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Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations

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Abstract In this paper, we prove the existence of global weak solutions for 3D compressible Navier–Stokes equations with degenerate viscosity. The method is based on the Bresch and Desjardins (Commun Math Phys 238:211–223 2003) entropy conservation. The main contribution of this paper is to derive the Mellet and Vasseur (Commun Partial Differ Equ 32:431–452, 2007) type inequality for weak solutions, even if it is not verified by the first level of approximation. This provides existence of global solutions in time, for the compressible barotropic Navier–Stokes equations. The result holds for any $\gamma > 1$ in two dimensional space, and for $1 < \gamma < 3$ in three dimensional space, in both case with large initial data possibly vanishing on the vacuum. This solves an open problem proposed by Lions (Mathematical topics in fluid mechanics. Vol. 2. Compressible models, 1998).

Mathematics Subject Classification 35Q35 · 76N10

1 Introduction

The existence of global weak solutions of compressible Navier–Stokes equations with degenerate viscosity has been a long standing open problem. The objective of this current paper is to establish the existence of global weak solutions to the following 3D compressible Navier–Stokes equations:

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$$\begin{aligned} &\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ &(\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P - 2\nu\mathrm{div}(\rho\mathbb{D}\mathbf{u}) = 0, \end{aligned} \tag{1.1}$$

with initial data

$$\rho|_{t=0} = \rho_0(x), \quad \rho\mathbf{u}|_{t=0} = \rho_0\mathbf{u}_0, \tag{1.2}$$

where $P = \rho^\gamma$, $\gamma > 1$, denotes the pressure, ρ is the density of fluid, $\mathbf{u}$ stands for the velocity of fluid, $\mathbb{D}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + \nabla^T\mathbf{u}]$ is the strain tensor. For the sake of simplicity we will consider the case of bounded domain with periodic boundary conditions, namely $\Omega = \mathbb{T}^3$.

In the case $\gamma = 2$ in two dimensional space, this corresponds to the shallow water equations, where $\rho(t,x)$ stands for the height of the water at position x, and time t, and $\mathbf{u}(t,x)$ is the 2D velocity at the same position, and same time. In this case, the physical viscosity was formally derived as in (1.1) (see Gent [15]). In this context, the global existence of weak solutions to equations (1.1) is proposed as an open problem by Lions in [27]. A careful derivation of the shallow water equations with the following viscosity term

$$2\nu\mathrm{div}(\rho\mathbb{D}\mathbf{u}) + 2\nu\nabla(\rho\mathrm{div}\mathbf{u})$$

can be found in the recent work by Marche [28]. Bresch and Noble [5,6] provided the mathematical derivation of viscous shallow-water equations with the above viscosity. However, this viscosity cannot be covered by the BD entropy.

Compared with the incompressible flows, dealing with the vacuum is a very challenging problem in the study of the compressible flows. Kazhikhov and Shelukhin [25] established the first existence result on the compressible Navier–Stokes equations in one dimensional space. Due to the difficulty from the vacuum, the initial density should be bounded away from zero in their work. It has been extended by Serre [34] and Hoff [20] for the discontinuous initial data, and by Mellet and Vasseur [33] in the case of density dependent viscosity coefficient, see also in spherically symmetric case [10,11,18]. For the multidimensional case, Matsumura and Nishida [29–31] first established the global existence with the small initial data, and later by Hoff [21–23] for discontinuous initial data. To remove the difficulty from the vacuum, Lions in [27] introduced the concept of renormalized solutions to establish the global existence of weak solutions for $\gamma > \frac{9}{5}$ concerning large initial data that may vanish, and then Feireisl et al. [13] and Feireisl [14] extended the existence results to $\gamma > \frac{3}{2}$, and even to Navier–Stokes-Fourier system. In all above works, the viscosity coefficients were assumed to be fixed positive numbers. This is important to control the gradient of the velocity, in the context of solutions close to an equilibrium, a breakthrough was obtained by Danchin [8,9]. However, the regularity and the uniqueness of the weak solutions for large data remain largely open for the compressible Navier–Stokes equations, even as in two dimensional space, see Vaigant and Kazhikhov [37] (see also Germain [16], and Haspot [19], where criteria for regularity or uniqueness are proposed).

The problem becomes even more challenging when the viscosity coefficients depend on the density. Indeed, the Navier–Stokes equations (1.3) is highly degenerated at the vacuum because the velocity cannot even be defined when the density vanishes. It is very difficult to deduce any estimate of the gradient on the velocity field due to the vacuum. This is the essential difference from the compressible Navier–Stokes equations with the non-density dependent viscosity coefficients. The first tool of handling this difficulty is due to Bresch et al., see [3], where the authors developed a new mathematical entropy to show the structure of the diffusion terms providing some regularity for the density. An early version of this entropy can be found in 1D for constant viscosity in [35,36]. The result was later extended for the case with an additional quadratic friction term $r\rho|\mathbf{u}|\mathbf{u}$, refer to Bresch and Desjardins [1,2] and the recent results by Bresch et al. [4] and by Zatorska [39]. Unfortunately, those bounds are not enough to treat the compressible Navier–Stokes equations without additional control on the vacuum, as the introduction of capillarity, friction, or cold pressure.

The primary obstacle to prove the compactness of the solutions to (1.3) is the lack of strong convergence for $\sqrt{\rho}\mathbf{u}$ in L^2. We cannot pass to the limit in the term $\rho\mathbf{u}\otimes\mathbf{u}$ without the strong convergence of $\sqrt{\rho}\mathbf{u}$ in L^2. This is an other essential difference with the case of non-density dependent viscosity. To solve this problem, a new estimate is established in Mellet and Vasseur [32], providing a $L^\infty(0, T; L\log L(\Omega))$ control on $\rho|\mathbf{u}|^2$. This new estimate provides the weak stability of smooth solutions of (1.3).

The classical way to construct global weak solutions of (1.3) would consist in constructing smooth approximation solutions, verifying the priori estimates, including the Bresch–Desjardins entropy, and the Mellet–Vasseur inequality. However, those extra estimates impose a lot of structure on the approximating system. Up to now, no such approximation scheme has been discovered. In [1,2], Bresch and Desjardins propose a very nice construction of approximations, controlling both the usual energy and BD entropy. This allows the construction of weak solutions, when additional terms—as drag terms, or cold pressure, for instance—are added. Note that their result holds true even in dimension 3. However, their construction does not provide the control of the $\rho\mathbf{u}$ in $L^\infty(0, T; L\log L(\Omega))$.

The objective of our current work is to investigate the issue of existence of solutions for the compressible Navier–Stokes equations (1.1) with large initial data in 3D. Jungel [24] studied the compressible Navier–Stokes equations with the Bohm potential $\kappa\rho\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$, and obtained the existence of a particular weak solution. Moreover, he deduced an estimate of $\nabla\rho^{\frac{1}{4}}$ in $L^4((0,T)\times\Omega)$, which is very useful in this current paper. In [17], Gisclon and Lacroix-Violet showed the existence of usual weak solutions for the compressible quantum Navier–Stokes equations with the addition of a cold pressure. Independently, we proved the existence of weak solutions to the compressible quantum Navier–Stokes equations with damping terms, see [38]. This result is very similar to [17]. Actually, it is written in [17] that they can handle in a similar way the case with the drag force. Unfortunately, the case with the cold pressure is not suitable for our purpose.

Building up from the result [38] (a variant of [17]), we establish the logarithmic estimate for the weak solutions similar to [32]. For this, we first derive a "renormalized" estimate on $\rho\varphi(|\mathbf{u}|)$, for φ nice enough, for solutions of [38] with the additional drag forces. It is showed to be independent on the strength of those drag forces, allowing to pass into the limit when those forces vanish. Since this estimate cannot be derived from the approximation scheme of [38], it has to be carefully derived on weak solutions. After passing into the limit κ goes to 0, we can recover the logarithmic estimate, taking a suitable function φ. This is reminiscent to showing the conservation of the energy for weak solutions to incompressible Navier–Stokes equations. This conservation is true for smooth solutions. However, it is a long standing open problem, whether Leray–Hopf weak solutions are also conserving energy.

Equation (1.1) can be seen as a particular case of the following Navier–Stokes

$$\begin{aligned} \rho_t + \operatorname{div}(\rho\mathbf{u}) &= 0 \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P - 2\operatorname{div}(\mu(\rho)\mathbb{D}\mathbf{u}) - \nabla(\lambda(\rho)\operatorname{div}\mathbf{u}) &= 0, \end{aligned} \tag{1.3}$$

where the viscosity coefficients $\mu(\rho)$ and $\lambda(\rho)$ depend on the density, and may vanish on the vacuum. When the coefficients verify the following condition:

$$\lambda(\rho) = 2\rho\mu^{'}(\rho) - 2\mu(\rho)$$

the system still formally verifies the BD estimate. However, the construction of Bresch and Desjardins in [2] is more subtle in this case. Up to now, construction of weak solutions are known, only verifying a fixed combination of the classical energy and BD entropy (see [4]) in the case with additional terms. Those solutions verify the decrease of this so-called κ-entropy,[1] but not the decrease of Energy and BD entropy by themselves. The extension of our result, in this context, is considered in [7].

[1] Note that κ here is not related to the κ term in (1.6).

Without loss of generality, we will fix $2\nu = 1$ from now on. Note, that we can recover the general case by the simple change of variables $(\bar{\rho}, \bar{\mathbf{u}})(t, x) = (\rho, \mathbf{u})(2\nu t, 2\nu x)$.

The basic energy inequality associated to (1.1) reads as

$$E(t) + \int_0^T \int_\Omega \rho |\mathbb{D}\mathbf{u}|^2 \, dx\, dt \le E_0, \tag{1.4}$$

where

$$E(t) = E(\rho, \mathbf{u})(t) = \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma - 1}\rho^\gamma \right) dx,$$

and

$$E_0 = E(\rho, \mathbf{u})(0) = \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma - 1}\rho_0^\gamma \right) dx.$$

Remark that those a priori estimates are not enough to show the stability of the solutions of (1.1), in particular, for the compactness of ρ^γ. Fortunately, a particular mathematical structure was found in [1,3], which yields the bound of $\nabla \rho^{\frac{\gamma}{2}}$ in $L^2(0, T; L^2(\Omega))$. More precisely, we have the following Bresch-Desjardins entropy

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u} + \nabla \ln \rho|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \int_0^T \int_\Omega |\nabla \rho^{\frac{\gamma}{2}}|^2 \, dx\, dt \\ &\quad + \int_0^T \int_\Omega \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 \, dx\, dt \le \int_\Omega \left(\rho_0|\mathbf{u}_0|^2 + |\nabla\sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) dx. \end{aligned}$$

Thus, the initial data should be given in such way that

$$\begin{aligned} &\rho_0 \in L^1(\Omega) \cap L^\gamma(\Omega), \quad \rho_0 \ge 0 \text{ a.e. in } \Omega, \quad \nabla\sqrt{\rho_0} \in L^2(\Omega), \\ &\rho_0|\mathbf{u}_0|^2 \in L^1(\Omega). \end{aligned} \tag{1.5}$$

Remark 1.1 The initial condition $\nabla\sqrt{\rho_0} \in L^2(\Omega)$ comes from the Bresch-Desjardins entropy.

The primary obstacle to prove the compactness of the solutions to (1.6) with $r_0 = r_1 = 0$ is the lack of strong convergence for $\sqrt{\rho}\mathbf{u}$ in L^2. Jüngel proved in [24] the existence of particular weak solutions with test function $\rho\varphi$, which has been introduced in [3]. The main idea of his paper is to rewrite quantum Navier–Stokes equations as a viscous quantum Euler system by means of the effective velocity. He also proved Inequality (1.9) in [24]. This is crucial to get a key lemma in this current paper. Motivated by the works of [1,3,24], we proved in [38] the existence of weak solutions to (1.6), and Inequality (1.9). The r_0 and r_1 terms provide compactness on $\rho\mathbf{u}\otimes\mathbf{u}$ in L^1, and the strong convergence of $\sqrt{\rho}\mathbf{u}$ in L^2. Let us recall the following existence result from [38].

Proposition 1.1 *For any $\kappa \geq 0$, there exists a global weak solution to the following system*

$$\begin{aligned}&\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0,\\&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho^\gamma - \operatorname{div}(\rho\mathbb{D}\mathbf{u})\\&\quad = -r_0\mathbf{u} - r_1\rho|\mathbf{u}|^2\mathbf{u} + \kappa\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right),\end{aligned} \tag{1.6}$$

with the initial data (1.2) *and satisfying* (1.5) *and* $-r_0\int_\Omega \log_-\rho_0\,dx < \infty$. *In particular, we have the energy inequality*

$$\begin{aligned}&E(t) + \int_0^T\int_\Omega \rho|\mathbb{D}\mathbf{u}|^2\,dx\,dt + r_0\int_0^T\int_\Omega |\mathbf{u}|^2\,dx\,dt\\&+r_1\int_0^T\int_\Omega \rho|\mathbf{u}|^4\,dx\,dt \leq E_0,\end{aligned} \tag{1.7}$$

where

$$E(t) = E(\rho,\mathbf{u})(t) = \int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{\kappa}{2}|\nabla\sqrt{\rho}|^2\right)dx,$$

and

$$E_0 = E(\rho,\mathbf{u})(0) = \int_\Omega\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma-1}\rho_0^\gamma + \frac{\kappa}{2}|\nabla\sqrt{\rho_0}|^2\right)dx,$$

and the BD-entropy

$$\begin{aligned}&\int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}+\nabla\ln\rho|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\kappa}{2}|\nabla\sqrt{\rho}|^2 - r_0\log\rho\right)dx\\&+\int_0^T\int_\Omega|\nabla\rho^{\frac{\gamma}{2}}|^2\,dx\,dt + \int_0^T\int_\Omega\rho|\nabla\mathbf{u}-\nabla^T\mathbf{u}|^2\,dx\,dt\\&+\kappa\int_0^T\int_\Omega\rho|\nabla^2\log\rho|^2\,dx\,dt\\&\leq 2\int_\Omega\left(\rho_0|\mathbf{u}_0|^2 + |\nabla\sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\kappa}{2}|\nabla\sqrt{\rho_0}|^2 - r_0\log_-\rho_0\right)dx + 2E_0,\end{aligned} \tag{1.8}$$

where $\log_- g = \log\min(g,1)$.

We have the following inequality for any weak solution $(\rho, \mathbf{u})$

$$\kappa^{\frac{1}{2}}\|\sqrt{\rho}\|_{L^2(0,T;H^2(\Omega))} + \kappa^{\frac{1}{4}}\|\nabla\rho^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \leq C, \tag{1.9}$$

where C *only depends on the initial data.*

Moreover, the weak solution $(\rho, \mathbf{u})$ *has the following properties*

$$\rho\mathbf{u} \in C([0,T]; L^{\frac{3}{2}}_{weak}(\Omega)), \quad (\sqrt{\rho})_t \in L^2((0,T)\times\Omega); \tag{1.10}$$

If we use $(\rho_\kappa, \mathbf{u}_\kappa)$ *to denote the weak solution for* $\kappa > 0$*, then*

$$\sqrt{\rho_\kappa}\mathbf{u}_\kappa \to \sqrt{\rho}\mathbf{u} \quad \textit{strongly in } L^2((0,T)\times\Omega), \quad \textit{as} \ \ \kappa \to 0, \tag{1.11}$$

where $(\rho, \mathbf{u})$ *in* (1.11) *is a weak solution to Eq.* (1.6) *with initial data* (1.2) *for* $\kappa = 0$.

Remark 1.2 The energy inequality (1.7) yields the following estimates

$$\begin{aligned}
&\|\sqrt{\rho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq E_0 < \infty,\\
&\|\rho\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq E_0 < \infty,\\
&\|\sqrt{\kappa}\nabla\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))} \leq E_0 < \infty,\\
&\|\sqrt{\rho}\mathbb{D}\mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \leq E_0 < \infty,\\
&\|\sqrt{r_0}\mathbf{u}\|_{L^2(0,T;L^2(\Omega))} \leq E_0 < \infty,\\
&\|\sqrt[4]{r_1\rho}\mathbf{u}\|_{L^4(0,T;L^4(\Omega))} \leq E_0 < \infty.
\end{aligned} \tag{1.12}$$

The BD entropy (1.8) yields the following bounds on the density ρ:

$$\|\nabla\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))} \leq C < \infty, \tag{1.13}$$

$$\|\sqrt{\kappa\rho}\nabla^2\log\rho\|_{L^2(0,T;L^2(\Omega))} \leq C < \infty, \tag{1.14}$$

$$\|\nabla\rho^{\frac{\gamma}{2}}\|_{L^2(0,T;L^2(\Omega))} \leq C < \infty, \tag{1.15}$$

and

$$\begin{aligned}
\|\sqrt{\rho}\nabla\mathbf{u}\|^2_{L^2(0,T;L^2(\Omega))} \leq &\int_\Omega\left(\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla\sqrt{\rho_0}|^2 - r_0\log_-\rho_0\right)dx\\
&+2E_0 < \infty,
\end{aligned} \tag{1.16}$$

where C is bounded by the initial data, uniformly on r_0, r_1 and κ.

In fact, (1.13) yields

$$\sqrt{\rho} \in L^\infty(0,T;L^6(\Omega)), \tag{1.17}$$

in three dimensional space.

Remark 1.3 Inequality (1.9) is a consequence of the bound on (1.14). This was used already in [24]. The estimate for the full system (1.6) is proved in [38].

Remark 1.4 The weak formulation of momentum equation in (1.1) reads as

$$\begin{aligned}
&\int_\Omega \rho\mathbf{u}\cdot\psi\,dx|_{t=0}^{t=T} - \int_0^T\int_\Omega \rho\mathbf{u}\psi_t\,dx\,dt - \int_0^T\int_\Omega \rho\mathbf{u}\otimes\mathbf{u}:\nabla\psi\,dx\,dt\\
&\quad - \int_0^T\int_\Omega \rho^\gamma \mathrm{div}\psi\,dx\,dt - \int_0^T\int_\Omega \rho\mathbb{D}\mathbf{u}:\nabla\psi\,dx\,dt\\
&\quad = -r_0\int_0^T\int_\Omega \mathbf{u}\psi\,dx\,dt - r_1\int_0^T\int_\Omega \rho|\mathbf{u}|^2\mathbf{u}\psi\,dx\,dt\\
&\quad - 2\kappa\int_0^T\int_\Omega \Delta\sqrt{\rho}\nabla\sqrt{\rho}\psi\;dx\;dt\\
&\quad - \kappa\int_0^T\int_\Omega \Delta\sqrt{\rho}\sqrt{\rho}\mathrm{div}\psi\,dx\,dt.
\end{aligned} \tag{1.18}$$

for any test function ψ.

Our first main result reads as follows:

Theorem 1.1 *For any $\delta \in (0,2)$, there exists a constant C depending only on δ, such that the following holds true. There exists a weak solution $(\rho,\mathbf{u})$ to (1.6) with $\kappa = 0$ verifying all the properties of Proposition 1.1, and satisfying the following Mellet–Vasseur type inequality for every $T > 0$, and almost every $t < T$:*

$$\begin{aligned}
&\int_\Omega \rho(t,x)(1+|\mathbf{u}(t,x)|^2)\ln(1+|\mathbf{u}(t,x)|^2)\,dx\\
&\quad \le \int_\Omega \rho_0(1+|\mathbf{u}_0|^2)\ln(1+|\mathbf{u}_0|^2)\,dx\\
&\qquad + 8\int_\Omega\left(\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla\sqrt{\rho_0}|^2 - r_0\log_-\rho_0\right)dx + 16E_0\\
&\qquad + C\int_0^T\left(\int_\Omega (\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}}\left(\int_\Omega \rho(2+\ln(1+|\mathbf{u}|^2))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}\,dt,
\end{aligned}$$

where $\gamma > 1$ *in two dimensional space, and* $1 < \gamma < 3$ *in three dimensional space.*

Remark 1.5 The right hand side of the above inequality can be bounded by the initial data. In particular, it does not depend on r_0 and r_1. This theorem will provide the strong convergence of $\sqrt{\rho}\mathbf{u}$ in space $L^2(0, T; \Omega)$ when r_0, r_1 converge to 0. It is the key tool to obtain the existence of weak solutions, by following the different steps of [32].

Remark 1.6 The condition on γ are the same in Mellet and Vasseur [32]. They are needed to get a finite bound of the right hand side of the inequality. For δ small enough we need $\rho \in L^p(0, T; L^p(\Omega))$ with $p > 2\gamma - 1$. The estimate of Lemma 4.1 gives $\rho \in L^p(0, T; L^p(\Omega))$ for any $1 \leq p < \infty$ in dimension 2, and $\rho \in L^{\frac{5\gamma}{3}}(0, T; L^{\frac{5\gamma}{3}}(\Omega))$ in dimension 3. Note that $2\gamma - 1 < \frac{5\gamma}{3}$ for $\gamma < 3$.

We give the definition of the weak solution $(\rho, \mathbf{u})$ to the initial value problem (1.1)–(1.2) in the following sense: for any $t \in [0, T]$,

- (1.2) holds in $\mathcal{D}'(\Omega)$,
- (1.4) holds for almost every $t \in [0, T]$,
- (1.1) holds in $\mathcal{D}'((0, T) \times \Omega))$ and the following regularities are satisfied

$$
\begin{aligned}
&\rho \geq 0, \quad \rho \in L^\infty([0, T]; L^\gamma(\Omega)),\\
&\rho(1 + |\mathbf{u}|^2)\ln(1 + |\mathbf{u}|^2) \in L^\infty(0, T; L^1(\Omega)),\\
&\nabla\rho^{\frac{\gamma}{2}} \in L^2(0, T; L^2(\Omega)), \quad \nabla\sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)),\\
&\sqrt{\rho}\mathbf{u} \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\rho}\nabla\mathbf{u} \in L^2(0, T; L^2(\Omega)).
\end{aligned}
$$

Remark 1.7 The regularity $\nabla\sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))$ and $\nabla\rho^{\frac{\gamma}{2}} \in L^2(0, T; L^2(\Omega))$ come from the Bresch–Desjardins entropy.

As a sequence of Theorem 1.1, our second main result reads as follows:

Theorem 1.2 *Let* $(\rho_0,\ \rho_0 u_0)$ *satisfy* (1.5) *and*

$$
\int_\Omega \rho_0(1 + |\mathbf{u}_0|^2)\ln(1 + |\mathbf{u}_0|^2)\,dx < \infty.
$$

Then, for any $T > 0$*, for* $\gamma > 1$ *in two dimensional space, and* $1 < \gamma < 3$ *in three dimensional space, there exists a weak solution of* (1.1)–(1.2) *on* $(0, T)$.

We cannot obtain directly the estimate of Theorem 1.1 from (1.6) with $\kappa = 0$, because we do not have enough regularity on the solutions. But, the estimate is not true for the solutions of (1.6) for $\kappa > 0$. The idea is to obtain a control on

$$\int_\Omega \rho(t,x)\varphi_n(\phi(\rho)\mathbf{u}(t,x))\,dx$$

at the level $\kappa > 0$, for a φ_n, suitable bounded approximation of $(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)$, and a suitable cut-off function ϕ of ρ, controlling both the large and small ρ. The first step (see Sect. 2) consists in showing that we can control (uniformly with respect to κ) this quantity, for any *weak* solutions of (1.6) with $\kappa > 0$. This has to be done in several steps, taking into account the minimal regularity of the solutions, the weak control of the solutions close to the vacuum, and the extra capillarity higher order terms. In the limit κ goes to zero, the cut-off function ϕ has to converge to one in a special rate associated to κ (see Sects. 3, 4). For any weak limit to (1.6) obtained by limit κ converges to 0, this provides a (uniform in n, r_0, and r_1) bound to:

$$\int_\Omega \rho(t,x)\varphi_n(\phi(\rho)\mathbf{u}(t,x))\,dx.$$

Note that the bound is not uniform in n, for κ fixed. However, it becomes uniform in n at the limit κ converges to 0. In Sect. 5, we pass into the limit n goes to infinity, obtaining a uniform bound with respect to r_0 and r_1 of

$$\int_\Omega \rho(t,x)(1+|\mathbf{u}(t,x)|^2)\ln(1+|\mathbf{u}(t,x)|^2)\,dx.$$

Section 6 is devoted to the limit r_1 and r_0 converges to 0. The uniform estimate above provides the strong convergence of $\sqrt{\rho}\mathbf{u}$ needed to obtain the existence of global weak solutions to (1.1) with large initial data.

2 Approximation of the Mellet–Vasseur type inequality

In this section, we construct an approximation of the Mellet–Vasseur type inequality for any weak solution to the following level of approximation system

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho^\gamma - \operatorname{div}(\rho\mathbb{D}\mathbf{u})\\
&\quad = -r_0\mathbf{u} - r_1\rho|\mathbf{u}|^2\mathbf{u} + \kappa\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right),
\end{aligned} \tag{2.1}$$

with Initial data (1.5), verifying in addition that $\rho_0 \geq \frac{1}{m_0}$ for $m_0 > 0$ and $\sqrt{\rho_0}\mathbf{u}_0 \in L^\infty(\Omega)$. This restriction on the initial data will be useful later to get the strong convergence of $\sqrt{\rho}\mathbf{u}$ when t converges to 0. This restriction will be canceled at the very end, (see Sect. 6).

In the same line of Bresch–Desjardins [1,2,24], we constructed the weak solutions to the system (1.6) for any $\kappa \geq 0$ by the natural energy estimates and the Bresch–Desjardins entropy, see [38]. The term $r_1\rho|\mathbf{u}|^2\mathbf{u}$ turns out to be essential to show the strong convergence of $\sqrt{\rho}\mathbf{u}$ in $L^2(0,T;L^2(\Omega))$. Unfortunately, it is not enough to ensure the strong convergence of $\sqrt{\rho}\mathbf{u}$ in $L^2(0,T;L^2(\Omega))$ when r_0 and r_1 vanish.

We define two C^∞, nonnegative cut-off functions ϕ_m and ϕ_K as follows:

$$\phi_m(\rho)=1 \quad \text{for any } \rho>\frac{1}{m}, \quad \phi_m(\rho)=0 \quad \text{for any } \rho<\frac{1}{2m}, \tag{2.2}$$

where $m>0$ is any real number, and $|\phi_m'|\leq 2m$; and $\phi_K(\rho)\in C^\infty(\mathbb{R})$ is a nonnegative function such that

$$\phi_K(\rho)=1 \quad \text{for any } \rho<K, \quad \phi_K(\rho)=0 \text{ for any } \rho>2K, \tag{2.3}$$

where $K>0$ is any real number, and $|\phi_K'|\leq \frac{2}{K}$.

We define $\mathbf{v}=\phi(\rho)\mathbf{u}$, and $\phi(\rho)=\phi_m(\rho)\phi_K(\rho)$. The following lemma will be very useful to construct the approximation of the Mellet–Vasseur type inequality. The structure of the κ quantum term in [24] is essential to get this lemma in 3D. It seems not possible to get it from the Korteweg term of [1] in 3D.

Lemma 2.1 *For any fixed $\kappa>0$, we have*

$$\|\nabla\mathbf{v}\|_{L^2(0,T;L^2(\Omega))}\leq C,$$

where the constant C depend on $\kappa>0, r_1, K$ and m; and

$$\rho_t\in L^4(0,T;L^{6/5}(\Omega))+L^2(0,T;L^{3/2}(\Omega)) \quad \textit{uniformly in } \kappa.$$

Proof For any fixed $\kappa>0$, (1.9) gives us

$$\|\nabla\rho^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\leq C.$$

For $\mathbf{v}$, we calculate it as follows

$$\nabla\mathbf{v}=\nabla(\phi(\rho)\mathbf{u})=(\phi'(\rho)\nabla\rho)\mathbf{u}+\phi(\rho)\nabla\mathbf{u},$$

and hence

$$
\begin{aligned}
&\|(\phi'(\rho)\nabla\rho)\mathbf{u}+\phi(\rho)\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega))}\\
&\quad\le C\|\rho^{\frac14}\mathbf{u}\nabla\rho^{\frac14}\|_{L^2(0,T;L^2(\Omega))}+C\|\sqrt{\rho}\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega))}\\
&\quad\le C\|\rho^{\frac14}\mathbf{u}\|_{L^4(0,T;L^4(\Omega))}\|\nabla\rho^{\frac14}\|_{L^4(0,T;L^4(\Omega))}+C\|\sqrt{\rho}\nabla\mathbf{u}\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

where we used the definition of the function $\phi(\rho)$. Indeed, there exists $C>0$ such that

$$
\left|\phi'(\rho)\sqrt{\rho}\right|+\left|\frac{\phi(\rho)}{\sqrt{\rho}}\right|\le C
$$

for any $\rho>0$.

Meanwhile,

$$
\begin{aligned}
\rho_t&=-\nabla\rho\cdot\mathbf{u}-\rho\mathrm{div}\mathbf{u}\\
&=-2\nabla\sqrt{\rho}\cdot\rho^{\frac14}\mathbf{u}\rho^{\frac14}-\sqrt{\rho}\sqrt{\rho}\mathrm{div}\mathbf{u}=S_1+S_2.
\end{aligned}
$$

Using (1.12), (1.13) and (1.17), one obtains

$$
S_1\in L^4(0,T;L^r(\Omega))\quad\text{for } 1\le r\le\frac{6}{5}.
$$

By (1.12) and (1.17), we conclude

$$
S_2\in L^2(0,T;L^s(\Omega))\quad\text{for } 1\le s\le\frac{3}{2}.
$$

Thus, we have

$$
\rho_t\in L^4(0,T;L^r(\Omega))+L^2(0,T;L^s(\Omega)).
$$

□

We introduce a new nonnegative cut-off function φ_n which is in $C^1(\mathbb{R}^3)$:

$$
\varphi_n(\mathbf{u})=\tilde{\varphi}_n(|\mathbf{u}|^2), \tag{2.4}
$$

where $\tilde{\varphi}_n$ is given on $\mathbb{R}^+$ by

$$
\tilde{\varphi}_n''(y)\begin{cases}=\frac{1}{1+y} & \text{if } 0\le y\le n,\\ =-\frac{1}{1+y} & \text{if } n<y<C_n,\\ =0 & \text{if } y\ge C_n,\end{cases} \tag{2.5}
$$

with $\tilde{\varphi}_n'(0)=0$, $\tilde{\varphi}_n(0)=0$, and $C_n=e(1+n)^2-1$.

Here we gather the properties of the function $\tilde{\varphi}_n$ in the following lemma:

Lemma 2.2 *Let φ_n and $\tilde{\varphi}_n$ be defined as above. Then they verify*

- (a) *For any $\mathbf{u} \in \mathbb{R}^3$, we have*

$$\varphi_n''(\mathbf{u}) = 2\left(2\tilde{\varphi}_n''(|\mathbf{u}|^2)\mathbf{u} \otimes \mathbf{u} + \mathbf{I}\tilde{\varphi}_n'(|\mathbf{u}|^2)\right), \tag{2.6}$$

  where $\mathbf{I}$ is 3×3 identity matrix.
- (b) $|\tilde{\varphi}_n''(y)| \le \frac{1}{1+y}$ *for any $n > 0$ and any $y \ge 0$.*
- (c)

$$\tilde{\varphi}_n'(y) \begin{cases} = 1 + \ln(1+y) & \text{if } 0 \le y \le n, \\ = 0 & \text{if } y \ge C_n, \\ \ge 0, \text{ and } \le 1 + \ln(1+y) & \text{if } n < y \le C_n. \end{cases} \tag{2.7}$$

  In one word, $0 \le \tilde{\varphi}_n' \le 1 + \ln(1+y)$ for any $y \ge 0$, and it is compactly supported.
- *(d) For any given $n > 0$, we have*

$$|\varphi_n''(\mathbf{u})| \le 6 + 2\ln(1+n) \tag{2.8}$$

  for any $\mathbf{u} \in \mathbb{R}^3$.
- *(e)*

$$\tilde{\varphi}_n(y) = \begin{cases} (1+y)\ln(1+y) & \text{if } 0 \le y < n, \\ 2(1+\ln(1+n))y - (1+y)\ln(1+y) + 2(\ln(1+n) - n), & \text{if } n \le y \le C_n, \\ e(1+n)^2 - 2n - 2 & \text{if } y \ge C_n, \end{cases} \tag{2.9}$$

  $\tilde{\varphi}_n(y)$ is a nondecreasing function with respect to y for any fixed n, and it is a nondecreasing function with respect to n for any fixed y, and

$$\tilde{\varphi}_n(y) \to (1+y)\ln(1+y) \text{ a.e.} \tag{2.10}$$

  as $n \to \infty$.

The proof is easy. We give it in the appendix for the sake of completeness.

The first step of constructing the approximation of the Mellet–Vasseur type inequality is the following lemma:

Lemma 2.3 *For any weak solution to (2.1) constructed in Proposition 1.1, and any $\psi(t) \in \mathfrak{D}(-1, +\infty)$, we have*

$$\begin{aligned} &-\int_0^T \int_\Omega \psi_t \rho \varphi_n(\mathbf{v})\, dx\, dt + \int_0^T \int_\Omega \psi(t)\varphi_n'(\mathbf{v})\mathbf{F}\, dx\, dt \\ &\quad + \int_0^T \int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\mathbf{v}))\, dx\, dt \\ &\quad = \int_\Omega \rho_0\varphi_n(\mathbf{v}_0)\psi(0)\, dx, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} &\mathbb{S} = \rho\phi(\rho)\left(\mathbb{D}\mathbf{u} + \kappa\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathbb{I}\right), \quad \textit{and} \\ &\mathbf{F} = \rho^2\mathbf{u}\phi'(\rho)\mathrm{div}\mathbf{u} + 2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\phi(\rho) + \rho\nabla\phi(\rho)\mathbb{D}\mathbf{u} + r_0\mathbf{u}\phi(\rho) \\ &\quad + r_1\rho|\mathbf{u}|^2\mathbf{u}\phi(\rho) + \kappa\sqrt{\rho}\nabla\phi(\rho)\Delta\sqrt{\rho} + 2\kappa\phi(\rho)\nabla\sqrt{\rho}\Delta\sqrt{\rho}, \end{aligned} \tag{2.12}$$

where $\mathbb{I}$ is an identical matrix.

In this proof, κ, m and K are fixed. So the dependence of the constants appearing in this proof will not be specified.

Multiplying $\phi(\rho)$ on both sides of the second equation of (2.1), we have

$$\begin{aligned} &(\rho\mathbf{v})_t - \rho\mathbf{u}\phi'(\rho)\rho_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{v}) - \rho\mathbf{u}\otimes\mathbf{u}\nabla\phi(\rho) + 2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\phi(\rho) \\ &\quad - \mathrm{div}(\phi(\rho)\rho\mathbb{D}\mathbf{u}) + \rho\nabla\phi(\rho)\mathbb{D}\mathbf{u} + r_0\mathbf{u}\phi(\rho) + r_1\rho|\mathbf{u}|^2\mathbf{u}\phi(\rho) \\ &\quad - \kappa\nabla(\sqrt{\rho}\phi(\rho)\Delta\sqrt{\rho}) \\ &\quad + \kappa\sqrt{\rho}\nabla\phi(\rho)\Delta\sqrt{\rho} + 2\kappa\phi(\rho)\nabla\sqrt{\rho}\Delta\sqrt{\rho} = 0. \end{aligned}$$

Remark 2.1 Both $\nabla\sqrt{\rho}$ and ρ_t are functions, so the above equality are justified by regularizing ρ and passing into the limit.

Remark 2.2 At the very end of the proof, functions (5.4) will be used as ψ functions. Note that they are non-increasing. The functions $(-\psi_t)$ can be seen as regularizations of the Dirac mass $\delta(t-\tilde{t})$ at $t = \tilde{t}$, for a fixed $\tilde{t} > 0$.

We can rewrite the above equation as follows

$$(\rho\mathbf{v})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{v}) - \mathrm{div}\mathbb{S} + \mathbf{F} = 0, \tag{2.13}$$

where we used

$$\begin{aligned} &\rho\mathbf{u}\phi'(\rho)\rho_t + \rho\mathbf{u}\otimes\mathbf{u}\phi'(\rho)\nabla\rho = \rho\mathbf{u}\phi'(\rho)(\rho_t + \nabla\rho\cdot\mathbf{u}) \\ &\quad = -\rho^2\mathbf{u}\phi'(\rho)\mathrm{div}\mathbf{u}, \end{aligned}$$

and $\mathbb{S}$ and $\mathbf{F}$ are as in (2.12). Since $\sqrt{\rho}\phi(\rho)$ and $\rho\phi(\rho)$ is bounded, and by means of (1.9) and (1.12)–(1.16), we find

$$\|\mathbf{F}\|_{L^{\frac{4}{3}}(0,T;L^1(\Omega))} \leq C, \quad \|\mathbb{S}\|_{L^2(0,T;L^2(\Omega))} \leq C.$$

Note that those bounds depend on K and κ.

We first introducing a test function $\psi(t) \in \mathfrak{D}(0,+\infty)$. Essentially this function vanishes for t close $t = 0$. We will later extend the result for $\psi(t) \in \mathfrak{D}(-1,+\infty)$. We define a new function $\Phi = \overline{\psi(t)\varphi_n'(\overline{\mathbf{v}})}$, where $\overline{f(t,x)} = f * \eta_k(t,x)$, k is a small enough number. Note that, $\psi(t)$ is compactly supported in $(0,\infty)$, Φ is well defined on $(0,\infty)$ for k small enough. We use it to test (2.13) to have

$$\int_0^T \int_\Omega \overline{\psi(t)\varphi_n'(\overline{\mathbf{v}})}[(\rho\mathbf{v})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{v}) - \operatorname{div}\mathbb{S} + \mathbf{F}]\,dx\,dt = 0,$$

which in turn gives us

$$\int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{[(\rho\mathbf{v})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{v}) - \operatorname{div}\mathbb{S} + \mathbf{F}]}\,dx\,dt = 0. \tag{2.14}$$

The first term in (2.14) can be calculated as follows

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{(\rho\mathbf{v})_t}\,dx\,dt \\
&= \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})(\rho\overline{\mathbf{v}})_t\,dx\,dt + \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})[\overline{(\rho\mathbf{v})_t} - (\rho\overline{\mathbf{v}})_t]\,dx\,dt \\
&= \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})(\rho_t\overline{\mathbf{v}} + \rho\overline{\mathbf{v}}_t)\,dx\,dt + R_1 \\
&= \int_0^T \int_\Omega \psi(t)\rho_t\varphi_n'(\overline{\mathbf{v}})\overline{\mathbf{v}}\,dx\,dt + \int_0^T \int_\Omega \psi(t)\rho\varphi_n(\overline{\mathbf{v}})_t\,dx\,dt + R_1,
\end{aligned} \tag{2.15}$$

where

$$R_1 = \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})[\overline{(\rho\mathbf{v})_t} - (\rho\overline{\mathbf{v}})_t]\,dx\,dt.$$

Thanks to the first equation in (2.1), the second term in (2.14) is given by

$$\int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{v})}\,dx\,dt$$
$$= \int_0^T \int_\Omega \psi(t)\rho_t\varphi_n(\overline{\mathbf{v}})\,dx\,dt - \int_0^T \int_\Omega \psi(t)\rho_t\varphi_n'(\overline{\mathbf{v}})\overline{\mathbf{v}} + R_2, \quad (2.16)$$

and

$$R_2 = \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})[\operatorname{div}(\rho\mathbf{u}\otimes\overline{\mathbf{v}}) - \overline{\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{v})}].$$

From (2.14)–(2.16), we deduce

$$\int_0^T \int_\Omega \psi(t)(\rho\varphi_n(\overline{\mathbf{v}}))_t\,dx\,dt + R_1 + R_2 - \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{\operatorname{div}\mathbb{S}}\,dx\,dt$$
$$+ \int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{\mathbf{F}} = 0. \quad (2.17)$$

Note that, $\overline{\mathbf{v}}$ converges to $\mathbf{v}$ almost everywhere and

$$\rho\varphi_n(\overline{\mathbf{v}})\psi_t \to \rho\varphi_n(\mathbf{v})\psi_t \quad \text{in } L^1((0,T)\times\Omega).$$

So, up to a subsequence, we find

$$\int_0^T \int_\Omega (\rho\varphi_n(\overline{\mathbf{v}}))\psi_t\,dx\,dt \to \int_0^T \int_\Omega (\rho\varphi_n(\mathbf{v}))\psi_t\,dx\,dt \quad \text{as } k\to 0. \quad (2.18)$$

$\varphi_n'(\overline{\mathbf{v}})$ converges to $\varphi_n'(\mathbf{v})$ almost everywhere, and it is uniformly bounded in $L^\infty(0,T;\Omega)$, thus

$$\int_0^T \int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{\mathbf{F}} \to \int_0^T \int_\Omega \psi(t)\varphi_n'(\mathbf{v})\mathbf{F} \quad \text{as } k\to 0. \quad (2.19)$$

Noticing that

$$\nabla\mathbf{v} \in L^2(0,T;L^2(\Omega)),$$

this yields

$$\overline{\nabla\mathbf{v}} \to \nabla\mathbf{v} \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Since $\overline{\mathbb{S}}$ converges to $\mathbb{S}$ strongly in $L^2(0,T;L^2(\Omega))$, and $\varphi_n''(\overline{\mathbf{v}})$ converges to $\varphi_n''(\mathbf{v})$ almost everywhere and uniformly bounded in $L^\infty((0,T)\times\Omega)$, the following ones

$$\int_0^T\int_\Omega \psi(t)\varphi_n'(\overline{\mathbf{v}})\overline{\mathrm{div}\mathbb{S}}\,dx\,dt = -\int_0^T\int_\Omega \psi(t)\overline{\mathbb{S}} : \nabla(\varphi_n'(\overline{\mathbf{v}}))\,dx\,dt, \quad (2.20)$$

converges to

$$-\int_0^T\int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\mathbf{v}))\,dx\,dt. \quad (2.21)$$

To handle R_1 and R_2, we use the following lemma due to Lions, see [26].

Lemma 2.4 *Let $f\in W^{1,p}(\mathbb{R}^N)$, $g\in L^q(\mathbb{R}^N)$ with $1\le p,q\le\infty$, and $\frac{1}{p}+\frac{1}{q}\le 1$. Then, we have*

$$\|\mathrm{div}(fg)*w_\varepsilon-\mathrm{div}(f(g*w_\varepsilon))\|_{L^r(\mathbb{R}^N)}\le C\|f\|_{W^{1,p}(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}$$

for some $C\ge 0$ independent of ε, f and g, r is determined by $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. In addition,

$$\mathrm{div}(fg)*w_\varepsilon-\mathrm{div}(f(g*w_\varepsilon))\to 0 \quad in\ L^r(\mathbb{R}^N)$$

as $\varepsilon\to 0$ if $r<\infty$.

This lemma includes the following statement.

Lemma 2.5 *Let $f_t\in L^p(0,T)$, $g\in L^q(0,T)$ with $1\le p,q\le\infty$, and $\frac{1}{p}+\frac{1}{q}\le 1$. Then, we have*

$$\|(fg)_t*w_\varepsilon-(f(g*w_\varepsilon))_t\|_{L^r(0,T)}\le C\|f_t\|_{L^p(0,T)}\|g\|_{L^q(0,T)}$$

for some $C\ge 0$ independent of ε, f and g, r is determined by $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. In addition,

$$(fg)_t*w_\varepsilon-(f(g*w_\varepsilon))_t\to 0 \quad in\ L^r(0,T)$$

as $\varepsilon\to 0$ if $r<\infty$.

With Lemma 2.4 and Lemma 2.5 in hand, we are ready to handle the terms R_1 and R_2. For $\kappa>0$, by Lemma 2.1 and Poincaré inequality, we have $\mathbf{v}\in L^2(0,T;L^6(\Omega))$. Lemma 2.1 gives us

$$\rho_t\in L^4(0,T;L^{6/5}(\Omega))+L^2(0,T;L^{3/2}(\Omega)).$$

Thus, applying Lemma 2.5, we have

$$|R_1| \leq \int_0^T \int_\Omega \left|\psi(t)\varphi_n'(\overline{\mathbf{v}})[\overline{(\rho\mathbf{v})_t} - (\rho\overline{\mathbf{v}})_t]\right| \, dx\, dt$$
$$\leq C(\psi) \int_0^T \int_\Omega \left|\varphi_n'(\overline{\mathbf{v}})[\overline{(\rho\mathbf{v})_t} - (\rho\overline{\mathbf{v}})_t]\right| \, dx\, dt \to 0 \quad \text{as } k \to 0. \tag{2.22}$$

Similarly, Lemma 2.4 gives us

$$R_2 \to 0 \quad \text{as} \quad k \to 0. \tag{2.23}$$

Letting k goes to zero in (2.17), and using (2.18)–(2.23), we derive

$$\begin{aligned} &-\int_0^T \int_\Omega \psi_t \rho\varphi_n(\mathbf{v})\, dx\, dt + \int_0^T \int_\Omega \psi(t)\varphi_n'(\mathbf{v})\mathbf{F}\, dx\, dt \\ &+ \int_0^T \int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\mathbf{v}))\, dx\, dt = 0, \end{aligned} \tag{2.24}$$

for any test function $\psi \in \mathfrak{D}(0, \infty)$.

Now, we need to consider the test function $\psi(t) \in \mathfrak{D}(-1, \infty)$. For this, we need the continuity of $\rho(t)$ and $(\sqrt{\rho}\mathbf{u})(t)$ in the strong topology at $t = 0$.

In fact, Proposition 1.1 gives us

$$(\sqrt{\rho})_t \in L^2(0, T; L^2(\Omega)), \quad \sqrt{\rho} \in L^2(0, T; H^2(\Omega)),$$

which implies

$$\sqrt{\rho} \in C([0, T]; L^2(\Omega)) \quad \text{and} \quad \nabla\sqrt{\rho} \in C(0, T; L^2(\Omega)),$$

thanks to Theorem 3 on page 287, see [12]. Similarly, we have

$$\rho \in C([0, T]; L^2(\Omega)) \tag{2.25}$$

due to

$$\|\nabla\rho\|_{L^2(0,T;L^2(\Omega))} \leq C\|\nabla\rho^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\|\rho^{\frac{3}{4}}\|_{L^4(0,T;L^4(\Omega))}.$$

Meanwhile, we have

$$\sqrt{\rho} \in L^\infty(0, T; L^p(\Omega)) \quad \text{for any } 1 \leq p \leq 6,$$

and hence

$$\sqrt{\rho} \in C([0, T]; L^p(\Omega)) \quad \text{for any } 1 \leq p \leq 6. \tag{2.26}$$

On the other hand, we see

$$
\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{t\to 0} \frac{1}{2}\int_\Omega |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx \\
&\quad\le \operatorname*{ess\,lim\,sup}_{t\to 0}\left(\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + \kappa|\nabla\sqrt{\rho}|^2\right)dx\right. \\
&\qquad \left. - \int_\Omega \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1}\right) + \kappa|\nabla\sqrt{\rho_0}|^2\,dx\right) \\
&\qquad + \operatorname*{ess\,lim\,sup}_{t\to 0}\left(\int_\Omega \sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0 - \sqrt{\rho}\mathbf{u})\,dx + \int_\Omega\left(\frac{\rho_0^\gamma}{\gamma-1} - \frac{\rho^\gamma}{\gamma-1}\right)\right) \\
&\qquad - \kappa \operatorname*{ess\,lim\,sup}_{t\to 0}\left|\nabla\sqrt{\rho_0} - \nabla\sqrt{\rho}\right|^2\,dx \\
&\qquad + 2\kappa \operatorname*{ess\,lim\,sup}_{t\to 0}\int_\Omega \nabla\sqrt{\rho_0}\cdot(\nabla\sqrt{\rho_0} - \nabla\sqrt{\rho})\,dx.
\end{aligned}
\tag{2.27}
$$

Note that,

$$
\operatorname*{ess\,lim\,sup}_{t\to 0}\int_\Omega \nabla\sqrt{\rho_0}\cdot(\nabla\sqrt{\rho_0} - \nabla\sqrt{\rho})\,dx = 0,
\tag{2.28}
$$

and using (1.7), (2.26) and the convexity of $\rho \mapsto \rho^\gamma$, one obtains

$$
\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{t\to 0}\int_\Omega |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx \\
&\quad\le 2\operatorname*{ess\,lim\,sup}_{t\to 0}\int_\Omega \sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0 - \sqrt{\rho}\mathbf{u})\,dx \\
&\quad= 2\operatorname*{ess\,lim\,sup}_{t\to 0}\left(\int_\Omega \sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0 - \sqrt{\rho}\mathbf{u}\phi_m(\rho))\,dx\right. \\
&\qquad\left. + \int_\Omega \sqrt{\rho_0}\mathbf{u}_0(1-\phi_m(\rho))\sqrt{\rho}\mathbf{u}\,dx\right) \\
&\quad= B_1 + B_2.
\end{aligned}
$$

By Proposition 1.1, we have

$$
\rho\mathbf{u} \in C([0,T]; L^{\frac{3}{2}}_{\text{weak}}(\Omega)).
\tag{2.29}
$$

We consider B_1 as follows

$$B_1 = 2\text{ess}\limsup_{t\to 0}\left(\int_\Omega \sqrt{\rho_0}\mathbf{u}_0\left(\frac{\phi_m(\rho)}{\sqrt{\rho}}(\rho_0\mathbf{u}_0-\rho\mathbf{u})\right)dx\right.$$
$$\left.-\int_\Omega \sqrt{\rho_0}\rho_0|\mathbf{u}_0|^2\left(\frac{\phi_m(\rho)}{\sqrt{\rho}}-\frac{\phi_m(\rho_0)}{\sqrt{\rho_0}}\right)dx\right),$$

and use (2.26) and (2.29) to have $B_1 = 0$.

Since $m \geq m_0$, and $\rho_0 \geq \frac{1}{m_0}$, we find

$$|B_2| \leq \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^\infty(\Omega)}\|\sqrt{\rho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\text{ess}\limsup_{t\to 0}\|1-\phi_m(\rho)(t)\|_{L^2(\Omega)} = 0.$$

Thus, we have

$$\text{ess}\limsup_{t\to 0}\int_\Omega |\sqrt{\rho}\mathbf{u}-\sqrt{\rho_0}\mathbf{u}_0|^2\,dx = 0,$$

which in turn gives us

$$\sqrt{\rho}\mathbf{u} \in C([0,T];L^2(\Omega)). \tag{2.30}$$

From (2.25) and (2.30), we deduce

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_\Omega \rho\varphi_n(\mathbf{v})\,dx\,dt = \int_\Omega \rho_0\varphi_n(\mathbf{v}_0)\,dx.$$

Choosing the following test function for (2.24),

$$\psi_\tau(t) = \psi(t) \quad \text{for } t \geq \tau, \quad \psi_\tau(t) = \psi(\tau)\frac{t}{\tau} \quad \text{for } t \leq \tau,$$

we have

$$-\int_\tau^T\int_\Omega \psi_t\rho\varphi_n(\mathbf{v})\,dx\,dt + \int_0^T\int_\Omega \psi_\tau(t)\varphi_n'(\mathbf{v})\mathbf{F}\,dx\,dt$$
$$+\int_0^T\int_\Omega \psi_\tau(t)\mathbb{S}:\nabla(\varphi_n'(\mathbf{v}))\,dx\,dt = \frac{\psi(\tau)}{\tau}\int_0^\tau\int_\Omega \rho\varphi_n(\mathbf{v})\,dx\,dt.$$

Passing into the limit as $\tau \to 0$, this gives us

$$-\int_0^T\int_\Omega \psi_t\rho\varphi_n(\mathbf{v})\,dx\,dt + \int_0^T\int_\Omega \psi(t)\varphi_n'(\mathbf{v})\mathbf{F}\,dx\,dt$$
$$+\int_0^T\int_\Omega \psi(t)\mathbb{S}:\nabla(\varphi_n'(\mathbf{v}))\,dx\,dt = \int_\Omega \rho_0\psi(0)\varphi_n(\mathbf{v}_0)\,dx. \tag{2.31}$$

3 Recover the limits as $m \to \infty$

In this section, we want to recover the limits from (2.11) as $m \to \infty$. Here, we should remark that $(\rho, \mathbf{u})$ is any fixed weak solution to (2.1) verifying Proposition 1.1 with $\kappa > 0$. For any fixed weak solution $(\rho, \mathbf{u})$, $\phi_m(\rho)$ converges to 1 almost everywhere for (t, x), and it is uniform bounded in $L^\infty(0, T; \Omega)$, and

$$r_0\phi_K(\rho)\mathbf{u} \in L^2(0, T; L^2(\Omega)).$$

Thus, we find

$$\mathbf{v}_m = \phi_m\phi_K\mathbf{u} \to \phi_K\mathbf{u} \quad \text{almost everywhere for } (t, x)$$

as $m \to \infty$. The Dominated Convergence Theorem allows us to have

$$\mathbf{v}_m \to \phi_K\mathbf{u} \quad \text{in } L^2(0, T; L^2(\Omega))$$

as $m \to \infty$, and hence,

$$\varphi_n(\mathbf{v}_m) \to \varphi_n(\phi_K\mathbf{u}) \quad \text{in } L^p((0, T) \times \Omega)$$

for any $1 \leq p < \infty$. Thus, we can show that

$$\int_0^T \int_\Omega \psi'(t)(\rho\varphi_n(\mathbf{v}_m))\,dx\,dt \to \int_0^T \int_\Omega \psi'(t)(\rho\varphi_n(\phi_K(\rho)\mathbf{u}))\,dx\,dt$$

and

$$\int_\Omega \rho_0\varphi_n(\mathbf{v}_{m0}) \to \int_\Omega \rho_0\varphi_n(\phi_K(\rho_0)\mathbf{u}_0)$$

as $m \to \infty$.

Meanwhile, for any fixed ρ, we have

$$\phi'_m(\rho) \to 0 \quad \text{almost everywhere for } (t, x)$$

as $m \to \infty$.

Calculating $|\phi'_m(\rho)| \leq 2m$ as $\frac{1}{2m} \leq \rho \leq \frac{1}{m}$, and otherwise, $\phi'_m(\rho) = 0$, thus,

$$|\rho\phi'_m(\rho)| \leq 1 \quad \text{for all } \rho.$$

To pass into the limits in (2.31) as $m \to \infty$, we rely on the following Lemma:

Lemma 3.1 *If*

$$\|a_m\|_{L^\infty(0,T;\Omega)} \le C, \quad a_m \to a \quad \text{a.e. for } (t,x) \text{ and in } L^p((0,T)\times\Omega) \\ \text{for any } 1 \le p < \infty,$$

$f \in L^1((0,T)\times\Omega)$*, then we have*

$$\int_0^T \int_\Omega \phi_m(\rho) a_m f \, dx\, dt \to \int_0^T \int_\Omega a f \, dx\, dt \quad \text{as } m \to \infty,$$

and

$$\int_0^T \int_\Omega \left|\rho \phi_m'(\rho) a_m f\right| \, dx\, dt \to 0 \quad \text{as } m \to \infty.$$

Proof We have

$$|\phi_m(\rho) a_m f - af| \le |\phi_m(\rho) f - f||a_m| + |a_m f - af| = I_1 + I_2.$$

For I_1: $\phi_m(\rho) f \to f$ a.e. for (t,x) and

$$|\phi_m(\rho) f - f| \le 2|f| \quad \text{a.e. for } (t,x),$$

by Lebesgue's Dominated Convergence Theorem, we find

$$\int_0^T \int_\Omega |\phi_m(\rho) f - f| \, dx\, dt \to 0$$

as $m \to \infty$, which in turn yields

$$\begin{aligned} &\int_0^T \int_\Omega |\phi_m(\rho) a_m f - a_m f| \, dx\, dt \\ &\le \|a_m\|_{L^\infty(0,T;\Omega)} \int_0^T \int_\Omega |\phi_m(\rho) f - f| \, dx\, dt \\ &\to 0 \end{aligned}$$

as $m \to \infty$. Following the same line, we show

$$\int_0^T \int_\Omega |a_m f - af| \, dx\, dt \to 0$$

as $m \to \infty$. Thus we have

$$\int_0^T \int_\Omega \phi_m(\rho) a_m f \, dx \, dt \to \int_0^T \int_\Omega af \, dx \, dt$$

as $m \to \infty$.

We now consider $\int_0^T \int_\Omega |\rho\phi_m'(\rho)a_m f| \, dx \, dt$. Notice that $|\rho\phi_m'(\rho)| \leq C$, and $\rho\phi_m'(\rho)$ converges to 0 almost everywhere, so $\left|\rho\phi_m'(\rho)a_m f\right| \leq C|f|$, and by means of Lebesgue's Dominated Convergence Theorem, we have

$$\int_0^T \int_\Omega |\rho\phi_m'(\rho)a_m f| \, dx \, dt \to 0$$

as $m \to \infty$. □

Calculating

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(t)\mathbb{S}_m : \nabla(\varphi_n'(\mathbf{v}_m)) \, dx \, dt \\
&= \int_0^T \int_\Omega \psi(t)\mathbb{S}_m \varphi_n''(\mathbf{v}_m) \left(\nabla\phi_m\phi_K\mathbf{u} + \phi_m\nabla\phi_K\mathbf{u} + \phi_m\phi_K\nabla\mathbf{u}\right) \, dx \, dt \\
&= \int_0^T \int_\Omega \phi_m(\rho)a_{m1} f_1 \, dx \, dt + \int_0^T \int_\Omega \rho\phi_m'(\rho)a_{m2} f_2 \, dx \, dt, \qquad (3.1)
\end{aligned}$$

where

$$\begin{aligned}
a_{m1} &= \phi_m(\rho)\varphi_n''(\mathbf{v}_m), \\
f_1 &= \psi(t)\rho\phi_K(\rho)\left(\mathbb{D}\mathbf{u} + \kappa\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathbb{I}\right)(\mathbf{u}\nabla\phi_K(\rho) + \phi_K(\rho)\nabla\mathbf{u}),
\end{aligned}$$

and

$$\begin{aligned}
a_{m2} &= \varphi_n''(\mathbf{v}_m)\phi_m(\rho)\phi_K(\rho)\mathbf{u} = \varphi_n''(\mathbf{v}_m)\mathbf{v}_m, \\
f_2 &= \psi(t)\phi_K(\rho)\left(\mathbb{D}\mathbf{u} + \kappa\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathbb{I}\right)\nabla\rho \\
&= 2\psi(t)\phi_K(\rho)(\kappa\Delta\sqrt{\rho}\nabla\sqrt{\rho} + \sqrt{\rho}\mathbb{D}\mathbf{u}\nabla\sqrt{\rho}).
\end{aligned}$$

So applying Lemma 3.1 to (3.1), one obtains

$$\int_0^T \int_\Omega \psi(t)\mathbb{S}_m : \nabla(\varphi_n'(\mathbf{v}_m)) \, dx \, dt \to \int_0^T \int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\phi_K(\rho)\mathbf{u})) \, dx \, dt$$

as $m \to \infty$, where $\mathbb{S} = \phi_K(\rho)\rho(\mathbb{D}\mathbf{u} + \kappa\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathbb{I})$.

Letting $\mathbf{F}_m = \mathbf{F}_{m1} + \mathbf{F}_{m2}$, where

$$\begin{aligned}\mathbf{F}_{m1} &= \rho^2\mathbf{u}\phi'(\rho)\mathrm{div}\mathbf{u} + \rho\nabla\phi(\rho)\mathbb{D}\mathbf{u} + \kappa\sqrt{\rho}\nabla\phi(\rho)\Delta\sqrt{\rho}\\ &= \rho\left(\phi_m'(\rho)\phi_K(\rho)+\phi_m(\rho)\phi_K'(\rho)\right)\left(\rho\mathbf{u}\mathrm{div}\mathbf{u}+\nabla\rho\cdot\mathbb{D}\mathbf{u}+\kappa\nabla\rho\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right),\end{aligned}$$

where

$$\phi_K(\rho)\left(\rho\mathbf{u}\mathrm{div}\mathbf{u} + \nabla\rho\cdot\mathbb{D}\mathbf{u} + \kappa\nabla\rho\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \in L^1((0,T)\times\Omega),$$
$$\rho\phi_K'(\rho)\left(\rho\mathbf{u}\mathrm{div}\mathbf{u} + \nabla\rho\cdot\mathbb{D}\mathbf{u} + \kappa\nabla\rho\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \in L^1((0,T)\times\Omega),$$

and

$$\mathbf{F}_{m2} = \phi_m(\rho)\phi_K(\rho)(2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}} + r_0\mathbf{u} + r_1\rho|\mathbf{u}|^2\mathbf{u} + 2\kappa\nabla\sqrt{\rho}\Delta\sqrt{\rho}),$$

where

$$\phi_K(\rho)\left(2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}} + r_0\mathbf{u} + r_1\rho|\mathbf{u}|^2\mathbf{u} + 2\kappa\nabla\sqrt{\rho}\Delta\sqrt{\rho}\right) \in L^1((0,T)\times\Omega).$$

Using Lemma 3.1, we obtain

$$\int_0^T\int_\Omega \psi(t)\varphi_n'(\mathbf{v}_m)\mathbf{F}_m\,dx\,dt \to \int_0^T\int_\Omega \psi(t)\varphi_n'(\phi_K(\rho)\mathbf{u})\mathbf{F}\,dx\,dt,$$

where

$$\begin{aligned}\mathbf{F} = \rho^2\mathbf{u}\phi_K'(\rho)\mathrm{div}\mathbf{u} + 2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\phi_K(\rho) + \rho\nabla\phi_K(\rho)\mathbb{D}\mathbf{u} + r_0\mathbf{u}\phi_K(\rho)\\ + r_1\rho|\mathbf{u}|^2\mathbf{u}\phi_K(\rho) + \kappa\sqrt{\rho}\nabla\phi_K(\rho)\Delta\sqrt{\rho} + 2\kappa\phi_K(\rho)\nabla\sqrt{\rho}\Delta\sqrt{\rho}.\end{aligned}$$

Thus, letting $m \to \infty$ in (2.31), and using the above convergence in this section, we find

$$\begin{aligned}&-\int_0^T\int_\Omega \psi'(t)(\rho\varphi_n(\phi_K(\rho)\mathbf{u}))\,dx\,dt + \int_0^T\int_\Omega \psi(t)\varphi_n'(\phi_K(\rho)\mathbf{u})\mathbf{F}\,dx\,dt\\ &+\int_0^T\int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\phi_K(\rho)\mathbf{u}))\,dx\,dt = \int_\Omega \psi(0)\rho_0\varphi_n(\phi_K(\rho_0)\mathbf{u}_0)\,dx,\end{aligned}$$

which in turn gives us the following lemma:

Lemma 3.2 *For any weak solution to (2.1) verifying in Proposition 1.1, we have*

$$-\int_0^T\int_\Omega \psi'(t)(\rho\varphi_n(\phi_K(\rho)\mathbf{u}))\,dx\,dt + \int_0^T\int_\Omega \psi(t)\varphi_n'(\phi_K(\rho)\mathbf{u})\mathbf{F}\,dx\,dt$$
$$+\int_0^T\int_\Omega \psi(t)\mathbb{S} : \nabla(\varphi_n'(\phi_K(\rho)\mathbf{u}))\,dx\,dt = \int_\Omega \psi(0)\rho_0\varphi_n(\phi_K(\rho_0)\mathbf{u}_0)\,dx, \tag{3.2}$$

where $\mathbb{S} = \phi_K(\rho)\rho(\mathbb{D}\mathbf{u} + \kappa\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathbb{I})$, *and*

$$\mathbf{F} = \rho^2\mathbf{u}\phi_K'(\rho)\mathrm{div}\mathbf{u} + 2\rho^{\frac{\gamma}{2}}\nabla\rho^{\frac{\gamma}{2}}\phi_K(\rho) + \rho\nabla\phi_K(\rho)\mathbb{D}\mathbf{u} + r_0\mathbf{u}\phi_K(\rho)$$
$$+ r_1\rho|\mathbf{u}|^2\mathbf{u}\phi_K(\rho) + \kappa\sqrt{\rho}\nabla\phi_K(\rho)\Delta\sqrt{\rho} + 2\kappa\phi_K(\rho)\nabla\sqrt{\rho}\Delta\sqrt{\rho},$$

where $\mathbb{I}$ *is an identical matrix.*

4 Recover the limits as $\kappa \to 0$ and $K \to \infty$.

The objective of this section is to recover the limits in (3.2) as $\kappa \to 0$ and $K \to \infty$. In this section, we assume that $K = \kappa^{-\frac{3}{4}}$, thus $K \to \infty$ when $\kappa \to 0$. First, we address the following lemma.

Lemma 4.1 *Let* $\kappa \to 0$ *and* $K \to \infty$, *and denote* $\mathbf{v}_\kappa = \phi_K(\rho_\kappa)\mathbf{u}_\kappa$, *we have*

$$\rho_\kappa^\gamma \quad \text{is bounded in } L^r((0,T)\times\Omega) \text{ for any } 1 \le r < \infty \text{ in } 2D,$$
$$\text{and any } 1 \le r \le \frac{5}{3} \text{ in } 3D. \tag{4.1}$$

For any $g \in C^1(\mathbb{R}^+)$ *with g bounded, and* $0 < \alpha < \infty$ *in 2D,* $0 < \alpha < \frac{5\gamma}{3}$ *in 3D, we have*

$$\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2) \to \rho^\alpha g(|\mathbf{u}|^2); \quad \text{strongly in } L^1((0,T)\times\Omega).$$

In particular, we have, for any fixed n,

$$\rho_\kappa\varphi_n(\mathbf{v}_\kappa) \to \rho\varphi_n(\mathbf{u}) \quad \text{strongly in } L^1((0,T)\times\Omega), \tag{4.2}$$

and

$$\rho_\kappa^{2\gamma-1}(1+\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)) \to \rho^{2\gamma-1}(1+\tilde{\varphi}_n'(|\mathbf{u}|^2)) \quad \text{strongly in } L^1((0,T)\times\Omega), \tag{4.3}$$

for $1 < \gamma < 3$.

Proof In 2D, we deduce from (1.13)

$$\rho_\kappa \in L^\infty(0, T : L^p(\Omega)) \quad \text{for any } 1 \le p < \infty.$$

Thus, ρ_κ^γ is bounded in $L^r((0,T)\times\Omega)$ for $1 \le r < \infty$.

In 3D, we deduce that

$$\rho_\kappa^\gamma \in L^\infty(0, T : L^1(\Omega)) \cap L^1(0, T; L^3(\Omega)).$$

Applying Holder inequality, we have

$$\|\rho_\kappa^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \le \|\rho_\kappa^\gamma\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{2}{5}} \|\rho_\kappa^\gamma\|_{L^1(0,T;L^3(\Omega))}^{\frac{3}{5}}.$$

Thus, ρ_κ^γ is bounded in $L^{\frac{5}{3}}((0,T)\times\Omega)$.

We have that $(\rho_\kappa)_t$ is uniformly bounded in

$$L^4(0, T; L^{6/5}(\Omega)) + L^2(0, T; L^{3/2}(\Omega)),$$

thanks to Lemma 2.1. Since $\nabla\sqrt{\rho_\kappa}$ is uniformly bounded in $L^\infty(L^2)$, we have also $\sqrt{\rho_\kappa}$ uniformly bounded in $L^\infty(L^6)$. Those two estimates give

$$\|\nabla\rho_\kappa\|_{L^\infty(0,T;L^{3/2}(\Omega))} \le C.$$

Applying Aubin–Lions Lemma, one obtains

$$\rho_\kappa \to \rho \quad \text{strongly in } L^p(0, T; L^{3/2}(\Omega)) \quad \text{for } p < \infty.$$

When $\kappa \to 0$, we have $\sqrt{\rho_\kappa}\mathbf{u}_\kappa \to \sqrt{\rho}\mathbf{u}$ strongly in $L^2(0, T; L^2(\Omega))$ from Proposition 1.1, (also see [38]). Thus, up to a subsequence, for almost every (t, x) such that $\rho(t, x) \neq 0$, we have

$$\mathbf{u}_\kappa(t, x) = \frac{\sqrt{\rho_\kappa}\mathbf{u}_\kappa}{\sqrt{\rho_\kappa}} \to \mathbf{u}(t, x),$$

and

$$\mathbf{v}_\kappa \to \mathbf{u}(t, x),$$

as $\kappa \to 0$. For almost every (t, x) such that $\rho(t, x) = 0$,

$$\left|\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2)\right| \le C\rho_\kappa^\alpha(t, x) \to 0 = \rho^\alpha g(|\mathbf{u}|^2) \tag{4.4}$$

as $\kappa \to 0$.

Hence, $\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2)$ converges to $\rho^\alpha g(|\mathbf{u}|^2)$ almost everywhere. Since g is bounded and (4.1), $\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2)$ is uniformly bounded in $L^r((0,T)\times\Omega)$ for some $r>1$. Hence,

$$\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2) \to \rho^\alpha g(|\mathbf{u}|^2) \quad \text{in } L^1((0,T)\times\Omega).$$

By the uniqueness of the limit, the convergence holds for the whole sequence.

Applying this result with $\alpha=1$ and $g(|\mathbf{v}_\kappa|^2)=\varphi_n(\mathbf{v}_\kappa)$, we deduce (4.2).

Since $\gamma>1$ in 2D, we can take $\alpha=2\gamma-1<2\gamma$; and take $\gamma<3$ in 3D, we have $2\gamma-1<\frac{5\gamma}{3}$. Thus we use the above result with $\alpha=2\gamma-1$ and $g(|\mathbf{v}_\kappa|^2)=1+\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)$ to obtain (4.3). □

With the lemma in hand, we are ready to recover the limits in (3.2) as $\kappa\to 0$ and $K\to\infty$. We have the following lemma.

Lemma 4.2 *Let $K=\kappa^{-\frac{3}{4}}$, and $\kappa\to 0$, for any $\psi\geq 0$ and $\psi'\leq 0$, we have*

$$\begin{aligned}
&-\int_0^T\int_\Omega \psi'(t)\rho\varphi_n(\mathbf{u})\,dx\,dt\\
&\quad\leq 8\|\psi\|_{L^\infty}\left(\int_\Omega\left(\rho_0|\mathbf{u}_0|^2+\frac{\rho_0^\gamma}{\gamma-1}+|\nabla\sqrt{\rho_0}|^2-r_0\log_-\rho_0\right)dx+2E_0\right)\\
&\qquad+C(\|\psi\|_{L^\infty})\int_0^T\int_\Omega(1+\tilde{\varphi}_n'(|\mathbf{u}|^2))\rho^{2\gamma-1}\,dx\,dt+\psi(0)\int_\Omega\rho_0\varphi_n(\mathbf{u}_0)\,dx
\end{aligned} \tag{4.5}$$

Proof Here, we use $(\rho_\kappa,\mathbf{u}_\kappa)$ to denote the weak solutions to (2.1) verifying Proposition 1.1 with $\kappa>0$.

By Lemma 4.1, we can handle the first term in (3.2), that is,

$$\int_0^T\int_\Omega\psi'(t)(\rho_\kappa\varphi_n(\mathbf{v}_\kappa))\,dx\,dt\to\int_0^T\int_\Omega\psi'(t)(\rho\varphi_n(\mathbf{u}))\,dx\,dt \tag{4.6}$$

and

$$\psi(0)\int_\Omega\rho_0\varphi_n'(\mathbf{v}_{\kappa,0})\,dx\to\psi(0)\int_\Omega\rho_0\varphi_n'(\mathbf{u}_0)\,dx \tag{4.7}$$

as $\kappa\to 0$ and $K=\kappa^{-\frac{3}{4}}\to\infty$.

$$\begin{aligned}
&\int_0^T \int_\Omega \psi(t)\varphi_n'(\mathbf{v}_\kappa) \cdot \nabla \rho_\kappa^\gamma \phi_K(\rho_\kappa)\,dx\,dt \\
&\quad = -\int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma \phi_K(\rho_\kappa)\varphi_n'' : \nabla \mathbf{v}_\kappa\,dx\,dt \\
&\qquad - \int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma \varphi_n'(\mathbf{v}_\kappa) \cdot \nabla \phi_K(\rho_\kappa)\,dx\,dt \\
&\quad = P_1 + P_2.
\end{aligned} \tag{4.8}$$

We can control P_2 as follows

$$\begin{aligned}
|P_2| &\le \|\psi\|_{L^\infty} \int_0^T \int_\Omega |\rho_\kappa^\gamma||\varphi_n'||\nabla \phi_K(\rho_\kappa)|\,dx\,dt \\
&\le C(n, \|\psi\|_{L^\infty})\kappa^{-\frac{1}{4}} \|\phi_K' \sqrt{\rho_\kappa}\|_{L^\infty} \|\rho_\kappa^{\gamma+\frac{1}{4}}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))} \\
&\quad \times \left(\kappa^{\frac{1}{4}} \|\nabla \rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\right) \\
&\le \frac{2}{\sqrt{K}} C(n, \|\psi\|_{L^\infty})\kappa^{-\frac{1}{4}} \|\rho_\kappa^{\gamma+\frac{1}{4}}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))} \\
&\quad \left(\kappa^{\frac{1}{4}} \|\nabla \rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\right) \\
&\le \frac{2C}{\sqrt{K}} \kappa^{-\frac{1}{4}} = 2C\kappa^{\frac{1}{8}} \to 0
\end{aligned} \tag{4.9}$$

as $\kappa \to 0$, where we used $\frac{4}{3}(\gamma + \frac{1}{4}) \le \frac{5\gamma}{3}$ for any $\gamma > 1$.

Calculating P_1,

$$\begin{aligned}
P_1 &= -\int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma \phi_K(\rho_\kappa)\varphi_n'' : \nabla \mathbf{v}_\kappa\,dx\,dt \\
&= -\int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma (\phi_K(\rho_\kappa))^2 \varphi_n'' : \nabla \mathbf{u}_\kappa\,dx\,dt \\
&\quad - \int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma \phi_K(\rho_\kappa)\varphi_n'' : (\nabla \phi_K(\rho_\kappa) \otimes \mathbf{u}_\kappa)\,dx\,dt \\
&= P_{11} + P_{12}.
\end{aligned} \tag{4.10}$$

We bound P_{12} as follows

$$\begin{aligned}
|P_{12}| &= -\int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma \phi_K(\rho_\kappa)\varphi_n'' : (\nabla \phi_K(\rho_\kappa) \otimes \mathbf{u}_\kappa)\,dx\,dt \\
&= -\int_0^T \int_\Omega \psi(t)\rho_\kappa^\gamma (\nabla \phi_K(\rho_\kappa))^T \varphi_n''(\mathbf{v}_\kappa)\mathbf{v}_\kappa\,dx\,dt
\end{aligned}$$

$$\le C(n, \|\psi\|_{L^\infty})\kappa^{-\frac{1}{4}}\|\rho_\kappa^{\gamma+\frac{1}{4}}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))}\left(\kappa^{\frac{1}{4}}\|\nabla\rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\right)$$
$$\|\phi_K'\sqrt{\rho_\kappa}\|_{L^\infty}$$
$$\le \frac{2C}{\sqrt{K}}\kappa^{-\frac{1}{4}} = 2C\kappa^{\frac{1}{8}} \to 0 \tag{4.11}$$

as $\kappa \to 0$, where we used $|\phi_K'(\rho_\kappa)\sqrt{\rho_\kappa}| \le \frac{2}{\sqrt{K}}$, $\|\varphi_n''(\mathbf{v}_k)\mathbf{v}_k\|_{L^\infty(0,T;L^\infty(\Omega))} \le C(n)$ since φ_n'' is compactly supported, and $\frac{4}{3}(\gamma+\frac{1}{4}) \le \frac{5\gamma}{3}$ for any $\gamma > 1$.

Thanks to part a of Lemma 2.2, we have

$$\varphi_n''(\mathbf{v}_\kappa) : \nabla\mathbf{u}_\kappa = 4\tilde{\varphi}_n''(|\mathbf{v}_\kappa|^2)\nabla\mathbf{u}_\kappa : (\mathbf{v}_\kappa \otimes \mathbf{v}_\kappa) + 2\mathrm{div}\mathbf{u}_\kappa\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2).$$

Using Part b of Lemma 2.2, we find that

$$\begin{aligned}|\tilde{\varphi}_n''(|\mathbf{v}_\kappa|^2)\nabla\mathbf{u}_\kappa : (\mathbf{v}_\kappa \otimes \mathbf{v}_\kappa)| &\le |\tilde{\varphi}_n''(|\mathbf{v}_\kappa|^2)||\nabla\mathbf{u}_k||v_k|^2\\ &\le |\nabla\mathbf{u}_k|\frac{|v_k|^2}{1+|v_k|^2} \le |\nabla\mathbf{u}_k|,\end{aligned}$$

where we denote $|\nabla\mathbf{u}_k|^2 = \sum_{ij}|\partial_i\mathbf{u}_j|^2$. Hence

$$\begin{aligned}|P_{11}| &\le 4\int_0^T\int_\Omega \psi(t)|\phi_K|^2|\rho_\kappa^\gamma||\nabla\mathbf{u}_\kappa|\,dx\,dt\\ &\quad+2\int_0^T\int_\Omega \psi(t)|\phi_K|^2|\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)||\rho_\kappa^\gamma||\mathrm{div}\mathbf{u}_\kappa|\,dx\,dt\\ &\le 4\|\psi\|_{L^\infty}\int_0^T\int_\Omega \rho_\kappa|\nabla\mathbf{u}_\kappa|^2\,dx\,dt + C(\|\psi\|_{L^\infty})\int_0^T\int_\Omega \rho_\kappa^{2\gamma-1}\,dx\,dt\\ &\quad+2\int_0^T\int_\Omega \psi(t)|\phi_K|^2|\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)||\rho_\kappa^\gamma||\mathrm{div}\mathbf{u}_\kappa|\,dx\,dt,\end{aligned} \tag{4.12}$$

and the term

$$\begin{aligned}&2\int_0^T\int_\Omega \psi(t)|\phi_K|^2|\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)||\rho_\kappa^\gamma||\mathrm{div}\mathbf{u}_\kappa|\,dx\,dt\\ &\le 2\int_0^T\int_\Omega \psi(t)|\phi_K|^2|\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)|\rho_\kappa|\mathbb{D}\mathbf{u}_\kappa|^2\,dx\,dt\\ &\quad+C(\|\psi\|_{L^\infty})\int_0^T\int_\Omega |\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)|\rho_\kappa^{2\gamma-1}\,dx\,dt.\end{aligned} \tag{4.13}$$

Thus,

$$
\begin{aligned}
|P_{11}| \le 4\|\psi\|_{L^\infty} \int_0^T \int_\Omega \rho_\kappa |\nabla \mathbf{u}_\kappa|^2 \, dx \, dt \\
+ 2 \int_0^T \int_\Omega \psi(t) |\phi_K|^2 |\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)| \rho_\kappa |\mathbb{D}\mathbf{u}_\kappa|^2 \, dx \, dt \\
+ C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)) \rho_\kappa^{2\gamma-1} \, dx \, dt. \qquad (4.14)
\end{aligned}
$$

The first right hand side term will be controlled by

$$
4\|\psi\|_{L^\infty} \left(\int_\Omega \left(\rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} + |\nabla \sqrt{\rho_0}|^2 - r_0 \log_- \rho_0 \right) dx + 2E_0 \right)
$$

due to (1.16); and the second right hand side term will be absorbed by the dispersion term A_1 in (4.23). By Lemma 4.1, we have

$$
\int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2) \rho_\kappa^{2\gamma-1} \, dx \, dt \to \int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{u}|^2)) \rho^{2\gamma-1} \, dx \, dt \qquad (4.15)
$$

as $\kappa \to 0$.

Note that

$$
\int_0^T \int_\Omega \psi(t) \varphi_n'(\mathbf{v}_\kappa)(r_0 \mathbf{u}_\kappa + r_1 \rho_\kappa |\mathbf{u}_\kappa|^2 \mathbf{u}_\kappa) \, dx \, dt \ge 0, \qquad (4.16)
$$

so this term can be dropped directly.

We treat the other terms in $\mathbf{F}$ one by one,

$$
\begin{aligned}
&\int_0^T \int_\Omega \left| \psi(t) \varphi_n'(\mathbf{v}_\kappa) \rho_\kappa^2 \mathbf{u}_\kappa \phi_K'(\rho_\kappa) \mathrm{div} \mathbf{u}_\kappa \right| \, dx \, dt \\
&\le C(n, \psi) \|\rho_\kappa^{\frac{1}{4}} \mathbf{u}_\kappa\|_{L^4((0,T;L^4(\Omega))} \|\sqrt{\rho_\kappa} \mathrm{div} \mathbf{u}_\kappa\|_{L^2((0,T;L^2(\Omega))} \\
&\quad \times \|\phi_K'(\rho_\kappa) \sqrt{\rho_\kappa}\|_{L^\infty} \|\rho_\kappa^{\frac{3}{4}}\|_{L^4((0,T;L^4(\Omega))} \le C(n, \psi) \kappa^{\frac{3}{8}} \to 0 \qquad (4.17)
\end{aligned}
$$

as $\kappa \to 0$, where we used Sobolev inequality, and $|\phi_K'(\rho_\kappa)\sqrt{\rho_\kappa}| \le \frac{2}{\sqrt{K}}$;

$$
\begin{aligned}
&\int_0^T \int_\Omega \left|\psi(t)\varphi_n'(\mathbf{v}_\kappa)\rho_\kappa \nabla\phi_K(\rho_\kappa)\mathbb{D}\mathbf{u}_\kappa\right| \, dx\, dt \\
&\quad \le C(n,\psi)\frac{|\phi_K'(\rho_\kappa)\sqrt{\rho_\kappa}|}{\kappa^{\frac{1}{4}}}\left(\kappa^{\frac{1}{4}}\|\nabla\rho_\kappa^{\frac{1}{4}}\|_{L^4((0,T;L^4(\Omega))}\right) \\
&\quad\ \ \|\sqrt{\rho_\kappa}\mathbb{D}\mathbf{u}_\kappa\|_{L^2((0,T;L^2(\Omega))}\|\rho_\kappa^{\frac{3}{4}}\|_{L^4((0,T;L^4(\Omega))} \\
&\quad \le C(n,\psi)\kappa^{\frac{1}{8}} \to 0
\end{aligned} \tag{4.18}
$$

as $\kappa \to 0$;

$$
\begin{aligned}
&\kappa\int_0^T \int_\Omega \left|\psi(t)\varphi_n'(\mathbf{v}_\kappa)\sqrt{\rho_\kappa}\nabla\phi_K(\rho_\kappa)\Delta\sqrt{\rho_\kappa}\right| \, dx\, dt \\
&\quad \le 2C(n,\psi)\kappa^{\frac{1}{4}}\left(\kappa^{\frac{1}{4}}\|\nabla\rho_\kappa^{\frac{1}{4}}\|_{L^4((0,T;L^4(\Omega))}\right) \\
&\quad\ \ \|\sqrt{\kappa}\Delta\sqrt{\rho_\kappa}\|_{L^2((0,T;L^2(\Omega))}\|\rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \\
&\quad \le 2C(n,\psi)\kappa^{\frac{1}{4}} \to 0
\end{aligned} \tag{4.19}
$$

as $\kappa \to 0$, where we used $|\rho_\kappa\phi_K'(\rho_\kappa)| \le 1$. Finally

$$
\begin{aligned}
&\kappa\int_0^T \int_\Omega \left|\psi'(t)\varphi_n(\mathbf{v}_\kappa)\phi_K(\rho_\kappa)\nabla\sqrt{\rho_\kappa}\Delta\sqrt{\rho_\kappa}\right| \, dx\, dt \\
&\quad \le 2C(n,\psi)\kappa^{\frac{1}{4}}\left(\kappa^{\frac{1}{4}}\|\nabla\rho_\kappa^{\frac{1}{4}}\|_{L^4((0,T;L^4(\Omega))}\right) \\
&\quad\ \ \|\sqrt{\kappa}\Delta\sqrt{\rho_\kappa}\|_{L^2((0,T;L^2(\Omega))}\|\rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \\
&\quad \le 2C(n,\psi)\kappa^{\frac{1}{4}} \to 0
\end{aligned} \tag{4.20}
$$

as $\kappa \to 0$.

For the term $\mathbb{S}_\kappa = \phi_K(\rho_\kappa)\rho_\kappa(\mathbb{D}\mathbf{u}_\kappa + \kappa\frac{\Delta\sqrt{\rho_\kappa}}{\sqrt{\rho_\kappa}}\mathbb{I}) = \mathbb{S}_1 + \mathbb{S}_2$, we calculate as follows

$$
\begin{aligned}
&\int_0^T \int_\Omega \psi(t)\mathbb{S}_1 : \nabla(\varphi_n'(\mathbf{v}_\kappa))\, dx\, dt \\
&\quad = \int_0^T \int_\Omega \psi(t)\phi_K(\rho_\kappa)\rho_\kappa\mathbb{D}\mathbf{u}_\kappa : \nabla(\varphi_n'(\mathbf{v}_\kappa))\, dx\, dt \\
&\quad = \int_0^T \int_\Omega \psi(t)[\nabla\mathbf{u}_\kappa\varphi_n''(\mathbf{v}_\kappa)\rho_\kappa] : \mathbb{D}\mathbf{u}_\kappa(\phi_K(\rho_\kappa))^2\, dx\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega \psi(t)\rho_\kappa\phi_K(\rho_\kappa)\left(\mathbf{u}_\kappa^T\varphi_n''(\mathbf{v}_\kappa)\right)\mathbb{D}\mathbf{u}_\kappa\nabla(\phi_K(\rho_\kappa))\,dx\,dt\\
&= A_1 + A_2. \qquad (4.21)
\end{aligned}
$$

For A_1, by part a. of Lemma 2.2, we have

$$
\begin{aligned}
A_1 &= \int_0^T\int_\Omega \psi(t)[\nabla\mathbf{u}_\kappa\varphi_n''(\mathbf{v}_\kappa)\rho_\kappa]:\mathbb{D}\mathbf{u}_\kappa(\phi_K(\rho_\kappa))^2\,dx\,dt\\
&= 2\int_0^T\int_\Omega \psi(t)\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)(\phi_K(\rho_\kappa))^2\rho_\kappa\mathbb{D}\mathbf{u}_\kappa:\nabla\mathbf{u}_\kappa\,dx\,dt\\
&\quad+4\int_0^T\int_\Omega \psi(t)\rho_\kappa(\phi_K(\rho_\kappa))^2\tilde{\varphi}_n''(|\mathbf{v}_\kappa|^2)(\nabla\mathbf{u}_\kappa\mathbf{v}_\kappa\otimes\mathbf{v}_\kappa):\mathbb{D}\mathbf{u}_\kappa\,dx\,dt\\
&= A_{11}+A_{12}. \qquad (4.22)
\end{aligned}
$$

Notice that

$$
\mathbb{D}\mathbf{u}_\kappa:\nabla\mathbf{u}_\kappa = |\mathbb{D}\mathbf{u}_\kappa|^2,
$$

thus

$$
\begin{aligned}
A_1 &\geq 2\int_0^T\int_\Omega \psi(t)\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)(\phi_K(\rho_\kappa))^2\rho_\kappa|\mathbb{D}\mathbf{u}_\kappa|^2\,dx\,dt\\
&\quad-4\|\psi\|_{L^\infty}\int_0^T\int_\Omega \rho_\kappa|\nabla\mathbf{u}_\kappa|^2\,dx\,dt, \qquad (4.23)
\end{aligned}
$$

where we control A_{12}

$$
\begin{aligned}
A_{12} &\leq 4\int_0^T\int_\Omega |\psi(t)|\frac{|\mathbf{v}_\kappa|^2}{1+|\mathbf{v}_\kappa|^2}\rho_\kappa|\nabla\mathbf{u}_\kappa|^2\,dx\,dt\\
&\leq 4\|\psi\|_{L^\infty}\int_0^T\int_\Omega \rho_\kappa|\nabla\mathbf{u}_\kappa|^2\,dx\,dt\\
&\leq 4\|\psi\|_{L^\infty}\left(\int_\Omega\left(\rho_0|\mathbf{u}_0|^2+\frac{\rho_0^\gamma}{\gamma-1}+|\nabla\sqrt{\rho_0}|^2-r_0\log_-\rho_0\right)dx+2E_0\right),
\end{aligned}
$$

thanks to (1.16). For A_2, thanks to (2.8), we can control it as follows

$$
\begin{aligned}
|A_2| &\leq C(n,\psi)\|\sqrt{\rho_\kappa}\mathbb{D}\mathbf{u}_\kappa\|_{L^2((0,T;L^2(\Omega))}\|\rho_\kappa^{\frac{3}{4}}\|_{L^4((0,T;L^4(\Omega))}\\
&\quad\times(\kappa^{\frac14}\|\nabla\rho_\kappa^{\frac14}\|_{L^4((0,T;L^4(\Omega))})\frac{\|\phi_K'\sqrt{\rho_\kappa}\|_{L^\infty((0,T)\times\Omega)}}{\kappa^{\frac14}}\\
&\leq \frac{C}{\sqrt{K}\kappa^{\frac14}} = C\kappa^{\frac18}\to 0 \qquad (4.24)
\end{aligned}
$$

as $\kappa \to 0$. Note that the first right hand side term of (4.23) has a positive sign and control the limit using from the pressure (4.14). We need to treat the term related to $\mathbb{S}_2$,

$$\begin{aligned} &\kappa \int_0^T \int_\Omega \psi(t)\mathbb{S}_2 : \nabla(\varphi_n'(\mathbf{v}_\kappa))\,dx\,dt \\ &= \kappa \int_0^T \int_\Omega \psi(t)\nabla \mathbf{u}_\kappa \varphi_n''(\mathbf{v}_\kappa) : \sqrt{\rho_\kappa}\phi_K(\rho_\kappa)^2 \Delta\sqrt{\rho_\kappa}\,dx\,dt \\ &\quad + \kappa \int_0^T \int_\Omega \psi(t)\mathbf{u}_\kappa \phi_K(\rho_\kappa)\varphi_n''(\mathbf{v}_\kappa)\nabla\rho_\kappa\sqrt{\rho_\kappa}\phi_K'(\rho_\kappa)\Delta\sqrt{\rho_\kappa}\,dx\,dt \\ &= B_1 + B_2, \end{aligned} \tag{4.25}$$

we control B_1 as follows

$$\begin{aligned} |B_1| &\le C(n,\psi)\|\sqrt{\rho_\kappa}\nabla\mathbf{u}_\kappa\|_{L^2(0,T;L^2(\Omega))}\|\sqrt{\kappa}\Delta\sqrt{\rho_\kappa}\|_{L^2(0,T;L^2(\Omega))}\sqrt{\kappa} \\ &\le C\kappa^{\frac{1}{2}} \to 0 \end{aligned} \tag{4.26}$$

as $\kappa \to 0$, where we used $\|\phi_K^2\|_{L^\infty(0,T;L^\infty(\Omega))} \le C$.

For B_2, we have

$$\begin{aligned} |B_2| &\le C(n,\psi)\kappa^{\frac{1}{4}}\left(\kappa^{\frac{1}{4}}\|\nabla\rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\right) \\ &\quad \times \|\rho_\kappa^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))}\|\sqrt{\kappa}\Delta\sqrt{\rho_\kappa}\|_{L^2(0,T;L^2(\Omega))}\|\phi_K'(\rho_\kappa)\rho_\kappa\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\le C\kappa^{\frac{1}{4}} \to 0 \end{aligned} \tag{4.27}$$

as $\kappa \to 0$.

With (4.6)–(4.27), in particularly, letting $\kappa \to 0$ in (3.2), dropping the positive terms on the left side, we have the following inequality

$$\begin{aligned} &-\int_0^T \int_\Omega \psi'(t)\rho\varphi_n(\mathbf{u})\,dx\,dt \\ &\le 8\|\psi\|_{L^\infty}\left(\int_\Omega \left(\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla\sqrt{\rho_0}|^2 - r_0\log_-\rho_0\right)dx + 2E_0\right) \\ &\quad + \psi(0)\int_\Omega \rho_0\varphi_n(\mathbf{u}_0)\,dx + C(\|\psi\|_{L^\infty})\int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{u}|^2))\rho^{2\gamma-1}\,dx\,dt, \end{aligned}$$

which in turn gives us Lemma 4.2. □

5 Limit when $n \to \infty$

This section is dedicated to the proof of Theorem 1.1. For this, we obtain the Mellet–Vasseur type inequality for the weak solution of the compressible Navier–Stokes equation with drag forces. This is obtained by letting $n \to \infty$. Note that the weak solution $(\rho, \mathbf{u})$ does not depend on n. We start from Lemma 4.2. Our task is to bound the right term of (4.5),

$$\begin{aligned}
&C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{u}|^2))\rho^{2\gamma-1}\,dx\,dt \\
&\le C(\|\psi\|_{L^\infty}) \int_0^T \left(\int_\Omega (\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\,dx\right)^{\frac{2-\delta}{2}} \\
&\quad \times \left(\int_\Omega \rho(1 + \tilde{\varphi}_n'(|\mathbf{u}|^2))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}\,dt \\
&\le C(\|\psi\|_{L^\infty}) \int_0^T \left(\int_\Omega (\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}} \\
&\quad \times \left(\int_\Omega \rho(2 + \ln(1 + |\mathbf{u}|^2)))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}\,dt,
\end{aligned} \tag{5.1}$$

where we used part c of Lemma 2.2. By (4.5) and (5.1), we have

$$\begin{aligned}
&-\int_0^T \int_\Omega \psi'(t)\rho\varphi_n(\mathbf{u})\,dx\,dt \le \int_\Omega \rho_0\varphi_n(\mathbf{u}_0)\,dx \\
&+ 8\|\psi\|_{L^\infty}\left(\int_\Omega \left(\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla\sqrt{\rho_0}|^2 - r_0 \log_- \rho_0\right)dx + 2E_0\right) \\
&+ C(\|\psi\|_{L^\infty}) \int_0^T \left(\int_\Omega (\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}} \left(\int_\Omega \rho(2 + \ln(1 + |\mathbf{u}|^2)))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}\,dt.
\end{aligned}$$

Thanks to part e. of Lemma 2.2 and Monotone Convergence Theorem, we have

$$-\int_0^T \int_\Omega \psi'(t)\rho\varphi_n(\mathbf{u})\,dx\,dt \to -\int_0^T \int_\Omega \psi'(t)\rho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\,dx\,dt \tag{5.2}$$

as $n \to \infty$.

Letting $n \to \infty$, we have

$$
\begin{aligned}
&-\int_0^T \int_\Omega \psi'(t)\rho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\,dx\,dt \\
&\quad \le \psi(0)\int_\Omega \rho_0(1+|\mathbf{u}_0|^2)\ln(1+|\mathbf{u}_0|^2)\,dx \\
&\qquad + 8\|\psi\|_{L^\infty}\left(\int_\Omega\left(\rho_0|\mathbf{u}_0|^2+\frac{\rho_0^\gamma}{\gamma-1}+|\nabla\sqrt{\rho_0}|^2-r_0\log_-\rho_0\right)dx+2E_0\right) \\
&\qquad + C\int_0^T\left(\int_\Omega(\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}}\left(\int_\Omega\rho(2+\ln(1+|\mathbf{u}|^2))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}dt.
\end{aligned}
\tag{5.3}
$$

Taking

$$
\psi(t)\begin{cases} =1 & \text{if } t\le\tilde{t}-\frac{\epsilon}{2}\\ =\frac{1}{2}-\frac{t-\tilde{t}}{\epsilon} & \text{if } \tilde{t}-\frac{\epsilon}{2}\le t\le\tilde{t}+\frac{\epsilon}{2}\\ =0 & \text{if } t\ge\tilde{t}+\frac{\epsilon}{2},\end{cases}
\tag{5.4}
$$

then (5.3) gives for every $\tilde{t}\ge\frac{\epsilon}{2}$,

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_{\tilde{t}-\frac{\epsilon}{2}}^{\tilde{t}+\frac{\epsilon}{2}}\left(\int_\Omega\rho(1+|\mathbf{u}|^2)\ln(1+|\mathbf{u}|^2)\,dx\right)dt \\
&\quad \le \int_\Omega\rho_0(1+|\mathbf{u}_0|^2)\ln(1+|\mathbf{u}_0|^2)\,dx \\
&\qquad + 8\left(\int_\Omega\left(\rho_0|\mathbf{u}_0|^2+\frac{\rho_0^\gamma}{\gamma-1}+|\nabla\sqrt{\rho_0}|^2-r_0\log_-\rho_0\right)dx+2E_0\right) \\
&\qquad + C\int_0^T\left(\int_\Omega(\rho^{2\gamma-1-\frac{\delta}{2}})^{\frac{2}{2-\delta}}\right)^{\frac{2-\delta}{2}}\left(\int_\Omega\rho(2+\ln(1+|\mathbf{u}|^2))^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}dt.
\end{aligned}
$$

This gives Theorem 1.1 thanks to the Lebesgue point Theorem.

6 Weak solution without drag forces

The objective of this section is to apply Theorem 1.1 to prove Theorem 1.2. This provides the existence of global weak solutions to (1.1)–(1.2) by letting $r_0 \to 0$ and $r_1 \to 0$. Let $r = r_0 = r_1$, we use $(\rho_r, \mathbf{u}_r)$ to denote the weak solutions to (2.1) verifying Proposition 1.1 with $\kappa = 0$.

By (1.7) and (1.8), one obtains the following estimates,

$$\begin{aligned}
&\|\sqrt{\rho_r}\mathbf{u}_r\|_{L^\infty(0,T;L^2(\Omega))} \le C;\\
&\|\rho_r\|_{L^\infty(0,T;L^1\cap L^\gamma(\Omega))} \le C;\\
&\|\sqrt{\rho_r}\nabla\mathbf{u}_r\|_{L^2(0,T;L^2(\Omega))} \le C;\\
&\|\nabla\sqrt{\rho_r}\|_{L^\infty(0,T;L^2(\Omega))} \le C;\\
&\|\nabla\rho_r^{\gamma/2}\|_{L^2(0,T;L^2(\Omega))} \le C.
\end{aligned} \tag{6.1}$$

Theorem 1.1 gives us

$$\sup_{t\in[0,T]}\int_\Omega \rho_r|\mathbf{u}_r|^2\ln(1+|\mathbf{u}_r|^2)\,dx \le C. \tag{6.2}$$

Note that (6.1) and (6.2) are uniformly on r. Meanwhile, we have the following estimates from (1.7),

$$\begin{aligned}
&\int_0^T\int_\Omega r|\mathbf{u}_r|^2\,dx\,dt \le C,\\
&\int_0^T\int_\Omega r\rho_r|\mathbf{u}_r|^4\,dx\,dt \le C.
\end{aligned} \tag{6.3}$$

We construct solutions to the Navier–Stokes equations without drag forces by passing to the limits as $r \to 0$. Following the same line as in [32], we can show the convergence of the density and the pressure, prove the strong convergence of $\sqrt{\rho_r}\mathbf{u}_r$ in space $L^2_{loc}((0,T)\times\Omega)$, and the convergence of the diffusion terms. We remark that Theorem 1.1 is the key tool to show the strong convergence of $\sqrt{\rho_r}\mathbf{u}_r$. Here, we list all related convergence from [32]. In particular,

$$\begin{aligned}
&\sqrt{\rho_r} \to \sqrt{\rho} \quad \text{almost everywhere and strongly in } L^2_{loc}((0,T)\times\Omega)),\\
&\rho_r \to \rho \quad \text{in } C^0(0,T;L^{\frac{3}{2}}_{loc}(\Omega));
\end{aligned} \tag{6.4}$$

the convergence of pressure

$$\rho_r^\gamma \to \rho^\gamma \quad \text{strongly in } L^1_{loc}((0,T)\times\Omega); \tag{6.5}$$

the convergence of the momentum and $\sqrt{\rho_r}\mathbf{u}_r$

$$\begin{aligned}
&\rho_r\mathbf{u}_r \to \rho\mathbf{u} \quad \text{strongly in } L^2(0,T;L^p_{loc}(\Omega)) \quad \text{for } p\in[1,3/2);\\
&\sqrt{\rho_r}\mathbf{u}_r \to \sqrt{\rho}\mathbf{u} \quad \text{strongly in } L^2_{loc}((0,T)\times\Omega);
\end{aligned} \tag{6.6}$$

and the convergence of the diffusion terms

$$\begin{aligned} \rho_r \nabla \mathbf{u}_r &\to \rho \nabla \mathbf{u} \quad \text{in } \mathfrak{D}', \\ \rho_r \nabla^T \mathbf{u}_r &\to \rho \nabla^T \mathbf{u} \quad \text{in } \mathfrak{D}'. \end{aligned} \tag{6.7}$$

It remains to prove that terms $r\mathbf{u}_r$ and $r\rho_r|\mathbf{u}_r|^2\mathbf{u}_r$ tend to zero as $r \to 0$. Let ψ be any test function, then we estimate the term $r\mathbf{u}_r$

$$\begin{aligned} \left| \int_0^T \int_\Omega r\mathbf{u}_r \psi \, dx \, dt \right| &\leq \int_0^T \int_\Omega r^{\frac{1}{2}} r^{\frac{1}{2}} |\mathbf{u}_r||\psi| \, dx \, dt \\ &\leq \sqrt{r} \|\sqrt{r}\mathbf{u}_r\|_{L^2((0,T)\times\Omega)} \|\psi\|_{L^2((0,T)\times\Omega)} \to 0 \end{aligned} \tag{6.8}$$

as $r \to 0$, due to (6.3).

We also estimate $r\rho_r|\mathbf{u}_r|^2\mathbf{u}_r$ as follows

$$\begin{aligned} &\left| \int_0^T \int_\Omega r\rho_r|\mathbf{u}_r|^2\mathbf{u}_r \psi \, dx \, dt \right| \\ &\leq \sqrt{r} \|\sqrt{r}\sqrt{\rho_r}|\mathbf{u}_r|^2\|_{L^2((0,T)\times\Omega)} \|\sqrt{\rho_r}\mathbf{u}_r\|_{L^\infty(0,T;L^2(\Omega))} \|\psi\|_{L^\infty((0,T)\times\Omega)} \to 0 \end{aligned} \tag{6.9}$$

as $r \to 0$.

The global weak solutions to (2.1) verifying Proposition 1.1 with $\kappa = 0$ is in the following sense, that is, $(\rho_r, \mathbf{u}_r)$ satisfy the following weak formulation

$$\begin{aligned} &\int_\Omega \rho_r \mathbf{u}_r \cdot \psi \, dx|_{t=0}^{t=T} - \int_0^T \int_\Omega \rho_r \mathbf{u}_r \psi_t \, dx \, dt - \int_0^T \int_\Omega \rho_r \mathbf{u}_r \otimes \mathbf{u}_r : \nabla\psi \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \rho_r^\gamma \operatorname{div}\psi \, dx \, dt - \int_0^T \int_\Omega \rho \mathbb{D}\mathbf{u}_r : \nabla\psi \, dx \, dt \\ &= -r \int_0^T \int_\Omega \mathbf{u}_r \psi \, dx \, dt - r \int_0^T \int_\Omega \rho_r |\mathbf{u}_r|^2 \mathbf{u}_r \psi \, dx \, dt, \end{aligned} \tag{6.10}$$

where ψ is any test function.

Letting $r \to 0$ in the weak formulation (6.10), and applying (6.4)–(6.9), one obtains that

$$\begin{aligned} &\int_\Omega \rho \mathbf{u} \cdot \psi \, dx|_{t=0}^{t=T} - \int_0^T \int_\Omega \rho \mathbf{u} \psi_t \, dx \, dt - \int_0^T \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \nabla\psi \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \rho^\gamma \operatorname{div}\psi \, dx \, dt - \int_0^T \int_\Omega \rho \mathbb{D}\mathbf{u} : \nabla\psi \, dx \, dt = 0. \end{aligned} \tag{6.11}$$

Thus we proved Theorem 1.2 for any initial value $(\rho_0, \mathbf{u}_0)$ verifying (1.5) with the additional condition $\rho_0 \geq \frac{1}{m_0}$ and $\sqrt{\rho_0}\mathbf{u}_0 \in L^\infty(\Omega)$. This last condition can be dropped using [32].

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Appendix1: Proof of the Lemma 2.2

Proof We prove each statement one by one as follows:

- (a) Thanks to (2.4), we have $\varphi_n'(\mathbf{u}) = 2\tilde{\varphi}_n'(|\mathbf{u}|^2)\mathbf{u}$, and

$$\varphi_n''(\mathbf{u}) = 2\left(2\tilde{\varphi}_n''(|\mathbf{u}|^2)\mathbf{u}\otimes\mathbf{u} + \mathbf{I}\tilde{\varphi}_n'(|\mathbf{u}|^2)\right),$$

  where $\mathbf{I}$ is 3×3 identity matrix.
- (b) The statement of (b) follows directly from (2.5).
- (c) Integrating (2.5) with initial data $\tilde{\varphi}_n'(0) = 0$, one obtains

$$\tilde{\varphi}_n'(y) = \begin{cases} 1 + \ln(1+y) & \text{if } 0 \leq y < n, \\ 1 + 2\ln(1+n) - \ln(1+y), & \text{if } n \leq y \leq C_n \\ 0 & \text{if } y \geq C_n, \end{cases} \tag{7.1}$$

  Since

$$1 + 2\ln(1+n) - \ln(1+C_n) = 0,$$

  for any $y \geq 0$, (7.1) implies

$$\tilde{\varphi}_n'(y) \geq 0.$$

  For any $n \leq y \leq C_n$, we have

$$\begin{aligned} 1 + 2\ln(1+n) - \ln(1+y) &\leq 1 + 2\ln(1+y) - \ln(1+y) \\ &= 1 + \ln(1+y). \end{aligned}$$

  In one word, for any $y \geq 0$, we have

$$0 \leq \tilde{\varphi}_n'(y) \leq 1 + \ln(1+y).$$

- (d) By (a)–(c), it follows

$$\begin{aligned} |\varphi_n''(\mathbf{u})| &\leq 4|\tilde{\varphi}_n''||\mathbf{u}|^2 + 2|\tilde{\varphi}_n'| \leq 4\frac{|\mathbf{u}|^2}{1+|\mathbf{u}|^2} + 2(1+\ln(1+n)) \\ &\leq 6 + 2\ln(1+n). \end{aligned}$$

- (e) Integrating (7.1) with initial data $\tilde{\varphi}_n(0) = 0$, it gives (2.9). Moreover, thanks to (c), $\tilde{\varphi}_n(y)$ is an increasing function with respect to y for any fixed n. We have that $\tilde{\varphi}_n(y)$ is a nondecreasing function with respect to n for any fixed y.

□

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