

Degree of L^2 -Alexander torsion for 3-manifolds

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Abstract For an irreducible orientable compact 3-manifold N with empty or incompressible toral boundary, the full L^2 -Alexander torsion $\tau^{(2)}(N, \phi)(t)$ associated to any real first cohomology class ϕ of N is represented by a function of a positive real variable t . The paper shows that $\tau^{(2)}(N, \phi)$ is continuous, everywhere positive, and asymptotically monomial in both ends. Moreover, the degree of $\tau^{(2)}(N, \phi)$ equals the Thurston norm of ϕ . The result confirms a conjecture of J. Dubois, S. Friedl, and W. Lück and addresses a question of W. Li and W. Zhang. Associated to any admissible homomorphism $\gamma: \pi_1(N) \rightarrow G$, the L^2 -Alexander torsion $\tau^{(2)}(N, \gamma, \phi)$ is shown to be continuous and everywhere positive provided that G is residually finite and (N, γ) is weakly acyclic. In this case, a generalized degree can be assigned to $\tau^{(2)}(N, \gamma, \phi)$. Moreover, the generalized degree is bounded by the Thurston norm of ϕ .

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1 Introduction

Let N be an irreducible orientable compact 3-manifold with empty or incompressible toral boundary. Given a homomorphism $\gamma: \pi_1(N) \rightarrow G$ to a countable target group G and a cohomology class $\phi \in H^1(N; \mathbb{R})$, the triple

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$(\pi_1(N), \gamma, \phi)$ is said to be *admissible* if the homomorphism $\pi_1(N) \rightarrow \mathbb{R}$ induced by ϕ factors through γ . Associated to any given admissible triple, the L^2 -Alexander torsion has been introduced by Dubois et al. [7]. It is a function

$$\tau^{(2)}(N, \gamma, \phi) : \mathbb{R}_+ \rightarrow [0, +\infty),$$

uniquely defined up to multiplication by a function of the form $t \mapsto t^r$ where $r \in \mathbb{R}$. In this paper, we use a dotted equal symbol to mean two functions being equal to each other up to such a monic power function factor. When γ is taken to be $\text{id}_{\pi_1(N)} : \pi_1(N) \rightarrow \pi_1(N)$, the corresponding function is called the *full L^2 -Alexander torsion* with respect to ϕ , denoted by $\tau^{(2)}(N, \phi)(t)$. In [7, 9], the following properties about the full L^2 -Alexander torsion are proved:

1. For all $c \in \mathbb{R}$,

$$\tau^{(2)}(N, c\phi)(t) \doteq \tau^{(2)}(N, \phi)(t^c).$$

- 2.

$$\tau^{(2)}(N, -\phi)(t) \doteq \tau^{(2)}(N, \phi)(t).$$

3. For any fibered class $\phi \in H^1(N; \mathbb{Z})$,

$$\tau^{(2)}(N, \phi)(t) \doteq \begin{cases} 1 & t \in (0, e^{-h(\phi)}) \\ t^{x_N(\phi)} & t \in (e^{h(\phi)}, +\infty) \end{cases}$$

where $h(\phi)$ denotes the entropy of the monodromy, and $x_N(\phi)$ denotes the Thurston norm.

4. Denoting by $\text{Vol}(N)$ the simplicial volume of N ,

$$\tau^{(2)}(N, \phi)(1) = e^{\frac{\text{Vol}(N)}{6\pi}}.$$

5. If $\text{Vol}(N)$ equals 0,

$$\tau^{(2)}(N, \phi)(t) \doteq \begin{cases} 1 & t \in (0, 1] \\ t^{x_N(\phi)} & t \in [1, +\infty) \end{cases}$$

For knot complements, the full L^2 -Alexander torsion recovers the L^2 -Alexander invariant introduced earlier by Li and Zhang [18, 19]. If γ is virtually abelian, the L^2 -Alexander torsion is closely related to the twisted Alexander polynomial through a certain function associated to the Mahler measure [7]. We refer the reader to the survey [8] for more relations between the L^2 -Alexander torsion and other flavors of Alexander-type invariants.

It is generally anticipated that the degree of Alexander-type invariants conveys topological information about of the cohomology class ϕ of N . For example, the degree of twisted Alexander polynomials can be used to detect the Thurston norm of ϕ due to Friedl and Vidussi [14]. Various comparison results are also known, cf. [6, 12, 15, 25, 30, 31]. For L^2 -Alexander torsion, a fundamental problem is to define the degree in the first place. The following version has been proposed by Dubois et al. [7, Sect. 1.2] (there simply called the degree):

Definition 1.1 Let $f: \mathbb{R}_+ \rightarrow [0, +\infty)$ be a function. Suppose that f is asymptotically monomial in both ends, namely, as $t \rightarrow +\infty$, the following asymptotic formula holds for some constants $C_{+\infty} \in \mathbb{R}_+$ and $d_{+\infty} \in \mathbb{R}$:

$$f \sim C_{+\infty} \cdot t^{d_{+\infty}},$$

and the same property holds with $+\infty$ replaced by $0+$. Here the notation $f \sim g$ means that the ratio between the functions on both sides tends to 1. For such f , the *asymptote degree* of f is defined to be the value:

$$\deg^a(f) = d_{+\infty} - d_{0+} \in \mathbb{R}.$$

The main goal of this paper is to establish the existence of the asymptote degree for the full L^2 -Alexander torsion of 3-manifolds, and confirm in this case that the degree equals the Thurston norm:

Theorem 1.2 *Let N be an irreducible orientable compact 3-manifold with empty or incompressible toral boundary. Given any cohomology class $\phi \in H^1(N; \mathbb{R})$, the following properties hold true for any representative of the full L^2 -Alexander torsion $\tau^{(2)}(N, \phi)$.*

1. *The function $\tau^{(2)}(N, \phi)(t)$ is continuous and everywhere positive, defined for all $t \in \mathbb{R}_+$. In fact, the function $\tau^{(2)}(N, \phi)(t) \cdot \max\{1, t\}^m$ is multiplicatively convex for any sufficiently large positive constant m , where the bound depends on N and ϕ .*
2. *As the parameter t tends to $+\infty$,*

$$\tau^{(2)}(N, \phi)(t) \sim C(N, \phi) \cdot t^{d_{+\infty}}$$

for some constant $d_{+\infty} \in \mathbb{R}$ and some constant

$$C(N, \phi) \in \left[1, e^{\text{Vol}(N)/6\pi}\right].$$

The same asymptotic formula holds true for with $+\infty$ replaced by $0+$.

3. Hence the asymptote degree of $\tau^{(2)}(N, \phi)$ is valid. Furthermore,

$$\text{deg}^a \left(\tau^{(2)}(N, \phi) \right) = x_N(\phi).$$

4. The leading coefficient function

$$\begin{aligned} H^1(N; \mathbb{R}) &\rightarrow \left[1, e^{\text{Vol}(N)/6\pi} \right] \\ \phi &\mapsto C(N, \phi) \end{aligned}$$

is upper semicontinuous.

In particular, Theorem 1.2 confirms Conjecture 1.1(1) of Dubois–Friedl–Lück [8]. In fact, many aspects of Theorem 1.2 have also been conjectured, at least for knot complements, cf. [8, Sect. 5.8]. In particular, the first part of Theorem 1.2 addresses the question (Q2) of Li–Zhang [19].

The full L^2 –Alexander torsion apparently loses information about the fiberedness of cohomology classes in general. In fact, we have already observed that the full L^2 –Alexander torsion of graph manifolds is completely determined by the Thurston norm, [7, Theorem 1.2], [17]. However, we exhibit an example at the end of this paper to indicate that nontrivial leading coefficients could occur (Sect. 9). The example might suggest that the leading coefficient $C(N, \phi)$ retains some information about the cohomology class ϕ which is volume (of the 3-dimensional hyperbolic type) in nature. For a primitive classes $\phi \in H^1(N; \mathbb{Z})$, we hence wonder if $C(N, \phi)$ measures certain volume of the guts if one decomposes N along a maximal collection of mutually non-parallel Thurston-norm-realizing subsurfaces dual to ϕ .

It is possible to prove an analogous comparison theorem for more general L^2 –Alexander torsions. To this end, we introduce another degree under less strict requirements.

Definition 1.3 Let $f: \mathbb{R}_+ \rightarrow [0, +\infty)$ be a function. Suppose that the following supremum and infimum exist in \mathbb{R} :

$$\text{deg}_{+\infty}^b(f) = \inf \left\{ D_{+\infty} \in \mathbb{R} : \lim_{t \rightarrow +\infty} f(t) \cdot t^{-D_{+\infty}} = 0 \right\},$$

and

$$\text{deg}_{0+}^b(f) = \sup \left\{ D_{0+} \in \mathbb{R} : \lim_{t \rightarrow 0+} f(t) \cdot t^{-D_{0+}} = 0 \right\}.$$

For such f , the *growth bound degree* of f is defined to be the value:

$$\text{deg}^b(f) = \text{deg}_{+\infty}^b - \text{deg}_{0+}^b \in \mathbb{R}.$$

By saying that a pair (N, γ) is *weakly acyclic*, we mean that there are no non-vanishing L^2 -Betti numbers for the covering space of N that corresponds to $\text{Ker}(\gamma)$, regarded as an $\text{Im}(\gamma)$ -space, cf. [23, Sect. 6.5].

Theorem 1.4 *Let N be an irreducible orientable compact 3-manifold with empty or incompressible toral boundary, and $\gamma : \pi_1(N) \rightarrow G$ be a homomorphism. Suppose that G is finitely generated and residually finite, and (N, γ) is weakly acyclic. Then the following properties hold true for any representative of the L^2 -Alexander torsion $\tau^{(2)}(N, \gamma, \phi)$ of any admissible triple (N, γ, ϕ) over \mathbb{R} .*

1. *The function $\tau^{(2)}(N, \gamma, \phi)(t)$ is continuous and everywhere positive, defined for all $t \in \mathbb{R}_+$. In fact, the function $\tau^{(2)}(N, \gamma, \phi)(t) \cdot \max\{1, t\}^m$ is multiplicatively convex for any sufficiently large positive constant m , where the bound depends on (N, γ, ϕ) .*
2. *The growth bound degree of $\tau^{(2)}(N, \gamma, \phi)$ is valid. Furthermore,*

$$\text{deg}^b \left(\tau^{(2)}(N, \gamma, \phi) \right) \leq x_N(\phi).$$

3. *The degree function*

$$\begin{aligned} H^1(G; \mathbb{R}) &\rightarrow \mathbb{R} \\ \xi &\mapsto \text{deg}^b \left(\tau^{(2)}(N, \gamma, \phi + \gamma^* \xi) \right) \end{aligned}$$

is Lipschitz continuous.

In a weaker form, Theorem 1.4 generalizes the virtually abelian case which has been done in [7]. For example, if N is a compact orientable surface bundle over the circle and γ is a homomorphism of $\pi_1(N)$ onto a residually finite group G such that $\gamma^* : H^1(G; \mathbb{R}) \rightarrow H^1(N; \mathbb{R})$ is onto, then the assumptions of Theorem 1.4 are satisfied.

Remark 1.5 Completely independently from work of this paper, Friedl and Lück have also proved the equality between the (growth bound) degree of the full L^2 -Alexander torsion and the Thurston norm [13]. In fact, their work implies Theorem 1.4 (2) as well. Moreover, their work relies on a systematic study of twisting L^2 -invariants by Lück [24]. We point out that both [13, 24] keep track of the Euler structures more closely than this paper does, which should be important for potential applications. For example, with a fixed Euler structure, the L^2 -Alexander torsion becomes a genuine function in the pair (ϕ, t) , so it would make sense to study its continuity and other properties.

In the rest of the introduction, we discuss some ingredients involved in the proof of Theorem 1.2. Theorem 1.4 can be proved along the way. After

choosing some CW complex structure of N convenient for calculation as used in [7], we may manipulate $\tau(N, \phi)(t)$ into an alternating product, where the factors are regular Fuglede–Kadison determinants of the L^2 –Alexander twist of square matrices over $\mathbb{Z}\pi_1(N)$. Except the one coming from the boundary homomorphism between dimension 2 and dimension 1, the factors are all very simple and well understood. Therefore, the proof of Theorem 1.2 can be reduced to the study of the regular Fuglede–Kadison determinant for an L^2 –Alexander twist of a single matrix A . Associated to the admissible triple $(\pi_1(N), \text{id}_{\pi_1(N)}, \phi)$, the factor corresponding to A is a non-negative function defined for $t \in \mathbb{R}_+$ of the form

$$V(t) = \det_{\mathcal{N}(G)}^{\chi} (\kappa(\phi, \text{id}_{\pi_1(N)}, t)(A)),$$

where A is a square matrix over $\mathbb{Z}\pi_1(N)$ (cf. Sect. 2 for the notations).

The first ingredient is to show that $V(t)$ is a multiplicatively convex function with bounded exponent. See Sect. 4 for the terminology. In fact, we show in Theorem 5.1 that the asserted property holds true for general admissible triples (π, γ, ϕ) over \mathbb{R} and square matrices A over $\mathbb{C}\pi$, as long as the target group G of γ is residually finite. The exponent bound can be easily perceived, and can be easily proved once the multiplicative convexity is available. When G is finitely generated and virtually abelian, the multiplicative convexity can be verified by computation using Mahler measure of multivariable Laurent polynomials. Therefore, to approach the residually finite case, it is natural to consider a cofinal tower of virtually abelian quotients G , denoted as

$$G \rightarrow \cdots \rightarrow \Gamma_n \rightarrow \cdots \rightarrow \Gamma_2 \rightarrow \Gamma_1,$$

which gives rise to a sequence of L^2 –Alexander twist homomorphisms $\kappa(\gamma_n, \phi, t)$, where $\gamma_n: \pi \rightarrow \Gamma_n$ is the induced homomorphism. For any given $t \in \mathbb{R}_+$, the spectra of the matrices $A_n(t) = \kappa(\gamma_n, \phi, t)(A)$ could become increasingly dense near 0, as n tends to ∞ , so it should not be expected in general that the sequence of functions $\det_{\mathcal{N}(\Gamma_n)}^{\chi}(A_n(t))$ converged pointwise to $V_G(t) = \det_{\mathcal{N}(G)}^{\chi}(A(t))$. By introducing a positive ϵ -perturbation of the positive operator $A_n(t)^*A_n(t)$, namely,

$$H_{n,\epsilon}(t) = A_n(t)^*A_n(t) + \epsilon \cdot \mathbf{1},$$

the issue of small spectrum values can be bypassed. However, one has to be careful because of the fact that the L^2 –Alexander twist does not commute with the operation of taking self-adjoint. For example, $H_{n,\epsilon}(t)$ is in general not a family of L^2 –Alexander twisted operators, so the regular determinant of $H_{n,\epsilon}(t)$ does not need to be multiplicatively convex in the parameter $t \in \mathbb{R}_+$. Instead of arguing that way, for any fixed $T \in \mathbb{R}_+$, we look at the functions

$$W_{n,\epsilon}(s, T) = \det_{\mathcal{N}(\Gamma_n)}^r(\kappa(\gamma_n * \phi, \text{id}_{\Gamma_n}, s)(H_{n,\epsilon}(T)))$$

in a new parameter $s \in \mathbb{R}_+$. As $n \rightarrow \infty$ and then $\epsilon \rightarrow 0+$, we show that $W_{n,\epsilon}(1, T)$ converges to $W_{\infty,0}(1, T)$, while the limit superior of $W_{n,\epsilon}(s, T)$ does not exceed $W_{\infty,0}(s, T)$. Using the fact that $W_{n,\epsilon}(s, T)$ are multiplicatively convex in $s \in \mathbb{R}_+$, it can be implied that $V_G(t)$ is multiplicatively convex as well.

The growth bound degree is applicable to any (nowhere zero) multiplicatively convex function with bounded exponent. It can be equivalently characterized as the width of the range of all possible exponents (or ‘multiplicative slopes’) between pairs of points. As a consequence of Theorem 5.1, we are able to show that the growth bound degree $\text{deg}^b(V)$ depends Lipschitz-continuously on the cohomology class $\phi \in H^1(N; \mathbb{R})$ (Theorem 6.1).

The second ingredient is a criterion to confirm that $V(t)$ is asymptotically monomial as t tends to $+\infty$ or $0+$, or in other words, that $\text{deg}^a(V)$ equals $\text{deg}^b(V)$. To motivate the conditions, consider the sequence of determinant functions

$$V_n(t) = \det_{\mathcal{N}(\Gamma_n)}^r(\kappa(\gamma_n, \phi, t)(A))$$

associated to the cofinal tower of virtually abelian quotients Γ_n above. Using techniques of [22], what one can show is that for every $t \in \mathbb{R}_+$, as $n \rightarrow \infty$, the supremum limit of $V_n(t)$ does not exceed $V(t)$. On the other hand, the functions $V_n(t)$ are all multiplicatively convex and asymptotically monomial in both ends. As $t \rightarrow +\infty$, suppose

$$V_n(t) \sim C_{+\infty,n} \cdot t^{d_{+\infty,n}},$$

and similarly we introduce the notations $C_{0+,n}$ and $d_{0+,n}$ for $t \rightarrow 0+$. As $n \rightarrow \infty$, if the degrees $\text{deg}^b(V_n) = \text{deg}^a(V_n) = d_{+\infty,n} - d_{0+,n}$ converge to the growth bound degree $\text{deg}^b(V)$, and if the coefficients $C_{+\infty,n}$ and $C_{0+,n}$ are uniformly bounded below by some constant $L \in \mathbb{R}_+$, then it can be implied by the geometry of the log–log plots of the functions that $V(t)$ must be asymptotically monomial in both ends as well (Lemma 7.3).

For our proof of Theorem 1.2, the convergence of growth bound degrees can be guaranteed by the virtual RFRS property of 3-manifold groups, at least after excluding the case of graph manifolds, which has been treated by [7, Theorem 1.2], [17]. In fact, combined with the continuity of degree that we have already mentioned, the method of [7, Theorem 9.1] can be applied to produce a cofinal tower of virtually abelian quotients such that the growth bound degree of each $V_n(t)$ and $V(t)$ is equal to the Thurston norm of ϕ . On the other hand, based on the fact that A is a square matrix over $\mathbb{Z}\pi_1(N)$, computation shows that the coefficients $C_{+\infty,n}$ and $C_{0+,n}$ are all radicals of

the Mahler measure of certain multivariable Laurent polynomial over \mathbb{Z} . This yields a uniform lower bound 1 for all the coefficients. Therefore, the criterion is applicable to our situation, and we can complete the proof of Theorem 1.2.

The reader may find a compact package of key technical results, including Theorems 5.1, 6.1, and Lemma 7.4, which are specific about the regular Fuglede–Kadison determinant function for L^2 –Alexander twists of square matrices over group rings. We have endeavored to formulate those results in an encapsulated, adaptable fashion, in order to facilitate further study of L^2 –Alexander invariants in various other situations.

The organization of this paper is as follows. In Sect. 2, we recall some terminology that is used in this paper. In Sect. 3, we introduce regular Fuglede–Kadison determinants and discuss its limiting behavior. In Sect. 4, we introduce multiplicatively convex functions and mention some basic properties. After these preparing sections, we study the regular Fuglede–Kadison determinants of matrices under L^2 –Alexander twists in Sects. 5, 6, and 7: The multiplicative convexity and the existence of the growth bound degree is shown in Sect. 5. The continuity of degree is derived in Sect. 6. The criterion for monomial asymptotics is introduced in Sect. 7. In Sect. 8, we apply the ingredients to L^2 –Alexander torsions of 3-manifolds, and prove Theorems 1.2 and 1.4. In Sect. 9, we give an example regarding nontrivial leading coefficients.

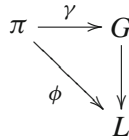
2 Preliminaries

In this section, we recall some terminology of Dubois et al. [7]. We also briefly recall some fundamental facts in 3-manifold topology. For background in L^2 -invariants, including group von Neumann algebras and Fuglede–Kadison determinants, we refer the reader to the book of Lück [23].

2.1 Admissible triples

Admissibility conditions have been introduced by S. Harvey for study of higher-order Alexander polynomials [16, Definition 1.4]. In this paper, we adopt the following notations, according to [7].

Definition 2.1 Let $L \subset \mathbb{R}$ be any additive group of real numbers, for example, \mathbb{Z} , \mathbb{Q} , or \mathbb{R} . Given a countable group π , and a homomorphism $\phi \in \text{Hom}(\pi, L)$, and a homomorphism $\gamma: \pi \rightarrow G$ to any countable group G , we say that (π, ϕ, γ) forms *an admissible triple over L* if ϕ factors through γ . That is, for some homomorphism $G \rightarrow L$, the following diagram commutes:



Given any positive real parameter $t \in \mathbb{R}_+$, there is a homomorphism of rings:

$$\kappa(\phi, \gamma, t): \mathbb{Z}\pi \longrightarrow \mathbb{R}G$$

defined uniquely by

$$\kappa(\phi, \gamma, t)(g) = t^{\phi(g)}\gamma(g)$$

for all $g \in \pi$ via linear extension over \mathbb{Z} . Then for any positive integer p , $\kappa(\phi, \gamma, t)$ naturally extends to be a homomorphism of algebras:

$$\kappa(\phi, \gamma, t): \text{Mat}_{p \times p}(\mathbb{C}\pi) \rightarrow \text{Mat}_{p \times p}(\mathbb{C}G)$$

by applying $\kappa(\phi, \gamma, t)$ to entries accordingly.

Note that $\kappa(\phi, \gamma, t)$ is not a homomorphism of $*$ -algebras in general. In fact,

$$\kappa(\phi, \gamma, t)(A)^* = \kappa(\phi, \gamma, t^{-1})(A^*).$$

Recall that for any square matrix $A = (a_{ij})_{p \times p}$ over $\mathbb{C}G$, as an operator of $\ell^2(G)^{\oplus p}$, the adjoint operator can be given by $A^* = (a_{ji}^*)_{p \times p}$, where the involution of an element $a = \sum_k a_k g_k \in \mathbb{C}G$ is given by $a^* = \sum_k \bar{a}_k g_k^{-1} \in \mathbb{C}G$.

Every admissible triple (π, ϕ, γ) over L sits naturally in an affine family of admissible triples parametrized by $\text{Hom}(G, L)$. Specifically, for any homomorphism

$$\xi \in \text{Hom}(G, L),$$

we have a new admissible triple $(\pi, \phi + \gamma^*\xi, \gamma)$, where $\phi + \gamma^*\xi: \pi \rightarrow L$ is the homomorphism defined by

$$(\phi + \gamma^*\xi)(g) = \phi(g) + \xi(\gamma(g))$$

for all $g \in \pi$. To speak of continuity, we consider the space $\text{Hom}(G, L)$ to be equipped with the compact-open topology, regarding G to be a discrete group and L have the subspace topology of \mathbb{R} .

Lemma 2.2 *If $\gamma: \pi \rightarrow G$ induces an isomorphism $\gamma_*: H_1(\pi; \mathbb{R}) \rightarrow H_1(G; \mathbb{R})$, then (π, γ, ϕ) is admissible for every homomorphism $\phi: \pi \rightarrow \mathbb{R}$.*

Proof In this case, the composition

$$\pi \xrightarrow{\gamma} G \longrightarrow H_1(G; \mathbb{R}) \xrightarrow{\gamma_*^{-1}} H_1(\pi; \mathbb{R}) \xrightarrow{\phi_*} \mathbb{R}$$

recovers the homomorphism ϕ . □

2.2 L^2 -Alexander torsion

Let X be a connected finite CW complex. The universal cover \widehat{X} of X is a CW complex equipped with a free action of the deck transformation group $\pi_1(X)$. We equip the chain complex $C_*(\widehat{X})$ with a left $\mathbb{Z}\pi_1(X)$ action induced by the deck transformation. On the other hand, given any admissible triple $(\pi_1(X), \gamma, \phi)$ over \mathbb{R} , and given a parameter value $t \in \mathbb{R}_+$, we may equip the Hilbert space $\ell^2(G)$ with a right $\mathbb{Z}\pi_1(X)$ -module structure via the representation:

$$\kappa(\phi, \gamma, t): \mathbb{Z}\pi_1(X) \longrightarrow \mathbb{R}G.$$

In this paper, we treat $\ell^2(G)$ as a right $\mathbb{R}G$ -module and a left Hilbert $\mathcal{N}(G)$ -module. Here we denote by

$$\mathcal{N}(G) = \mathcal{B}(\ell^2(G))^G$$

the group von Neumann algebra of G which consists of all the bounded operators that commutes with the right multiplication by elements of G . Twisting the chain complex of \widehat{X} by the module $\ell^2(G)$ via the representation $\kappa(\phi, \gamma, t)$ gives rise to a (left) Hilbert $\mathcal{N}(G)$ -chain complex:

$$\ell^2(G) \otimes_{\mathbb{Z}\pi_1(X)} C_*(\widehat{X})$$

and the twisted boundary homomorphism is defined by $\mathbf{1} \otimes \partial_*$. In fact, the twisted complex is finitely generated and free over $\mathcal{N}(G)$. In other words, by choosing a lift of each cell of X in \widehat{X} , each chain module of the complex can be identified with a direct sum of the regular Hilbert $\mathcal{N}(G)$ -modules:

$$\ell^2(G) \otimes_{\mathbb{Z}\pi_1(X)} C_k(\widehat{X}) \cong \ell^2(G)^{\oplus p_k}.$$

In this paper, we restrict ourselves to finitely generated, free Hilbert $\mathcal{N}(G)$ -chain complexes which are *weakly acyclic and of determinant class*. This

means that the ℓ^2 -Betti numbers are all trivial and all the Fuglede–Kadison determinants of the boundary homomorphisms take values in $(0, +\infty)$. In such case, the L^2 -Alexander torsion of X at t with respect to γ and ϕ is defined to be the multiplicatively alternating product of the Fuglede–Kadison determinants of the boundary homomorphisms:

$$\tau^{(2)}(X, \gamma, \phi)(t) \doteq \prod_{k \in \mathbb{Z}} \det_{\mathcal{N}(G)}(\mathbf{1} \otimes \partial_k)^{(-1)^k}.$$

Here the dotted equal means that we treat the L^2 -Alexander torsion as a function in the parameter $t \in \mathbb{R}_+$. In fact, choosing another collection of lifts may result in a change of the value of the right-hand side by a multiplicative factor t^r , for some exponent $r \in \mathbb{R}$ independent of t , so the function $\tau^{(2)}(X, \gamma, \phi)$ is well defined only up to a monic power function factor. We remark that our notational convention follows [7], and the exponential of the L^2 -torsion according to [23, Definition 3.29] is the multiplicative inverse of the $\tau^{(2)}$ above. To be convenient, a value 0 is artificially assigned to $\tau^{(2)}(X, \gamma, \phi)(t)$ if the twisted complex fails to be weakly acyclic or of determinant class. With this convention, the L^2 -Alexander torsion associated to (X, γ, ϕ) is a function determined up to a monic power function factor:

$$\tau^{(2)}(X, \gamma, \phi): \mathbb{R}_+ \rightarrow [0, +\infty).$$

Let N be a compact smooth manifold, possibly with boundary, and $\gamma: \pi_1(N) \rightarrow G$ be a homomorphism. The L^2 -Alexander torsion of N with respect to any admissible triple $(\pi_1(N), \gamma, \phi)$, denoted as $\tau^{(2)}(N, \gamma, \phi)$, is understood to be the L^2 -Alexander torsion of any finite CW complex structure of N . This notion does not depend on the choice of the CW structure [7, Sect. 4.2]. When γ is taken to be $\text{id}_{\pi_1(N)}: \pi_1(N) \rightarrow \pi_1(N)$, the triple $(\pi_1(N), \gamma, \phi)$ is admissible for every class $\phi \in H^1(N; \mathbb{R})$. The corresponding L^2 -Alexander torsion is called the *full L^2 -Alexander torsion* with respect to ϕ , denoted as $\tau^{(2)}(N, \phi)$.

2.3 Thurston norm and virtual fibering

Let N be an irreducible compact orientable 3-manifold with empty or incompressible toral boundary. The *Thurston norm*, named after Thurston who discovered it in [29], is a seminorm of the vector space:

$$x_N: H_2(N, \partial N; \mathbb{R}) \rightarrow [0, +\infty),$$

which takes \mathbb{Z} values on the integral lattice $H_2(N, \partial N; \mathbb{Z})$. It measures certain complexity of second relative homology classes. The precise definition

of x_N is not needed for the discussion of this paper, but we briefly recall it here for the reader's reference. For a possibly disconnected, compact, oriented surface Σ , the *complexity* $\chi_-(\Sigma)$ is defined to be the sum of the quantities $\chi_-(\Sigma_i) = \max\{-\chi(\Sigma), 0\}$ over all the components Σ_i of Σ . For an integral class $\phi \in H_2(N, \partial N; \mathbb{Z})$, the Thurston norm $x_N(\phi)$ is defined to be the minimum of the complexities $\chi_-(\Sigma)$ for Σ running over all the properly embedded compact oriented subsurfaces of N whose fundamental cycles represent ϕ . In this case, it is known that the properties $x_N(k\phi) = |k| x_N(\phi)$ and $x_N(\phi) + x_N(\psi) \leq x_N(\phi + \psi)$ hold for all $k \in \mathbb{Z}$ and $\phi, \psi \in H_2(N, \partial N; \mathbb{Z})$. Therefore, x_N extends linearly over $H_2(N, \partial N; \mathbb{Q})$ and then continuously over $H_2(N, \partial N; \mathbb{R})$ to become a seminorm. In general, the definition works for any orientable compact 3-manifold with possibly empty boundary, and the Thurston norm x_N may vanish on a subspace of $H_2(N, \partial N; \mathbb{R})$. However, it is known to be non-degenerate if the 3-manifold N supports a complete hyperbolic structure of finite volume in its interior.

The unit ball $B_x(N)$ of x_N in $H_2(N, \partial N; \mathbb{R})$ is a (possibly noncompact) convex polyhedron, symmetric about the origin, and supported by finitely many linear faces carried by rational affine hyperplanes. If N fibers over the circle via a map $N \rightarrow S^1$, any fiber of the fibration represents a homology class $[\Sigma] \in H_2(N, \partial N; \mathbb{Z})$, which depends only on the fibration. We can canonically identify $[\Sigma]$ with a cohomology class $\phi \in H^1(N; \mathbb{Z}) \cong [N, S^1]$, by Poincaré Duality (after fixing an orientation of N). As we have assumed N to be irreducible with incompressible boundary, $x_N(\phi)$ equals $-\chi(\Sigma)$. Any such ϕ is called a *fibered class*. Thurston has shown that every fibered class is contained in the open cone over a top-dimensional face of $\partial B_x(N)$, and every integral class of that cone is a fibered class. Such open cones are hence called the *fibered cones* of x_N .

In general, N may possess no fibered cones at all. However, given any class $\phi \in H^1(N; \mathbb{R})$, we can usually pass to a finite cover $p: \tilde{N} \rightarrow N$, so that $p^*\phi \in H^1(\tilde{N}; \mathbb{R})$ is *quasi-fibered*, namely, $p^*\phi$ lies on the (point-set theoretic) boundary of a fibered cone possessed by \tilde{N} . To be precise, the virtual quasi-fibered property holds true for every class $\phi \in H^1(N; \mathbb{R})$ if $\pi_1(N)$ is virtually residually finite rationally solvable (or RFRS), due to a theorem of Ian Agol [1]. Based on the confirmations of the Virtual Haken Conjecture and the Virtual Fibered Conjecture due to the works of Ian Agol [2], Daniel Wise [32], and many other authors, it has been known that $\pi_1(N)$ is virtually RFRS if and only if N supports a complete Riemannian metric of nonpositive curvature in its interior [20, 26, 27]. For example, if the simplicial volume $\text{Vol}(N)$ is positive, or in other words, if N contains at least one hyperbolic piece in its geometric decomposition, then the virtual quasi-fibered property is possessed by N . We refer the reader to the survey [3] for more background about virtual properties of 3-manifolds.

3 Regular Fuglede–Kadison determinant

Let G be a countable discrete group. For any $p \times p$ matrix A over $\mathcal{N}(G)$, the regular Fuglede–Kadison determinant of A is defined to be

$$\det_{\mathcal{N}(G)}^{\times}(A) = \begin{cases} \det_{\mathcal{N}(G)}(A) & \text{if } A \text{ is full rank of determinant class} \\ 0 & \text{otherwise} \end{cases}$$

This gives rise to a function:

$$\det_{\mathcal{N}(G)}^{\times} : \text{Mat}_{p \times p}(\mathcal{N}(G)) \rightarrow [0, +\infty).$$

Regular Fuglede–Kadison determinants have been used in [7]. In the rest of the section we study the semicontinuity of this quantity under two kinds of limiting processes.

Lemma 3.1 *If a sequence of $p \times p$ matrices $\{A_n\}_{n \in \mathbb{N}}$ over $\mathcal{N}(G)$ converges to $A \in \text{Mat}_{p \times p}(\mathcal{N}(G))$ with respect to the norm topology, then*

$$\limsup_{n \rightarrow \infty} \det_{\mathcal{N}(G)}^{\times}(A_n) \leq \det_{\mathcal{N}(G)}^{\times}(A).$$

Moreover, if A is a positive operator, then

$$\lim_{\epsilon \rightarrow 0^+} \det_{\mathcal{N}(G)}^{\times}(A + \epsilon \cdot \mathbf{1}) = \det_{\mathcal{N}(G)}^{\times}(A).$$

Proof Since $\det_{\mathcal{N}(G)}^{\times}(A^*A)$ equals $\det_{\mathcal{N}(G)}^{\times}(A)^2$, it suffices to show the inequality for positive operators $\{A_n\}_{n \in \mathbb{N}}$ and A . For any arbitrary constant $\epsilon > 0$, the positive operators $(A_n + \epsilon \cdot \mathbf{1})$ and $(A + \epsilon \cdot \mathbf{1})$ are invertible, so the regular Fuglede–Kadison determinant agrees with the Fuglede–Kadison determinant. Since the Fuglede–Kadison determinant is continuous on the subgroup of invertible matrices $\text{GL}(p, \mathcal{N}(G))$ with respect to the norm topology [5, Theorem 1.10(d)],

$$\lim_{n \rightarrow \infty} \det_{\mathcal{N}(G)}^{\times}(A_n + \epsilon \cdot \mathbf{1}) = \det_{\mathcal{N}(G)}^{\times}(A + \epsilon \cdot \mathbf{1}).$$

On the other hand, by [23, Lemma 3.15(6)], or as a trivial fact if A_n fails to be injective,

$$\det_{\mathcal{N}(G)}^{\times}(A_n) \leq \det_{\mathcal{N}(G)}^{\times}(A_n + \epsilon \cdot \mathbf{1}).$$

Therefore,

$$\limsup_{n \rightarrow \infty} \det_{\mathcal{N}(G)}^{\times}(A_n) \leq \limsup_{n \rightarrow \infty} \det_{\mathcal{N}(G)}^{\times}(A_n + \epsilon \cdot \mathbf{1}) = \det_{\mathcal{N}(G)}^{\times}(A + \epsilon \cdot \mathbf{1}).$$

As $\epsilon > 0$ is arbitrary, it suffices to prove

$$\lim_{\epsilon \rightarrow 0^+} \det_{\mathcal{N}(G)}^r(A + \epsilon \cdot \mathbf{1}) = \det_{\mathcal{N}(G)}^r(A).$$

In fact, if A is injective, the last limit follows from [23, Lemma 3.15(5)]. Otherwise, $\det_{\mathcal{N}(G)}^r(A)$ equals 0. Denoting by $b \in (0, p]$ the von Neumann dimension $\dim_{\mathcal{N}(G)} \text{Ker}(A)$, it is easy to estimate

$$0 \leq \det_{\mathcal{N}(G)}^r(A + \epsilon \cdot \mathbf{1}) \leq \epsilon^b (\|A\| + \epsilon)^{p-b}.$$

We again have:

$$\lim_{\epsilon \rightarrow 0^+} \det_{\mathcal{N}(G)}^r(A + \epsilon \cdot \mathbf{1}) = 0 = \det_{\mathcal{N}(G)}^r(A).$$

This completes the proof. □

Lemma 3.2 *Let*

$$G \rightarrow \dots \rightarrow \Gamma_n \rightarrow \dots \rightarrow \Gamma_2 \rightarrow \Gamma_1,$$

be a cofinal tower of quotients of G , and denote by $\psi_n : G \rightarrow \Gamma_n$ the quotient homomorphisms. Suppose that all the target groups Γ_n are finitely generated and residually finite. Let A_G be a square matrix over $\mathbb{C}G$. Then

$$\limsup_{n \rightarrow \infty} \det_{\mathcal{N}(\Gamma_n)}^r(\psi_{n*} A_G) \leq \det_{\mathcal{N}(G)}^r(A_G).$$

Moreover, for any constant $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \det_{\mathcal{N}(\Gamma_n)}^r(\psi_{n*} (A_G^* A_G + \epsilon \cdot \mathbf{1})) = \det_{\mathcal{N}(G)}^r(A_G^* A_G + \epsilon \cdot \mathbf{1}).$$

Here the tower being cofinal means that

$$\bigcap_{n \in \mathbb{N}} \text{Ker} \psi_n = \{ \text{id}_G \}.$$

Proof Assuming that the ‘moreover’ part has been proved, we can derive the first inequality as follows. For any constant $\epsilon > 0$,

$$\begin{aligned} \det_{\mathcal{N}(\Gamma_n)}^r(\psi_{n*} A_G) &= \det_{\mathcal{N}(\Gamma_n)}^r(\psi_{n*} (A_G^* A_G))^{1/2} \\ &\leq \det_{\mathcal{N}(\Gamma_n)}^r(\psi_{n*} (A_G^* A_G + \epsilon \cdot \mathbf{1}))^{1/2}. \end{aligned}$$

The last expression tends to the regular Fuglede–Kadison determinant of A_G as ϵ tends to $0+$, by Lemma 3.1. This implies the asserted inequality

$$\limsup_{n \rightarrow \infty} \det_{\mathcal{N}(\Gamma_n)}^{\mathcal{F}}(\psi_{n*} A_G) \leq \det_{\mathcal{N}(G)}^{\mathcal{F}}(A_G).$$

It remains to prove the asserted limit in the ‘moreover’ part. For simplicity, given any constant $\epsilon > 0$, we rewrite the matrices as

$$H_\infty = A_G^* A_G + \epsilon \cdot \mathbf{1} \in \text{Mat}_{p \times p}(\mathbb{C}G)$$

and

$$H_n = \psi_{n*} H_\infty \in \text{Mat}_{p \times p}(\mathbb{C}\Gamma_n).$$

Note that the self-adjoint operators H_n acting on $\ell^2(\Gamma_n)^{\oplus p}$ are positive with spectra bounded uniformly ϵ from 0 and the same holds for H_∞ . In this case, approximation of determinants should follow from well known techniques. In the rest of the proof, we derive the approximation

$$\lim_{n \rightarrow \infty} \det_{\mathcal{N}(\Gamma_n)}^{\mathcal{F}}(H_n) = \det_{\mathcal{N}(G)}^{\mathcal{F}}(H_\infty).$$

from a theorem of W. Lück [22, Theorem 3.4(3)], which is originally done for cofinal towers of finite quotients.

It is convenient to argue by contradiction, assuming that the limit of the left-hand side did not exist or did not equal to the right-hand side. In either case, possibly after passing to a subsequence, we assume that there exists a constant $\delta > 0$ such that the following gap estimate holds for all $n \in \mathbb{N}$:

$$\left| \det_{\mathcal{N}(\Gamma_n)}^{\mathcal{F}}(H_n) - \det_{\mathcal{N}(G)}^{\mathcal{F}}(H_\infty) \right| \geq 2\delta.$$

By induction, we show that there exists a cofinal tower of finite quotients of G

$$G \rightarrow \cdots \rightarrow \Gamma'_n \rightarrow \cdots \rightarrow \Gamma'_2 \rightarrow \Gamma'_1,$$

with the following properties: For all $n \in \mathbb{N}$, we have that Γ'_n is a further quotient of Γ_n , and moreover,

$$\left| \det_{\mathcal{N}(\Gamma'_n)}^{\mathcal{F}}(H'_n) - \det_{\mathcal{N}(\Gamma_n)}^{\mathcal{F}}(H_n) \right| < \delta,$$

where H'_n is the induced matrix of H_n over $\mathbb{C}\Gamma'_n$. For n equal to 1, take a cofinal tower of finite quotients of Γ_1 :

$$\Gamma_1 \rightarrow \cdots \rightarrow \Gamma_{1,j} \rightarrow \cdots \rightarrow \Gamma_{1,2} \rightarrow \Gamma_{1,1}.$$

Denote the induced matrix of H_n over $\mathbb{C}\Gamma_{1,j}$ by $H_{1,j}$. Since $H_{1,j}$ is positive with spectrum bounded at least ϵ from 0, Lück’s theorem implies

$$\lim_{j \rightarrow \infty} \det_{\mathcal{N}(\Gamma_{1,j})}^{\times} (H_{1,j}) = \det_{\mathcal{N}(\Gamma_1)}^{\times} (H_1),$$

so we choose Γ'_1 to be the quotient $\Gamma_{1,j}$ for a sufficiently large j . Suppose by induction that Γ'_n has been constructed for some $n \in \mathbb{N}$. To construct Γ'_{n+1} , we take a tower of finite quotients

$$\Gamma_{n+1} \rightarrow \cdots \rightarrow \Gamma_{n+1,j} \rightarrow \cdots \rightarrow \Gamma_{n+1,2} \rightarrow \Gamma_{n+1,1}.$$

in the same fashion as above, but also require the first term $\Gamma_{n+1,1}$ to be Γ'_n . The same construction thus yields some sufficiently large j such that $\Gamma_{n+1,j}$ can be chosen as Γ'_{n+1} . This completes the induction.

Provided with the new tower, Lück’s theorem again implies

$$\lim_{n \rightarrow \infty} \det_{\mathcal{N}(\Gamma'_n)}^{\times} (H'_n) = \det_{\mathcal{N}(\Gamma_\infty)}^{\times} (H_\infty).$$

Therefore, for sufficiently large n ,

$$\left| \det_{\mathcal{N}(\Gamma_n)}^{\times} (H_n) - \det_{\mathcal{N}(\Gamma_\infty)}^{\times} (H_\infty) \right| < 2\delta.$$

This contradicts the assumed gap estimation, and hence completes the proof. □

4 Multiplicatively convex function

In this section, we give an introduction to multiplicatively convex functions. In subsequent sections, such functions arise naturally as we take the regular Fuglede–Kadison determinants of matrices under L^2 –Alexander twists.

Definition 4.1 Let $(a, b) \subset \mathbb{R}_+$ be an interval of positive real numbers. A function $f : (a, b) \rightarrow [0, +\infty)$ is said to be *multiplicatively convex* if for all points $t_0, t_1 \in (a, b)$ and every constant $\lambda \in (0, 1)$,

$$f \left(t_0^\lambda \cdot t_1^{1-\lambda} \right) \leq f(t_0)^\lambda \cdot f(t_1)^{1-\lambda}.$$

The product of two multiplicatively convex functions is again multiplicatively convex. Furthermore, if $f(t)$ is multiplicatively convex, then for any constant $r \in \mathbb{R}_+$, both $f(t^{\pm r})$ and $f(t)^r$ are multiplicatively convex as well.

Lemma 4.2 *If a function $f: \mathbb{R}_+ \rightarrow [0, +\infty)$ is multiplicatively convex, then f is continuous. Moreover, f is either the constant function 0 or nowhere zero.*

Proof If f equals zero at some point c , it is clear from the definition that f has to be the constant function 0. When f is nowhere zero, then $\log \circ f \circ \exp$ is a convex function on \mathbb{R} . In either case, f is continuous. \square

Lemma 4.3 *If $f: \mathbb{R}_+ \rightarrow [0, +\infty)$ is multiplicatively mid-point convex and upper semi-continuous, namely,*

- *For every pair of points $t_0, t_1 \in \mathbb{R}_+$, $f(\sqrt{t_0 t_1}) \leq \sqrt{f(t_0) \cdot f(t_1)}$, and*
- *For every point $t_0 \in \mathbb{R}_+$, $\limsup_{t \rightarrow t_0} f(t) \leq f(t_0)$,*

then f is multiplicatively convex.

Proof Given any $t_0 \in \mathbb{R}_+$, let $\{t_n \in \mathbb{R}_+\}_{n \in \mathbb{N}}$ be a sequence of points such that t_n converges to t_0 and $f(t_n)$ converges to $\liminf_{t \rightarrow t_0} f(t)$. We have

$$f(t_0)^2 \leq \limsup_{n \rightarrow \infty} f(t_n) f(t_0^2/t_n) \leq \liminf_{t \rightarrow t_0} f(t) \cdot \limsup_{t \rightarrow t_0} f(t) \leq f(t_0)^2.$$

Then $\liminf_{t \rightarrow t_0} f(t) = \limsup_{t \rightarrow t_0} f(t) = f(t_0)$. It follows that f is continuous. It is clear that f is everywhere positive unless f is constantly zero. When f is everywhere positive, we may take $F = \log \circ f \circ \exp$ which is mid-point convex and continuous, so it is well known that F is convex, or equivalently, that f is multiplicatively convex. \square

Definition 4.4 Let $(a, b) \subset \mathbb{R}_+$ be an interval of positive real numbers. A nowhere zero multiplicatively convex function $f: (a, b) \rightarrow (0, +\infty)$ is said to have *bounded exponent* if there exists some positive constant R such that for all pairs of distinct points $t_0, t_1 \in (a, b)$,

$$\left| \frac{\log f(t_1) - \log f(t_0)}{\log t_1 - \log t_0} \right| \leq R.$$

For multiplicatively convex functions, the growth bound degree (Definition 1.3) can be characterized by the limit exponents:

Lemma 4.5 *Suppose that $f: \mathbb{R}_+ \rightarrow [0, +\infty)$ is a nowhere zero multiplicatively convex function. Then the growth bound degree $\deg^b(f) \in \mathbb{R}$ exists if and only if f has bounded exponent. Moreover, in this case, the following equalities hold true:*

$$\deg_{+\infty}^b(f) = \lim_{t_0, t_1 \rightarrow +\infty} \frac{\log f(t_0) - \log f(t_1)}{\log t_0 - \log t_1},$$

and

$$\text{deg}_{0+}^b(f) = \lim_{t_0, t_1 \rightarrow 0+} \frac{\log f(t_0) - \log f(t_1)}{\log t_0 - \log t_1}.$$

Proof We show that the equalities hold if $\text{deg}^b \in \mathbb{R}_+$ exists. If there exists $D_{+\infty} \in \mathbb{R}$ such that $\lim_{t \rightarrow +\infty} f(t) \cdot t^{-D_{+\infty}} = 0$, then $\log f(t)$ is less than or equal to $D_{+\infty} \log t$ for all sufficiently large $t \in \mathbb{R}_+$. For all $t_0, t_1 \in \mathbb{R}_+$, by the multiplicative convexity of f ,

$$\begin{aligned} \frac{\log f(t_0) - \log f(t_1)}{\log t_0 - \log t_1} &\leq \limsup_{t \rightarrow +\infty} \frac{\log f(t) - \log f(t_1)}{\log t - \log t_1} \\ &\leq \limsup_{t \rightarrow +\infty} \frac{D_{+\infty} \log t - \log f(t_1)}{\log t - \log t_1} = D_{+\infty}. \end{aligned}$$

Denote by

$$d_{+\infty} = \lim_{t_0, t_1 \rightarrow +\infty} \frac{\log f(t_0) - \log f(t_1)}{\log t_0 - \log t_1} \in \mathbb{R}.$$

It is easy to see that for any constant $\delta > 0$,

$$\lim_{t \rightarrow +\infty} f(t) \cdot t^{-(d_{+\infty} + \delta)} = 0.$$

Consequently,

$$\text{deg}_{+\infty}^b(f) = d_{+\infty}.$$

The equality for $0+$ can be proved in a similar way. We have shown the ‘only-if’ direction. The existence of exponent bound leads to the existence of $d_{+\infty}$ and d_{0+} in \mathbb{R} , so

$$d_{0+} - 1 < \text{deg}_{0+}^b(f) \leq \text{deg}_{+\infty}^b(f) < d_{+\infty} + 1.$$

This shows the ‘if’ direction. □

Example 4.6

1. A *monomial function* on an interval (a, b) is a function of the form $f(t) = Ct^r$ for some constants $C \in \mathbb{R}_+$ and $r \in \mathbb{R}$. Such a function is *multiplicatively linear* in the sense that for all points $t_0, t_1 \in (a, b)$ and for every constant $\lambda \in (0, 1)$,

$$f\left(t_0^{1-\lambda} \cdot t_1^\lambda\right) = f(t_0)^{1-\lambda} \cdot f(t_1)^\lambda.$$

2. A *piecewise monomial function* on an interval (a, b) is a continuous function $f: (a, b) \rightarrow (0, +\infty)$ such that for finitely many points $a = c_0 < c_1 < \dots < c_{n-1} < c_n = b$, the function is a monomial $C_i t^{r_i}$ on the subinterval (c_{i-1}, c_i) where i runs over $1, \dots, n$. Such a continuous function is multiplicatively convex if and only if $r_1 \leq r_2 \leq \dots \leq r_n$.
3. Given any Laurent polynomial

$$p(z) = D \cdot z^n \cdot \prod_{i=1}^l (z - b_i) \in \mathbb{C}[z, z^{-1}],$$

with a leading coefficient $D \in \mathbb{C}^\times$ and nontrivial zeros $b_i \in \mathbb{C}^\times$, the function

$$M(p(z); t) = |D| \cdot t^n \cdot \prod_{i=1}^l \max(t, |b_i|),$$

of the variable $t \in \mathbb{R}_+$, is piecewise monomial and multiplicatively convex.

5 Multiplicative convexity and exponent bound

In this section, we show that residually finite L^2 -Alexander twists result in multiplicatively convex determinant functions with bounded exponents.

Theorem 5.1 *Given any admissible triple (π, ϕ, γ) over \mathbb{R} and any square matrix A over $\mathbb{C}\pi$, denote by $V: \mathbb{R}_+ \rightarrow [0, +\infty)$ the regular Fuglede-Kadison determinant function*

$$V(t) = \det_{\mathcal{N}(G)}^{\mathbb{F}}(\kappa(\phi, \gamma, t)(A)),$$

where G is the target group of γ and $\kappa(\phi, \gamma, t)$ is the induced change of coefficients.

Suppose that G is finitely generated and residually finite. Then $V(t)$ is either constantly zero or multiplicatively convex with exponent bounded. Moreover, there exists a constant $R(A, \phi) \in [0, +\infty)$ depending only on A and ϕ so that

$$\deg^b(V) \leq R(A, \phi).$$

The rest of this section is devoted to the proof of Theorem 5.1.

5.1 The degree bound

For any $p \times p$ matrix A over $\mathbb{C}\pi$, we can decompose A as a unique sum:

$$A = \sum_{g \in \pi} g \cdot A_g$$

where A_g are $p \times p$ matrices over \mathbb{C} and only finitely many A_g are nonzero. Given any homomorphism $\phi \in \text{Hom}(\pi, \mathbb{R})$, we define

$$R(A, \phi) = p \cdot \left(\max_{A_g \neq 0} \phi(g) - \min_{A_g \neq 0} \phi(g) \right).$$

The quantity $R(A, \phi)$ behaves well under operations of the matrix and the cohomology class. In fact, we observe the following elementary properties. The proof is straightforward so we omit it in this paper.

Lemma 5.2 1. For all $A \in \text{Mat}_{p \times p}(\mathbb{C}) \subset \text{Mat}_{p \times p}(\mathbb{C}\pi)$, $R(A, \phi) = 0$.
 2. For all $A, B \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$,

$$R(AB, \phi) \leq R(A, \phi) + R(B, \phi)$$

and

$$R(A + B, \phi) \leq \max(R(A, \phi), R(B, \phi))$$

3. For all $A \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$, and $c \in \mathbb{R}$, and $\phi, \psi \in \text{Hom}(\pi, \mathbb{R})$,

$$R(A, c\phi) = |c| \cdot R(A, \phi)$$

and

$$R(A, \phi + \psi) \leq R(A, \phi) + R(A, \psi).$$

4. Let $\gamma: \pi \rightarrow G$ be a group homomorphism. For all $A \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$ and $\xi \in H^1(G; \mathbb{R})$,

$$R(A, \gamma^*\xi) \geq R(\gamma_*A, \xi)$$

The following lemma can be combined with Lemma 4.5 to yield the degree bound, once we have shown that $V(t)$ is multiplicatively convex.

Lemma 5.3 Given any admissible triple (π, ϕ, γ) over \mathbb{R} and any square matrix A over $\mathbb{C}\pi$, write

$$V(t) = \det_{\mathcal{N}(G)}^r(\kappa(\phi, \gamma, t)(A))$$

where G is the target group of γ . Then the following statement holds true.

For every constant $R' > R(A, \phi)$, there exist constants $D_{+\infty}, D_{0+} \in \mathbb{R}$ such that $D_{+\infty} - D_{0+} < R'$. Moreover,

$$\lim_{t \rightarrow +\infty} V(t) \cdot t^{-D_{+\infty}} = 0,$$

and

$$\lim_{t \rightarrow 0+} V(t) \cdot t^{-D_{0+}} = 0.$$

Proof We adopt the notations at the beginning of this subsection. Given $R' > R(A, \phi)$, we denote by 5δ the difference $R' - R(A, \phi)$. Take

$$D_{+\infty} = 2\delta + p \cdot \max_{A_g \neq \mathbf{0}} \phi(g),$$

and

$$D_{0+} = -2\delta + p \cdot \min_{A_g \neq \mathbf{0}} \phi(g).$$

For sufficiently large $t \in \mathbb{R}_+$, the operator norm of $t^{-D_{+\infty}+\delta} \cdot \kappa(\phi, \gamma, t)(A)$ is bounded by 1. Therefore,

$$\begin{aligned} 0 &\leq \limsup_{t \rightarrow +\infty} V(t) \cdot t^{-D_{+\infty}} \leq 1^p \cdot \lim_{t \rightarrow +\infty} \det_{\mathcal{N}(G)}^r(t^{-\delta} \cdot \mathbf{1}) \\ &= \lim_{t \rightarrow +\infty} t^{-p\delta} = 0. \end{aligned}$$

This yields the asserted limit for $t \rightarrow +\infty$. The limit for $t \rightarrow 0+$ can be proved in a similar way. \square

5.2 Multiplicative convexity for virtually abelian twists

In this section, we prove Theorem 5.1 under the assumption that G is finitely generated and virtually abelian.

Given an admissible triple (π, ϕ, γ) over \mathbb{R} and a parameter value $t \in \mathbb{R}_+$, for any $p \times p$ matrix A of $\mathbb{C}\pi$, we define

$$A_G(t) = \kappa(\phi, \gamma, t)(A) \in \text{Mat}_{p \times p}(\mathbb{C}G)$$

and write

$$V(t) = \det_{\mathcal{N}(G)}^r(A_G(t)).$$

Proposition 5.4 *Let (π, ϕ, γ) is an admissible triple over \mathbb{R} . Suppose that G is finitely generated and virtually abelian. Then for every matrix $A \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$, the function $V(t)$ is multiplicatively convex.*

The following lemma treats the essential case where G is finitely generated and free abelian.

Lemma 5.5 *Let (π, ϕ, γ) be an admissible triple over \mathbb{R} . Suppose that γ is an isomorphism onto a finitely generated free abelian group G . Then for every $A \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$, the function $V(t)$ is multiplicatively convex.*

Proof For any admissible triple (π, ϕ, γ) over \mathbb{R} , the image $\phi(\pi)$ is finitely generated as G is finitely generated and free abelian. Take a basis $r_1, \dots, r_d \in \mathbb{R}_+$ of the \mathbb{Q} -vector space spanned by $\phi(\pi) \subset \mathbb{R}$. Possibly after dividing each r_i by a positive integer, we can decompose ϕ as a sum:

$$\phi = r_1\phi_1 + \dots + r_d\phi_d$$

where ϕ_i are homomorphisms in $\text{Hom}(\pi, \mathbb{Z})$. We fix such a basis for the rest of the proof. Consider a multivariable version of twist as follows. Given any vector $\vec{t} = (t_1, \dots, t_d) \in \mathbb{R}_+^d$, there is a homomorphism of rings:

$$\kappa(\phi, \gamma, \vec{t}) : \mathbb{Z}\pi \longrightarrow \mathbb{R}G$$

defined uniquely by

$$\kappa(\phi, \gamma, \vec{t})(g) = t_1^{\phi_1(g)} \dots t_d^{\phi_d(g)} \gamma(g)$$

for all $g \in \pi$ via linear extension over \mathbb{Z} . There are induced homomorphisms between matrix algebras over $\mathbb{C}\pi$ and $\mathbb{C}G$ as before. We define

$$A_G(\vec{t}) = \kappa(\phi, \gamma, \vec{t})(A) \in \text{Mat}_{p \times p}(\mathbb{C}G).$$

Denote

$$W(\vec{t}) = \det_{\mathcal{N}(G)}^r(A_G(\vec{t})).$$

Then

$$V(t) = W((t^{r_1}, \dots, t^{r_d})).$$

On the other hand, we identify $A_G(\vec{t})$ as a family of $p \times p$ matrices over the multivariable Laurent polynomial ring $\mathbb{C}\left[z_1^{\pm 1}, \dots, z_l^{\pm 1}\right]$, where l is the rank of G . Denote by $\vec{1}$ the diagonal vector $(1, \dots, 1) \in \mathbb{R}_+^d$. If we write the Laurent polynomial matrix at $\vec{1}$ as:

$$A_G(\vec{1}) = A_G(\vec{1})(z_1, \dots, z_l),$$

then at \vec{t} the Laurent polynomial matrix can be computed by:

$$A_G(\vec{t}) = A_G(\vec{1})(\tilde{t}_1 z_1, \dots, \tilde{t}_l z_l)$$

where, for j running over $1, \dots, l$,

$$\tilde{t}_j = t_1^{\phi_1(z_j)} \dots t_d^{\phi_d(z_j)}.$$

In fact, the relation can be checked by looking at the monomials in each entry of $A_G(\vec{1})$. The effect of the twist is that in any monomial, each z_j that appears contributes an exponent $\phi_i(z_j)$ to the associated coefficient t_i .

The value of $W(\vec{t})$ can be computed by the (multiplicative) Mahler measure of the usual determinant of the Laurent polynomial matrix $A_G(\vec{t})$. Precisely, the usual determinant gives rise to a Laurent polynomial for the square matrix at $\vec{1}$:

$$p_A(z_1, \dots, z_l) = \text{Det}_{\mathbb{C}\left[z_1^{\pm 1}, \dots, z_l^{\pm 1}\right]} \left(A_G(\vec{1}) \right),$$

so

$$p_A(\tilde{t}_1 z_1, \dots, \tilde{t}_l z_l) = \text{Det}_{\mathbb{C}\left[z_1^{\pm 1}, \dots, z_l^{\pm 1}\right]} \left(A_G(\vec{t}) \right).$$

By [7, Lemma 2.6], (cf. [23, Exercise 3.8] and [28, Sect. 1.2]), if p_A is not the zero polynomial,

$$\begin{aligned} W(\vec{t}) &= M(p_A(\tilde{t}_1 z_1, \dots, \tilde{t}_l z_l)) \\ &= \exp \left[\frac{1}{(2\pi)^l} \cdot \int_0^{2\pi} \dots \int_0^{2\pi} \log \left| p_A(\tilde{t}_1 e^{i\theta_1}, \dots, \tilde{t}_l e^{i\theta_l}) \right| d\theta_1, \dots, d\theta_l \right]. \end{aligned}$$

Note that if p_A is the zero polynomial, then $W(\vec{t})$ and $V(t)$ are constantly zero, so the multiplicative convexity of $V(t)$ holds in this trivial case. We assume in the rest of the proof that p_A is not the zero polynomial.

First consider the case when (π, ϕ, γ) is an admissible triple over \mathbb{Q} . In this case, d is at most 1. We can assume that d equals 1, since otherwise $V(t)$ is a constant function. There is a splitting short exact sequence of free abelian groups:

$$1 \longrightarrow \gamma(\text{Ker}(\phi)) \longrightarrow G \xrightarrow{\phi \circ \gamma^{-1}} \phi(\pi) \longrightarrow 1.$$

We may choose a basis of the free abelian group G such that $\phi(z_l) = mr_1$ for some nonzero integer m and $\phi(z_i) = 0$ for all other z_i . For any given values $\theta_1, \dots, \theta_{l-1} \in [0, 2\pi]$, we introduce the notations

$$q_{\theta_1, \dots, \theta_{l-1}}(z) = p_A \left(e^{i\theta_1}, \dots, e^{i\theta_{l-1}}, z \right) \in \mathbb{C}[z, z^{-1}],$$

and

$$v_{\theta_1, \dots, \theta_{l-1}}(t) = \log M \left(q_{\theta_1, \dots, \theta_{l-1}} \left(t^{mr_1} z \right) \right).$$

Then

$$\begin{aligned} \log V(t) &= \log W(t^{r_1}) \\ &= \frac{1}{(2\pi)^l} \cdot \int_0^{2\pi} \dots \int_0^{2\pi} \log \left| p_A \left(e^{i\theta_1}, \dots, e^{i\theta_{l-1}}, t^{mr_1} e^{i\theta_l} \right) \right| d\theta_1 \dots d\theta_l \\ &= \frac{1}{(2\pi)^{l-1}} \cdot \int_0^{2\pi} \dots \int_0^{2\pi} v_{\theta_1, \dots, \theta_{l-1}}(t) d\theta_1 \dots d\theta_{l-1}. \end{aligned}$$

For any one-variable Laurent polynomial $q \in \mathbb{C}[z, z^{-1}]$, the Mahler measure can be computed using Jensen’s formula:

$$M(q(z)) = |D| \cdot \prod_{i=1}^l \max(1, |b_i|),$$

where the constants $D \in \mathbb{C}$ and $n \in \mathbb{Z}$ and $b_i \in \mathbb{C}$ are given by any factorization

$$q(z) = D \cdot z^n \cdot \prod_{i=1}^l (z - b_i) \in \mathbb{C}[z, z^{-1}].$$

It is evident that for any such q , the following function in $t \in \mathbb{R}_+$ is multiplicatively convex:

$$M(q(t^{mr_1}z)) = |D| \cdot t^{nmr_1} \cdot \prod_{i=1}^l \max(t^{mr_1}, |b_i|),$$

possibly constantly zero if q is 0. Therefore, for all pairs of distinct points $T_0, T_1 \in \mathbb{R}_+$, and all constants $0 < \lambda < 1$, we have the comparison:

$$(1 - \lambda) \cdot v_{\theta_1, \dots, \theta_{l-1}}(T_0) + \lambda \cdot v_{\theta_1, \dots, \theta_{l-1}}(T_1) \geq v_{\theta_1, \dots, \theta_{l-1}}(T_0^{1-\lambda} \cdot T_1^\lambda).$$

Integrating both sides and taking the exponential yields

$$V(T_0)^{1-\lambda} \cdot V(T_1)^\lambda \geq V(T_0^{1-\lambda} \cdot T_1^\lambda).$$

In other words, $V(t)$ is multiplicatively convex.

For the general case over \mathbb{R} , denote by \vec{r} the vector $(r_1, \dots, r_d) \in \mathbb{R}_+^d$. Take a sequence of rational vectors $\{\vec{r}^{(n)} \in \mathbb{Q}_+^d\}$ which converges to \vec{r} in \mathbb{R}_+^d as n tends to infinity. Observe that for each $\vec{r}^{(n)}$, the function

$$V_n(t) = W\left(t^{r_1^{(n)}}, \dots, t^{r_d^{(n)}}\right)$$

is equal to the regular Fuglede–Kadison determinant of the matrix

$$\kappa(\phi^{(n)}, \gamma, t)(A) \in \text{Mat}_{p \times p}(\mathbb{C}G),$$

where

$$\phi^{(n)} = r_1^{(n)}\phi_1 + \dots + r_d^{(n)}\phi_d$$

is a homomorphism in $\text{Hom}(\pi, \mathbb{Q})$. Then $V_n(t)$ are multiplicatively convex by the rational case that we have proved. On the other hand, as \vec{r} varies over \mathbb{R}_+^d , the coefficients of the Laurent polynomials $p_A(\tilde{t}_1 z_1, \dots, \tilde{t}_l z_l)$ varies continuously, so the Mahler measures of the Laurent polynomials vary continuously by Boyd [4]. In particular, for every $t \in \mathbb{R}_+$,

$$\lim_{n \rightarrow \infty} V_n(t) = V(t).$$

Given any constants $T_0, T_1 \subset \mathbb{R}_+$ and $0 < \lambda < 1$, we have shown the multiplicative convexity for the rational case:

$$V_n(T_0)^{1-\lambda} \cdot V_n(T_1)^\lambda \geq V_n \left(T_0^{1-\lambda} \cdot T_1^\lambda \right).$$

Taking the limit as $n \rightarrow \infty$,

$$V(T_0)^{1-\lambda} \cdot V(T_1)^\lambda \geq V \left(T_0^{1-\lambda} \cdot T_1^\lambda \right).$$

In other words, the function $V(t)$ is multiplicatively convex. This completes the proof. □

Proof of Proposition 5.4 Take a free abelian subgroup \tilde{G} of $\tilde{\pi}$ of finite index, which is hence finitely generated. Denote by $\tilde{\pi}$ the preimage $\gamma^{-1}(\tilde{G})$. Take restrictions $\tilde{\phi}, \tilde{\gamma}$ of given homomorphisms to $\tilde{\pi}$ accordingly. The restriction of A to $\mathbb{C}\tilde{\pi}$, denoted as $\text{res}_{\tilde{\pi}}^{\tilde{\pi}} A$, is a square matrix over $\mathbb{C}\tilde{\pi}$ of size $p \cdot [\pi : \tilde{\pi}]$. We observe that the operation of restriction commutes with $\kappa(\gamma, \phi, t)$ and $*$. Denote by $\tilde{V}(t)$ the corresponding determinant function for the admissible triple $(\tilde{\pi}, \tilde{\phi}, \tilde{\gamma})$ and the matrix $\text{res}_{\tilde{\pi}}^{\tilde{\pi}} A$. By basic properties of regular Fuglede-Kadison determinants,

$$\begin{aligned} V(t) &= \det_{\mathcal{N}(G)}^{\mathbb{r}}(A_G(t)) \\ &= \det_{\mathcal{N}(\gamma(\pi))}^{\mathbb{r}} \left(\text{res}_G^{\gamma(\pi)}(A_G(t)) \right) \\ &= \det_{\mathcal{N}(\tilde{G})}^{\mathbb{r}} \left(\text{res}_{\tilde{G}}^{\tilde{G}}(A_G(t)) \right)^{1/[\gamma(\pi):\tilde{G}]} \\ &= \det_{\mathcal{N}(\tilde{G})}^{\mathbb{r}} \left(\left(\text{res}_{\tilde{\pi}}^{\tilde{\pi}} A \right)_{\tilde{G}}(t) \right)^{1/[\pi:\tilde{\pi}]} \\ &= \tilde{V}(t)^{1/[\pi:\tilde{\pi}]}. \end{aligned}$$

Note that $\tilde{V}(t)$ is constantly zero if and only if $V(t)$ is constantly zero. Suppose that $\tilde{V}(t)$ is not constantly zero. By Lemma 5.4, the function $\tilde{V}(t)$ is multiplicatively convex, so $V(t)$ is multiplicatively convex as well. This completes the proof. □

5.3 Multiplicative convexity for residually finite twists

Let (π, ϕ, γ) be an admissible triple over \mathbb{R} . Suppose that the target group G of γ is finitely generated and residually finite. Take a cofinal tower of normal finite index subgroups of G :

$$G \geq N_1 \geq N_2 \geq \dots \geq N_n \geq \dots$$

Here the tower being cofinal means that

$$\bigcap_{n=1}^{\infty} N_n = \{\text{id}_G\}.$$

Fix a homomorphism $G \rightarrow \mathbb{R}$ via which ϕ factors through γ . Denote by K_n the kernel of $N_n \rightarrow H_1(N_n; \mathbb{Q})$, which remains normal in G . Let

$$\Gamma_n = G/K_n.$$

There are induced homomorphisms by the composition of γ and the quotient $G \rightarrow \Gamma_n$, denoted as

$$\gamma_n: \pi \rightarrow \Gamma_n.$$

It is clear that Γ_n are all finitely generated and virtually abelian. Therefore, we obtain a tower of admissible triples over \mathbb{R} :

$$\{(\pi, \phi, \gamma_n)\}_{n \in \mathbb{N}}$$

with finitely generated virtually abelian targets.

Given any $p \times p$ matrix A over $\mathbb{C}\pi$, and any value of parameter $T \in \mathbb{R}_+$, and any constant $\epsilon \in [0, +\infty)$, we introduce a positive operator on $\ell^2(\Gamma_n)^{\oplus p}$:

$$H_{n,\epsilon}(T) = (\kappa(\phi, \gamma_n, T)(A))^* (\kappa(\phi, \gamma_n, T)(A)) + \epsilon \cdot \mathbf{1}$$

which is expressed as a $p \times p$ matrix over $\mathbb{C}\Gamma_n$. When the subscript n is replaced with the symbol ∞ , we adopt the convention that $\Gamma_\infty = G$ and $\gamma_\infty = \gamma$.

Proof of Theorem 5.1 Given an admissible triple (π, ϕ, γ) over \mathbb{R} and a square matrix A over $\mathbb{C}\pi$. We adopt the assumptions and notations of this subsection. Possibly after replacing G with the image of γ , which does not affect the value of the determinant, we may further assume that γ is surjective. Then there are uniquely induced homomorphisms $\gamma_{n*}\phi \in \text{Hom}(\Gamma_n, \mathbb{R})$ whose pull-backs through γ are ϕ , and $(\Gamma_n, \text{id}_{\Gamma_n}, \gamma_{n*}\phi)$ are admissible triples. For parameters $s, T, t \in \mathbb{R}_+$, we write

$$W_{n,\epsilon}(s, T) = \det_{\mathcal{N}(\Gamma_n)}^{\mathbb{R}}(\kappa(\gamma_{n*}\phi, \text{id}_{\Gamma_n}, s)(H_{n,\epsilon}(T))),$$

and

$$V_n(t) = \det_{\mathcal{N}(\Gamma_n)}^{\mathbb{R}}(\kappa(\phi, \gamma_n, t)(A)).$$

Observe that $\kappa(\gamma_n * \phi, \text{id}_{\Gamma_n}, s) \circ \kappa(\phi, \gamma_n, t)$ equals $\kappa(\phi, \gamma_n, st)$. Therefore, for any given $T_0, T_1 \in \mathbb{R}_+$, we have the relations:

$$W_{n,0} \left(1, \sqrt{T_0 T_1} \right) = V_n \left(\sqrt{T_0 T_1} \right)^2$$

and

$$W_{n,0} \left(\sqrt{T_1/T_0}, \sqrt{T_0 T_1} \right) = V_n(T_0) V_n(T_1),$$

which hold for both $n \in \mathbb{N}$ and ∞ . Note that $W_{n,\epsilon}(1, T)$ is always the regular Fuglede–Kadison determinant for a positive operator, but the twisted matrix in the expression of $W_{n,\epsilon}(s, T)$ is not self-adjoint in general.

We claim that the following comparison holds for all $s, T \in \mathbb{R}_+$:

$$W_{\infty,0}(1, T) \leq W_{\infty,0}(s, T).$$

In fact, by Lemma 5.4, the function $W_{n,\epsilon}(s, T)$ is multiplicatively convex in $s \in \mathbb{R}_+$ for all $n \in \mathbb{N}$ and $\epsilon \in [0, +\infty)$. Observe that $H_{n,\epsilon}(T)$ is self-adjoint, so the anti-commutativity of $\kappa(\phi, \gamma_n, s)$ and $*$ yields $W_{n,\epsilon}(s, T) = W_{n,\epsilon}(s^{-1}, T)$. This implies that for all $\epsilon \in [0, +\infty)$ and $n \in \mathbb{N}$,

$$W_{n,\epsilon}(1, T) \leq W_{n,\epsilon}(s, T).$$

Given any arbitrary $\epsilon > 0$, Lemma 3.2 and the above imply

$$W_{\infty,\epsilon}(1, T) = \lim_{n \rightarrow \infty} W_{n,\epsilon}(1, T) \leq \limsup_{n \rightarrow \infty} W_{n,\epsilon}(s, T) \leq W_{\infty,\epsilon}(s, T).$$

As ϵ tends to $0+$, Lemma 3.1 and the above imply

$$W_{\infty,0}(1, T) = \lim_{\epsilon \rightarrow 0+} W_{\infty,\epsilon}(1, T) \leq \limsup_{\epsilon \rightarrow 0+} W_{\infty,\epsilon}(s, T) \leq W_{\infty,0}(s, T).$$

This proves the claim.

Note that the family of operators $\kappa(\phi, \gamma, s)(A)$ is continuous in $s \in \mathbb{R}_+$ with respect to the norm topology. Lemma 3.1 implies that $V_\infty(t)$ is upper semicontinuous in $t \in \mathbb{R}_+$. On the other hand, the claim implies that $V_\infty(t)$ is multiplicatively mid-point convex in $t \in \mathbb{R}_+$. By Lemma 4.3, the function $V_\infty(t)$, or $V(t)$ as in the statement of Theorem 5.1, is multiplicatively convex.

Provided with the multiplicative convexity, assuming that $V(t)$ is nowhere zero, the exponent bound and the degree estimate

$$\text{deg}^b(V) \leq R(A, \phi)$$

follow from Lemmas 4.5 and 5.3. This completes the proof of Theorem 5.1. \square

6 Continuity of degree

In this section, we show that the growth bound degree of the regular Fuglede–Kadison determinant of L^2 -Alexander twists varies continuously as we deform the cohomology class.

Theorem 6.1 *Given any admissible triple (π, ϕ, γ) over \mathbb{R} and any square matrix A over $\mathbb{C}\pi$, denote by G the target group of γ . For any vector $\xi \in H^1(G; \mathbb{R})$, denote by*

$$V_\xi(t) = \det_{\mathcal{N}(G)}^r(\kappa(\phi + \gamma^*\xi, \gamma, t)(A))$$

the determinant function of A associated with the deformed admissible triple $(\pi, \phi + \gamma^*\xi, \gamma)$.

Suppose that G is finitely generated and residually finite. Then the function $V_\xi(t)$ is constantly zero at every vector $\xi \in H^1(G; \mathbb{R})$ whenever it is constantly zero somewhere. Apart from that exception, for all pairs of vectors $\xi, \eta \in H^1(G; \mathbb{R})$,

$$|\deg^b(V_\xi) - \deg^b(V_\eta)| \leq 2R(A, \gamma^*(\xi - \eta)).$$

In particular, the assignment with the degree $\xi \mapsto \deg^b(V_\xi(t))$ defines a Lipschitz continuous function on $H^1(G; \mathbb{R})$ valued in $[0, +\infty)$.

The continuity of degree is a consequence of Theorem 5.1. The rest of this section is devoted to the proof of Theorem 6.1.

We may assume without loss of generality that $\eta \in H^1(G; \mathbb{R})$ is trivial. In fact, otherwise we can replace the reference class ϕ by $\phi + \gamma^*\eta$. Hence ξ and η are replaced by $\xi - \eta$ and 0 respectively.

We adopt the following notations. Given any matrix $A \in \text{Mat}_{p \times p}(\mathbb{C}\pi)$, denote

$$A_G(t) = \kappa(\phi, \gamma, t)(A) \in \text{Mat}_{p \times p}(\mathbb{C}G).$$

For any vector $\xi \in H^1(G; \mathbb{R}) \cong \text{Hom}(G; \mathbb{R})$, we consider the canonical admissible triple (G, ξ, id_G) , so for every constant $s \in \mathbb{R}_+$, there is a matrix deformed from $A_G(t)$, namely:

$$A_G(t, s) = \kappa(\xi, \text{id}_G, s)(A_G(t)) \in \text{Mat}_{p \times p}(\mathbb{C}G).$$

We introduce

$$W(t, s) = \det_{\mathcal{N}(G)}^r (A_G(t, s)).$$

Note that

$$W(t, 1) = V_0(t)$$

and

$$W(t, t) = V_\xi(t).$$

Lemma 6.2 *If the function $V_0(t)$ is constantly zero, then for all vectors $\xi \in H^1(G; \mathbb{R})$, the function $V_\xi(t)$ is constantly zero as well.*

Proof Suppose $V_0(t)$ is constantly zero. Given any constant $T_0 \in \mathbb{R}_+$, apply Theorem 5.1 to the family of matrices $A_G(T_0, s)$, we see that $W(T_0, s)$ is multiplicatively convex in the parameter $s \in \mathbb{R}_+$. At $s = 1$, we have $W(T_0, 1) = V_0(T_0) = 0$. This implies that $W(T_0, s)$ is constantly zero in s by Lemma 4.2. In particular, $V_\xi(T_0) = W(T_0, T_0) = 0$. As $T_0 \in \mathbb{R}_+$ is arbitrary, it follows that $V_\xi(t)$ is constantly zero. \square

Now it suffices to assume that the functions $V_\xi(t)$ are nowhere zero, for all $\xi \in H^1(G; \mathbb{R})$. By Theorem 5.1, $V_\xi(t)$ are multiplicatively convex and have bounded exponent.

Lemma 6.3

1. $|\text{deg}_{+\infty}^b(V_\xi) - \text{deg}_{+\infty}^b(V_0)| \leq R(A, \gamma^*\xi);$
2. $|\text{deg}_{0+}^b(V_\xi) - \text{deg}_{0+}^b(V_0)| \leq R(A, \gamma^*\xi).$

Proof We prove the first estimate and the second can be proved in the same way.

Given any constant $T_0 \in \mathbb{R}_+$ and $K > 0$, it follows from the multiplicative convexity of $W(T_0^{1+K}, s)$ in the parameter $s \in \mathbb{R}_+$ that

$$\left| \frac{\log W(T_0^{1+K}, T_0^{1+K}) - \log W(T_0^{1+K}, 1)}{\log T_0^{1+K} - \log 1} \right| \leq R(A_G(T_0^{1+K}), \xi) \leq R(A, \gamma^*\xi),$$

so

$$\left| \log W(T_0^{1+K}, T_0^{1+K}) - \log W(T_0^{1+K}, 1) \right| \leq R(A, \gamma^*\xi) \cdot (1 + K) \log T_0.$$

Similarly,

$$|\log W(T_0, T_0) - \log W(T_0, 1)| \leq R(A, \gamma^* \xi) \cdot \log T_0.$$

By the multiplicative convexity of $W(t, 1) = V_0(t)$, for any arbitrary $\delta > 0$, the following estimate holds for sufficiently large $T_0 > 1$ and any arbitrary $K > 0$:

$$\left| \frac{\log W(T_0^{1+K}, 1) - \log W(T_0, 1)}{\log T_0^{1+K} - \log T_0} - \deg_{+\infty}^b(V_0) \right| < \delta,$$

so

$$\left| \log W(T_0^{1+K}, 1) - \log W(T_0, 1) - \deg_{+\infty}^b(V_0) K \log T_0 \right| < \delta \cdot K \log T_0.$$

Therefore, for any arbitrary $\delta > 0$, the following estimate holds for sufficiently large $T_0 > 1$ and any arbitrary $K > 0$:

$$\begin{aligned} & \left| \log W(T_0^{1+K}, T_0^{1+K}) - \log W(T_0, T_0) - \deg_{+\infty}^b(V_0) K \log T_0 \right| \\ & < R(A, \gamma^* \xi) \cdot (2 + K) \log T_0 + \delta \cdot K \log T_0, \end{aligned}$$

or equivalently,

$$\begin{aligned} & \left| \frac{\log V_\xi(T_0^{1+K}) - \log V_\xi(T_0)}{\log T_0^{1+K} - \log T_0} - \deg_{+\infty}^b(V_0) \right| \\ & < R(A, \gamma^* \xi) \cdot \left(1 + \frac{2}{K} \right) + \delta. \end{aligned}$$

Take the limit as $T_0 \rightarrow +\infty$, and then take the limit as $K \rightarrow +\infty$:

$$|\deg_{+\infty}^b(V_\xi) - \deg_{+\infty}^b(V_0)| \leq R(A, \gamma^* \xi) + \delta.$$

As $\delta > 0$ is an arbitrary constant, the estimate

$$|\deg_{+\infty}^b(V_\xi) - \deg_{+\infty}^b(V_0)| \leq R(A, \gamma^* \xi)$$

follows. The second estimate can be done similarly using $1/T_0$ instead of T_0 . \square

Combining the estimates of Lemma 6.3, we obtain

$$|\deg^b(V_\xi) - \deg^b(V_0)| \leq 2R(A, \gamma^* \xi).$$

This completes the proof of Theorem 6.1.

7 Asymptotics for integral matrices

In this section, we give a criterion for checking under special circumstances that the regular Fuglede–Kadison determinant of L^2 –Alexander twists is asymptotically monomial.

Definition 7.1 Let (π, γ, ϕ) be an admissible triple with a countable target group G , and

$$G \rightarrow \cdots \rightarrow \Gamma_n \rightarrow \cdots \rightarrow \Gamma_2 \rightarrow \Gamma_1$$

be a cofinal tower of quotients of G . Denote by $\psi_n: G \rightarrow \Gamma_n$ the quotient homomorphisms. A sequence of admissible triples

$$\{(\pi, \gamma_n, \phi)\}_{n \in \mathbb{N}}$$

with target groups $\{\Gamma_n\}_{n \in \mathbb{N}}$ is said to form a *cofinal tower of quotients* of (π, γ, ϕ) if $\gamma_n = \psi_n \circ \gamma$ holds for every $n \in \mathbb{N}$. For simplicity, we often speak of cofinal towers of admissible triples without explicitly mentioning the cofinal tower of quotients of G .

In the statement of the theorem below, we adopt the notation

$$V_n(t) = \det_{\mathcal{N}(\Gamma_n)}^r(\kappa(\phi, \gamma_n, t)(A)).$$

The notation $V_G(t)$ is understood similarly.

Theorem 7.2 *Let (π, γ_G, ϕ) be an admissible triple over \mathbb{R} with a finitely generated target group G . Let A be a square matrix over $\mathbb{Z}\pi$.*

Suppose that there exists a sequence of admissible triples $\{(\pi, \gamma_n, \phi)\}_{n \in \mathbb{N}}$ over \mathbb{R} satisfying all the following conditions:

- *The target groups Γ_n of γ_n are finitely generated and virtually abelian.*
- *The sequence of admissible triples $\{(\pi, \gamma_n, \phi)\}_{n \in \mathbb{N}}$ forms a cofinal tower of quotients of (π, γ_G, ϕ) .*
- *The sequence of degrees $\{\text{deg}^b(V_n)\}_{n \in \mathbb{N}}$ converges to $\text{deg}^b(V_G)$ in $[0, +\infty)$.*

In particular, note that $V_G(t)$ should not be constantly zero. Then, as $t \rightarrow +\infty$,

$$V_G(t) \sim C_{+\infty} \cdot t^{\text{deg}_{+\infty}^b(V_G)}$$

for some constant

$$C_{+\infty} \in [1, V_G(1)].$$

The same statement holds true with $+\infty$ replaced by $0+$.

We point out that among the three conditions the convergence of degrees is usually the hardest to satisfy or to verify. The $\mathbb{Z}\pi$ -matrix assumption is responsible for the lower bound 1 of the coefficients $C_{+\infty}$ and C_{0+} in an essential way. In particular, the argument does not apply to matrices over $\mathbb{C}\pi$ to yield similar monomial asymptoticity.

The rest of this section is devoted to the proof of 7.2.

Lemma 7.3 *Let \hat{f} be a nowhere zero multiplicatively convex function on \mathbb{R}_+ with bounded exponent. Suppose that there exists a sequence of nowhere zero multiplicatively convex functions on \mathbb{R}_+ with bounded exponent $\{f_n\}_{n \in \mathbb{N}}$ satisfying all the following conditions:*

- *There exists a uniform constant $L \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and for all pairs of distinct points $t_0, t_1 \in \mathbb{R}_+$,*

$$\frac{\log f_n(t_0) \log t_1 - \log f_n(t_1) \log t_0}{\log t_1 - \log t_0} \geq L.$$

- *For every point $t \in \mathbb{R}_+$,*

$$\limsup_{n \rightarrow \infty} f_n(t) \leq \hat{f}(t).$$

-

$$\lim_{n \rightarrow \infty} \deg^b(f_n) = \deg^b(\hat{f}).$$

Then as $t \rightarrow +\infty$,

$$\hat{f}(t) \sim C_{+\infty} \cdot t^{\deg_{+\infty}^b(\hat{f})}$$

for some constant

$$C_{+\infty} \in [e^L, \hat{f}(1)].$$

The same statement holds true with $+\infty$ replaced by $0+$.

Proof To understand the geometric meaning of the terms in presence, consider the log–log plot of a function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, namely, the parametrized curve

$$\mathcal{P}_f(t) = (\log t, \log f(t)), \quad t \in \mathbb{R}_+$$

on the Cartesian XY plane. The line through a pair of distinct points $\mathcal{P}_f(t_0)$ and $\mathcal{P}_f(t_1)$ has the slope

$$\alpha_f(t_0, t_1) = \frac{\log f(t_1) - \log f(t_0)}{\log t_1 - \log t_0},$$

and it has the Y-intercept

$$\beta_f(t_0, t_1) = \frac{\log f(t_0) \log t_1 - \log f(t_1) \log t_0}{\log t_1 - \log t_0}.$$

If f is multiplicatively convex with bounded exponent, then \mathcal{P}_f is a convex graph. The constants $\text{deg}_{+\infty}^b(f)$ and $\text{deg}_{0+}^b(f)$ are exactly the supremum and the infimum for slope of chords of \mathcal{P}_f , respectively (Lemma 4.5). For any such f , it is easy to see that for $t \rightarrow +\infty$, the asymptotic formula

$$f(t) \sim C_{+\infty} \cdot t^{\text{deg}_{+\infty}^b(f)}$$

holds for some constant $C_{+\infty} \in \mathbb{R}_+$ if and only if the following limit exists in \mathbb{R} :

$$\beta_{+\infty}(f) = \lim_{t_0, t_1 \rightarrow +\infty} \beta_f(t_0, t_1),$$

(which otherwise diverges to $-\infty$). Moreover, $\log C_{+\infty}$ must be $\beta_{+\infty}(f)$ if the asymptotic formula holds. The same criterion holds for $0+$ in place of $+\infty$. We also observe that if $\beta_f(t_0, t_1)$ is uniformly bounded below by some constant $L \in \mathbb{R}$ for all pairs of distinct parameters $t_0, t_1 \in \mathbb{R}_+$, then equivalently, the curve \mathcal{P}_f is contained entirely in the wedge region $\mathcal{V}(L, f)$ supported on the two rays emanating from the point $(0, L)$ along the directions $(-1, -\text{deg}_{0+}^b(f))$ and $(1, \text{deg}_{+\infty}^b(f))$.

To prove Lemma 7.3, we observe from the geometric meaning that the limit Y-intercept $C_{+\infty}$ is at most $\hat{f}(1)$. It remains to bound $C_{+\infty}$ from below by e^L , or equivalently, to show that the log–log plot of the function \hat{f} is contained in the wedge region $\mathcal{V}(L, \hat{f})$.

We argue by contradiction, supposing that there were a point $P = \mathcal{P}_{\hat{f}}(T_0)$ lying outside $\mathcal{V}(L, \hat{f})$. By the first condition, the curves \mathcal{P}_n of f_n are all contained in their own wedge regions $\mathcal{V}(L, f_n)$. In particular, the second condition implies that $T_0 \neq 1$. Let $3\delta \cdot |\log T_0|$ be the vertical distance of P from $\mathcal{V}(L, \hat{f})$. For all sufficiently large n , the second condition implies that the right side of $\mathcal{V}(L, f_n)$ is at most $\delta \cdot |\log T_0|$ above P . Then the third condition forces the slope of the left side of $\mathcal{V}(L, f_n)$ to be at least δ less than that of $\mathcal{V}(L, \hat{f})$ for all sufficiently large n . Consequently, for some parameter value $T_1 \in \mathbb{R}_+$ that

is sufficiently close to $0+$, the curve point $Q = \mathcal{P}_{\hat{f}}(T_1)$ must stay uniformly below the left sides of all those $\mathcal{V}(L, f_n)$, for instance, of distance at least 1. However, we see that the second condition is violated at the point Q : We have shown that the curves \mathcal{P}_n would have been at least distance 1 above Q for all sufficiently large n . The contradiction completes the proof. \square

Lemma 7.4 *Let (π, ϕ, γ) be an admissible triple over \mathbb{R} with a target group G . Let A be a square matrix over $\mathbb{Z}\pi$. Suppose that G is finitely generated and virtually abelian. Then for all pairs of distinct points $t_0, t_1 \in \mathbb{R}_+$,*

$$\frac{\log V_G(t_0) \log t_1 - \log V_G(t_1) \log t_0}{\log t_1 - \log t_0} \geq 0,$$

unless $V_G(t)$ is constantly zero.

Proof By Theorem 5.1, the function $V_G(t)$ is either constantly zero or multiplicatively convex with bounded exponent. It suffices to consider the latter case. By the geometric meaning of the expression explained in the proof of Lemma 7.3, we can equivalently prove that $V_G(t)$ is asymptotically monomial in both ends with the coefficient no less than 1.

We start by a few reductions. Observe that whether or not the asserted inequality holds true does not change under passage from G to any finite index subgroup \tilde{G} of $\gamma(\pi)$. Indeed, by basic properties of regular Fuglede–Kadison determinants,

$$\begin{aligned} V_G(t) &= \det_{\mathcal{N}(G)}^r(\kappa(\gamma, \phi, t)(A)) \\ &= \det_{\mathcal{N}(\gamma(\pi))}^r(\kappa(\gamma, \phi, t)(A)) \\ &= \det_{\mathcal{N}(\tilde{G})}^r\left(\kappa(\gamma, \phi, t)\left(\text{res}_{\gamma(\pi)}^{\tilde{G}}(A)\right)\right)^{\frac{1}{[\gamma(\pi):\tilde{G}]}} \\ &= V_{\tilde{G}}(t)^{\frac{1}{[\gamma(\pi):\tilde{G}]}}, \end{aligned}$$

Therefore, possibly after replacing G with a finite index subgroup \tilde{G} of $\gamma(\pi)$, and replacing π with $\gamma(\pi)$, we may assume without loss of generality that γ is an isomorphism, and G is a finitely generated free abelian group.

After these reductions, we denote by l the rank of G and identify $\mathbb{C}G$ with the Laurent polynomial ring $\mathbb{C}[z_1^{\pm 1}, \dots, z_l^{\pm 1}]$. Choose a basis $r_1, \dots, r_d \in \mathbb{R}_+$ of the \mathbb{Q} -vector space spanned by $\phi(\pi)$ such that elements of $\phi(\pi)$ are \mathbb{Z} -linear combinations of r_i . Then we can uniquely decompose ϕ as a sum:

$$\phi = r_1\phi_1 + \dots + r_d\phi_d$$

where ϕ_i are homomorphisms in $\text{Hom}(\pi, \mathbb{Z})$.

As in the proof of Lemma 5.5, the function $V_G(t)$ can be expressed in terms of a multivariable determinant function:

$$V_G(t) = W \left((t^{r_1}, \dots, t^{r_d}) \right),$$

where for any vector $\vec{t} = (t_1, \dots, t_d) \in \mathbb{R}_+^d$,

$$\begin{aligned} W(\vec{t}) &= \det_{\mathcal{N}(G)}^{\mathbb{F}}(A_G(\vec{t})) \\ &= M(p_A(\tilde{t}_1 z_1, \dots, \tilde{t}_l z_l)) \\ &= \exp \left[\frac{1}{(2\pi)^l} \cdot \int_0^{2\pi} \dots \int_0^{2\pi} \log \left(\left| p_A(\tilde{t}_1 e^{i\theta_1}, \dots, \tilde{t}_l e^{i\theta_l}) \right| \right) d\theta_1, \dots, d\theta_l \right], \end{aligned}$$

and for each j ,

$$\tilde{t}_j = t_1^{\phi_1(z_j)} \dots t_d^{\phi_d(z_j)}.$$

Recall the notations there that the Laurent polynomial matrix

$$A_G(\vec{t}) = \kappa(\phi, \gamma, \vec{t})(A) \in \text{Mat}_{p \times p} \left(\mathbb{C} \left[z_1^{\pm 1}, \dots, z_l^{\pm 1} \right] \right).$$

is defined using the homomorphism of matrix algebras $\kappa(\phi, \gamma, \vec{t})$ determined by the formula

$$\kappa(\phi, \gamma, \vec{t})(g) = t_1^{\phi_1(g)} \dots t_d^{\phi_d(g)} \gamma(g)$$

for all $g \in \pi$. The usual determinant of the Laurent polynomial matrix $A_G(\vec{t})$ at the diagonal vector $\vec{1} = (1, \dots, 1) \in \mathbb{Z}^d$ gives rise to the Laurent polynomial

$$p_A(z_1, \dots, z_l) = \text{Det}_{\mathbb{C} \left[z_1^{\pm 1}, \dots, z_l^{\pm 1} \right]} \left(A_G(\vec{1}) \right).$$

The idea is to govern the asymptotics of $V_G(t)$ by the fact that p_A is a Laurent polynomial over \mathbb{Z} , since A is assumed to be over $\mathbb{Z}\pi$. To this end, expand the Laurent polynomial p_A as

$$p_A(z_1, \dots, z_l) = \sum_{\vec{v} \in \mathbb{Z}^l} a_{\vec{v}} z_1^{v_1} \dots z_l^{v_l}$$

where v_i are the entries of $\vec{v} \in \mathbb{Z}^l$. Only finitely many coefficients $a_{\vec{v}}$ in the summation are nonzero. For any vector $\vec{v} \in \mathbb{Z}^l$, denote

$$\Phi \vec{v} = (\phi_1(z_1^{v_1} \dots z_l^{v_l}), \dots, \phi_d(z_1^{v_1} \dots z_l^{v_l})) \in \mathbb{Z}^d.$$

Denote by $\vec{r} \in \mathbb{R}_+^d$ the vector (r_1, \dots, r_d) . Let $\vec{w}_{\text{top}} \in \mathbb{Z}^d$ be the unique vector at which the maximum of the following set is achieved:

$$\left\{ \langle \vec{r}, \vec{w} \rangle \in \mathbb{R} : \sum_{\Phi \vec{v} = \vec{w}} a_{\vec{v}} z_1^{v_1} \dots z_l^{v_l} \neq 0 \right\}.$$

The uniqueness is a consequence of the linear independence of r_1, \dots, r_d over \mathbb{Q} . The integrand for $V_G(t)$, denoted as $\omega(t, \vec{\theta})$, can be calculated by:

$$\begin{aligned} \omega(t, \vec{\theta}) &= \log \left| p_A \left(t^{r_1 \phi_1(z_1) + \dots + r_d \phi_d(z_1)} e^{i\theta_1}, \dots, t^{r_1 \phi_1(z_l) + \dots + r_d \phi_d(z_l)} e^{i\theta_l} \right) \right| \\ &= \log \left| \sum_{\vec{w} \in \mathbb{Z}^d} \sum_{\Phi \vec{v} = \vec{w}} a_{\vec{v}} t^{\langle \vec{r}, \Phi \vec{v} \rangle} e^{i \langle \vec{\theta}, \vec{v} \rangle} \right| \\ &= \log \left| \sum_{\Phi \vec{v} = \vec{w}_{\text{top}}} a_{\vec{v}} t^{\langle \vec{r}, \vec{w}_{\text{top}} \rangle} e^{i \langle \vec{\theta}, \vec{v} \rangle} + \sum_{\Phi \vec{v} \neq \vec{w}_{\text{top}}} a_{\vec{v}} t^{\langle \vec{r}, \Phi \vec{v} \rangle} e^{i \langle \vec{\theta}, \vec{v} \rangle} \right| \\ &= \log \left| \sum_{\Phi \vec{v} = \vec{w}_{\text{top}}} a_{\vec{v}} e^{i \langle \vec{\theta}, \vec{v} \rangle} + \sum_{\Phi \vec{v} \neq \vec{w}_{\text{top}}} a_{\vec{v}} t^{\langle \vec{r}, \Phi \vec{v} - \vec{w}_{\text{top}} \rangle} e^{i \langle \vec{\theta}, \vec{v} \rangle} \right| \\ &\quad + \langle \vec{r}, \vec{w}_{\text{top}} \rangle \cdot \log t. \end{aligned}$$

Accordingly, the integral

$$\log V_G(t) = \frac{1}{(2\pi)^l} \int_0^{2\pi} \dots \int_0^{2\pi} \omega(t, \vec{\theta}) \, d\theta_1 \dots d\theta_l$$

breaks into the sum of two terms. The first term gives rise to the logarithmic Mahler measure of the Laurent polynomial

$$q_t(z_1, \dots, z_l) = \sum_{\Phi \vec{v} = \vec{w}_{\text{top}}} a_{\vec{v}} z_1^{v_1} \dots z_l^{v_l} + \sum_{\Phi \vec{v} \neq \vec{w}_{\text{top}}} a_{\vec{v}} t^{\langle \vec{r}, \Phi \vec{v} - \vec{w}_{\text{top}} \rangle} z_1^{v_1} \dots z_l^{v_l}.$$

By the way \vec{w}_{top} is selected, as t tends to $+\infty$, the coefficients of q_t converge to those of its chief part

$$q_{+\infty}(z_1, \dots, z_l) = \sum_{\Phi \vec{v} = \vec{w}_{\text{top}}} a_{\vec{v}} z_1^{v_1} \dots z_l^{v_l}.$$

Thus, by the continuity of Mahler measure [4], the first term of $\log V_G(t)$ approximates the logarithmic Mahler measure of $q_{+\infty}$ as $t \rightarrow +\infty$. The

second term is just the integration against $\langle \vec{r}, \vec{w}_{\text{top}} \rangle \cdot \log t$, which is constant with respect to $\vec{\theta}$. Putting together, as $t \rightarrow +\infty$,

$$\log V_G(t) = \log M(q_{+\infty}) + \langle \vec{r}, \vec{w}_{\text{top}} \rangle \cdot \log t + o(1).$$

The calculation yields the asymptotic formula:

$$V_G(t) \sim C_{+\infty} \cdot t^{\langle \vec{r}, \vec{w}_{\text{top}} \rangle}$$

as $t \rightarrow +\infty$. The coefficient satisfies the asserted estimation

$$C_{+\infty} = M(q_{+\infty}) \geq 1,$$

because $q_{+\infty}$ is a Laurent polynomial over \mathbb{Z} , cf. [10, Lemma 3.7]. The same argument works for $V_G(t^{-1})$ as well, which proves the $0+$ direction. We conclude that $V_G(t)$ is asymptotically monomial in both ends with the coefficient greater than or equal to 1. This completes the proof. \square

Proof of Theorem 7.2 We adopt the notations of the statement. By Theorem 5.1 and Lemma 4.2, the third assumption implies that the function $V_G(t)$ is positive for all $t \in \mathbb{R}_+$. By Lemma 3.2, the second condition of Lemma 7.3 is satisfied for $V_G(t)$ and $\{V_n(t)\}_{n \in \mathbb{N}}$. By Lemma 7.4, the functions $\{V_n(t)\}_{n \in \mathbb{N}}$ satisfy the first condition of Lemma 7.3. The third condition of Lemma 7.3 has been guaranteed by the assumption of Theorem 7.2. Therefore, Lemma 7.3 implies that $V_G(t)$ is asymptotically monomial in both ends with the coefficient lying in the interval $[1, V_G(1)]$. This completes the proof of Theorem 7.2. \square

8 L^2 -Alexander torsion of 3-manifolds

In this section, we study the L^2 -Alexander torsion of 3-manifolds using the tools that we have developed. In Sect. 8.1, we recall a formula for calculation used by [7]. We prove Theorem 1.4 in Sect. 8.2, and Theorem 1.2 in Sect. 8.3.

8.1 Efficient cellular presentation

To calculate L^2 -Alexander torsion of 3-manifolds, the following formula has been used by [7, Proposition 9.1], and we state it in some more details.

Lemma 8.1 *Suppose that N is an irreducible orientable compact 3-manifold with empty or incompressible toral boundary. There exist a (possibly empty) collection of elements $u_1, v_1, \dots, u_l, v_l \in \pi_1(N)$ and a square matrix A over $\mathbb{Z}\pi_1(N)$ such that the following holds true. The homological classes*

$[u_i] - [v_i]$ are nontrivial in $H_1(N; \mathbb{Q})$. Furthermore, for every homomorphism $\gamma : \pi_1(N) \rightarrow G$ which induces an isomorphism under $H_1(-; \mathbb{R})$, and for every cohomology class $\phi \in H^1(N; \mathbb{R})$,

$$\begin{aligned} \tau^{(2)}(N, \gamma, \phi)(t) &\doteq \det_{\mathcal{N}(G)}^{\mathbb{R}}(\kappa(\gamma, \phi, t)(A)) \cdot \prod_{i=1}^l \det_{\mathcal{N}(G)}^{\mathbb{R}}(\kappa(\gamma, \phi, t)(u_i - v_i))^{-1} \\ &= \det_{\mathcal{N}(G)}^{\mathbb{R}}(\kappa(\gamma, \phi, t)(A)) \cdot \prod_{i=1}^l \max \left\{ t^{\phi(u_i)}, t^{\phi(v_i)} \right\}^{-1}. \end{aligned}$$

The products are considered to be 1 if l equals zero. Moreover, given any primitive cohomology $\phi_0 \in H^1(N; \mathbb{Z}) \cong \text{Hom}(\pi_1(N), \mathbb{Z})$ in the first place, we may require in addition that $\phi_0(u_i) \neq \phi_0(v_i)$ for $i = 1, \dots, l$, and that A has the form:

$$A_0 + \mu \cdot \begin{pmatrix} \mathbf{1}_{k \times k} & 0 \\ 0 & 0 \end{pmatrix},$$

where A_0 is a square matrix over $\mathbb{Z}\text{Ker}(\phi_0)$, and $\phi_0(\mu) = 1$, and

$$k - l = x_N(\phi_0).$$

Proof We may assume that $H_1(N; \mathbb{R})$ is nontrivial since otherwise the L^2 -Alexander torsion is constant. Take any primitive cohomology class $\phi_0 \in H^1(N; \mathbb{Z})$, for example, as specified in the moreover part. We employ the construction of Friedl in [11, Sect. 4] to produce a $\pi_1(N)$ -equivariant CW complex structure on the universal cover of N . To be precise, there exist finitely many properly embedded oriented compact subsurface $\Sigma_1, \dots, \Sigma_s$ and accordingly $r_1, \dots, r_s \in \mathbb{N}$, satisfying the following properties:

- $r_1[\Sigma_1] + \dots + r_s[\Sigma_s] \in H_2(N, \partial N; \mathbb{Z})$ is dual to ϕ_0 .
- $-r_1\chi(\Sigma_1) - \dots - r_s\chi(\Sigma_s) = x(\phi_0)$.
- Σ_i are mutually disjoint and the complement of their union in N is connected.

The calculation here is the same as [7, Proposition 9.1] except that instead of computing square matrices induced by $\kappa(\gamma, \phi_0, t)$ there, we compute those induced by $\kappa(\gamma, \phi, t)$ for any class $\phi \in H^1(N; \mathbb{R})$. For example, the determinant contribution from a block

$$\begin{bmatrix} 1 & -v_i \\ 1 & -z_i \end{bmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{Z}\pi_1(N)),$$

where i runs over $1, \dots, s$ becomes:

$$\begin{aligned} \det_{\mathcal{N}(G)}^r \left(\kappa(\gamma, \phi, t) \begin{bmatrix} 1 & -v_i \\ 1 & -z_i \end{bmatrix} \right) &= \det_{\mathcal{N}(G)}^r \begin{bmatrix} 1 & -t^{\phi(v_i)} \gamma(v_i) \\ 1 & -t^{\phi(z_i)} \gamma(z_i) \end{bmatrix} \\ &= \det_{\mathcal{N}(G)}^r \\ &\quad \times \begin{bmatrix} 1 - t^{\phi(v_i z_i^{-1})} \gamma(v_i z_i^{-1}) & -t^{\phi(v_i)} \gamma(v_i) \\ 0 & -t^{\phi(z_i)} \gamma(z_i) \end{bmatrix} \\ &= \det_{\mathcal{N}(G)}^r (\kappa(\gamma, \phi, t)(z_i - v_i)). \end{aligned}$$

The elements v_i and z_i arising from Friedl’s construction satisfy $\phi_0(v_i) = r_i$ and $\phi_0(z_i) = 0$. Since (π, γ, ϕ_0) is an admissible triple and $\phi_0(v_i) - \phi_0(z_i) = r_i \neq 0$, the element $\gamma(v_i z_i^{-1})$ must have infinite order in G . Then [7, Lemma 2.8] yields

$$\begin{aligned} \det_{\mathcal{N}(G)}^r (\kappa(\gamma, \phi, t)(z_i - v_i)) &= \det_{\mathcal{N}(G)}^r \left(t^{\phi(z_i)} \gamma(z_i) - t^{\phi(v_i)} \gamma(v_i) \right) \\ &= t^{\phi(z_i)} \cdot \max \left\{ 1, t^{\phi(v_i z_i^{-1})} \right\} \\ &= \max \left\{ t^{\phi(z_i)}, t^{\phi(v_i)} \right\}. \end{aligned}$$

The point here is that we do not need to require $\phi(v_i) - \phi(z_i) \neq 0$ for all ϕ . With the modification above, we see that

$$u_1, v_1, \dots, u_s, v_s \in \pi_1(N)$$

can be taken to be $z_1, v_1, \dots, z_s, v_s$. Similarly, we take

$$u_{s+1}, v_{s+1}, \dots, u_{2s}, v_{2s} \in \pi_1(N)$$

to be $x_1, v_1, \dots, x_s, v_s$ in the notations of [7, Proposition 9.1], where $\phi_0(x_i) = 0$ for all $i = 1, \dots, s$. This gives rise to a total number of $l = 2s$ pairs of u_i and v_i . The matrix A is a square matrix over $\mathbb{Z}\pi_1(N)$ of the form

$$\begin{bmatrix} \mathbf{1}_{n_1 \times n_1} & -v_1 \cdot \mathbf{1}_{n_1 \times n_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & \mathbf{1}_{n_s \times n_s} & -v_s \cdot \mathbf{1}_{n_s \times n_s} & 0 \\ * & \cdots & \cdots & * & * & * & * \end{bmatrix},$$

where $n_i = -\chi(\Sigma_i) + 2$, and $*$ stand for (not necessarily square) blocks with entries in $\mathbb{Z}\text{Ker}(\phi_0)$, and $\phi_0(v_i) = r_i$.

One can further manipulate the matrix A into the asserted form without affecting the regular Fuglede–Kadison determinant under $\kappa(\phi, \gamma, t)$. This can be done by adding diagonal $\mathbf{1}_{1 \times 1}$ blocks and performing elementary transformations using well known tricks, so we omit the details, cf. [7, Proposition 9.3]. \square

8.2 Degree for residually finite twists

In this subsection, we prove Theorem 1.4. Throughout this subsection, let N be an irreducible orientable compact 3-manifold with empty or incompressible toral boundary, and $\gamma: \pi_1(N) \rightarrow G$ be a homomorphism. Suppose that G is finitely generated and residually finite and (N, γ) is weakly acyclic.

For any admissible triple $(\pi_1(N), \gamma, \phi)$ over \mathbb{R} , denote by

$$\tau^{(2)}(N, \gamma, \phi): \mathbb{R}_+ \rightarrow [0, +\infty)$$

any representative of the associated L^2 -Alexander torsion.

Lemma 8.2 *Given any admissible triple $(\pi_1(N), \gamma, \phi)$ over \mathbb{R} ,*

$$\tau^{(2)}(N, \gamma, \phi)(1) > 0.$$

Proof As (N, γ) is weakly acyclic, it follows from the definition that $\tau^{(2)}(N, \gamma, \phi)(1)$ is the L^2 -torsion of the pair (N, γ) , namely, the L^2 -torsion of the covering space of N which corresponds to $\text{Ker}(\gamma)$ equipped with the action of $\text{Im}(\gamma)$. The latter can be computed through a weakly acyclic Hilbert chain complex of which the boundary operators are represented by matrices over $\mathbb{Z}\text{Im}(\gamma)$. As G is residually finite, [22, Theorem 3.4(2)] implies that $\tau^{(2)}(N, \gamma)$ is a multiplicatively alternating product of positive constants which are no smaller than 1, hence must be nonzero. \square

Lemma 8.3 *Let $u_1, v_1, \dots, u_l, v_l \in \pi_1(N)$ be a collection of elements and A be a square matrix over $\mathbb{Z}\pi_1(N)$ as asserted by Lemma 8.1. Given any admissible triple $(\pi_1(N), \gamma, \phi)$ over \mathbb{R} , the following formula holds valid and true:*

$$\text{deg}^b(\tau^{(2)}(N, \gamma, \phi)) = \text{deg}^b\left(\det_{N(G)}^{\mathbb{F}}(\kappa(\phi, \gamma, t)(A))\right) - \sum_{i=1}^l |\phi(u_i) - \phi(v_i)|.$$

Proof The function $\det_{N(G)}^{\mathbb{F}}(\kappa(\phi, \gamma, t)(A))$ of $t \in \mathbb{R}_+$ is multiplicatively convex by Theorem 5.1. In fact, it is nowhere zero and hence with bounded exponent by Lemmas 4.2, 8.1, and 8.2. Thus it is valid to speak of $\text{deg}^b(\tau^{(2)}(N, \gamma, \phi))$ and the formula follows immediately from Lemma 8.1. \square

Proof of Theorem 1.4 We continue to adopt the assumptions of this subsection. It follows from Lemmas 8.1, 8.2, and Theorem 5.1 that $\tau^{(2)}(N, \gamma, \phi)$ is everywhere positive and continuous in $t \in \mathbb{R}_+$. For any constants $a, b \in \mathbb{R}$, note that the function $\max\{t^a, t^b\}^{-1}$ can always be turned into a multiplicatively convex function by multiplying a sufficiently high power of $\max\{1, t\}$, for example, by making the power at least $|a - b|$. It further follows that $\tau^{(2)}(N, \phi) \cdot \max\{1, t\}^m$ is multiplicatively convex with bounded exponent for any sufficiently large positive constant m . The Lipschitz continuity of $\text{deg}^b(\tau^{(2)}(N, \gamma, \phi + \gamma^*\xi))$ as a function of $\xi \in H^1(G; \mathbb{R})$ is a consequence of Theorem 6.1. Therefore, it remains to show that for all admissible triple (N, γ, ϕ) , the following comparison holds true:

$$\text{deg}^b(\tau^{(2)}(N, \gamma, \phi)) \leq x_N(\phi).$$

To this end, we first prove the comparison for any admissible triple (N, γ, ϕ_0) where ϕ_0 is a primitive class in $H^1(N; \mathbb{Z})$. Let $u_1, v_1, \dots, u_l, v_l \in \pi_1(N)$ be a collection of elements and A be a square matrix over $\mathbb{Z}\pi_1(N)$ as guaranteed by the ‘moreover’ part of Lemma 8.1. It is clear that for any arbitrary $\delta > 0$,

$$\lim_{t \rightarrow 0^+} \det_{\mathcal{N}(G)}^r(\kappa(\gamma, \phi_0, t)(A)) \cdot t^\delta = 0,$$

and

$$\lim_{t \rightarrow +\infty} \det_{\mathcal{N}(G)}^r(\kappa(\gamma, \phi_0, t)(A)) \cdot t^{-k-\delta} = 0,$$

so

$$\text{deg}^b\left(\det_{\mathcal{N}(G)}^r(\kappa(\gamma, \phi_0, t)(A))\right) \leq k.$$

On the other hand, the integrality of ϕ_0 and the property that $\phi_0(u_i) \neq \phi_0(v_i)$ imply

$$\sum_{i=1}^l |\phi_0(u_i) - \phi_0(v_i)| \geq l.$$

Then Lemma 8.3 yields the comparison

$$\text{deg}^b\left(\tau^{(2)}(N, \gamma, \phi_0)\right) \leq k - l = x_N(\phi_0).$$

For admissible triples over \mathbb{Q} , the comparison follows immediately from the integral case by considering an integral multiple of ϕ . For admissible triples over \mathbb{R} , the comparison follows from the continuity of degree together with the continuity of Thurston norm.

This completes the proof of Theorem 1.4. □

8.3 Degree for the full twist

In this subsection, we prove Theorem 1.2. Suppose that N is an irreducible orientable compact 3-manifold with empty or incompressible toral boundary. When N contains no hyperbolic piece in its geometric decomposition, N is a graph manifold, possibly a Seifert fibered space. Theorem 1.2 in this case is an immediate consequence of [7, Theorem 1.2], [17].

Therefore, throughout this section, we assume that N contains at least one hyperbolic piece, or in other words, N is either hyperbolic or so-called mixed. Note that N is aspherical so the ℓ^2 -Betti numbers of N all vanish, by Lott-Lück [21]. For any class $\phi \in H^1(\pi_1(N); \mathbb{R})$, any representative of the associated full L^2 -Alexander torsion

$$\tau^{(2)}(N, \phi): \mathbb{R}_+ \rightarrow [0, +\infty)$$

is everywhere positive and continuous, and $\deg^b(\tau^{(2)}(N, \phi)) \in \mathbb{R}$ is at most $x_N(\phi)$ (Theorem 1.4). It remains to determine the asymptotics as the parameter t tends to $+\infty$ or $0+$.

Recall that a class $\phi \in H^1(N; \mathbb{R})$ is said to be *quasi-fibered* if ϕ is the limit of a sequence of fibered classes in $H^1(N; \mathbb{Q})$.

Lemma 8.4 *Let G be a finitely generated, residually finite group. For every homomorphism $\gamma: \pi_1(N) \rightarrow G$ which induces an isomorphism under $H_1(-; \mathbb{R})$, and for every quasi-fibered class $\phi \in H^1(N; \mathbb{R})$,*

$$\deg^b\left(\tau^{(2)}(N, \gamma, \phi)\right) = x_N(\phi).$$

Proof Note that $(\pi_1(N), \gamma, \phi)$ is always admissible regardless of ϕ by Lemma 2.2. If $\phi \in H_1(N; \mathbb{Q})$ is a rational, fibered class, the conclusion follows from [7, Theorem 1.3]. In fact, for such ϕ , the L^2 -Alexander torsion $\tau^{(2)}(N, \gamma, \phi)$ is known to be asymptotically monomial, (indeed, eventually monomial, by [7, Theorem 1.3]) so in this case,

$$\deg^b\left(\tau^{(2)}(N, \gamma, \phi)\right) = \deg^a\left(\tau^{(2)}(N, \gamma, \phi)\right) = x_N(\phi),$$

cf. Definitions 1.1 and 1.3.

For any quasi-fibered class $\phi \in H_1(N; \mathbb{R})$, we take a sequence of rational, fibered classes $\{\phi_n\}_{n \in \mathbb{N}}$ which converges to ϕ . Then by the continuity of degree (Theorem 1.4 (3)) and the formula of Lemma 8.1, we see that

$$\begin{aligned} \deg^b\left(\tau^{(2)}(N, \gamma, \phi_n)\right) &= \lim_{n \rightarrow \infty} \deg^b\left(\tau^{(2)}(N, \gamma, \phi_n)\right) \\ &= \lim_{n \rightarrow \infty} x_N(\phi_n) = x_N(\phi). \end{aligned}$$

This completes the proof. □

Let $u_1, v_1, \dots, u_l, v_l \in \pi_1(N)$ be a collection of elements and A be a square matrix over $\mathbb{Z}\pi_1(N)$ as asserted by Lemma 8.1.

Lemma 8.5 *Given any class $\phi \in H^1(N; \mathbb{R})$, there exists a tower of quotients of $\pi_1(N)$*

$$\pi_1(N) \rightarrow \dots \rightarrow \Gamma_n \rightarrow \dots \rightarrow \Gamma_2 \rightarrow \Gamma_1$$

with all the following properties:

- The quotients Γ_n are finitely generated and virtually abelian.
- The homomorphisms $\gamma_n: \pi_1(N) \rightarrow \Gamma_n$ induce isomorphisms under $H_1(-; \mathbb{R})$.
- The sequence of admissible triples $\{(\pi_1(N), \gamma_n, \phi)\}_{n \in \mathbb{N}}$ forms a cofinal tower of quotients of $(\pi_1(N), \gamma_\infty, \phi)$, where γ_∞ denotes $\text{id}_{\pi_1(N)}: \pi_1(N) \rightarrow \pi_1(N)$.

Furthermore, the tower can be required to satisfy:

$$\text{deg}^b(V_n) = \text{deg}^b(V_\infty)$$

for all $n \in \mathbb{N}$, where

$$V_n(t) = \det_{\mathcal{N}(\Gamma_n)}^{\mathbb{F}}(\kappa(\phi, \gamma_n, t)(A)),$$

and the notation $V_\infty(t)$ is understood similarly.

Proof As we have assumed for this section that N is either hyperbolic or mixed, there exists a regular finite cover $p: \tilde{N} \rightarrow N$ which corresponds to a finite index subgroup $\tilde{\pi}$ of $\pi_1(N)$, such that $p^*\phi \in H^1(\tilde{N}; \mathbb{R})$ is quasi-fibered. This follows from a combination of Agol’s RFRS criterion for virtual fibering [1] and the virtual specialness of hyperbolic and mixed 3-manifolds [2, 27, 32], cf. [7, Sect. 10.1]. Observe that for any further subgroup of finite index in $\tilde{\pi}$ which is normal in $\pi_1(N)$, the corresponding finite cover again carries the pull-back of ϕ as a quasi-fibered class.

Take a cofinal tower of normal finite-index subgroups of $\pi_1(N)$,

$$\pi_1(N) \geq \Pi_1 \geq \Pi_2 \geq \dots \geq \Pi_n \geq \dots$$

Possibly after intersecting the terms with $\tilde{\pi}$, we may require that Π_n are all contained in $\tilde{\pi}$. For all $n \in \mathbb{N}$, define

$$\Gamma_n = \pi_1(N) / (\text{Ker}(\Pi_n \rightarrow H_1(\Pi_n; \mathbb{Q}))).$$

All the asserted properties of Lemma 8.5 hold obviously true for the tower of quotients $\{\Gamma_n\}$, except maybe the ‘furthermore’ part.

To check the equality of degree, denote by

$$p_n : \tilde{N}_n \rightarrow N$$

the finite cover corresponding to the image of Π_n in Γ_n . Taking restriction to $\pi_1(\tilde{N}_n)$ gives rise to new admissible triples $(\pi_1(\tilde{N}_n), \tilde{\gamma}_n, p_n^*\phi)$. By the dotted equality of Lemma 8.1, and basic properties of regular Fuglede–Kadison determinants, and Lemma 8.4, for all $n \in \mathbb{N}$,

$$\begin{aligned} \text{deg}^b \left(\tau^{(2)}(N, \gamma_n, \phi) \right) &= \frac{1}{[\tilde{N} : N]} \cdot \text{deg}^b \left(\tau^{(2)} \left(\tilde{N}_n, \tilde{\gamma}_n, p_n^*\phi \right) \right) \\ &= \frac{1}{[\tilde{N} : N]} \cdot x_{\tilde{N}_n} (p_n^*\phi) = x_N(\phi). \end{aligned}$$

Note that the calculation above does not require the target group to be virtually abelian. Therefore, the same calculation for $\tau^{(2)}(N, \gamma_\infty, \phi)$ yields the equality

$$\text{deg}^b \left(\tau^{(2)}(N, \gamma_\infty, \phi) \right) = x_N(\phi).$$

It follows from Lemma 8.3 that

$$\text{deg}^b(V_n) = \text{deg}^b(V_\infty)$$

for all $n \in \mathbb{N}$. □

Proof of Theorem 1.2 We continue to adopt the assumptions of this subsection. It suffices to prove the statements (2)–(4).

Given N hyperbolic or mixed and any $\phi \in H^1(N; \mathbb{R})$, we take a tower of quotients as guaranteed by Lemma 8.5. By Theorem 7.2, we see that the function (now dropping the subscript ∞)

$$V(t) = \det_{\mathcal{N}(\Gamma_n)}^r \left(\kappa \left(\phi, \text{id}_{\pi_1(N)}, t \right) (A) \right)$$

is asymptotically monomial in both ends. In fact, as $t \rightarrow +\infty$,

$$V(t) \sim C_{+\infty} \cdot t^{\text{deg}_{+\infty}^b(V)}$$

for some constant

$$C_{+\infty} \in \left[1, e^{\text{Vol}(N)/6\pi} \right],$$

and the same statement holds true with $+\infty$ replaced by $0+$. Here the upper bound comes from

$$V(1) = \tau^{(2)}(N, \phi)(1) = \tau^{(2)}(N) = e^{\text{Vol}(N)/6\pi}.$$

Therefore, $\tau^{(2)}(N, \phi)$ is also asymptotically monomial in both ends with the same estimation of coefficients. In particular, the asymptote degree of $\tau^{(2)}(N, \phi)$ is valid, and

$$\text{deg}^a \left(\tau^{(2)}(N, \phi) \right) = \text{deg}^b \left(\tau^{(2)}(N, \phi) \right) = x_N(\phi).$$

By the symmetry of L^2 -Alexander torsion for 3-manifolds [9], we further imply

$$C_{+\infty} = C_{0+}.$$

This allows us to refer to both of them by one notation:

$$C(N, \phi) \in \left[1, e^{\text{Vol}(N)/6\pi} \right].$$

It remains to argue that $C(N, \phi)$ depends upper semi-continuously on $\phi \in H^1(N; \mathbb{R})$. In fact, suppose that $\{\phi_n \in H^1(N; \mathbb{R})\}_{n \in \mathbb{N}}$ is a sequence of cohomology classes which converges to ϕ . We write

$$V(\phi_n, t) = \det_{\mathcal{N}(\Gamma_n)}^x \left(\kappa(\phi_n, \text{id}_{\pi_1(N)}, t)(A) \right).$$

By Lemma 3.1, for all $t \in \mathbb{R}_+$,

$$\limsup_{n \rightarrow \infty} V(\phi_n, t) \leq V(t).$$

By the continuity of degree (Theorem 6.1),

$$\lim_{n \rightarrow \infty} \text{deg}^b(V(\phi_n, t)) = \text{deg}^b(V(t))$$

Then it follows from Lemma 7.3 that

$$C(N, \phi) \geq \limsup_{n \rightarrow \infty} C(N, \phi_n).$$

In other words, the leading coefficient $C(N, \phi)$ is upper semicontinuous as a function of $\phi \in H^1(N; \mathbb{R})$.

This completes the proof of Theorem 1.2. □

9 Example

We conclude our discussion with an example regarding nontrivial leading coefficients. Specifically, we construct an oriented closed 3-manifold N such that the leading coefficient $C(N, \phi)$ of the full L^2 -Alexander torsion $\tau^{(2)}(N, \phi)$ gives rise to values other than the asserted bounds, as ϕ varies over $H^1(N; \mathbb{R})$.

The oriented closed 3-manifold

$$N = K \cup \bigcup_{i \in \mathbb{Z}/3\mathbb{Z}} J_i$$

is constructed by gluing a product piece K and three figure-eight knot complements J_i as follows. Let

$$K \cong \Sigma_{0,3} \times S^1$$

be the product of the thrice holed sphere and the circle. We mark the boundary components of $\Sigma_{0,3}$ in cyclic order. For each $i \in \mathbb{Z}/3\mathbb{Z}$, denote by $\partial_i K \cong \partial_i \Sigma_{0,3} \times S^1$ the i -th boundary component of K accordingly. For each $i \in \mathbb{Z}/3\mathbb{Z}$, take a copy of a figure-eight knot complement

$$J_i \cong S^3 \setminus \text{Nhd}^\circ(\mathbf{4}_1).$$

We remind the reader that the interior of the figure-eight complement J_i is a punctured torus bundle over the circle with a pseudo-Anosov monodromy, and it has a unique complete hyperbolic structure of volume $\text{Vol}(J_i) = 2v_3$, where $v_3 \approx 1.01494$ is the volume of the regular ideal hyperbolic tetrahedron. Denote by μ_i and λ_i the longitude and the meridian of J_i accordingly, so that the boundary of J_i has a canonical product structure $\partial J_i \cong \lambda_i \times \mu_i$. Endow K and J_i with canonical orientations so that the boundary is oriented accordingly. The oriented closed 3-manifold N is obtained by gluing K and J_i along the boundary in such a way that $\partial_i K$ is identified with $-\partial J_i$ via an isomorphism that takes the factor $\partial_i \Sigma_{0,3}$ to λ_i and the factor S^1 to $-\mu_i$.

Note that the inclusion maps induce an embedding

$$\begin{aligned} H^1(N; \mathbb{R}) &\rightarrow H^1(J_0; \mathbb{R}) \oplus H^1(J_1; \mathbb{R}) \oplus H^1(J_2; \mathbb{R}) \\ \phi &\mapsto (\phi_0, \phi_1, \phi_2) \end{aligned}$$

By identifying $H^1(J_i; \mathbb{R})$ with \mathbb{R} , we can identify $H^1(N; \mathbb{R})$ with the 2-subspace of the 3-space given by the linear equation:

$$\phi_0 + \phi_1 + \phi_2 = 0.$$

By the fibration structure of the figure-eight complement, it is easy to argue topologically that the Thurston norm of any cohomology class ϕ in $H^1(N; \mathbb{R})$ is given by the formula:

$$x_N(\phi) = |\phi_0| + |\phi_1| + |\phi_2|.$$

The unit ball $B_x(N)$ of x_N is hence the region bounded by the regular hexagon whose vertices are $(\pm\frac{1}{2}, \mp\frac{1}{2}, 0)$, $(0, \pm\frac{1}{2}, \mp\frac{1}{2})$, and $(\mp\frac{1}{2}, 0, \pm\frac{1}{2})$. There are no fibered cones because the restriction of every primitive class $\phi \in H^1(N; \mathbb{Z})$ to K vanishes on the Seifert fiber $[S^1] \in H_1(K; \mathbb{Z})$, which means no subsurface that is dual to ϕ could be transverse to the Seifert fibration everywhere (or so-called horizontal) restricted to K .

The full L^2 -Alexander torsion of N associated with any cohomology class $\phi \in H^1(N; \mathbb{R})$ can be calculated by the formula:

$$\tau^{(2)}(N, \phi) \doteq \tau^{(2)}(J_0, \phi_0) \cdot \tau^{(2)}(J_1, \phi_1) \cdot \tau^{(2)}(J_2, \phi_2).$$

This follows from [23, Theorem 3.35(1)] (see [23, Theorem 3.93(2)] for a similar calculation). Note that in our case, the pieces K and J_i are weakly acyclic glued along tori which contribute nothing to the L^2 -torsion of the twisted chain complex. There ought to be a factor $\tau^{(2)}(K, \phi_K)$ corresponding to the restriction of ϕ to K on the right-hand side, but that factor is represented by 1 according to [17], cf. [7, Theorem 1.2]. For each $i \in \mathbb{Z}/3\mathbb{Z}$, it follows from the fiberedness of the figure-eight knot complement that the leading coefficient

$$C(J_i, \phi_i) = \begin{cases} e^{v_3/3\pi} & \phi_i = 0 \\ 1 & \phi_i \neq 0 \end{cases}$$

Therefore, for any cohomology class $\phi = (\phi_1, \phi_2, \phi_3) \in H^1(N; \mathbb{R})$, the leading coefficient of $\tau^{(2)}(N, \phi)$ is given by the formula:

$$C(N, \phi) = e^{\frac{\delta(\phi) \cdot v_3}{3\pi}}$$

where $\delta(\phi)$ denotes the number of zero coordinates in (ϕ_0, ϕ_1, ϕ_2) subject to the constraint $\phi_0 + \phi_1 + \phi_2 = 0$. To summarize, the leading coefficient $C(N, \phi)$ equals $e^{\text{Vol}(N)/6\pi}$ at the origin, and $e^{\text{Vol}(N)/18\pi}$ along the six radial rays through the vertices of $B_x(N)$ (except at the origin), and 1 in the rest part of $H^1(N; \mathbb{R})$.

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