_m))$. We denote the space of ϕ by M and the WD_E-invariant bilinear form on M by B. Let

$$C_\phi = \big\{g \in \mathrm{GL}(M) \mid B(gx, gy) = B(x, y) \text{ for any } x, y \in M, \\ \text{and } g\phi(w) = \phi(w)g \text{ for any } w \in WD_E\}$$

be the centralizer of $\mathrm{Im}(\phi)$ in $\mathrm{Aut}(M, B)$. Also we put

$$C_\phi^+ = \begin{cases} C_\phi \cap \mathrm{SL}(M) & \text{if } E = F \text{ and } m \text{ is even,} \\ C_\phi & \text{otherwise.} \end{cases}$$

Finally, we define the component groups A_ϕ and A_ϕ^+ of ϕ by

$$A_\phi = \pi_0(C_\phi) \quad \text{and} \quad A_\phi^+ = \pi_0(C_\phi^+),$$

respectively.

Let $\phi \in \Phi(H(V_m))$. For an irreducible representation ϕ_0 of WD_E, we denote the multiplicity of ϕ_0 in ϕ by $m_\phi(\phi_0)$. We can decompose

$$\phi = m_1\phi_1 + \cdots + m_r\phi_r + \phi' + {}^c\phi'^\vee,$$

where $\phi_1, \ldots, \phi_r$ are distinct irreducible (conjugate) self-dual representations of WD_E of the same type as ϕ, $m_i = m_\phi(\phi_i)$, and ϕ' is a sum of irreducible representations of WD_E which are not (conjugate) self-dual of the same type as ϕ. Then by [10, § 4], A_ϕ is described as follows:

$$A_\phi = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

Namely, A_ϕ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank r with a canonical basis $\{a_i\}$ indexed by the summands ϕ_i of ϕ. For $a = a_{i_1} + \cdots + a_{i_k} \in A_\phi$ with $1 \le i_1 < \cdots < i_k \le r$, we put

$$\phi^a = \phi_{i_1} \oplus \cdots \oplus \phi_{i_k}.$$

Also, we denote

$$z_\phi := \sum_{i=1}^{r} m_\phi(\phi_i) \cdot a_i = \sum_{i=1}^{s} m_i \cdot a_i \in A_\phi.$$

This is the image of $-\mathbf{1}$ in A_ϕ. We call z_ϕ the central element in A_ϕ. The determinant map $\det\colon \mathrm{GL}(M) \to \mathbb{C}^\times$ gives a homomorphism

$$\det\colon A_\phi \to \mathbb{Z}/2\mathbb{Z}, \quad \sum_{i=1}^{r} \varepsilon_i a_i \mapsto \sum_{i=1}^{r} \varepsilon_i \cdot \dim(\phi_i),$$

where $\varepsilon_i \in \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$. Then the group A_ϕ^+ can be described as follows ([10, Theorem 8.1]):

$$A_\phi^+ = \begin{cases} \ker(\det) & \text{if } E = F \text{ and } m \text{ is even}, \\ A_\phi & \text{otherwise}. \end{cases}$$

We say that a parameter ϕ is discrete if $m_i = 1$ for any $i = 1, \dots, r$ and $\phi' = 0$, i.e., ϕ is a multiplicity-free sum of irreducible (conjugate) self-dual representations of WD_E of the same type as ϕ. We denote by $\Phi_{\mathrm{disc}}(H(V_m))$ the subset of equivalence classes of discrete parameters. Then we have a sequence

$$\Phi_{\mathrm{disc}}(H(V_m)) \subset \Phi_{\mathrm{temp}}(H(V_m)) \subset \Phi(H(V_m)).$$

B.2 Local Langlands correspondence for connected classical groups

In this subsection, we introduce $\Pi(H(V_m))$ and state some properties of the local Langlands correspondence which we need.

First, we consider orthogonal groups. So we assume that $E = F$ and $\epsilon = +1$, and we write $H(V_m) = \mathrm{O}(V_m)$. We define equivalence relations $\sim_{\det}$ on $\mathrm{Irr}(\mathrm{O}(V_m))$ and $\sim_\varepsilon$ on $\mathrm{Irr}(\mathrm{SO}(V_m))$ by

$$\sigma \sim_{\det} \sigma \otimes \det \quad \text{and} \quad \sigma_0 \sim_\varepsilon \sigma_0^\varepsilon$$

for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$ and $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_m))$. Here, we fix an element $\varepsilon \in \mathrm{O}(V_m) \setminus \mathrm{SO}(V_m)$ and define σ_0^ε by $\sigma_0^\varepsilon(h) = \sigma_0(\varepsilon^{-1} h \varepsilon)$ for $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_m))$ and $h \in \mathrm{SO}(V_m)$. Note that $\sigma|\mathrm{SO}(V_m) \cong (\sigma \otimes \det)|\mathrm{SO}(V_m)$ for $\sigma \in \mathrm{Irr}(\mathrm{O}(V_m))$, and $\mathrm{Ind}_{\mathrm{SO}(V_m)}^{\mathrm{O}(V_m)}(\sigma_0) \cong \mathrm{Ind}_{\mathrm{SO}(V_m)}^{\mathrm{O}(V_m)}(\sigma_0^\varepsilon)$ for $\sigma_0 \in \mathrm{Irr}(\mathrm{SO}(V_m))$. The restriction and the induction give a canonical bijection

$$\mathrm{Irr}(\mathrm{O}(V_m))/\sim_{\det} \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V_m))/\sim_\varepsilon .$$

In [1], Arthur has parametrized not $\mathrm{Irr}(\mathrm{SO}(V_m))$ but $\mathrm{Irr}(\mathrm{SO}(V_m))/\sim_\varepsilon$. Via the above bijection, we translate the parametrization for $\mathrm{Irr}(\mathrm{O}(V_m))/\sim_{\det}$.

We return the general setting. Let E be either F or a quadratic extension of F, V_m be an ϵ-Hermitian space of dimension m for fixed $\epsilon \in \{\pm 1\}$, and $H(V_m)$ be the isometry group of V_m. We define $\Pi(H(V_m))$ by

$$\Pi(H(V_m)) = \begin{cases} \mathrm{Irr}(H(V_m))/\sim_{\det} & \text{if } E = F \text{ and } \epsilon = +1, \\ \mathrm{Irr}(H(V_m)) & \text{otherwise.} \end{cases}$$

For $\pi \in \mathrm{Irr}(H(V_m))$, we denote the image of π under the canonical map $\mathrm{Irr}(H(V_m)) \to \Pi(H(V_m))$ by $[\pi]$. Also, we denote the image of $\mathrm{Irr}_*(H(V_m))$ in $\Pi(H(V_m))$ by $\Pi_*(H(V_m))$ for $* = $ disc or temp.

If $E \neq F$ or $\epsilon = +1$, then there exist exactly two Witt towers $\mathcal{V}$ and $\mathcal{V}'$ such that $V_m \in \mathcal{V}$ and

$$\begin{cases} \dim(V_m) \equiv \dim(V'_{m'}) \bmod 2 & \text{if } E \neq F, \\ \mathrm{disc}(V_m) = \mathrm{disc}(V'_{m'}) & \text{if } E = F \text{ and } \epsilon = +1 \end{cases}$$

for $V'_{m'} \in \mathcal{V}'$. Let $\mathcal{V}^+$ be the Witt tower whose anisotropic space is

$$\begin{cases} 0 & \text{if } E \neq F \text{ and } m \text{ is even,} \\ (E, 1) & \text{if } E \neq F, m \text{ is odd and } \epsilon = +1, \\ (E, \delta) & \text{if } E \neq F, m \text{ is odd and } \epsilon = -1, \\ 0 & \text{if } E = F, m \text{ is even and } \mathrm{disc}(V_m) = 1, \\ (F(\sqrt{d}), \mathrm{tr}_{F(\sqrt{d})/F}) & \text{if } E = F, m \text{ is even and } d := \mathrm{disc}(V_m) \neq 1 \text{ in } F^\times/F^{\times 2}, \\ (F, 2\mathrm{disc}(V_m)) & \text{if } E = F \text{ and } m \text{ is odd.} \end{cases}$$

We denote the other Witt tower by $\mathcal{V}^-$. A pure inner form of $H(V_m)$ is $H(V_m^+)$ or $H(V_m^-)$, where $V_m^\pm \in \mathcal{V}^\pm$. If $E = F$ and $\epsilon = -1$, a pure inner form of $H(V_m)$ is $H(V_m)$ itself only.

Now we are ready to describe the desiderata for the Langlands correspondence.

Desideratum B.1 (1) *There exists a canonical surjection*

$$\bigsqcup_{V_m^\bullet} \Pi(H(V_m^\bullet)) \to \Phi(H(V_m)),$$

where $V_m^\bullet$ runs over the spaces such that $H(V_m^\bullet)$ is a pure inner form of $H(V_m)$. For $\phi \in \Phi(H(V_m))$, we denote by Π_ϕ^0 the inverse image of ϕ under this map, and call Π_ϕ^0 the L-packet of ϕ.

(2) *There exists a bijection*

$$\iota \colon \Pi_\phi^0 \to \widehat{A_\phi^+},$$

which satisfies certain character identities. Here, we denote by $\widehat{A_\phi^+}$ the Pontryagin dual of A_ϕ^+.

(3) *Let $[\pi] \in \Pi_\phi^0$ with $\iota([\pi]) = \eta$. Then $[\pi] \in \Pi(H(V_m^-))$ if and only if $z_\phi \in A_\phi^+$ and $\eta(z_\phi) = -1$.*

(4) *We have*

$$\bigsqcup_{V_m^\bullet} \Pi_* \left(H(V_m^\bullet)\right) = \bigsqcup_{\phi \in \Phi_*(H(V_m))} \Pi_\phi^0$$

for $* \in \{\mathrm{disc}, \mathrm{temp}\}$.

(5) *Assume that $\phi = \phi_\tau + \phi_0 + {}^c\phi_\tau^\vee$, where ϕ_0 is an element in $\Phi_{\mathrm{temp}}(H(V_{m_0}))$ and ϕ_τ is an irreducible tempered representation of WD_E which corresponds to $\tau \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_k(E))$. Then the induced representation*

$$\mathrm{Ind}_Q^{H(V_m)} (\tau \otimes \pi_0)$$

is a direct sum of tempered representations of $H(V_m)$, where Q is a parabolic subgroup of $H(V_m)$ with Levi subgroup $L_Q = \mathrm{GL}_k(E) \times H(V_{m_0})$ and π_0 is (a representative of) an element in $\Pi^0_{\phi_0}$. The L-packet Π^0_ϕ is given by

$$\Pi^0_\phi = \left\{[\pi] \,\middle|\, \pi \subset \mathrm{Ind}_Q^{H(V_m)}(\tau \otimes \pi_0), [\pi_0] \in \Pi^0_{\phi_0}\right\}.$$

Moreover if $\pi \subset \mathrm{Ind}_Q^{H(V_m)}(\tau \otimes \pi_0)$, then $\iota([\pi])|A^+_{\phi_0} = \iota([\pi_0])$.

(6) *Assume that*

$$\phi = \phi_{\tau_1}|\cdot|^{s_1} + \cdots + \phi_{\tau_r}|\cdot|^{s_r} + \phi_0 + {}^c\left(\phi_{\tau_1}|\cdot|^{s_1} + \cdots + \phi_{\tau_r}|\cdot|^{s_r}\right)^\vee,$$

where ϕ_0 is an element in $\Phi_{\mathrm{temp}}(H(V_{m_0}))$, ϕ_{τ_i} is an irreducible tempered representation of WD_E which corresponds to $\tau_i \in \mathrm{Irr}_{\mathrm{temp}}(\mathrm{GL}_{k_i}(E))$, and $s_1 \geq \cdots \geq s_r > 0$. Then the L-packet Π^0_ϕ consists of (the equivalence classes of) the unique irreducible quotients π of the standard modules

$$\tau_1|\det|_F^{s_1} \times \cdots \times \tau_r|\det|_F^{s_r} \rtimes \pi_0,$$

where π_0 runs over (representatives of) elements of $\Pi^0_{\phi_0}$. Moreover if π is the unique irreducible quotient of $\tau_1|\det|_F^{s_1} \times \cdots \times \tau_r|\det|_F^{s_r} \rtimes \pi_0$, then $\iota([\pi])|A^+_{\phi_0} = \iota([\pi_0])$.

(7) *The local Langlands correspondence respects the standard γ-factor. Namely, we have*

$$\gamma(s, \pi, \chi, \psi) = \gamma(s, \phi \otimes \chi, \psi_E)$$

for $\pi \in \mathrm{Irr}(H(V_m))$ whose parameter is ϕ, and any character χ of $E^\times$. Here, we put $\psi_E = \psi \circ \mathrm{tr}_{E/F}$.

(8) *The Plancherel measures are invariants of an L-packet. Namely, if π_1, π_2 have the same parameter ϕ, then we have*

$$\mu_\psi(\tau_s \otimes \pi_1) = \mu_\psi(\tau_s \otimes \pi_2)$$

for any $\tau \in \mathrm{Irr}(\mathrm{GL}_k(E))$. In particular, by a result of Shahidi [42], we have

$$\mu_\psi(\tau_s \otimes \pi) = \gamma\left(s, \phi_\tau \otimes \phi^\vee, \psi_E\right) \cdot \gamma\left(-s, \phi_\tau^\vee \otimes \phi, \psi_E^{-1}\right)$$
$$\cdot \gamma\left(2s, R \circ \phi_\tau, \psi\right) \cdot \gamma\left(-2s, R \circ \phi_\tau^\vee, \psi^{-1}\right)$$

for any π whose parameter is $\phi \in \Phi(H(V_m))$, where

$$R = \begin{cases} \mathrm{Asai}^{+} & \text{if } E \neq F \text{ and } m \text{ is even}, \\ \mathrm{Asai}^{-} & \text{if } E \neq F \text{ and } m \text{ is odd}, \\ \mathrm{Sym}^2 & \text{if } E = F, \epsilon = +1 \text{ and } m \text{ is odd}, \\ \wedge^2 & \text{otherwise}. \end{cases}$$

The desiderata B.1 (7) and (8), at least for quasi-split classical groups, should follow from [1] and [35], supplemented by some results of many others. For non-quasi-split unitary groups, see also [23] and [32, § 1.4, Theorem 1.4.1].

Remark B.2 The bijection $\iota \colon \Pi_\phi^0 \to \widehat{A_\phi^+}$ may not be canonical. It is determined by a choice of a Whittaker datum of a quasi-split pure inner form $H(V_m^\bullet)$. If m is odd, then $H(V_m^\bullet)$ has a unique Whittaker datum, so that ι is canonical. Otherwise, we choose the Whittaker datum such that

$$\iota = \begin{cases} J_{\psi^E} & \text{in [12]} & \text{if } E \neq F \text{ and } \epsilon = +1, \\ J_{\psi} & \text{in [12]} & \text{if } E \neq F \text{ and } \epsilon = -1, \\ \iota_{\mathfrak{w}_1} & \text{in [2]} & \text{if } E = F \text{ and } \epsilon = +1, \\ \iota_{\mathfrak{w}_1'} & \text{in [2]} & \text{if } E = F \text{ and } \epsilon = -1. \end{cases}$$

Here, in the first case, we fix a nonzero element $\delta \in E$ such that $\mathrm{tr}_{E/F}(\delta) = 0$ and put $\psi^E(x) = \psi(\frac{1}{2}\mathrm{tr}_{E/F}(\delta x))$ for $x \in E$.

Remark B.3 If $H(V_m) = \mathrm{Sp}(V_m)$ is a symplectic group, then $z_\phi \notin A_\phi^+$ so that

$$A_\phi = A_\phi^+ \oplus (\mathbb{Z}/2\mathbb{Z}) z_\phi$$

for each $\phi \in \Phi(\mathrm{Sp}(V_m))$. Hence we may identify $\widehat{A_\phi^+}$ with

$$\left\{\eta \in \widehat{A_\phi} \;\middle|\; \eta(z_\phi) = 1\right\} \subset \widehat{A_\phi}.$$

If $H(V_m)$ is not an orthogonal group, we have $\Pi(H(V_m)) = \mathrm{Irr}(H(V_m))$. In this case, we set $\Pi_\phi = \Pi_\phi^0$ for $\phi \in \Phi(H(V_m))$. Using Remark B.3, unless $H(V_m)$ is an orthogonal group, we may regard ι as an injection

$$\iota\colon \Pi_\phi \hookrightarrow \widehat{A_\phi}.$$

If $\pi \in \Pi_\phi$ and $\iota(\pi) = \eta \in \widehat{A_\phi}$, we call (ϕ, η) the L-parameter for π.

The L-parameter for the contragredient representation $\pi^\vee$ of π is described by Kaletha [22].

Proposition B.4 ([22, Theorem 4.9]) *Let $\pi \in \mathrm{Irr}(H(V_m))$ with L-parameter (ϕ_π, η_π). We denote the L-parameter for $\pi^\vee$ by $(\phi_{\pi^\vee}, \eta_{\pi^\vee})$. Then we have $\phi_{\pi^\vee} = \phi_\pi^\vee$. In particular, the component groups A_{ϕ_π} and $A_{\phi_\pi^\vee}$ are canonically identified. Moreover, we have $\eta_{\pi^\vee} = \eta_\pi \cdot \eta_0$, where η_0 is given by*

$$\eta_0(a) = \begin{cases} \omega_{E/F}(-1)^{\dim(\phi_\pi^a)} & \text{if } E \neq F \text{ and } m \text{ is even}, \\ \det(\phi_\pi^a)(-1) & \text{if } E = F \text{ and } \epsilon = -1, \\ 1 & \text{otherwise} \end{cases}$$

for $a \in A_{\phi_\pi}$.

B.3 Local Langlands correspondence for full orthogonal groups

In this subsection, we explain the parametrization of $\mathrm{Irr}(\mathrm{O}(V_m))$. Through this subsection, we assume $E = F$ and $\epsilon = +1$, so that $H(V_m) = \mathrm{O}(V_m)$. For $\phi \in \Phi(\mathrm{O}(V_m))$, we define the L-packet Π_ϕ of $\mathrm{O}(V_m)$, which is a subset of $\sqcup_{V_m^\bullet} \mathrm{Irr}(\mathrm{O}(V_m^\bullet))$ by the inverse image of Π_ϕ^0 under the canonical map

$$\bigsqcup_{V_m^\bullet} \mathrm{Irr}(\mathrm{O}(V_m^\bullet)) \rightarrow \bigsqcup_{V_m^\bullet} \Pi(\mathrm{O}(V_m^\bullet)) = \bigsqcup_{V_m^\bullet} \mathrm{Irr}(\mathrm{O}(V_m^\bullet))/\sim_{\det}.$$

In the rest of this subsection, we parametrize Π_ϕ.

First, we assume that m is odd. Then $\mathrm{O}(V_m) = \mathrm{SO}(V_m) \times \{\pm \mathbf{1}_{V_m}\}$. Any representation $\pi \in \mathrm{Irr}(\mathrm{O}(V_m))$ is determined by its image $[\pi]$ in $\mathrm{Irr}(\mathrm{O}(V_m))/\sim_{\det}$ and its central character $\omega_\pi : \{\pm \mathbf{1}_{V_m}\} \rightarrow \mathbb{C}^\times$. Hence we have a bijection

$$\Pi_\phi \rightarrow \widehat{A_\phi} \times \{\pm 1\}, \ \pi \mapsto \big(\iota([\pi]), \omega_\pi(-\mathbf{1}_{V_m^\bullet})\big).$$

If $\pi \in \Pi_\phi$ corresponds to $(\eta, \nu) \in \widehat{A_\phi} \times \{\pm 1\}$, we call the triple (ϕ, η, ν) the L-parameter for π.

Next, we assume that m is even. For $\phi \in \Phi(\mathrm{O}(V_m))$, we have an inclusion $A_\phi^+ \subset A_\phi$, so that we obtain a canonical surjection

$$\widehat{A_\phi} \twoheadrightarrow \widehat{A_\phi^+}.$$

Proposition B.5 *For $\phi \in \Pi_\phi$, we have $\#\Pi_\phi = \#\widehat{A_\phi}$. Moreover, the following are equivalent:*

- $[A_\phi : A_\phi^+] = 2$;
- $\pi \otimes \det \not\cong \pi$ *for some* $\pi \in \Pi_\phi$;
- $\pi \otimes \det \not\cong \pi$ *for any* $\pi \in \Pi_\phi$.

Proof This follows from [2, Proposition 3.2]. □

We fix $\epsilon \in \mathrm{O}(V_m) \setminus \mathrm{SO}(V_m)$ as in [3], which depends on the choice of Whittaker datum. Then [1, Theorem 2.2.4] gives a bijection

$$\iota\colon \Pi_\phi \to \widehat{A_\phi}$$

which satisfies a similar condition of Desiderata B.1 (2) – (8), and such that the diagram

$$\begin{array}{ccc} \Pi_\phi & \xrightarrow{\iota} & \widehat{A_\phi} \\ \downarrow & & \downarrow \\ \Pi_\phi^0 & \xrightarrow{\iota} & \widehat{A_\phi^+} \end{array}$$

is commutative. More precisely, see [3]. If $\pi \in \Pi_\phi$ and $\iota(\pi) = \eta$, we call (ϕ, η) the L-parameter for π.

B.4 Local Langlands correspondence for metaplectic groups

In this subsection, we explain the parametrization of $\mathrm{Irr}(\mathrm{Mp}(W_{2n}))$. Let (W_{2n}, V_m) be as in Sect. 2.2. Through this subsection, we assume $E = F$, $\epsilon = +1$ and $m = 2n+1$, so that $G(W_{2n}) = \mathrm{Mp}(W_{2n})$ and $H(V_m) = \mathrm{O}(V_{2n+1})$.

First, we recall a result of Gan–Savin.

Theorem B.6 ([13, Theorem 1.1 and Corollary 1.2]) *Let $c \in F^\times/F^{\times 2}$. The theta correspondence gives a natural bijection (depending on the choice of ψ)*

$$\mathrm{Irr}(\mathrm{Mp}(W_{2n})) \to \bigsqcup_{V_{2n+1}^\bullet} \mathrm{Irr}\left(\mathrm{O}(V_{2n+1}^\bullet)\right) / \sim_{\det} = \bigsqcup_{V_{2n+1}^\bullet} \Pi\left(\mathrm{O}(V_{2n+1}^\bullet)\right),$$

where the union is taken over all the isomorphism classes of orthogonal spaces $V_{2n+1}^\bullet$ over F with $\dim(V_{2n+1}^\bullet) = 2n+1$ and $\mathrm{disc}(V_{2n+1}^\bullet) = c$.

We describe this map more precisely. Since we are in the p-adic setting, there are exactly two isomorphism classes V_{2n+1} and V'_{2n+1} such that $\dim(V_{2n+1}) = \dim(V'_{2n+1}) = 2n+1$ and $\mathrm{disc}(V_{2n+1}) = \mathrm{disc}(V'_{2n+1}) = c$. For $\pi \in \mathrm{Irr}(\mathrm{Mp}(W_{2n}))$, exactly one of two theta lifts $\Theta_{V_{2n+1},W_{2n}}(\pi)$ and $\Theta_{V'_{2n+1},W_{2n}}(\pi)$ is nonzero. If $\Theta_{V^\bullet_{2n+1},W_{2n}}(\pi)$ is nonzero, then the image of π under this map is $[\theta_{V^\bullet_{2n+1},W_{2n}}(\pi)]$. Also, the inverse image can be described as follows: For $\sigma \in \mathrm{Irr}(\mathrm{O}(V^\bullet_{2n+1}))$, exactly one of two theta lifts $\Theta_{W_{2n},V^\bullet_{2n+1}}(\sigma)$ and $\Theta_{W_{2n},V^\bullet_{2n+1}}(\sigma \otimes \det)$ is nonzero, and the image of $[\sigma] \in \Pi(\mathrm{O}(V^\bullet_{2n+1}))$ under the inverse map is the nonzero small theta lift $\theta_{W_{2n},V^\bullet_{2n+1}}(\sigma)$ or $\theta_{W_{2n},V^\bullet_{2n+1}}(\sigma \otimes \det)$.

Corollary B.7 *The theta correspondence for* $(\mathrm{Mp}(W_{2n}), \mathrm{O}(V^\bullet_{2n+1}))$ *with* $\mathrm{disc}(V^\bullet_{2n+1}) = 1$ *and the local Langlands correspondence for* $\mathrm{O}(V^\bullet_{2n+1})$ *gives a surjection (depending on ψ)*

$$\mathrm{Irr}(\mathrm{Mp}(W_{2n})) \to \Phi(\mathrm{O}(V_{2n+1})).$$

For $\phi \in \Phi(\mathrm{O}(V_{2n+1}))$, we denote by Π_ϕ the inverse image of ϕ under this map, and call Π_ϕ the L-packet of ϕ. Moreover, the composition of ι for $\mathrm{O}(V_{2n+1})$ and the relevant theta lift gives a bijection (depending on ψ)

$$\iota \colon \Pi_\phi \to \widehat{A_\phi}.$$

We define $\Phi(\mathrm{Mp}(W_{2n})) := \Phi(\mathrm{O}(V_{2n+1}))$. For $* = \mathrm{disc}$ or temp, we put $\Phi_*(\mathrm{Mp}(W_{2n})) := \Phi_*(\mathrm{O}(V_{2n+1}))$. Then by [13, Theorem 1.3], we see that Desideratum B.1 (1), (2), (4), (5), (6), (7) and (8) for $R = \mathrm{Sym}^2$ are satisfied.

We also need to know the effect of theta correspondence on L-parameters for the pair for $(\mathrm{Mp}(W_{2n}), \mathrm{O}(V_{2n+1}))$ with $\mathrm{disc}(V_{2n+1}) = c$. Then $\chi_V = \chi_c$, where χ_c is the quadratic character of $F^\times$ associated to $c \in F^\times/F^{\times 2}$.

Theorem B.8 *We write $c = \mathrm{disc}(V_{2n+1})$. Let $\pi \in \mathrm{Irr}(\mathrm{Mp}(W_{2n}))$ and $\sigma \in \mathrm{Irr}(\mathrm{O}(V_{2n+1}))$ with L-parameters (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$, respectively. Assume that $\sigma = \theta_{V_{2n+1},W_{2n}}(\pi)$. Then we have the following:*

(1) *We have*

$$\phi_\sigma = \phi_\pi \otimes \chi_c.$$

In particular, we have a canonical identification $A_{\phi_\pi} = A_{\phi_\sigma}$.

(2) *The characters η_π and η_σ are related by*

$$\eta_\sigma(a)/\eta_\pi(a) = \varepsilon(\phi_\pi^a) \cdot \varepsilon\left(\phi_\pi^a \otimes \chi_c\right) \cdot \chi_c(-1)^{\frac{1}{2}\dim(\phi_\pi^a)} \in \{\pm 1\}$$

for any $a \in A_{\phi_\pi} = A_{\phi_\sigma}$.

(3) *Let* $(\phi_{\pi^\vee}, \eta_{\pi^\vee})$ *be the* L*-parameter for* $\pi^\vee \in \mathrm{Irr}(\mathrm{Mp}(W_{2n}))$. *Then we have*

$$\phi_{\pi^\vee} = \phi_\pi \otimes \chi_{-1} \quad \textit{and}$$
$$\eta_{\pi^\vee}(a)/\eta_\pi(a) = \varepsilon\left(\phi_\pi^a\right) \cdot \varepsilon\left(\phi_\pi^a \otimes \chi_{-1}\right) \cdot \chi_{-1}(-1)^{\frac{1}{2}\dim(\phi_\pi^a)} \in \{\pm 1\}$$

for any $a \in A_{\phi_\pi} = A_{\phi_{\pi^\vee}}$.

Proof This follows from [13, Theorem 1.5]. See also [2, § 3.6]. □

Appendix C: Gross–Prasad conjecture

To prove main theorems, we used two highly non-trivial results. One is the Gross–Prasad conjecture, which gives an answer for restriction problems. The other is Prasad's conjectures, which describe the local theta correspondence for (almost) equal rank cases. In this appendix, we state the Gross–Prasad conjecture (GP).

The Gross–Prasad conjecture consists of four cases; orthogonal, hermitian, symplectic-metaplectic, and skew-hermitian cases. For each case, the statements are slightly different. So we state each case separately. We refer the reader to [10, §6 and §18] for a discussion of the various subtleties in the definition of the characters which appear in the statements of conjecture.

First, we state the GP conjecture for the orthogonal cases.

Theorem C.1 (GP conjecture for the orthogonal cases) *For an orthogonal space* $V_m^\bullet$, *we put* $V_{m+1}^\bullet = V_m^\bullet \oplus L_{(-1)^{m+1}}$, *where* $L_{(-1)^{m+1}}$ *is the orthogonal space of dimension* 1 *and discriminant* $(-1)^{m+1}$. *We set* V_{even} *and* V_{odd} *so that*

$$\{V_{\mathrm{even}},\ V_{\mathrm{odd}}\} = \{V_m,\ V_{m+1}\} \quad \textit{and} \quad \dim(V_{\mathrm{even}}) \in 2\mathbb{Z}.$$

For $\phi \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{even}}))$, $\phi' \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_{\mathrm{odd}}))$ *and* $\nu \in \{\pm 1\}$, *there exists a unique pair* $(\sigma, \sigma') \in \Pi_\phi \times \Pi_{\phi'}$ *such that*

- $\sigma \otimes \sigma'$ *is a representation of* $\mathrm{O}(V_m^\bullet) \times \mathrm{O}(V_{m+1}^\bullet)$ *for some* $V_m^\bullet$;
- *the central character of* σ' *corresponds to* ν;
- $\mathrm{Hom}_{\mathrm{O}(V_m^\bullet)}(\sigma \otimes \sigma', \mathbb{C}) \neq 0$.

Moreover, $\iota(\sigma)$ *and* $\iota(\sigma')$ *are given by*

$$\begin{cases} \iota(\sigma)(a) = \varepsilon\left(\phi^a \otimes \phi'\right) \cdot \det(\phi^a)(-1)^{\frac{1}{2}\dim(\phi')} \cdot \nu^{\dim(\phi^a)}, \\ \iota(\sigma')(a') = \varepsilon\left(\phi \otimes \phi'^{a'}\right) \cdot \det(\phi)(-1)^{\frac{1}{2}\dim(\phi'^{a'})} \end{cases}$$

for $a \in A_\phi$ *and* $a' \in A_{\phi'}$.

The GP conjecture for the special orthogonal cases was proven in [47–50]. In [3], the authors extended this result to the full orthogonal cases under an assumption on LLC for $\mathrm{O}(V_{2n})$.

Secondly, we state the GP conjecture for the hermitian cases.

Theorem C.2 (GP conjecture for the hermitian cases) *Suppose that* $E \neq F$. *For a hermitian space* $V_m^\bullet$, *we put* $V_{m+1}^\bullet = V_m^\bullet \oplus L_{(-1)^m}$, *where* $L_{(-1)^m}$ *is the hermitian space of dimension* 1 *and discriminant* $(-1)^m$. *For* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}(V_m))$ *and* $\phi' \in \Phi_{\mathrm{temp}}(\mathrm{U}(V_{m+1}))$, *there exists a unique pair* $(\sigma, \sigma') \in \Pi_\phi \times \Pi_{\phi'}$ *such that* $\sigma \otimes \sigma'$ *is a representation of* $\mathrm{U}(V_m^\bullet) \times \mathrm{U}(V_{m+1}^\bullet)$ *for some* $V_m^\bullet$, *and*

$$\mathrm{Hom}_{\mathrm{U}(V_m^\bullet)}\left(\sigma \otimes \sigma', \mathbb{C}\right) \neq 0.$$

Moreover, $\iota(\sigma)$ *and* $\iota(\sigma')$ *are given by*

$$\begin{cases} \iota(\sigma)(a) = \omega_{E/F}(-1)^{(m+1)\dim(\phi^a)} \cdot \varepsilon\left(\phi^a \otimes \phi', \psi_2^E\right), \\ \iota(\sigma')(a') = \omega_{E/F}(-1)^{m \dim(\phi'^{a'})} \cdot \varepsilon\left(\phi \otimes \phi'^{a'}, \psi_2^E\right) \end{cases}$$

for $a \in A_\phi$ *and* $a' \in A_{\phi'}$.

The GP conjecture for the hermitian cases was proven in [4–6].

Thirdly, we state the GP conjecture for the symplectic-metaplectic cases.

Theorem C.3 (GP conjecture for the symplectic-metaplectic cases) *Let* W_n *be a symplectic space. For* $c \in F^\times$, *we denote by* ω_{ψ_c} *be the Weil representation of* $\mathrm{Mp}(W_n \otimes L_1)$ *associated to the additive character* $\psi_c(x) := \psi(cx)$ *of* F, *where* L_1 *is the orthogonal space of dimension* 1 *and discriminant* 1. *For* $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Sp}(W_n))$ *and* $\phi' \in \Phi_{\mathrm{temp}}(\mathrm{Mp}(W_n))$, *there exists a unique pair* $(\pi, \pi') \in \Pi_\phi \times \Pi_{\phi'}$ *such that*

$$\mathrm{Hom}_{\mathrm{Mp}(W_n)}(\pi \otimes \pi', \omega_{\psi_c}) \neq 0.$$

Moreover, $\iota(\pi)$ *and* $\iota(\pi')$ *are given by*

$$\begin{cases} \iota(\pi)(a) = \varepsilon\left(\phi^a \chi_c \otimes \phi'\right) \cdot \varepsilon\left(\phi \chi_c \otimes \phi'\right)^{\det(a)} \\ \qquad\qquad \cdot \det(\phi^a)(-1)^{\frac{1}{2}\dim(\phi')} \cdot \det(\phi^a)(c), \\ \iota(\pi')(a') = \varepsilon\left(\phi \chi_c \otimes \phi'^{a'}\right) \cdot \varepsilon(\phi'^{a'}) \cdot \chi_c(-1)^{\frac{1}{2}\dim(\phi'^{a'})} \end{cases}$$

for $a \in A_\phi$ *and* $a' \in A_{\phi'}$.

The GP conjecture for the symplectic-metaplectic cases was proven by [2] when $c = 1$. For general c, it follows from [10, Proposition 18.1] and the case when $c = 1$.

Finally, we state the GP conjecture for the skew-hermitian cases.

Theorem C.4 (GP conjecture for the skew-hermitian cases) *Suppose that $E \neq F$. Let W_n be a skew-hermitian space. For a character χ of $E^\times$ such that $\chi|F^\times = \omega_{E/F}$, we denote by $\omega_{\psi,\chi}$ the Weil representation of $\mathrm{U}(W_n^\bullet)$ associated to ψ and χ. For $\phi, \phi' \in \Phi_{\mathrm{temp}}(\mathrm{U}(W_n))$, there exists a unique pair $(\pi, \pi') \in \Pi_\phi \times \Pi_{\phi'}$ such that π and π' are representations of the same group $\mathrm{U}(W_n^\bullet)$ and*

$$\mathrm{Hom}_{\mathrm{U}(W_n^\bullet)}\left(\pi \otimes \pi', \omega_{\psi,\chi}\right) \neq 0.$$

Moreover, $\iota(\pi)$ and $\iota(\pi')$ are given by

$$\begin{cases} \iota(\pi)(a) = \varepsilon\left(\phi^a \otimes \phi' \otimes \chi^{-1}, \psi_2^E\right), \\ \iota(\pi')(a') = \varepsilon\left(\phi \otimes \phi'^{a'} \otimes \chi^{-1}, \psi_2^E\right) \end{cases}$$

for $a \in A_\phi$ and $a' \in A_{\phi'}$.

The GP conjecture for the skew-hermitian cases was proven by [12]. We also use the following form.

Corollary C.5 *Let the notation be as above. For $\phi, \phi' \in \Phi_{\mathrm{temp}}(\mathrm{U}(W_n))$, there exists a unique pair $(\pi, \pi') \in \Pi_\phi \times \Pi_{\phi'}$ such that π and π' are representations of the same group $\mathrm{U}(W_n^\bullet)$ and*

$$\mathrm{Hom}_{\mathrm{U}(W_n^\bullet)}\left(\pi \otimes \pi', \overline{\omega_{\psi,\chi}}\right) \neq 0.$$

Moreover, $\iota(\pi)$ and $\iota(\pi')$ are given as follows:

$$\begin{cases} \iota(\pi)(a) = \omega_{E/F}(-1)^{\dim(\phi^a)} \cdot \varepsilon\left(\phi^a \otimes \phi' \otimes \chi, \psi_2^E\right), \\ \iota(\pi')(a') = \omega_{E/F}(-1)^{\dim(\phi'^{a'})} \cdot \varepsilon\left(\phi \otimes \phi'^{a'} \otimes \chi, \psi_2^E\right) \end{cases}$$

for $a \in A_\phi$ and $a' \in A_{\phi'}$.

Proof Since π and π' are tempered, we have $\pi^\vee = \overline{\pi}$ and $\pi'^\vee = \overline{\pi'}$. The assertion follows from Theorem C.4 and Proposition B.4. □

We also need the following lemma.

Lemma C.6 *Let V_m be a Hermitian space of dimension m and W_n be a skew-Hermitian space of dimension n. Put $V_{m+1} = V_m \oplus L$ for some line L. If $E = F$, we set $G(W_n)$ and $G'(W_n)$ to be $\mathrm{Sp}(W_n)$ or $\mathrm{Mp}(W_n')$ such that $\{G(W_n), G(W_n')\} = \{\mathrm{Sp}(W_n), \mathrm{Mp}(W_n)\}$. Let $\omega = \omega_{\psi_c}$ or $\omega_{\psi,\chi}$.*

(1) *For $\sigma \in \mathrm{Irr}_{\mathrm{temp}}(H(V_{m+1}))$, there exists $\sigma' \in \mathrm{Irr}_{\mathrm{temp}}(H(V_m))$ such that $\mathrm{Hom}_{H(V_m)}(\sigma \otimes \sigma', \mathbb{C}) \neq 0$.*
(2) *For $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(W_n))$, there exists $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G'(W_n))$ such that $\mathrm{Hom}_{G(W_n)}(\pi \otimes \pi', \omega) \neq 0$.*

Proof The proof is similar to that of Lemma 12.5 in [13]. The absolutely convergence of double integrals which we need are proven in [21] for orthogonal cases, [16] for hermitian cases, [52] for symplectic-metaplectic cases, and [51] for skew-hermitian cases. □

Appendix D: Prasad's conjectures

In this appendix, we state Prasad's conjectures [40], which are the other highly non-trivial results.

Let (V_m, W_n) be as in Sect. 2.2. We have fixed a non-trivial additive character ψ of F, and $\delta \in E^\times$ such that $\mathrm{tr}_{E/F}(\delta) = 0$ if $E \neq F$. Recall that we put

$$\psi_c^E(x) = \psi\left(\frac{c}{2}\mathrm{tr}_{E/F}(\delta x)\right)$$

for $x \in E$ and $c \in F^\times$. If $c = 1$, we simply write $\psi^E = \psi_1^E$. For a representation ϕ of WD_E, we write $\varepsilon(\phi, \psi_c^E) = \varepsilon(1/2, \phi, \psi_c^E)$.

First, we state Prasad's conjecture for the equal rank case:

Theorem D.1 (Prasad's conjecture for the equal rank case) *Assume that $E \neq F$ and $m = n$. Hence $G(W_n) = \mathrm{U}(W_n)$ and $H(V_m^\pm) = \mathrm{U}(V_n^\pm)$. Let $\pi \in \mathrm{Irr}(\mathrm{U}(W_n))$ with L-parameter (ϕ, η). Then we have the following:*

(1) *There is a unique pure inner form $\mathrm{U}(V_n^\bullet)$ of $\mathrm{U}(V_n)$ such that $\Theta_{V_n^\bullet, W_n}(\pi)$ is nonzero.*
(2) *For given $\mathrm{U}(V_n^\bullet)$, the theta lift $\Theta_{V_n^\bullet, W_n}(\pi)$ is nonzero if and only if*

$$\varepsilon(\phi \otimes \chi_V^{-1}, \psi_2^E) = \omega_{E/F}\left(\delta^{-n} \cdot \mathrm{disc}(V_n^\bullet) \cdot \mathrm{disc}(W_n)\right).$$

(3) *Suppose $\Theta_{V_n^\bullet, W_n}(\pi)$ is nonzero. Let $(\theta(\phi), \theta(\eta))$ be the L-parameter of $\theta_{V_n^\bullet, W_n}(\pi)$. Then $\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_W$. In particular, we have a canonical identification $A_\phi = A_{\theta(\phi)}$. Moreover, we have*

$$\theta(\eta)(a)/\eta(a) = \varepsilon\left(\phi^a \otimes \chi_V^{-1}, \psi_2^E\right)$$

for $a \in A_\phi = A_{\theta(\phi)}$.

Next, we state Prasad's conjecture for the almost equal rank case. If $E = F$ and $\epsilon = -1$, then $G(W_n) = \mathrm{O}(W_n)$ and $H(V_m) = \mathrm{Sp}(V_m)$. Recall that for $\pi \in \mathrm{Irr}(\mathrm{O}(W_n))$, we may consider the two theta lifts $\Theta_{V_m,W_n}(\pi)$ and $\Theta_{V_m,W_n}(\pi \otimes \det)$.

Theorem D.2 (Prasad's conjecture for the almost equal rank case) *Assume that $l = n - m + \epsilon_0 = -1$. Let $\pi \in \mathrm{Irr}(G(W_n))$ with L-parameter (ϕ, η). Then we have the following:*

(i) *Suppose that ϕ does not contain χ_V.*
  (a) *For any pure inner form $H(V_m^\bullet)$ of $H(V_m)$, the theta lift $\Theta_{V_m^\bullet,W_n}(\pi)$ is nonzero.*
  (b) *Let $(\theta(\phi), \theta(\eta))$ be the L-parameter of $\theta_{V_m^\bullet,W_n}(\pi)$. Then we have $\theta(\phi) = (\phi \otimes \chi_V^{-1}\chi_W) \oplus \chi_W$. Hence there is a canonical injection $A_\phi \hookrightarrow A_{\theta(\phi)}$.*
  (c) *We have $[A_{\theta(\phi)} : A_\phi] = 2$.*
  (d) *The character $\theta(\eta)$ of $A_{\theta(\phi)}$ satisfies*

$$\theta(\eta)|A_\phi = \eta.$$

(ii) *Suppose that ϕ contains χ_V.*
  (a) *Exactly one of two theta lifts $\Theta_{V_m,W_n}(\pi)$ and $\Theta_{V'_m,W_n}(\pi)$ (or $\Theta_{V_m,W_n}(\pi)$ and $\Theta_{V_m,W_n}(\pi \otimes \det)$) is nonzero.*
  (b) *$\Theta_{V_m^\bullet,W_n}(\pi)$ is nonzero if and only if*

$$\begin{cases} \eta(z_\phi + e_1) = 1 & \text{if } G(W_n) = \mathrm{O}(W_n) \text{ and } H(V_m) = \mathrm{Sp}(V_m), \\ V_m^\bullet \in \mathcal{V}^{\eta(z_\phi)} & \text{otherwise}. \end{cases}$$

  Here, e_1 is the element in A_ϕ corresponding to χ_V.
  (c) *Suppose that $\Theta_{V_m^\bullet,W_n}(\pi)$ is nonzero. Let $(\theta(\phi), \theta(\eta))$ be the L-parameter of $\theta_{V_m^\bullet,W_n}(\pi)$. Then $\theta(\phi) = (\phi \otimes \chi_V^{-1}\chi_W) \oplus \chi_W$. Hence there is a canonical injection $A_\phi \hookrightarrow A_{\theta(\phi)}$.*
  (d) *We have $[A_{\theta(\phi)} : A_\phi] = 1$.*
  (e) *The character $\theta(\eta)$ of $A_{\theta(\phi)}$ satisfies*

$$\theta(\eta)|A_\phi = \eta.$$

Prasad's conjectures (Theorems D.1 and D.2) are established by [12] when $E \neq F$. When $E = F$, Theorem D.2 is proven by [2] and [3].

By the conservation relation (Proposition 2.5), for any $\pi \in \mathrm{Irr}(G(W_n))$, we have

$$m^{\mathrm{down}}(\pi) \leq n + \epsilon_0 + 1.$$

If $m^{\mathrm{down}}(\pi) = n + \epsilon_0 + 1$, then $m^{\mathrm{up}}(\pi) = m^{\mathrm{down}}(\pi) = n + \epsilon_0 + 1$. Namely, both of two theta lifts $\Theta_{V_m^{\bullet}, W_n}(\pi)$ with $m = n + \epsilon_0 + 1$ are nonzero. In this case, ϕ does not contain χ_V by Theorem D.2.

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