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Quivers with relations for symmetrizable Cartan matrices I: Foundations

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Abstract We introduce and study a class of Iwanaga–Gorenstein algebras defined via quivers with relations associated with symmetrizable Cartan matrices. These algebras generalize the path algebras of quivers associated with symmetric Cartan matrices. We also define a corresponding class of generalized preprojective algebras. For these two classes of algebras we obtain generalizations of classical results of Gabriel, Dlab–Ringel, and Gelfand–Ponomarev. In particular, we obtain new representation theoretic realizations of all finite root systems without any assumption on the ground field.

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1 Introduction and main results

1.1 Quivers, Coxeter functors and preprojective algebras

Let Q be a finite connected acyclic quiver, and let $H = KQ$ be the path algebra of Q with coefficients in a field K. The following five results, all proved in the 1970's, form an essential part of the foundations of modern representation theory of finite-dimensional algebras.

(1) Gabriel's Theorem [30]: The quiver Q is representation-finite if and only if Q is a Dynkin quiver of type A_n, D_n, E_6, E_7, E_8. In this case, there is a bijection between the isomorphism classes of indecomposable representations of Q and the set of positive roots of the corresponding simple complex Lie algebra.

(2) Bernstein, Gelfand and Ponomarev's [11] discovery of Coxeter functors

$$C^{\pm}(-) = F_{i_n}^{\pm} \circ \cdots \circ F_{i_1}^{\pm} : \mathrm{rep}(H) \to \mathrm{rep}(H),$$

which are defined as compositions of reflection functors. They lead to a more conceptual proof of Gabriel's Theorem. Applied to the indecomposable projective (resp. injective) representations they yield a family of indecomposable representations, called *preprojective* (resp. *preinjective*) representations.

(3) Gabriel's Theorem [32] saying that there are functorial isomorphisms $TC^{\pm}(-) \cong \tau^{\pm}(-)$, where T is a twist functor and $\tau(-)$ is the Auslander–Reiten translation (see also the comment below on an earlier result by Brenner and Butler).

(4) Auslander, Platzeck and Reiten's Theorem [3] saying that the functors F_k^+ and $\operatorname{Hom}_H(T, -)$, where F_k^+ is a BGP-reflection functor and T is the associated APR-tilting module, are equivalent. This result can be considered as the starting point of tilting theory.

(5) Gelfand and Ponomarev's [35] discovery of the preprojective algebra $\Pi(Q)$ of a quiver, and their result that $\Pi(Q)$, seen as a module over H, is isomorphic to the direct sum of all preprojective H-modules, hence the name *preprojective algebra*. The algebra $\Pi(Q)$ is isomorphic to the tensor algebra $T_H(\operatorname{Ext}_H^1(\mathrm{D}(H), H))$, where D denotes the duality with respect to the base field K, see [7,22,48].

The above results hold for arbitrary ground fields K. At the price of quite strong assumptions on K they were generalized from quivers to the more general setup of modulated graphs. (One needs to assume the existence of finite field extensions of K with prescribed degrees.) For the finite type situation, this extended the theory from the simply laced root systems of types A_n, D_n, E_6, E_7 and E_8 to the non-simply laced root systems B_n, C_n, F_4 and G_2. The definition of a modulated graph (also called *species*) and of its representations is due to Gabriel [31]. The theory itself has been developed by Dlab and Ringel, who generalized (1), (2) and (5) to modulated graphs [25–28,46]. Brenner and Butler [14] proved an earlier result closely related to (3), which is also valid for modulated graphs. (They don't treat $C^{\pm}$ as endofunctors, and the twist automorphism T does not appear.)

1.2 Hereditary, selfinjective and Iwanaga–Gorenstein algebras

In this section, by an *algebra* we mean a finite-dimensional K-algebra.

An algebra A is *hereditary* if all A-modules have projective and injective dimension at most 1. The representation theory of quivers and species corresponds to the representation theory of finite-dimensional hereditary algebras.

An algebra A is *selfinjective* if the classes of projective and injective A-modules coincide. This implies that all modules (except the projective-injectives) have infinite projective and injective dimension. Despite being opposite homological extremes, hereditary and selfinjective algebras are often intimately linked. For example the path algebra KQ is always hereditary, and in contrast, if Q is a Dynkin quiver, then the closely related preprojective algebra $\Pi(Q)$ is selfinjective. Also, the classification of representation-finite selfinjective algebras shows striking similarities to the classification of representation-finite hereditary algebras.

An algebra A is *m-Iwanaga–Gorenstein* if

$$\operatorname{inj.dim}(A) \leq m \quad \text{and} \quad \operatorname{proj.dim}(\mathrm{D}(A)) \leq m.$$

These algebras were first studied by Iwanaga [39,40]. In this case, [4, Lemma 6.9] implies that $\mathrm{inj.\,dim}(A) = \mathrm{proj.\,dim}(\mathrm{D}(A))$, and by [40, Theorem 5] for any A-module M the following are equivalent:

(i) $\mathrm{proj.\,dim}(M) \le m$;
(ii) $\mathrm{inj.\,dim}(M) \le m$;
(iii) $\mathrm{proj.\,dim}(M) < \infty$;
(iv) $\mathrm{inj.\,dim}(M) < \infty$.

Note that with this definition a given algebra can be m-Iwanaga–Gorenstein for different values of m. An algebra is selfinjective if and only if it is 0-Iwanaga–Gorenstein. All hereditary algebras and also all selfinjective algebras are 1-Iwanaga–Gorenstein. Now let A be a 1-Iwanaga–Gorenstein algebra. Then there are two subcategories of the category $\mathrm{rep}(A)$ of finite-dimensional A-modules which are of interest:

(a) The subcategory

$$\mathcal{H}(A) := \{M \in \mathrm{rep}(A) \mid \ \mathrm{proj.\,dim}(M) \le 1 \text{ and } \mathrm{inj.\,dim}(M) \le 1\}.$$

(b) The subcategory

$$\mathcal{GP}(A) := \{M \in \mathrm{rep}(A) \mid \mathrm{Ext}_A^1(M, A) = 0\}$$

of *Gorenstein-projective* modules.

Let $\mathcal{P}(A)$ be the subcategory of projective A-modules. We have

$$\mathcal{P}(A) = \mathcal{H}(A) \cap \mathcal{GP}(A).$$

For each $M \in \mathrm{rep}(A)$ there are short exact sequences

$$0 \to H_M \to G_M \to M \to 0$$

and

$$0 \to M \to H^M \to G^M \to 0$$

with $H_M, H^M \in \mathcal{H}(A)$ and $G_M, G^M \in \mathcal{GP}(A)$, see [2, 8.1].

The category $\mathcal{H}(A)$ carries the homological features of module categories of hereditary algebras, whereas $\mathcal{GP}(A)$ is a Frobenius category, thus displaying the homological features of module categories of selfinjective algebras. We have $\mathcal{GP}(A) = \mathcal{P}(A)$ and $\mathcal{H}(A) = \mathrm{rep}(A)$ if and only if A is hereditary, and in the other extreme we have $\mathcal{GP}(A) = \mathrm{rep}(A)$ and $\mathcal{H}(A) = \mathcal{P}(A)$ if and only if A is selfinjective.

The stable category of $\mathcal{GP}(A)$ is a triangulated category, which is triangle equivalent to the *singularity category*

$$D_{\mathrm{sg}}(A) := D^b(A)/K^b(\mathrm{proj}(A))$$

defined and studied by Buchweitz [17], see also [45]. (Here $D^b(A)$ denotes the derived category of bounded complexes of finite-dimensional A-modules, and $K^b(\mathrm{proj}(A))$ is the homotopy category of bounded complexes of finite-dimensional projective A-modules.) It follows that $D_{\mathrm{sg}}(A) = 0$ if and only if A is hereditary.

Thus the class of 1-Iwanaga–Gorenstein algebras can be seen as an intermediary class sitting between the hereditary and the selfinjective algebras, and the singularity category $D_{\mathrm{sg}}(A)$ can be considered as a measure of how far A is from being hereditary.

1.3 1-Iwanaga–Gorenstein algebras attached to Cartan matrices

To each symmetrizable generalized Cartan matrix C and an orientation Ω of C we attach an infinite series of 1-Iwanaga–Gorenstein algebras $H = H(C, D, \Omega)$ indexed by the different symmetrizers D of C. These algebras are defined by quivers with relations over an arbitrary field K.

If C is symmetric and connected, then (C, Ω) corresponds to a connected acyclic quiver Q, and the series of algebras H consists of the algebras of the form

$$A_m \otimes_K KQ, \qquad (m \geq 1),$$

where $A_m := K[X]/(X^m)$ is a truncated polynomial ring. Representations of such algebras are nothing else than representations of Q over the ground rings A_m.

In the general case of a symmetrizable matrix C, the algebras H can be identified with tensor algebras of modulations of the oriented valued graph Γ corresponding to (C, Ω). However, in contrast to the classical notion of a modulation, the rings attached to vertices of Γ are truncated polynomial rings instead of division rings.

We also introduce a series of algebras $\Pi = \Pi(C, D)$, again defined by quivers with relations, which can be regarded as preprojective algebras of quivers (or more generally of modulated graphs) over truncated polynomial rings.

We show that analogues of all five results mentioned in Sect. 1.1 also hold for our algebras H and Π. However certain definitions must be adapted. For example, we say that H has *finite τ-representation type* if its Auslander–

Reiten quiver has only finitely many τ-orbits consisting entirely of modules of finite homological dimension. The analogue of (1) states that H has finite τ-representation type if and only if C is of Dynkin type. In this case, there is a bijection between the isomorphism classes of indecomposable modules sitting on these τ-orbits and the positive roots of the simple Lie algebra associated with C. So for each Cartan matrix C of Dynkin type we get an infinite family of new representation theoretic incarnations of the root system of C. Let us stress that even in the non-simply laced case these incarnations are defined without any assumption on the ground field K. To prove this theorem, we define analogues of the reflection functors and Coxeter functors of (2), and we give an analogue of Gabriel's Theorem (3) for the subcategory of H-modules of finite homological dimension. This yields alternative descriptions of the preprojective algebra Π similar to (5). We also obtain an analogue of (4) describing the reflection functors in terms of certain tilting H-modules.

In the rest of this section we give precise definitions of the algebras H and Π, and we state our main results in more detail. We then point out previous appearances of some of the algebras H and Π in the literature.

1.4 Definition of *H* and Π

A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is a *symmetrizable generalized Cartan matrix* provided the following hold:

(C1) $c_{ii} = 2$ for all i;
(C2) $c_{ij} \leq 0$ for all $i \neq j$;
(C3) $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$.
(C4) There is a diagonal integer matrix $D = \operatorname{diag}(c_1, \dots, c_n)$ with $c_i \geq 1$ for all i such that DC is symmetric.

The matrix D appearing in (C4) is called a *symmetrizer* of C. The symmetrizer D is *minimal* if $c_1 + \cdots + c_n$ is minimal. From now on, by a *Cartan matrix* we always mean a symmetrizable generalized Cartan matrix. In this case, define for all $c_{ij} < 0$

$$g_{ij} := |\gcd(c_{ij}, c_{ji})|, \quad f_{ij} := |c_{ij}|/g_{ij}, \quad k_{ij} := \gcd(c_i, c_j).$$

Note that we have

$$g_{ij} = g_{ji}, \quad k_{ij} = k_{ji}, \quad c_i = k_{ij} f_{ji}.$$

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix. An *orientation of* C is a subset $\Omega \subset \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ such that the following hold:

(i) $\{(i, j), (j, i)\} \cap \Omega \neq \varnothing$ if and only if $c_{ij} < 0$;

(ii) For each sequence $((i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$ we have $i_1 \neq i_{t+1}$.

For an orientation Ω of C let $Q := Q(C, \Omega) := (Q_0, Q_1, s, t)$ be the quiver with the set of vertices $Q_0 := \{1, \dots, n\}$ and with the set of arrows

$$Q_1 := \{\alpha_{ij}^{(g)} : j \to i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \to i \mid 1 \leq i \leq n\}.$$

(Thus we have $s(\alpha_{ij}^{(g)}) = j$ and $t(\alpha_{ij}^{(g)}) = i$ and $s(\varepsilon_i) = t(\varepsilon_i) = i$, where $s(a)$ and $t(a)$ denote the starting and terminal vertex of an arrow a, respectively.) If $g_{ij} = 1$, we also write α_{ij} instead of $\alpha_{ij}^{(1)}$. We call Q a *quiver of type* C. Let $Q^\circ := Q^\circ(C, \Omega)$ be the quiver obtained from Q by deleting all loops ε_i. Clearly, Q° is an acyclic quiver. Having said that, one might want to call Ω an *acyclic orientation*. Future research might require to modify the definition of an orientation and drop the acyclicity assumption.

Throughout let K be a field. For a quiver $Q = Q(C, \Omega)$ and a symmetrizer $D = \operatorname{diag}(c_1, \dots, c_n)$ of C, let

$$H := H(C, D, \Omega) := KQ/I$$

where KQ is the path algebra of Q, and I is the ideal of KQ defined by the following relations:

(**H**1) For each i we have the *nilpotency relation*

$$\varepsilon_i^{c_i} = 0.$$

(**H**2) For each $(i, j) \in \Omega$ and each $1 \leq g \leq g_{ij}$ we have the *commutativity relation*

$$\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.$$

The following remarks are straightforward.

(i) H is a finite-dimensional K-algebra.
(ii) H depends on the chosen symmetrizer D. But note that the relation (H2) does not depend on D.
(iii) The relation (H2) becomes redundant for all $(i, j) \in \Omega$ with $k_{ij} = 1$.
(iv) If C is symmetric and if D is minimal, then H is isomorphic to the path algebra KQ°.

The behaviour of H under change of the symmetrizer D is studied in [33].

The *opposite orientation* of an orientation Ω is defined as $\Omega^* := \{(j, i) \mid (i, j) \in \Omega\}$. Let $\overline{\Omega} := \Omega \cup \Omega^*$. For later use, let us define

$$\begin{aligned}\Omega(i, -) &:= \{j \in Q_0 \mid (i, j) \in \Omega\}, \quad & \Omega(-, j) &:= \{i \in Q_0 \mid (i, j) \in \Omega\},\\ \overline{\Omega}(i, -) &:= \{j \in Q_0 \mid (i, j) \in \overline{\Omega}\}, \quad & \overline{\Omega}(-, j) &:= \{i \in Q_0 \mid (i, j) \in \overline{\Omega}\}.\end{aligned}$$

Observe that $\overline{\Omega}(i, -) = \overline{\Omega}(-, i)$.

For $(i, j) \in \overline{\Omega}$ define

$$\operatorname{sgn}(i, j) := \begin{cases} 1 & \text{if } (i, j) \in \Omega, \\ -1 & \text{if } (i, j) \in \Omega^*. \end{cases}$$

For $Q = Q(C, \Omega)$ and a symmetrizer $D = \operatorname{diag}(c_1, \dots, c_n)$ of C, we define an algebra

$$\Pi := \Pi(C, D, \Omega) := K\overline{Q}/\overline{I}$$

as follows. The *double quiver* $\overline{Q} = \overline{Q}(C)$ is obtained from Q by adding a new arrow $\alpha_{ji}^{(g)} : i \to j$ for each arrow $\alpha_{ij}^{(g)} : j \to i$ of Q°. (Note that we did not add any new loops to the quiver Q.) The ideal $\overline{I}$ of the path algebra $K\overline{Q}$ is defined by the following relations:

(P1) For each i we have the *nilpotency relation*

$$\varepsilon_i^{c_i} = 0.$$

(P2) For each $(i, j) \in \overline{\Omega}$ and each $1 \le g \le g_{ij}$ we have the *commutativity relation*

$$\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}.$$

(P3) For each i we have the *mesh relation*

$$\sum_{j \in \overline{\Omega}(-,i)} \sum_{g=1}^{g_{ji}} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i, j) \varepsilon_i^{f} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} = 0.$$

We call Π a *preprojective algebra* of type C. Here are again some straightforward remarks:

(i) Up to isomorphism, the algebra $\Pi := \Pi(C, D) := \Pi(C, D, \Omega)$ does not depend on the orientation Ω of C.

(ii) In general, Π can be infinite-dimensional.

(iii) Π depends on the chosen symmetrizer D. But note that the relations (P2) and (P3) do not depend on D.

(iv) If C is symmetric and if D is minimal, then Π is isomorphic to the classical preprojective algebra $\Pi(Q^\circ)$ associated with Q°.

For an example illustrating the above definitions, see below Sect. 13.1.

1.5 Main results

Let $e_1, \dots, e_n$ be the idempotents in H (resp. Π) corresponding to the vertices of Q (resp. $\overline{Q}$). Define $H_i := e_i H e_i$. Clearly, H_i is isomorphic to the truncated polynomial ring $K[\varepsilon_i]/(\varepsilon_i^{c_i})$. For each representation M of H or Π we get an H_i-module structure on $M_i := e_i M$. The following definition is of central importance.

Definition 1.1 A module $M \in \operatorname{rep}(H)$ or $M \in \operatorname{rep}(\Pi)$ is called *locally free* if M_i is a free H_i-module for every i.

Let $\operatorname{rep}_{\text{l.f.}}(H)$ (resp. $\operatorname{rep}_{\text{l.f.}}(\Pi)$) be the subcategory of all locally free $M \in \operatorname{rep}(H)$ (resp. $M \in \operatorname{rep}(\Pi)$).

Theorem 1.2 *The algebra H is a* 1*-Iwanaga–Gorenstein algebra. For $M \in$* $\operatorname{rep}(H)$ *the following are equivalent*:

(i) proj. $\dim(M) \leq 1$;
(ii) inj. $\dim(M) \leq 1$;
(iii) proj. $\dim(M) < \infty$;
(iv) inj. $\dim(M) < \infty$;
(v) *M is locally free.*

Let M be a locally free module. For each $i \in Q_0$ let r_i be the rank of the free H_i-module M_i. Thus $\dim_K(M_i) = r_i c_i$. We put

$$\underline{\operatorname{rank}}(M) := (r_1, \dots, r_n).$$

Let τ be the Auslander–Reiten translation for the algebra H, and let τ^- be the inverse Auslander–Reiten translation. An indecomposable H-module M is *preprojective* (resp. *preinjective*) if there exists some $k \geq 0$ such that $M \cong \tau^{-k}(P)$ (resp. $M \cong \tau^k(I)$) for some indecomposable projective H-module P (resp. indecomposable injective H-module I). Let us warn the reader that the usual definition of a preprojective or preinjective module M requires some additional conditions on the Auslander–Reiten component containing M.

In general, the Auslander–Reiten translates $\tau^k(M)$ of an indecomposable locally free H-module M are not locally free, see the example in Sect. 13.5.

An indecomposable H-module M is called *τ-locally free*, if $\tau^k(M)$ is locally free for all $k \in \mathbb{Z}$.

A module M over an algebra A is called *rigid* if $\operatorname{Ext}_A^1(M, M) = 0$.

The following result is an analogue for the algebras $H = H(C, D, \Omega)$ of Gabriel's Theorem (1) for quivers and of its generalization by Dlab and Ringel to modulated graphs.

Theorem 1.3 *There are only finitely many isomorphism classes of τ-locally free H-modules if and only if C is of Dynkin type. In this case, the following hold:*

- (i) *The map $M \mapsto \underline{\operatorname{rank}}(M)$ yields a bijection between the set of isomorphism classes of τ-locally free H-modules and the set $\Delta^+(C)$ of positive roots of the semisimple complex Lie algebra associated with C.*
- (ii) *For an indecomposable H-module M the following are equivalent:*
  - (a) *M is preprojective;*
  - (b) *M is preinjective;*
  - (c) *M is τ-locally free;*
  - (d) *M is locally free and rigid.*

Crawley-Boevey [21] studied representations of quivers over principal ideal domains. There are some striking analogies between Theorem 1.3 and his results.

Note that the algebras H are usually representation infinite, even if C is a Cartan matrix of Dynkin type with a minimal symmetrizer D. Already for C of type B_3 with minimal symmetrizer, there exist indecomposable locally free H-modules M with $\underline{\operatorname{rank}}(M) \notin \Delta^+(C)$, see Sect. 13.7. Furthermore, for C of type B_5 with minimal symmetrizer there exists a K^*-family of pairwise non-isomorphic indecomposable locally free H-modules, all having the same dimension vector.

Inspired by the classical theory for path algebras and modulated graphs we define *Coxeter functors*

$$C^+, C^- \colon \operatorname{rep}(H) \to \operatorname{rep}(H)$$

as products of reflection functors, see Sect. 8. Let

$$T \colon \operatorname{rep}(H) \to \operatorname{rep}(H)$$

be the *twist automorphism* induced from the algebra automorphism $H \to H$ defined by $\varepsilon_i \mapsto \varepsilon_i$ and $\alpha_{ij}^{(g)} \mapsto -\alpha_{ij}^{(g)}$. In other words, T sends a representation $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ of H to $(M_i, -M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$. The following theorem, analogous to Gabriel's Theorem (3), relates Coxeter functors to the

Auslander–Reiten translation, and provides the main step in proving Theorem 1.3.

Theorem 1.4 *For* $M \in \operatorname{rep}(H)$ *there are functorial isomorphisms*

$$\mathrm{D}\operatorname{Ext}^1_H(M, H) \cong TC^+(M) \text{ and } \operatorname{Ext}^1_H(\mathrm{D}(H), M) \cong TC^-(M).$$

Furthermore, if $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$*, then there are functorial isomorphisms*

$$\tau(M) \cong TC^+(M) \text{ and } \tau^-(M) \cong TC^-(M).$$

Vice versa, if $\tau(M) \cong TC^+(M)$ *or* $\tau^-(M) \cong TC^-(M)$ *for some* $M \in \operatorname{rep}(H)$*, then* $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$.

Recall that H is a 1-Iwanaga–Gorenstein algebra, and that $\mathcal{GP}(H)$ denotes the subcategory of Gorenstein-projective H-modules.

Corollary 1.5 *For* $M \in \operatorname{rep}(H)$ *the following are equivalent:*

(i) $C^+(M) = 0$;
(ii) $M \in \mathcal{GP}(H)$.

For an algebra A and an A-A-bimodule M, let $T_A(M)$ denote the corresponding tensor algebra. Theorem 1.4 implies the following description of the preprojective algebra $\Pi = \Pi(C, D)$ associated with H.

Theorem 1.6 $\Pi \cong T_H(\operatorname{Ext}^1_H(\mathrm{D}(H), H))$.

The algebra Π contains H as a subalgebra in an obvious way. Let ${}_H\Pi$ be the algebra Π considered as a left module over H. The following result says that ${}_H\Pi$ is isomorphic to the direct sum of all preprojective H-modules. This justifies that Π is called a *preprojective algebra*.

Theorem 1.7 *We have* ${}_H\Pi \cong \bigoplus_{m \geq 0} \tau^{-m}({}_HH)$. *In particular,* Π *is finite-dimensional if and only if* C *is of Dynkin type.*

Finally, we obtain the following analogue for locally free Π-modules of the classical important Ext-symmetry of preprojective algebras.

Theorem 1.8 *For* $M, N \in \operatorname{rep}_{\mathrm{l.f.}}(\Pi)$ *we have a functorial isomorphism*

$$\operatorname{Ext}^1_\Pi(M, N) \cong \mathrm{D}\operatorname{Ext}^1_\Pi(N, M).$$

1.6 Previous appearances of $H(C, D, \Omega)$

1.6.1. Let Q be a quiver without oriented cycles. Ringel and Zhang [51] study representations of Q over the algebra $A := K[X]/(X^2)$ of dual numbers. This can be interpreted as the category of Λ-modules with $\Lambda := A \otimes_K KQ$. It is shown in [51] that Λ is a 1-Iwanaga–Gorenstein algebra, and that the stable category of $\mathcal{GP}(\Lambda)$ is triangle equivalent to the orbit category $D^b(KQ)/[1]$ of the bounded derived category $D^b(KQ)$ of the path algebra KQ modulo the shift functor [1]. In our setup, if we take symmetric Cartan matrices C with symmetrizer $D = \operatorname{diag}(2, \dots, 2)$, then the class of algebras $H(C, D, \Omega)$ coincides with the class of algebras studied by Ringel and Zhang. Fan [29] studies the Hall algebra of representations of Q over $K[X]/(X^m)$ with $m \geq 1$. Again this is a special case of our setup with C symmetric and $D = \operatorname{diag}(m, \dots, m)$. For Q a quiver of type A_2, $A = K[X]/(X^m)$ and $\Lambda := A \otimes_K KQ$ the category $\mathcal{GP}(\Lambda)$ is studied in work of Ringel and Schmidmeier [50]. Note also that in this case we have $\Lambda \cong T_2(A)$, where $T_2(A)$ is the algebra of upper triangular 2×2-matrices with entries in A. More generally, the algebras $T_2(A)$ with A a Nakayama algebra have been studied by Skowroński [53], and the algebras $T_n(A)$ have been studied by Leszczyński and Skowroński [42].

1.6.2. A general framework for studying cluster structures arising from 2-Calabi–Yau categories with loops has been provided by [16]. As an example they study the cluster category $\mathcal{C} := D^b(\mathcal{T}_n)/\tau^-[1]$ of the mesh category of a tube $\mathcal{T}_n$ of rank $n \geq 2$. The endomorphism algebras of the maximal rigid objects in $\mathcal{C}$ have been studied by Vatne and Yang [54,56]. It turns out that there exists a maximal rigid object T in $\mathcal{C}$ such that $\operatorname{End}_{\mathcal{C}}(T)$ is isomorphic to one of our algebras $H(C, D, \Omega)$, where C is of Dynkin type C_{n-1} and D is minimal. (We identify the types $C_1 = A_1$ and $C_2 = B_2$.)

1.6.3 Let Q be a Dynkin quiver of type E_8, and let $F := S^4\Sigma^{-4}$, where S is the Serre functor and Σ is the translation functor for the bounded derived category $D^b(KQ)$ of the path algebra KQ. Ladkani [41, Section 2.6] studies the orbit category $\mathcal{C} := D^b(KQ)/F$. He shows that $\mathcal{C}$ is a triangulated 2-Calabi–Yau category containing exactly 6 cluster-tilting objects. Ladkani shows that $\mathcal{C}$ categorifies a cluster algebra of Dynkin type G_2. He also shows that the cluster tilting-objects in $\mathcal{C}$ have an endomorphism algebra isomorphic to $A = KQ/I$, where Q is a quiver of the form

$$\begin{tikzcd}
1 & 2 \arrow[l] \arrow[loop above, "\varepsilon"] & \text{or} & 1 \arrow[r] & 2 \arrow[loop above, "\varepsilon"]
\end{tikzcd}$$

and I is generated by ε^3. Note that the algebras A are isomorphic to the algebras $H(C, D, \Omega)$ with C of type G_2 and D minimal.

1.7 Previous appearances of $\Pi(C, D)$

1.7.1 Let C be a Cartan matrix of Dynkin type. In [38], an algebra $A = A(C)$ was introduced, by means of an infinite quiver with potential. Certain finite-dimensional A-modules (the *generic kernels* $K_{k,m}^{(i)}$, see [38, Definition 4.5]) were shown to encode the q-characters of the Kirillov–Reshetikhin modules of the quantum loop algebra $U_q(L\mathfrak{g})$, where $\mathfrak{g}$ is the complex simple Lie algebra with Cartan matrix C. The connection with the algebras considered in this article is the following: Let $\widetilde{\Pi}(C)$ denote the algebra $K\overline{Q}/\widetilde{I}$, where $\widetilde{I}$ is the two-sided ideal defined by the relations (P2) and (P3) only. (Thus, the nilpotency relation (P1) is omitted.) Then $A(C)$ is a truncation of a $\mathbb{Z}$-covering of $\widetilde{\Pi}(C^*)$, where C^* is the transposed Cartan matrix, in other words, the Cartan matrix of the Langlands dual $\mathfrak{g}^L$ of $\mathfrak{g}$. In particular, for $m \ll 0$ the generic kernels $K_{c_i,m}^{(i)}$ of $A(C)$ coincide with the indecomposable projective $\Pi(C^*, D)$-modules regarded as $\mathbb{Z}$-graded $\widetilde{\Pi}(C^*)$-modules (compare for instance [38, Section 6.5] to Fig. 11 below). This generalizes [38, Example 4.7].

1.7.2 The algebras $\widetilde{\Pi}(C)$ mentioned in Sect. 1.7.1 were defined and studied independently by Cecotti [18, Section 3.4] and Cecotti and del Zotto [19, Section 5.1]. In [18] they are called *generalized preprojective algebras.*

1.7.3 For (C, Ω) let

$$W(C, \Omega) := \sum_{(j,i)\in\overline{\Omega}} \sum_{g=1}^{g_{ji}} \operatorname{sgn}(i, j)\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)}.$$

Then the cyclic derivatives of the potential $W(C, \Omega)$ yield the defining relations (P2) and (P3) of $\Pi(C, D)$, compare [18,19,38], where these relations are also encoded via potentials.

1.7.4 After the first version of this article appeared on arXiv, we were informed by D. Yamakawa of the following connection between $\Pi(C, D)$ and some quiver varieties for quivers with multiplicities introduced in [55]. Suppose that C is a generalized Cartan matrix of the form $C = (c_{ij}) = 2I_n - AD$, where I_n is the unit matrix, $A = (a_{ij})$ is a symmetric matrix with $a_{ij} \in \mathbb{N}$ and $a_{ii} = 0$, and $D = \operatorname{diag}(c_1, \dots, c_n)$ with positive integers c_i. Then C is symmetrizable with symmetrizer D. (Note that not every symmetrizable Cartan matrix is of this form. For instance Cartan matrices of type C_n are of this form, but not Cartan matrices of type B_n.) Assume further that $k_{ij} = \gcd(c_i, c_j) = 1$ whenever $c_{ij} < 0$. Then the defining relations (P2) of $\Pi(C, D)$ are redundant, and the mesh relations (P3) can be regarded as the vanishing of the moment map of some Hamiltonian space considered in [55]. As a result, in this case, the isomorphism classes of locally free $\Pi(C, D)$-modules of rank vector $\mathbf{r}$ are parametrized by the set theoretical quotient $\mathcal{N}^{set}(\mathbf{r}, 0)$ of [55].

1.8 Future directions

This article intends to provide the foundation for generalizing many of the connections between path algebras, preprojective algebras, Lie algebras and cluster algebras from the symmetric to the symmetrizable case.

In particular, since the algebras H and Π are defined via quivers with relations, one can study their module varieties over an arbitrary field K. Taking $K = \mathbb{C}$, one can hope to generalize Lusztig's nilpotent varieties and Nakajima quiver varieties from the symmetric to the symmetrizable case.

As a first step in this direction, in [34] we construct the enveloping algebra of the positive part of an arbitrary simple finite-dimensional complex Lie algebra as an algebra of constructible functions on varieties of locally free H-modules.

1.9 Outline

The article is organized as follows. In Sect. 2 we recall some definitions and basic facts on Cartan matrices, quadratic forms and Weyl groups. A description of the projective and injective H-modules and some fundamental results on locally free H-modules are obtained in Sect. 3. In particular, Sect. 3 contains the proof of Theorem 1.2 (combine Proposition 3.5 and Corollary 3.7). In Sect. 4 we show that the quadratic form q_C associated with a Cartan matrix C coincides with the restriction of the homological Euler form of H to the subcategory of locally free H-modules. The representation theory of the algebras H and Π can be reformulated in terms of a generalization of the representation theory of modulated graphs. This point of view, which is of central importance for proving (and in part also for formulating) our main results, is explained in Sect. 5. An interpretation of H and Π as tensor algebras is discussed in Sect. 6. Section 7 provides a bimodule resolution of H. In Sect. 8 we introduce a trace pairing for H_i-modules and relate it with the adjunction isomorphisms. Section 9 contains some fundamental properties of generalizations of BBK-reflection functors to our algebras Π. (The letters BBK stand for Baumann and Kamnitzer [8,9] and Bolten [12]. Independently from each other they introduced reflection functors for the classical preprojective algebras associated with quivers.) The reflection functors for Π restrict to reflection functors for H. We show that the latter are generalized versions of APR-tilting functors. The intimate relation between Coxeter functors and the Auslander–Reiten translation for H is studied in Sect. 10. Theorem 1.4 follows from Theorem 10.1 and Proposition 11.9. We also prove some crucial properties of the algebras Π. In particular, Theorem 1.6 corresponds to Corollary 10.6. In Sect. 11 we use the previous constructions for proving Theorem 1.3 (see Theorem 11.10). The proof of Theorem 1.7 can be found in Sect. 11.3. We also obtain some first results on the Auslander–Reiten theory of H. Section 12 contains the construction of a bimodule resolution for Π. This resolution plays an important part in the study of locally free representations. In particular, Sect. 12.2 contains the proof of Theorem 1.8 (see Theorem 12.6). Finally, Sect. 13 contains a collection of examples.

1.10 Notation

By a *subcategory* we always mean a full subcategory which is closed under isomorphisms and direct summands. By an *algebra* we mean an associative K-algebra with 1. For a K-algebra A let $\operatorname{mod}(A)$ be the category of finite-dimensional left A-modules. If $A = KQ/I$ is the path algebra of a quiver Q modulo some ideal I, then $\operatorname{rep}(A)$ denotes the category of finite-dimensional representations of (Q, I). By definition these are the representations of Q which are annihilated by I. We often identify $\operatorname{mod}(A)$ and $\operatorname{rep}(A)$. Let $\operatorname{proj}(A)$ and $\operatorname{inj}(A)$ be the full subcategories of $\operatorname{rep}(A)$ with objects the projective and injective A-modules, respectively. Let $\mathrm{D} := \operatorname{Hom}_K(-, K)$ be the usual K-duality. For a finite-dimensional algebra A let $\tau(-) = \tau_A(-)$ be the Auslander–Reiten translation of A. For a module X we denote by $\operatorname{add}(X)$ the subcategory of modules which are isomorphic to finite direct sums of direct summands of X. As a general reference for the representation theory of finite-dimensional algebras we refer to the books [5,47]. The composition of maps $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $gf\colon X \to Z$. For arrows $\alpha\colon i \to j$ and $\beta\colon j \to k$ in a quiver, we write their composition as $\beta\alpha\colon i \to k$. By $\mathbb{N}$ we denote the natural numbers including 0.

2 Cartan matrices and the Weyl group

2.1 Cartan matrices and valued graphs

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix, and let $D = \operatorname{diag}(c_1, \dots, c_n)$ be a symmetrizer of C. The *valued graph* $\Gamma(C)$ of C has vertices $1, \dots, n$ and an (unoriented) edge between i and j if and only if $c_{ij} < 0$. An edge $i \frac{\quad\quad}{} j$ has *value* $(|c_{ji}|, |c_{ij}|)$. In this case, we display this valued edge as

$$i \xrightarrow{\;(|c_{ji}|,|c_{ij}|)\;} j$$

and we just write $i \frac{\quad\quad}{} j$ if $(|c_{ji}|, |c_{ij}|) = (1, 1)$.

A Cartan matrix C is *connected* if $\Gamma(C)$ is a connected graph. In this case, the symmetrizer D is uniquely determined up to multiplication with a positive integer. More precisely, if D is a minimal symmetrizer of a connected Cartan matrix C, then the other symmetrizers of C are given by mD with $m \geq 1$.

2.2 The quadratic form

Define the quadratic form $q_C\colon \mathbb{Z}^n \to \mathbb{Z}$ of C by

$$q_C := \sum_{i=1}^{n} c_i X_i^2 - \sum_{i<j} c_i |c_{ij}| X_i X_j .$$

(Recall that $c_i|c_{ij}| = c_j|c_{ji}|$.) The quadratic form q_C plays a crucial role in the representation theory of the quivers of Cartan type C and more generally of the species (see for example [26]) of type C.

A Cartan matrix C is of *Dynkin* or *Euclidean type* if q_C is positive definite or positive semidefinite, respectively. It is well known that C is of Dynkin type if and only if $\Gamma(C)$ is a disjoint union of Dynkin graphs. (The Dynkin graphs are listed in Sect. 13.2.)

2.3 The Weyl group

As before let $C = (c_{ij})$ be a Cartan matrix, and let $\alpha_1, \dots, \alpha_n$ be the positive simple roots of the Kac–Moody algebra $\mathfrak{g}(C)$ associated with C. For $1 \le i, j \le n$ define

$$s_i(\alpha_j) := \alpha_j - c_{ij}\alpha_i .$$

This yields a reflection $s_i\colon \mathbb{Z}^n \to \mathbb{Z}^n$ on the *root lattice*

$$\mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

where we identify α_i with the ith standard basis vector of $\mathbb{Z}^n$. The *Weyl group* $W(C)$ of $\mathfrak{g}(C)$ is the subgroup of $\mathrm{Aut}(\mathbb{Z}^n)$ generated by $s_1, \dots, s_n$. The Weyl group is finite if and only if C is of Dynkin type.

2.4 Roots

Let

$$\Delta_{\mathrm{re}}(C) := \bigcup_{i=1}^{n} W(\alpha_i)$$

be the set of *real roots* of C.

Let

$$(-,-)_C\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

be the *symmetric bilinear form* of C defined by $(\alpha_i, \alpha_j)_C := c_i c_{ij}$. The *fundamental region* of C is defined by

$$F := \{d \in \mathbb{N}^n \mid d \neq 0,\ \operatorname{supp}(d) \text{ connected}, (d, \alpha_i)_C \leq 0 \text{ for all } 1 \leq i \leq n\},$$

where $\operatorname{supp}(d)$ is the full subgraph of $\Gamma(C)$ given by the vertices i with $d_i \neq 0$. Then

$$\Delta_{\mathrm{im}}(C) := W(F) \cup W(-F)$$

is the set of *imaginary roots* of C.

Let

$$\Delta^+_{\mathrm{re}}(C) := \Delta_{\mathrm{re}}(C) \cap \mathbb{N}^n \quad \text{and} \quad \Delta^+_{\mathrm{im}}(C) := \Delta_{\mathrm{im}}(C) \cap \mathbb{N}^n$$

be the set of *positive real roots* and *positive imaginary roots*, respectively. It turns out that

$$\Delta_{\mathrm{re}}(C) = \Delta^+_{\mathrm{re}}(C) \cup -\Delta^+_{\mathrm{re}}(C) \quad \text{and} \quad \Delta_{\mathrm{im}}(C) = \Delta^+_{\mathrm{im}}(C) \cup -\Delta^+_{\mathrm{im}}(C).$$

Finally, let

$$\Delta(C) := \Delta_{\mathrm{re}}(C) \cup \Delta_{\mathrm{im}}(C)$$

be the set of *roots* of C, and

$$\Delta^+(C) := \Delta(C) \cap \mathbb{N}^n = \Delta^+_{\mathrm{re}}(C) \cup \Delta^+_{\mathrm{im}}(C)$$

is the set of *positive roots*.

By definition, for $w \in W(C)$ and $d \in \Delta(C)$ we have $w(d) \in \Delta(C)$. We have $q_C(d) = c_i$ if $d \in W(\alpha_i)$ is a real root, and $q_C(d) \leq 0$ if d is an imaginary root. The following are equivalent:

(i) C is of Dynkin type;
(ii) $\Delta(C)$ is finite;
(iii) $\Delta_{\mathrm{re}}(C) = \Delta(C)$.

2.5 Coxeter transformations

For an orientation Ω of C and some $1 \leq i \leq n$ let

$$s_i(\Omega) := \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cup \{(s, r) \in \Omega^* \mid i \in \{r, s\}\}.$$

If i is a sink or source in $Q^\circ(C, \Omega)$, then $s_i(\Omega)$ is again an orientation of C. A sequence $\mathbf{i} = (i_1, \ldots, i_n)$ is a *$+$-admissible sequence* for (C, Ω) if $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$, i_1 is a sink in $Q^\circ(C, \Omega)$ and i_k is a sink in the acyclic quiver $Q^\circ(C, s_{i_{k-1}} \cdots s_{i_1}(\Omega))$ for $2 \leq k \leq n$. For such a sequence $\mathbf{i}$ let

$$\beta_{\mathbf{i},k} := \beta_k := \begin{cases} \alpha_{i_1} & \text{if} \quad k = 1, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if} \quad 2 \leq k \leq n \end{cases}$$

where $s_1, \ldots, s_n \in W(C)$. Similarly, define

$$\gamma_{\mathbf{i},k} := \gamma_k := \begin{cases} s_{i_n} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if} \quad 1 \leq k \leq n-1, \\ \alpha_{i_n} & \text{if} \quad k = n. \end{cases}$$

Let

$$c^+ := s_{i_n} s_{i_{n-1}} \cdots s_{i_1} : \mathbb{Z}^n \to \mathbb{Z}^n \text{ and } c^- := s_{i_1} s_{i_2} \cdots s_{i_n} : \mathbb{Z}^n \to \mathbb{Z}^n.$$

be the *Coxeter transformations*. For $k \in \mathbb{Z}$ we set

$$c^k := \begin{cases} (c^+)^k & \text{if} \quad k > 0, \\ (c^-)^{-k} & \text{if} \quad k < 0, \\ \text{id} & \text{if } k = 0. \end{cases}$$

We get

$$\begin{aligned} c^+(\beta_{\mathbf{i},k}) &= (s_{i_n} s_{i_{n-1}} \cdots s_{i_1})(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})) \\ &= s_{i_n} s_{i_{n-1}} \cdots s_{i_k}(\alpha_{i_k}) \\ &= -s_{i_n} s_{i_{n-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) \\ &= -\gamma_{\mathbf{i},k}. \end{aligned}$$

The following two lemmas are well known. For example, they are a consequence of the study of preprojective and preinjective representations of species without oriented cycles.

Lemma 2.1 *Suppose C is not of Dynkin type. Then the elements $c^{-r}(\beta_i)$ and $c^s(\gamma_j)$ with $r, s \geq 0$ and $1 \leq i, j \leq n$ are pairwise different elements in $\Delta^+_{\mathrm{re}}(C)$.*

Let C be of Dynkin type. For $1 \leq i \leq n$ let $p_i \geq 1$ be minimal with $c^{-p_i}(\beta_i) \notin \mathbb{N}^n$, and let $q_j \geq 1$ be minimal such that $c^{q_j}(\gamma_j) \notin \mathbb{N}^n$. It is well known that such p_i and q_j exist. The elements $c^{-r}(\beta_i)$ with $1 \leq i \leq n$ and $0 \leq r \leq p_i - 1$ are pairwise different, and the elements $c^s(\gamma_j)$ with $1 \leq j \leq n$ and $0 \leq s \leq q_j - 1$ are pairwise different.

Lemma 2.2 *Assume that C is of Dynkin type. Then*

$$\Delta^+(C) = \{c^{-r}(\beta_i) \mid 1 \leq i \leq n,\ 0 \leq r \leq p_i - 1\} = \{c^s(\gamma_j) \mid 1 \leq j \leq n,\ 0 \leq s \leq q_j - 1\}.$$

3 Locally free H-modules

For the whole section, let $H = H(C, D, \Omega)$ and $Q = Q(C, \Omega)$ as defined in Sect. 1.4.

3.1 Modules defined by idempotents

Let $M \in \operatorname{rep}(H)$. For $1 \leq i \leq n$ let $M_i := e_i M$. The H-module structure on M is described by the spaces M_i with $1 \leq i \leq n$ and by K-linear maps $M(\alpha)\colon M_{s(\alpha)} \to M_{t(\alpha)}$ with α running through the arrows of the quiver Q. (Of course, these maps need to satisfy the defining relations for H.)

For a non-empty subset J of $\{1, \dots, n\}$ let $e := \sum_{j \in J} e_j$. Thus e is an idempotent in H. We get a vector space decomposition

$$eM = \bigoplus_{j \in J} M_j.$$

For $1 \leq i \leq n$ we set

$$(eM)_i := \begin{cases} M_i & \text{if} \quad i \in J, \\ 0 & \text{otherwise,} \end{cases}$$

and for each arrow α of Q we define a map $(eM)(\alpha)\colon (eM)_{s(\alpha)} \to (eM)_{t(\alpha)}$ by

$$(eM)(\alpha) := \begin{cases} M(\alpha) & \text{if } s(\alpha), \quad t(\alpha) \in J, \\ 0 & \text{otherwise.} \end{cases}$$

This defines an H-module structure on eM. This follows from the nature of the defining relations for H. Namely, for a given i, there are no relations passing through i, i.e. any of the relations either starts in i, ends in i, or does not involve i.

3.2 Description of the projective and injective modules

The algebra H is by definition a path algebra modulo an admissible ideal generated by zero relations and commutativity relations. This implies that each indecomposable projective H-module $P_i := He_i$ has a basis B_i with the following properties: For each path p in Q and each $b \in B_i$ we have $p \cdot b \in B_i \cup \{0\}$. In particular, we can visualize P_i by drawing a graph with vertices the elements in B_i, and an arrow $b \xrightarrow{a} b'$ if for an arrow $a \in Q_1$ and $b, b' \in B_i$ we have $a \cdot b = b'$. We say that P_i has a *multiplicative basis*. Similarly, the indecomposable injective H-modules $I_i := \mathrm{D}(e_i H)$ have a multiplicative basis.

Let i be a vertex of Q. We define an H-module ${}_{-}H_i$ as follows: A basis of $e_i\,{}_{-}H_i$ is given by vectors $a_{i,c}$ with $1 \leq c \leq c_i$, and for $j \in \Omega(-, i)$ a basis of $e_j\,{}_{-}H_i$ is given by vectors $b_{j,c}^{f,g}$ with $1 \leq c \leq c_j$, $1 \leq f \leq f_{ji}$ and $1 \leq g \leq g_{ji}$, and for $s \notin \Omega(-, i)$ define $e_s\,{}_{-}H_i := 0$. The arrows of Q act as follows:

$$\varepsilon_i a_{i,c} := \begin{cases} a_{i,c-1} & \text{if} \quad c \geq 2, \\ 0 & \text{if} \quad c = 1, \end{cases}$$

$$\varepsilon_j b_{j,c}^{f,g} := \begin{cases} b_{j,c-1}^{f,g} & \text{if} \quad c \geq 2, \\ 0 & \text{if} \quad c = 1, \end{cases}$$

and for $0 \leq k < k_{ji}$, $0 \leq f < f_{ji}$ and $1 \leq g \leq g_{ji}$ we have

$$\alpha_{ji}^{(g)} a_{i,c_i - f - k f_{ji}} := b_{j,c_j - k f_{ij}}^{f_{ji} - f, g}.$$

For $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ (and therefore $k_{ji} = 3$) we display a part of the module $(e_i + e_j)\,{}_{-}H_i$ in Fig. 1.

The module ${}_{-}H_i$ has one *i-column* with basis $(a_{i,1}, \dots, a_{i,c_i})$, and for each $j \in \Omega(-, i)$ it has a *j-column* with basis $(b_{j,1}^{f,g}, \dots, b_{j,c_j}^{f,g})$ for each $1 \leq f \leq f_{ji}$ and each $1 \leq g \leq g_{ji}$. By definition we have

$$\dim {}_{-}H_i = c_i + \sum_{j \in \Omega(-,i)} f_{ji} g_{ji} c_j.$$

Fig. 1 Construction of the H-module $(e_i + e_j)_{-}H_i$ with $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ and $k_{ji} = 3$

$$\begin{tikzcd}
a_{i,9} \arrow[r, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & b^{3,g}_{j,6} \arrow[dd, "\varepsilon_j"] & & \\
a_{i,8} \arrow[rr, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & & b^{2,g}_{j,6} \arrow[ddd, "\varepsilon_j"] & \\
& b^{3,g}_{j,5} \arrow[ddd, "\varepsilon_j"] & & \\
a_{i,7} \arrow[rrr, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & & & b^{1,g}_{j,6} \arrow[ddd, "\varepsilon_j"] \\
& & b^{2,g}_{j,5} \arrow[ddd, "\varepsilon_j"] & \\
a_{i,6} \arrow[r, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & b^{3,g}_{j,4} \arrow[ddd, "\varepsilon_j"] & & \\
& & & b^{1,g}_{j,5} \arrow[ddd, "\varepsilon_j"] \\
a_{i,5} \arrow[rr, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & & b^{2,g}_{j,4} \arrow[ddd, "\varepsilon_j"] & \\
& b^{3,g}_{j,3} \arrow[ddd, "\varepsilon_j"] & & \\
a_{i,4} \arrow[rrr, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & & & b^{1,g}_{j,4} \arrow[ddd, "\varepsilon_j"] \\
& & b^{2,g}_{j,3} \arrow[ddd, "\varepsilon_j"] & \\
a_{i,3} \arrow[r, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & b^{3,g}_{j,2} \arrow[ddd, "\varepsilon_j"] & & \\
& & & b^{1,g}_{j,3} \arrow[ddd, "\varepsilon_j"] \\
a_{i,2} \arrow[rr, "\alpha_{ji}^{(g)}"] \arrow[dd, "\varepsilon_i"'] & & b^{2,g}_{j,2} \arrow[ddd, "\varepsilon_j"] & \\
& b^{3,g}_{j,1} & & \\
a_{i,1} \arrow[rrr, "\alpha_{ji}^{(g)}"] & & & b^{1,g}_{j,2} \arrow[dd, "\varepsilon_j"] \\
& & b^{2,g}_{j,1} & \\
& & & b^{1,g}_{j,1}
\end{tikzcd}$$

The number of j-columns of ${}_{-}H_i$ is $f_{ji}g_{ji} = |c_{ji}|$.

Suppose j is a sink in Q°. The module $P_j = e_j P_j$ has a basis $a_{j,1}, \dots, a_{j,c_j}$ such that

$$\varepsilon_j a_{j,c} := \begin{cases} a_{j,c-1} & \text{if} \quad c \geq 2, \\ 0 & \text{if } c = 1, \end{cases} .$$

Then $(a_{j,1}, \dots, a_{j,c_j})$ is the *j-column* of P_j.

Next, assume that i is a vertex of Q such that for each $j \in \Omega(-, i)$ the projective module P_j is already constructed, and P_j has a distinguished basis including a j-column $(a_{j,1}, \dots, a_{j,c_j})$, which forms a basis of $e_j P_j$.

Then P_i is constructed as follows: We take for each $j \in \Omega(-, i)$, $1 \leq f \leq f_{ji}$ and $1 \leq g \leq g_{ji}$ a copy $P_j^{f,g}$ of P_j and identify the j-column $(b_{j,1}^{f,g}, \dots, b_{j,c_j}^{f,g})$ of ${}_{-}H_i$ with the j-column $(a_{j,1}, \dots, a_{j,c_j})$ of $P_j^{f,g}$. The resulting module is our indecomposable projective H-module P_i, and by definition its *i-column* is the i-column $(a_{i,1}, \dots, a_{i,c_i})$ of the module ${}_{-}H_i$.

The indecomposable injective H-modules I_j are constructed dually by gluing modules ${}_jH_{-}$, which are dual to the modules ${}_{-}H_i$. Again, for $c_i = 9$, $c_j = 6$, $f_{ji} = 3$, $f_{ij} = 2$ and $k_{ji} = 3$ we display a part of the module $(e_i + e_j)\,{}_jH_{-}$ in Fig. 2.

Recall that the notion of a locally free module can be found in Definition 1.1. Let $S_1, \dots, S_n$ be the simple H-modules with $\underline{\dim}(S_i) = \alpha_i$, and let $E_1, \dots, E_n$ be the (indecomposable) locally free H-modules with $\underline{\mathrm{rank}}(E_i) = \alpha_i$. (Here $\alpha_1, \dots, \alpha_n$ is the standard basis of $\mathbb{Z}^n$.) We refer to the E_i as the *generalized simple* H-modules. Thus E_i corresponds to the regular representation of H_i. More precisely, we have $E_i = e_i E_i$, and $e_i E_i$ has a basis $a_{i,1}, \dots, a_{i,c_i}$ such that

$$\varepsilon_i a_{i,c} := \begin{cases} a_{i,c-1} & \text{if} \quad c \geq 2, \\ 0 & \text{if } c = 1. \end{cases}$$

In particular, if i is a sink in $Q^\circ(C, \Omega)$, then $E_i = P_i$. Dually, if i is a source in $Q^\circ(C, \Omega)$, then $E_i = I_i$.

It follows from our construction of P_i and I_i that these modules are locally free. Furthermore, we get the following result, which again follows directly from our construction.

Proposition 3.1 *For every $i \in Q_0$, the canonical exact sequence*

$$0 \to \bigoplus_{j \in \Omega(-,i)} P_j^{|c_{ji}|} \to P_i \to E_i \to 0$$

Fig. 2 Construction of the H-module $(e_i + e_j)_j H_-$ with $c_i = 9, c_j = 6, f_{ji} = 3, f_{ij} = 2$ and $k_{ji} = 3$

$$\begin{tikzcd}
b_{i,9}^{2,g} \arrow[d, "\varepsilon_i"] & & \\
b_{i,8}^{2,g} \arrow[dd, "\varepsilon_i"] & & \\
& b_{i,9}^{1,g} \arrow[dd, "\varepsilon_i"] & \\
b_{i,7}^{2,g} \arrow[dd, "\varepsilon_i"] \arrow[rr, "\alpha_{ji}^{(g)}"] & & a_{j,6} \arrow[ddd, "\varepsilon_j"] \\
& b_{i,8}^{1,g} \arrow[dd, "\varepsilon_i"] & \\
b_{i,6}^{2,g} \arrow[dd, "\varepsilon_i"] & & \\
& b_{i,7}^{1,g} \arrow[dd, "\varepsilon_i"] \arrow[r, "\alpha_{ji}^{(g)}"] & a_{j,5} \arrow[ddd, "\varepsilon_j"] \\
b_{i,5}^{2,g} \arrow[dd, "\varepsilon_i"] & & \\
& b_{i,6}^{1,g} \arrow[dd, "\varepsilon_i"] & \\
b_{i,4}^{2,g} \arrow[dd, "\varepsilon_i"] \arrow[rr, "\alpha_{ji}^{(g)}"] & & a_{j,4} \arrow[ddd, "\varepsilon_j"] \\
& b_{i,5}^{1,g} \arrow[dd, "\varepsilon_i"] & \\
b_{i,3}^{2,g} \arrow[dd, "\varepsilon_i"] & & \\
& b_{i,4}^{1,g} \arrow[dd, "\varepsilon_i"] \arrow[r, "\alpha_{ji}^{(g)}"] & a_{j,3} \arrow[ddd, "\varepsilon_j"] \\
b_{i,2}^{2,g} \arrow[dd, "\varepsilon_i"] & & \\
& b_{i,3}^{1,g} \arrow[dd, "\varepsilon_i"] & \\
b_{i,1}^{2,g} \arrow[rr, "\alpha_{ji}^{(g)}"] & & a_{j,2} \arrow[dd, "\varepsilon_j"] \\
& b_{i,2}^{1,g} \arrow[d, "\varepsilon_i"] & \\
& b_{i,1}^{1,g} \arrow[r, "\alpha_{ji}^{(g)}"] & a_{j,1}
\end{tikzcd}$$

is a minimal projective resolution of E_i, and the canonical exact sequence

$$0 \to E_i \to I_i \to \bigoplus_{j \in \Omega(i,-)} I_j^{|c_{ji}|} \to 0$$

is a minimal injective resolution of E_i.

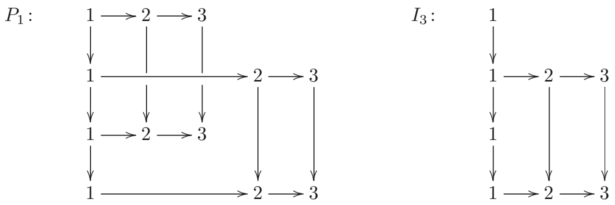


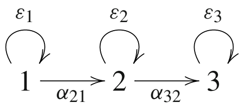
Fig. 3 A projective and an injective H-module for type C_3

3.3 Example

Let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \operatorname{diag}(4, 2, 2)$ and $\Omega = \{(2, 1), (3, 2)\}$. Thus C is a Cartan matrix of type C_3. Then $H = H(C, D, \Omega)$ is given by the quiver



with relations $\varepsilon_1^4 = 0$, $\varepsilon_2^2 = \varepsilon_3^2 = 0$, $\varepsilon_2\alpha_{21} = \alpha_{21}\varepsilon_1^2$ and $\varepsilon_3\alpha_{32} = \alpha_{32}\varepsilon_2$. The indecomposable projective H-module P_1 and the indecomposable injective H-module I_3 are displayed in Fig. 3. The modules P_2 and P_3 are submodules of P_1, and I_1 and I_2 are factor modules of I_3.

3.4 The rank vectors of projective and injective modules

Assume that $\mathbf{i} = (i_1, \dots, i_n)$ is a $+$-admissible sequence for (C, Ω). Without loss of generality, assume that $i_k = k$ for $1 \le k \le n$. Recall that we defined some positive roots $\beta_k, \gamma_k \in \Delta_{\mathrm{re}}^+(C)$ in Sect. 2.5.

Lemma 3.2 *We have* $\underline{\operatorname{rank}}(P_k) = \beta_k$.

Proof By our construction of the indecomposable projective H-modules P_i we get

$$\begin{aligned} \underline{\operatorname{rank}}(P_k) &= \underline{\operatorname{rank}}(E_k) + \sum_{j \in \Omega(-,k)} g_{jk} f_{jk} \underline{\operatorname{rank}}(P_j) \\ &= \underline{\operatorname{rank}}(E_k) + \sum_{j \in \Omega(-,k)} |c_{jk}| \underline{\operatorname{rank}}(P_j). \end{aligned}$$

For $k = 1$ we have $P_k \cong E_k$. Thus we have $\underline{\mathrm{rank}}(P_k) = \alpha_1 = \beta_1$. For $k \geq 2$ we have

$$\begin{aligned}\beta_k &= s_1 \cdots s_{k-2}(\alpha_k - c_{k-1,k}\alpha_{k-1})\\ &= s_1 \cdots s_{k-2}(\alpha_k) - c_{k-1,k}\beta_{k-1}\\ &= \alpha_k - \sum_{j \in \Omega(-,k)} c_{jk}\beta_j\\ &= \alpha_k + \sum_{j \in \Omega(-,k)} |c_{jk}|\beta_j\end{aligned}$$

The claim follows by induction. □

The proof of the next result is similar to the proof of Lemma 3.2.

Lemma 3.3 *We have* $\underline{\mathrm{rank}}(I_k) = \gamma_k$.

As a consequence of Lemmas 3.2 and 3.3 we get the following result.

Proposition 3.4 *We have* $\underline{\mathrm{rank}}(P_i), \underline{\mathrm{rank}}(I_i) \in \Delta_{\mathrm{re}}^{+}(C)$.

3.5 The Coxeter matrix

The *Cartan matrix* C_H of H is the $(n \times n)$-matrix with kth column the dimension vector $\underline{\dim}(P_k)$, $1 \leq k \leq n$. (This is not to be confused with the Cartan matrix C.) It follows that the kth row of C_H is $\underline{\dim}(I_k)$, $1 \leq k \leq n$, see for example [48, Section 2.4, p.70]. The matrix C_H is invertible over $\mathbb{Q}$ (but not necessarily over $\mathbb{Z}$). (We can choose a numbering of the vertices of $Q(C, \Omega)$ such that C_H is an upper triangular matrix with only non-zero entries on the diagonal.) The *Coxeter matrix* of H is defined as

$$\Phi_H := -C_H^T C_H^{-1}$$

where C_H^T denotes the transpose of C_H. It follows that

$$\Phi_H(\underline{\dim}(P_k)) = -\underline{\dim}(I_k)$$

(Here we treat $\underline{\dim}(P_k)$ as a column vector.)

Next, let $C_{H,P}$ be the $(n \times n)$-matrix with kth column the rank vector $\underline{\mathrm{rank}}(P_k)$, and let $C_{H,I}$ be the $(n \times n)$-matrix with kth row the rank vector $\underline{\mathrm{rank}}(I_k)$, $1 \leq k \leq n$. We have

$$C_{H,P} = D^{-1}C_H \text{ and } C_{H,I} = C_H D^{-1}.$$

We get

$$D^{-1}\Phi_H D = -D^{-1}C_H^T C_H^{-1} D = -C_{H,I}^T C_{H,P}^{-1}$$

and this matrix satisfies

$$D^{-1}\Phi_H D(\underline{\mathrm{rank}}(P_k)) := -\underline{\mathrm{rank}}(I_k).$$

Thus by Lemmas 3.2 and 3.3 we can identify $D^{-1}\Phi_H D$ with the Coxeter transformation c^+.

3.6 Homological characterization of locally free modules

Proposition 3.5 *For* $M \in \mathrm{rep}(H)$ *the following are equivalent*:

(i) proj. $\dim(M) \leq 1$;
(ii) inj. $\dim(M) \leq 1$;
(iii) proj. $\dim(M) < \infty$;
(iv) inj. $\dim(M) < \infty$;
(v) M *is locally free.*

Proof For $M = 0$, all properties (i), ..., (v) hold. Thus we assume that M is non-zero.

Let M be locally free. Then there exists a vertex i of the quiver $Q(C, \Omega)$ of H such that $e_i M \neq 0$ and $e_j M = 0$ for all $j \in \Omega(-, i)$. (Here we used that $Q^\circ(C, \Omega)$ is acyclic.) It follows that $e_i M$ is a submodule of M, and $(1 - e_i)M$ is isomorphic to the factor module $M/e_i M$ of M. So we get a short exact sequence

$$0 \to e_i M \to M \to (1 - e_i)M \to 0$$

of H-modules. Note that $e_i M$ and $(1 - e_i)M$ are both locally free. If $(1 - e_i)M = 0$, then $e_i M = M$. In this case, we have $M \cong E_i^m$ for some $m \geq 1$, and Proposition 3.1 yields that (i), ..., (iv) hold for M. If $(1 - e_i)M \neq 0$, then by induction on the dimension we get that (i), ..., (iv) hold for $e_i M$ and for $(1 - e_i)M$. Now one uses long exact homology sequences associated with the short exact sequence above to show that (i), ..., (iv) also hold for M.

Next, assume that M is not locally free. Let i be a vertex of Q such that $e_i M$ is not a free H_i-module. Any projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \tag{3.1}$$

of M yields a projective resolution of H_i-modules

$$\cdots \to e_i P_2 \to e_i P_1 \to e_i P_0 \to e_i M \to 0. \tag{3.2}$$

But H_i is a selfinjective algebra, and $e_i M$ is not a projective H_i-module. Thus the resolution (3.2) and therefore also the resolution (3.1) has to be infinite. This implies $\operatorname{proj.dim}(M) = \infty$. Dually, one shows that $\operatorname{inj.dim}(M) = \infty$. □

For a finite-dimensional algebra A, let $\tau = \tau_A$ denote its Auslander–Reiten translation. Recall that for $X, Y \in \operatorname{mod}(A)$ there are functorial isomorphisms

$$\operatorname{Ext}^1_A(X, Y) \cong \mathrm{D}\,\overline{\operatorname{Hom}}_A(Y, \tau(X)) \cong \mathrm{D}\,\underline{\operatorname{Hom}}_A(\tau^-(Y), X),$$

see for example [48, Section 2.4] for details. These isomorphisms are often referred to as *Auslander–Reiten formulas*. If $\operatorname{proj.dim}(X) \leq 1$, we get a functorial isomorphism

$$\operatorname{Ext}^1_A(X, Y) \cong \mathrm{D}\operatorname{Hom}_A(Y, \tau(X)),$$

and if $\operatorname{inj.dim}(Y) \leq 1$, then

$$\operatorname{Ext}^1_A(X, Y) \cong \mathrm{D}\operatorname{Hom}_A(\tau^-(Y), X).$$

Recall that an A-module X is *τ-rigid* (resp. *τ^--rigid*) if $\operatorname{Hom}_A(X, \tau(X)) = 0$ (resp. $\operatorname{Hom}_A(\tau^-(X), X) = 0$) [1]. Clearly, if X is τ-rigid or τ^--rigid, then X is rigid.

Corollary 3.6 *For $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$ the following are equivalent:*

(i) *M is rigid;*
(ii) *M is τ-rigid;*
(iii) *M is τ^--rigid.*

Combining Propositions 3.1 and 3.5 yields the following result.

Corollary 3.7 *The algebra H is a 1-Iwanaga–Gorenstein algebra.*

Lemma 3.8 *The subcategory $\operatorname{rep}_{\mathrm{l.f.}}(H)$ is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms.*

Proof Let

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be a short exact sequence in $\operatorname{rep}(H)$. For each $1 \leq i \leq n$ this induces a short exact sequence

$$0 \to e_i X \to e_i Y \to e_i Z \to 0$$

of H_i-modules. Recall that $M \in \operatorname{rep}(H)$ is locally free if and only if $e_i M$ is a projective (and therefore also an injective) H_i-module for all i. It follows that if any two of the three modules $e_i X$, $e_i Y$ and $e_i Z$ are projective H_i-modules, then the third module is also projective as an H_i-modules. This finishes the proof. □

For the following definitions, see for example [6]. Let A be a finite-dimensional K-algebra, and let $\mathcal{U}$ be a subcategory of $\operatorname{mod}(A)$. Then $\mathcal{U}$ is a *resolving subcategory* if the following hold:

(i) ${}_A A \in \mathcal{U}$;
(ii) $\mathcal{U}$ is closed under extensions (i.e. for a short exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, if $X, Z \in \mathcal{U}$, then $Y \in \mathcal{U}$);
(iii) $\mathcal{U}$ is closed under kernels of epimorphisms.

Dually, $\mathcal{U}$ is *coresolving* if

(i) $\mathrm{D}(A_A) \in \mathcal{U}$;
(ii) $\mathcal{U}$ is closed under extensions;
(iii) $\mathcal{U}$ is closed under cokernels of monomorphisms.

For $X \in \operatorname{mod}(A)$ a homomorphism $f\colon X \to U$ is a *left $\mathcal{U}$-approximation* of X if $U \in \mathcal{U}$ and

$$\operatorname{Hom}_A(U, U') \xrightarrow{\operatorname{Hom}_A(f,U')} \operatorname{Hom}_A(X, U') \to 0$$

is exact for all $U' \in \mathcal{U}$. Dually, a homomorphism $g\colon U \to X$ is a *right $\mathcal{U}$-approximation* of X if $U \in \mathcal{U}$ and

$$\operatorname{Hom}_A(U', U) \xrightarrow{\operatorname{Hom}_A(U',g)} \operatorname{Hom}_A(U', X) \to 0$$

is exact for all $U' \in \mathcal{U}$. The subcategory $\mathcal{U}$ is *covariantly finite* if every $X \in \operatorname{mod}(A)$ has a left $\mathcal{U}$-approximation. Dually, $\mathcal{U}$ is *contravariantly finite* if every $X \in \operatorname{mod}(A)$ has a right $\mathcal{U}$-approximation. Finally, $\mathcal{U}$ is *functorially finite* if $\mathcal{U}$ is covariantly and contravariantly finite.

Theorem 3.9 *The subcategory* $\operatorname{rep}_{\text{l.f.}}(H)$ *is resolving, coresolving and functorially finite. In particular,* $\operatorname{rep}_{\text{l.f.}}(H)$ *has Auslander–Reiten sequences.*

Proof By Lemma 3.8 and Proposition 3.1 we get that $\operatorname{rep}_{\text{l.f.}}(H)$ is a resolving and coresolving subcategory of $\operatorname{rep}(H)$. Furthermore, by Proposition 3.5 we know that $\operatorname{rep}_{\text{l.f.}}(H)$ coincides with the subcategory of all H-modules with projective dimension 1. Thus $\operatorname{rep}_{\text{l.f.}}(H)$ is covariantly finite by [4, Proposition 4.2]. Since $\operatorname{rep}_{\text{l.f.}}(H)$ also coincides with the subcategory of all H-modules with injective dimension 1, the dual of [4, Proposition 4.2] yields that $\operatorname{rep}_{\text{l.f.}}(H)$ is contravariantly finite. Thus $\operatorname{rep}_{\text{l.f.}}(H)$ is functorially finite in $\operatorname{rep}(H)$. Now it follows from [6, Theorem 2.4] that $\operatorname{rep}_{\text{l.f.}}(H)$ has Auslander–Reiten sequences. □

4 The homological bilinear form

As before, let $H = H(C, D, \Omega)$. For $M, N \in \operatorname{rep}_{\text{l.f.}}(H)$ define

$$\begin{aligned} \langle M, N\rangle_H &:= \dim \operatorname{Hom}_H(M, N) - \dim \operatorname{Ext}^1_H(M, N),\\ (M, N)_H &:= \langle M, N\rangle_H + \langle N, M\rangle_H,\\ q_H(M) &:= \langle M, M\rangle_H. \end{aligned}$$

Proposition 4.1 *For $M, N \in \operatorname{rep}_{\text{l.f.}}(H)$ we have*

$$\langle M, N\rangle_H = \sum_{i=1}^n c_i a_i b_i - \sum_{(j,i)\in\Omega} c_i |c_{ij}| a_i b_j$$

where $\underline{\operatorname{rank}}(M) = (a_1, \dots, a_n)$ *and* $\underline{\operatorname{rank}}(N) = (b_1, \dots, b_n)$.

Proof Let $Q = Q(C, \Omega)$. Let i_1 be a sink in Q°, and let i_n be a source of Q°. We get short exact sequences

$$0 \to E_{i_1}^{a_{i_1}} \xrightarrow{f_1} M \xrightarrow{f_2} M' \to 0 \tag{4.1}$$

and

$$0 \to N' \xrightarrow{g_1} N \xrightarrow{g_2} E_{i_n}^{b_{i_n}} \to 0 \tag{4.2}$$

where f_1 is the obvious canonical inclusion, f_2 is the canonical projection onto $\operatorname{Cok}(f_1)$, g_2 is the obvious canonical projection, and g_1 is the canonical inclusion of $\operatorname{Ker}(g_2)$. Applying $\operatorname{Hom}_H(-, N)$ to sequence (4.1) and $\operatorname{Hom}_H(M, -)$ to the sequence (4.2) yields the long exact cohomology sequences

$$\begin{aligned} 0 \to \operatorname{Hom}_H(M', N) \to \operatorname{Hom}_H(M, N) \to \operatorname{Hom}_H(E_{i_1}^{a_{i_1}}, N)\\ \to \operatorname{Ext}^1_H(M', N) \to \operatorname{Ext}^1_H(M, N) \to \operatorname{Ext}^1_H(E_{i_1}^{a_{i_1}}, N) \to 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} 0 \to \operatorname{Hom}_H(M, N') \to \operatorname{Hom}_H(M, N) \to \operatorname{Hom}_H(M, E_{i_n}^{b_{i_n}})\\ \to \operatorname{Ext}^1_H(M, N') \to \operatorname{Ext}^1_H(M, N) \to \operatorname{Ext}^1_H(M, E_{i_n}^{b_{i_n}}) \to 0. \end{aligned} \tag{4.4}$$

For the exactness of the first cohomology sequence we used that $\text{inj. dim}(N) \leq 1$, and for the second sequence we needed that $\text{proj. dim}(M) \leq 1$, compare Proposition 3.5. The first sequence implies that

$$\langle M, N \rangle_H = \langle M', N \rangle_H + \langle E_{i_1}^{a_{i_1}}, N \rangle_H,$$

and the second sequence yields

$$\langle M, N \rangle_H = \langle M, N' \rangle_H + \langle M, E_{i_n}^{b_{i_n}} \rangle_H.$$

Thus by induction we get

$$\langle M, N \rangle_H = \sum_{1 \leq i, j \leq n} a_i b_j \langle E_i, E_j \rangle_H.$$

For $1 \leq j \leq n$ we have

$$\dim \operatorname{Hom}_H(E_i, E_j) = \dim \operatorname{Hom}_H(P_i, E_j) = \begin{cases} c_i & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the minimal projective resolution of E_i has the form

$$0 \to \bigoplus_{j \in \Omega(-,i)} P_j^{|c_{ji}|} \to P_i \to E_i \to 0. \tag{4.5}$$

Applying $\operatorname{Hom}_H(-, E_j)$ for $1 \leq j \leq n$ yields

$$\dim \operatorname{Ext}_H^1(E_i, E_j) = \begin{cases} c_j |c_{ji}| & \text{if } j \in \Omega(-, i), \\ 0 & \text{otherwise.} \end{cases}$$

Since $c_j c_{ji} = c_i c_{ij}$, the result follows. □

Proposition 4.1 shows that for $M, N \in \operatorname{rep}_{\text{l.f.}}(H)$ the number $\langle M, N \rangle_H$ depends only on the rank vectors $\underline{\operatorname{rank}}(M)$ and $\underline{\operatorname{rank}}(N)$. This implies:

Corollary 4.2 *The map $(M, N) \mapsto \langle M, N \rangle_H$ descends to the Grothendieck group $\mathbb{Z}^n$ of $\operatorname{rep}_{\text{l.f.}}(H)$ and induces a bilinear form $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ still denoted by $\langle -, - \rangle_H$. This bilinear form is characterized by $\langle \alpha_i, \alpha_j \rangle_H = \langle E_i, E_j \rangle_H$, where $\alpha_1, \ldots, \alpha_n$ is the standard basis of $\mathbb{Z}^n$.*

Let

$$(-, -)_H \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

be the symmetrization of $\langle -, - \rangle_H$ defined by $(a, b)_H := \langle a, b \rangle_H + \langle b, a \rangle_H$, and let $q_H : \mathbb{Z}^n \to \mathbb{Z}$ be the quadratic form defined by $q_H(a) := \langle a, a \rangle_H$. The forms q_H and $\langle -, - \rangle_H$ are called the *homological bilinear forms* of H.

Corollary 4.3 *We have* $q_H = q_C$ *and* $(-, -)_H = (-, -)_C$.

Proof By definition we have

$$q_C := \sum_{i=1}^{n} c_i X_i^2 - \sum_{i<j} c_i |c_{ij}| X_i X_j,$$

and we know from Proposition 4.1 that

$$q_H = \sum_{i=1}^{n} c_i X_i^2 - \sum_{(j,i)\in\Omega} c_i |c_{ij}| X_i X_j.$$

Note that q_H does not depend on the orientation Ω, since $c_i c_{ij} = c_j c_{ji}$ for all i, j. Thus we have $q_H = q_C$. Similarly, one also shows easily that $(-, -)_H = (-, -)_C$. □

5 An analogy to the representation theory of modulated graphs

The constructions and results of this section form a crucial part of this article. For example, it contains the foundation for defining reflection functors and Coxeter functors for the algebras $H(C, D, \Omega)$.

5.1 The bimodules ${}_jH_i$

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix with symmetrizer $D = \operatorname{diag}(c_1, \dots, c_n)$, and let Ω be an orientation of C, and let Ω^* be the opposite orientation. Let $H := H(C, D, \Omega)$ and $H^* := H(C, D, \Omega^*)$. Recall that for $1 \le i \le n$ we have

$$H_i := e_i H e_i = K[\varepsilon_i]/(\varepsilon_i^{c_i}).$$

In the following we write $\otimes_i$ for a tensor product $\otimes_{H_i}$ over H_i. If there is no danger of misunderstanding, we also just write $\otimes$ instead of $\otimes_i$.

For $(j, i) \in \Omega$ we define

$$\begin{aligned} {}_jH_i &:= H_j \operatorname{Span}_K(\alpha_{ji}^{(g)} \mid 1 \le g \le g_{ji}) H_i \\ &= \operatorname{Span}_K(\varepsilon_j^{f_j} \alpha_{ji}^{(g)} \varepsilon_i^{f_i} \mid f_j, f_i \ge 0, 1 \le g \le g_{ji}). \end{aligned}$$

Our considerations in Sect. 3.2 show that ${}_jH_i$ is an H_j-H_i-bimodule, which is free as a left H_j-module and free as a right H_i-module. Let ${}_iH_j$ be the corresponding H_i-H_j-bimodule coming from H^*. We get

$$ {}_jH_i = \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ji}-1} H_j(\alpha_{ji}^{(g)}\varepsilon_i^f) = \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ij}-1} (\varepsilon_j^f\alpha_{ji}^{(g)})H_i $$

and

$$ {}_iH_j = \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ij}-1} H_i(\alpha_{ij}^{(g)}\varepsilon_j^f) = \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ji}-1} (\varepsilon_i^f\alpha_{ij}^{(g)})H_j. $$

So we have

$$ \begin{aligned} {}_{H_j}({}_jH_i) &\cong H_j^{|c_{ji}|} \cong ({}_iH_j)_{H_j}, \\ {}_{H_i}({}_iH_j) &\cong H_i^{|c_{ij}|} \cong ({}_jH_i)_{H_i}. \end{aligned} $$

Define

$$ \begin{aligned} {}_jL_i &:= \{\alpha_{ji}^{(g)}, \alpha_{ji}^{(g)}\varepsilon_i, \dots, \alpha_{ji}^{(g)}\varepsilon_i^{f_{ji}-1} \mid 1 \le g \le g_{ij}\}, \\ {}_iL_j &:= \{\alpha_{ij}^{(g)}, \alpha_{ij}^{(g)}\varepsilon_j, \dots, \alpha_{ij}^{(g)}\varepsilon_j^{f_{ij}-1} \mid 1 \le g \le g_{ij}\}, \\ {}_jR_i &:= \{\alpha_{ji}^{(g)}, \varepsilon_j\alpha_{ji}^{(g)}, \dots, \varepsilon_j^{f_{ij}-1}\alpha_{ji}^{(g)} \mid 1 \le g \le g_{ij}\}, \\ {}_iR_j &:= \{\alpha_{ij}^{(g)}, \varepsilon_i\alpha_{ij}^{(g)}, \dots, \varepsilon_i^{f_{ji}-1}\alpha_{ij}^{(g)} \mid 1 \le g \le g_{ij}\}. \end{aligned} $$

Then ${}_jL_i$ (resp. ${}_jR_i$) is a basis of ${}_jH_i$ as a left H_j-modules (resp. as a right H_i-module). We have $|{}_jL_i| = |{}_iR_j| = |c_{ji}|$ and $|{}_iL_j| = |{}_jR_i| = |c_{ij}|$.

Let $({}_jL_i)^*$ and $({}_jR_i)^*$ be the dual basis of $\operatorname{Hom}_{H_j}({}_jH_i, H_j)$ and $\operatorname{Hom}_{H_i}({}_jH_i, H_i)$, respectively. For $b \in {}_jL_i$ or $b \in {}_jR_i$ let b^* be the corresponding dual basis vector. Similarly, define $({}_iL_j)^*$ and $({}_iR_j)^*$.

There is an H_i-H_j-bimodule isomorphism

$$ \rho\colon {}_iH_j \to \operatorname{Hom}_{H_j}({}_jH_i, H_j) $$

given by

$$ \rho\left(\varepsilon_i^{f_{ji}-1-f}\alpha_{ij}^{(g)}\right) = (\alpha_{ji}^{(g)}\varepsilon_i^f)^* $$

for $0 \leq f \leq f_{ji} - 1$ and $1 \leq g \leq g_{ij}$. Indeed, for the left H_i-module structure on $\mathrm{Hom}_{H_j}({}_jH_i, H_j)$ one has

$$\varepsilon_i \cdot (\alpha_{ji}^{(g)} \varepsilon_i^f)^* = \begin{cases} (\alpha_{ji}^{(g)} \varepsilon_i^{f-1})^* & \text{if} \quad f > 0, \\ (\alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1})^* \cdot \varepsilon_j^{f_{ij}} & \text{if} \quad f = 0. \end{cases}$$

Similarly there is an H_i-H_j-bimodule isomorphism

$$\lambda\colon {}_iH_j \to \mathrm{Hom}_{H_i}({}_jH_i, H_i)$$

given by

$$\lambda\left(\alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}-1-f}\right) = (\varepsilon_j^f \alpha_{ji}^{(g)})^*$$

for $0 \leq f \leq f_{ij} - 1$ and $1 \leq g \leq g_{ij}$. In particular, we get $\rho({}_iR_j) = ({}_jL_i)^*$ and $\lambda({}_iL_j) = ({}_jR_i)^*$. In the following, we sometimes identify the spaces $\mathrm{Hom}_{H_j}({}_jH_i, H_j)$, ${}_iH_j$ and $\mathrm{Hom}_{H_i}({}_jH_i, H_i)$ via ρ and λ. For example, for $b \in {}_jL_i$, we consider $b^* \in \mathrm{Hom}_{H_j}({}_jH_i, H_j)$ as an element in ${}_iH_j$.

If N_j is an H_j-module, then we have a natural isomorphism of H_i-modules

$$\mathrm{Hom}_{H_j}({}_jH_i, N_j) \to {}_iH_j \otimes_j N_j$$

defined by

$$f \mapsto \sum_{b \in {}_jL_i} b^* \otimes_j f(b).$$

Now, if in addition M_i is an H_i-module, the adjunction map gives an isomorphism of K-vector spaces:

$$\mathrm{Hom}_{H_j}({}_jH_i \otimes_i M_i, N_j) \to \mathrm{Hom}_{H_i}(M_i, \mathrm{Hom}_{H_j}({}_jH_i, N_j)).$$

Combining these two maps we get a functorial isomorphism of K-vector spaces

$$\mathrm{ad}_{ji} := \mathrm{ad}_{ji}(M_i, N_j)\colon \mathrm{Hom}_{H_j}({}_jH_i \otimes_i M_i, N_j) \to \mathrm{Hom}_{H_i}(M_i, {}_iH_j \otimes_j N_j)$$

given by

$$f \mapsto \left(f^\vee\colon m \mapsto \sum_{b \in {}_jL_i} b^* \otimes_j f(b \otimes_i m)\right).$$

The inverse ad_{ji}^{-1} of ad_{ji} is given by

$$g \mapsto \left(g^{\vee} : h \otimes_i m \mapsto \sum_{b \in {}_j L_i} b^*(h)(g(m))_b \right)$$

where the elements $(g(m))_b \in N_j$ are uniquely determined by

$$g(m) = \sum_{b \in {}_j L_i} b^* \otimes_j (g(m))_b.$$

Here we used that each element x in ${}_iH_j \otimes_j N_j$ can be written uniquely as a sum of the form

$$x = \sum_{b \in {}_j L_i} b^* \otimes_j x_b.$$

5.2 Representation theory of modulated graphs

The tuple $(H_i, {}_iH_j, {}_jH_i)$ defined in Sect. 5.1 is called a *modulation* of C and is denoted by $\mathcal{M}(C, D)$.

For an orientation Ω of C, a *representation* $M = (M_i, M_{ij})$ of $(\mathcal{M}(C, D), \Omega)$ is given by a finite-dimensional H_i-module M_i for each $1 \leq i \leq n$ and an H_i-linear map

$$M_{ij} : {}_iH_j \otimes_j M_j \to M_i$$

for each $(i, j) \in \Omega$. A morphism $f : M \to N$ of representations $M = (M_i, M_{ij})$ and $N = (N_i, N_{ij})$ of $(\mathcal{M}(C, D), \Omega)$ is a tuple $f = (f_i)_i$ of H_i-linear maps $f_i : M_i \to N_i$ for $1 \leq i \leq n$ such that for each $(i, j) \in \Omega$ the diagram

$$\begin{tikzcd}
{}_iH_j \otimes_j M_j \arrow[r, "1 \otimes_j f_j"] \arrow[d, "M_{ij}"] & {}_iH_j \otimes_j N_j \arrow[d, "N_{ij}"] \\
M_i \arrow[r, "f_i"] & N_i
\end{tikzcd}$$

commutes. One easily checks that the representations of $(\mathcal{M}(C, D), \Omega)$ form an abelian category $\mathrm{rep}(C, D, \Omega)$.

For $(M_i, M_{ij}) \in \mathrm{rep}(C, D, \Omega)$ define a representation

$$(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$$

of $H(C, D, \Omega)$ as follows: Define a K-linear map $M(\varepsilon_i)\colon M_i \to M_i$ by

$$M(\varepsilon_i)(m) := \varepsilon_i m.$$

(Here we use that M_i is an H_i-module.) Let $(i, j) \in \Omega$. Recall that ${}_iH_j$ has an H_i-basis

$${}_iL_j = \{\alpha_{ij}^{(g)}, \alpha_{ij}^{(g)}\varepsilon_j, \dots, \alpha_{ij}^{(g)}\varepsilon_j^{f_{ij}-1} \mid 1 \leq g \leq g_{ij}\}.$$

Define a K-linear map $M(\alpha_{ij}^{(g)})\colon M_j \to M_i$ by

$$M(\alpha_{ij}^{(g)})(m) := M_{ij}(\alpha_{ij}^{(g)} \otimes_j m).$$

Now one can check that the relations (H1) and (H2) are satisfied. In other words, $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is a representation of $H(C, D, \Omega)$.

Conversely, let $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ be a representation of $H(C, D, \Omega)$. Note that M_i is an H_i-module via the map $M(\varepsilon_i)$. For $(i, j) \in \Omega$ define an H_i-linear map

$$M_{ij}\colon {}_iH_j \otimes_j M_j \to M_i$$

by

$$M_{ij}(\alpha_{ij}^{(g)}\varepsilon_j^f \otimes m) := (M(\alpha_{ij}^{(g)}) \circ M(\varepsilon_j)^f)(m).$$

Then $(M_i, M_{ij}) \in \operatorname{rep}(C, D, \Omega)$.

These two constructions yield obviously mutually inverse bijections between the representations of $(\mathcal{M}(C, D), \Omega)$ and $H(C, D, \Omega)$. It is also clear how to associate to a morphism in $\operatorname{rep}(C, D, \Omega)$ a morphism in $\operatorname{rep}(H(C, D, \Omega))$ and vice versa. Now it is straightforward to verify the following statement.

Proposition 5.1 *The categories* $\operatorname{rep}(C, D, \Omega)$ *and* $\operatorname{rep}(H(C, D, \Omega))$ *are isomorphic.*

Thus the representation theory of the algebras $H(C, D, \Omega)$ shows a striking analogy to the representation theory of modulated graphs in the sense of Dlab and Ringel [26]. The main difference is that in Dlab and Ringel's theory, the rings H_i would be division rings, whereas in our case they are commutative symmetric algebras, or more precisely, truncations of polynomial rings. Generalizations of the representation theory of modulated graphs have been formulated already in [43].

5.3 Representations of $\Pi(C, D)$

Next, we want to interpret the category $\operatorname{rep}(\Pi(C, D))$ of finite-dimensional representations of $\Pi(C, D)$ as a category of representations of modulated graphs. Let $\operatorname{rep}(C, D, \overline{\Omega})$ be the category with objects $M = (M_i, M_{ij}, M_{ji})$ with $(i, j) \in \Omega$ such that $(M_i, M_{ij}) \in \operatorname{rep}(C, D, \Omega)$ and $(M_i, M_{ji}) \in \operatorname{rep}(C, D, \Omega^*)$. Given two such objects M and N a tuple $f = (f_i)_i$ is a homomorphism $f\colon M \to N$ if f is both a homomorphism $(M_i, M_{ij}) \to (N_i, N_{ij})$ in $\operatorname{rep}(C, D, \Omega)$ and a homomorphism $(M_i, M_{ji}) \to (N_i, N_{ji})$ in $\operatorname{rep}(C, D, \Omega^*)$.

For an object $M = (M_i, M_{ij}, M_{ji})$ in $\operatorname{rep}(C, D, \overline{\Omega})$ let

$$M_{i,\mathrm{in}} := (\operatorname{sgn}(i, j)M_{ij})_j \colon \bigoplus_{j\in\overline{\Omega}(i,-)} {}_iH_j \otimes M_j \to M_i$$

and

$$M_{i,\mathrm{out}} := (M_{ji}^{\vee})_j \colon M_i \to \bigoplus_{j\in\overline{\Omega}(-,i)} {}_iH_j \otimes M_j.$$

These are both H_i-module homomorphisms. (Recall that $M_{ji}^{\vee} = \operatorname{ad}_{ji}(M_{ji})$, see Sect. 5.1.) Set

$$\widetilde{M_i} := \bigoplus_{j\in\overline{\Omega}(i,-)} {}_iH_j \otimes M_j.$$

Since $\overline{\Omega}(i, -) = \overline{\Omega}(-, i)$, we have

$$\bigoplus_{j\in\overline{\Omega}(i,-)} {}_iH_j \otimes M_j = \bigoplus_{k\in\overline{\Omega}(-,i)} {}_iH_k \otimes M_k.$$

Thus we get a diagram

$$\widetilde{M_i} \xrightarrow{M_{i,\mathrm{in}}} M_i \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M_i}.$$

Proposition 5.2 *The category* $\operatorname{rep}(\Pi(C, D))$ *is isomorphic to the full subcategory of* $\operatorname{rep}(C, D, \overline{\Omega})$ *with objects* $M = (M_i, M_{ij}, M_{ji})$ *such that*

$$M_{i,\mathrm{in}} \circ M_{i,\mathrm{out}} = 0$$

for all i.

Proof For an object $M = (M_i, M_{ij}, M_{ji})$ in $\operatorname{rep}(C, D, \overline{\Omega})$, the composition

$$M_{i,\mathrm{in}} \circ M_{i,\mathrm{out}} = \sum_{j \in \overline{\Omega}(-,i)} \operatorname{sgn}(i, j) M_{ij} \circ M_{ji}^{\vee}$$

is in $\operatorname{End}_{H_i}(M_i)$ and maps an element $m \in M_i$ to

$$M_{i,\mathrm{in}} \circ M_{i,\mathrm{out}}(m) = \sum_{j \in \overline{\Omega}(-,i)} \operatorname{sgn}(i, j) \sum_{b \in {}_jL_i} M_{ij} \left(b^* \otimes_j M_{ji}(b \otimes_i m)\right).$$

Let $b \in {}_jL_i$. Thus we have $b = \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f}$ for some $0 \leq f \leq f_{ji} - 1$. This implies that $b^* = \varepsilon_i^f \alpha_{ij}^{(g)} \in {}_iR_j$. It follows that

$$\begin{aligned} &\operatorname{sgn}(i, j) M_{ij} \left(b^* \otimes_j M_{ji}(b \otimes_i m)\right) \\ &\quad = \operatorname{sgn}(i, j) M(\varepsilon_i)^f M(\alpha_{ij}^{(g)}) M(\alpha_{ji}^{(g)}) M(\varepsilon_i)^{f_{ji}-1-f}(m). \end{aligned}$$

In view of the defining relation (P3) of $\Pi(C, D)$, this yields the result. □

6 The algebras *H* and Π are tensor algebras

Let A be a K-algebra, and let $M = {}_AM_A$ be an A-A-bimodule. The *tensor algebra* $T_A(M)$ is defined as

$$T_A(M) := \bigoplus_{k \geq 0} M^{\otimes k}$$

where $M^0 := A$, and $M^{\otimes k}$ is the k-fold tensor product of M for $k \geq 1$. The multiplication of $T_A(M)$ is defined as follows: For $r, s \geq 1$, $m_i, m_i' \in M$ and $a, a' \in A$ let

$$(m_1 \otimes \cdots \otimes m_r) \cdot (m_1' \otimes \cdots \otimes m_s') := (m_1 \otimes \cdots \otimes m_r \otimes m_1' \otimes \cdots \otimes m_s')$$

and

$$a(m_1 \otimes \cdots \otimes m_r)a' := (am_1 \otimes \cdots \otimes m_r a').$$

Recall that the modules over a tensor algebra $T_A(M)$ are given by the A-module homomorphisms $M \otimes_A X \to X$, where X is an A-module.

Let A be a K-algebra, A_0 a subalgebra and A_1 an A_0-A_0-subbimodule of A. Following [10] we say that A is *freely generated by* A_1 *over* A_0 if the following holds: For every K-algebra B and any pair (f_0, f_1) with $f_0\colon A_0 \to B$

an algebra homomorphism, and $f_1\colon A_1 \to B$ an A_0-A_0-bimodule homomorphism (with the A_0-A_0-bimodule structure on B given by f_0) there exists a unique K-algebra homomorphism $f\colon A \to B$ which extends f_0 and f_1. The following two lemmas can be found in [10, Section 1].

Lemma 6.1 *For any K-algebra A and any A-A-bimodule M the tensor algebra $T_A(M)$ is freely generated by M over A.*

Lemma 6.2 *Let A be a K-algebra which is freely generated by A_1 over A_0. Then A is isomorphic to the tensor algebra $T_{A_0}(A_1)$.*

Let Q be a finite quiver, and let $w\colon Q_1 \to \{0, 1\}$ be a map assigning to each arrow of Q a *degree*. Then the path algebra KQ is naturally $\mathbb{N}$-graded: Each path gets as degree the sum of the degrees of its arrows. By definition the paths of length 0 have degree 0. Let $r_1, \dots, r_m$ be a set of relations for KQ which are homogeneous with respect to this grading. Suppose that there is some $1 \le l \le m$ such that $\deg(r_i) = 0$ for $1 \le i \le l$ and $\deg(r_j) = 1$ for $l + 1 \le j \le m$.

Let $A := KQ/I$, where I is the ideal generated by $r_1, \dots, r_m$. Clearly, A is again $\mathbb{N}$-graded. Let A_i be the subspace of elements with degree i. Observe that A_1 is naturally an A_0-A_0-bimodule. Now Lemmas 6.1, 6.2 yield the following result.

Proposition 6.3 *A is isomorphic to the tensor algebra $T_{A_0}(A_1)$.*

As before, let $H = H(C, D, \Omega)$. Define

$$S := \prod_{i=1}^{n} H_i \text{ and } B := \bigoplus_{(i,j)\in\Omega} {}_iH_j.$$

Clearly, B is an S-S-bimodule.

Proposition 6.4 $H \cong T_S(B)$.

Proof The algebra H is graded by defining $\deg(\varepsilon_i) := 0$ and $\deg(\alpha_{ij}^{(g)}) := 1$ for all $(i, j) \in \Omega$ and all g. The defining relations for H are homogeneous, S is the subalgebra of elements of degree 0, and B is the subspace of elements of degree 1. Now we can apply Proposition 6.3. □

Let $\Pi = \Pi(C, D, \Omega)$ be the preprojective algebra. Define $\deg(\varepsilon_i) := 0$ for all i, and for $(i, j) \in \Omega$ let $\deg(\alpha_{ij}^{(g)}) := 0$ and $\deg(\alpha_{ji}^{(g)}) := 1$ for all g. Let

$$\Pi_1 := \Pi(C, D, \Omega)_1$$

be the subspace of Π consisting of the elements of degree 1. Note that Π_1 is an H-H-bimodule. Again we can apply Proposition 6.3 and get the following result.

Proposition 6.5 $\Pi \cong T_H(\Pi_1)$.

Define

$$\overline{B} := \bigoplus_{(i,j)\in\overline{\Omega}} {}_iH_j.$$

Next, for $1 \leq i \leq n$ let

$$\rho_i := \sum_{j\in\overline{\Omega}(-,i)} \operatorname{sgn}(i,j) \sum_{b\in {}_jL_i} b^* \otimes_j b \in T_S(\overline{B}).$$

Every $b \in {}_jL_i$ is of the form $b = \alpha_{ji}^{(g)}\varepsilon_i^{f_{ji}-1-f}$ for some $0 \leq f \leq f_{ji}-1$ and $1 \leq g \leq g_{ij}$. Then $b^* \in {}_iR_j$ is equal to $\varepsilon_i^f\alpha_{ij}^{(g)}$. Thus ρ_i translates to the defining relation (P3)

$$\sum_{j\in\overline{\Omega}(-,i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i,j)\varepsilon_i^f\alpha_{ij}^{(g)}\alpha_{ji}^{(g)}\varepsilon_i^{f_{ji}-1-f} = 0.$$

of Π.

The algebra $T_S(\overline{B})/(\rho_1,\dots,\rho_n)$ is an analogue of Dlab and Ringel's [28] definition of a preprojective algebra of a modulated graph.

Proposition 6.6 $\Pi \cong T_S(\overline{B})/(\rho_1,\dots,\rho_n)$.

Proof Similarly as in the proof of Proposition 6.4 one shows that $T_S(\overline{B})$ is isomorphic to the path algebra $K\overline{Q}$ modulo the defining relations (P1) and (P2) of Π.

Let M be a module over the tensor algebra $T_S(\overline{B})$. Then M is defined by the structure maps

$$M_{ij} : {}_iH_j \otimes_j M_j \to M_i$$

for each $(i,j) \in \overline{\Omega}$. This yields maps

$$M_{iji} := M_{ij} \circ (\operatorname{id}_{{}_iH_j} \otimes_j M_{ji}) : {}_iH_j \otimes_j {}_jH_i \otimes_i M_i \to M_i.$$

Now M is a module over $T_S(\overline{B})/(\rho_1,\dots,\rho_n)$ if and only if for each vertex i and each $m \in M_i$ we have $\rho_i m = 0$. This is equivalent to

$$\sum_{j\in\overline{\Omega}(-,i)} \operatorname{sgn}(i,j)M_{iji}(\sum_{b\in {}_jL_i} b^* \otimes_j b \otimes_i m) = 0.$$

It follows from the definitions that

$$\sum_{j\in\overline{\Omega}(-,i)} \operatorname{sgn}(i,j)M_{iji}(\sum_{b\in{}_jL_i} b^* \otimes_j b \otimes_i m) = (M_{i,\mathrm{in}} \circ M_{i,\mathrm{out}})(m).$$

Now Proposition 5.2 yields the result. □

7 Projective resolutions of *H*-modules

Proposition 7.1 *We have a short exact sequence of H-H-bimodules*

$$P_\bullet: \qquad 0 \to \bigoplus_{(j,i)\in\Omega} He_j \otimes_j {}_jH_i \otimes_i e_iH \xrightarrow{d} \bigoplus_{k=1}^{n} He_k \otimes_k e_kH \xrightarrow{\mathrm{mult}} H \to 0 \tag{7.1}$$

where

$$d(p \otimes_j h \otimes_i q) := ph \otimes_i q - p \otimes_j hq.$$

Proof We know that $H = T_S(B)$. The sequence $P_\bullet$ is isomorphic to the sequence

$$0 \to H \otimes_S B \otimes_S H \xrightarrow{d} H \otimes_S H \xrightarrow{\mathrm{mult}} H \to 0$$

of H-H-bimodules, where $d(h \otimes b \otimes h') := (hb \otimes h' - h \otimes bh')$. Now the statement follows from [52, Theorems 10.1 and 10.5]. □

The components of $P_\bullet$ are projective as left H-modules and as right H-modules. However, the components are not projective as H-H-bimodules. (Except, if S is semisimple, then the first two components are in fact projective bimodules.) In any case, viewed as a short exact sequence of left or right modules, $P_\bullet$ splits as an exact sequence of projective modules.

Corollary 7.2 *If $M \in \mathrm{rep}_{\mathrm{l.f.}}(H)$, then $P_\bullet \otimes_H M$ is a projective resolution of M. Explicitly, $P_\bullet \otimes_H M$ looks as follows*

$$0 \to \bigoplus_{(j,i)\in\Omega} He_j \otimes_j {}_jH_i \otimes_i M_i \xrightarrow{d\otimes M} \bigoplus_{k=1}^{n} He_k \otimes_k M_k \xrightarrow{\mathrm{mult}} M \to 0 \tag{7.2}$$

where

$$(d \otimes M)(p \otimes_j h \otimes_i m) = ph \otimes_i m - p \otimes_j M_{ji}(h \otimes_i m).$$

(Here $M_{ji}\colon {}_jH_i \otimes_i M_i \to M_j$ *is the* H_j*-linear structure map of* M *associated with* $(j, i) \in \Omega$.*)*

Proof By the remarks above, $P_\bullet \otimes_H M$ is always exact. If M is locally free, then $e_kH \otimes_H M = e_kM$ and ${}_jH_i \otimes_i e_iH \otimes_H M = {}_jH_i \otimes_i M_i$ are free H_k- resp. H_j-modules. Thus the relevant components of $P_\bullet \otimes_H M$ are indeed projective. □

8 The trace pairing

8.1 The trace pairing for homomorphisms between free H_i-modules

For each $i = 1, 2, \dots, n$ we have the K-linear map

$$t_i^{\max}\colon H_i \to K$$

defined by

$$\sum_{j=0}^{c_i-1} \lambda_j \varepsilon_i^j \mapsto \lambda_{c_i-1}.$$

For free H_i-modules U and V, with V finitely generated, the *trace pairing* is the non-degenerate, bilinear form

$$\mathrm{tr} := \mathrm{tr}_{U,V}\colon \operatorname{Hom}_{H_i}(U, V) \times \operatorname{Hom}_{H_i}(V, U) \to K$$

defined by

$$(f, g) \mapsto t_i^{\max}(\mathrm{Tr}_{H_i}(f \circ g)).$$

It induces an isomorphism

$$\operatorname{Hom}_{H_i}(U, V) \to \mathrm{D}\operatorname{Hom}_{H_i}(V, U),\ f \mapsto \mathrm{tr}(f, -).$$

Note, that for $U = \bigoplus_{j\in J} H_iu_j$ and $V = \bigoplus_{k=1}^r H_iv_k$ we have

$$\operatorname{Hom}_{H_i}(U, V) = \prod_{j\in J} \bigoplus_{k=1}^{r} \operatorname{Hom}_{H_i}(H_iu_j, H_iv_k) \text{ and}$$

$$\begin{aligned}\mathrm{D}\,\mathrm{Hom}_{H_i}(V, U) &= \mathrm{D}\left(\bigoplus_{j\in J}\bigoplus_{k=1}^{r}\mathrm{Hom}_{H_i}(H_i v_k, H_i u_j)\right)\\ &= \prod_{j\in J}\bigoplus_{k=1}^{r}\mathrm{D}\,\mathrm{Hom}_{H_i}(H_i v_k, H_i u_j).\end{aligned}$$

Let W be another finitely generated free H_i-module. The following lemma is easily verified:

Lemma 8.1 *For $f \in \mathrm{Hom}_{H_i}(V, W)$ the following diagram of natural morphisms commutes:*

$$\begin{tikzcd}
\mathrm{Hom}_{H_i}(U, V) \arrow[r, "\mathrm{Hom}(U,f)"] \arrow[d] & \mathrm{Hom}_{H_i}(U, W) \arrow[d] \\
\mathrm{D}\,\mathrm{Hom}_{H_i}(V, U) \arrow[r, "\mathrm{D}\,\mathrm{Hom}(f,U)"] & \mathrm{D}\,\mathrm{Hom}_{H_i}(W, U)
\end{tikzcd}$$

In other words, under the trace pairing the transpose of $\mathrm{Hom}_{H_i}(f, U)$ *is identified with* $\mathrm{Hom}_{H_i}(U, f)$.

8.2 Adjunction and trace pairing

Recall from Sect. 5.1 that for $(j, i) \in \overline{\Omega}$ we have isomorphisms of H_i-H_j-bimodules

$$\mathrm{ad}_{ji}\colon\ \mathrm{Hom}_{H_j}({}_jH_i, H_j) \to \mathrm{Hom}_{H_i}(H_i, {}_iH_j),$$

where we abbreviate ${}_jH_i \otimes_i H_i = {}_jH_i$ and ${}_iH_j \otimes_j H_j = {}_iH_j$.

Lemma 8.2 *The diagram of natural isomorphisms*

$$\begin{tikzcd}
\mathrm{Hom}_{H_j}({}_jH_i, H_j) \arrow[r, "\mathrm{ad}_{ji}"] \arrow[d] & \mathrm{Hom}_{H_i}(H_i, {}_iH_j) \arrow[d] \\
\mathrm{D}\,\mathrm{Hom}_{H_j}(H_j, {}_jH_i) \arrow[r, "\mathrm{D}(\mathrm{ad}_{ji})"] & \mathrm{D}\,\mathrm{Hom}_{H_i}({}_iH_j, H_i)
\end{tikzcd}$$

with the vertical isomorphisms induced by the respective trace pairings, commutes.

Proof We have to show that for any $\phi \in \operatorname{Hom}_{H_j}({}_jH_i, H_j)$ and $\psi \in \operatorname{Hom}_{H_i}({}_iH_j, H_i)$ we have

$$\operatorname{tr}_{H_i, {}_iH_j}(\operatorname{ad}_{ji}(\phi), \psi) = \operatorname{tr}_{{}_jH_i, H_j}(\phi, \operatorname{ad}_{ij}(\psi)) \tag{8.1}$$

To this end, write

$$\phi = \sum_{g=1}^{g_{ij}} \sum_{k=0}^{f_{ji}-1} (\alpha_{ji}^{(g)} \epsilon_i^k)^* \phi^{(g,k)} \quad \text{with} \quad \phi^{(g,k)} = \sum_{l=0}^{c_j-1} \phi_l^{(g,k)} \epsilon_j^l \in H_j \quad \text{and}$$
$$\psi = \sum_{g=1}^{g_{ij}} \sum_{k'=0}^{f_{ij}-1} (\alpha_{ij}^{(g)} \epsilon_j^{k'})^* \psi^{(g,k')} \quad \text{with} \quad \psi^{(g,k')} = \sum_{l'=0}^{c_i-1} \psi_{l'}^{(g,k')} \epsilon_i^{l'} \in H_i,$$

where we use heavily the notation from Sects. 1.4 and 5.1. Now a straightforward, though tedious, calculation shows that both sides of (8.1) yield

$$\sum_{g=1}^{g_{ij}} \sum_{k'=0}^{f_{ij}-1} \sum_{k=0}^{f_{ji}-1} \sum_{l=0}^{k_{ij}-1} \psi_{k+f_{ji}l}^{(g,k')} \, \phi_{k'+f_{ij}(k_{ij}-1-l)}^{(g,k)}.$$

□

As a direct consequence we obtain the following:

Proposition 8.3 *Let M be a free H_i-module, and N a finitely generated free H_j-module, and denote by $\mathrm{D}(\operatorname{ad}_{ij})\colon \mathrm{D}\operatorname{Hom}_{H_j}(N, {}_jH_i \otimes_i M) \to \mathrm{D}\operatorname{Hom}_{H_i}({}_iH_j \otimes_j N, M)$ the transpose of $\operatorname{ad}_{ij} = \operatorname{ad}_{ij}(N, M)$. We have a commutative diagram*

$$\begin{tikzcd}
\operatorname{Hom}_{H_j}({}_jH_i \otimes_i M, N) \arrow[r, "\operatorname{ad}_{ji}"] \arrow[d] & \operatorname{Hom}_{H_i}(M, {}_iH_j \otimes_j N) \arrow[d] \\
\mathrm{D}\operatorname{Hom}_{H_j}(N, {}_jH_i \otimes_i M) \arrow[r, "\mathrm{D}(\operatorname{ad}_{ij})"] & \mathrm{D}\operatorname{Hom}_{H_i}({}_iH_j \otimes_j N, M)
\end{tikzcd}$$

where the vertical arrows are the isomorphisms induced by the trace pairings.

Proof In fact, this follows from Lemma 8.2 since ad_{ji} and the isomorphisms induced by the trace pairing are in fact natural transformations between the corresponding bifunctors defined on pairs of free modules. □

9 Reflection functors

In this section, let $H = H(C, D, \Omega)$ and $\Pi = \Pi(C, D)$.

9.1 Reflection functors for Π

We keep the notations of Sect. 5.3. Let $M \in \operatorname{rep}(\Pi)$. Thus we have $M = (M_i, M_{ij}, M_{ji})$, where (i, j) runs over Ω, such that $M_{i,\mathrm{in}} \circ M_{i,\mathrm{out}} = 0$ for each i. Hence for every i, we have $M_{i,\mathrm{out}}(M_i) \subseteq \operatorname{Ker}(M_{i,\mathrm{in}})$.

Generalizing the construction in [8, Section 2.2], see also [12], we fix some vertex i and construct a new Π-module by replacing the diagram

$$\widetilde{M}_i \xrightarrow{M_{i,\mathrm{in}}} M_i \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M}_i$$

by

$$\widetilde{M}_i \xrightarrow{\overline{M}_{i,\mathrm{out}} M_{i,\mathrm{in}}} \operatorname{Ker}(M_{i,\mathrm{in}}) \xrightarrow{\mathrm{can}} \widetilde{M}_i$$

where $\overline{M}_{i,\mathrm{out}} \colon M_i \to \operatorname{Ker}(M_{i,\mathrm{in}})$ is induced by $M_{i,\mathrm{out}}$ and can is the canonical inclusion. Gluing this new datum with the remaining part of M gives a new Π-module $\Sigma_i^+(M)$.

Similarly, replacing

$$\widetilde{M}_i \xrightarrow{M_{i,\mathrm{in}}} M_i \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M}_i$$

by

$$\widetilde{M}_i \xrightarrow{\mathrm{can}} \operatorname{Cok}(M_{i,\mathrm{out}}) \xrightarrow{M_{i,\mathrm{out}} \overline{M}_{i,\mathrm{in}}} \widetilde{M}_i$$

where $\overline{M}_{i,\mathrm{in}} \colon \operatorname{Cok}(M_{i,\mathrm{out}}) \to M_i$ is induced by $M_{i,\mathrm{in}}$ and can is the canonical projection. Gluing this new datum with the remaining part of M gives a new Π-module denoted by $\Sigma_i^-(M)$.

The above constructions are obviously functorial. It is straightforward to show that Σ_i^+ is left exact, and Σ_i^- is right exact. Both functors are covariant, K-linear and additive.

The commutative diagram

$$\begin{tikzcd}
\widetilde{M}_i \arrow[r, "\mathrm{can}"] \arrow[d, equal] & \operatorname{Cok}(M_{i,\mathrm{out}}) \arrow[r, "M_{i,\mathrm{out}}\overline{M}_{i,\mathrm{in}}"] \arrow[d, "\overline{M}_{i,\mathrm{in}}"] & \widetilde{M}_i \arrow[d, equal] \\
\widetilde{M}_i \arrow[r, "M_{i,\mathrm{in}}"] \arrow[d, equal] & M_i \arrow[r, "M_{i,\mathrm{out}}"] \arrow[d, "\overline{M}_{i,\mathrm{out}}"] & \widetilde{M}_i \arrow[d, equal] \\
\widetilde{M}_i \arrow[r, "\overline{M}_{i,\mathrm{out}}M_{i,\mathrm{in}}"] & \operatorname{Ker}(M_{i,\mathrm{in}}) \arrow[r, "\mathrm{can}"] & \widetilde{M}_i
\end{tikzcd}$$

of H_i-module homomorphisms summarizes the situation and also shows the existence of canonical homomorphisms $\Sigma_i^-(M) \to M \to \Sigma_i^+(M)$.

For $M \in \operatorname{rep}(\Pi)$ let $\operatorname{sub}_i(M)$ be the largest submodule U of M such that $e_i U = U$, and let $\operatorname{soc}_i(M)$ be the largest submodule V of M such that V is isomorphic to a direct sum of copies of S_i. For example, we have $\operatorname{sub}_i(E_i) = E_i$ and $\operatorname{soc}_i(E_i) \cong S_i$. Dually, let $\operatorname{fac}_i(M)$ be the largest factor module M/U of M such that $e_i(M/U) = M/U$, and let $\operatorname{top}_i(M)$ be the largest factor module M/V of M such that M/V is isomorphic to a direct sum of copies of S_i. All these constructions are functorial.

The proof of the following proposition follows almost word by word the proof of Baumann and Kamnitzer [8, Proposition 2.5], who deal with classical preprojective algebras associated with Dynkin quivers. One difference is that we need to work with sub_i and fac_i instead of soc_i and top_i.

Proposition 9.1 *For each i the following hold:*

(i) *The pair* (Σ_i^-, Σ_i^+) *is a pair of adjoint functors, i.e. there is a functorial isomorphism*

$$\operatorname{Hom}_\Pi(\Sigma_i^-(M), N) \cong \operatorname{Hom}_\Pi(M, \Sigma_i^+(N)).$$

(ii) *The adjunction morphisms* $\mathrm{id} \to \Sigma_i^+\Sigma_i^-$ *and* $\Sigma_i^-\Sigma_i^+ \to \mathrm{id}$ *can be inserted in functorial short exact sequences*

$$0 \to \operatorname{sub}_i \to \mathrm{id} \to \Sigma_i^+\Sigma_i^- \to 0$$

and

$$0 \to \Sigma_i^-\Sigma_i^+ \to \mathrm{id} \to \operatorname{fac}_i \to 0.$$

Proof To establish (i), it is enough to define a pair of mutually inverse bijections between $\operatorname{Hom}_\Pi(\Sigma_i^-(M), N)$ and $\operatorname{Hom}_\Pi(M, \Sigma_i^+(N))$ for any Π-modules M and N, which are functorial in M and N. The construction looks as follows. Consider a morphism $f\colon M \to \Sigma_i^+(N)$. By definition, this is a collection of H_j-module homomorphisms

$$f_j\colon M_j \to (\Sigma_i^+(N))_j$$

with $1 \le j \le n$ such that the diagram

$$\begin{tikzcd}
{}_iH_j \otimes_j M_j \arrow[r, "1\otimes f_j"] \arrow[d, "M_{ij}"] & {}_iH_j \otimes_j (\Sigma_i^+(N))_j \arrow[d, "(\Sigma_i^+(N))_{ij}"] \\
M_i \arrow[r, "f_i"] & (\Sigma_i^+(N))_i
\end{tikzcd}$$

commutes for all $(i, j) \in \overline{\Omega}$. Recall that

$$\widetilde{M}_i = \bigoplus_{j \in \overline{\Omega}(i,-)} {}_iH_j \otimes_j M_j.$$

Set

$$\widetilde{f}_i := \bigoplus_{j \in \overline{\Omega}(i,-)} 1 \otimes f_j \colon \widetilde{M}_i \to \widetilde{N}_i.$$

In the diagram

$$\begin{tikzcd}
\widetilde{M}_i \arrow[r, "M_{i,\mathrm{in}}"] \arrow[d, "\widetilde{f}_i"] & M_i \arrow[r, "M_{i,\mathrm{out}}"] \arrow[d, "f_i"] & \widetilde{M}_i \arrow[r, "\pi"] \arrow[d, "\widetilde{f}_i"] & \mathrm{Cok}(M_{i,\mathrm{out}}) \arrow[r, "M_{i,\mathrm{out}}\overline{M}_{i,\mathrm{in}}"] \arrow[d, dashed, "g_i"] & \widetilde{M}_i \arrow[d, "\widetilde{f}_i"] \\
\widetilde{N}_i \arrow[r, "\overline{N}_{i,\mathrm{out}}N_{i,\mathrm{in}}"] & \mathrm{Ker}(N_{i,\mathrm{in}}) \arrow[r, "\iota"] & \widetilde{N}_i \arrow[r, "N_{i,\mathrm{in}}"] & N_i \arrow[r, "N_{i,\mathrm{out}}"] & \widetilde{N}_i
\end{tikzcd}$$

the two left squares commute.

There is thus a unique map g_i making the third square commutative. (Observe that $N_{i,\mathrm{in}}\widetilde{f}_i M_{i,\mathrm{out}} = N_{i,\mathrm{in}}\iota f_i = 0$. Thus $N_{i,\mathrm{in}}\widetilde{f}_i$ factors through the cokernel of $M_{i,\mathrm{out}}$.)

The fourth square also commutes. Thus if we set $g_j := f_j$ for all vertices $j \neq i$ we get a homomorphism $g \colon \Sigma_i^-(M) \to N$. Conversely, consider a homomorphism $g \colon \Sigma_i^-(M) \to N$ and set

$$\widetilde{g}_i := \bigoplus_{j \in \overline{\Omega}(-,i)} 1 \otimes g_j \colon \widetilde{M}_i \to \widetilde{N}_i.$$

In the diagram

$$\begin{tikzcd}
\widetilde{M}_i \arrow[r, "M_{i,\mathrm{in}}"] \arrow[d, "\widetilde{g}_i"] & M_i \arrow[r, "M_{i,\mathrm{out}}"] \arrow[d, dashed, "f_i"] & \widetilde{M}_i \arrow[r, "\pi"] \arrow[d, "\widetilde{g}_i"] & \mathrm{Cok}(M_{i,\mathrm{out}}) \arrow[r, "M_{i,\mathrm{out}}\overline{M}_{i,\mathrm{in}}"] \arrow[d, "g_i"] & \widetilde{M}_i \arrow[d, "\widetilde{g}_i"] \\
\widetilde{N}_i \arrow[r, "\overline{N}_{i,\mathrm{out}}N_{i,\mathrm{in}}"] & \mathrm{Ker}(N_{i,\mathrm{in}}) \arrow[r, "\iota"] & \widetilde{N}_i \arrow[r, "N_{i,\mathrm{in}}"] & N_i \arrow[r, "N_{i,\mathrm{out}}"] & \widetilde{N}_i
\end{tikzcd}$$

the two right squares commute. Thus there is a unique map f_i making the second square commutative. The first square then also commutes. Thus if we set $f_j := g_j$ for all the vertices $j \neq i$, we get a morphism $f \colon M \to \Sigma_i^+(N)$.

To establish (ii), one checks that $\Sigma_i^-\Sigma_i^+(M)$ is the Π-module obtained by replacing in M the part

$$\widetilde{M_i} \xrightarrow{M_{i,\mathrm{in}}} M_i \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M_i}$$

with

$$\widetilde{M_i} \xrightarrow{M_{i,\mathrm{in}}} \mathrm{Im}(M_{i,\mathrm{in}}) \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M_i}$$

and that $\Sigma_i^+\Sigma_i^-(M)$ is the Π-module obtained by replacing in M the part

$$\widetilde{M_i} \xrightarrow{M_{i,\mathrm{in}}} M_i \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M_i}$$

with

$$\widetilde{M_i} \xrightarrow{M_{i,\mathrm{in}}} \mathrm{Im}(M_{i,\mathrm{out}}) \xrightarrow{M_{i,\mathrm{out}}} \widetilde{M_i}.$$

It remains to observe that as vector spaces, $\mathrm{fac}_i(M) \cong \mathrm{Cok}(M_{i,\mathrm{in}})$ and $\mathrm{sub}_i(M) \cong \mathrm{Ker}(M_{i,\mathrm{out}})$. □

For the following corollary, observe that $\mathrm{sub}_i(M) = 0$ if and only if $\mathrm{soc}_i(M) = 0$. Dually, $\mathrm{fac}_i(M) = 0$ if and only if $\mathrm{top}_i(M) = 0$.

Corollary 9.2 *The functors $\Sigma_i^+ : \mathcal{T}_i \to \mathcal{S}_i$ and $\Sigma_i^- : \mathcal{S}_i \to \mathcal{T}_i$ define inverse equivalences of the subcategories*

$$\mathcal{T}_i := \left\{M \in \mathrm{rep}(\Pi) \mid \mathrm{top}_i(M) = 0\right\}$$

and

$$\mathcal{S}_i := \{M \in \mathrm{rep}(\Pi) \mid \mathrm{soc}_i(M) = 0\}.$$

Corollary 9.3 *For $M, N \in \mathrm{rep}(\Pi)$ the following hold:*

(i) *If $M, N \in \mathcal{T}_i$, then Σ_i^+ induces an isomorphism*

$$\mathrm{Ext}^1_\Pi(M, N) \cong \mathrm{Ext}^1_\Pi(\Sigma_i^+(M), \Sigma_i^+(N)).$$

(ii) *If $M, N \in \mathcal{S}_i$, then Σ_i^- induces an isomorphism*

$$\mathrm{Ext}^1_\Pi(M, N) \cong \mathrm{Ext}^1_\Pi(\Sigma_i^-(M), \Sigma_i^-(N)).$$

Proposition 9.4 *For $M \in \mathrm{rep}(\Pi)$ the following are equivalent:*

(i) $\mathrm{top}_i(M) = 0$;
(ii) $M \cong \Sigma_i^- \Sigma_i^+(M)$.

Furthermore, if $M \in \mathrm{rep}_{\mathrm{l.f.}}(\Pi)$, *then* (i) *and* (ii) *are equivalent to the following:*

(iii) $\underline{\mathrm{rank}}(\Sigma_i^+(M)) = s_i(\underline{\mathrm{rank}}(M))$.

Dually, the following are equivalent:

(i) $\mathrm{soc}_i(M) = 0$;
(ii) $M \cong \Sigma_i^+ \Sigma_i^-(M)$.

Furthermore, if $M \in \mathrm{rep}_{\mathrm{l.f.}}(\Pi)$, *then* (i) *and* (ii) *are equivalent to the following:*

(iii) $\underline{\mathrm{rank}}(\Sigma_i^-(M)) = s_i(\underline{\mathrm{rank}}(M))$.

Proof The equivalence of (i) and (ii) follows directly from Proposition 9.1 and Corollary 9.2.

Suppose (i) holds for some vertex i of $Q(C, \Omega)$. Let $a = (a_1, \dots, a_n) = \underline{\mathrm{rank}}(M)$. Recall that we have the H_i-module homomorphism

$$M_{i,\mathrm{in}} \colon \bigoplus_{j \in \overline{\Omega}(i,-)} {}_iH_j \otimes_j M_j \to M_i.$$

Since $\mathrm{top}_i(M) = 0$, the map $M_{i,\mathrm{in}}$ is surjective. This implies that $\Sigma_i^+(M)$ is again locally free with

$$\left(\underline{\mathrm{rank}}(\Sigma_i^+(M)))_i = \sum_{j \in \overline{\Omega}(i,-)} |c_{ij}| a_j - a_i = (s_i(\underline{\mathrm{rank}}(\Sigma_i^+(M)))\right)_i.$$

(Here we used that ${}_iH_j \otimes_j M_j$ is a free H_i-module of rank $|c_{ij}|a_j$.) Thus (iii) holds.

Vice versa, the equality

$$(\underline{\mathrm{rank}}(\Sigma_i^+(M)))_i = (s_i(\underline{\mathrm{rank}}(\Sigma_i^+(M))))_i$$

implies that $M_{i,\mathrm{in}}$ is surjective. Thus (iii) implies (i). □

Let $M = (M_i, M_{ij}, M_{ji}) \in \mathrm{rep}(C, D, \overline{\Omega})$. We say that $M_{i,\mathrm{in}}$ (resp. $M_{i,\mathrm{out}}$) *splits* if the image of $M_{i,\mathrm{in}}$ (resp. $M_{i,\mathrm{out}}$) is a free H_i-module. The following lemma is straightforward.

Lemma 9.5 *Let* $M \in \mathrm{rep}(\Pi)$ *be locally free. For each* i *the following hold:*

(i) $\Sigma_i^+(M)$ *is locally free if and only if* $M_{i,\mathrm{in}}$ *splits.*
(ii) $\Sigma_i^-(M)$ *is locally free if and only if* $M_{i,\mathrm{out}}$ *splits.*

9.2 Reflection functors for *H*

We keep the notations of Sect. 5.2. Recall that for an orientation Ω of C and some $1 \leq i \leq n$ we defined

$$s_i(\Omega) := \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cup \{(s, r) \in \Omega^* \mid i \in \{r, s\}\}.$$

Define

$$s_i(H) := s_i(H(C, D, \Omega)) := H(C, D, s_i(\Omega)). \tag{9.1}$$

Now let k be a sink in $Q^\circ(C, \Omega)$. Then Σ_k^+ obviously restricts to a *reflection functor*

$$F_k^+ \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H))$$

which can also be described as follows. Let $M = (M_i, M_{ij}) \in \operatorname{rep}(C, D, \Omega)$. Recall that

$$M_{k,\mathrm{in}} = (\operatorname{sgn}(k, j) M_{kj})_j \colon \bigoplus_{j \in \Omega(k,-)} {}_kH_j \otimes M_j \to M_k.$$

(Note that $\Omega(k, -) = \overline{\Omega}(k, -)$, since k is a sink.) Let $N_k := \operatorname{Ker}(M_{k,\mathrm{in}})$. We obtain an exact sequence

$$0 \to N_k \to \bigoplus_{j \in \Omega(k,-)} {}_kH_j \otimes_j M_j \xrightarrow{M_{k,\mathrm{in}}} M_k.$$

Let us denote by $(N_{jk}^\vee)_j$ the inclusion map $N_k \to \bigoplus_{j \in \Omega(k,-)} {}_kH_j \otimes_j M_j$. Then we have $F_k^+(M) = (N_r, N_{rs})$ with $(r, s) \in s_k(\Omega)$, where

$$N_r := \begin{cases} M_r & \text{if} \quad r \neq k, \\ N_k & \text{if} \quad r = k \end{cases} \quad \text{and}$$

$$N_{rs} := \begin{cases} M_{rs} & \text{if} \quad (r, s) \in \Omega \quad \text{and} \quad r \neq k, \\ (N_{rs}^\vee)^\vee & \text{if} \quad (r, s) \in \Omega^* \quad \text{and} \quad s = k. \end{cases}$$

Similarly, if k is a source in $Q^\circ(C, \Omega)$, then Σ_k^- restricts to a *reflection functor*

$$F_k^- \colon \operatorname{rep}(H) \to \operatorname{rep}(s_k(H)).$$

Proposition 9.6 *Let M be locally free and rigid in* rep(H). *Then $F_k^\pm(M)$ is locally free and rigid.*

Proof Without loss of generality assume that 1 is a sink and n is a source in $Q^\circ(C, \Omega)$. Let M be a locally free and rigid H-module. To get a contradiction, assume that $F_1^+(M)$ is not locally free and that M is of minimal dimension with this property. Recall that $F_1^+(E_1) = 0$. Thus by the minimality of its dimension, M does not have any direct summand isomorphic to E_1. We can also assume that $e_i M \neq 0$ for all $1 \leq i \leq n$. (Otherwise M can be considered as a module over an algebra $H(C', D', \Omega')$ with fewer vertices.) Since $F_1^+(M)$ is not locally free, we get $\mathrm{Hom}_H(M, E_1) \neq 0$, see Lemma 9.5.

We have a short exact sequence

$$0 \to M' \to M \to e_n M \to 0 \tag{9.2}$$

where $M' := (e_1 + \cdots + e_{n-1})M$. Clearly, M' and $e_n M$ are both locally free. In particular, $e_n M \cong E_n^s$ for some $s \geq 1$. We have $\mathrm{Hom}_H(M', e_n M) = 0$. Thus applying $\mathrm{Hom}_H(M', -)$ to (9.2) we get an embedding $\mathrm{Ext}_H^1(M', M') \to \mathrm{Ext}_H^1(M', M)$. Applying $\mathrm{Hom}_H(-, M)$ to (9.2) and using that $\mathrm{Ext}_H^2(e_n M, M) = 0$ we get $\mathrm{Ext}_H^1(M', M) = 0$. This shows that M' is rigid. Applying $\mathrm{Hom}_H(-, E_1)$ to the sequence (9.2) yields that $\mathrm{Hom}_H(M', E_1) \neq 0$. Since M' is locally free and rigid, the minimality of M implies that $M' \cong E_1^r \oplus U$ for some $r \geq 1$ and some locally free and rigid module U with $\mathrm{Hom}_H(U, E_1) = 0$. This yields short exact sequences

$$0 \to U \xrightarrow{f} M \xrightarrow{g} V \to 0 \tag{9.3}$$

and

$$0 \to E_1^r \to V \to e_n M \to 0 \tag{9.4}$$

where f is the obvious embedding, and g is the obvious projection onto $V := \mathrm{Cok}(f)$. Note that V is also locally free, and that $\mathrm{Hom}_H(U, E_n) = 0$. Applying $\mathrm{Hom}_H(U, -)$ to (9.4) implies that $\mathrm{Hom}_H(U, V) = 0$. Now we apply $\mathrm{Hom}_H(M, -)$ and $\mathrm{Hom}_H(-, V)$ to (9.3) and get similarly as before that V is rigid. Applying $\mathrm{Hom}_H(-, E_1)$ to (9.3) implies $\mathrm{Hom}_H(M, E_1) \cong \mathrm{Hom}_H(V, E_1) \neq 0$. By the minimality of M it now follows that $U = 0$ and $V = M$. Thus we have $n = 2$.

Since M is locally free, Proposition 3.5 implies that it has a minimal projective resolution of the form

$$0 \to P'' \to P' \to M \to 0.$$

Since M is rigid and locally free, we know that M is also τ-rigid, see Corollary 3.6. Thus by [1, Proposition 2.5] we get that $\mathrm{add}(P') \cap \mathrm{add}(P'') = 0$. (It can be easily checked that [1, Proposition 2.5] is true for arbitrary ground fields.) We obviously have $\mathrm{Hom}_H(M, E_2) \neq 0$, since 2 is a source in $Q^\circ(C, \Omega)$. By assumption we have $\operatorname{Hom}_H(M, E_1) \neq 0$. This implies that $\operatorname{Hom}_H(M, S_i) \neq 0$ for the simple H-modules S_i, where $i = 1, 2$. Thus P' contains both P_1 and P_2 as direct summands. Since $\operatorname{add}(P') \cap \operatorname{add}(P'') = 0$, it follows that $P'' = 0$. In other words, M is projective. But $P = E_1$ is the only indecomposable projective H-module with $\operatorname{Hom}_H(P, E_1) \neq 0$. Thus M contains a direct summand isomorphic to E_1, a contradiction.

Altogether we proved that $\operatorname{Hom}_H(M, E_1) = 0$ for any locally free and rigid H-module M, which does not have a direct summand isomorphic to E_1. In this case, we have $\operatorname{top}_1(M) = 0$ and $M_{1,\mathrm{in}}$ is surjective. Thus $F_1^+(M)$ is rigid by Corollary 9.3, and $F_1^+(M)$ is locally free by Lemma 9.5. The corresponding statement for $F_n^-(M)$ is proved dually. This finishes the proof. □

9.3 Reflection functors and APR-tilting

As before, let $H = H(C, D, \Omega)$ and $Q = Q(C, \Omega)$. Let i be a sink in Q°, and define

$$T := {}_H H/E_i \oplus \tau_H^-(E_i) \text{ and } B := \operatorname{End}_H(T)^{\mathrm{op}}.$$

We also assume that i is not a source in Q°. We have $E_i = He_i$. For $1 \leq j \leq n$ set $P_j := He_j$ and $I_j := \mathrm{D}(e_j H)$. Let S_j be the corresponding simple H-module with $S_j \cong \operatorname{top}(P_j) \cong \operatorname{soc}(I_j)$. Furthermore, let

$$T_j := \begin{cases} He_j & \text{if} \quad j \neq i, \\ \tau_H^-(E_i) & \text{if} \quad j = i. \end{cases}$$

Thus we have $T = T_1 \oplus \cdots \oplus T_n$. The indecomposable projective B-modules are (up to isomorphism) $\operatorname{Hom}_H(T, T_j)$ where $1 \leq j \leq n$. Let e_j denote the primitive idempotent in B obtained by composing the canonical projection $T \to T_j$ with the canonical inclusion $T_j \to T$. Then $\operatorname{Hom}_H(T, T_j) \cong Be_j$. Finally, let S_j denote the simple B-module associated with e_j. We are using the same notation for the idempotents in H, $s_i(H)$ and B and also for the associated simple modules. However, the context will always save us from confusion.

Recall that a finite-dimensional module T over a finite-dimensional algebra A is a *tilting module* if the following hold:

(i) proj. $\dim(T) \leq 1$;
(ii) $\operatorname{Ext}_A^1(T, T) = 0$;
(iii) The number of isomorphism classes of indecomposable direct summands of T is equal to the number of isomorphism classes of simple A-modules.

The following result is inspired by Auslander, Platzeck and Reiten's [3] ground breaking interpretation of BGP-reflection functors as homomorphism functors of certain tilting modules. Their result can be seen as the beginning of tilting theory.

Theorem 9.7 *With the notation above, the H-module T is a tilting module, and the functors*

$$F_i^+(-)\colon \operatorname{rep}(H) \to \operatorname{rep}(s_i(H)) \textit{ and } \operatorname{Hom}_H(T, -)\colon \operatorname{rep}(H) \to \operatorname{rep}(B)$$

are equivalent, *i.e. there exists an equivalence*

$$S\colon \operatorname{rep}(s_i(H)) \to \operatorname{rep}(B)$$

such that we have an isomorphism of functors $S \circ F_i^+ \cong \operatorname{Hom}_H(T, -)$.

As a preparation for the proof of Theorem 9.7 we construct a minimal injective resolution of the H-module E_i, see (9.6) below. This yields via the inverse Nakayama functor a minimal projective resolution (9.7) of $\tau_H^-(E_i)$.

Let E_i' be the right H-module such that $\mathrm{D}(E_i') \cong E_i$. Similarly to Proposition 3.1 there is a minimal projective resolution

$$0 \to \bigoplus_{j \in \Omega(i,-)} {}_iH_j \otimes_j e_jH \xrightarrow{\mu} e_iH \to E_i' \to 0 \tag{9.5}$$

of the right H-module E_i', where the map $e_iH \to E_i'$ is the canonical projection, and for $j \in \Omega(i, -)$ the jth component of the map μ is just the multiplication map

$$H_{ij}\colon {}_iH_j \otimes_j e_jH \to e_iH.$$

(The maps H_{ij} are the structure maps of the (left) H-module structure on H.) Note that ${}_iH_j \otimes_j e_jH \cong e_jH^{|c_{ji}|}$.

There is a chain of H_j-H_i-bimodule isomorphisms

$${}_jH_i \cong \operatorname{Hom}_{H_i}({}_iH_j, H_i) \cong \mathrm{D}\operatorname{Hom}_{H_j}({}_jH_i, H_j) \cong \mathrm{D}({}_iH_j),$$

where the first and third isomorphisms are constructed as in Sect. 5.1, and the second isomorphism is defined as in Lemma 8.2. Thus for $r \in {}_iR_j$ we can consider r^* now as an element in $\mathrm{D}({}_iH_j)$.

Under the isomorphism

$$\mathrm{D}(e_jH) \otimes_j \mathrm{D}({}_iH_j) \to \mathrm{D}({}_iH_j \otimes_j e_jH)$$

defined by

$$\iota \otimes \eta \mapsto \big(h \otimes e_j h' \mapsto \eta(h)\iota(e_j h')\big)$$

the dual of the multiplication map H_{ij} is identified with the map

$$H_{ij}^* \colon \mathrm{D}(e_i H) \to \mathrm{D}(e_j H) \otimes_j \mathrm{D}({}_i H_j)$$

defined by

$$\varphi \mapsto \sum_{r \in_i R_j} \varphi(r \cdot -) \otimes r^*.$$

We have $\mathrm{D}(e_i H) = I_i$ and $\mathrm{D}(e_j H) \otimes_j \mathrm{D}({}_i H_j) \cong I_j \otimes_j {}_j H_i \cong I_j^{|c_{ji}|}$. Applying the duality D to (9.5) we get a minimal injective resolution

$$0 \to E_i \to I_i \xrightarrow{(H_{ij}^*)_{j \in \Omega(i,-)}} \bigoplus_{j \in \Omega(i,-)} I_j \otimes_j {}_j H_i \to 0. \tag{9.6}$$

For $j \in \Omega(i, -)$ set

$$\theta_{ij} := \nu_H^{-1}(H_{ij}^*) \colon E_i \to P_j \otimes_j {}_j H_i$$

which is given by $he_i \mapsto \sum_{r \in_i R_j} hr \otimes r^*$. Here $\nu_H^{-1} \colon \mathrm{inj}(H) \to \mathrm{proj}(H)$ is the inverse Nakayama functor obtained via restriction from $\mathrm{Hom}_H(\mathrm{D}(H_H), -)$. Let $\theta := (\theta_{ij})_j$ where $j \in \Omega(i, -)$. Recall that $E_i = P_i$, since i is a sink in Q°. We get that the exact sequence

$$0 \to E_i \xrightarrow{\theta} \bigoplus_{j \in \Omega(i,-)} P_j \otimes_j {}_j H_i \to \tau_H^-(E_i) \to 0 \tag{9.7}$$

is a minimal projective resolution of $\tau_H^-(E_i)$. Note that (9.7) is not an Auslander–Reiten sequence if E_i is not simple.

Lemma 9.8 *The following hold:*

(i) $\tau_H^-(E_i)$ *is locally free;*
(ii) $\mathrm{Hom}_H(\tau_H^-(E_i), H) = 0$*;*
(iii) $\mathrm{End}_H(\tau_H^-(E_i)) \cong \mathrm{End}_H(E_i) \cong H_i$.

Proof The existence of the exact sequence (9.7) implies that $\tau_H^-(E_i)$ is locally free. Since $\mathrm{inj.\,dim}(E_i) \le 1$, we have $\mathrm{Hom}_H(\tau_H^-(E_i), H) = 0$. The H-module E_i is indecomposable non-injective with $\mathrm{proj.\,dim}(\tau_H^-(E_i)) = 1$

and inj. $\dim(E_i) = 1$. Thus by the Auslander–Reiten formulas we get $\operatorname{End}_H(\tau_H^-(E_i)) \cong \operatorname{End}_H(E_i) \cong H_i$. This finishes the proof. □

Lemma 9.9 *The H-module T is a tilting module.*

Proof By the above considerations we know that

$$0 \to E_i \xrightarrow{\theta} \bigoplus_{j \in \Omega(i,-)} P_j \otimes_j {}_j H_i \to \tau_H^-(E_i) \to 0$$

is a minimal projective resolution of $\tau_H^-(E_i)$. It follows that proj. $\dim(T) = 1$. Furthermore, we have

$$\begin{aligned}\operatorname{Ext}_H^1(T,T) &= \operatorname{Ext}_H^1(\tau_H^-(E_i), T)\\ &\cong \mathrm{D}\operatorname{Hom}_H(T, E_i)\\ &= \mathrm{D}\operatorname{Hom}_H(\tau_H^-(E_i), E_i)\\ &= 0.\end{aligned}$$

The equality in the first line of the above equations holds since $\tau_H^-(E_i)$ is the only non-projective indecomposable direct summand of T. The isomorphism in the second line follows by the Auslander–Reiten formulas and the fact that proj. $\dim(\tau_H^-(E_i)) \leq 1$. The equality in the third line holds since i is a sink. Finally, the equality in the fourth line follows from Lemma 9.8(ii). The module H has exactly n pairwise non-isomorphic indecomposable direct summands. This finishes the proof. □

Lemma 9.10 *We have ${}_B\,\mathrm{D}(T) \cong \mathrm{D}(B)/\,\mathrm{D}(e_i B) \oplus \tau_B(\mathrm{D}(e_i B))$.*

Proof We have

$${}_B\,\mathrm{D}(T) \cong \operatorname{Hom}_H(T, \mathrm{D}(H)) = \bigoplus_{j=1}^{n} \operatorname{Hom}_H(T, \mathrm{D}(e_j H)).$$

For $j \neq i$ there are B-module isomorphisms

$$\operatorname{Hom}_H(T, \mathrm{D}(e_j H)) \cong \mathrm{D}\operatorname{Hom}_H(He_j, T) = \mathrm{D}\operatorname{Hom}_H(T_j, T) \cong \mathrm{D}(e_j B).$$

It follows that

$${}_B\,\mathrm{D}(T) \cong \mathrm{D}(B)/\,\mathrm{D}(e_i B) \oplus \operatorname{Hom}_H(T, \mathrm{D}(e_i H)).$$

Since $\operatorname{Ext}_H^1(T, E_i) \neq 0$, the Connecting Lemma [36, Section 2.3] implies that

$$\operatorname{Hom}_H(T, \mathrm{D}(e_i H)) \cong \tau_B(\operatorname{Ext}_H^1(T, E_i)).$$

We have

$$\operatorname{Ext}^1_H(T, E_i) \cong \mathrm{D}\operatorname{Hom}_H(\tau_H^-(E_i), T) \cong \mathrm{D}(e_i B).$$

The first isomorphism is obtained from the Auslander–Reiten formulas and the fact that inj. $\dim(E_i) \leq 1$. The second isomorphism is obvious. Note that both isomorphism are B-module isomorphisms. This finishes the proof. □

For any of the algebras $A \in \{H, s_i(H), B\}$ and any simple A-module S_j let

$$\begin{aligned}\mathcal{T}_j^A &:= \{M \in \operatorname{rep}(A) \mid \operatorname{Hom}_A(M, S_j) = 0\},\\ \mathcal{S}_j^A &:= \{M \in \operatorname{rep}(A) \mid \operatorname{Hom}_A(S_j, M) = 0\}.\end{aligned}$$

Using the notation from Sect. 9.1, for $A \in \{H, s_i(H)\}$ we have $\mathcal{T}_j^A = \mathcal{T}_j \cap \operatorname{rep}(A)$ and $\mathcal{S}_j^A = \mathcal{S}_j \cap \operatorname{rep}(A)$.

Lemma 9.11 *The functors* $F := \operatorname{Hom}_H(T, -)$ *and* $G := T \otimes_B -$ *induce mutually quasi-inverse equivalences* $F\colon \mathcal{T} \to \mathcal{Y}$ *and* $G\colon \mathcal{Y} \to \mathcal{T}$, *where*

$$\begin{aligned}\mathcal{T} &:= \{M \in \operatorname{rep}(H) \mid \operatorname{Ext}^1_H(T, M) = 0\}\\ &= \{M \in \operatorname{rep}(H) \mid \mathrm{D}\operatorname{Hom}_H(M, E_i) = 0\}\\ &= \mathcal{T}_i^H\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y} &:= \{N \in \operatorname{rep}(B) \mid \operatorname{Tor}^B_1(T, N) = 0\}\\ &= \{N \in \operatorname{rep}(B) \mid \mathrm{D}\operatorname{Ext}^1_B(N, \mathrm{D}(T)) = 0\}\\ &= \{N \in \operatorname{rep}(B) \mid \operatorname{Hom}_B(\mathrm{D}(e_i B), N) = 0\}\\ &= \mathcal{S}_i^B.\end{aligned}$$

Furthermore, we have $F(\operatorname{rep}(H)) \subseteq \mathcal{Y}$ *and* $G(\operatorname{rep}(B)) \subseteq \mathcal{T}$.

Proof This follows mainly from classical tilting theory (Brenner-Butler Therorem, see for example [36, Section 2]) and the Auslander–Reiten formulas. The third equality in the description of $\mathcal{Y}$ uses the Auslander–Reiten formulas in combination with Lemma 9.10. For $j \neq i$ we have

$$\operatorname{Hom}_B(\mathrm{D}(e_i B), \mathrm{D}(e_j B)) \cong \mathrm{D}\operatorname{Ext}^1_B(\mathrm{D}(e_j B), \mathrm{D}(T)) = 0.$$

Thus every composition factor of $\mathrm{D}(e_i B)$ is isomorphic to S_i. This implies the fourth equality in the description of $\mathcal{Y}$. □

Corollary 9.12 *Via restriction of the functors F_i^+, $F \circ F_i^-$ and F we get a commutative diagram*

$$\begin{tikzcd}
\mathcal{T}_i^H \arrow[r, "F_i^+"] \arrow[d, equal] & \mathcal{S}_i^{s_i(H)} \arrow[d, "F \circ F_i^-"] \\
\mathcal{T}_i^H \arrow[r, "F"] & \mathcal{S}_i^B
\end{tikzcd}$$

of equivalences of subcategories.

Proof Combine Lemma 9.11 and Corollary 9.2. □

Corollary 9.13 *Via restriction of the functor $F \circ F_i^-$ we get a commutative diagram*

$$\begin{tikzcd}
\mathrm{rep}(H) \arrow[r, "F_i^+"] \arrow[d, equal] & \mathcal{S}_i^{s_i(H)} \arrow[d, "F \circ F_i^-"] \\
\mathrm{rep}(H) \arrow[r, "F"] & \mathcal{S}_i^B
\end{tikzcd}$$

with $F \circ F_i^-$ an equivalence of subcategories.

Proof This follows from Corollary 9.12, the definition of F_i^+ and the last statement in Lemma 9.11. □

Lemma 9.14 *The following hold:*

(i) $\mathrm{proj}(s_i(H)) \subseteq \mathcal{S}_i^{s_i(H)}$;
(ii) $\mathrm{proj}(B) \subseteq \mathcal{S}_i^B$.

Proof Since i is a source in $Q(C, s_i(\Omega))^\circ$, we get that $\mathrm{proj}(s_i(H)) \subseteq \mathcal{S}_i^{s_i(H)}$. Part (ii) is obvious, since the modules $\mathrm{Hom}_T(T, T_j)$ are (up to isomorphism) the indecomposable projective B-modules. □

Theorem 9.7 follows now from [3, Lemma 2.2]. (Corollary 9.13 and Lemma 9.14 and the fact that $\mathcal{S}_i^A$ is closed under submodules for $A \in \{s_i(H), B\}$ ensure that the assumptions of [3, Lemma 2.2] are satisfied.)

As a consequence, via restriction of $F \circ F_i^-$, we get an equivalence of subcategories

$$\mathrm{proj}(s_i(H)) \to \mathrm{proj}(B).$$

In particular, we have $F_i^+(T_j) \cong s_i(H)e_j$ for all $1 \leq j \leq n$. We also get an algebra isomorphism $s_i(H) \cong B$.

We leave it as an exercise to formulate a dual version of Theorem 9.7.

9.4 Coxeter functors

Let $Q = Q(C, \Omega)$.

Given a $+$-admissible sequence $(i_1, \dots, i_n)$ for (C, Ω) let

$$C^+ := F^+_{i_n} \circ \cdots \circ F^+_{i_1} \colon \operatorname{rep}(H) \to \operatorname{rep}(H).$$

Dually, one defines *$-$-admissible sequences* $(j_1, \dots, j_n)$ and $C^- := F^-_{j_n} \circ \cdots \circ F^-_{j_1}$. We call C^+ and C^- *Coxeter functors*. Similarly as in the classical case one proves the following result, compare [11].

Lemma 9.15 *The functors C^+ and C^- do not depend on the chosen admissible sequences for (C, Ω).*

The next lemma is a consequence of Proposition 9.1, Corollary 9.2 and Lemma 9.5.

Lemma 9.16 *Let M be an indecomposable locally free H-module. Let $(i_1, \dots, i_n)$ be a $+$-admissible sequence for (C, Ω). Assume that $F^+_{i_s} \cdots F^+_{i_1}(M)$ is locally free and non-zero for some $1 \le s \le n$. Then we have*

$$F^+_{i_k} \cdots F^+_{i_1}(M) \cong (F^-_{i_{k+1}} \cdots F^-_{i_s})(F^+_{i_s} \cdots F^+_{i_{k+1}}) F^+_{i_k} \cdots F^+_{i_1}(M)$$

for $1 \le k \le s-1$, and $F^+_{i_k} \cdots F^+_{i_1}(M)$ is indecomposable and locally free for $1 \le k \le s$.

There is also an obvious dual of Lemma 9.16.

10 Coxeter functors and Auslander–Reiten translations

10.1 Overview

As before we fix $H = H(C, D, \Omega)$. Our aim is to compare the Coxeter functors C^+ and C^- introduced in Sect. 9.4 with the Auslander–Reiten translations τ and τ^-.

Without loss of generality we assume that for each $(i, j) \in \Omega$ we have $i < j$. Thus,

$$C^+ = F^+_n \circ \cdots \circ F^+_1.$$

Recall that we defined the twist automorphism T of H by

$$T(\varepsilon_i) = \varepsilon_i, \qquad T(\alpha_{ij}^{(g)}) = -\alpha_{ij}^{(g)}, \qquad (i \in Q_0,\ (i, j) \in \Omega,\ 1 \le g \le g_{ij}).$$

The twist by T defines an automorphism of $\operatorname{rep}(H)$ which we denote also by T. More explicitly, for $M = (M_i, M_{ij}) \in \operatorname{rep}(H)$ we have $(TM)_i = M_i$ and $(TM)_{ij} = -M_{ij}$.

Following [32, Section 5], we start in Sect. 10.2 by constructing a new algebra $\widetilde{H}$ containing two subalgebras $H_{(0)}$ and $H_{(1)}$ canonically isomorphic to H. Denoting by

$$\operatorname{Res}_a\colon\ \operatorname{rep}(\widetilde{H}) \to \operatorname{rep}(H_{(a)}), \quad (a \in \{0, 1\})$$

the corresponding restriction functors, we will show that $C^+ \cong \operatorname{Res}_1 \circ \operatorname{Res}_0^*$, where

$$\operatorname{Res}_0^*\colon\ \operatorname{rep}(H_{(0)}) \to \operatorname{rep}(\widetilde{H})$$

is right adjoint to Res_0. This will follow from a factorization

$$\operatorname{Res}_0^* = \operatorname{Res}^*_{(n-1,n)} \circ \cdots \circ \operatorname{Res}^*_{(1,2)} \circ \operatorname{Res}^*_{(0,1)}$$

similar to the definition of C^+, and from a comparison of the functors $\operatorname{Res}^*_{(i-1,i)}$ and F_i^+ obtained in Lemma 10.2.

After that, we will give a different description of the adjoint functor Res_0^*, which will allow to show that, for $M \in \operatorname{rep}_{\text{l.f.}}(H)$, the H-module $\operatorname{Res}_1 \circ \operatorname{Res}_0^*(M)$ is the kernel of a certain map d_M^*. On the other hand, it follows from Corollary 7.2 that $\tau(TM)$ is the kernel of the map $\operatorname{D}\operatorname{Hom}_H(d \otimes TM, H)$. We will then show that, under the trace pairings, the maps d_M^* and $\operatorname{D}\operatorname{Hom}_H(d \otimes TM, H)$ can be identified, hence

$$C^+(M) \cong \operatorname{Res}_1 \circ \operatorname{Res}_0^*(M) \cong \tau(TM).$$

A more detailed statement of our results will be given in Theorem 10.1, whose proof is carried out in Sects. 10.3–10.5.

The remaining sections present direct applications of Theorem 10.1. In 10.6 we give another description of the preprojective algebra $\Pi = \Pi(C, D)$ as a tensor algebra. In 10.7 we adapt to our setting a description of the category $\operatorname{rep}(\Pi)$ due to Ringel in the classical case, in terms of H-module homomorphisms $M \to TC^+(M)$. Finally, in Sect. 10.8 we show that the subcategory of Gorenstein-projective H-modules coincides with the kernel of the Coxeter functor C^+.

10.2 An analogue of the Gabriel–Riedtmann construction

The following is an adaptation of [32, Section 5] to our situation.

10.2.1 The algebra $\widetilde{H}$

To our fixed datum (C, D, Ω) we attach a new algebra $\widetilde{H}$ defined by a quiver with relations. The quiver $\widetilde{Q}$ has set of vertices

$$\widetilde{Q}_0 := \{(i, a) \mid i \in Q_0,\ a \in \{0, 1\}\},$$

and set of arrows

$$\begin{aligned}\widetilde{Q}_1 := &\{\alpha^{(g)}_{(i,a)(j,a)} \colon (j, a) \to (i, a) \mid (i, j) \in \Omega,\ 1 \le g \le g_{ij},\ a \in \{0, 1\}\}\\ &\cup\ \{\alpha^{(g)}_{(j,0)(i,1)} \colon (i, 1) \to (j, 0) \mid (i, j) \in \Omega,\ 1 \le g \le g_{ij}\}\\ &\cup\ \{\varepsilon_{(i,a)} \colon (i, a) \to (i, a) \mid (i, a) \in \widetilde{Q}_0\}.\end{aligned}$$

Accordingly we put

$$\widetilde{\Omega} := \{((i, a), (j, a)), ((j, 0), (i, 1)) \mid (i, j) \in \Omega, a \in \{0, 1\}\}.$$

Let

$$\widetilde{H} := K\widetilde{Q}/\widetilde{I}$$

where $\widetilde{I}$ is the ideal of $K\widetilde{Q}$ defined by the following relations:

($\widetilde{\mathbf{H}}$1) For each $(i, a) \in \widetilde{Q}_0$ we have

$$\varepsilon^{c_i}_{(i,a)} = 0.$$

($\widetilde{\mathbf{H}}$2) For each $((i, a), (j, b)) \in \widetilde{\Omega}$ and each $1 \le g \le g_{ij}$ we have

$$\varepsilon^{f_{ji}}_{(i,a)}\alpha^{(g)}_{(i,a)(j,b)} = \alpha^{(g)}_{(i,a)(j,b)}\varepsilon^{f_{ij}}_{(j,b)}.$$

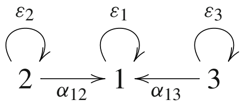
($\widetilde{\mathbf{H}}$3) For each $i \in Q_0$ we have

$$\begin{aligned}&\sum_{j\in\Omega(i,-)}\sum_{g=1}^{g_{ij}}\sum_{f=0}^{f_{ji}-1} \varepsilon^{f}_{(i,0)}\alpha^{(g)}_{(i,0)(j,0)}\alpha^{(g)}_{(j,0)(i,1)}\varepsilon^{f_{ji}-1-f}_{(i,1)}\\ &+ \sum_{j\in\Omega(-,i)}\sum_{g=1}^{g_{ij}}\sum_{f=0}^{f_{ji}-1} \varepsilon^{f}_{(i,0)}\alpha^{(g)}_{(i,0)(j,1)}\alpha^{(g)}_{(j,1)(i,1)}\varepsilon^{f_{ji}-1-f}_{(i,1)} = 0.\end{aligned}$$

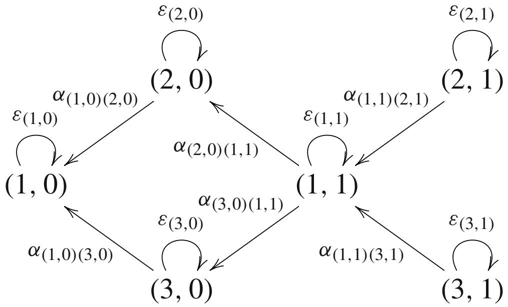
When C is symmetric and D is minimal the algebra $\widetilde{H}$ coincides with the bounded quiver denoted by $\widetilde{QQ}$ in [32, Section 5.3].

10.2.2 Example

Let $H = H(C, D, \Omega)$ be defined by the quiver



with relations $\varepsilon_2 = 0$, $\varepsilon_1^2 = \varepsilon_3^2 = 0$ and $\varepsilon_1\alpha_{13} = \alpha_{13}\varepsilon_3$. Here C is a Cartan matrix of Dynkin type B_3, and D is the minimal symmetrizer. Then $\widetilde{H}$ is defined by the quiver



bound by the relations

$$\begin{aligned}
&\varepsilon_{(2,a)} = 0,\\
&\varepsilon_{(1,a)}^2 = \varepsilon_{(3,a)}^2 = 0,\\
&\varepsilon_{(1,a)}\alpha_{(1,a)(3,a)} = \alpha_{(1,a)(3,a)}\varepsilon_{(3,a)},\\
&\varepsilon_{(3,0)}\alpha_{(3,0)(1,1)} = \alpha_{(3,0)(1,1)}\varepsilon_{(1,1)},
\end{aligned}$$

with $a \in \{0, 1\}$, and

$$\begin{aligned}
&\alpha_{(2,0)(1,1)}\alpha_{(1,1)(2,1)} = 0,\\
&\varepsilon_{(1,0)}\alpha_{(1,0)(2,0)}\alpha_{(2,0)(1,1)} + \alpha_{(1,0)(2,0)}\alpha_{(2,0)(1,1)}\varepsilon_{(1,1)}\\
&\ + \ \varepsilon_{(1,0)}\alpha_{(1,0)(3,0)}\alpha_{(3,0)(1,1)} + \alpha_{(1,0)(3,0)}\alpha_{(3,0)(1,1)}\varepsilon_{(1,1)} = 0,\\
&\varepsilon_{(3,0)}\alpha_{(3,0)(1,1)}\alpha_{(1,1)(3,1)} + \alpha_{(3,0)(1,1)}\alpha_{(1,1)(3,1)}\varepsilon_{(3,1)} = 0.
\end{aligned}$$

10.2.3 $\widetilde{H}$ as a quotient of a tensor algebra

It will be useful to have a more intrinsic description of $\widetilde{H}$ in the spirit of Sect. 6. Define $\widetilde{C} = (\widetilde{c}_{(i,a),(j,b)}) \in \mathbb{Z}^{\widetilde{Q}_0 \times \widetilde{Q}_0} = M_{2n}(\mathbb{Z})$ by

$$\widetilde{c}_{(i,a),(j,b)} := \begin{cases} c_{ij} & \text{if} \quad (a=b), \\ & \text{or } (a,b)=(0,1) \quad \text{and} \quad (i,j)\in\Omega^*, \\ & \text{or } (a,b)=(1,0) \quad \text{and} \quad (i,j)\in\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\widetilde{C}$ is a Cartan matrix with symmetrizer $\widetilde{D} = \operatorname{diag}(c_1, \dots, c_n, c_1, \dots, c_n)$, where $D = \operatorname{diag}(c_1, \dots, c_n)$ is our symmetrizer for C, and $\widetilde{\Omega}$ is an orientation of $\widetilde{C}$. Moreover, if $\widetilde{c}_{(i,a),(j,b)} < 0$ then

$$\widetilde{g}_{(i,a),(j,b)} = g_{ij}, \quad \widetilde{f}_{(i,a),(j,b)} = f_{ij}, \quad \widetilde{k}_{(i,a),(j,b)} = k_{ij}.$$

As before, one defines the corresponding algebra $H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega})$. Let

$$H_{(i,a)} = K[\varepsilon_{(i,a)}]/(\varepsilon_{(i,a)}^{c_i}).$$

We have isomorphisms

$$\eta_{(i,a)} \colon H_i \to H_{(i,a)}$$

defined by $\varepsilon_i \mapsto \varepsilon_{(i,a)}$, and as before for each $((i,a),(j,b)) \in \widetilde{\Omega}$ we get an $H_{(i,a)}$-$H_{(j,b)}$-bimodule ${}_{(i,a)}H_{(j,b)}$ and an $H_{(j,b)}$-$H_{(i,a)}$-bimodule ${}_{(j,b)}H_{(i,a)}$. There are bimodule isomorphisms

$${}_{(i,a)}H_{(j,b)} \cong \begin{cases} {}_iH_j & \text{if } a=b \text{ and } (i,j)\in\Omega, \\ {}_iH_j & \text{if } (a,b)=(0,1) \text{ and } (i,j)\in\Omega^*. \end{cases}$$

via $\eta_{(i,a)}$ and $\eta_{(j,b)}$.

Set

$$\widetilde{S} := \prod_{(i,a)\in\widetilde{Q}_0} H_{(i,a)}.$$

Then

$$\widetilde{B} := \bigoplus_{(\mathbf{i},\mathbf{j})\in\widetilde{\Omega}} {}_{\mathbf{i}}H_{\mathbf{j}}$$

is an $\widetilde{S}$-$\widetilde{S}$-bimodule, and we have an isomorphism

$$T_{\widetilde{S}}(\widetilde{B}) \cong H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega}).$$

In case $(i, j) \in \Omega$ we abbreviate ${}_iR_j^0$ for the standard right basis of ${}_{(i,0)}H_{(j,0)}$ and ${}_iL_j^1$ for the standard left basis of ${}_{(i,1)}H_{(j,1)}$. Moreover, in this case we can identify in a obvious way ${}_{(j,0)}H_{(i,1)}$ with $\mathrm{Hom}_{H_{(j,0)}}({}_{(i,0)}H_{(j,0)}, H_{(j,0)})$ and obtain an $H_{(j,0)}$-basis $(r_+^*)_{r \in_i R_j^0}$ of ${}_{(j,0)}H_{(i,1)}$ which is under this identification dual to ${}_iR_j^0$. Similarly, we obtain a dual $H_{(i,1)}$-basis $(\ell_-^*)_{\ell \in_i L_j^1}$ of ${}_{(j,0)}H_{(i,1)}$.

For $j \in Q_0$, define

$$\widetilde{\rho}_j := \sum_{\substack{i \in \Omega(-,j) \\ \ell \in_i L_j^1}} \ell_-^* \otimes \ell + \sum_{\substack{k \in \Omega(j,-) \\ r \in_j R_k^0}} r \otimes r_+^*. \tag{10.1}$$

We have $\widetilde{\rho}_j \in e_{(j,0)}\widetilde{B} \otimes_{\widetilde{S}} \widetilde{B}e_{(j,1)}$. Now, arguing as in Sect. 6, we obtain:

$$\widetilde{H} \cong T_{\widetilde{S}}(\widetilde{B})/(\widetilde{\rho}_j \mid j \in Q_0).$$

Similarly to the case of preprojective algebras, for $M \in \mathrm{rep}(H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega}))$ and $j \in Q_0$ we can define maps

$$\widetilde{M}_{j,\mathrm{in}} = (M_{(j,0),(k,c)})_{(k,c)} \colon \bigoplus_{(k,c) \in \widetilde{\Omega}((j,0),-)} {}_{(j,0)}H_{(k,c)} \otimes_{H_{(k,c)}} M_{(k,c)} \to M_{(j,0)},$$

$$\widetilde{M}_{j,\mathrm{out}} = (\mathrm{ad}_{(i,a),(j,1)}(M_{(i,a),(j,1)}))_{(i,a)} \colon M_{(j,1)} \to \bigoplus_{(i,a) \in \widetilde{\Omega}(-,(j,1))} {}_{(j,1)}H_{(i,a)} \otimes_{H_{(i,a)}} M_{(i,a)}.$$

Note that $\widetilde{\Omega}((j,0),-) = \widetilde{\Omega}(-,(j,1))$ and thus, if we identify by a slight abuse $H_{(j,0)}$ with $H_{(j,1)}$, we can write

$$\bigoplus_{(k,c) \in \widetilde{\Omega}((j,0),-)} {}_{(j,0)}H_{(k,c)} \otimes_{H_{(k,c)}} M_{(k,c)} = \bigoplus_{(i,a) \in \widetilde{\Omega}(-,(j,1))} {}_{(j,1)}H_{(i,a)} \otimes_{H_{(i,a)}} M_{(i,a)}.$$

With this setup $M \in \mathrm{rep}(H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega}))$ belongs to $\mathrm{rep}(\widetilde{H})$ if and only if

$$\widetilde{M}_{j,\mathrm{in}} \circ \widetilde{M}_{j,\mathrm{out}} = 0 \tag{10.2}$$

for all $j \in Q_0$.

10.2.4 The subalgebras $H_{(0)}$ and $H_{(1)}$

For $a = 0, 1$ set

$$\mathbb{1}_a := \sum_{i\in I} e_{(i,a)}, \qquad H_{(a)} := \mathbb{1}_a \widetilde{H} \mathbb{1}_a.$$

Clearly, $H_{(a)}$ is a (non-unitary) subalgebra of $\widetilde{H}$, and we have natural isomorphisms $\eta_a \colon H \to H_{(a)}$ with $\eta_a(\varepsilon_i) = \varepsilon_{(i,a)}$ and $\eta_a(\alpha_{ij}^{(g)}) = \alpha_{(i,a)(j,a)}^{(g)}$.

We obtain for $a \in \{0, 1\}$ exact restriction functors

$$\mathrm{Res}_a \colon \mathrm{rep}(\widetilde{H}) \to \mathrm{rep}(H_{(a)}), \quad M \mapsto \mathbb{1}_a \widetilde{H} \otimes_{\widetilde{H}} M = \mathbb{1}_a M.$$

We will use several times the elementary fact that the functor

$$\mathrm{Res}_0^* \colon \mathrm{rep}(H_{(0)}) \to \mathrm{rep}(\widetilde{H}), \quad N \mapsto \mathrm{Hom}_{H_{(0)}}(\mathbb{1}_0 \widetilde{H}, N)$$

is uniquely characterized up to isomorphism as the right adjoint of Res_0. It is not hard to see that

$$\mathrm{Res}_0 \circ \mathrm{Res}_0^* \cong \mathrm{id}_{\mathrm{rep}(H_{(0)})}\,.$$

10.2.5 The H-H-bimodule X

Define

$$X = X(C, D, \Omega) := \mathbb{1}_0 \widetilde{H} \mathbb{1}_1.$$

We regard X as an H-H-bimodule via the maps η_0 and η_1, that is,

$$hxh' := \eta_0(h) x \eta_1(h'), \qquad (h, h' \in H,\ x \in X).$$

Similarly, using η_0 and η_1 we can regard $\mathrm{Res}_1 \circ \mathrm{Res}_0^*$ as a functor from $\mathrm{rep}(H)$ to $\mathrm{rep}(H)$. Then it is easy to see that we have an isomorphism of functors:

$$\mathrm{Res}_1 \circ \mathrm{Res}_0^* \cong \mathrm{Hom}_H(X, -).$$

Theorem 10.1 *The following hold:*

(a) *For each $M \in \mathrm{rep}(H)$ we have a functorial isomorphism*

$$\mathrm{Hom}_H(X, M) \cong C^+(M).$$

(b) *For each* $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$ *we have functorial isomorphisms*

$$\operatorname{Hom}_H(X, TM) \cong TC^+(M) \cong \tau(M).$$

(c) *For each* $M \in \operatorname{rep}(H)$ *we have a functorial isomorphism*

$$X \otimes_H M \cong C^-(M).$$

(d) *For each* $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$ *we have functorial isomorphisms*

$$X \otimes_H TM \cong TC^-(M) \cong \tau^-(M).$$

10.3 Proof of Theorem 10.1(a)

We follow the hints from [32, Section 5.5]. For $l \in Q_0 \cup \{0\}$ we define idempotents in $\widetilde{H}$

$$\mathbb{1}^{(l)} := \sum_{i>l} e_{(i,0)} + \sum_{i\leq l} e_{(i,1)}, \qquad \mathbb{1}_0^{(l)} := \mathbb{1}_0 + \sum_{i\leq l} e_{(i,1)},$$

and the corresponding (non-unitary) subalgebras

$$H^{(l)} := \mathbb{1}^{(l)}\widetilde{H}\mathbb{1}^{(l)}, \qquad \widetilde{H}^{(l)} := \mathbb{1}_0^{(l)}\widetilde{H}\mathbb{1}_0^{(l)}.$$

Clearly, $H^{(0)} = \widetilde{H}^{(0)} = H_{(0)}$, $H^{(n)} = H_{(1)}$, $\widetilde{H}^{(n)} = \widetilde{H}$, and an easy calculation shows that, using the notation of Eq. (9.1),

$$H^{(l)} = s_l \cdots s_2 s_1(H^{(0)}), \qquad (l \in Q_0).$$

Moreover $\mathbb{1}^{(l)} \in \widetilde{H}^{(l)}$ and thus $H^{(l)} \subset \widetilde{H}^{(l)} \supset \widetilde{H}^{(l-1)}$ for $l \in Q_0$. We study the corresponding restriction functors:

$$\begin{aligned} \operatorname{Res}^{(l)}\colon \operatorname{rep}(\widetilde{H}) &\to \operatorname{rep}(H^{(l)}), & M &\mapsto \mathbb{1}^{(l)}\widetilde{H} \otimes_{\widetilde{H}} M, \\ \operatorname{Res}_{(l,m)}\colon \operatorname{rep}(\widetilde{H}^{(m)}) &\to \operatorname{rep}(\widetilde{H}^{(l)}), & M &\mapsto \mathbb{1}_0^{(l)}\widetilde{H}^{(m)} \otimes_{\widetilde{H}^{(m)}} M, \quad (l < m). \end{aligned}$$

Obviously, $\operatorname{Res}_{(l,m)}$ admits a right adjoint

$$\operatorname{Res}^*_{(l,m)}(-) = \operatorname{Hom}_{\widetilde{H}^{(l)}}(\mathbb{1}_0^{(l)}\widetilde{H}^{(m)}, -)$$

and

$$\operatorname{Res}_0 = \operatorname{Res}_{(0,1)} \circ \operatorname{Res}_{(1,2)} \circ \cdots \circ \operatorname{Res}_{(n-1,n)}.$$

Thus we have

$$\operatorname{Res}_0^* = \operatorname{Res}_{(n-1,n)}^* \circ \cdots \circ \operatorname{Res}_{(1,2)}^* \circ \operatorname{Res}_{(0,1)}^* .$$

Lemma 10.2 *With the above notations we have functorial isomorphisms*

$$\operatorname{Res}^{(i)} \circ \operatorname{Res}_{(i-1,i)}^*(M) \cong F_i^+ \circ \operatorname{Res}^{(i-1)}(M)$$

for all $M \in \operatorname{rep}(\widetilde{H}^{(i-1)})$ *and* $i \in Q_0 = \{1, \dots, n\}$.

Proof Note that naturally

$$\operatorname{Res}_{(i-1,i)} \circ \operatorname{Res}_{(i-1,i)}^*(M) \cong M$$

for all $M \in \operatorname{rep}(\widetilde{H}^{(i-1)})$. Now there is a unique functor

$$R_{(i-1,i)}^* \colon \operatorname{rep}(\widetilde{H}^{(i-1)}) \to \operatorname{rep}(\widetilde{H}^{(i)})$$

satisfying the two following conditions for all $M \in \operatorname{rep}(\widetilde{H}^{(i-1)})$:

$$\operatorname{Res}_{(i-1,i)} \circ R_{(i-1,i)}^*(M) = M, \qquad \operatorname{Res}^{(i)} \circ R_{(i-1,i)}^*(M) = F_i^+ \circ \operatorname{Res}^{(i-1)}(M).$$

Indeed, the first condition fixes the restriction of $R_{(i-1,i)}^*(M)$ to $\widetilde{H}^{(i-1)}$ and the second one fixes the restriction of $R_{(i-1,i)}^*(M)$ to $H^{(i)}$. Because of the definitions of $H^{(i)}$ and $\widetilde{H}^{(i-1)}$, this determines completely the structure of $R_{(i-1,i)}^*(M)$, and gives uniqueness. Note that the quivers of $\widetilde{H}^{(i-1)}$ and of $H^{(i)}$ contain some common arrows, but the representations M and $F_i^+ \circ \operatorname{Res}^{(i-1)}(M)$ are the same for those arrows, by definition of F_i^+. So $R_{(i-1,i)}^*(M)$ is indeed a representation of $H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega})$, supported on the vertices and arrows of $\widetilde{H}^{(i)}$. Finally, this representation satisfies the relation (10.2) for $j = i$, because again of the definition of F_i^+, so $R_{(i-1,i)}^*(M) \in \operatorname{rep}(\widetilde{H}^{(i)})$.

To prove the lemma, we have to show that the above functor $R_{(i-1,i)}^*$ is isomorphic to $\operatorname{Res}_{(i-1,i)}^*$, or equivalently, that $R_{(i-1,i)}^*$ is right adjoint to $\operatorname{Res}_{(i-1,i)}$. To do so, let $N \in \operatorname{rep}(\widetilde{H}^{(i)})$ and $M \in \operatorname{rep}(\widetilde{H}^{(i-1)})$, and consider the natural map

$$\operatorname{Hom}_{\widetilde{H}^{(i)}}(N, R_{(i-1,i)}^*(M)) \to \operatorname{Hom}_{\widetilde{H}^{(i-1)}}(\operatorname{Res}_{(i-1,i)}(N), M)$$

obtained by restricting $f \colon N \to R_{(i-1,i)}^*(M)$ to $\operatorname{Res}_{(i-1,i)}(N)$. We have to show that this restriction is in fact bijective. That is, for $g \in \operatorname{Hom}_{\widetilde{H}^{(i-1)}}(\operatorname{Res}_{(i-1,i)}(N), M)$ we have to show that there exists a unique $g_{(i,1)} \in$

$\mathrm{Hom}_{H_i}(N_{(i,1)}, \mathrm{Ker}(\widetilde{M}_{i,\mathrm{in}}))$ which lifts g to an element of $\mathrm{Hom}_{\widetilde{H}^{(i)}}(N, R^*_{(i-1,i)}(M))$.

Now, let

$$N_{i,+} := \bigoplus_{(k,c)\in\widetilde{\Omega}((i,0),-)} {}_{(i,0)}H_{(k,c)} \otimes_{H_{(k,c)}} N_{(k,c)}$$

denote the domain of $\widetilde{N}_{i,\mathrm{in}}$, and similarly let $M_{i,+}$ denote the domain of $\widetilde{M}_{i,\mathrm{in}}$. By the definition of $\widetilde{H}^{(i-1)}$-homomorphisms, we have a commutative diagram

$$\begin{tikzcd}
 & N_{(i,1)} \arrow[r, "\widetilde{N}_{i,\mathrm{out}}"] & N_{i,+} \arrow[r, "\widetilde{N}_{i,\mathrm{in}}"] \arrow[d, "g_{i,+}"] & N_{(i,0)} \arrow[d, "g_{(i,0)}"] \\
0 \arrow[r] & \mathrm{Ker}(\widetilde{M}_{i,\mathrm{in}}) \arrow[r] & M_{i,+} \arrow[r, "\widetilde{M}_{i,\mathrm{in}}"] & M_{(i,0)}
\end{tikzcd}$$

where the bottom row is exact by construction, and in the top row the composition is zero since N is a $\widetilde{H}^{(i)}$-module. Thus $\widetilde{M}_{i,\mathrm{in}} \circ g_{i,+} \circ \widetilde{N}_{i,\mathrm{out}} = 0$. By the universal property of $\mathrm{Ker}(\widetilde{M}_{i,\mathrm{in}})$ there exists a unique morphism of H_i-modules $g_{(i,1)}\colon N_{(i,1)} \to \mathrm{Ker}(\widetilde{M}_{i,\mathrm{in}})$ which makes the left-hand square commutative. □

We can now finish the proof of Theorem 10.1(a). Using n times Lemma 10.2, for $M \in \mathrm{rep}(H)$ (regarded as a representation of H_0) we have

$$\begin{aligned}
\mathrm{Hom}_H(X, M) &= \mathrm{Res}_1 \circ \mathrm{Res}_0^*(M) \\
&= \mathrm{Res}^{(n)} \circ \mathrm{Res}^*_{(n-1,n)} \circ \cdots \circ \mathrm{Res}^*_{(0,1)}(M) \\
&= F_n^+ \circ \mathrm{Res}^{(n-1)} \circ \mathrm{Res}^*_{(n-2,n-1)} \circ \cdots \circ \mathrm{Res}^*_{(0,1)}(M) \\
&= \ \cdots \\
&= F_n^+ \circ F_{n-1}^+ \circ \cdots \circ F_1^+ \circ \mathrm{Res}^{(0)}(M) \\
&= C^+(M).
\end{aligned}$$

10.4 Proof of Theorem 10.1(b)

We follow the idea of [32, Section 5.4], and start by giving an alternative description of Res_0^*. This is done by constructing in two steps a functor $R_0^*\colon \mathrm{rep}(H_{(0)}) \to \mathrm{rep}(\widetilde{H})$, and then showing that R_0^* is right adjoint to Res_0.

Let $M \in \operatorname{rep}(H_{(0)})$. We first define $\widetilde{M} \in \operatorname{rep}(H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega}))$ by requiring that

$$\begin{aligned}
&\operatorname{Res}_0(\widetilde{M}) = M,\\
&\operatorname{Res}_1(\widetilde{M}) = \bigoplus_{(k,l)\in\Omega} \operatorname{Hom}_{H_{(l,0)}}({}_{(l,0)}H_{(k,1)} \otimes_{H_{(k,1)}} e_{(k,1)}H_1, M_{(l,0)}).
\end{aligned}$$

(Note that $H_{(0)}$ and $H_{(1)}$ can also be regarded as subalgebras of $H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega})$, so we allow ourselves, by some abuse of notation, to continue to denote the restriction functors $\operatorname{rep}(H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega})) \to \operatorname{rep}(H_{(a)})$ by Res_a.) It remains to define, for $(i, j) \in \Omega$, the structure map

$$\widetilde{M}_{(j,0),(i,1)} \colon {}_{(j,0)}H_{(i,1)} \otimes_{H_{(i,1)}} \widetilde{M}_{(i,1)} \longrightarrow \widetilde{M}_{(j,0)} = M_{(j,0)}.$$

This is given by the following composition:

$$\begin{aligned}
&{}_{(j,0)}H_{(i,1)} \otimes_{H_{(i,1)}} \left(\bigoplus_{(k,l)\in\Omega} \operatorname{Hom}_{H_{(l,0)}}({}_{(l,0)}H_{(k,1)} \otimes e_{(k,1)}H_1 e_{(i,1)}, M_{(l,0)}) \right)\\
&\xrightarrow{\text{proj.}} {}_{(j,0)}H_{(i,1)} \otimes_{H_{(i,1)}} \operatorname{Hom}_{H_{(j,0)}}({}_{(j,0)}H_{(i,1)} \otimes e_{(i,1)}H_1 e_{(i,1)}, M_{(j,0)})\\
&= {}_{(j,0)}H_{(i,1)} \otimes_{H_{(i,1)}} \operatorname{Hom}_{H_{(j,0)}}({}_{(j,0)}H_{(i,1)}, M_{(j,0)}) \xrightarrow{\text{eval.}} M_{(j,0)},
\end{aligned}$$

where the first map is the projection on the direct summand indexed by $(k, l) = (i, j)$ and the second map is the evaluation $h \otimes \varphi \mapsto \varphi(h)$.

Secondly, we define a subrepresentation $R_0^*(M)$ of $\widetilde{M}$ as follows. We set

$$(R_0^*(M))_{(i,0)} = \widetilde{M}_{(i,0)} = M_{(i,0)}, \qquad (i \in Q_0),$$

and we define $(R_0^*(M))_{(h,1)}$ as the subspace of $\widetilde{M}_{(h,1)}$ consisting of all

$$(\mu^h_{k,l})_{(k,l)\in\Omega} \in \bigoplus_{(k,l)\in\Omega} \operatorname{Hom}_{H_{(l,0)}}({}_{(l,0)}H_{(k,1)} \otimes e_{(k,1)}H_{(1)}e_{(h,1)}, M_{(l,0)})$$

such that, for all $l \in Q_0$ and $n^{(1)} \in e_{(l,1)}H_{(1)}e_{(h,1)}$ the following relation holds:

$$\sum_{\substack{k\in\Omega(-,l)\\ \ell\in_k L^1_l}} \mu^h_{k,l}(\ell^*_- \otimes \ell \cdot n^{(1)}) + \sum_{\substack{m\in\Omega(l,-)\\ r\in_l R^0_m}} M_{(l,0),(m,0)}(r \otimes \mu^h_{l,m}(r^*_+ \otimes n^{(1)})) = 0. \tag{10.3}$$

Here, we use the notation from Sect. 10.2.3. It is straightforward to check that $R_0^*(M)$ is an $H(\widetilde{C}, \widetilde{D}, \widetilde{\Omega})$-subrepresentation of $\widetilde{M}$. Moreover, $R_0^*(M)$ is in fact a representation of $\widetilde{H}$. To see this, we check the defining relations (10.1) with the help of the special case $n^{(1)} = e_{(l,1)}$ of Eq. (10.3). In fact, if we apply $\widetilde{\rho}_j$ to $\mu^{(j)} = (\mu^j_{k,l})_{(k,l)\in\Omega} \in R_0^*(M)_{(j,1)}$ we deduce from the definitions that

$$\begin{aligned}
&\sum_{\substack{i\in\Omega(-,j)\\ \ell\in{}_iL_j^1}} \widetilde{M}_{(j,0),(i,1)}(\ell_-^* \otimes \widetilde{M}_{(i,1),(j,1)}(\ell \otimes \mu^{(j)}))\\
&\quad + \sum_{\substack{k\in\Omega(j,-)\\ r\in{}_jR_k^0}} M_{(j,0),(k,0)}(r \otimes \widetilde{M}_{(k,0),(j,1)}(r_+^* \otimes \mu^{(j)}))\\
&= \sum_{\substack{i\in\Omega(-,j)\\ \ell\in{}_iL_j^1}} \mu^j_{i,j}(\ell_-^* \otimes \ell \cdot e_{(j,1)})\\
&\quad + \sum_{\substack{k\in\Omega(j,-)\\ r\in{}_jR_k^0}} M_{(j,0),(k,0)}(r \otimes \mu^j_{j,k}(r_+^* \otimes e_{(j,1)})) = 0,
\end{aligned}$$

as required.

Thus, we have obtained a functor $R_0^*\colon \operatorname{rep}(H_{(0)}) \to \operatorname{rep}(\widetilde{H})$, $M \mapsto R_0^*(M)$. It will follow from the next lemma that R_0^* is isomorphic to Res_0^*.

Lemma 10.3 *R_0^* is right adjoint to* Res_0.

Proof Let $N \in \operatorname{rep}(\widetilde{H})$ and $M \in \operatorname{rep}(H_{(0)})$. Consider $\chi \in \operatorname{Hom}_{\widetilde{H}}(N, R_0^*(M))$. Thus, χ is given by a family of maps

$$\chi^{(i,a)} \in \operatorname{Hom}_{H_{(i,a)}}(N_{(i,a)}, (R_0^*(M))_{(i,a)}), \quad ((i,a) \in \widetilde{Q}_0),$$

subject to the usual commutativity relations. By the construction of $R_0^*(M)$ we have more explicitly for all $i \in Q_0$ and $n_{(i,1)} \in N_{(i,1)}$:

$$\begin{aligned}
&\chi^{(i,0)} \in \operatorname{Hom}_{H_{(i,0)}}(N_{(i,0)}, M_{(i,0)}),\\
&\chi^{(i,1)}(n_{(i,1)}) \in \bigoplus_{(k,l)\in\Omega} \operatorname{Hom}_{H_{(l,0)}}({}_{(l,0)}H_{(k,1)} \otimes e_{(k,1)}H_1e_{(i,1)}, M_{(l,0)}).
\end{aligned}$$

Let us denote by $\chi^{(i,1)}_{k,l}(-, n_{(i,1)})$ the (k,l)-component of $\chi^{(i,1)}(n_{(i,1)})$. These maps are subject to the following relations for $(i,j) \in \Omega$, $\ell^{(a)} \in {}_iL_j^a$, $r \in {}_iR_j^0$:

$$\chi^{(i,0)}(N_{(i,0),(j,0)}(\ell^{(0)} \otimes n_{(j,0)})) = M_{(i,0),(j,0)}(\ell^{(0)} \otimes \chi^{(j,0)}(n_{(j,0)})), \tag{10.4}$$

$$\chi^{(j,0)}(N_{(j,0),(i,1)}(r_+^* \otimes n_{(i,1)})) = \chi_{i,j}^{(i,1)}(r_+^* \otimes e_{(i,1)}, n_{(i,1)}), \tag{10.5}$$

$$\chi_{k,l}^{(i,1)}(-, N_{(i,1),(j,1)}(\ell^{(1)} \otimes n_{(j,1)})) = \chi_{k,l}^{(j,1)}(- \cdot \ell^{(1)}, n_{(j,1)}). \tag{10.6}$$

Equation (10.4) means that we have indeed a well-defined restriction

$$r_{N,M}\colon \operatorname{Hom}_{\widetilde{H}}(N, R_0^*(M)) \to \operatorname{Hom}_{H_{(0)}}(\operatorname{Res}_0(N), M).$$

Combining (10.5) and (10.6) we see that the maps $\chi_{k,l}^{(j,1)}$ for $(k,l) \in \Omega$ and $j \in Q_0$ are determined by the maps $\chi^{(i,0)}$ with $i \in Q_0$, in other words $r_{N,M}$ is injective.

By the same token we see that for each $\chi^{(0)} \in \operatorname{Hom}_{H_{(0)}}(\operatorname{Res}_0(N), M)$ there exists $\widetilde{\chi} \in \operatorname{Hom}_{H(\widetilde{C},\widetilde{D},\widetilde{\Omega})}(N, \widetilde{M})$ which restricts to $\chi^{(0)}$. We leave it as an exercise to show that if $N \in \operatorname{rep}(\widetilde{H})$ then $\operatorname{Im}(\widetilde{\chi}) \subset R_0^*(M)$. Thus, $r_{N,M}$ is bijective. □

Proposition 10.4 *For $M \in \operatorname{rep}_{\mathrm{l.f.}}(H)$ we have*

$$\tau(TM) \cong \operatorname{Res}_1 \circ R_0^*(M),$$

where in the right-hand side $H_{(0)}$ and $H_{(1)}$ are identified with H by means of the isomorphisms η_0 and η_1.

Proof Since M is locally free, TM is also locally free and Corollary 7.2 provides a projective resolution:

$$0 \to \bigoplus_{(j,i)\in\Omega} He_j \otimes {}_jH_i \otimes (TM)_i \xrightarrow{d\otimes TM} \bigoplus_{k=1}^{n} He_k \otimes (TM)_k \xrightarrow{\text{mult}} TM \to 0$$

Therefore, by definition of the Auslander–Reiten translation τ, we know that $\tau(TM)$ is isomorphic to $\operatorname{Ker}(\mathrm{D}\operatorname{Hom}_H(d \otimes_H TM, H))$.

On the other hand, the construction of $R_0^*(M)$ shows that $\operatorname{Res}_1 \circ R_0^*(M)$ can be identified with the kernel of the map

$$d_M^*\colon \bigoplus_{(j,i)\in\Omega} \operatorname{Hom}_{H_i}({}_jH_i^* \otimes e_jH, M_i) \longrightarrow \bigoplus_{k=1}^{n} \operatorname{Hom}_{H_k}(e_kH, M_k)$$

whose (j,i)-component is defined by

$$\varphi_{(j,i)} \mapsto \sum_{\ell\in {}_jL_i} \varphi_{(j,i)}(\ell^* \otimes_j \ell \cdot -) + \sum_{r\in {}_jR_i} M_{ji}(r \otimes \varphi_{(j,i)}(r^* \otimes_j -)). \tag{10.7}$$

Indeed, $\mathrm{Res}_1 \circ R_0^*(M)$ is the subspace of $\mathrm{Res}_1(\widetilde{M})$ defined by Eq. (10.3). Our goal is to identify under the trace pairing the map $\mathrm{D}\,\mathrm{Hom}_H(d \otimes TM, H)$ with d_M^*. For $(j, i) \in \Omega$, the restriction $d \otimes TM \colon He_j \otimes {}_jH_i \otimes M_i \to He_i \otimes M_i \oplus He_j \otimes M_j$ is given by

$$(d \otimes TM)(p \otimes h \otimes m) = ph \otimes m + p \otimes M_{ji}(h \otimes m). \tag{10.8}$$

(Note the plus sign, coming from the twist map T). Using adjunction we have

$$\mathrm{Hom}_H(He_i \otimes M_i, H) \cong \mathrm{Hom}_{H_i}(M_i, \mathrm{Hom}_H(He_i, H)) \cong \mathrm{Hom}_{H_i}(M_i, e_iH),$$

so under the trace pairing we get

$$\begin{aligned} &\mathrm{D}\,\mathrm{Hom}_H(He_i \otimes M_i \oplus He_j \otimes M_j, H) \\ &\quad \cong \mathrm{Hom}_{H_i}(e_iH, M_i) \oplus \mathrm{Hom}_{H_j}(e_jH, M_j). \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathrm{D}\,\mathrm{Hom}_H(He_j \otimes {}_jH_i \otimes M_i) \\ &\quad \cong \mathrm{Hom}_{H_j}(e_jH, {}_jH_i \otimes M_i) \cong \mathrm{Hom}_{H_i}({}_iH_j \otimes e_jH, M_i), \end{aligned}$$

where the second isomorphism is given by ad_{ij}^{-1}. Hence $\mathrm{D}\,\mathrm{Hom}_H(d \otimes TM, H)$ can be identified with a map from $\oplus_{(j,i)\in\Omega} \mathrm{Hom}_{H_i}({}_iH_j \otimes e_jH, M_i)$ to $\oplus_k \mathrm{Hom}_{H_k}(e_kH, M_k)$, and comparison between (10.7) and (10.8) shows that this map is indeed d_M^*. □

Now we are ready to prove part (b) of Theorem 10.1. By Lemma 10.3 and the uniqueness of adjoint functors we have a functorial isomorphism $R_0^*(-) \cong \mathrm{Res}_0^*(-)$. Hence, by Proposition 10.4, if $M \in \mathrm{rep}_{\mathrm{l.f.}}(H)$ we have

$$\mathrm{Hom}_H(X, TM) = \mathrm{Res}_1 \circ \mathrm{Res}_0^*(TM) \cong \mathrm{Res}_1 \circ R_0^*(TM) \cong \tau(T^2M) = \tau(M).$$

This proves Theorem 10.1 (b).

10.5 Proof of Theorem 10.1(c),(d)

Clearly, (c) follows from (a) since C^- is left adjoint to C^+. In order to show (d), let $M, N \in \mathrm{rep}_{\mathrm{l.f.}}(H)$. Recall, that this implies that both, M and N, have projective and injective dimension at most 1. We obtain functorial isomorphisms

$$\mathrm{Hom}_H(\tau^-(M), N) \cong \mathrm{Hom}_H(M, \tau(N))$$

$$\begin{aligned} &\cong \mathrm{Hom}_H(M, C^+(TN)) \\ &\cong \mathrm{Hom}_H(C^-(TM), N). \end{aligned}$$

The first isomorphism is obtained from the Auslander–Reiten formulas, the second follows from (b), and the third isomorphism is just the adjunction map.

With the usual H-H-bimodule structure on $\mathrm{D}(H)$ we obtain a functorial isomorphism of right H-modules

$$\mathrm{D}(X) \cong \mathrm{Hom}_H(X, \mathrm{D}(H))$$

for all (left) H-modules X. Now, in our situation $\mathrm{D}(H)$ is locally free, thus taking $N = \mathrm{D}(H)$ in the above chain of functorial isomorphisms we get

$$\tau^-(M) \cong C^-(TM) \cong X^T \otimes_H M$$

where the last isomorphism comes from (c). This proves Theorem 10.1 (d).

10.6 Another description of $\Pi(C, D)$ as a tensor algebra

Let $\Pi = \Pi(C, D)$. Recall from Sect. 6 that Π_1 is the subspace of Π of elements of degree 1. Let X^T be the twisted version of the H-H-bimodule X, where the bimodule structure is defined by

$$hxh' := hxT(h'), \qquad (h, h' \in H,\ x \in X).$$

Theorem 10.5 *We have isomorphisms of H-H-bimodules*

$$\Pi_1 \cong X^T \cong \mathrm{Ext}^1_H(\mathrm{D}(H), H).$$

Proof Note that the bimodule isomorphism $\Pi_1 \cong X^T$ follows directly from the definitions. On the other hand, we have by Theorem 10.1(d) for locally free modules M a functorial isomorphism

$$X^T \otimes_H M \cong \tau^-(M) \cong \mathrm{Ext}^1_H(\mathrm{D}(H), M). \tag{10.9}$$

Note that the functor

$$\mathrm{Ext}^1_H(\mathrm{D}(H), -)\colon\ \mathrm{rep}(H) \to \mathrm{rep}(H)$$

is right exact since proj. $\dim(\mathrm{D}(H)) \leq 1$. For $M \in \mathrm{rep}(H)$ let

$$P_1 \to P_0 \to M \to 0 \tag{10.10}$$

be a projective presentation of M. Applying the right exact functors $X^T \otimes_H -$ and $\operatorname{Ext}^1_H(\mathrm{D}(H), -)$ to (10.10) yields a functorial commutative diagram

$$\begin{tikzcd}
X^T \otimes_H P_1 \arrow[r] \arrow[d, "\eta_{P_1}"] & X^T \otimes_H P_0 \arrow[r] \arrow[d, "\eta_{P_0}"] & X^T \otimes_H M \arrow[r] \arrow[d, "\eta_M"] & 0 \\
\operatorname{Ext}^1_H(\mathrm{D}(H), P_1) \arrow[r] & \operatorname{Ext}^1_H(\mathrm{D}(H), P_0) \arrow[r] & \operatorname{Ext}^1_H(\mathrm{D}(H), M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since the restrictions of $X^T \otimes_H -$ and $\operatorname{Ext}^1_H(\mathrm{D}(H), -)$ to $\operatorname{rep}_{\mathrm{l.f.}}(H)$ are isomorphic, we get that η_{P_0} and η_{P_1} are isomorphisms. This implies that η_M is an isomorphism as well. It follows that the functors $X^T \otimes_H -$ and $\operatorname{Ext}^1_H(\mathrm{D}(H), -)$ are isomorphic. From the canonical isomorphism of H-H-bimodules $\operatorname{Ext}^1_H(\mathrm{D}(H), H) \otimes_H H \cong \operatorname{Ext}^1_H(\mathrm{D}(H), H)$ we conclude that the right exact functors $\operatorname{Ext}^1_H(\mathrm{D}(H), -)$ and $\operatorname{Ext}^1_H(\mathrm{D}(H), H) \otimes_H -$ are isomorphic. This implies that $\operatorname{Ext}^1_H(\mathrm{D}(H), H)$ and X^T are isomorphic as H-H-bimodules. □

Corollary 10.6 *We have K-algebra isomorphisms*

$$\Pi \cong T_H(X^T) \cong T_H(\operatorname{Ext}^1_H(\mathrm{D}(H), H)).$$

Proof Combine Theorem 10.5 and Proposition 6.5. □

Corollary 10.7 *For $M \in \operatorname{rep}(H)$ there are functorial isomorphisms*

$$\operatorname{Hom}_H(X^T, M) \cong \mathrm{D}\operatorname{Ext}^1_H(M, H) \text{ and } X^T \otimes_H M \cong \operatorname{Ext}^1_H(\mathrm{D}(H), M).$$

Proof We get the second isomorphism from the proof of Theorem 10.5. The first isomorphism follows then by adjunction. □

10.7 The morphism categories $\mathcal{C}(1, TC^+)$ and $\mathcal{C}(TC^-, 1)$

Let $\Pi = \Pi(C, D)$. Following a definition due to Ringel [49] , we define a category $\mathcal{C}(1, TC^+)$ as follows. Its objects are the H-module homomorphisms $M \to TC^+(M)$, where $M \in \operatorname{rep}(H)$ and the morphisms in $\mathcal{C}(1, TC^+)$ are given by commutative diagrams

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "h"'] & TC^+(M) \arrow[d, "TC^+(h)"] \\
N \arrow[r, "g"] & TC^+(N).
\end{tikzcd}$$

Similarly, let $\mathcal{C}(TC^-, 1)$ be the category with objects the H-module homomorphisms $TC^-(M) \to M$.

Theorem 10.8 *The categories* $\operatorname{rep}(\Pi), \mathcal{C}(1, TC^+)$ *and* $\mathcal{C}(TC^-, 1)$ *are isomorphic.*

Proof It follows from Proposition 9.1 that (TC^-, TC^+) is a pair of adjoint functors $\operatorname{rep}(H) \to \operatorname{rep}(H)$. Now [49, Lemma 1] implies that the categories $\mathcal{C}(1, TC^+)$ and $\mathcal{C}(TC^-, 1)$ are isomorphic. It follows from Theorem 10.1(c) that there is a functorial isomorphism $X^T \otimes_H - \cong TC^-(-)$, and by Corollary 10.6 we have $\Pi \cong T_H(X^T)$. Now [49, Lemma 12] gives an isomorphism of categories $\mathcal{C}(TC^-, 1) \cong \operatorname{rep}(\Pi(C, D))$. □

One can also adapt Ringel's proof of [49, Theorem B] to obtain a more direct proof of Theorem 10.8.

10.8 The kernel of the Coxeter functor

As before, let $H = H(C, D, \Omega)$. Recall that H is a 1-Iwanaga–Gorenstein algebra with the subcategory

$$\mathcal{GP}(H) = \{M \in \operatorname{rep}(H) \mid \operatorname{Ext}^1_H(M, H) = 0\}$$

of Gorenstein-projective modules.

As an immediate consequence of Theorem 10.1(a), Corollary 10.7, and the definition of $C^+(-)$ we get the following result. Here, the map $M_{i,\mathrm{in}}$ is defined as in Sect. 5.3, since we can regard the H-module M also as a module over $\Pi(C, D)$.

Theorem 10.9 *For an H-module M the following are equivalent:*

(i) $M \in \mathcal{GP}(H)$;
(ii) $C^+(M) = 0$;
(iii) $M_{i,\mathrm{in}}$ *is injective for all* $1 \le i \le n$.

If C is symmetric, then the equivalence of (i) and (iii) in Theorem 10.9 is a special case of [44, Theorem 5.1]. For C symmetric and $D = \operatorname{diag}(2, \dots, 2)$ the category $\mathcal{GP}(H)$ has been studied in detail in [51].

11 τ-Locally free H-modules

11.1 Preprojective, preinjective, and regular H-modules

Let M be an indecomposable H-module. Recall that M is *τ-locally free* if $\tau^k(M)$ is locally free for all $k \in \mathbb{Z}$. Furthermore, M is called *preprojective* (resp. *preinjective*) if there exists some $k \ge 0$ such that $M \cong \tau^{-k}(P)$ (resp.

$M \cong \tau^k(I)$) for some indecomposable projective H-module P (resp. indecomposable injective H-module I). A τ-locally free H-module M is *τ-locally free regular* if M is neither preprojective nor preinjective. (An indecomposable module M over a finite-dimensional algebra is called *regular* if $\tau^k(M) \neq 0$ for all $k \in \mathbb{Z}$.)

Let $\mathcal{C}$ be a K-linear category. The *stable category* $\underline{\mathcal{C}}$ (resp. $\overline{\mathcal{C}}$) is the quotient category of $\mathcal{C}$ modulo the ideal of all morphisms factoring through projective (resp. injective) objects.

Proposition 11.1 *The restriction of* $\tau(-)$ *yields an equivalence of stable categories*

$$\underline{\operatorname{rep}}_{\text{l.f.}}(H) \to \overline{\mathcal{F}}(H)$$

where $\mathcal{F}(H) := \{M \in \operatorname{rep}(H) \mid \operatorname{Hom}_H(\mathrm{D}(H), M) = 0\}$, *and* $\tau^-(-)$ *yields an equivalence of stable categories*

$$\overline{\operatorname{rep}}_{\text{l.f.}}(H) \to \underline{\mathcal{G}}(H)$$

where $\mathcal{G}(H) := \{M \in \operatorname{rep}(H) \mid \operatorname{Hom}_H(M, H) = 0\}$.

Proof Combine Proposition 3.5 with [4, Lemma 4.1] and its dual. □

Corollary 11.2 *For an indecomposable* $M \in \operatorname{rep}(H)$ *the following are equivalent:*

(i) $M \in \operatorname{rep}_{\text{l.f.}}(H)$;
(ii) $\operatorname{Hom}_H(\tau^-(M), H) = 0$;
(iii) $\operatorname{Hom}_H(\mathrm{D}(H), \tau(M)) = 0$.

Corollary 11.3 *For an indecomposable* $M \in \operatorname{rep}(H)$ *the following are equivalent:*

(i) *M is τ-locally free;*
(ii) $\operatorname{Hom}_H(\tau^-(\tau^k(M)), H) = 0$ *for all* $k \in \mathbb{Z}$;
(iii) $\operatorname{Hom}_H(\mathrm{D}(H), \tau(\tau^k(M))) = 0$ *for all* $k \in \mathbb{Z}$.

Proof Recall that for an indecomposable module M we have $\tau(\tau^-(M)) \cong M$ if and only if M is not injective, and $\tau^-(\tau(M)) \cong M$ if and only if M is not projective. Now the statement is a direct consequence of Corollary 11.2. □

Proposition 11.4 *Let* $M \in \operatorname{rep}_{\text{l.f.}}(H)$ *be indecomposable and rigid. Then M is τ-locally free and $\tau^k(M)$ is rigid for all* $k \in \mathbb{Z}$.

Proof By Theorem 10.1 we know that $\tau^k(M) \cong T^k C^k(M)$. Now the result follows from Proposition 9.6. □

Recall from Sect. 3.5 the Coxeter matrix Φ_H.

Proposition 11.5 *For a τ-locally free module $M \in \operatorname{rep}(H)$ the following hold:*

(i) *If $\tau^k(M) \neq 0$ for some $k \in \mathbb{Z}$, then $\underline{\operatorname{rank}}(\tau^k(M)) = (D^{-1}\Phi_H D)^k (\underline{\operatorname{rank}}(M))$.*
(ii) *If $\tau^k(M) \neq 0$ for some $k \in \mathbb{Z}$ and $\underline{\operatorname{rank}}(M)$ is contained in $\Delta_{\mathrm{re}}^+(C)$ or $\Delta_{\mathrm{im}}^+(C)$, then $\underline{\operatorname{rank}}(\tau^k(M))$ is in $\Delta_{\mathrm{re}}^+(C)$ or $\Delta_{\mathrm{im}}^+(C)$, respectively.*

Proof Part (i) follows from [48, Section 2.4, p.75] combined with Corollary 11.3 and the fact that τ-locally free H-modules and their τ^k-translates have projective and injective dimension at most 1. To prove (ii), let i be a sink (resp. source) in $Q^\circ(C, \Omega)$. Then for a τ-locally free module M with $M \ncong E_i$ the map $M_{i,\mathrm{in}}$ is surjective (resp. $M_{i,\mathrm{out}}$ is injective). Now the result follows from Proposition 9.4 and Theorem 10.1(b),(d). □

Proposition 11.6 *Let M be a preprojective or preinjective H-module. Then the following hold:*

(i) *M is τ-locally free and rigid;*
(ii) *$\underline{\operatorname{rank}}(M) \in \Delta_{\mathrm{re}}^+(C)$;*
(iii) *If M and N are preprojective or preinjective H-modules with $\underline{\dim}(M) = \underline{\dim}(N)$, then $M \cong N$.*

Proof By definition we have $M \cong \tau^{-k}(P_i)$ or $M \cong \tau^k(I_i)$ for some $k \geq 0$ and some $1 \leq i \leq n$. The modules P_i and I_i are indecomposable, locally free and rigid. Thus by Proposition 11.4 the module M is τ-locally free and rigid. We know from Sect. 3.4 that $\underline{\operatorname{rank}}(P_i), \underline{\operatorname{rank}}(I_i) \in \Delta_{\mathrm{re}}^+(C)$. Now part (ii) follows from Proposition 11.5(ii), and (iii) is a consequence of Lemmas 2.1 and 2.2. □

Lemma 11.7 *Assume C is connected and not of Dynkin type. Let X be a preprojective, Y a τ-locally free regular and Z a preinjective H-module. Then we have* $\operatorname{Hom}_H(Z, Y) = 0$, $\operatorname{Hom}_H(Y, X) = 0$ *and* $\operatorname{Hom}_H(Z, X) = 0$.

Proof We have $X \cong \tau^{-k}(P_i)$ for some $1 \leq i \leq n$ and some $k \geq 0$. We get $\operatorname{Hom}_H(Y, X) \cong \operatorname{Hom}_H(\tau^k(Y), P_i)$ and $\operatorname{Hom}_H(Z, X) \cong \operatorname{Hom}_H(\tau^k(Z), P_i)$. Now Corollary 11.3 yields that these homomorphism spaces are zero. Similarly, one shows that $\operatorname{Hom}_H(Z, Y) = 0$. □

A sequence $((i_1, p_1), \dots, (i_t, p_t))$ with $1 \leq i_k \leq n$ and $p_i \in \{+, -\}$ is *admissible* for (C, Ω) if the following hold:

(i) Either i_1 is a sink in $Q^\circ(C, \Omega)$ and $p_1 = +$, or i_1 is a source in $Q^\circ(C, \Omega)$ and $p_1 = -$;
(ii) For each $2 \leq k \leq n$, either i_k is a sink in $Q^\circ(C, s_{i_{k-1}} \cdots s_{i_1}(\Omega))$ and $p_k = +$, or i_k is a source in $Q^\circ(C, s_{i_{k-1}} \cdots s_{i_1}(\Omega))$ and $p_k = -$.

Proposition 11.8 *For an indecomposable locally free* $M \in \operatorname{rep}(H)$ *the following are equivalent:*

- (i) M *is* τ*-locally free;*
- (ii) *For each admissible sequence* $((i_1, p_1), \dots, (i_t, p_t))$ *for* (C, Ω) *the module*

$$F_{i_t}^{p_t} \cdots F_{i_1}^{p_1}(M)$$

is locally free.

Proof Assume M is τ-locally free. Let i be a sink in $Q^\circ(C, \Omega)$. We want to show that $F_i^+(M)$ is τ-locally free. If $M \cong P_i = E_i$, then $F_i^+(M) = 0$, which is trivially τ-locally free. Thus we can assume that $M \ncong E_i$. If $M \cong P_j$ for some $j \neq i$, then $\operatorname{top}_i(M) = 0$. In particular, $M_{i,\mathrm{in}}$ is surjective. Thus $F_i^+(M)$ is locally free. Now Lemma 9.16 yields that $F_i^+(M)$ is indecomposable. Next, assume that M is not projective. In other words, we have $\tau(M) \neq 0$. There clearly exists a $+$-admissible sequence $(i_1, \dots, i_n)$ for (C, Ω) with $i_1 = i$. Using that M is τ-locally free and applying Theorem 10.1 we get

$$\tau(M) \cong TC^+(M) \cong TF_{i_n}^+ \cdots F_{i_1}^+(M).$$

By Lemma 9.16, the module $F_i^+(M)$ is indecomposable and locally free. We can now assume that $\tau^k(F_i^+(M)) \neq 0$ for all $k \in \mathbb{Z}$. (Otherwise, the indecomposable module $F_i^+(M)$ is preprojective or preinjective and therefore τ-locally free.)

Let $k > 0$. There exists a $+$-admissible sequence $(j_1, \dots, j_n)$ for $(C, s_i(\Omega))$ with $j_n = i$. It follows that $(i, j_1, \dots, j_{n-1})$ is a $+$-admissible sequence for (C, Ω). We have

$$\tau(F_i^+(M)) \cong T(F_{j_n}^+ \cdots F_{j_1}^+(F_i^+(M)) \cong F_i^+(\tau(M)),$$

and this module is indecomposable and locally free since $\tau(M)$ is τ-locally free. Now it follows by induction that $\tau^k(F_i^+(M)) \cong F_i^+(\tau^k(M))$ is indecomposable and locally free for each $k > 0$.

Next, let $k < 0$. Then there exists a $-$-admissible sequence $(j_1, \dots, j_n)$ for $(C, s_i(\Omega))$ with $j_1 = i$. Then $(j_2, \dots, j_n, i)$ is a $-$-admissible sequence for (C, Ω). We get

$$\tau^-(F_i^+(M)) \cong F_{j_n}^- \cdots F_{j_2}^- F_i^- F_i^+(M) \cong F_{j_n}^- \cdots F_{j_2}^-(M)$$

and

$$F_i^+(\tau^-(M)) \cong F_i^+ F_i^- F_{j_n}^- \cdots F_{j_2}^-(M) \cong F_{j_n}^- \cdots F_{j_2}^-(M).$$

(For the last isomorphism we used the dual of Lemma 9.16.) Again by induction we get that $\tau^k(F_i^+(M)) \cong F_i^+(\tau^k(M))$ is indecomposable and locally free for each $k < 0$.

Altogether we showed that $F_i^+(M)$ is τ-locally free. Dually, one shows that $F_j^-(M)$ is τ-locally free for each source j in $Q^\circ(C, \Omega)$. This implies (ii).

To show the other direction, assume that (ii) holds. It follows that $T^k C^k(M)$ is locally free for all $k \in \mathbb{Z}$. Now we can apply Theorem 10.1 and get $\tau^k(M) \cong T^k C^k(M)$. Thus M is τ-locally free. □

Proposition 11.9 *For an H-module M the following are equivalent:*

(i) *M is locally free;*
(ii) $\tau(M) \cong TC^+(M)$*;*
(iii) $\tau^-(M) \cong TC^-(M)$.

Proof By Theorem 10.1(b) we know that (i) implies (ii). Now suppose that (ii) holds. Let $f : M \to N$ be a monomorphism from M to a locally free H-module N. (For example we could just take the injective envelope of M.) Since TC^+ is a left exact functor we get an exact sequence

$$0 \to TC^+(M) \to TC^+(N).$$

By Theorem 10.1(b) and assumption (ii) we get an exact sequence

$$0 \to \tau(M) \to \tau(N).$$

By Proposition 11.1 we have $\mathrm{Hom}_H(\mathrm{D}(H), \tau(N)) = 0$, since N is locally free. Applying $\mathrm{Hom}_H(\mathrm{D}(H), -)$ to the exact sequence above gives $\mathrm{Hom}_H(\mathrm{D}(H), \tau(M)) = 0$. Again by Proposition 11.1 this implies that M is locally free. The equivalence of (i) and (iii) is proved dually. □

11.2 Finite type classification for τ-locally free modules

Theorem 11.10 *For a Cartan matrix C of Dynkin type the following hold*:

(i) *The map $M \mapsto \underline{\mathrm{rank}}(M)$ yields a bijection between the set of isomorphism classes of τ-locally free H-modules and the set $\Delta^+(C)$ of positive roots of the Lie algebra associated with C.*
(ii) *For an indecomposable H-module M the following are equivalent:*
  (a) *M is preprojective;*
  (b) *M is preinjective;*
  (c) *M is τ-locally free;*
  (d) *M is locally free and rigid.*

Proof Let $\mathbf{i} = (i_1, \dots, i_n)$ be a $+$-admissible sequence for (C, Ω). We have $\underline{\mathrm{rank}}(P_{i_k}) = \beta_{\mathbf{i},k}$ for $1 \le k \le n$, compare Sect. 3.4. Since C is of Dynkin type, we get all elements of $\Delta^+(C)$ by applying the Coxeter transformation c^{-s} to the $\beta_{\mathbf{i},k}$ with $s \ge 0$ and $1 \le k \le n$, compare Lemma 2.2. In particular, the preprojective H-modules and the preinjective H-modules coincide. Now Proposition 11.6 implies part (i). We also get that (a) and (b) in part (ii) are equivalent, and that (a) and (b) implies (c) and (d). By Proposition 11.4 we know that (d) implies (c). Now let $M \in \mathrm{rep}(H)$ be τ-locally free. Then there exists an injective H-module I_i with $\mathrm{Hom}_H(M, I_i) \neq 0$. Since C is of Dynkin type, we know that $P_j = \tau^k(I_i)$ for some $k \ge 0$ and some $1 \le j \le n$. If $\tau^s(M) = 0$ for some $s \ge 0$, then M is preprojective and we are done. Thus assume that $\tau^s(M) \neq 0$ for all $s \ge 0$. Then we have $\mathrm{Hom}_H(M, I_i) \cong \mathrm{Hom}_H(\tau^k(M), \tau^k(I_i)) = \mathrm{Hom}_H(\tau^k(M), P_j) \neq 0$. (Note that all modules appearing here have projective and injective dimension at most one. Thus the stable homomorphism spaces are equal to the ordinary homomorphism spaces.) It follows that $\mathrm{Hom}_H(\tau^-(\tau^{k+1}(M)), P_j) \neq 0$, a contradiction to Corollary 11.3. This finishes the proof. □

Combining the results in Sect. 11.1, Theorem 11.10 and Lemmas 2.1 and 2.2 we get the following result.

Theorem 11.11 *There are finitely many isomorphism classes of τ-locally free H-modules if and only if C is of Dynkin type.*

11.3 The algebra Π as a module over *H*

Let $\Pi = \Pi(C, D)$.

Theorem 11.12 ${}_H\Pi \cong \bigoplus_{m \ge 0} \tau^{-m}({}_H H)$.

Proof By Proposition 11.6 we know that $\tau^{-m}(H)$ is locally free for all $m \ge 0$. Thus we have $\tau^-(H) \cong \mathrm{Ext}^1_H(\mathrm{D}(H), H)$ and

$$\begin{aligned} \tau^-(\tau^{-(m-1)}(H)) &\cong \mathrm{Ext}^1_H(\mathrm{D}(H), \tau^{-(m-1)}(H)) \\ &\cong \mathrm{Ext}^1_H(\mathrm{D}(H), H) \otimes_H \tau^{-(m-1)}(H) \\ &\cong \mathrm{Ext}^1_H(\mathrm{D}(H), H) \otimes_H \mathrm{Ext}^1_H(\mathrm{D}(H), H)^{\otimes(m-1)} \end{aligned}$$

where the last isomorphism follows by induction. By Corollary 10.6 we know that $\Pi \cong T_H(\mathrm{Ext}^1_H(\mathrm{D}(H), H))$. The result follows. □

Corollary 11.13 Π *is finite-dimensional if and only if C is of Dynkin type.*

Proof This follows directly from Theorems 11.10 and 11.12 and the fact that $\Delta^+(C)$ is finite if C is of Dynkin type. □

11.4 Regular components of the Auslander–Reiten quiver

A connected component $\mathcal{C}$ of the Auslander–Reiten quiver of H is *τ-locally free regular* if it consists only of τ-locally free regular modules. (A connected component of the Auslander–Reiten quiver of a finite-dimensional algebra is called *regular* if it consists only of regular modules.)

Proposition 11.14 *For a connected component $\mathcal{C}$ of the Auslander–Reiten quiver of H the following are equivalent:*

(i) *$\mathcal{C}$ contains a τ-locally free regular module;*
(ii) *$\mathcal{C}$ is τ-locally free regular.*

Proof Trivially, (ii) implies (i). For the other direction assume that M is a τ-locally free regular module in $\mathcal{C}$. Let $0 \to \tau(M) \to E \to M \to 0$ be the Auslander–Reiten sequence ending in M. Applying $\tau^k(-)$ yields again an Auslander–Reiten sequence

$$0 \to \tau^{k+1}(M) \to \tau^k(E) \to \tau^k(M) \to 0$$

for each $k \in \mathbb{Z}$. Here we used that $\tau^{k+1}(M)$ and $\tau^k(M)$ and therefore also $\tau^k(E)$ have projective and injective dimension equal to 1. It follows that $\tau^k(N)$ is locally free for each indecomposable direct summand N of E. Now (ii) follows by induction. □

Let $\mathcal{C}$ be a connected component of the Auslander–Reiten quiver of H. Suppose $\mathcal{C}$ contains an indecomposable projective module P_i with $c_i \geq 2$. Then $\operatorname{rad}(P_i)$ is obviously not locally free. Thus $\operatorname{rad}(P_i)$ contains an indecomposable direct summand R, which is not locally free. Since the inclusion $\operatorname{rad}(P_i) \to P_i$ is a sink map, there is an arrow $[R] \to [P_i]$ in the Auslander–Reiten quiver of H. Thus $\mathcal{C}$ contains a module, which is not locally free.

Ringel [47] proved that the regular components of the Auslander–Reiten quiver of a wild hereditary algebra are always of type $\mathbb{Z}A_\infty$. An alternative proof is due to Crawley-Boevey [20, Section 2] and can easily be adapted to obtain the following theorem.

Theorem 11.15 *Assume that C is connected and neither of Dynkin nor of Euclidean type. Let $\mathcal{C}$ be a τ-locally free regular component of the Auslander–Reiten quiver of H. Then $\mathcal{C}$ is a component of type $\mathbb{Z}A_\infty$.*

12 Projective resolutions and Ext-group symmetries of Π-modules

12.1 Projective resolutions of $\Pi(C, D)$-modules

Let $H = H(C, D, \Omega)$ and $\Pi = \Pi(C, D)$. Let $\operatorname{Rep}_{\mathrm{l.f.}}(\Pi)$ denote the category of *all* locally free Π-modules, possibly of infinite rank. For typographic reasons we use in this section the convention $I = \{1, 2, \dots, n\}$. Recall that we have defined the S-S-bimodule

$$\overline{B} := \bigoplus_{(i,j)\in\overline{\Omega}} {}_iH_j,$$

and that we can identify Π and $T_S(\overline{B})/J$, where J is the ideal of $T_S(\overline{B})$ which is generated by the elements $\rho_i \in \overline{B} \otimes_S \overline{B}$ for $i \in I$. For the next result we follow closely the ideas from [24, Lemma 3.1].

Proposition 12.1 *There is an exact sequence of* Π-Π-*bimodules*

$$\underbrace{\bigoplus_{i\in I} \Pi e_i \otimes e_i \Pi \xrightarrow{f} \bigoplus_{(j,i)\in\overline{\Omega}} \Pi e_j \otimes_j {}_jH_i \otimes_i e_i \Pi \xrightarrow{g} \bigoplus_{i\in I} \Pi e_i \otimes_i e_i \Pi}_{P_\bullet} \xrightarrow{h} \Pi \to 0 \tag{12.1}$$

where

$$\begin{aligned} f(e_i \otimes e_i) &:= \rho_i \otimes e_i + e_i \otimes \rho_i, \\ g(e_j \otimes h \otimes e_i) &:= he_i \otimes e_i - e_j \otimes e_j h, \\ h(m \otimes m') &:= mm'. \end{aligned}$$

Proof Observe first, that the above complex can be written more compactly as

$$\Pi \otimes_S \Pi \xrightarrow{f} \Pi \otimes_S \otimes \overline{B} \otimes_S \Pi \xrightarrow{g} \Pi \otimes_S \Pi \xrightarrow{h} \Pi \to 0$$

Note that we have a surjective Π-Π-bimodule homomorphism

$$\bigoplus_{i=1}^{n} \Pi e_i \otimes_i e_i \Pi \xrightarrow{r} J/J^2$$

defined by $e_i \otimes e_i \mapsto \rho_i$. Moreover, we have a canonical map

$$J/J^2 \xrightarrow{\text{can}} \Pi \otimes_S \overline{B} \otimes_S \Pi$$

given by $\overline{x} \mapsto \tilde{x}_l \otimes 1 + 1 \otimes \tilde{x}_r$ coming from the compositions of

$$J \xrightarrow{i_{J,l}} \bigoplus_{k\geq 1} (\overline{B}^{\otimes k} \otimes_S \overline{B}) \xrightarrow{\text{proj}\otimes\text{id}} \Pi \otimes_S \overline{B}$$

and

$$J \xrightarrow{i_{J,r}} \bigoplus_{k\geq 1}(\overline{B} \otimes_S \overline{B}^{\otimes k}) \xrightarrow{\mathrm{id}\otimes\mathrm{proj}} \overline{B} \otimes_S \Pi,$$

respectively, where the maps $i_{J,l}$ and $i_{J,r}$ are the obvious inclusions. Note that both compositions vanish on J^2. It is easy to see that $f = \mathrm{can} \circ r$. Thus we only have to show that the sequence

$$J/J^2 \xrightarrow{\mathrm{can}} \Pi \otimes_S \overline{B} \otimes_S \overline{B} \xrightarrow{u} \Pi \otimes_S \Pi \xrightarrow{\mathrm{mult}} \Pi \to 0$$

where $u(1 \otimes b \otimes 1) := b \otimes 1 - 1 \otimes b$ is exact. This is a special case of a combination of results by Schofield [52, Theorems 10.1,10.3,10.5]. □

Corollary 12.2 *For each $M \in \mathrm{Rep}_{\mathrm{l.f.}}(\Pi)$ the complex $P_\bullet \otimes_\Pi M$ is the beginning of a projective resolution of M.*

Proof The components of $P_\bullet$ are projective as left and as right modules. In fact, for example $H_i^m \otimes_i e_i\Pi \cong (e_i\Pi)^m$ is a projective right Π-module. Now, ${}_jH_i$ is a free right H_i-module and Πe_j is a free right H_j-module and thus $\Pi e_j \otimes_j {}_jH_i$ is also a free right H_i-module. Altogether, $\Pi e_j \otimes_j {}_jH_i \otimes_i e_i\Pi$ is a projective right Π-module. A similar argument shows that $\Pi e_i \otimes_{H_i} e_i\Pi$ is projective as a right Π-module. Thus as a sequence of right modules the sequence $P_\bullet$ splits. This implies that the sequence $P_\bullet \otimes_\Pi M$ is exact. Now, if $M \in \mathrm{Rep}_{\mathrm{l.f.}}(\Pi)$, then the relevant components of $P_\bullet \otimes_\Pi M$ are evidently projective. □

Let us write the complex $P_\bullet \otimes_\Pi M$ explicitly:

$$\begin{aligned}\bigoplus_{i\in I} \Pi e_i \otimes_i e_i M \xrightarrow{f_M} \bigoplus_{(j,i)\in\overline{\Omega}} \Pi e_j \otimes_j {}_jH_i \otimes_i e_i M \\ \xrightarrow{g_M} \bigoplus_{i\in I} \Pi e_i \otimes_i e_i M \xrightarrow{h_M} M \to 0 \end{aligned} \qquad (12.2)$$

and the maps f_M, g_M, h_M act on generators as follows:

$$\begin{aligned} f_M(e_j \otimes m_j) = \sum_{\substack{i\in\Omega(-,j)\\ l\in_i L_j}} \big(l^* \otimes l \otimes m_j + e_j \otimes l^* \otimes M_{ij}(l \otimes m_j)\big) \\ - \sum_{\substack{k\in\Omega(j,-)\\ r\in_j R_k}} \big(r \otimes r^* \otimes m_j + e_j \otimes r \otimes M_{kj}(r^* \otimes m_j)\big), \end{aligned}$$

$$\begin{aligned} g_M(e_i \otimes h \otimes m_j) &= h \otimes m_j - e_i \otimes M_{ij}(h \otimes m_j), \\ h_M(e_i \otimes m_i) &= m_i. \end{aligned}$$

Proposition 12.3 *Suppose that the Cartan matrix C has no components of Dynkin type. Then in the complex $P_\bullet$ for Π we have* $\mathrm{Ker}(f) = 0$. *In particular, for all $M \in$* $\mathrm{Rep}_{\mathrm{l.f.}}(\Pi)$ *we have* $\mathrm{proj.\,dim}(M) \leq 2$.

Proof In a first step we show that $\mathrm{Ker}(E_i' \otimes_\Pi f) = 0$ for the generalized simple right Π-modules E_i' concentrated at the vertex $i \in I$. To this end we adapt the relevant part of the proof of [15, Proposition 4.2] to our setting: Choose an orientation Ω such that in $H = H(C, D, \Omega)$ the projective H-module $P_i = He_i$ is the generalized simple H-module E_i. We have a short exact sequence of H-modules

$$0 \to P_i \to \oplus_{j \in \Omega(i,-)} P_j \otimes_j {}_jH_i \to \tau^-(P_i) \to 0, \tag{12.3}$$

see the proof of Theorem 9.7. Note, that this sequence is possibly not an Auslander–Reiten sequence. Applying $\mathrm{Hom}_H(-, \Pi)$ to (12.3) we obtain the sequence of right Π-modules

$$\begin{aligned} 0 \to \mathrm{Hom}_H(\tau^-(P_i), \Pi) &\to \oplus_{j \in \Omega(i,-)i} H_j \otimes_j \Pi \\ &\to e_i\Pi \to \mathrm{Ext}^1_H(\tau^-(P_i), \Pi) \to 0 \end{aligned} \tag{12.4}$$

Now, by Theorem 11.12

$${}_H\Pi = \oplus_{j \in I} \oplus_{k \in \mathbb{N}} \tau^{-k}(P_j),$$

with all summands indecomposable, locally free preprojective modules. In particular, in our situation all summands have injective dimension 1. Thus, $\mathrm{Ext}^1_H(\tau^-(P_i), \tau^{-k}(P_j)) = \mathrm{D}\,\mathrm{Hom}_H(\tau^{-k}(P_j), P_i) = 0$ unless $(j, k) = (i, 0)$. We conclude that we have an isomorphism of right Π-modules $\mathrm{Ext}^1_H(\tau^-(P_i), \Pi) \cong E_i'$. Next

$$\mathrm{Hom}_H(\tau^-(P_i), \Pi) \cong \mathrm{Hom}_H(P_i, \tau({}_H\Pi)) = e_i\Pi$$

where the last equality holds since by our hypothesis on C we have $\tau({}_H\Pi) = {}_H\Pi$. Alltogether, we can identify now the exact sequence (12.4) with

$$0 \to e_i\Pi \xrightarrow{E_i' \otimes f} \oplus_{j \in \Omega(i,-)i} H_j \otimes_j \Pi \xrightarrow{E_i' \otimes g} e_i\Pi \to E_i' \to 0$$

Now, let $U := \mathrm{Ker}(f)$. Then U is projective as a left Π-module since $P_\bullet \xrightarrow{h} \Pi \to 0$ is a (split) exact sequence of projective left modules. Next we observe that $E_i' \otimes_\Pi U = 0$ since $\operatorname{Cok}(f)$ is projective as a left module. Now, let $\mathfrak{m} \subset \Pi$ be the ideal which is generated by $\overline{B}$. Thus $\Pi/\mathfrak{m}^j \in \operatorname{rep}_{\text{l.f.}}(\Pi)$ is filtered by the generalized simples E_i' for all $j \geq 1$ and $\Pi \subset \varprojlim_j(\Pi/\mathfrak{m}^j)$. Thus, since U is projective as a left module we get

$$U = \Pi \otimes_\Pi U \subset (\varprojlim_j(\Pi/\mathfrak{m}^j)) \otimes_\Pi U \subset \varprojlim_j(\Pi/\mathfrak{m}^j \otimes_\Pi U) = 0.$$

This finishes the proof. □

12.2 Symmetry of extension groups

Let $\Pi = \Pi(C, D)$, and let $M = (M_i, M_{ij}, M_{ji})$ and $N = (N_i, N_{ij}, N_{ji})$ be in $\operatorname{Rep}_{\text{l.f.}}(\Pi)$. Let $Q_\bullet(M, N)$ be *the complex*

$$\bigoplus_{k\in I} \operatorname{Hom}_{H_k}(M_k, N_k) \xleftarrow{\tilde{f}_{M,N}} \bigoplus_{(i,j)\in\overline{\Omega}} \operatorname{Hom}_{H_i}({}_iH_j \otimes_j M_j, N_i) \xleftarrow{\tilde{g}_{M,N}} \bigoplus_{k\in I} \operatorname{Hom}_{H_k}(M_k, N_k) \tag{12.5}$$

where $\tilde{f}_{M,N}$ is defined by

$$\left(\tilde{f}_{M,N}((\psi_{ij})_{(i,j)\in\overline{\Omega}})\right)_k := \sum_{j\in\overline{\Omega}(-,k)} \operatorname{sgn}(j,k)(N_{kj} \circ \operatorname{ad}_{j,k}(\psi_{j,k}) - \psi_{k,j} \circ \operatorname{ad}_{j,k}(M_{jk}))$$

and $\tilde{g}_{M,N}$ is defined by

$$(\tilde{g}_{M,N}((\phi_k)_{k\in I}))_{(i,j)} := N_{ij} \circ (\operatorname{id}_{{}_iH_j} \otimes \phi_j) - \phi_i \circ M_{ij}.$$

If N is of finite rank, via the trace pairing from Sect. 8.1, we can identify the K-dual of the shifted complex $Q_\bullet(N, M)[2]$ with the following complex $Q_\bullet(N, M)^*$:

$$\bigoplus_{k\in I} \operatorname{Hom}_{H_k}(M_k, N_k) \xleftarrow{\tilde{g}^*_{N,M}} \bigoplus_{(i,j)\in\overline{\Omega}} \operatorname{Hom}_{H_i}(M_i, {}_iH_j \otimes_j N_j) \xleftarrow{\tilde{f}^*_{N,M}} \bigoplus_{k\in I} \operatorname{Hom}_{H_k}(M_k, N_k) \tag{12.6}$$

(Recall that we have for example natural isomorphisms

$$\operatorname{Hom}_{H_k}(M_k, N_k) \cong \mathrm{D}\operatorname{Hom}_{H_k}(N_k, M_k)$$

since N_k is free of finite rank by hypothesis.) We define moreover

$$\begin{aligned}\operatorname{ad}_{M,N} : & \bigoplus_{(i,j)\in\overline{\Omega}} \operatorname{Hom}_{H_j}({}_jH_i \otimes_i M_i, N_j) \\ & \xrightarrow{\oplus_{(i,j)\in\overline{\Omega}} \operatorname{sgn}(i,j)\operatorname{ad}_{ji}} \bigoplus_{(i,j)\in\overline{\Omega}} \operatorname{Hom}_{H_i}(M_i, {}_iH_j \otimes_j N_j).\end{aligned}$$

We know that $\operatorname{ad}_{M,N}$ is an isomorphism.

Lemma 12.4 *For* $M, N \in \operatorname{Rep}_{\mathrm{l.f.}}(\Pi)$ *the following hold:*

(a) *The complex* $\operatorname{Hom}_\Pi(P_\bullet \otimes_\Pi M, N)$ *is isomorphic to* $Q_\bullet(M, N)$.
(b) *If N is of finite rank*

$$(\mathrm{id}_{\oplus \operatorname{Hom}_{H_k}(M_k,N_k)}, \operatorname{ad}_{M,N}, \mathrm{id}_{\oplus \operatorname{Hom}_{H_k}(M_k,N_k)})$$

is an isomorphism between the complexes $Q_\bullet(M, N)$ *and* $Q_\bullet(N, M)^*$.

Proof Part (a) is straightforward. For (b) we show that $\tilde{f}^*_{N,M} = \operatorname{ad}_{M,N} \circ \tilde{g}_{M,N}$. To this end we evaluate for $(i, j) \in \overline{\Omega}$ the (i, j)-component of $\tilde{f}^*_{N,M}$ on $(\lambda_k)_{k\in I} \in \oplus_{k\in I} \operatorname{Hom}_{H_k}(M_k, N_k)$, according to our discussion of the trace pairing in Sects. 8.1 and 8.2:

$$\begin{aligned}(\tilde{f}^*_{N,M}((\lambda_k)_k))_{ij} &= \operatorname{sgn}(i,j)(\operatorname{ad}^*_{i,j}(\lambda_j \circ M_{ji}) - \operatorname{ad}_{j,i}(N_{ji}) \circ \lambda_i) \\ &= \operatorname{sgn}(i,j)(\operatorname{ad}_{j,i}(\lambda_j \circ M_{ji}) - \operatorname{ad}_{j,i}(N_{ji}) \circ \lambda_i) \\ &= \operatorname{sgn}(i,j)(\operatorname{ad}_{j,i}(\lambda_j \circ M_{ji} - N_{ji} \circ (\mathrm{id}_{H_i} \otimes \lambda_i)))\end{aligned}$$

where the second equality follows from Proposition 8.3, and the third equality is just the definition of ad_{ji}. The proof of $\tilde{g}^*_{N,M} \circ \operatorname{ad}_{M,N} = \tilde{f}_{M,N}$ is similar. □

Proposition 12.5 *For* $M, N \in \operatorname{Rep}_{\mathrm{l.f.}}(\Pi)$ *we have the following functorial isomorphisms:*

(a) $\operatorname{Ker}(\tilde{g}_{M,N}) = \operatorname{Hom}_\Pi(M, N)$ *and* $\operatorname{Ker}(\tilde{f}_{M,N}) / \operatorname{Im}(\tilde{g}_{M,N}) \cong \operatorname{Ext}^1_\Pi(M, N)$.
(b) $\operatorname{Cok} \tilde{f}_{M,N} = \operatorname{Ext}^2_\Pi(M, N)$ *if C has no component of Dynkin type.*
(c) $\operatorname{Hom}_\Pi(N, M) \cong \mathrm{D}\operatorname{Cok}(\tilde{f}_{M,N})$ *if M is of finite rank.*

Proof Part (a) follows from the functorial isomorphism of (12.5) with $\operatorname{Hom}_\Pi(P_\bullet \otimes_\Pi M, N)$. Part (b) follows by the same token since in this situation by Proposition 12.3 the map f_M in (12.2) is injective.

For (c) we just note that by (a) and Lemma 12.4 (b) we have $\mathrm{Ker}(\tilde{f}^*_{M,N}) = \mathrm{Ker}(\tilde{g}_{N,M}) = \mathrm{Hom}_\Pi(N, M)$. ☐

Theorem 12.6 *Let* $M \in \mathrm{Rep}_{\mathrm{l.f.}}(\Pi)$ *and* $N \in \mathrm{rep}_{\mathrm{l.f.}}(\Pi)$.

(a) *There is a functorial isomorphism*

$$\mathrm{Ext}^1_\Pi(M, N) \cong \mathrm{D}\,\mathrm{Ext}^1_\Pi(N, M).$$

(b) *If the Cartan matrix* C *has no component of Dynkin type, we have more generally functorial isomorphisms*

$$\mathrm{Ext}^{2-i}_\Pi(M, N) \cong \mathrm{D}\,\mathrm{Ext}^i_\Pi(N, M) \textit{ for } i = 0, 1, 2.$$

(c) *If* M *is also of finite rank we have*

$$\dim \mathrm{Ext}^1_\Pi(M, N) = \dim \mathrm{Hom}_\Pi(M, N) + \dim \mathrm{Hom}_\Pi(N, M) - (M, N)_H.$$

Proof Recall, that by Proposition 12.5 (a) and (b) we have naturally $H^i(Q_\bullet(M, N)) \cong \mathrm{Ext}^i_\Pi(M, N)$ for $i = 0, 1$ and also for $i = 2$ in case C has no Dynkin component. In our situation (N of finite rank), the complex $Q_\bullet(N, M)^*$ is, via the trace pairing, identified to the K-dual shifted complex $\mathrm{D}\,Q_\bullet(N, M)[2]$. Thus, by the same token we have $H^{2-i}(Q_\bullet(N, M)^*) \cong \mathrm{D}\,\mathrm{Ext}^i_\Pi(N, M)$ for $i = 0, 1$, and also for $i = 2$ in the Dynkin-free case. Now, by Lemma 12.4 (b) the complexes $Q_\bullet(M, N)$ and $Q_\bullet(N, M)^*$ are naturally isomorphic, which implies (a) and (b). For (c), we observe that by Proposition 12.5 we obtain from the complex (12.5) the equality

$$\begin{aligned}
&\dim \mathrm{Hom}_\Pi(N, M) - \dim \mathrm{Ext}^1_\Pi(M, N) + \dim \mathrm{Hom}_\Pi(M, N)\\
&\quad = 2 \cdot \sum_{k\in I} \dim \mathrm{Hom}_{H_k}(M_k, N_k) - \sum_{(i,j)\in\overline{\Omega}} \dim \mathrm{Hom}_{H_j}({}_jH_i \otimes_i M_i, N_j)\\
&\quad = (M, N)_H,
\end{aligned}$$

which is equivalent to our claim. ☐

The last statement in Theorem 12.6 generalizes Crawley-Boevey's formula in [23, Lemma 1].

Corollary 12.7 *Suppose that the Cartan matrix* C *is connected of Dynkin type. Then* Π *is a selfinjective algebra. Moreover, in this situation we have* $\mathrm{Ker}(f) = \mathrm{Hom}_\Pi(\mathrm{D}(\Pi), \Pi)$ *as a* Π-Π-*bimodule, where* f *is the last morphism in the complex* $P_\bullet$ *of Proposition* 12.1.

Proof By Corollary 11.13, Π is in this situation a finite-dimensional basic K-algebra. So, for the first claim we have to show only that Π, as a left module, is injective. In any case, we can find a short exact sequence of left Π-modules

$$0 \to \Pi \xrightarrow{\iota} Q \xrightarrow{\pi} R \to 0$$

with Q injective. Since Π and Q are locally free, R is also locally free. Now, $\mathrm{Hom}_\Pi(\iota, \Pi)$ is surjective since $\mathrm{Ext}^1_\Pi(R, \Pi) = \mathrm{D}\,\mathrm{Ext}^1_\Pi(\Pi, R) = 0$ by Theorem 12.6. Thus, there exists $\rho \in \mathrm{Hom}_\Pi(Q, \Pi)$ with $\rho\iota = \mathrm{id}_\Pi$. In other words, Π is a direct summand of the injective module Q.

For the second claim we note that we have here natural identifications $f = \mathrm{D}(\tilde{f}_{\Pi,\mathrm{D}(\Pi)}) = \tilde{g}_{\mathrm{D}(\Pi),\Pi}$. Now, $\mathrm{Ker}(\tilde{g}_{\mathrm{D}(\Pi),\Pi}) = \mathrm{Hom}_\Pi(\mathrm{D}(\Pi), \Pi)$ by Proposition 12.5 (a). □

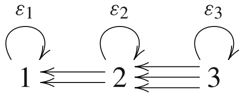
Remark 12.8 For (classical) preprojective algebras Π associated to a Dynkin quiver it seems to be folklore that $\mathrm{Ker}(f) \cong \mathrm{D}(\Pi)$. This is not in contradiction with the above statement. In fact, the Nakayama automorphism, viewed as an element of the group of outer automorphism has order 2 (except for a few cases over fields of characteristic 2 when it is the identity), see [15, Theorem 4.8]. and thus $\mathrm{D}(\Pi) \cong \mathrm{Hom}_\Pi(\mathrm{D}(\Pi), \Pi)$ as a bimodule. We expect that for our generalized preprojective algebras a similar statement holds.

13 Examples

13.1. The algebras $H(C, D)$ and $\Pi(C, D)$

$$C = \begin{pmatrix} 2 & -4 & 0 \\ -6 & 2 & -3 \\ 0 & -9 & 2 \end{pmatrix}$$

is a Cartan matrix, and $D = \mathrm{diag}(9, 6, 2)$ is the minimal symmetrizer of C. Let $\Omega = \{(1, 2), (2, 3)\}$. This is an orientation of C. We have $f_{12} = 2$, $f_{21} = 3$, $f_{23} = 1$, $f_{32} = 3$, $g_{12} = 2$ and $g_{23} = 3$. The algebra $H = H(C, D, \Omega)$ is given by the quiver



with relations

$$\varepsilon_1^9 = 0, \varepsilon_2^6 = 0, \varepsilon_3^2 = 0,$$
$$\varepsilon_1^3 \alpha_{12}^{(g)} = \alpha_{12}^{(g)} \varepsilon_2^2, \quad (g = 1, 2),$$

$$\varepsilon_2^3\alpha_{23}^{(g)} = \alpha_{23}^{(g)}\varepsilon_3, \quad (g = 1, 2, 3).$$

(Recall that $\alpha_{ij}^{(g)}$ denotes an arrow $j \to i$.)

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the double quiver $\overline{Q}(C)$ with relations

$$\begin{aligned}
&\varepsilon_1^9 = 0, \quad \varepsilon_2^6 = 0, \quad \varepsilon_3^2 = 0,\\
&\varepsilon_1^3\alpha_{12}^{(g)} = \alpha_{12}^{(g)}\varepsilon_2^2, \quad \varepsilon_2^2\alpha_{21}^{(g)} = \alpha_{21}^{(g)}\varepsilon_1^3, \quad (g = 1, 2),\\
&\varepsilon_2^3\alpha_{23}^{(g)} = \alpha_{23}^{(g)}\varepsilon_3, \quad \varepsilon_3\alpha_{32}^{(g)} = \alpha_{32}^{(g)}\varepsilon_2^3, \quad (g = 1, 2, 3),\\
&\sum_{g=1}^{2}\left(\alpha_{12}^{(g)}\alpha_{21}^{(g)}\varepsilon_1^2 + \varepsilon_1\alpha_{12}^{(g)}\alpha_{21}^{(g)}\varepsilon_1 + \varepsilon_1^2\alpha_{12}^{(g)}\alpha_{21}^{(g)}\right) = 0,\\
&\sum_{g=1}^{2}\left(-\alpha_{21}^{(g)}\alpha_{12}^{(g)}\varepsilon_2 - \varepsilon_2\alpha_{21}^{(g)}\alpha_{12}^{(g)}\right)\\
&\quad + \sum_{g=1}^{3}\left(\alpha_{23}^{(g)}\alpha_{32}^{(g)}\varepsilon_2^2 + \varepsilon_2\alpha_{23}^{(g)}\alpha_{32}^{(g)}\varepsilon_2 + \varepsilon_2^2\alpha_{23}^{(g)}\alpha_{32}^{(g)}\right) = 0,\\
&\sum_{g=1}^{3} -\alpha_{32}^{(g)}\alpha_{23}^{(g)} = 0.
\end{aligned}$$

13.2 Cartan matrices of Dynkin type

Figure 4 shows a list of valued graphs called *Dynkin graphs*. By definition each of the graphs A_n, B_n, C_n and D_n has n vertices. The graphs A_n, D_n, E_6, E_7 and E_8 are the *simply laced Dynkin graphs*. A Cartan matrix C is of *Dynkin type* if the valued graph $\Gamma(C)$ is isomorphic (as a valued graph) to a disjoint union of Dynkin graphs.

13.3 Finite representation type

Let $H = H(C, D, \Omega)$ with $D = \mathrm{diag}(c_1, \dots, c_n)$. Without loss of generality assume that C is connected. We only sketch the proof of the following proposition.

Proposition 13.1 *The algebra H is representation-finite if and only if we are in one of the following cases*:

(i) *C is of Dynkin type A_n, C_n, D_n, E_6, E_7, E_8, B_2, B_3 or G_2, and D is minimal;*

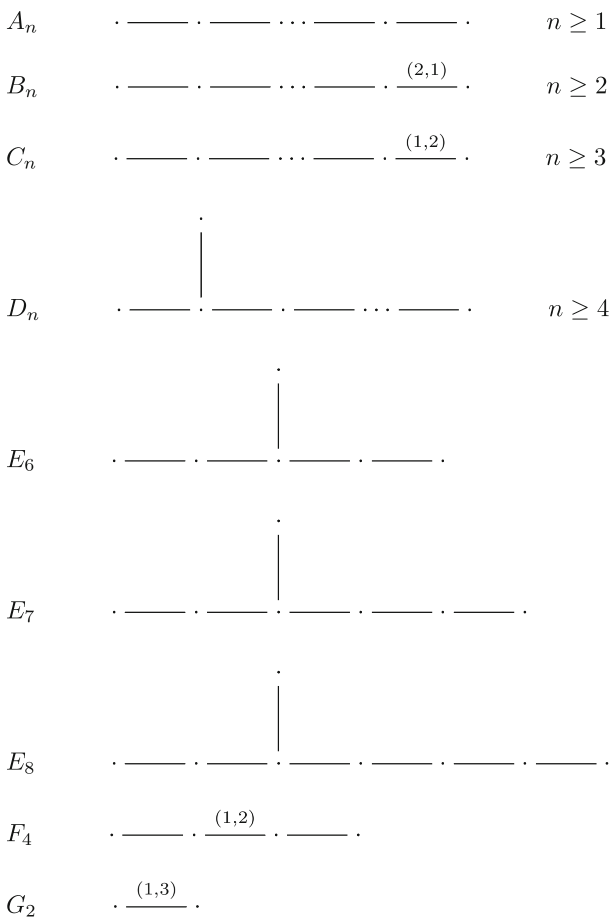


Fig. 4 Dynkin graphs

(ii) *C is of Dynkin type A_1;*
(iii) *C is of Dynkin type A_2, and we have $(c_1, c_2) = (2, 2)$ or $(c_1, c_2) = (3, 3)$;*
(iv) *C is of Dynkin type A_3, and we have $(c_1, c_2, c_3) = (2, 2, 2)$.*

Proof Assume that D is minimal. For Dynkin types A_n, D_n, E_6, E_7, E_8, the algebra H is representation-finite by Gabriel's Theorem. For type C_n, the algebra H is a representation-finite string algebra. The Auslander–Reiten quiver of H for types B_2, B_3 and G_2 can be computed by covering theory and the knitting algorithm for preprojective components. They all turn out to be finite.

If C is of type A_1, then the symmetrizers are $D = (m)$ with $m \geq 1$. Then $H \cong K[\varepsilon_1]/(\varepsilon_1^m)$ is just a truncated polynomial ring, which is obviously representation-finite.

If C is of type A_2 and $(c_1, c_2) = (2, 2)$ or $(c_1, c_2) = (3, 3)$, then H is a representation-finite algebra, see Bongartz and Gabriel's list *Maximal algebras with* 2 *simples modules* in [13, Section 7].

If C is of type A_3 with $(c_1, c_2, c_3) = (2, 2, 2)$, then one can again use covering theory and the knitting algorithm to check that H is representation-finite.

It is straightforward to check that these are all representation-finite cases. (One first compiles the list of all minimal algebras H, which are not mentioned in (i), (ii), (iii) and (iv). These are the algebras $H = H(C, D, \Omega)$ of types

- A_2 with $D = \operatorname{diag}(4, 4)$;
- A_3 with $D = \operatorname{diag}(3, 3, 3)$;
- A_4 with $D = \operatorname{diag}(2, 2, 2, 2)$;
- B_2 with $D = \operatorname{diag}(4, 2)$;
- B_4 with D minimal;
- D_4 with $D = \operatorname{diag}(2, 2, 2, 2)$;
- F_4 with D minimal.

Then one uses covering theory and the Happel–Vossieck list (see [37]) to check that these minimal algebras are representation infinite.) □

13.4 Notation

In the following sections we discuss several examples. We also display the Auslander–Reiten quivers of some representation-finite algebras H. The τ-locally free H-modules are marked with a double frame, the locally free H-modules, which are not τ-locally free, are marked with a single solid frame, and the Gorenstein-projective H-modules, which are not projective, have a dashed frame.

13.5 Dynkin type A_2

Let

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \operatorname{diag}(2, 2)$ and $\Omega = \{(1, 2)\}$. Thus C is a Cartan matrix of Dynkin type A_2 with a non-minimal symmetrizer. We have $f_{12} = f_{21} = 1$. Thus $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"] \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

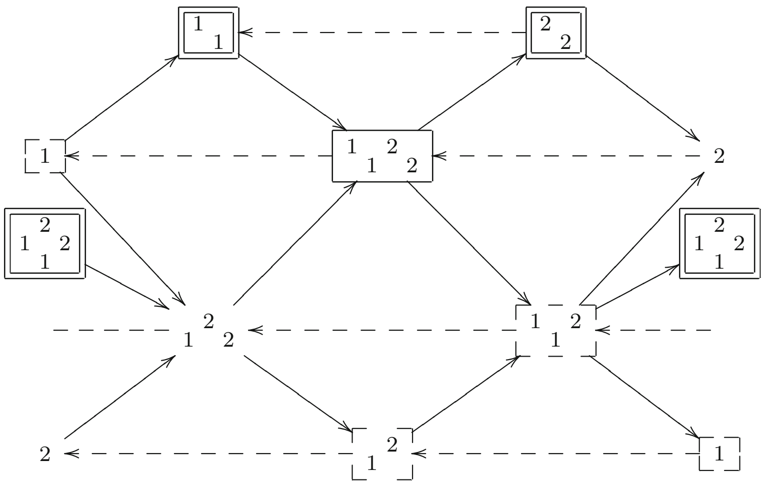
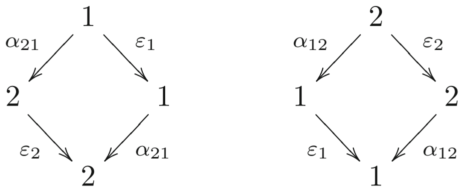


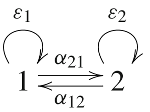
Fig. 5 The Auslander–Reiten quiver of $H(C, D, \Omega)$ of type A_2 with $D = \mathrm{diag}(2, 2)$

Fig. 6 The indecomposable projective $\Pi(C, D)$-modules for type A_2 with $D = (2, 2)$



with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$ and $\varepsilon_1\alpha_{12} = \alpha_{12}\varepsilon_2$. The Auslander–Reiten quiver of H is displayed in Fig. 5. The numbers in the figure correspond to composition factors and basis vectors. (The three modules in the left most column have to be identified with the three modules in the right most column.) Note that $P_2 \cong I_1$ is projective-injective.

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver



with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$, $\varepsilon_1\alpha_{12} = \alpha_{12}\varepsilon_2$, $\varepsilon_2\alpha_{21} = \alpha_{21}\varepsilon_1$, $\alpha_{12}\alpha_{21} = 0$ and $-\alpha_{21}\alpha_{12} = 0$. The indecomposable projective Π-modules are shown in Fig. 6. (The arrows indicate when an arrow of the algebra Π acts with a non-zero scalar on a basis vector.)

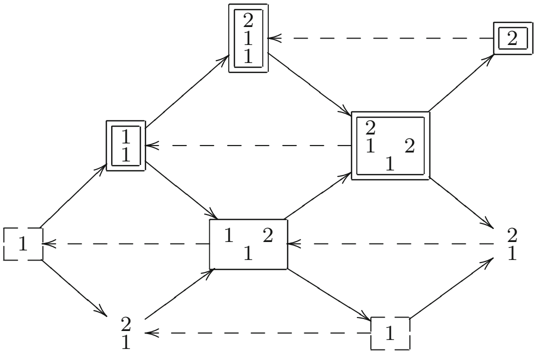


Fig. 7 The Auslander–Reiten quiver of $H(C, D, \Omega)$ of type B_2 with D minimal

13.6 Dynkin type B_2

Let

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

with symmetrizer $D = \mathrm{diag}(2, 1)$ and $\Omega = \{(1, 2)\}$. The graph $\Gamma(C)$ looks as follows:

$$\begin{tikzcd} 1 \arrow[r, dash, "{(2,1)}"] & 2 \end{tikzcd}$$

Thus C is a Cartan matrix of Dynkin type B_2. We have $f_{12} = 1$ and $f_{21} = 2$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"] \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = 0$ and $\varepsilon_2 = 0$. The Auslander–Reiten quiver of H is shown in Fig. 7. The numbers in the figure correspond to composition factors and basis vectors. (In the last two rows the two modules on the left have to be identified with the corresponding two modules on the right.)

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] \arrow[r, shift left, "\alpha_{21}"] & 2 \arrow[l, shift left, "\alpha_{12}"] \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = 0$, $\varepsilon_2 = 0$, $\alpha_{12}\alpha_{21}\varepsilon_1 + \varepsilon_1\alpha_{12}\alpha_{21} = 0$ and $-\alpha_{21}\alpha_{12} = 0$. The indecomposable projective Π-modules are shown in Fig. 8. (The arrows

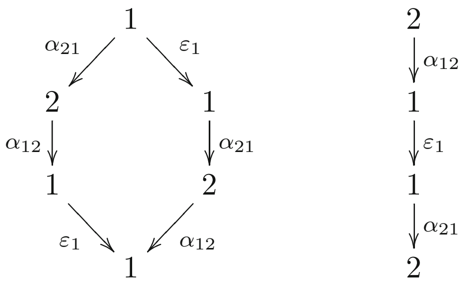


Fig. 8 The indecomposable projective $\Pi(C, D)$-modules for type B_2 with D minimal

indicate when an arrow of the algebra Π acts with a non-zero scalar on a basis vector.)

13.7 Dynkin type B_3

Let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

with symmetrizer $D = \operatorname{diag}(2, 2, 1)$ and $\Omega = \{(1, 2), (2, 3)\}$. The graph $\Gamma(C)$ looks as follows:

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash, "{(2,1)}"] & 3 \end{tikzcd}$$

Thus C is a Cartan matrix of Dynkin type B_3. We have $f_{12} = f_{21} = 1$, $f_{23} = 1$ and $f_{32} = 2$. Thus $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"'] \arrow[loop above, "\varepsilon_2"] & 3 \arrow[l, "\alpha_{23}"'] \arrow[loop above, "\varepsilon_3"] \end{tikzcd}$$

with relations $\varepsilon_1^2 = \varepsilon_2^2 = 0$, $\varepsilon_3 = 0$ and $\varepsilon_1\alpha_{12} = \alpha_{12}\varepsilon_2$. The Auslander–Reiten quiver of H is shown in Fig. 9. As vertices we have the graded dimension vectors (arising from the obvious $\mathbb{Z}$-covering of H) of the indecomposable H-modules. (In the last three rows the three modules on the left have to be identified with the corresponding three modules on the right.) The indecomposable H-module M with graded dimension vector

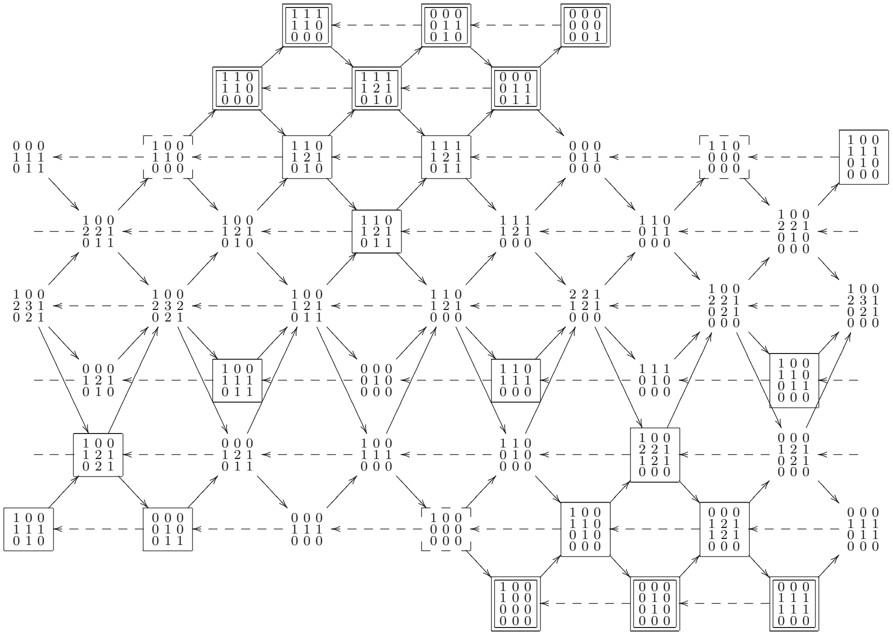


Fig. 9 The Auslander–Reiten quiver of $H(C, D, \Omega)$ of type B_3 with D minimal

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{matrix}$$

is locally free. (It is a direct summand of an extension of locally free modules.) We have $\underline{\mathrm{rank}}(M) = (1, 2, 1)$. In the root lattice of C this corresponds to $\alpha_1 + 2\alpha_2 + \alpha_3$. Thus we have $\underline{\mathrm{rank}}(M) \notin \Delta^+(C)$.

13.8 Dynkin type C_3

Let

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \mathrm{diag}(1, 1, 2)$ and $\Omega = \{(1, 2), (2, 3)\}$. The graph $\Gamma(C)$ looks as follows:

$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash, "{(1,2)}"] & 3 \end{tikzcd}$$

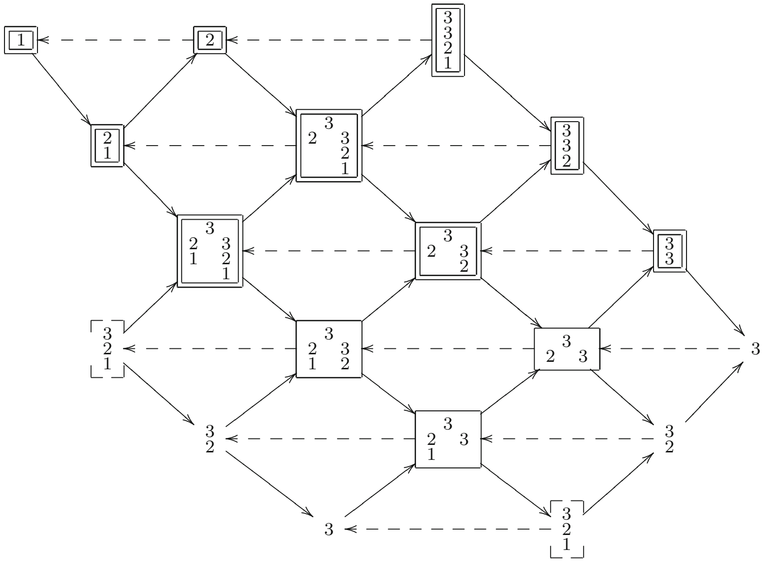


Fig. 10 The Auslander–Reiten quiver of $H(C, D, \Omega)$ of type C_3 with D minimal

Thus C is a Cartan matrix of Dynkin type C_3. We have $f_{12} = f_{21} = 1$, $f_{23} = 2$ and $f_{32} = 1$. Then $H = H(C, D, \Omega)$ is given by the quiver

$$\begin{tikzcd}
1 \arrow[loop above, "\varepsilon_1"] & 2 \arrow[l, "\alpha_{12}"'] \arrow[loop above, "\varepsilon_2"] & 3 \arrow[l, "\alpha_{23}"'] \arrow[loop above, "\varepsilon_3"]
\end{tikzcd}$$

with relations $\varepsilon_1 = \varepsilon_2 = 0$ and $\varepsilon_3^2 = 0$. The Auslander–Reiten quiver of H is shown in Fig. 10. The numbers in the figure correspond to composition factors and basis vectors. (In the last three rows the three modules on the left have to be identified with the corresponding three modules on the right.)

The preprojective algebra $\Pi = \Pi(C, D)$ is given by the quiver

$$\begin{tikzcd}
1 \arrow[loop above, "\varepsilon_1"] \arrow[r, shift left, "\alpha_{21}"] & 2 \arrow[l, shift left, "\alpha_{12}"] \arrow[loop above, "\varepsilon_2"] \arrow[r, shift left, "\alpha_{32}"] & 3 \arrow[l, shift left, "\alpha_{23}"] \arrow[loop above, "\varepsilon_3"]
\end{tikzcd}$$

with relations $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3^2 = 0$, $\alpha_{12}\alpha_{21} = 0$, $-\alpha_{21}\alpha_{12} + \alpha_{23}\alpha_{32} = 0$ and $-\alpha_{32}\alpha_{23}\varepsilon_3 - \varepsilon_3\alpha_{32}\alpha_{23} = 0$. The indecomposable projective Π-modules are shown in Fig. 11.

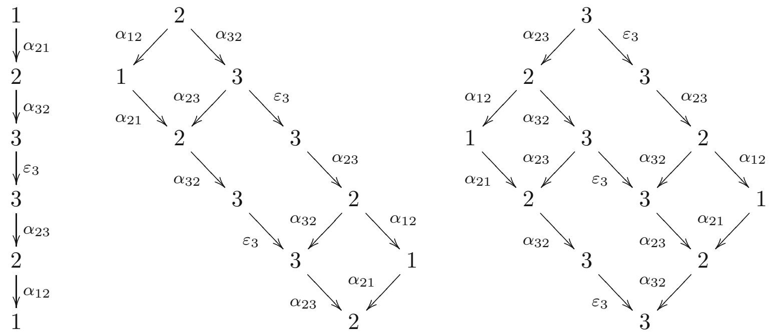


Fig. 11 The indecomposable projective $\Pi(C, D)$-modules for type C_3 with D minimal

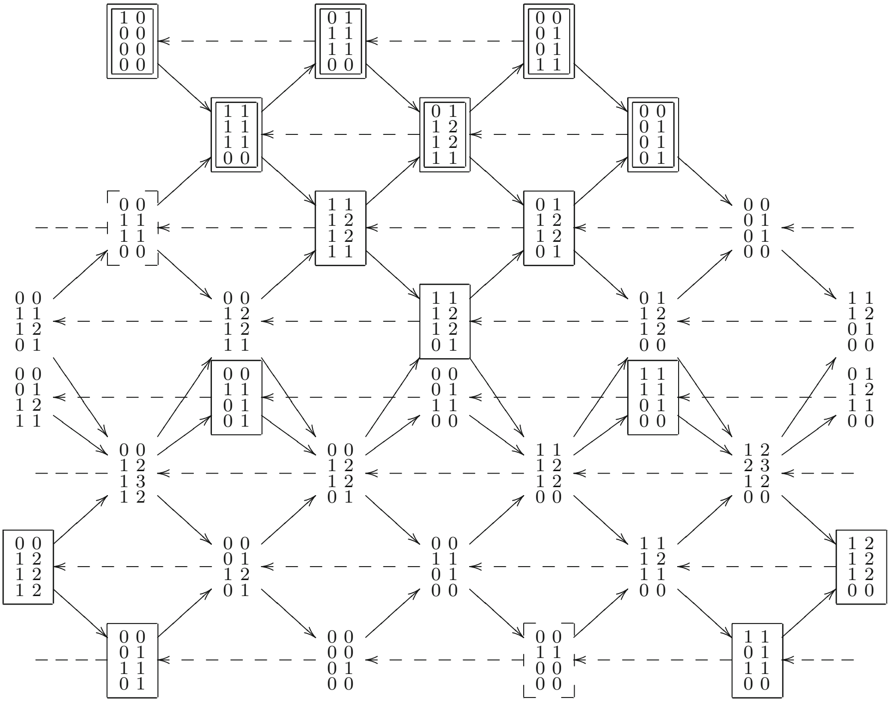


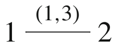
Fig. 12 The Auslander–Reiten quiver of $H(C, D, \Omega)$ of type G_2 with D minimal

13.9 Dynkin type G_2

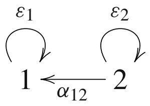
Let

$$C = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

with symmetrizer $D = \mathrm{diag}(1, 3)$ and $\Omega = \{(1, 2)\}$. The graph $\Gamma(C)$ looks as follows:



Thus C is a Cartan matrix of Dynkin type G_2. We have $f_{12} = 3$ and $f_{21} = 1$. Thus $H = H(C, D, \Omega)$ is given by the quiver



with relations $\varepsilon_1 = 0$ and $\varepsilon_2^3 = 0$. The Auslander–Reiten quiver of H is displayed in Fig. 12. As vertices we have the graded dimension vectors (arising from the obvious $\mathbb{Z}$-covering of H) of the indecomposable H-modules. (The three modules in the left most column have to be identified with the three modules in the right most column.)

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