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The Gross–Prasad conjecture and local theta correspondence

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Abstract We establish the Fourier–Jacobi case of the local Gross–Prasad conjecture for unitary groups, by using local theta correspondence to relate the Fourier–Jacobi case with the Bessel case established by Beuzart-Plessis. To achieve this, we prove two conjectures of Prasad on the precise description of the local theta correspondence for (almost) equal rank unitary dual pairs in terms of the local Langlands correspondence. The proof uses Arthur's multiplicity formula and thus is one of the first examples of a concrete application of this "global reciprocity law".

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Contents

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1 Introduction

In [15,16,23,24], a restriction problem in the representation theory of classical groups was studied and a precise conjecture was formulated for this restriction problem. This so-called *Gross–Prasad* (GP) conjecture has generated much interest in recent years.

1.1 Restriction problem

In this paper, we shall focus on the restriction problem for unitary groups. Thus, let F be a nonarchimedean local field of characteristic 0 and residue characteristic p, and let E be a quadratic field extension of F. Let V_{n+1} be a Hermitian space of dimension $n+1$ over E and W_n a skew-Hermitian space of dimension n over E. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1, so that we have a natural inclusion of their corresponding unitary groups $\mathrm{U}(V_n) \hookrightarrow \mathrm{U}(V_{n+1})$. In particular, if we set

$$G_n = \mathrm{U}(V_n) \times \mathrm{U}(V_{n+1}) \quad \text{or} \quad \mathrm{U}(W_n) \times \mathrm{U}(W_n)$$

and

$$H_n = \mathrm{U}(V_n) \quad \text{or} \quad \mathrm{U}(W_n),$$

then we have a diagonal embedding

$$\Delta : H_n \hookrightarrow G_n.$$

Let π be an irreducible smooth representation of G_n. In the Hermitian case, one is interested in determining

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \mathbb{C}).$$

We shall call this the *Bessel* case (B) of the GP conjecture. In the skew-Hermitian case, the restriction problem requires another piece of data: a Weil representation ω_{ψ,χ,W_n}, where ψ is a nontrivial additive character of F and

χ is a character of $E^\times$ whose restriction to $F^\times$ is the quadratic character $\omega_{E/F}$ associated to E/F by local class field theory. Then one is interested in determining

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \omega_{\psi,\chi,W_n}).$$

We shall call this the *Fourier–Jacobi* case (FJ) of the GP conjecture. To unify notation, we shall let $\nu = \mathbb{C}$ or ω_{ψ,χ,W_n} in the respective cases.

By surprisingly recent results of Aizenbud–Gourevitch–Rallis–Schiffmann [1] and Sun [56], it is known that the above Hom spaces have dimension at most 1. Thus the main issue is to determine when the Hom space is nonzero. In [15], an answer for this issue is formulated in the framework of the local Langlands correspondence, in its enhanced form due to Vogan [58] which takes into account all pure inner forms.

1.2 Local Langlands correspondence

More precisely, a pure inner form of $\mathrm{U}(V_n)$ is simply a group of the form $\mathrm{U}(V_n')$, where V_n' is a Hermitian space of dimension n over E; likewise in the skew-Hermitian case. Thus, a pure inner form of G_n is a group of the form

$$G_n' = \mathrm{U}(V_n') \times \mathrm{U}(V_{n+1}') \quad \text{or} \quad \mathrm{U}(W_n') \times \mathrm{U}(W_n'').$$

We say that such a pure inner form is *relevant* if

$$V_n' \subset V_{n+1}' \quad \text{or} \quad W_n' = W_n'',$$

and

$$V_{n+1}'/V_n' \cong V_{n+1}/V_n$$

in the Hermitian case. If G_n' is relevant, we set

$$H_n' = \mathrm{U}(V_n') \quad \text{or} \quad \mathrm{U}(W_n'),$$

so that we have a diagonal embedding

$$\Delta : H_n' \hookrightarrow G_n'.$$

Now suppose that ϕ is an L-parameter for the group G_n. Then ϕ gives rise to a Vogan L-packet Π_ϕ consisting of certain irreducible smooth representations of G_n and its (not necessarily relevant) pure inner forms G'_n. Moreover, after fixing a Whittaker datum for G_n, there is a natural bijection

$$\Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi),$$

where S_ϕ is the component group associated to ϕ. Thus an irreducible smooth representation of G_n is labelled by a pair (ϕ, η), where ϕ is an L-parameter for G_n and η is an irreducible character of S_ϕ.

By the recent work of Arthur [2], Mok [44], and Kaletha–Mínguez–Shin–White [33], together with the stabilization of the twisted trace formula established by Waldspurger and Mœglin–Waldspurger [43], the local Langlands correspondence for unitary groups is now unconditional, except that the general case of the weighted fundamental lemma has not been written; the work of Chaudouard–Laumon [8] is limited to the case of split groups.

1.3 Gross–Prasad conjecture

With this short preparation, the GP conjecture can be loosely stated as follows:

Gross–Prasad conjecture (i) *Given a generic L-parameter ϕ for G_n, there is a unique representation $\pi(\phi, \eta)$ in the Vogan L-packet Π_ϕ such that $\pi(\phi, \eta)$ is a representation of a relevant pure inner form G'_n and such that*

$$\mathrm{Hom}_{\Delta H'_n}(\pi(\phi, \eta), \nu) \neq 0.$$

(ii) *There is a precise recipe for the distinguished character η (which we will recall in Sect. 3.2 below).*

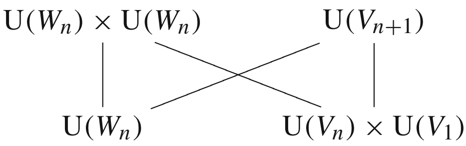
In a stunning series of papers [61–64], Waldspurger has established the Bessel case of the GP conjecture for special orthogonal groups in the case of tempered L-parameters; the case of general generic L-parameters is then dealt with by Mœglin–Waldspurger [42]. Beuzart-Plessis [4–6] has since extended Waldspurger's techniques to settle the Bessel case of the GP conjecture for unitary groups in the tempered case.

1.4 Purpose of this paper

The purpose of this paper is to establish the Fourier–Jacobi case of the GP conjecture, as well as two conjectures of Prasad concerning local theta correspondence in the (almost) equal rank case.

Let us describe the main idea of the proof. For simplicity, we restrict ourselves to the case of tempered L-parameters here. The Bessel and Fourier–Jacobi cases of the GP conjecture are related by the local theta correspondence.

More precisely, there is a see-saw diagram



and the associated see-saw identity reads:

$$\mathrm{Hom}_{\mathrm{U}(W_n)}(\Theta_{\psi,\chi,V_n,W_n}(\sigma)\otimes\omega_{\psi,\chi,V_1,W_n},\pi)$$
$$\cong\mathrm{Hom}_{\mathrm{U}(V_n)}(\Theta_{\psi,\chi,V_{n+1},W_n}(\pi),\sigma)$$

for irreducible smooth representations π of $\mathrm{U}(W_n)$ and σ of $\mathrm{U}(V_n)$. Hence the left-hand side of the see-saw identity concerns the Fourier–Jacobi case (FJ) whereas the right-hand side concerns the Bessel case (B). It is thus apparent that precise knowledge of the local theta correspondence for unitary groups of (almost) equal rank will give the precise relation of (FJ) to (B).

More precisely, one would need to know:

- (Θ) For irreducible tempered representations π and σ, the big theta lifts $\Theta_{\psi,\chi,V_{n+1},W_n}(\pi)$ and $\Theta_{\psi,\chi,V_n,W_n}(\sigma)$ are irreducible (if nonzero).
- (P1) If σ has parameter (ϕ,η) and $\Theta_{\psi,\chi,V_n,W_n}(\sigma)$ has parameter (ϕ',η'), then (ϕ',η') can be precisely described in terms of (ϕ,η).
- (P2) Likewise, if π has parameter (ϕ,η) and $\Theta_{\psi,\chi,V_{n+1},W_n}(\pi)$ has parameter (ϕ',η'), then (ϕ',η') can be precisely described in terms of (ϕ,η).

In fact, in [47,48], Prasad has formulated precise conjectures regarding (P1) and (P2) for the theta correspondence for $\mathrm{U}(V_n)\times\mathrm{U}(W_n)$ and $\mathrm{U}(V_{n+1})\times\mathrm{U}(W_n)$ respectively; we shall recall his conjectures precisely in Sect. 4. We shall also denote by (weak P1) the part of the conjecture (P1) concerning only the correspondence of L-parameters $\phi\mapsto\phi'$; likewise we have (weak P2). Then we recall that in our earlier paper [17], we have shown:

Proposition 1.1 *The statements* (Θ)*,* (weak P1) *and* (weak P2) *hold.*

Using Proposition 1.1, the first observation of this paper is:

Proposition 1.2 *Assume* (B) *and* (P2)*. Then* (FJ) *and* (P1) *follow.*

In view of Proposition 1.2 and the work of Beuzart-Plessis [4–6], it remains to show the statement (P2), and our main result is:

Theorem 1.3 *The conjecture* (P2)*, and hence* (FJ) *and* (P1)*, holds.*

Let us make a few comments about the results:

- In fact, we prove (P1) and (P2) for all (not necessarily tempered nor generic) L-parameters.
- We mention a related result of Mœglin [41] about the local theta correspondence for symplectic-orthogonal dual pairs of arbitrary rank. She considered A-packets for a large class of A-parameters, including all tempered L-parameters, and then determined the analog of the correspondence $(\phi, \eta) \mapsto (\phi', \eta')$ in the sense of Arthur, assuming that the correspondence is known for supercuspidal (and slightly more general) representations.
- It is interesting to note that in Proposition 1.2, the roles of (P1) and (P2) can be switched. In other words, it is also sufficient to prove (P1) in order to prove (FJ). We shall explain in the next subsection why we prefer to prove (P2).
- In [15], both the Bessel (B) and Fourier–Jacobi (FJ) cases of the GP conjecture were formulated for pairs of spaces $V_n \subset V_{n+2k+1}$ or $W_n \subset W_{n+2k}$ for any nonnegative integer k and for any generic L-parameters for $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+2k+1})$ or $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+2k})$. Beuzart-Plessis [4–6] has in fact verified (B) for all tempered L-parameters for $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+2k+1})$. In §9, we check that the argument as in [42] gives (B) for all generic L-parameters for $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+2k+1})$ and then show that Theorem 1.3 continues to hold for all generic L-parameters for $\mathrm{U}(W_n) \times \mathrm{U}(W_n)$.
- On the other hand, it was shown in [15, Theorem 19.1] that the GP conjecture in the case of generic L-parameters for $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+2k})$ (for all $k > 0$) follows from that for $\mathrm{U}(W_n) \times \mathrm{U}(W_n)$. Namely, we can deduce from Theorem 1.3 the following:

Corollary 1.4 *The Fourier–Jacobi case of the GP conjecture holds for all generic L-parameters for* $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+2k})$ *for any* $k \geq 0$.

1.5 Prasad's conjectures

Given Proposition 1.1, the main work is to determine how η' depends on (ϕ, η) in (P1) and (P2). In fact, the precise determination of η' in (P1) is a very subtle issue, as it depends on certain local roots numbers. In the case of (P2), the dependence of η' on (ϕ, η) is more simplistic.

The proof of (P2) proceeds by the following steps:

- First, by our results in [17], the nontempered case can be reduced to the tempered case on smaller unitary groups.
- Next, we show that the tempered case can be reduced to the square-integrable case on smaller unitary groups. This is achieved by a nontrivial extension of the techniques in the PhD thesis of the second author [31] and uses the delicate details of the normalization of the intertwining operators involved in the local intertwining relation [2,33,44].

- Finally, we show the square-integrable case by a global argument. More precisely, we shall globalize an irreducible square-integrable representation π of $\mathrm{U}(W_n)$ to an irreducible cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ such that
  - Π_v is not square-integrable for all places outside the place of interest, so that (P2) is known for Π_v outside the place of interest,
  - Π has tempered A-parameter whose global component group is equal to the local component group of the L-parameter of π,
  - Π has nonzero global theta lift to a unitary group which globalizes $\mathrm{U}(V_{n+1})$.

  The desired result then follows for the place of interest by applying Arthur's multiplicity formula for the automorphic discrete spectrum, which can be viewed as a sort of product formula (see (6.3)).

We can now explain why we prefer to prove (P2) rather than (P1). Note that one could attempt to follow the same strategy of proof for the statement (P1). However, in the globalization step above, we need to ensure that Π has nonzero global theta lift to a certain unitary group. For the case of (P1), the nonvanishing of this global theta lift is controlled by the nonvanishing of $L(\frac{1}{2}, \Pi)$, and it is well-known that the nonvanishing of this central critical value is a very subtle issue with arithmetic implications. On the other hand, for the statement (P2), the nonvanishing of the global theta lift of Π is governed by the nonvanishing of $L(1, \Pi)$. Now it is certainly much easier to ensure the nonvanishing of $L(1, \Pi)$ compared to $L(\frac{1}{2}, \Pi)$. For example, if Π has tempered A-parameter, then one knows that $L(1, \Pi) \neq 0$. It is for this reason that we prove (P2) rather than (P1).

1.6 3 Birds and 2 stones

To summarise, in proving our main theorem, we have killed "3 birds" [i.e. (FJ), (P1) and (P2)] with "2 stones" [i.e. (B) and Arthur's multiplicity formula], though it is probably more accurate to describe the latter as two cannon balls. We stress however that no animals (besides the two authors) have suffered in the preparation of this article.

Notation

Let F be a nonarchimedean local field of characteristic 0 and residue characteristic p. We fix an algebraic closure $\bar{F}$ of F. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group of F and W_F the Weil group of F. Let $|\cdot|_F$ be the normalized absolute value on F. We fix a nontrivial additive character ψ of F.

Let E be a quadratic field extension of F and $\omega_{E/F}$ the quadratic character of $F^\times$ associated to E/F by local class field theory. Let c denote the nontrivial Galois automorphism of E over F. Let $\mathrm{Tr}_{E/F}$ and $\mathrm{N}_{E/F}$ be the trace and norm maps from E to F. We choose an element $\delta \in E^\times$ such that $\mathrm{Tr}_{E/F}(\delta) = 0$. We write $|\cdot| = |\cdot|_E$ for the normalized absolute value on E. Let ψ_E be the nontrivial additive character of E defined by $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$.

If G is a linear algebraic group over F, we identify G with its group of F-rational points $G(F)$. For any totally disconnected locally compact group G, let $\mathbb{1}_G$ be the trivial representation of G and $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible smooth representations of G. For any set X, let 1_X be the identity map of X. For any positive integer n, let 1_n be the identity matrix in GL_n.

2 Local Langlands correspondence

In this section, we summarize some properties of the local Langlands correspondence for unitary groups.

2.1 Hermitian and skew-Hermitian spaces

Fix $\varepsilon = \pm 1$. Let V be a finite dimensional vector space over E equipped with a nondegenerate ε-Hermitian c-sesquilinear form

$$\langle \cdot, \cdot \rangle_V : V \times V \longrightarrow E.$$

Thus we have

$$\begin{aligned} \langle av, bw \rangle_V &= ab^c \langle v, w \rangle_V, \\ \langle w, v \rangle_V &= \varepsilon \cdot \langle v, w \rangle_V^c \end{aligned}$$

for $v, w \in V$ and $a, b \in E$. Put $n = \dim V$ and $\mathrm{disc}\, V = (-1)^{(n-1)n/2} \cdot \det V$, so that

$$\mathrm{disc}\, V \in \begin{cases} F^\times / \mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = +1; \\ \delta^n \cdot F^\times / \mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = -1. \end{cases}$$

We define $\epsilon(V) = \pm 1$ by

$$\epsilon(V) = \begin{cases} \omega_{E/F}(\mathrm{disc}\, V) & \text{if } \varepsilon = +1; \\ \omega_{E/F}(\delta^{-n} \cdot \mathrm{disc}\, V) & \text{if } \varepsilon = -1. \end{cases}$$

Given a positive integer n, there are precisely two isometry classes of n-dimensional ε-Hermitian spaces V, which are distinguished from each other by their signs $\epsilon(V)$. Note that $\epsilon(V)$ depends on the choice of δ if $\varepsilon = -1$ and n is odd. Let $\mathrm{U}(V)$ be the unitary group of V, i.e. the connected reductive linear algebraic group over F defined by

$$\mathrm{U}(V) = \{g \in \mathrm{GL}(V) \mid \langle gv, gw\rangle_V = \langle v, w\rangle_V \quad \text{for } v, w \in V\}.$$

If $n = 0$, we interpret $\mathrm{U}(V)$ as the trivial group $\{1\}$.

2.2 L-parameters and component groups

Let W_E be the Weil group of E and $WD_E = W_E \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group of E. We say that a continuous homomorphism $\phi : WD_E \to \mathrm{GL}_n(\mathbb{C})$ is a representation of WD_E if

- ϕ is semisimple,
- the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic.

We say that ϕ is tempered if the image of W_E is bounded. Let $\phi^\vee$ be the contragredient representation of ϕ defined by $\phi^\vee(w) = {}^t\phi(w)^{-1}$. Fix $s \in W_F \setminus W_E$ and define a representation ϕ^c of WD_E by $\phi^c(w) = \phi(sws^{-1})$. Then the equivalence class of ϕ^c is independent of the choice of s. We say that ϕ is conjugate self-dual if there is a nondegenerate bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which satisfies

$$B(\phi(w)x, \phi^c(w)y) = B(x, y)$$

for all $w \in WD_E$ and $x, y \in \mathbb{C}^n$. Namely, ϕ is conjugate self-dual if and only if ϕ^c is equivalent to $\phi^\vee$. For $b = \pm 1$, we say that ϕ is conjugate self-dual with sign b if there is a nondegenerate bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which satisfies the above condition and the condition that

$$B(y, x) = b \cdot B(x, \phi(s^2)y)$$

for all $x, y \in \mathbb{C}^n$. Note that the sign b depends not only on ϕ but also on B. We also say that ϕ is conjugate orthogonal (resp. conjugate symplectic) if it is conjugate self-dual with sign $+1$ (resp. -1). If ϕ is conjugate self-dual with sign b (with respect to a bilinear form B), then $\det\phi$ is conjugate self-dual with sign b^n. By [15, Lemma 3.4], a character χ of $E^\times$ (or rather the character of WD_E associated to χ by local class field theory) is conjugate orthogonal (resp. conjugate symplectic) if and only if $\chi|_{F^\times} = \mathbb{1}_{F^\times}$ (resp. $\chi|_{F^\times} = \omega_{E/F}$).

By [15, Sect. 8], an L-parameter for the unitary group $\mathrm{U}(V)$ is an n-dimensional conjugate self-dual representation ϕ of WD_E with sign $(-1)^{n-1}$.

We may decompose ϕ into a direct sum

$$\phi = \bigoplus_i m_i \phi_i$$

with pairwise inequivalent irreducible representations ϕ_i of WD_E and multiplicities m_i. We say that ϕ is square-integrable if it is multiplicity-free (so that $m_i = 1$ for all i) and ϕ_i is conjugate self-dual with sign $(-1)^{n-1}$ for all i.

For an L-parameter ϕ for $\mathrm{U}(V)$, fix a bilinear form B as above and let $\mathrm{Aut}(\phi, B)$ be the group of elements in $\mathrm{GL}_n(\mathbb{C})$ which centralize the image of ϕ and preserve B. Let

$$S_\phi = \mathrm{Aut}(\phi, B) / \mathrm{Aut}(\phi, B)^0$$

be the component group of ϕ, where $\mathrm{Aut}(\phi, B)^0$ is the identity component of $\mathrm{Aut}(\phi, B)$. As shown in [15, Sect. 8], S_ϕ has an explicit description of the form

$$S_\phi = \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j$$

with a canonical basis $\{a_j\}$, where the product ranges over all j such that ϕ_j is conjugate self-dual with sign $(-1)^{n-1}$. In particular, S_ϕ is an elementary abelian 2-group. We shall let z_ϕ denote the image of $-1 \in \mathrm{GL}_n(\mathbb{C})$ in S_ϕ. More explicitly, we have

$$z_\phi = (m_j a_j) \in \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j.$$

2.3 Local Langlands correspondence

The local Langlands correspondence for general linear groups, which was established by Harris–Taylor [26], Henniart [29], and Scholze [51], is a certain bijection between $\mathrm{Irr}(\mathrm{GL}_n(E))$ and equivalence classes of n-dimensional representations of WD_E. This bijection satisfies natural properties which determine it uniquely. For example, if π is an irreducible smooth representation of $\mathrm{GL}_n(E)$ with central character ω_π and ϕ is the n-dimensional representation of WD_E associated to π, then

- $\omega_\pi = \det \phi$,
- π is essentially square-integrable if and only if ϕ is irreducible,
- π is tempered if and only if ϕ is tempered.

The local Langlands correspondence (as enhanced by Vogan [58]) for unitary groups says that there is a canonical partition

$$\mathrm{Irr}(\mathrm{U}(V^+)) \sqcup \mathrm{Irr}(\mathrm{U}(V^-)) = \bigsqcup_{\phi} \Pi_\phi,$$

where V^+ and V^- are the n-dimensional ε-Hermitian spaces with $\epsilon(V^+) = +1$ and $\epsilon(V^-) = -1$, the disjoint union on the right-hand side runs over all equivalence classes of L-parameters ϕ for $\mathrm{U}(V^\pm)$, and Π_ϕ is a finite set of representations known as a Vogan L-packet. We may decompose Π_ϕ as

$$\Pi_\phi = \Pi_\phi^+ \sqcup \Pi_\phi^-,$$

where for $\epsilon = \pm 1$, Π_ϕ^ϵ consists of the representations of $\mathrm{U}(V^\epsilon)$ in Π_ϕ.

2.4 Whittaker data

To describe the L-packet Π_ϕ more precisely, it is necessary to choose a Whittaker datum, which is a conjugacy class of pairs (N, ψ_N), where

- N is the unipotent radical of a Borel subgroup of the quasi-split unitary group $\mathrm{U}(V^+)$,
- ψ_N is a generic character of N.

Then relative to this datum, there is a canonical bijection

$$J_{\psi_N} : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi).$$

When n is odd, such a datum is canonical. When n is even, as explained in [15, Sect. 12], it is determined by the choice of an $\mathrm{N}_{E/F}(E^\times)$-orbit of nontrivial additive characters

$$\begin{cases} \psi^E : E/F \to \mathbb{C}^\times & \text{if } \varepsilon = +1; \\ \psi : F \to \mathbb{C}^\times & \text{if } \varepsilon = -1. \end{cases}$$

According to this choice, we write

$$\begin{cases} J_{\psi^E} & \text{if } \varepsilon = +1; \\ J_\psi & \text{if } \varepsilon = -1 \end{cases}$$

for J_{ψ_N}. We formally adopt the same notation when n is odd. Suppose that $\varepsilon = +1$, so that V^+ and V^- are Hermitian spaces. Let $W^+ = \delta \cdot V^+$ be the space V^+ equipped with the skew-Hermitian form $\delta \cdot \langle \cdot, \cdot \rangle_{V^+}$. Similarly, we define the skew-Hermitian space $W^- = \delta \cdot V^-$. Then for $\epsilon = \pm 1$, $\mathrm{U}(V^\epsilon)$ and $\mathrm{U}(W^\epsilon)$ are physically equal. For a given ϕ, let J_{ψ^E} and J_ψ be the above bijections for $\mathrm{U}(V^\pm)$ and $\mathrm{U}(W^\pm)$ respectively. One has:

- if n is even, then

$$J_{\psi^E} = J_\psi \iff \psi^E(x) = \psi\left(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\right),$$

- if n is odd, then $J_{\psi^E} = J_\psi$.

Having fixed the Whittaker datum (N, ψ_N), we shall write $\pi(\phi, \eta)$ or simply $\pi(\eta)$ for the irreducible smooth representation in Π_ϕ corresponding to $\eta \in \mathrm{Irr}(S_\phi)$ under the bijection J_{ψ_N}. If ϕ is tempered, then for any Whittaker datum (N, ψ'_N), there is a unique (N, ψ'_N)-generic representation of $\mathrm{U}(V^+)$ in Π_ϕ by [5, Lemme 7.10.1], and the irreducible characters of S_ϕ associated to these generic representations under the bijection J_{ψ_N} are described as follows:

- The unique (N, ψ_N)-generic representation of $\mathrm{U}(V^+)$ in Π_ϕ corresponds to the trivial character of S_ϕ.
- When n is even, there are precisely two Whittaker datum. If (N, ψ'_N) is not conjugate to (N, ψ_N), then by [32, Sect. 3], the unique (N, ψ'_N)-generic representation of $\mathrm{U}(V^+)$ in Π_ϕ corresponds to the character η_- of S_ϕ given by

$$\eta_-(a_j) = (-1)^{\dim \phi_j}.$$

The character η_- has a role even when n is odd. Indeed, if n is odd, we may take $V^- = a \cdot V^+$, i.e. the space V^+ equipped with the Hermitian form $a \cdot \langle \cdot, \cdot \rangle_{V^+}$, where $a \in F^\times \smallsetminus \mathrm{N}_{E/F}(E^\times)$. Then $\mathrm{U}(V^+)$ and $\mathrm{U}(V^-)$ are physically equal. Under this identification, we have

$$\Pi_\phi^+ = \Pi_\phi^-$$

for any ϕ. Let $\pi = \pi(\phi, \eta)$ be a representation of $\mathrm{U}(V^+)$ in Π_ϕ. If we regard π as a representation of $\mathrm{U}(V^-)$ via the above identification, then it has associated character $\eta \cdot \eta_-$. In particular, if ϕ is tempered, then the unique (N, ψ_N)-generic representation of $\mathrm{U}(V^-)$ in Π_ϕ corresponds to η_-.

2.5 Properties of the local Langlands correspondence

We highlight some properties of the local Langlands correspondence which are used in this paper:

- $\pi(\phi, \eta)$ is a representation of $\mathrm{U}(V^\epsilon)$ if and only if $\eta(z_\phi) = \epsilon$.
- $\pi(\phi, \eta)$ is square-integrable if and only if ϕ is square-integrable.
- $\pi(\phi, \eta)$ is tempered if and only if ϕ is tempered.

- If ϕ is tempered but not square-integrable, then we can write

$$\phi = \phi_1 \oplus \phi_0 \oplus (\phi_1^c)^\vee,$$

where

- ϕ_1 is a k-dimensional irreducible representation of WD_E for some positive integer k,
- ϕ_0 is a tempered L-parameter for $\mathrm{U}(V_0^\pm)$, where $V_0^\pm$ are the ε-Hermitian spaces of dimension $n-2k$ over E.

Note that there is a natural embedding $S_{\phi_0} \hookrightarrow S_\phi$. Let $\eta_0 \in \mathrm{Irr}(S_{\phi_0})$ and put $\epsilon = \eta_0(z_{\phi_0})$. We can write

$$V^\epsilon = X \oplus V_0^\epsilon \oplus X^*,$$

where X and X^* are k-dimensional totally isotropic subspaces of V^ϵ such that $X \oplus X^*$ is nondegenerate and orthogonal to V_0^ϵ. Let P be the maximal parabolic subgroup of $\mathrm{U}(V^\epsilon)$ stabilizing X and M its Levi component stabilizing X^*, so that

$$M \cong \mathrm{GL}(X) \times \mathrm{U}(V_0^\epsilon).$$

Let τ be the irreducible (unitary) square-integrable representation of $\mathrm{GL}(X)$ associated to ϕ_1, and let $\pi_0 = \pi(\phi_0, \eta_0)$ be the irreducible tempered representation of $\mathrm{U}(V_0^\epsilon)$ in Π_{ϕ_0} corresponding to η_0. Then the induced representation $\mathrm{Ind}_P^{\mathrm{U}(V^\epsilon)}(\tau \otimes \pi_0)$ has a decomposition

$$\mathrm{Ind}_P^{\mathrm{U}(V^\epsilon)}(\tau \otimes \pi_0) = \bigoplus_\eta \pi(\phi, \eta),$$

where the sum ranges over all $\eta \in \mathrm{Irr}(S_\phi)$ such that $\eta|_{S_{\phi_0}} = \eta_0$. Moreover, if ϕ_1 is conjugate self-dual, let

$$R(w, \tau \otimes \pi_0) \in \mathrm{End}_{\mathrm{U}(V^\epsilon)}(\mathrm{Ind}_P^{\mathrm{U}(V^\epsilon)}(\tau \otimes \pi_0))$$

be the normalized intertwining operator defined in Sect. 7.3 below, where w is the unique nontrivial element in the relative Weyl group for M. Then the restriction of $R(w, \tau \otimes \pi_0)$ to $\pi(\phi, \eta)$ is the scalar multiplication by

$$\begin{cases} \epsilon^k \cdot \eta(a_1) & \text{if } \phi_1 \text{ has sign } (-1)^{n-1}; \\ \epsilon^k & \text{if } \phi_1 \text{ has sign } (-1)^n, \end{cases} \tag{2.1}$$

where a_1 corresponds to ϕ_1. These properties follow from the definition of η, induction in stages [33, Sect. 2.7], and the local intertwining relation [44, Theorem 3.4.3], [33, Theorem 2.6.2]. We also remark that the factor ϵ^k arises from the splitting $s' : W_\psi(M, G) \to \pi_0(N_\psi(M, G))$ defined in [33, Sect. 2.4.1], which can be explicated by using an analog of Lemma 7.2 below for the dual group.

- In general, we can write

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus (\phi_r^c)^\vee \oplus \cdots \oplus (\phi_1^c)^\vee,$$

where

- for $i = 1, \ldots, r$, ϕ_i is a k_i-dimensional representation of WD_E of the form $\phi_i = \phi_i' \otimes |\cdot|^{e_i}$ for some tempered representation ϕ_i' of WD_E and real number e_i such that

$$e_1 > \cdots > e_r > 0,$$

- ϕ_0 is a tempered L-parameter for $\mathrm{U}(V_0^\pm)$, where $V_0^\pm$ are the ε-Hermitian spaces of dimension $n - 2(k_1 + \cdots + k_r)$ over E.

Note that the natural map $S_{\phi_0} \to S_\phi$ is an isomorphism. Let $\eta \in \mathrm{Irr}(S_\phi)$ and put $\epsilon = \eta(z_\phi)$. We can write

$$V^\epsilon = X_1 \oplus \cdots \oplus X_r \oplus V_0^\epsilon \oplus X_r^* \oplus \cdots \oplus X_1^*,$$

where X_i and X_i^* are k_i-dimensional totally isotropic subspaces of V^ϵ such that $X_i \oplus X_i^*$ are nondegenerate, mutually orthogonal, and orthogonal to V_0^ϵ. Let P be the parabolic subgroup of $\mathrm{U}(V^\epsilon)$ stabilizing the flag

$$X_1 \subset X_1 \oplus X_2 \subset \cdots \subset X_1 \oplus \cdots \oplus X_r$$

and M its Levi component stabilizing the flag

$$X_1^* \subset X_1^* \oplus X_2^* \subset \cdots \subset X_1^* \oplus \cdots \oplus X_r^*,$$

so that

$$M \cong \mathrm{GL}(X_1) \times \cdots \times \mathrm{GL}(X_r) \times \mathrm{U}(V_0^\epsilon).$$

Then $\pi(\phi, \eta)$ is the unique irreducible quotient of the standard module

$$\mathrm{Ind}_P^{\mathrm{U}(V^\epsilon)}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0),$$

where for $i = 1, \ldots, r$, τ_i is the irreducible essentially tempered representation of $\mathrm{GL}(X_i)$ associated to ϕ_i, and $\pi_0 = \pi(\phi_0, \eta_0)$ is the irreducible tempered representation of $\mathrm{U}(V_0^\epsilon)$ in Π_{ϕ_0} corresponding to $\eta_0 := \eta|_{S_{\phi_0}} \in \mathrm{Irr}(S_{\phi_0})$.

- If $\pi = \pi(\phi, \eta)$, then the contragredient representation $\pi^\vee$ of π has L-parameter $\phi^\vee$ and associated character $\eta_{\pi^\vee} = \eta \cdot \nu$, where

$$\nu(a_j) = \begin{cases} \omega_{E/F}(-1)^{\dim \phi_j} & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

  Note that the component groups S_ϕ and $S_{\phi^\vee}$ are canonically identified. In the case of unitary groups, this property follows from a result of Kaletha [32, Sect. 4].

3 Gross–Prasad conjecture

In this section, we explicate the statement of the Gross–Prasad conjecture for unitary groups. In particular, we recall the definition of the distinguished character η of the component group.

3.1 Pairs of spaces

For $\epsilon = \pm 1$, let V_n^ϵ denote the n-dimensional Hermitian space with $\epsilon(V_n^\epsilon) = \epsilon$ and W_n^ϵ the n-dimensional skew-Hermitian space with $\epsilon(W_n^\epsilon) = \epsilon$, so that $W_n^\epsilon = \delta \cdot V_n^\epsilon$. For the Gross–Prasad conjecture, we consider the pair of spaces:

$$V_n^+ \subset V_{n+1}^+ \quad \text{or} \quad W_n^+ = W_n^+.$$

Then the relevant pure inner form (other than itself) is

$$V_n^- \subset V_{n+1}^- \quad \text{or} \quad W_n^- = W_n^-$$

and observe that

$$V_{n+1}^\epsilon / V_n^\epsilon \cong L_{(-1)^n},$$

where for $a \in F^\times$, L_a denotes the Hermitian line with form $a \cdot \mathrm{N}_{E/F}$. We have the groups

$$G_n^\epsilon = \mathrm{U}(V_n^\epsilon) \times \mathrm{U}(V_{n+1}^\epsilon) \quad \text{or} \quad \mathrm{U}(W_n^\epsilon) \times \mathrm{U}(W_n^\epsilon)$$

and

$$H_n^\epsilon = \mathrm{U}(V_n^\epsilon) \quad \text{or} \quad \mathrm{U}(W_n^\epsilon),$$

and the embedding

$$\Delta : H_n^\epsilon \hookrightarrow G_n^\epsilon.$$

We also have the Langlands–Vogan parametrization (depending on the choice of the Whittaker datum) relative to the fixed pair of spaces. For an L-parameter $\phi = \phi^\diamondsuit \times \phi^\heartsuit$ for $G_n^\pm$, the component group is:

$$S_\phi = S_{\phi^\diamondsuit} \times S_{\phi^\heartsuit}.$$

In particular, under the local Langlands correspondence, the representation $\pi(\eta) \in \Pi_\phi$ is a representation of a relevant pure inner form if and only if

$$\eta(z_{\phi^\diamondsuit}, z_{\phi^\heartsuit}) = 1,$$

and $\pi(\eta)$ is a representation of G_n^ϵ if and only if

$$\eta(z_{\phi^\diamondsuit}, 1) = \eta(1, z_{\phi^\heartsuit}) = \epsilon.$$

3.2 The distinguished character η

We shall now define a distinguished character $\eta \in \mathrm{Irr}(S_\phi)$ when $\phi = \phi^\diamondsuit \times \phi^\heartsuit$. Writing

$$S_{\phi^\diamondsuit} = \prod_i (\mathbb{Z}/2\mathbb{Z}) a_i \quad \text{and} \quad S_{\phi^\heartsuit} = \prod_j (\mathbb{Z}/2\mathbb{Z}) b_j,$$

we thus need to specify the signs $\eta(a_i) = \pm 1$ and $\eta(b_j) = \pm 1$. We consider the Bessel and Fourier–Jacobi cases separately.

- *Bessel case.* We fix a nontrivial character ψ^E of E/F which determines the local Langlands correspondence for the even unitary group in $G_n^\epsilon = \mathrm{U}(V_n^\epsilon) \times \mathrm{U}(V_{n+1}^\epsilon)$. We set $\psi_{-2}^E(x) = \psi^E(-2x)$ and define:

$$\begin{cases} \eta^\spadesuit(a_i) = \epsilon\left(\frac{1}{2}, \phi_i^\diamondsuit \otimes \phi^\heartsuit, \psi_{-2}^E\right); \\ \eta^\spadesuit(b_j) = \epsilon\left(\frac{1}{2}, \phi^\diamondsuit \otimes \phi_j^\heartsuit, \psi_{-2}^E\right). \end{cases}$$

- *Fourier–Jacobi case.* In this case, we need to fix a nontrivial character ψ of F and a character χ of $E^\times$ with $\chi|_{F^\times} = \omega_{E/F}$ to specify the Weil representation $\nu = \omega_{\psi,\chi,W_n^\epsilon}$ of $\mathrm{U}(W_n^\epsilon)$. The recipe for the distinguished character $\eta^\clubsuit$ of S_ϕ depends on the parity of $n = \dim W_n^\epsilon$.
  - If n is odd, recall that $\det W_n^+ \in \delta \cdot \mathrm{N}_{E/F}(E^\times)$ and define

$$\begin{cases} \eta^\clubsuit(a_i) = \epsilon\left(\frac{1}{2}, \phi_i^\diamondsuit \otimes \phi^\heartsuit \otimes \chi^{-1}, \psi_2^E\right); \\ \eta^\clubsuit(b_j) = \epsilon\left(\frac{1}{2}, \phi^\diamondsuit \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi_2^E\right), \end{cases}$$

    where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

  - If n is even, the fixed character ψ is used to fix the local Langlands correspondence for $\mathrm{U}(W_n^\epsilon)$. We set

$$\begin{cases} \eta^\clubsuit(a_i) = \epsilon\left(\frac{1}{2}, \phi_i^\diamondsuit \otimes \phi^\heartsuit \otimes \chi^{-1}, \psi^E\right); \\ \eta^\clubsuit(b_j) = \epsilon\left(\frac{1}{2}, \phi^\diamondsuit \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi^E\right), \end{cases}$$

    where the ϵ-factors are defined using any nontrivial additive character ψ^E of E/F. (The result is independent of this choice.)

We refer the reader to [15, Sect. 18] for a discussion of the various subtleties in the definition of $\eta^\spadesuit$ or $\eta^\clubsuit$.

3.3 Conjectures (B) and (FJ)

Let us formally state the statements $(\mathrm{B})_n$ and $(\mathrm{FJ})_n$:

$(\mathrm{B})_n$ Given a *tempered* L-parameter ϕ for $G_n^\pm = \mathrm{U}(V_n^\pm) \times \mathrm{U}(V_{n+1}^\pm)$ and a representation $\pi(\eta) \in \Pi_\phi$ of a relevant pure inner form G_n^ϵ,

$$\mathrm{Hom}_{\Delta H_n^\epsilon}(\pi(\eta), \mathbb{C}) \neq 0 \Longleftrightarrow \eta = \eta^\spadesuit.$$

$(\mathrm{FJ})_n$ Given a *tempered* L-parameter ϕ for $G_n^\pm = \mathrm{U}(W_n^\pm) \times \mathrm{U}(W_n^\pm)$ and a representation $\pi(\eta) \in \Pi_\phi$ of a relevant pure inner form G_n^ϵ,

$$\mathrm{Hom}_{\Delta H_n^\epsilon}(\pi(\eta), \nu) \neq 0 \Longleftrightarrow \eta = \eta^\clubsuit.$$

We shall denote by (B) the collection of statements $(\mathrm{B})_n$ for all $n \geq 0$, and by (FJ) the collection of statements $(\mathrm{FJ})_n$ for all $n \geq 0$. We stress that both (B) and (FJ) are considered only for tempered representations in this paper (except in Sect. 9 where we treat the case of generic L-parameters).

4 Local theta correspondence and Prasad's conjectures

In this section, we explicate the statement of Prasad's conjectures on the local theta correspondence for unitary groups of (almost) equal rank.

4.1 Weil representations

Let V be a Hermitian space and W a skew-Hermitian space. To consider the theta correspondence for the reductive dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$, one requires certain additional data:

(i) a nontrivial additive character ψ of F;
(ii) a pair of characters χ_V and χ_W of $E^\times$ such that

$$\chi_V|_{F^\times} = \omega_{E/F}^{\dim V} \quad \text{and} \quad \chi_W|_{F^\times} = \omega_{E/F}^{\dim W}.$$

One way to fix such a pair is simply to fix a character χ of $E^\times$ such that $\chi|_{F^\times} = \omega_{E/F}$ and then set

$$\chi_V = \chi^{\dim V} \quad \text{and} \quad \chi_W = \chi^{\dim W}.$$

(iii) a trace zero element $\delta \in E^\times$.

To elaborate, the tensor product $V \otimes W$ has a natural symplectic form defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \mathrm{Tr}_{E/F}(\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W).$$

Then there is a natural map

$$\mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes W).$$

One has the metaplectic S^1-cover $\mathrm{Mp}(V \otimes W)$ of $\mathrm{Sp}(V \otimes W)$, and the character ψ (together with the form $\langle \cdot, \cdot \rangle$ on $V \otimes W$) determines a Weil representation ω_ψ of $\mathrm{Mp}(V \otimes W)$. The data $(\psi, \chi_V, \chi_W, \delta)$ then allows one to specify a splitting of the metaplectic cover over $\mathrm{U}(V) \times \mathrm{U}(W)$, as shown in [25,37]. In fact, by construction and [25, Lemma A.7], it does not depend on the choice of δ.

Hence, we have a Weil representation $\omega_{\psi,\chi_V,\chi_W,V,W}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$. The Weil representation $\omega_{\psi,\chi_V,\chi_W,V,W}$ depends only on the orbit of ψ under $\mathrm{N}_{E/F}(E^\times)$.

4.2 Local theta correspondence

Given an irreducible smooth representation π of $\mathrm{U}(W)$, the maximal π-isotypic quotient of $\omega_{\psi,\chi_V,\chi_W,V,W}$ is of the form

$$\Theta_{\psi,\chi_V,\chi_W,V,W}(\pi) \boxtimes \pi$$

for some smooth representation $\Theta_{\psi,\chi_V,\chi_W,V,W}(\pi)$ of $\mathrm{U}(V)$ of finite length. By the Howe duality, which was proved by Waldspurger [59] for $p \neq 2$ and by the first author and Takeda [20,21] for any p (so that the assumption $p \neq 2$ can be removed from the results of [17] stated below), the maximal semisimple quotient $\theta_{\psi,\chi_V,\chi_W,V,W}(\pi)$ of $\Theta_{\psi,\chi_V,\chi_W,V,W}(\pi)$ is either zero or irreducible. If χ_V and χ_W are clear from the context, we simply write $\Theta_{\psi,V,W}(\pi) = \Theta_{\psi,\chi_V,\chi_W,V,W}(\pi)$ and $\theta_{\psi,V,W}(\pi) = \theta_{\psi,\chi_V,\chi_W,V,W}(\pi)$.

In this paper, we consider the theta correspondence for $\mathrm{U}(V) \times \mathrm{U}(W)$ with

$$|\dim V - \dim W| \leq 1.$$

We will state two conjectures of Prasad which describe the local theta correspondence in terms of the local Langlands correspondence.

4.3 Equal rank case

We first consider the case $\dim V = \dim W = n$. We shall consider the theta correspondence for $\mathrm{U}(V_n^{\epsilon}) \times \mathrm{U}(W_n^{\epsilon'})$. The following summarises some results of [17]:

Theorem 4.1 *Let ϕ be an L-parameter for $\mathrm{U}(W_n^{\pm})$. Then we have:*

(i) *For any fixed $\pi \in \Pi_{\phi}^{\epsilon'}$, exactly one of $\Theta_{\psi,V_n^+,W_n^{\epsilon'}}(\pi)$ or $\Theta_{\psi,V_n^-,W_n^{\epsilon'}}(\pi)$ is nonzero.*

(ii) *$\Theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi) \neq 0$ if and only if*

$$\epsilon\left(\tfrac{1}{2}, \phi \otimes \chi_V^{-1}, \psi_2^E\right) = \epsilon \cdot \epsilon',$$

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

(iii) *If $\Theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi)$ is nonzero, then $\theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi)$ has L-parameter*

$$\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_W.$$

(iv) *The theta correspondence* $\pi \mapsto \theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi)$ *gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi_{\theta(\phi)}.$$

(v) *If* ϕ *is tempered and* $\Theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi)$ *is nonzero, then* $\Theta_{\psi,V_n^{\epsilon},W_n^{\epsilon'}}(\pi)$ *is irreducible.*

4.4 Conjecture (P1)

After the above theorem, the remaining question is to specify the bijection of Vogan L-packets given in (iv). We shall do this using the bijections

$$J_\psi : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi) \quad \text{and} \quad J_{\psi^E} : \Pi_{\theta(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}),$$

where

$$\psi^E(x) = \psi\left(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\right). \tag{4.1}$$

Note that the bijections J_ψ and J_{ψ^E} are independent of ψ and ψ^E when n is odd, but when n is even, they do depend on these additive characters and it is crucial for ψ and ψ^E to be related as in (4.1) for what follows to hold.

Having fixed the bijections J_ψ and J_{ψ^E}, we need to describe the bijection

$$\begin{aligned} \mathrm{Irr}(S_\phi) &\longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}) \\ \eta &\longleftrightarrow \theta(\eta) \end{aligned}$$

induced by the theta correspondence. Note that the component groups S_ϕ and $S_{\theta(\phi)}$ are canonically identified, since $\theta(\phi)$ is simply a twist of ϕ by a conjugate orthogonal character.

Now the first conjecture of Prasad states the following.

(P1)$_n$ Let ϕ be an L-parameter for $\mathrm{U}(W_n^{\pm})$ and let $\eta \in \mathrm{Irr}(S_\phi)$. Suppose that

$$S_\phi = S_{\theta(\phi)} = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i.$$

Then, relative to J_ψ and J_{ψ^E} as above,

$$\theta(\eta)(a_i)/\eta(a_i) = \epsilon\left(\tfrac{1}{2}, \phi_i \otimes \chi_V^{-1}, \psi_2^E\right),$$

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

We shall denote by (P1) the collection of all statements $(\mathrm{P1})_n$ for all $n \geq 0$. Note that we consider (P1) for all L-parameters, and not just tempered ones. However, we note:

Proposition 4.2 *Suppose that $(P1)_k$ holds for all tempered L-parameters for all $k < n$. Then $(P1)_k$ holds for all nontempered L-parameters for all $k \leq n$.*

Proof This follows from the analog of [19, Theorem 8.1(iii)] for unitary groups. □

Moreover, the following is a corollary of Theorem 4.1(ii):

Corollary 4.3 *The statement $(P1)_n$ holds if ϕ is irreducible.*

4.5 Almost equal rank case

Now we consider the case $\dim V = n + 1$ and $\dim W = n$. We shall consider the theta correspondence for $\mathrm{U}(V^{\epsilon}_{n+1}) \times \mathrm{U}(W^{\epsilon'}_{n})$. The following summarises some results of [17]:

Theorem 4.4 *Let ϕ be an L-parameter for $\mathrm{U}(W^{\pm}_n)$. Then we have:*

(i) *Suppose that ϕ does not contain χ_V.*
  (a) *For any $\pi \in \Pi^{\epsilon'}_{\phi}$, $\Theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ is nonzero and $\theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi \otimes \chi_V^{-1} \chi_W) \oplus \chi_W.$$

  (b) *For each $\epsilon = \pm 1$, the theta correspondence $\pi \mapsto \theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ gives a bijection*

$$\Pi_{\phi} \longleftrightarrow \Pi^{\epsilon}_{\theta(\phi)}.$$

(ii) *Suppose that ϕ contains χ_V.*
  (a) *For any fixed $\pi \in \Pi^{\epsilon'}_{\phi}$, exactly one of $\Theta_{\psi, V^{+}_{n+1}, W^{\epsilon'}_n}(\pi)$ or $\Theta_{\psi, V^{-}_{n+1}, W^{\epsilon'}_n}(\pi)$ is nonzero.*
  (b) *If $\Theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ is nonzero, then $\theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi \otimes \chi_V^{-1} \chi_W) \oplus \chi_W.$$

  (c) *The theta correspondence $\pi \mapsto \theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ gives a bijection*

$$\Pi_{\phi} \longleftrightarrow \Pi_{\theta(\phi)}.$$

(iii) *If ϕ is tempered and $\Theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ is nonzero, then $\Theta_{\psi, V^{\epsilon}_{n+1}, W^{\epsilon'}_n}(\pi)$ is irreducible.*

4.6 Conjecture (P2)

After the above theorem, it remains to specify the bijections given in (i)(b) and (ii)(c). As in the case of (P1), we shall do this using the bijections

$$J_\psi : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi) \quad \text{and} \quad J_{\psi^E} : \Pi_{\theta(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}),$$

where

$$\psi^E(x) = \psi\left(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\right).$$

Note that J_ψ is independent of ψ when n is odd, whereas J_{ψ^E} is independent of ψ^E when n is even.

Observe that:

- If ϕ does not contain χ_V, then

$$S_{\theta(\phi)} = S_\phi \times (\mathbb{Z}/2\mathbb{Z})a_0,$$

  where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ arises from the summand χ_W in $\theta(\phi)$. Thus, for each ϵ, one has a canonical bijection

$$\begin{aligned} \mathrm{Irr}(S_\phi) &\longleftrightarrow \mathrm{Irr}^\epsilon(S_{\theta(\phi)}) \\ \eta &\longleftrightarrow \theta(\eta) \end{aligned}$$

  induced by the theta correspondence, where $\mathrm{Irr}^\epsilon(S_{\theta(\phi)})$ is the set of irreducible characters η' of $S_{\theta(\phi)}$ such that $\eta'(z_{\theta(\phi)}) = \epsilon$.
- On the other hand, if ϕ contains χ_V, then $\phi \otimes \chi_V^{-1}\chi_W$ contains χ_W, so that

$$S_{\theta(\phi)} = S_\phi.$$

  Thus, one has a canonical bijection

$$\begin{aligned} \mathrm{Irr}(S_\phi) &\longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}) \\ \eta &\longleftrightarrow \theta(\eta) \end{aligned}$$

  induced by the theta correspondence.

Now we can state the second conjecture of Prasad.

$(\mathrm{P2})_n$ Let ϕ be an L-parameter for $\mathrm{U}(W_n^\pm)$ and let $\eta \in \mathrm{Irr}(S_\phi)$. Fix the bijections J_ψ and J_{ψ^E} as above.

- If ϕ does not contain χ_V, then $\theta(\eta)$ is the unique irreducible character in $\mathrm{Irr}^{\epsilon}(S_{\theta(\phi)})$ such that

$$\theta(\eta)|_{S_\phi} = \eta.$$

- On the other hand, if ϕ contains χ_V, then

$$\theta(\eta) = \eta.$$

We shall denote by (P2) the collection of all the statements $(\mathrm{P2})_n$ for all $n \geq 0$. Note that we consider (P2) for all L-parameters, and not just tempered ones. However, we note:

Proposition 4.5 *Suppose that* $(\mathrm{P2})_k$ *holds for all tempered L-parameters for all $k < n$. Then* $(\mathrm{P2})_k$ *holds for all nontempered L-parameters for all $k \leq n$.*

Proof This follows from [17, Proposition C.4(ii)]. □

5 (B) + (P2) $\Longrightarrow$ (FJ) + (P1)

In this section, we shall show that Conjectures (FJ) and (P1) follow from Conjectures (B) and (P2), together with Theorems 4.1 and 4.4.

Suppose that we are given tempered L-parameters $\phi^{\diamondsuit}$ and $\phi^{\heartsuit}$ for $\mathrm{U}(W_n^{\pm})$. Let

$$\pi^{\diamondsuit} = \pi(\eta^{\diamondsuit}) \in \Pi^{\epsilon'}_{\phi^{\diamondsuit}} \quad \text{and} \quad \pi^{\heartsuit} = \pi(\eta^{\heartsuit}) \in \Pi^{\epsilon'}_{\phi^{\heartsuit}}$$

be representations such that

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\pi^{\diamondsuit} \otimes \pi^{\heartsuit}, \omega_{\psi,\chi,W_n^{\epsilon'}}) \neq 0.$$

We first show that

$$\eta^{\diamondsuit} \otimes \eta^{\heartsuit} = \eta^{\clubsuit}.$$

Since the representations involved are unitary (as $\phi^{\diamondsuit}$ and $\phi^{\heartsuit}$ are tempered),

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\pi^{\diamondsuit} \otimes \pi^{\heartsuit}, \omega_{\psi,\chi,W_n^{\epsilon'}}) \neq 0$$

if and only if

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}((\pi^{\diamondsuit})^{\vee} \otimes \omega_{\psi,\chi,W_n^{\epsilon'}}, \pi^{\heartsuit}) \neq 0.$$

5.1 See-Saw

Now we consider the see-saw diagram (for an ϵ to be determined soon):

$$\begin{array}{ccc} \mathrm{U}(W_n^{\epsilon'}) \times \mathrm{U}(W_n^{\epsilon'}) & & \mathrm{U}(V_{n+1}^{\epsilon}) \\ | & \times & | \\ \mathrm{U}(W_n^{\epsilon'}) & & \mathrm{U}(V_n^{\epsilon}) \times \mathrm{U}(L_{(-1)^n}) \end{array} .$$

We shall consider the local theta correspondence for the above see-saw diagram. For this, we need to specify precisely the data used in setting up the theta correspondence. More precisely, for the dual pair $\mathrm{U}(V_{n+1}^{\epsilon}) \times \mathrm{U}(W_n^{\epsilon'})$, we shall use the characters

$$\chi_{V_{n+1}^{\epsilon}} = \chi^{n+(-1)^n} \quad \text{and} \quad \chi_{W_n^{\epsilon'}} = \chi^n,$$

and for the dual pair $\mathrm{U}(V_n^{\epsilon}) \times \mathrm{U}(W_n^{\epsilon'})$, we use

$$\chi_{V_n^{\epsilon}} = \chi_{W_n^{\epsilon'}} = \chi^n.$$

Then for the dual pair $\mathrm{U}(L_{(-1)^n}) \times \mathrm{U}(W_n^{\epsilon'})$, we have no choice but to use

$$\chi_{L_{(-1)^n}} = \chi^{(-1)^n} \quad \text{and} \quad \chi_{W_n^{\epsilon'}} = \chi^n.$$

In particular, the restriction of $\omega_{\psi, \chi_{L_{(-1)^n}}, \chi_{W_n^{\epsilon'}}, L_{(-1)^n}, W_n^{\epsilon'}}$ to $\mathrm{U}(W_n^{\epsilon'})$ is equal to

$$\begin{cases} \omega_{\psi, \chi, W_n^{\epsilon'}} & \text{if } n \text{ is even;} \\ \omega_{\psi, \chi, W_n^{\epsilon'}}^{\vee} & \text{if } n \text{ is odd.} \end{cases}$$

In any case, having fixed these normalizations, we shall suppress them from the notation for simplicity.

Because of the above differences for even and odd n, it will now be convenient to treat the even and odd cases separately.

5.2 Even case

Assume first that n is even. By Theorem 4.1, we may choose $\sigma \in \mathrm{Irr}(\mathrm{U}(V_n^{\epsilon}))$ such that

$$\Theta_{\psi, V_n^{\epsilon}, W_n^{\epsilon'}}(\sigma) = (\pi^{\diamond})^{\vee}.$$

This uniquely determines ϵ. Moreover, by Theorem 4.1, we know that σ has L-parameter

$$\phi_\sigma = (\phi^\diamondsuit)^\vee,$$

since the L-parameter of $(\pi^\diamondsuit)^\vee$ is $(\phi^\diamondsuit)^\vee$.

Taking the representation $\pi^\heartsuit$ on $\mathrm{U}(W_n^{\epsilon'})$ and the representation σ on $\mathrm{U}(V_n^\epsilon)$, the resulting see-saw identity reads:

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}((\pi^\diamondsuit)^\vee \otimes \omega_{\psi,\chi,W_n^{\epsilon'}}, \pi^\heartsuit) \\ &= \mathrm{Hom}_{\mathrm{U}(V_n^\epsilon)}(\Theta_{\psi,V_{n+1}^\epsilon,W_n^{\epsilon'}}(\pi^\heartsuit), \sigma). \end{aligned}$$

By Theorem 4.4,

$$\tau := \Theta_{\psi,V_{n+1}^\epsilon,W_n^{\epsilon'}}(\pi^\heartsuit)$$

has L-parameter

$$\phi_\tau = (\phi^\heartsuit \otimes \chi^{-1}) \oplus \chi^n.$$

Recall that we have used the character ψ to fix the local Langlands correspondence for $\mathrm{U}(W_n^{\epsilon'})$. The component group $S_{\phi^\heartsuit}$ is of the form

$$S_{\phi^\heartsuit} = \prod_j (\mathbb{Z}/2\mathbb{Z}) b_j$$

and there is a natural embedding $S_{\phi^\heartsuit} \hookrightarrow S_{\phi_\tau}$. Now, by (P2), the representation τ has associated character $\eta_\tau \in \mathrm{Irr}(S_{\phi_\tau})$ which satisfies:

$$\eta_\tau = \eta^\heartsuit \quad \text{on} \quad S_{\phi^\heartsuit}.$$

On the other hand, by (B), one knows exactly what η_τ is. Namely, (B) gives:

$$\begin{aligned} \eta_\tau(b_j) &= \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi^E\right) \\ &= \epsilon\left(\tfrac{1}{2}, \phi^\diamondsuit \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi^E\right) \\ &= \eta^\clubsuit(b_j), \end{aligned}$$

where ψ^E is any nontrivial character of E/F. Thus, we deduce that

$$\eta^\heartsuit = \eta^\clubsuit \quad \text{on} \quad S_{\phi^\heartsuit}.$$

Now of course we could reverse the role of $\pi^{\diamondsuit}$ and $\pi^{\heartsuit}$ in the above argument. Then we conclude that

$$\eta^{\diamondsuit} \otimes \eta^{\heartsuit} = \eta^{\clubsuit}$$

as desired.

5.3 Odd case

Now suppose that n is odd. Then we use the character

$$\psi^E(x) = \psi\left(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\right)$$

of E/F to specify the local Langlands correspondence for $\mathrm{U}(V_{n+1}^{\epsilon})$. By Theorem 4.1, we may choose $\sigma \in \mathrm{Irr}(\mathrm{U}(V_n^{\epsilon}))$ such that

$$\Theta_{\psi, V_n^{\epsilon}, W_n^{\epsilon'}}(\sigma) = \pi^{\diamondsuit}.$$

This uniquely determines ϵ. Moreover, by Theorem 4.1, we know that σ has L-parameter

$$\phi_\sigma = \phi^{\diamondsuit}.$$

Taking the representation $(\pi^{\heartsuit})^\vee$ on $\mathrm{U}(W_n^{\epsilon'})$ and the representation σ on $\mathrm{U}(V_n^{\epsilon})$, the resulting see-saw identity reads:

$$\begin{aligned} 0 \neq & \mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\pi^{\diamondsuit} \otimes \omega^\vee_{\psi,\chi,W_n^{\epsilon'}}, (\pi^{\heartsuit})^\vee) \\ = & \mathrm{Hom}_{\mathrm{U}(V_n^{\epsilon})}(\Theta_{\psi, V_{n+1}^{\epsilon}, W_n^{\epsilon'}}((\pi^{\heartsuit})^\vee), \sigma). \end{aligned}$$

By Theorem 4.4,

$$\tau := \Theta_{\psi, V_{n+1}^{\epsilon}, W_n^{\epsilon'}}((\pi^{\heartsuit})^\vee)$$

has L-parameter

$$\phi_\tau = ((\phi^{\heartsuit})^\vee \otimes \chi) \oplus \chi^n.$$

Now by (P2), the representation τ has associated character $\eta_\tau \in \mathrm{Irr}(S_{\phi_\tau})$ satisfying:

$$\eta_\tau = \eta^{\heartsuit} \quad \text{on} \quad S_{\phi^{\heartsuit}} = S_{(\phi^{\heartsuit})^\vee \otimes \chi}.$$

On the other hand, by (B), we know that

$$\begin{aligned}\eta_\tau(b_j) &= \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes (\phi_j^\heartsuit)^\vee \otimes \chi, \psi_{-2}^E\right)\\ &= \epsilon\left(\tfrac{1}{2}, \phi^\diamondsuit \otimes \phi_j^\heartsuit \otimes \chi^{-1}, \psi_2^E\right)\\ &= \eta^\clubsuit(b_j).\end{aligned}$$

Hence, we conclude that

$$\eta^\heartsuit = \eta^\clubsuit \quad \text{on} \quad S_{\phi^\heartsuit}.$$

Reversing the role of $\pi^\diamondsuit$ and $\pi^\heartsuit$ in the above argument, we conclude that

$$\eta^\diamondsuit \otimes \eta^\heartsuit = \eta^\clubsuit$$

as desired.

5.4 Proof of (FJ)

At this point, we have shown that if

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\pi^\diamondsuit \otimes \pi^\heartsuit, \omega_{\psi,\chi,W_n^{\epsilon'}}) \neq 0,$$

then $\eta^\diamondsuit \otimes \eta^\heartsuit$ is equal to the distinguished character $\eta^\clubsuit$. To complete the proof of (FJ), it remains to show that the above Hom space is nonzero for some ϵ' and pair of representations $(\pi^\diamondsuit, \pi^\heartsuit) \in \Pi_{\phi^\diamondsuit}^{\epsilon'} \times \Pi_{\phi^\heartsuit}^{\epsilon'}$. This will follow from the above see-saw diagram, Theorems 4.1 and 4.4. Let us illustrate this in the case when n is even; the case when n is odd is similar.

Consider the tempered L-parameters $\phi := (\phi^\heartsuit \otimes \chi^{-1}) \oplus \chi^n$ for $\mathrm{U}(V_{n+1}^\pm)$ and $\phi' := (\phi^\diamondsuit)^\vee$ for $\mathrm{U}(V_n^\pm)$. By (B), there is a pair of representations

$$(\tau, \tau') \in \Pi_\phi^\epsilon \times \Pi_{\phi'}^\epsilon$$

such that

$$\mathrm{Hom}_{\mathrm{U}(V_n^\epsilon)}(\tau, \tau') \neq 0.$$

By Theorem 4.4, we can find a unique $\pi^\heartsuit \in \Pi_{\phi^\heartsuit}^{\epsilon'}$ (which determines ϵ') such that

$$\tau = \Theta_{\psi, V_{n+1}^\epsilon, W_n^{\epsilon'}}(\pi^\heartsuit).$$

Now the see-saw identity gives

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathrm{U}(V_n^\epsilon)}(\Theta_{\psi,V_{n+1}^\epsilon,W_n^{\epsilon'}}(\pi^\heartsuit),\tau') \\ &= \mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\Theta_{\psi,V_n^\epsilon,W_n^{\epsilon'}}(\tau')\otimes\omega_{\psi,\chi,W_n^{\epsilon'}},\pi^\heartsuit). \end{aligned}$$

In particular,

$$\pi^\diamondsuit := \Theta_{\psi,V_n^\epsilon,W_n^{\epsilon'}}(\tau')^\vee \neq 0$$

and by Theorem 4.1, it has L-parameter $(\phi')^\vee = \phi^\diamondsuit$. Thus we see that for some $(\pi^\diamondsuit,\pi^\heartsuit) \in \Pi_{\phi^\diamondsuit}^{\epsilon'} \times \Pi_{\phi^\heartsuit}^{\epsilon'}$, we have

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}((\pi^\diamondsuit)^\vee\otimes\omega_{\psi,\chi,W_n^{\epsilon'}},\pi^\heartsuit) \neq 0$$

as desired. This completes the proof of (FJ).

5.5 Proof of (P1)

Now we come to the proof of (P1). In particular, we consider the theta correspondence for $\mathrm{U}(V_n^\epsilon) \times \mathrm{U}(W_n^{\epsilon'})$ relative to the Weil representation $\omega_{\psi,\chi_V,\chi_W,V_n^\epsilon,W_n^{\epsilon'}}$. Given an L-parameter ϕ for $\mathrm{U}(W_n^\pm)$, we would like to explicate the bijection

$$\theta : \mathrm{Irr}(S_\phi) \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)})$$

furnished by Theorem 4.1, with $\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_W$. Here, recall that

$$S_\phi = S_{\theta(\phi)} = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i.$$

Since we now have (B), (FJ) and (P2) at our disposal, we shall be able to determine θ using the see-saw diagram.

More precisely, we start with a tempered L-parameter ϕ and consider an irreducible tempered representation $\pi = \pi(\eta) \in \Pi_\phi^{\epsilon'}$. One knows by Theorem 4.1 that $\Theta_{\psi,V_n^\epsilon,W_n^{\epsilon'}}(\pi) \in \Pi_{\theta(\phi)}^\epsilon$ is a nonzero irreducible tempered representation of $\mathrm{U}(V_n^\epsilon)$ for a unique ϵ. By the analog of [19, Lemma 12.5] for unitary groups, one can find an irreducible tempered representation σ of $\mathrm{U}(V_{n-1}^\epsilon)$ such that

$$\mathrm{Hom}_{\mathrm{U}(V_{n-1}^\epsilon)}(\Theta_{\psi,V_n^\epsilon,W_n^{\epsilon'}}(\pi),\sigma) \neq 0.$$

By (B), one has

$$\theta(\eta)(a_i) = \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes \phi_i \otimes \chi_V^{-1}\chi_W, \psi_{-2}^E\right),$$

where ϕ_σ is the L-parameter of σ.

On the other hand, one has the see-saw diagram

$$\begin{array}{ccc} \mathrm{U}(W_n^{\epsilon'}) \times \mathrm{U}(W_n^{\epsilon'}) & & \mathrm{U}(V_n^{\epsilon}) \\ | & \times & | \\ \mathrm{U}(W_n^{\epsilon'}) & & \mathrm{U}(V_{n-1}^{\epsilon}) \times \mathrm{U}(L_{(-1)^{n-1}}) \end{array}.$$

We consider the theta correspondence for $\mathrm{U}(L_{(-1)^{n-1}}) \times \mathrm{U}(W_n^{\epsilon'})$ relative to the pair of characters $(\chi^{(-1)^{n-1}}, \chi_W)$, so that the theta correspondence for $\mathrm{U}(V_{n-1}^{\epsilon}) \times \mathrm{U}(W_n^{\epsilon'})$ is with respect to the pair $(\chi_V\chi^{(-1)^n}, \chi_W)$. We shall suppress these pairs of characters from the notation in the following. By Theorem 4.4, the representation

$$\tau := \Theta_{\psi, V_{n-1}^{\epsilon}, W_n^{\epsilon'}}(\sigma) \neq 0$$

is irreducible and tempered. Moreover, τ has L-parameter

$$\phi_\tau = (\phi_\sigma \otimes \chi_V\chi_W^{-1}\chi^{(-1)^n}) \oplus \chi_V\chi^{(-1)^n}. \tag{5.1}$$

It will now be convenient to consider the even and odd cases separately.

5.6 Even case

Assume first that n is even. By the see-saw identity, one has

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\Theta_{\psi, V_{n-1}^{\epsilon}, W_n^{\epsilon'}}(\sigma) \otimes \omega_{\psi,\chi,W_n^{\epsilon'}}^\vee, \pi) \\ &= \mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\tau \otimes \pi^\vee, \omega_{\psi,\chi,W_n^{\epsilon'}}). \end{aligned}$$

It follows by (FJ) that

$$\eta(a_i) \cdot \omega_{E/F}(-1)^{\dim \phi_i} = \eta_{\pi^\vee}(a_i) = \epsilon\left(\tfrac{1}{2}, \phi_\tau \otimes \phi_i^\vee \otimes \chi^{-1}, \psi_2^E\right),$$

where the local root number appearing here is independent of the choice of the additive character of E/F used since $\dim \phi_\tau = n$ is even. Hence, by (5.1), one has

$$\eta(a_i) = \epsilon\left(\tfrac{1}{2}, \phi_\sigma \otimes \phi_i^\vee \otimes \chi_V \chi_W^{-1}, \psi_2^E\right) \times \epsilon\left(\tfrac{1}{2}, \phi_i^\vee \otimes \chi_V, \psi_2^E\right) \cdot \omega_{E/F}(-1)^{\dim \phi_i}.$$

Noting that ϕ_i is conjugate symplectic, we may compute:

$$\begin{aligned}\theta(\eta)(a_i)/\eta(a_i) &= \epsilon\left(\tfrac{1}{2}, \phi_i^\vee \otimes \chi_V, \psi_2^E\right) \cdot \omega_{E/F}(-1)^{\dim \phi_i} \\ &= \epsilon\left(\tfrac{1}{2}, \phi_i^\vee \otimes \chi_V, \psi_{-2}^E\right) \\ &= \epsilon\left(\tfrac{1}{2}, \phi_i \otimes \chi_V^{-1}, \psi_2^E\right)\end{aligned}$$

as desired.

5.7 Odd case

Now suppose that n is odd. By the see-saw identity, one has

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\tau \otimes \omega_{\psi,\chi,W_n^{\epsilon'}}, \pi) \neq 0,$$

so that

$$\mathrm{Hom}_{\mathrm{U}(W_n^{\epsilon'})}(\tau^\vee \otimes \pi, \omega_{\psi,\chi,W_n^{\epsilon'}}) \neq 0.$$

By (FJ), one has

$$\begin{aligned}\eta(a_i) &= \epsilon\left(\tfrac{1}{2}, \phi_\tau^\vee \otimes \phi_i \otimes \chi^{-1}, \psi_2^E\right) \\ &= \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes \phi_i \otimes \chi_V^{-1}\chi_W, \psi_2^E\right) \cdot \epsilon\left(\tfrac{1}{2}, \phi_i \otimes \chi_V^{-1}, \psi_2^E\right),\end{aligned}$$

where the second equality follows from (5.1). On the other hand, we have seen that

$$\begin{aligned}\theta(\eta)(a_i) &= \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes \phi_i \otimes \chi_V^{-1}\chi_W, \psi_{-2}^E\right) \\ &= \epsilon\left(\tfrac{1}{2}, \phi_\sigma^\vee \otimes \phi_i \otimes \chi_V^{-1}\chi_W, \psi_2^E\right),\end{aligned}$$

where the second equality follows because $\dim \phi_\sigma^\vee = n-1$ is even. Hence, we conclude that

$$\theta(\eta)(a_i)/\eta(a_i) = \epsilon\left(\tfrac{1}{2}, \phi_i \otimes \chi_V^{-1}, \psi_2^E\right)$$

as desired.

We have thus shown Conjecture (P1) for tempered L-parameters. For non-tempered L-parameters, (P1) follows from the tempered case by Proposition 4.2.

To summarise, we have shown the following proposition:

Proposition 5.1 *Assume that* $(\mathrm{B})_k$ *and* $(\mathrm{P2})_k$ *hold for all tempered* L*-parameters for all* $k \leq n$*. Then* $(\mathrm{FJ})_k$ *and* $(\mathrm{P1})_k$ *also hold for all tempered* L*-parameters for all* $k \leq n$*.*

5.8 (B) + (P1) $\Longrightarrow$ (FJ) + (P2)

Instead of assuming (B) and (P2) as we have done above, one may assume (B) and (P1). Using the same arguments as above, together with Theorems 4.1 and 4.4, one can then deduce (FJ) and (P2). We state this formally as a proposition and leave the details of the proof to the reader.

Proposition 5.2 *Assume that* $(\mathrm{B})_k$ *and* $(\mathrm{P1})_k$ *hold for all tempered* L*-parameters for all* $k \leq n$*. Then* $(\mathrm{FJ})_k$ *and* $(\mathrm{P2})_k$ *also hold for all tempered* L*-parameters for all* $k \leq n$*.*

6 Proof of (P2)

After the previous section, and in view of the results of Beuzart-Plessis [4–6] (who proves (B)), it remains to prove $(\mathrm{P2})_n$. We shall prove $(\mathrm{P2})_n$ by using induction on n.

6.1 The base cases

For $(\mathrm{P2})_0$, there is nothing to prove. By [16,25] and [5], we know that $(\mathrm{B})_1$ and $(\mathrm{P1})_1$ hold. Hence it follows by Proposition 5.2 that $(\mathrm{P2})_1$ holds.

For $(\mathrm{P2})_2$, the nontempered case follows from the tempered case by Proposition 4.5. To show $(\mathrm{P2})_2$ for tempered L-parameters, it follows by Proposition 5.2 that it suffices to show $(\mathrm{P1})_2$ for tempered L-parameters. Now $(\mathrm{P1})_2$ was shown in [16, Theorem 11.2] by a global argument, appealing to the analog of $(\mathrm{P1})_2$ at archimedean places. However, we can also give a purely local proof here.

Suppose that ϕ is a tempered L-parameter for $\mathrm{U}(W_2^{\pm})$ and we are considering the theta correspondence for $\mathrm{U}(V_2^{\epsilon}) \times \mathrm{U}(W_2^{\epsilon'})$ with respect to a pair of characters (χ_V, χ_W). If ϕ is irreducible, then Corollary 4.3 guarantees that

$(\mathrm{P1})_2$ holds. Hence we shall assume that $\phi = \phi_1 \oplus \phi_2$ with 1-dimensional characters ϕ_i. If ϕ_1 or ϕ_2 is not conjugate symplectic, then S_ϕ is trivial and $(\mathrm{P1})_2$ follows from Theorem 4.1. Thus, we shall further assume that both ϕ_1 and ϕ_2 are conjugate symplectic, so that

$$S_\phi = \begin{cases} (\mathbb{Z}/2\mathbb{Z})a_1 \times (\mathbb{Z}/2\mathbb{Z})a_2 & \text{if } \phi_1 \neq \phi_2; \\ ((\mathbb{Z}/2\mathbb{Z})a_1 \times (\mathbb{Z}/2\mathbb{Z})a_2)/\Delta\mathbb{Z}/2\mathbb{Z} & \text{if } \phi_1 = \phi_2. \end{cases}$$

To unify notation in the two cases, we shall regard $\mathrm{Irr}(S_\phi)$ as a subset of the irreducible characters of $(\mathbb{Z}/2\mathbb{Z})a_1 \times (\mathbb{Z}/2\mathbb{Z})a_2$ even when $\phi_1 = \phi_2$.

Let $\pi = \pi(\eta) \in \Pi_\phi^{\epsilon'}$. By Theorem 4.1, we know that the theta lift of π to $\mathrm{U}(V_2^\epsilon)$ is nonzero for a uniquely determined ϵ given by

$$\epsilon = \epsilon\left(\tfrac{1}{2}, \phi \otimes \chi_V^{-1}, \psi_2^E\right) \cdot \epsilon',$$

and has L-parameter

$$\theta(\phi) = \phi \otimes \chi_V^{-1} \chi_W.$$

Set

$$\sigma = \Theta_{\psi, V_2^\epsilon, W_2^{\epsilon'}}(\pi) \in \Pi_{\theta(\phi)}^\epsilon$$

and let $\theta(\eta) \in \mathrm{Irr}(S_{\theta(\phi)})$ be the irreducible character associated to σ. Then we need to compute $\theta(\eta)(a_i)/\eta(a_i)$.

Consider the decomposition

$$V_2^\epsilon = V_1^\epsilon \oplus L_{-1},$$

and choose a character $\mu \in \mathrm{Irr}(\mathrm{U}(V_1^\epsilon))$ such that

$$\mathrm{Hom}_{\mathrm{U}(V_1^\epsilon)}(\sigma, \mu) \neq 0.$$

Then by $(\mathrm{B})_1$, one sees that

$$\begin{aligned} \theta(\eta)(a_i) &= \epsilon\left(\tfrac{1}{2}, \mu_E^{-1} \phi_i \chi_V^{-1} \chi_W, \psi_{-2}^E\right) \\ &= \epsilon\left(\tfrac{1}{2}, \mu_E^{-1} \phi_i \chi_V^{-1} \chi_W, \psi_2^E\right) \cdot \omega_{E/F}(-1), \end{aligned} \tag{6.1}$$

where μ_E is the character of $E^\times$ given by $\mu_E(x) = \mu(x/x^c)$.

On the other hand, consider the see-saw diagram

$$\begin{array}{ccc} \mathrm{U}(W_2^{\epsilon'}) \times \mathrm{U}(W_2^{\epsilon'}) & & \mathrm{U}(V_2^{\epsilon}) \\ | & \times & | \\ \mathrm{U}(W_2^{\epsilon'}) & & \mathrm{U}(V_1^{\epsilon}) \times \mathrm{U}(L_{-1}) \end{array} .$$

For a conjugate symplectic character χ of $E^\times$, we consider the theta correspondences for

$$\mathrm{U}(V_1^{\epsilon}) \times \mathrm{U}(W_2^{\epsilon'}) \quad \text{with respect to} \quad (\chi_V \chi, \chi_W)$$

and

$$\mathrm{U}(L_{-1}) \times \mathrm{U}(W_2^{\epsilon'}) \quad \text{with respect to} \quad (\chi^{-1}, \chi_W).$$

Set

$$\tau := \Theta_{\psi, \chi_V \chi, \chi_W, V_1^{\epsilon}, W_2^{\epsilon'}}(\mu) \quad \text{on} \quad \mathrm{U}(W_2^{\epsilon'}).$$

Then Theorem 4.4 implies that τ has L-parameter

$$\phi_\tau = \mu_E \chi_V \chi_W^{-1} \chi \oplus \chi_V \chi.$$

Now the see-saw identity then gives

$$0 \neq \mathrm{Hom}_{\mathrm{U}(V_1^{\epsilon})}(\sigma, \mu) = \mathrm{Hom}_{\mathrm{U}(W_2^{\epsilon'})}(\tau \otimes \omega^{\vee}_{\psi, \chi, W_2^{\epsilon'}}, \pi).$$

Since we do not know $(\mathrm{FJ})_2$ at this point, this nonvanishing does not give us the desired information about η. However, we note that

$$\mathrm{Hom}_{\mathrm{U}(W_2^{\epsilon'})}(\tau \otimes \omega^{\vee}_{\psi, \chi, W_2^{\epsilon'}}, \pi) = \mathrm{Hom}_{\mathrm{U}(W_2^{\epsilon'})}(\pi^{\vee} \otimes \omega^{\vee}_{\psi, \chi, W_2^{\epsilon'}}, \tau^{\vee}).$$

This allows one to exchange the roles of π and τ in a variant of the above see-saw diagram.

More precisely, since $\phi = \phi_1 \oplus \phi_2$ with conjugate symplectic characters ϕ_i, it follows by $(\mathrm{P2})_1$ (which we have shown) that the L-packet $\Pi_{\phi^\vee}$ can be constructed via theta lifts from $\mathrm{U}(V_1^{\pm})$. Namely, if we start with the L-parameter

$$\phi' := \phi_1^{-1} \phi_2 \chi_W \quad \text{for} \quad \mathrm{U}(V_1^{\pm})$$

and consider the theta correspondence for $\mathrm{U}(V_1^{\epsilon''}) \times \mathrm{U}(W_2^{\epsilon'})$ with respect to the pair (ϕ_2^{-1}, χ_W), then the theta lifts of $\Pi_{\phi'}$ give the L-packet $\Pi_{\phi^\vee}$. In particular, we see that

$$\pi^\vee = \Theta_{\psi,\phi_2^{-1},\chi_W,V_1^{\epsilon''},W_2^{\epsilon'}}(\mu')$$

for a unique $\mu' \in \Pi_{\phi'}^{\epsilon''}$ (which determines ϵ''). Indeed, $(\mathrm{P2})_1$ says that

$$\epsilon'' = \eta_{\pi^\vee}(a_1) = \eta(a_1) \cdot \omega_{E/F}(-1). \tag{6.2}$$

Thus, we may consider the see-saw diagram

$$\begin{array}{ccc} \mathrm{U}(W_2^{\epsilon'}) \times \mathrm{U}(W_2^{\epsilon'}) & & \mathrm{U}(V_2^{\epsilon''}) \\ | & \diagdown \quad \diagup & | \\ \mathrm{U}(W_2^{\epsilon'}) & & \mathrm{U}(V_1^{\epsilon''}) \times \mathrm{U}(L_{-1}) \end{array},$$

and the theta correspondences for

$$\mathrm{U}(V_1^{\epsilon''}) \times \mathrm{U}(W_2^{\epsilon'}) \quad \text{with respect to} \quad (\phi_2^{-1}, \chi_W)$$

and

$$\mathrm{U}(L_{-1}) \times \mathrm{U}(W_2^{\epsilon'}) \quad \text{with respect to} \quad (\chi^{-1}, \chi_W),$$

so that the theta correspondence for

$$\mathrm{U}(V_2^{\epsilon''}) \times \mathrm{U}(W_2^{\epsilon'})$$

is with respect to $(\phi_2^{-1}\chi^{-1}, \chi_W)$. The see-saw identity then reads:

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathrm{U}(W_2^{\epsilon'})}(\pi^\vee \otimes \omega^\vee_{\psi,\chi,W_2^{\epsilon'}}, \tau^\vee) \\ &= \mathrm{Hom}_{\mathrm{U}(V_1^{\epsilon''})}(\Theta_{\psi,\phi_2^{-1}\chi^{-1},\chi_W,V_2^{\epsilon''},W_2^{\epsilon'}}(\tau^\vee), \mu'). \end{aligned}$$

In particular, $\Theta_{\psi,\phi_2^{-1}\chi^{-1},\chi_W,V_2^{\epsilon''},W_2^{\epsilon'}}(\tau^\vee) \neq 0$ on $\mathrm{U}(V_2^{\epsilon''})$. By Theorem 4.1(ii), one deduces that

$$\begin{aligned}\epsilon'' \cdot \epsilon' &= \epsilon\left(\tfrac{1}{2}, \phi_\tau^\vee \otimes \phi_2\chi, \psi_2^E\right)\\ &= \epsilon\left(\tfrac{1}{2}, \mu_E^{-1}\phi_2\chi_V^{-1}\chi_W, \psi_2^E\right) \cdot \epsilon\left(\tfrac{1}{2}, \phi_2\chi_V^{-1}, \psi_2^E\right).\end{aligned}$$

By (6.1) and (6.2), and noting that $\epsilon' = \eta(a_1) \cdot \eta(a_2)$, we see that

$$\eta(a_2) = \theta(\eta)(a_2) \cdot \epsilon\left(\tfrac{1}{2}, \phi_2\chi_V^{-1}, \psi_2^E\right)$$

as desired. It then follows by Theorem 4.1(ii) that

$$\eta(a_1) = \theta(\eta)(a_1) \cdot \epsilon\left(\tfrac{1}{2}, \phi_1\chi_V^{-1}, \psi_2^E\right)$$

as well.

Thus, we have demonstrated $(\mathrm{P1})_2$, and hence $(\mathrm{P2})_2$.

6.2 Inductive step

Now we assume that $n \geq 3$ and $(\mathrm{P2})_k$ holds for all $k < n$. Proposition 4.5 implies that $(\mathrm{P2})_n$ holds for all nontempered L-parameters. We are thus reduced to the case of tempered L-parameters. Then we have the following theorem whose proof will be given in the next two sections:

Theorem 6.1 *If* $(\mathrm{P2})_k$ *holds for all tempered L-parameters for all $k < n$, then* $(\mathrm{P2})_n$ *holds for all tempered but non-square-integrable L-parameters.*

The proof of this theorem is an elaborate extension of the techniques developed in the PhD thesis of the second author [31]. Assuming this theorem for the moment, we are thus reduced to the case of square-integrable L-parameters.

6.3 Square-integrable case

We now consider $(\mathrm{P2})_n$ for a square-integrable L-parameter

$$\phi = \phi_1 \oplus \cdots \oplus \phi_r$$

for $\mathrm{U}(W_n^\pm)$. Thus ϕ is multiplicity-free and each ϕ_i is an n_i-dimensional irreducible conjugate self-dual representation of WD_E with sign $(-1)^{n-1}$. Recall that the component group S_ϕ is of the form

$$S_\phi = \prod_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i.$$

We shall first assume that $r > 1$. Then either $r \geq 3$ or else $r = 2$ in which case we may assume that $n_1 = \dim \phi_1 \geq 2$.

Let $\pi = \pi(\eta) \in \Pi_\phi^{\epsilon'}$ be an irreducible square-integrable representation of $\mathrm{U}(W_n^{\epsilon'})$ with associated character $\eta \in \mathrm{Irr}(S_\phi)$. We consider the theta correspondence for $\mathrm{U}(V_{n+1}^\epsilon) \times \mathrm{U}(W_n^{\epsilon'})$ with respect to the data (ψ, χ_V, χ_W), and suppose that

$$\pi' := \Theta_{\psi, V_{n+1}^\epsilon, W_n^{\epsilon'}}(\pi) \neq 0.$$

Then by Theorem 4.4, $\pi' = \pi'(\eta') \in \Pi_{\theta(\phi)}^\epsilon$ is an irreducible tempered representation of $\mathrm{U}(V_{n+1}^\epsilon)$ with associated character $\eta' \in \mathrm{Irr}(S_{\theta(\phi)})$. We want to determine η' in terms of η. Indeed, recall that there is a natural embedding

$$S_\phi \hookrightarrow S_{\theta(\phi)}$$

and we need to show that $\eta'(a_i) = \eta(a_i)$. We shall do so by a global argument.

6.4 Globalization

Let us begin the process of globalization which is the most delicate part of the argument. Choose a number field $\mathbb{F}$ and a quadratic field extension $\mathbb{E}$ of $\mathbb{F}$ such that

- $\mathbb{F}$ is totally complex;
- $\mathbb{E}_{v_0}/\mathbb{F}_{v_0} = E/F$ for a finite place v_0 of $\mathbb{F}$;
- there is a fixed finite place w of $\mathbb{F}$ which is split in $\mathbb{E}$.

Fix:

- a nontrivial additive character Ψ of $\mathbb{A}/F$ such that $\Psi_{v_0} = \psi$ (in its $\mathrm{N}_{E/F}(E^\times)$-orbit);
- a conjugate symplectic Hecke character χ of $\mathbb{A}_\mathbb{E}^\times$;
- a trace zero element $\delta \in \mathbb{E}^\times$ so that the signs of the skew-Hermitian spaces $W_n^\pm$ at the place v_0 are defined using δ.

Let S be a sufficiently large finite set of inert finite places of $\mathbb{F}$, not containing v_0, such that for all $v \notin S \cup \{v_0\}$, either v is split in $\mathbb{E}$ or else $\mathbb{E}_v/\mathbb{F}_v$, Ψ_v and χ_v are all unramified. Moreover, S can be made arbitrarily large.

If $\Sigma = \boxplus_{i=1}^r \Sigma_i$ is an isobaric sum of irreducible cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A}_\mathbb{E})$, we say that Σ is a tempered A-parameter for $\mathrm{U}(\mathbb{W}_n)$, where $\mathbb{W}_n$ is an n-dimensional skew-Hermitian space over $\mathbb{E}$, if

- $\sum_{i=1}^{r} n_i = n$,
- $\Sigma_i \neq \Sigma_j$ if $i \neq j$,
- the (twisted) Asai L-function $L(s, \Sigma_i, \mathrm{As}^{(-1)^{n-1}})$ has a pole at $s = 1$ for all i.

We shall globalize the L-parameter ϕ to a tempered A-parameter Σ as follows.

(i) At v_0, consider the given irreducible representation ϕ_i of WD_E. Since ϕ_i is conjugate self-dual with sign $(-1)^{n-1}$, it may not be an L-parameter for $\mathrm{U}(W_{n_i}^{\pm})$. Instead, the representation

$$\phi'_{i,v_0} := \phi_i \otimes \chi_{v_0}^{n_i - n}$$

is conjugate self-dual with sign $(-1)^{n_i-1}$, and thus defines an L-parameter for $\mathrm{U}(W_{n_i}^{\pm})$.

(ii) At $v \in S$, choose a representation $\phi_{i,v}$ of WD_E which is the multiplicity-free sum of 1-dimensional conjugate self-dual characters with sign $(-1)^{n-1}$. As above, $\phi_{i,v}$ is conjugate self-dual with sign $(-1)^{n-1}$ and thus may not be an L-parameter for $\mathrm{U}(W_{n_i,v}^{\pm})$, where $W_{n_i,v}^{\pm}$ are the n_i-dimensional skew-Hermitian spaces over E_v. We set

$$\phi'_{i,v} := \phi_{i,v} \otimes \chi_v^{n_i - n},$$

so that $\phi'_{i,v}$ is an L-parameter for $\mathrm{U}(W_{n_i,v}^{\pm})$. The local component group $S_{\phi'_{i,v}}$ of $\phi'_{i,v}$ is of the form

$$S_{\phi'_{i,v}} = (\mathbb{Z}/2\mathbb{Z})^{n_i}$$

and the Vogan L-packet $\Pi_{\phi'_{i,v}}$ consists of 2^{n_i} irreducible square-integrable representations of $\mathrm{U}(W_{n_i,v}^{\pm})$.

(iii) We require in addition that, for all $v \in S$,

$$\phi_v := \phi_{1,v} \oplus \cdots \oplus \phi_{r,v}$$

is not multiplicity-free, i.e. ϕ_v is not a square-integrable L-parameter for $\mathrm{U}(W_{n,v}^{\pm})$. To achieve this, we pick a character μ_v contained in $\phi_{1,v}$ and then ensure that μ_v is also contained in $\phi_{i_v,v}$ for some $i_v \geq 2$. It is here that we use the assumption that $r > 1$. Moreover, we may ensure that

$$i_v \neq i_{v'}$$

for some distinct $v, v' \in S$ if $r > 2$.

(iv) For each $v \in S$, there is a natural map

$$(\mathbb{Z}/2\mathbb{Z})^r = \prod_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i \longrightarrow S_{\phi_v}$$

which sends a_i to the image of the element $-1_{\phi_{i,v}}$ in S_{ϕ_v}. In view of (iii), for $\#S$ large enough (indeed, for $\#S \geq 2$), the induced diagonal map

$$(\mathbb{Z}/2\mathbb{Z})^r \longrightarrow \prod_{v \in S} S_{\phi_v}$$

is injective.

(v) Now for each $i = 1, \ldots, r$, we have a collection of square-integrable L-parameters $\phi'_{i,v}$ for $v \in S \cup \{v_0\}$. For each $v \in S \cup \{v_0\}$, pick an irreducible square-integrable representation $\pi_v \in \Pi^+_{\phi'_{i,v}}$. Let $\mathbb{W}^+_{n_i}$ be the n_i-dimensional skew-Hermitian space over $\mathbb{E}$ whose localization at each inert v is $W^+_{n_i,v}$, where we have used the trace zero element $\delta \in \mathbb{E}^\times$ to define the sign of a skew-Hermitian space over E_v. Then by a result of Shin [55, Theorem 5.13] (proved using the trace formula), one can find an irreducible cuspidal automorphic representation Π'_i of $\mathrm{U}(\mathbb{W}^+_{n_i})(\mathbb{A})$ such that

- $\Pi'_{i,v} = \pi_v$ for all $v \in S \cup \{v_0\}$;
- $\Pi'_{i,v}$ is unramified for all inert $v \notin S \cup \{v_0\}$;
- $\Pi'_{i,w}$ is an irreducible supercuspidal representation of $\mathrm{U}(\mathbb{W}^+_{n_i,w}) \cong \mathrm{GL}_{n_i}(\mathbb{F}_w)$.

(vi) By results of Mok [44], the representation Π'_i has tempered A-parameter Σ'_i, which is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_\mathbb{E})$ such that $L(s, \Sigma'_i, \mathrm{As}^{(-1)^{n_i-1}})$ has a pole at $s = 1$. The cuspidality of Σ'_i is a consequence of the fact that $\Pi'_{i,w}$ is supercuspidal at the split place w. If we set

$$\Sigma_i = \Sigma'_i \otimes \chi^{n-n_i},$$

then Σ_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_\mathbb{E})$ such that $L(s, \Sigma_i, \mathrm{As}^{(-1)^{n-1}})$ has a pole at $s = 1$. In particular, setting

$$\Sigma = \boxplus_{i=1}^{r} \Sigma_i,$$

we see that Σ is a tempered A-parameter for $\mathrm{U}(\mathbb{W}_n)$, where $\mathbb{W}_n$ is an n-dimensional skew-Hermitian space over $\mathbb{E}$.

6.5 Properties of Σ

We have completed the construction of a global tempered A-parameter Σ. Let us examine some crucial properties of Σ.

- (Local components) It follows by construction that the local components of the A-parameter Σ are given as follows:
  - at the place v_0, Σ_{v_0} has L-parameter ϕ;
  - at all places $v \in S$, Σ_v has L-parameter ϕ_v;
  - at all inert places $v \notin S \cup \{v_0\}$, Σ_v is unramified.

  In particular, we have found a globalization Σ of the given local L-parameter ϕ so that at all inert places $v \neq v_0$ of $\mathbb{F}$, Σ_v defines a non-square-integrable L-parameter for $\mathrm{U}(W_{n,v}^{\pm})$.
- (Whittaker data) We shall use the additive character $\Psi = \otimes_v \Psi_v$ to fix the Whittaker datum at each place v. Together with the fixed trace zero element $\delta \in \mathbb{E}^{\times}$, we have thus fixed the local Langlands correspondence for $\mathrm{U}(W_{n,v}^{\pm})$ for each v.
- (Component groups) The global component group S_{Σ} of the A-parameter Σ admits a natural map $S_{\Sigma} \to S_{\Sigma_v}$ for each place v. For $v = v_0$, this natural map is an isomorphism, so that we have a canonical identification:

$$S_{\Sigma} = S_{\Sigma_{v_0}} = \prod_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z}) a_i.$$

  On the other hand, in view of (iv) above, we see that the diagonal map

$$S_{\Sigma} \longrightarrow \prod_{v \neq v_0} S_{\Sigma_v}$$

  is injective. Thus, given any $\eta \in \mathrm{Irr}(S_{\phi}) = \mathrm{Irr}(S_{\Sigma_{v_0}})$, one can find $\eta_v \in \mathrm{Irr}(S_{\Sigma_v})$ for $v \neq v_0$ so that

$$\left(\eta \otimes \left(\bigotimes_{v \neq v_0} \eta_v\right)\right) \circ \Delta = \mathbb{1}_{S_{\Sigma}},$$

  where

$$\Delta : S_{\Sigma} \longrightarrow \prod_{v} S_{\Sigma_v}$$

is the diagonal map.

- (Arthur's multiplicity formula) Consider the global A-packet associated to Σ. For any collection $\eta_v \in \mathrm{Irr}(S_{\Sigma_v})$ of irreducible characters with associated representations $\pi(\eta_v)$ of local unitary groups $\mathrm{U}(W_{n,v}^{\epsilon'_v})$, consider the representation

$$\Pi := \bigotimes_v \pi(\eta_v)$$

of the adelic unitary group $\prod'_v \mathrm{U}(W_{n,v}^{\epsilon'_v})$. Arthur's multiplicity formula [33, Theorem 1.7.1] then states that the following are equivalent:

  - the adelic unitary group $\prod'_v \mathrm{U}(W_{n,v}^{\epsilon'_v})$ is equal to $\mathrm{U}(\mathbb{W}_n)(\mathbb{A})$ for a skew-Hermitian space $\mathbb{W}_n$ over $\mathbb{E}$ and Π occurs in the automorphic discrete spectrum

$$L^2_{\mathrm{disc}}(\mathrm{U}(\mathbb{W}_n)(\mathbb{F})\backslash \mathrm{U}(\mathbb{W}_n)(\mathbb{A}));$$

  - the character $(\otimes_v \eta_v) \circ \Delta$ of S_Σ is trivial.

By the above discussion combined with a result of Wallach [65], [9, Proposition 4.10], we may find an n-dimensional skew-Hermitian space $\mathbb{W}_n$ over $\mathbb{E}$ and an irreducible cuspidal automorphic representation Π of $\mathrm{U}(\mathbb{W}_n)(\mathbb{A})$ in the global A-packet associated to Σ such that $\Pi_{v_0} = \pi(\eta)$. For each v, we shall write the local component Π_v as $\pi(\eta_v)$.

6.6 Global theta correspondence

Now we shall construct a Hermitian space $\mathbb{V}_{n+1}$ of dimension $n+1$ over $\mathbb{E}$, and consider the global theta correspondence for $\mathrm{U}(\mathbb{V}_{n+1}) \times \mathrm{U}(\mathbb{W}_n)$. To define such a global theta correspondence, we shall use the fixed additive character Ψ of $\mathbb{A}/\mathbb{F}$, and we also need to fix a pair of Hecke characters $\chi_\mathbb{V}$ and $\chi_\mathbb{W}$ of $\mathbb{A}_\mathbb{E}^\times$ such that

$$\chi_\mathbb{V}|_{\mathbb{A}^\times} = \omega_{\mathbb{E}/\mathbb{F}}^{n+1} \quad \text{and} \quad \chi_\mathbb{W}|_{\mathbb{A}^\times} = \omega_{\mathbb{E}/\mathbb{F}}^{n},$$

where $\omega_{\mathbb{E}/\mathbb{F}}$ is the quadratic Hecke character of $\mathbb{A}^\times$ associated to $\mathbb{E}/\mathbb{F}$ by global class field theory. We pick these so that, in addition:

(a) at the place v_0, we have

$$\chi_{\mathbb{V},v_0} = \chi_V \quad \text{and} \quad \chi_{\mathbb{W},v_0} = \chi_W;$$

(b) at some place $v_1 \in S$, $\chi_{\mathbb{V},v_1}$ is not contained in the L-parameter associated to Σ_{v_1}.

Indeed, since $\mathbb{E}^\times/\mathbb{F}^\times \cong \mathrm{Ker}(\mathrm{N}_{\mathbb{E}/\mathbb{F}})$ is anisotropic, for given conjugate orthogonal characters μ_i of $\mathbb{E}_{v_i}^\times$, there is a conjugate orthogonal Hecke character μ of $\mathbb{A}_{\mathbb{E}}^\times$ such that $\mu_{v_i} = \mu_i$ for $i = 0, 1$. Thus, we can achieve (a) and (b) by replacing $\chi_{\mathbb{V}}$ and $\chi_{\mathbb{W}}$ by their twists by conjugate orthogonal Hecke characters of $\mathbb{A}_{\mathbb{E}}^\times$ if necessary. The condition (b) guarantees that at the place v_1, the representation Π_{v_1} has nonzero local theta lift to both $\mathrm{U}(V^+_{n+1,v_1})$ and $\mathrm{U}(V^-_{n+1,v_1})$ by Theorem 4.4(i)(a). Moreover, the conservation relation (proved by Sun–Zhu [57]) implies that the theta lifts of Π_{v_1} to $\mathrm{U}(V^+_{n-1,v_1})$ and $\mathrm{U}(V^-_{n-1,v_1})$ are both zero.

Now we note:

Lemma 6.2 *There is a Hermitian space $\mathbb{V}_{n+1}$ of dimension $n+1$ over $\mathbb{E}$ such that:*

- *at the place v_0, $\mathbb{V}_{n+1,v_0}$ is equal to the given Hermitian space V^ϵ_{n+1};*
- *for all places v, the representation Π_v has nonzero local theta lift to $\mathrm{U}(\mathbb{V}_{n+1,v})$ with respect to the theta lift defined by the data $(\Psi_v, \chi_{\mathbb{V}_v}, \chi_{\mathbb{W}_v})$.*

Proof For all $v \neq v_0, v_1$, we may pick $\mathbb{V}_{n+1,v}$ so that the local theta lift of Π_v to $\mathrm{U}(\mathbb{V}_{n+1,v})$ is nonzero, and then complete these to a coherent collection of Hermitian spaces by picking V^ϵ_{n+1} at v_0 and the uniquely determined Hermitian space at v_1. □

6.7 Completion of the proof

Consider the global theta lift $\Pi' := \Theta_{\Psi,\mathbb{V}_{n+1},\mathbb{W}_n}(\Pi)$ to $\mathrm{U}(\mathbb{V}_{n+1})(\mathbb{A})$. The condition (b) above ensures that Π' is cuspidal. To show that Π' is nonzero, we consider the standard L-function $L(s, \Pi)$ of Π defined using the doubling zeta integral of Piatetski-Shapiro–Rallis [40,46]. Observe that the partial L-function $L^{S\cup\{v_0\}}(s, \Pi)$ agrees with the partial standard L-function $L^{S\cup\{v_0\}}(s, \Sigma)$ of Σ, so that

$$L^{S\cup\{v_0\}}(1, \Pi) = L^{S\cup\{v_0\}}(1, \Sigma) = \prod_{i=1}^{r} L^{S\cup\{v_0\}}(1, \Sigma_i) \neq 0$$

since Σ_i is unitary and cuspidal. By [40, Proposition 5], the local standard L-factor $L(s, \Pi_v)$ at $v \in S \cup \{v_0\}$ is holomorphic and nonzero at $s = 1$ since Π_v is tempered. Hence

$$L(1, \Pi) \neq 0$$

and it follows by [18, Theorem 1.4] that Π' is nonzero. Thus Π' is an irreducible cuspidal automorphic representation of $\mathrm{U}(\mathbb{V}_{n+1})(\mathbb{A})$ such that $\Pi'_{v_0} = \pi'(\eta')$.

Recall that we have fixed the local Langlands correspondence for $\mathrm{U}(\mathbb{W}_{n,v})$ for each v using the Whittaker datum determined by the additive character Ψ_v together with the trace zero element δ. To fix the local Langlands correspondence for $\mathrm{U}(\mathbb{V}_{n+1,v})$ for each v, we shall use the Whittaker datum determined by the additive character $\Psi_v^{\mathbb{E}_v} = \Psi_v\left(\frac{1}{2}\,\mathrm{Tr}_{\mathbb{E}_v/\mathbb{F}_v}(\delta\,\cdot\,)\right)$. Then we may write

$$\Pi = \bigotimes_v \pi(\eta_v) \quad \text{and} \quad \Pi' = \bigotimes_v \pi'(\eta'_v)$$

with associated irreducible characters η_v and η'_v of the local component groups.

Recall that Π has tempered A-parameter. By Theorem 4.4, Π' also has tempered A-parameter. Hence, applying Arthur's multiplicity formula [33, Theorem 1.7.1] to Π and Π', we see that

$$\prod_v \eta_v(a_{i,v}) = 1 \quad \text{and} \quad \prod_v \eta'_v(a_{i,v}) = 1 \tag{6.3}$$

for all i, where $a_{i,v}$ is the image of a_i in S_{Σ_v}. However, for all places $v \neq v_0$, either v is split, or else the L-parameter of Π_v is not square-integrable. Thus, for all inert $v \neq v_0$, one knows that $(\mathrm{P2})_n$ holds. In particular,

$$\eta'_v(a_{i,v}) = \eta_v(a_{i,v})$$

for all $v \neq v_0$. Thus, we conclude that at the place v_0, we have

$$\eta'(a_i) = \eta(a_i)$$

as desired.

We have thus completed the proof of $(\mathrm{P2})_n$ when $r > 1$, i.e. when ϕ is reducible. To deal with the case when ϕ is irreducible, with $r = 1$, we can again appeal to a variation of the global argument as above. Namely, in the globalization step above, we may now take the L-parameter ϕ_v for $v \in S$ to be square-integrable L-parameters which are *reducible*. Then the rest of the argument is the same, using the fact that we have shown $(\mathrm{P2})_n$ for every place $v \neq v_0$. This completes the proof of $(\mathrm{P2})_n$.

7 Preparations for the proof of Theorem 6.1

To finish the proof of (P2), it now remains to prove Theorem 6.1. For this, we need to introduce more notation. Fix $\varepsilon = \pm 1$. In this and next sections, we shall let V and W be an ε-Hermitian space and a $(-\varepsilon)$-Hermitian space respectively. Put

$$m = \dim V \quad \text{and} \quad n = \dim W.$$

7.1 Parabolic subgroups

Let r be the Witt index of V and V_{an} an anisotropic kernel of V. Choose a basis $\{v_i, v_i^* \mid i = 1, \ldots, r\}$ of the orthogonal complement of V_{an} such that

$$\langle v_i, v_j \rangle_V = \langle v_i^*, v_j^* \rangle_V = 0, \quad \langle v_i, v_j^* \rangle_V = \delta_{i,j}$$

for $1 \le i, j \le r$. Let k be a positive integer with $k \le r$ and set

$$X = Ev_1 \oplus \cdots \oplus Ev_k, \quad X^* = Ev_1^* \oplus \cdots \oplus Ev_k^*.$$

Let V_0 be the orthogonal complement of $X \oplus X^*$ in V, so that V_0 is an ε-Hermitian space of dimension $m_0 = m - 2k$ over E. We shall write an element in the unitary group $\mathrm{U}(V)$ as a block matrix relative to the decomposition $V = X \oplus V_0 \oplus X^*$. Let $P = M_P U_P$ be the maximal parabolic subgroup of $\mathrm{U}(V)$ stabilizing X, where M_P is the Levi component of P stabilizing X^* and U_P is the unipotent radical of P. We have

$$\begin{aligned} M_P &= \{m_P(a) \cdot h_0 \mid a \in \mathrm{GL}(X),\ h_0 \in \mathrm{U}(V_0)\}, \\ U_P &= \{u_P(b) \cdot u_P(c) \mid b \in \mathrm{Hom}(V_0, X),\ c \in \mathrm{Herm}(X^*, X)\}, \end{aligned}$$

where

$$\begin{aligned} m_P(a) &= \begin{pmatrix} a & & \\ & 1_{V_0} & \\ & & (a^*)^{-1} \end{pmatrix}, \\ u_P(b) &= \begin{pmatrix} 1_X & b & -\frac{1}{2}bb^* \\ & 1_{V_0} & -b^* \\ & & 1_{X^*} \end{pmatrix}, \\ u_P(c) &= \begin{pmatrix} 1_X & & c \\ & 1_{V_0} & \\ & & 1_{X^*} \end{pmatrix}, \end{aligned}$$

and

$$\mathrm{Herm}(X^*, X) = \{c \in \mathrm{Hom}(X^*, X) \mid c^* = -c\}.$$

Here, the elements $a^* \in \mathrm{GL}(X^*)$, $b^* \in \mathrm{Hom}(X^*, V_0)$, and $c^* \in \mathrm{Hom}(X^*, X)$ are defined by requiring that

$$\begin{aligned} \langle ax, x' \rangle_V &= \langle x, a^* x' \rangle_V, \\ \langle bv, x' \rangle_V &= \langle v, b^* x' \rangle_V, \\ \langle cx', x'' \rangle_V &= \langle x', c^* x'' \rangle_V \end{aligned}$$

for $x \in X$, $x', x'' \in X^*$, and $v \in V_0$. In particular, $M_P \cong \mathrm{GL}(X) \times \mathrm{U}(V_0)$ and

$$1 \longrightarrow \mathrm{Herm}(X^*, X) \longrightarrow U_P \longrightarrow \mathrm{Hom}(V_0, X) \longrightarrow 1.$$

Put

$$\rho_P = \frac{m_0 + k}{2}, \quad w_P = \begin{pmatrix} & & -I_X \\ & 1_{V_0} & \\ -\varepsilon I_X^{-1} & & \end{pmatrix},$$

where $I_X \in \mathrm{Isom}(X^*, X)$ is defined by $I_X v_i^* = v_i$ for $1 \le i \le k$.

Similarly, let r' be the Witt index of W and choose a basis $\{w_i, w_i^* \mid i = 1, \dots, r'\}$ of the orthogonal complement of an anisotropic kernel of W such that

$$\langle w_i, w_j \rangle_W = \langle w_i^*, w_j^* \rangle_W = 0, \quad \langle w_i, w_j^* \rangle_W = \delta_{i,j}$$

for $1 \le i, j \le r'$. We assume that $k \le r'$ and set

$$Y = Ew_1 \oplus \cdots \oplus Ew_k, \quad Y^* = Ew_1^* \oplus \cdots \oplus Ew_k^*.$$

Let W_0 be the orthogonal complement of $Y \oplus Y^*$ in W, so that W_0 is a $(-\varepsilon)$-Hermitian space of dimension $n_0 = n - 2k$ over E. Let $Q = M_Q U_Q$ be the maximal parabolic subgroup of $\mathrm{U}(W)$ stabilizing Y, where M_Q is the Levi component of Q stabilizing Y^* and U_Q is the unipotent radical of Q. Then $M_Q \cong \mathrm{GL}(Y) \times \mathrm{U}(W_0)$ and

$$1 \longrightarrow \mathrm{Herm}(Y^*, Y) \longrightarrow U_Q \longrightarrow \mathrm{Hom}(W_0, Y) \longrightarrow 1,$$

where

$$\mathrm{Herm}(Y^*, Y) = \{c \in \mathrm{Hom}(Y^*, Y) \mid c^* = -c\}.$$

For $a \in \mathrm{GL}(Y)$, $b \in \mathrm{Hom}(W_0, Y)$, and $c \in \mathrm{Herm}(Y^*, Y)$, we define elements $m_Q(a) \in M_Q$ and $u_Q(b), u_Q(c) \in U_Q$ as above. Put

$$\rho_Q = \frac{n_0 + k}{2}, \quad w_Q = \begin{pmatrix} & & -I_Y \\ & 1_{W_0} & \\ \varepsilon I_Y^{-1} & & \end{pmatrix},$$

where $I_Y \in \mathrm{Isom}(Y^*, Y)$ is defined by $I_Y w_i^* = w_i$ for $1 \leq i \leq k$.

7.2 Haar measures

We need to choose Haar measures on various groups. In particular, we shall define Haar measures on U_P and U_Q in the following.

Recall the symplectic form

$$\langle \cdot, \cdot \rangle = \mathrm{Tr}_{E/F}(\langle \cdot, \cdot \rangle_V \otimes \langle \cdot, \cdot \rangle_W)$$

on $V \otimes W$ over F. We consider the following spaces and pairings:

- $(x, y) \mapsto \psi(\langle x, I_Y^{-1} y \rangle)$ for $x, y \in V \otimes Y$;
- $(x, y) \mapsto \psi(\langle x, I_Y y \rangle)$ for $x, y \in V_0 \otimes Y^*$;
- $(x, y) \mapsto \psi(\langle I_X^{-1} x, y \rangle)$ for $x, y \in X \otimes W_0$;
- $(x, y) \mapsto \psi(\langle I_X x, y \rangle)$ for $x, y \in X^* \otimes W_0$;
- $(x, y) \mapsto \psi(\langle I_X^{-1} x, I_Y y \rangle)$ for $x, y \in X \otimes Y^*$;
- $(x, y) \mapsto \psi(\langle I_X x, I_Y^{-1} y \rangle)$ for $x, y \in X^* \otimes Y$;
- $(x, y) \mapsto \psi(\langle I_X x, I_Y y \rangle)$ for $x, y \in X^* \otimes Y^*$.

On these spaces, we take the self-dual Haar measures with respect to these pairings. Put

$$e^{**} = v_1^* \otimes w_1^* + \cdots + v_k^* \otimes w_k^* \in X^* \otimes Y^*.$$

- We transfer the Haar measure on $V_0 \otimes Y^*$ to $\mathrm{Hom}(X^*, V_0)$ via the isomorphism $x \mapsto xe^{**}$ for $x \in \mathrm{Hom}(X^*, V_0)$.
- We transfer the Haar measure on $\mathrm{Hom}(X^*, V_0)$ to $\mathrm{Hom}(V_0, X)$ via the isomorphism $x \mapsto x^*$ for $x \in \mathrm{Hom}(V_0, X)$.
- Similarly, we define the Haar measure on $\mathrm{Hom}(W_0, Y)$.

Furthermore:

- We transfer the Haar measure on $X \otimes Y^*$ to $\mathrm{Hom}(X^*, X)$ via the isomorphism $x \mapsto xe^{**}$ for $x \in \mathrm{Hom}(X^*, X)$. This Haar measure on $\mathrm{Hom}(X^*, X)$ is self-dual with respect to the pairing $(x, y) \mapsto \psi(\langle I_X^{-1} xe^{**}, I_Y ye^{**} \rangle)$.

- We take the Haar measure $|2|_F^{-k^2/2}\,dx$ on $\mathrm{Herm}(X^*, X)$, where dx is the self-dual Haar measure on $\mathrm{Herm}(X^*, X)$ with respect to the pairing $(x, y) \mapsto \psi(\langle I_X^{-1} xe^{**}, I_Y ye^{**}\rangle)$.
- Similarly, we define the Haar measure on $\mathrm{Herm}(Y^*, Y)$.

Then:

- We take the Haar measure $du = db\,dc$ on U_P for $u = u_P(b)u_P(c)$ with $b \in \mathrm{Hom}(V_0, X)$ and $c \in \mathrm{Herm}(X^*, X)$.
- Similarly, we define the Haar measure on U_Q.

We note the following Fourier inversion formula:

Lemma 7.1 *For $\varphi \in \mathscr{S}(X \otimes Y^*)$, we have*

$$\int_{\mathrm{Herm}(Y^*,Y)} \left(\int_{\mathrm{Hom}(X^*,X)} \varphi(xe^{**})\psi(\langle xe^{**}, ce^{**}\rangle)dx \right) dc = \int_{\mathrm{Herm}(X^*,X)} \varphi(ce^{**})\,dc.$$

Proof We consider the nondegenerate symmetric bilinear form $(x, y) \mapsto \langle I_X^{-1} x, I_Y y\rangle$ on $X \otimes Y^*$ over F, and the subspaces

$$\mathrm{Herm}(X^*, X)e^{**} \quad \text{and} \quad I_X I_Y^{-1}\,\mathrm{Herm}(Y^*, Y)e^{**}$$

of $X \otimes Y^* = \mathrm{Hom}(X^*, X)e^{**}$. For $x \in \mathrm{Hom}(X^*, X)$ and $y \in \mathrm{Herm}(Y^*, Y)$, we have

$$\begin{aligned}
\langle I_X^{-1} xe^{**}, I_Y I_X I_Y^{-1} ye^{**}\rangle &= \langle I_X^{-1} xe^{**}, I_X ye^{**}\rangle \\
&= \langle I_X^* I_X^{-1} xe^{**}, ye^{**}\rangle \\
&= \varepsilon \cdot \langle xe^{**}, ye^{**}\rangle
\end{aligned}$$

since $I_X^* = \varepsilon I_X$. For $x \in \mathrm{Herm}(X^*, X)$ and $y \in \mathrm{Herm}(Y^*, Y)$, noting that $x^* = -x$, $y^* = -y$, and x commutes with y, we have

$$\begin{aligned}
\langle xe^{**}, ye^{**}\rangle &= \langle y^*e^{**}, x^*e^{**}\rangle \\
&= \langle ye^{**}, xe^{**}\rangle \\
&= -\langle xe^{**}, ye^{**}\rangle,
\end{aligned}$$

so that

$$\langle xe^{**}, ye^{**}\rangle = 0.$$

Since $\mathrm{Hom}(X^*, X)e^{**}$ is nondegenerate with respect to the above bilinear form, we see that $X \otimes Y^*$ decomposes as the orthogonal direct sum

$$X \otimes Y^* = \mathrm{Herm}(X^*, X)e^{**} \oplus I_X I_Y^{-1}\,\mathrm{Herm}(Y^*, Y)e^{**}.$$

These yield the desired Fourier inversion formula. □

7.3 Normalized intertwining operators

In this subsection, we define the normalized intertwining operator which is used to describe the local Langlands correspondence.

Let τ be an irreducible (unitary) square-integrable representation of $\mathrm{GL}(X)$ on a space $\mathscr{V}_\tau$ with central character ω_τ. For any $s \in \mathbb{C}$, we realize the representation $\tau_s := \tau \otimes |\det|^s$ on $\mathscr{V}_\tau$ by setting $\tau_s(a)v := |\det a|^s \tau(a)v$ for $a \in \mathrm{GL}(X)$ and $v \in \mathscr{V}_\tau$. Let σ_0 be an irreducible tempered representation of $\mathrm{U}(V_0)$ on a space $\mathscr{V}_{\sigma_0}$. We consider the induced representation

$$\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s \otimes \sigma_0)$$

of $\mathrm{U}(V)$, which is realized on the space of smooth functions $\Phi_s : \mathrm{U}(V) \to \mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0}$ such that

$$\Phi_s(u m_P(a) h_0 h) = |\det a|^{s+\rho_P} \tau(a)\sigma_0(h_0)\Phi_s(h)$$

for all $u \in U_P$, $a \in \mathrm{GL}(X)$, $h_0 \in \mathrm{U}(V_0)$, and $h \in \mathrm{U}(V)$. Let A_P be the split component of the center of M_P and $W(M_P) = \mathrm{Norm}_{\mathrm{U}(V)}(A_P)/M_P$ the relative Weyl group for M_P. Noting that $W(M_P) \cong \mathbb{Z}/2\mathbb{Z}$, we denote by w the nontrivial element in $W(M_P)$. For any representative $\tilde{w} \in \mathrm{U}(V)$ of w, we define an unnormalized intertwining operator

$$\mathcal{M}(\tilde{w}, \tau_s \otimes \sigma_0) : \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s \otimes \sigma_0) \longrightarrow \mathrm{Ind}_P^{\mathrm{U}(V)}(w(\tau_s \otimes \sigma_0))$$

by (the meromorphic continuation of) the integral

$$\mathcal{M}(\tilde{w}, \tau_s \otimes \sigma_0)\Phi_s(h) = \int_{U_P} \Phi_s(\tilde{w}^{-1} u h)\, du,$$

where $w(\tau_s \otimes \sigma_0)$ is the representation of M_P on $\mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0}$ given by

$$(w(\tau_s \otimes \sigma_0))(m) = (\tau_s \otimes \sigma_0)(\tilde{w}^{-1} m \tilde{w})$$

for $m \in M_P$.

Now, following [2,33,44], we shall normalize the intertwining operator $\mathcal{M}(\tilde{w}, \tau_s \otimes \sigma_0)$, depending on the choice of the Whittaker datum. Having fixed the additive character ψ and the trace zero element δ, we define the sign $\epsilon(V)$ and use the Whittaker datum relative to

$$\begin{cases} \psi^E = \psi\left(\frac{1}{2}\operatorname{Tr}_{E/F}(\delta \cdot)\right) & \text{if } \varepsilon = +1; \\ \psi & \text{if } \varepsilon = -1. \end{cases}$$

The definition of the normalized intertwining operator is very subtle because one has to choose the following data appropriately:

- a representative $\tilde{w}$;
- a normalizing factor $r(w, \tau_s \otimes \sigma_0)$;
- an intertwining isomorphism $\mathcal{A}_w$.

Following the procedure of [39, Sect. 2.1], [2, Sect. 2.3], [44, Sect. 3.3], [33, Sect. 2.3], we take the representative $\tilde{w} \in \mathrm{U}(V)$ of w defined by

$$\tilde{w} = w_P \cdot m_P((-1)^{m'} \cdot \kappa_V \cdot J) \cdot (-1_{V_0})^k,$$

where w_P is as in Sect. 7.1, $m' = [\frac{m}{2}]$,

$$\kappa_V = \begin{cases} -\delta & \text{if } m \text{ is even and } \varepsilon = +1; \\ 1 & \text{if } m \text{ is even and } \varepsilon = -1; \\ -1 & \text{if } m \text{ is odd and } \varepsilon = +1; \\ -\delta & \text{if } m \text{ is odd and } \varepsilon = -1, \end{cases}$$

and

$$J = \begin{pmatrix} & & & (-1)^{k-1} \\ & & \iddots & \\ & -1 & & \\ 1 & & & \end{pmatrix} \in \mathrm{GL}_k(E).$$

Here, we have identified $\mathrm{GL}(X)$ with $\mathrm{GL}_k(E)$ using the basis $\{v_1, \ldots, v_k\}$. This element $\tilde{w}$ arises as follows.

First assume that $\epsilon(V) = +1$. In particular, $\mathrm{U}(V)$ is quasi-split. We have $V_{\mathrm{an}} = \{0\}$ if m is even and $V_{\mathrm{an}} = E v_{\mathrm{an}}$ for some $v_{\mathrm{an}} \in V_{\mathrm{an}}$ such that

$$\langle v_{\mathrm{an}}, v_{\mathrm{an}} \rangle_V = \begin{cases} 1 & \text{if } \varepsilon = +1; \\ \delta & \text{if } \varepsilon = -1 \end{cases}$$

if m is odd. Via the decomposition

$$V = Ev_1 \oplus \cdots \oplus Ev_r \oplus V_{\mathrm{an}} \oplus Ev_r^* \oplus \cdots \oplus Ev_1^*,$$

we regard $\mathrm{U}(V)$ as a subgroup of $\mathrm{GL}_m(E)$, which induces an isomorphism $\mathrm{U}(V)(\bar{F}) \cong \mathrm{GL}_m(\bar{F})$. Let $\boldsymbol{spl} = (B, T, \{X_i\})$ be the F-splitting of $\mathrm{U}(V)$ consisting of the Borel subgroup B stabilizing the flag

$$Ev_1 \subset Ev_1 \oplus Ev_2 \subset \cdots \subset Ev_1 \oplus \cdots \oplus Ev_r,$$

the maximal torus T of diagonal matrices, and the set $\{X_i \mid i = 1, \ldots, m-1\}$ of simple root vectors given as follows:

- $X_i = E_{i,i+1}$ for $1 \leq i \leq r-1$;
- $X_i = -E_{i,i+1}$ for $m-r+1 \leq i \leq m-1$;
- if m is even, then

$$X_r = \begin{cases} \delta^{-1} \cdot E_{r,r+1} & \text{if } \varepsilon = +1, \\ E_{r,r+1} & \text{if } \varepsilon = -1; \end{cases}$$

- if m is odd, then $X_r = E_{r,r+1}$ and

$$X_{r+1} = \begin{cases} -E_{r+1,r+2} & \text{if } \varepsilon = +1, \\ \delta^{-1} \cdot E_{r+1,r+2} & \text{if } \varepsilon = -1. \end{cases}$$

Here, $E_{i,j} \in \mathrm{Lie}\,\mathrm{U}(V)(\bar{F}) \cong \mathrm{M}_m(\bar{F})$ is the matrix with one at the (i, j)th entry and zero elsewhere. Then $\boldsymbol{spl}$ and ψ give rise to the above Whittaker datum, whose restriction to M_P is preserved by the representative $\tilde{w}^{\mathrm{LS}}$ of w defined in [39, Sect. 2.1], [2, Sect. 2.3], [44, Sect. 3.3] with respect to $\boldsymbol{spl}$.

Lemma 7.2 *We have* $\tilde{w}^{\mathrm{LS}} = \tilde{w}$.

Proof First, we review the case of SL_2. We take an F-splitting of SL_2 consisting of the Borel subgroup of upper triangular matrices, the maximal torus of diagonal matrices, and a simple root vector

$$X = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

Let $\{H, X, Y\}$ be the $\mathfrak{sl}_2$-triple containing X, so that

$$Y = \begin{pmatrix} 0 & 0 \\ a^{-1} & 0 \end{pmatrix}.$$

If s is the simple reflection with respect to X, then the representative of s defined in [39, Sect. 2.1] is

$$\exp(X)\exp(-Y)\exp(X) = \begin{pmatrix} & a \\ -a^{-1} & \end{pmatrix}.$$

Now we compute $\tilde{w}^{\mathrm{LS}}$. Let $\iota_i : \mathrm{GL}(Ev_i \oplus Ev_{i+1}) \hookrightarrow \mathrm{GL}(X)$ and $\iota'_j : \mathrm{U}(Ev_j \oplus Ev_j^*) \hookrightarrow \mathrm{U}(V)$ be the natural embeddings. Let s_i be the simple reflection with respect to X_i and $\tilde{s}_i$ the representative of s_i as above. Put $w_i = s_i s_{m-i}$ and $\tilde{w}_i = \tilde{s}_i \tilde{s}_{m-i}$ for $1 \le i \le r-1$, and

$$w_r = \begin{cases} s_r & \text{if } m \text{ is even,} \\ s_r s_{r+1} s_r & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad \tilde{w}_r = \begin{cases} \tilde{s}_r & \text{if } m \text{ is even,} \\ \tilde{s}_r \tilde{s}_{r+1} \tilde{s}_r & \text{if } m \text{ is odd.} \end{cases}$$

More explicitly, we have

$$\tilde{w}_i = m_P\left(\iota_i\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)$$

for $1 \le i \le r-1$ and

$$\tilde{w}_r = \iota'_r\left(\begin{pmatrix} & 1 \\ \varepsilon & \end{pmatrix}\begin{pmatrix} \kappa_V & \\ & (\kappa_V^c)^{-1} \end{pmatrix}\right) \cdot (-1_{V_{\mathrm{an}}}).$$

Put

$$\begin{aligned} x_i &= w_{k-1} \cdots w_{i+1} w_i, & y_j &= w_j w_{j+1} \cdots w_{r-1} w_r w_{r-1} \cdots w_{j+1} w_j, \\ \tilde{x}_i &= \tilde{w}_{k-1} \cdots \tilde{w}_{i+1} \tilde{w}_i, & \tilde{y}_j &= \tilde{w}_j \tilde{w}_{j+1} \cdots \tilde{w}_{r-1} \tilde{w}_r \tilde{w}_{r-1} \cdots \tilde{w}_{j+1} \tilde{w}_j \end{aligned}$$

for $1 \le i \le k-1$ and $1 \le j \le k$. Let w_T be the representative of w in the Weyl group for T which preserves the set of roots of T in $B \cap M_P$. Then w_T has a reduced expression

$$w_T = y_k x_1 y_k x_2 \cdots y_k x_{k-1} y_k$$

and hence $\tilde{w}^{\mathrm{LS}}$ is defined by

$$\tilde{w}^{\mathrm{LS}} = \tilde{y}_k \tilde{x}_1 \tilde{y}_k \tilde{x}_2 \cdots \tilde{y}_k \tilde{x}_{k-1} \tilde{y}_k.$$

If we put $\tilde{x}'_i = \tilde{w}_{k-1}^{-1} \cdots \tilde{w}_{i+1}^{-1} \tilde{w}_i^{-1}$, then we have $\tilde{y}_k \tilde{x}_i = \tilde{x}'_i \tilde{y}_i$, so that

$$\tilde{w}^{\mathrm{LS}} = \tilde{x}'_1 \tilde{y}_1 \tilde{x}'_2 \tilde{y}_2 \cdots \tilde{x}'_{k-1} \tilde{y}_{k-1} \tilde{y}_k.$$

On the other hand, we have

$$\tilde{x}'_i = m_P \begin{pmatrix} 1_{i-1} & & \\ & & -1_{k-i} \\ & 1 & \end{pmatrix}$$

and

$$\tilde{y}_j = \iota'_j\left(\begin{pmatrix} & 1 \\ \varepsilon & \end{pmatrix}\begin{pmatrix} \kappa_V & \\ & (\kappa_V^c)^{-1} \end{pmatrix}\right) \cdot m_P \begin{pmatrix} 1_{j-1} & & \\ & (-1)^{r-j} & \\ & & -1_{k-j} \end{pmatrix} \cdot (-1_{V_0}).$$

In particular, $\tilde{x}'_i$ commutes with $\tilde{y}_j$ if $i > j$, so that

$$\tilde{w}^{\mathrm{LS}} = \tilde{x}'_1 \tilde{x}'_2 \cdots \tilde{x}'_{k-1} \tilde{y}_1 \tilde{y}_2 \cdots \tilde{y}_{k-1} \tilde{y}_k.$$

Since $\tilde{x}'_1 \cdots \tilde{x}'_{k-1} = m_P(J)$ and

$$\begin{aligned} \tilde{y}_1 \cdots \tilde{y}_k &= \prod_{j=1}^{k} \iota'_j\left(\begin{pmatrix} & 1 \\ \varepsilon & \end{pmatrix}\begin{pmatrix} \kappa_V & \\ & (\kappa_V^c)^{-1} \end{pmatrix}\right) \cdot m_P((-1)^{r-1} \cdot 1_k) \cdot (-1_{V_0})^k \\ &= \prod_{j=1}^{k} \iota'_j\begin{pmatrix} & 1 \\ \varepsilon & \end{pmatrix} \cdot m_P((-1)^{r-1} \cdot \kappa_V \cdot 1_k) \cdot (-1_{V_0})^k, \end{aligned}$$

the assertion follows. □

Next, we consider the case $\epsilon(V) = -1$. Let V^+ be the m-dimensional ε-Hermitian space with $\epsilon(V^+) = +1$. We may assume that $V^+ = X \oplus V_0^+ \oplus X^*$ for some m_0-dimensional ε-Hermitian space V_0^+ with $\epsilon(V_0^+) = +1$. Let P^+ be the maximal parabolic subgroup of $\mathrm{U}(V^+)$ stabilizing X and M_{P^+} its Levi component stabilizing X^*, so that $M_{P^+} \cong \mathrm{GL}(X) \times \mathrm{U}(V_0^+)$. Fix an isomorphism $V_0^+ \otimes_F \bar{F} \cong V_0 \otimes_F \bar{F}$ as ε-Hermitian spaces over $E \otimes_F \bar{F}$ and extend it to an isomorphism $V^+ \otimes_F \bar{F} \cong V \otimes_F \bar{F}$ whose restriction to $(X \otimes_F \bar{F}) \oplus (X^* \otimes_F \bar{F})$ is the identity map. This induces a pure inner twist (ξ, z), i.e. $\xi : \mathrm{U}(V^+) \to \mathrm{U}(V)$ is an inner twist and $z \in Z^1(\Gamma, \mathrm{U}(V^+))$ is a 1-cocyle such that $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1} = \mathrm{Ad}(z(\sigma))$ for all $\sigma \in \Gamma$. Then $P^+ = \xi^{-1}(P)$ and ξ induces an inner twist $\xi : M_{P^+} \to M_P$ whose restriction to $\mathrm{GL}(X)$ is the identity map. Moreover, z satisfies the assumption in [33, Sect. 2.4.1]. Let w^+ be the nontrivial element in the relative Weyl group for M_{P^+} and $\tilde{w}^+ \in \mathrm{U}(V^+)$ the representative of w^+ as above. Then the representative of w defined in [33, Sect. 2.3] is $\xi(\tilde{w}^+)$, which is equal to $\tilde{w}$.

We use the normalizing factor $r(w, \tau_s \otimes \sigma_0)$ defined as follows. Let $\lambda(E/F, \psi)$ be the Langlands λ-factor (see [14, Sect. 5]) and put

$$\lambda(w, \psi) = \begin{cases} \lambda(E/F, \psi)^{(k-1)k/2} & \text{if } m \text{ is even;} \\ \lambda(E/F, \psi)^{(k+1)k/2} & \text{if } m \text{ is odd.} \end{cases}$$

Let ϕ_τ and ϕ_0 be the L-parameters of τ and σ_0 respectively. Let As^+ be the Asai representation of the L-group of $\mathrm{Res}_{E/F}\,\mathrm{GL}_k$ and $\mathrm{As}^- = \mathrm{As}^+ \otimes \omega_{E/F}$ its twist (see [15, Sect. 7]). If we set

$$r(w, \tau_s \otimes \sigma_0) = \lambda(w, \psi) \cdot \gamma(s, \phi_\tau \otimes \phi_0^\vee, \psi_E)^{-1} \cdot \gamma(2s, \mathrm{As}^{(-1)^m} \circ \phi_\tau, \psi)^{-1},$$

then by [33, Lemmas 2.2.3 and 2.3.1], the normalized intertwining operator

$$\mathcal{R}(w, \tau_s \otimes \sigma_0) := |\kappa_V|^{k\rho_P} \cdot r(w, \tau_s \otimes \sigma_0)^{-1} \cdot \mathcal{M}(\tilde{w}, \tau_s \otimes \sigma_0)$$

is holomorphic at $s = 0$ and satisfies

$$\mathcal{R}(w, w(\tau_s \otimes \sigma_0)) \circ \mathcal{R}(w, \tau_s \otimes \sigma_0) = 1.$$

Here, the factor $|\kappa_V|^{k\rho_P}$ arises because the Haar measure on U_P defined in [33, Sect. 2.2] with respect to ***spl*** is equal to $|\kappa_V|^{k\rho_P}\, du$.

Now assume that $w(\tau \otimes \sigma_0) \cong \tau \otimes \sigma_0$, which is equivalent to $(\tau^c)^\vee \cong \tau$. We may take the unique isomorphism

$$\mathcal{A}_w : \mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0} \longrightarrow \mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0}$$

such that:

- $\mathcal{A}_w \circ (w(\tau \otimes \sigma_0))(m) = (\tau \otimes \sigma_0)(m) \circ \mathcal{A}_w$ for all $m \in M_P$;
- $\mathcal{A}_w = \mathcal{A}'_w \otimes 1_{\mathscr{V}_{\sigma_0}}$ with an isomorphism $\mathcal{A}'_w : \mathscr{V}_\tau \to \mathscr{V}_\tau$ such that $\Lambda \circ \mathcal{A}'_w = \Lambda$. Here, $\Lambda : \mathscr{V}_\tau \to \mathbb{C}$ is the unique (up to a scalar) Whittaker functional with respect to the Whittaker datum (N_k, ψ_{N_k}), where N_k is the group of unipotent upper triangular matrices in $\mathrm{GL}_k(E)$ and ψ_{N_k} is the generic character of N_k given by $\psi_{N_k}(x) = \psi_E(x_{1,2} + \cdots + x_{k-1,k})$.

Note that $\mathcal{A}_w^2 = 1_{\mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0}}$. We define a self-intertwining operator

$$R(w, \tau \otimes \sigma_0) : \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau \otimes \sigma_0) \longrightarrow \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau \otimes \sigma_0)$$

by

$$R(w, \tau \otimes \sigma_0)\Phi(h) = \mathcal{A}_w(\mathcal{R}(w, \tau \otimes \sigma_0)\Phi(h)).$$

By construction,

$$R(w, \tau \otimes \sigma_0)^2 = 1.$$

7.4 Weil representations

In this subsection, we recall some explicit formulas for the Weil representations.

Let $\mathbb{W}$ be a finite dimensional vector space over F equipped with a nondegenerate symplectic form $\langle \cdot, \cdot \rangle_{\mathbb{W}} : \mathbb{W} \times \mathbb{W} \to F$. Let $\mathscr{H}(\mathbb{W}) = \mathbb{W} \oplus F$ be the associated Heisenberg group, i.e. the multiplication law is given by

$$(w, t) \cdot (w', t') = \left(w + w', t + t' + \frac{1}{2} \langle w, w' \rangle_{\mathbb{W}} \right)$$

for $w, w' \in \mathbb{W}$ and $t, t' \in F$. Fix maximal totally isotropic subspaces $\mathbb{X}$ and $\mathbb{X}^*$ of $\mathbb{W}$ such that $\mathbb{W} = \mathbb{X} \oplus \mathbb{X}^*$. Let ρ be the Heisenberg representation of $\mathscr{H}(\mathbb{W})$ on $\mathscr{S}(\mathbb{X}^*)$ with central character ψ. Namely,

$$\rho((x + x', t))\varphi(x'_0) = \psi\left(t + \langle x'_0, x \rangle_{\mathbb{W}} + \tfrac{1}{2} \langle x', x \rangle_{\mathbb{W}} \right) \varphi(x'_0 + x')$$

for $\varphi \in \mathscr{S}(\mathbb{X}^*)$, $x \in \mathbb{X}$, $x', x'_0 \in \mathbb{X}^*$, and $t \in F$.

In Sect. 4.1, we have introduced the Weil representations for unitary groups. To define these representations, we have fixed the additive character ψ and the pair of characters (χ_V, χ_W). For simplicity, we write:

- ω for the Weil representation $\omega_{\psi,\chi_V,\chi_W,V,W}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$ on a space $\mathscr{S}$;
- ω_0 for the Weil representation $\omega_{\psi,\chi_V,\chi_W,V,W_0}$ of $\mathrm{U}(V) \times \mathrm{U}(W_0)$ on a space $\mathscr{S}_0$;
- ω_{00} for the Weil representation $\omega_{\psi,\chi_V,\chi_W,V_0,W_0}$ of $\mathrm{U}(V_0) \times \mathrm{U}(W_0)$ on a space $\mathscr{S}_{00}$.

We take a mixed model

$$\mathscr{S} = \mathscr{S}(V \otimes Y^*) \otimes \mathscr{S}_0$$

of ω, where we regard $\mathscr{S}$ as a space of functions on $V \otimes Y^*$ with values in $\mathscr{S}_0$. Similarly, we take a mixed model

$$\mathscr{S}_0 = \mathscr{S}(X^* \otimes W_0) \otimes \mathscr{S}_{00}$$

of ω_0, where we regard $\mathscr{S}_0$ as a space of functions on $X^* \otimes W_0$ with values in $\mathscr{S}_{00}$. Also, we write:

- ρ_0 for the Heisenberg representation of $\mathscr{H}(V \otimes W_0)$ on $\mathscr{S}_0$ with central character ψ;
- ρ_{00} for the Heisenberg representation of $\mathscr{H}(V_0 \otimes W_0)$ on $\mathscr{S}_{00}$ with central character ψ.

Using [37, Theorem 3.1], we can derive the following formulas for the Weil representations ω and ω_0. Put $\Delta = \delta^2 \in F^\times$. As in [49, Appendix], let $\gamma_F(\psi)$ be the Weil index of the character $x \mapsto \psi(x^2)$ of second degree and set

$$\gamma_F(a, \psi) = \frac{\gamma_F(\psi_a)}{\gamma_F(\psi)}$$

for $a \in F^\times$, where $\psi_a(x) = \psi(ax)$. Note that $\gamma_F(\Delta, \psi) = \lambda(E/F, \psi)^{-1}$. For $\varphi \in \mathscr{S}$ and $x \in V \otimes Y^*$, we have

$$\begin{aligned}
(\omega(h)\varphi)(x) &= \omega_0(h)\varphi(h^{-1}x), && h \in \mathrm{U}(V),\\
(\omega(g_0)\varphi)(x) &= \omega_0(g_0)\varphi(x), && g_0 \in \mathrm{U}(W_0),\\
(\omega(m_Q(a))\varphi)(x) &= \chi_V(\det a)|\det a|^{m/2}\varphi(a^*x), && a \in \mathrm{GL}(Y),\\
(\omega(u_Q(b))\varphi)(x) &= \rho_0((b^*x, 0))\varphi(x), && b \in \mathrm{Hom}(W_0, Y),\\
(\omega(u_Q(c))\varphi)(x) &= \psi\left(\tfrac{1}{2}\langle cx, x\rangle\right)\varphi(x), && c \in \mathrm{Herm}(Y^*, Y),\\
(\omega(w_Q)\varphi)(x) &= \gamma_V^{-k}\int_{V\otimes Y}\varphi(-I_Y^{-1}y)\psi(\langle y, x\rangle)\,dy,
\end{aligned}$$

where

$$\gamma_V = \begin{cases} \omega_{E/F}(\det V)\cdot\gamma_F(-\Delta, \psi)^m\cdot\gamma_F(-1, \psi)^{-m} & \text{if } \varepsilon = +1;\\ \chi_V(\delta)^{-1}\cdot\omega_{E/F}(\delta^{-m}\cdot\det V)\cdot\gamma_F(-\Delta, \psi)^m\cdot\gamma_F(-1, \psi)^{-m} & \text{if } \varepsilon = -1.\end{cases}$$

Also, for $\varphi_0 \in \mathscr{S}_0$ and $x \in X^* \otimes W_0$, we have

$$\begin{aligned}
(\omega_0(g_0)\varphi_0)(x) &= \omega_{00}(g_0)\varphi_0(g_0^{-1}x), && g_0 \in \mathrm{U}(W_0),\\
(\omega_0(h_0)\varphi_0)(x) &= \omega_{00}(h_0)\varphi_0(x), && h_0 \in \mathrm{U}(V_0),\\
(\omega_0(m_P(a))\varphi_0)(x) &= \chi_W(\det a)|\det a|^{n_0/2}\varphi_0(a^*x), && a \in \mathrm{GL}(X),\\
(\omega_0(u_P(b))\varphi_0)(x) &= \rho_{00}((b^*x, 0))\varphi_0(x), && b \in \mathrm{Hom}(V_0, X),\\
(\omega_0(u_P(c))\varphi_0)(x) &= \psi\left(\tfrac{1}{2}\langle cx, x\rangle\right)\varphi_0(x), && c \in \mathrm{Herm}(X^*, X),\\
(\omega_0(w_P)\varphi_0)(x) &= \gamma_W^{-k}\int_{X\otimes W_0}\varphi_0(-I_X^{-1}y)\psi(\langle y, x\rangle)\,dy,
\end{aligned}$$

where

$$\gamma_W = \begin{cases} \chi_W(\delta)^{-1} \cdot \omega_{E/F}(\delta^{-n} \cdot \det W) \cdot \gamma_F(-\Delta, \psi)^n \cdot \gamma_F(-1, \psi)^{-n} & \text{if } \varepsilon = +1; \\ \omega_{E/F}(\det W) \cdot \gamma_F(-\Delta, \psi)^n \cdot \gamma_F(-1, \psi)^{-n} & \text{if } \varepsilon = -1, \end{cases}$$

and

$$\begin{aligned} (\rho_0((y+y',0))\varphi_0)(x) &= \psi\Big(\langle x, y\rangle + \tfrac{1}{2}\langle y', y\rangle\Big)\varphi_0(x+y'), && y \in X \otimes W_0, \\ & && y' \in X^* \otimes W_0, \\ (\rho_0((y_0,0))\varphi_0)(x), &= \rho_{00}((y_0,0))\varphi_0(x), && y_0 \in V_0 \otimes W_0. \end{aligned}$$

7.5 Zeta integrals of Godement–Jacquet

In this subsection, we review the theory of local factors for GL_k developed by Godement–Jacquet [22].

Let τ be an irreducible smooth representation of $\mathrm{GL}_k(E)$ on a space $\mathscr{V}_\tau$ with central character ω_τ. For any character χ of $E^\times$, we realize the representation $\tau\chi := \tau \otimes (\chi \circ \det)$ on $\mathscr{V}_\tau$ by setting $(\tau\chi)(a)v := \chi(\det a)\tau(a)v$ for $a \in \mathrm{GL}_k(E)$ and $v \in \mathscr{V}_\tau$. Put $\tau_s := \tau|\cdot|^s$ for $s \in \mathbb{C}$. Let τ^c be the representation of $\mathrm{GL}_k(E)$ on $\mathscr{V}_\tau$ defined by $\tau^c(a) = \tau(a^c)$. We write

$$L(s,\tau) = L(s,\phi_\tau) \quad \text{and} \quad \epsilon(s,\tau,\psi_E) = \epsilon(s,\phi_\tau,\psi_E)$$

for the standard L-factor and ϵ-factor of τ, where ϕ_τ is the k-dimensional representation of WD_E associated to τ and ψ_E is the nontrivial additive character of E defined by $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. Then the standard γ-factor of τ is defined by

$$\gamma(s,\tau,\psi_E) = \epsilon(s,\tau,\psi_E) \cdot \frac{L(1-s,\tau^\vee)}{L(s,\tau)},$$

where $\tau^\vee$ is the contragredient representation of τ.

For $s \in \mathbb{C}$, $\phi \in \mathscr{S}(\mathrm{M}_k(E))$, and a matrix coefficient f of τ, put

$$Z(s,\phi,f) = \int_{\mathrm{GL}_k(E)} \phi(a) f(a) |\det a|^s \, da,$$

where we have fixed a Haar measure da on $\mathrm{GL}_k(E)$. This integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbb{C}$.

Moreover,

$$\frac{Z\left(s+\frac{k-1}{2},\phi,f\right)}{L(s,\tau)}$$

is an entire function of s. If τ is square-integrable, then $Z(s,\phi,f)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{k-1}{2}$ by [22, Proposition 1.3].

Let $\hat{\phi} \in \mathscr{S}(\mathrm{M}_k(E))$ be the Fourier transform of ϕ defined by

$$\hat{\phi}(x) = \int_{\mathrm{M}_k(E)} \phi(y)\psi_E(\mathrm{Tr}(xy))\,dy,$$

where dy is the self-dual Haar measure on $\mathrm{M}_k(E)$ with respect to the pairing $(x,y) \mapsto \psi_E(\mathrm{Tr}(xy))$. Let $\check{f}$ be the matrix coefficient of $\tau^\vee$ given by $\check{f}(a) = f(a^{-1})$. Then the local functional equation asserts that

$$Z\left(-s+\frac{k+1}{2},\hat{\phi},\check{f}\right) = \gamma(s,\tau,\psi_E)\cdot Z\left(s+\frac{k-1}{2},\phi,f\right).$$

8 Proof of Theorem 6.1

Now we can begin the proof of Theorem 6.1. This will be proved by an explicit construction of an equivariant map which realizes the theta correspondence. Recall from Sect. 7 that we have fixed $\varepsilon = \pm 1$, an m-dimensional ε-Hermitian space $V = X \oplus V_0 \oplus X^*$, and an n-dimensional $(-\varepsilon)$-Hermitian space $W = Y \oplus W_0 \oplus Y^*$.

8.1 Construction of equivariant maps

Recall that we have identified $\mathrm{GL}(X)$ with $\mathrm{GL}_k(E)$ using the basis $\{v_1,\ldots,v_k\}$. Similarly, we identify $\mathrm{GL}(Y)$ with $\mathrm{GL}_k(E)$ using the basis $\{w_1,\ldots,w_k\}$. Thus we can define an isomorphism $i : \mathrm{GL}(Y) \to \mathrm{GL}(X)$ via these identifications. Put

$$e = v_1 \otimes w_1^* + \cdots + v_k \otimes w_k^* \in X \otimes Y^*,$$
$$e^* = v_1^* \otimes w_1 + \cdots + v_k^* \otimes w_k \in X^* \otimes Y.$$

Then $i(a)^c e = a^* e$ and $(i(a)^c)^* e^* = a e^*$ for $a \in \mathrm{GL}(Y)$.

For $\varphi \in \mathscr{S} = \mathscr{S}(V \otimes Y^*) \otimes \mathscr{S}_0$, we define functions $\mathfrak{f}(\varphi)$, $\hat{\mathfrak{f}}(\varphi)$ on $\mathrm{U}(W) \times \mathrm{U}(V)$ with values in $\mathscr{S}_0$ by

$$\mathfrak{f}(\varphi)(gh) = (\omega(gh)\varphi)\begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{\mathfrak{f}}(\varphi)(gh) = \int_{X\otimes Y^*} (\omega(gh)\varphi)\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \psi(\varepsilon\langle x, e^*\rangle)\, dx$$

for $g \in \mathrm{U}(W)$ and $h \in \mathrm{U}(V)$. Here, we write an element in $V \otimes Y^*$ as a block matrix

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

with $y_1 \in X \otimes Y^*$, $y_2 \in V_0 \otimes Y^*$, and $y_3 \in X^* \otimes Y^*$. We also define functions $f(\varphi)$, $\hat{f}(\varphi)$ on $\mathrm{U}(W) \times \mathrm{U}(V)$ with values in $\mathscr{S}_{00}$ by

$$f(\varphi)(gh) = \mathrm{ev}(\mathfrak{f}(\varphi)(gh)),$$
$$\hat{f}(\varphi)(gh) = \mathrm{ev}(\hat{\mathfrak{f}}(\varphi)(gh)),$$

where $\mathrm{ev} : \mathscr{S}_0 = \mathscr{S}(X^* \otimes W_0) \otimes \mathscr{S}_{00} \to \mathscr{S}_{00}$ is the evaluation at $0 \in X^* \otimes W_0$. If $f = f(\varphi)$ or $\hat{f}(\varphi)$, then

$$\begin{aligned} f(uu'gh) &= f(gh), && u \in U_Q, \\ & && u' \in U_P, \\ f(g_0h_0gh) &= \omega_{00}(g_0h_0) f(gh), && g_0 \in \mathrm{U}(W_0), \\ & && h_0 \in \mathrm{U}(V_0), \\ f(m_Q(a)m_P(i(a)^c)gh) &= (\chi_V \chi_W^c)(\det a)|\det a|^{\rho_P+\rho_Q} f(gh), && a \in \mathrm{GL}(Y). \end{aligned}$$

Let τ be an irreducible (unitary) square-integrable representation of $\mathrm{GL}_k(E)$ on a space $\mathscr{V}_\tau$. We may regard τ as a representation of $\mathrm{GL}(X)$ or $\mathrm{GL}(Y)$ via the above identifications. Let π_0 and σ_0 be irreducible tempered representations of $\mathrm{U}(W_0)$ and $\mathrm{U}(V_0)$ on spaces $\mathscr{V}_{\pi_0}$ and $\mathscr{V}_{\sigma_0}$ respectively. Fix nonzero invariant nondegenerate bilinear forms $\langle\cdot,\cdot\rangle$ on $\mathscr{V}_\tau \times \mathscr{V}_{\tau^\vee}$, $\mathscr{V}_{\pi_0} \times \mathscr{V}_{\pi_0^\vee}$, and $\mathscr{V}_{\sigma_0} \times \mathscr{V}_{\sigma_0^\vee}$. Let

$$\langle\cdot,\cdot\rangle : (\mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0^\vee}) \times \mathscr{V}_{\tau^\vee} \longrightarrow \mathscr{V}_{\sigma_0^\vee}$$

be the induced map.

Now assume that

$$\sigma_0 = \Theta_{\psi,V_0,W_0}(\pi_0).$$

We fix a nonzero $\mathrm{U}(V_0) \times \mathrm{U}(W_0)$-equivariant map

$$\mathcal{T}_{00} : \omega_{00} \otimes \sigma_0^\vee \longrightarrow \pi_0.$$

For $\varphi \in \mathscr{S}$, $\Phi_s \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$, $g \in \mathrm{U}(W)$, $\check{v} \in \mathscr{V}_{\tau^\vee}$, and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$, put

$$\begin{aligned}\langle \mathcal{T}_s(\varphi \otimes \Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle \\ &= L\Big(s - s_0 + \tfrac{1}{2}, \tau\Big)^{-1} \\ &\quad \times \int_{U_P \mathrm{U}(V_0) \backslash \mathrm{U}(V)} \langle \mathcal{T}_{00}(\hat{f}(\varphi)(gh) \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0 \rangle \, dh,\end{aligned}$$

where we have fixed Haar measures on $\mathrm{U}(V)$ and $\mathrm{U}(V_0)$, and set

$$s_0 = \frac{m-n}{2} = \frac{m_0 - n_0}{2}.$$

Note that $\langle \Phi_s(h), \check{v}\rangle \in \mathscr{V}_{\sigma_0^\vee}$.

Lemma 8.1 *The integral $\langle \mathcal{T}_s(\varphi \otimes \Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle$ is absolutely convergent for* $\mathrm{Re}(s) > s_0 - \frac{1}{2}$ *and admits a holomorphic continuation to $\mathbb{C}$.*

Proof We may assume that $\varphi = \varphi' \otimes \varphi_0$ and $\Phi_s(1) = v \otimes v_0$, where $\varphi' \in \mathscr{S}(V \otimes Y^*)$, $\varphi_0 \in \mathscr{S}_0$, $v \in \mathscr{V}_\tau$, and $v_0 \in \mathscr{V}_{\sigma_0^\vee}$. By the Iwasawa decomposition, it suffices to consider the integral

$$\int_{\mathrm{GL}(X)} \langle \mathcal{T}_{00}(\hat{f}(\varphi)(m_P(a)) \otimes \langle \Phi_s(m_P(a)), \check{v}\rangle), \check{v}_0 \rangle |\det a|^{-2\rho_P} \, da. \quad (8.1)$$

Put

$$\phi(y) = \int_{X \otimes Y^*} \varphi' \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \psi(\varepsilon \langle x, y \rangle) \, dx$$

for $y \in X^* \otimes Y$. Then we have

$$\hat{f}(\varphi)(m_P(a)) = \chi_W(\det a) |\det a|^{k + n_0/2} \phi(a^* e^*) \cdot \mathrm{ev}(\varphi_0)$$

for $a \in \mathrm{GL}(X)$. Hence we have

$$\begin{aligned}(8.1) &= \langle \mathcal{T}_{00}(\mathrm{ev}(\varphi_0) \otimes v_0), \check{v}_0 \rangle \\ &\quad \times \int_{\mathrm{GL}(X)} \phi(a^* e^*) \langle \tau(a^c) v, \check{v} \rangle |\det a|^{s - s_0 + k/2} \, da.\end{aligned}$$

This completes the proof, in view of Sect. 7.5. □

Thus we obtain a $\mathrm{U}(V)\times\mathrm{U}(W)$-equivariant map

$$\mathcal{T}_s : \omega\otimes\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c\chi_W^c\otimes\sigma_0^\vee)\longrightarrow\mathrm{Ind}_Q^{\mathrm{U}(W)}(\tau_s\chi_V\otimes\pi_0).$$

Lemma 8.2 *If* $\mathrm{Re}(s) < s_0+\frac{1}{2}$, *then we have*

$$\begin{aligned}&\langle\mathcal{T}_s(\varphi\otimes\Phi_s)(g),\check{v}\otimes\check{v}_0\rangle\\&\quad= L\Big(s-s_0+\tfrac{1}{2},\tau\Big)^{-1}\cdot\gamma\Big(s-s_0+\tfrac{1}{2},\tau,\psi_E\Big)^{-1}\\&\qquad\times\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\langle\mathcal{T}_{00}(f(\varphi)(gh)\otimes\langle\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,dh.\end{aligned}$$

Proof We may assume that $\varphi=\varphi'\otimes\varphi_0$ and $\Phi_s(1)=v\otimes v_0$, where $\varphi'\in\mathcal{S}(V\otimes Y^*)$, $\varphi_0\in\mathcal{S}_0$, $v\in\mathcal{V}_\tau$, and $v_0\in\mathcal{V}_{\sigma_0^\vee}$. Put $f(a)=\langle\tau(a)v,\check{v}\rangle$ for $a\in\mathrm{GL}(X)$. Let $\phi\in\mathcal{S}(X^*\otimes Y)$ be as in the proof of Lemma 8.1. We define its Fourier transform $\hat{\phi}\in\mathcal{S}(X\otimes Y^*)$ by

$$\hat{\phi}(x)=\int_{X^*\otimes Y}\phi(y)\psi(-\varepsilon\langle x,y\rangle)\,dy.$$

By the Fourier inversion formula, we have

$$\hat{\phi}(x)=\varphi'\begin{pmatrix}x\\0\\0\end{pmatrix}.$$

Hence we have

$$f(\varphi)(m_P(a))=\chi_W(\det a)|\det a|^{n_0/2}\hat{\phi}(a^{-1}e)\cdot\mathrm{ev}(\varphi_0)$$

for $a\in\mathrm{GL}(X)$. If $s_0-\frac{1}{2}<\mathrm{Re}(s)<s_0+\frac{1}{2}$, then by the local functional equation of the zeta integrals of Godement–Jacquet (see Sect. 7.5), we have

$$\begin{aligned}&\int_{\mathrm{GL}(X)}\langle\mathcal{T}_{00}(\hat{f}(\varphi)(m_P(a))\otimes\langle\Phi_s(m_P(a)),\check{v}\rangle),\check{v}_0\rangle|\det a|^{-2\rho_P}\,da\\&\quad=\langle\mathcal{T}_{00}(\mathrm{ev}(\varphi_0)\otimes v_0),\check{v}_0\rangle\cdot\int_{\mathrm{GL}(X)}\phi(a^*e^*)f(a^c)|\det a|^{s-s_0+k/2}\,da\\&=\langle\mathcal{T}_{00}(\mathrm{ev}(\varphi_0)\otimes v_0),\check{v}_0\rangle\\&\quad\times\gamma\Big(s-s_0+\tfrac{1}{2},\tau,\psi_E\Big)^{-1}\cdot\int_{\mathrm{GL}(X)}\hat{\phi}(ae)\check{f}(a^c)|\det a|^{-s+s_0+k/2}\,da\\&=\langle\mathcal{T}_{00}(\mathrm{ev}(\varphi_0)\otimes v_0),\check{v}_0\rangle\\&\quad\times\gamma\Big(s-s_0+\tfrac{1}{2},\tau,\psi_E\Big)^{-1}\cdot\int_{\mathrm{GL}(X)}\hat{\phi}(a^{-1}e)f(a^c)|\det a|^{s-s_0-k/2}\,da\end{aligned}$$

$$= \gamma\Big(s - s_0 + \tfrac{1}{2}, \tau, \psi_E\Big)^{-1}$$
$$\times \int_{\mathrm{GL}(X)} \langle \mathcal{T}_{00}(f(\varphi)(m_P(a)) \otimes \langle \Phi_s(m_P(a)), \check{v}\rangle), \check{v}_0\rangle |\det a|^{-2\rho_P}\, da.$$

This completes the proof. □

Lemma 8.3 *Assume that $m \geq n$. Let $\Phi \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau^c \chi_W^c \otimes \sigma_0^\vee)$. If $\Phi \neq 0$, then there exists $\varphi \in \mathscr{S}$ such that*

$$\mathcal{T}_0(\varphi \otimes \Phi) \neq 0.$$

Proof Fix a special maximal compact subgroup K of $\mathrm{U}(V)$. We extend Φ to a holomorphic section Φ_s of $\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$ so that $\Phi_s|_K$ is independent of s. We have

$$L\Big(s - s_0 + \tfrac{1}{2}, \tau\Big)^{-1} \cdot \gamma\Big(s - s_0 + \tfrac{1}{2}, \tau, \psi_E\Big)^{-1} = L\Big(-s + s_0 + \tfrac{1}{2}, \tau^\vee\Big)^{-1}$$

up to an invertible function. Since τ is square-integrable and $s_0 \geq 0$, the right-hand side is holomorphic and nonzero at $s = 0$. By Lemma 8.2, it suffices to show that there exist $\varphi \in \mathscr{S}$, $\check{v} \in \mathscr{V}_{\tau^\vee}$, and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$ such that

$$\int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} \langle \mathcal{T}_{00}(f(\varphi)(h) \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0\rangle\, dh \tag{8.2}$$

is nonzero and independent of s for $\mathrm{Re}(s) \ll 0$.

Let $\varphi = \varphi' \otimes \varphi_0$, where $\varphi' \in \mathscr{S}(V \otimes Y^*)$ and $\varphi_0 \in \mathscr{S}_0$. Then we have

$$(8.2) = \int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} \varphi'(h^{-1}x_0) \Psi_s(h)\, dh,$$

where

$$x_0 = \begin{pmatrix} e \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_s(h) = \langle \mathcal{T}_{00}(\mathrm{ev}(\omega_0(h)\varphi_0) \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0\rangle.$$

We can choose φ_0, $\check{v}$, and $\check{v}_0$ so that $\Psi_s|_K$ is nonzero and independent of s. Since $h \mapsto h^{-1}x_0$ induces a homeomorphism

$$U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V) \xrightarrow{\sim} \mathrm{U}(V)x_0$$

and $\mathrm{U}(V)x_0$ is locally closed in $V \otimes Y^*$, there exists φ' such that

$$\mathrm{supp}\,\varphi' \cap \mathrm{U}(V)x_0 = Kx_0$$

and such that $\varphi'(k^{-1}x_0) = \overline{\Psi_s(k)}$ for all $k \in K$. Hence we have

$$(8.2) = \int_{U_P \mathrm{U}(V_0)\backslash U_P \mathrm{U}(V_0) K} \varphi'(h^{-1}x_0)\Psi_s(h)\,dh$$
$$= \int_{(U_P \mathrm{U}(V_0)\cap K)\backslash K} |\Psi_s(k)|^2\,dk \neq 0.$$

Since $\Psi_s|_K$ is independent of s, so is this integral. This completes the proof. □

8.2 Compatibilities with intertwining operators

Now we shall prove a key property of the equivariant map we have constructed.

Let $w \in W(M_P)$ and $w' \in W(M_Q)$ be the nontrivial elements in the relative Weyl groups. As in Sect. 7.3, we take the representatives $\tilde{w} \in \mathrm{U}(V)$ of w and $\tilde{w}' \in \mathrm{U}(W)$ of w' defined by

$$\tilde{w} = w_P \cdot m_P((-1)^{m'} \cdot \kappa_V \cdot J) \cdot (-1_{V_0})^k,$$
$$\tilde{w}' = w_Q \cdot m_Q((-1)^{n'} \cdot \kappa_W \cdot J) \cdot (-1_{W_0})^k,$$

where $m' = [\frac{m}{2}]$ and $n' = [\frac{n}{2}]$. Having fixed τ, π_0, and σ_0, we shall write

$$\mathcal{M}(\tilde{w}, s) = \mathcal{M}(\tilde{w}, \tau_s^c \chi_W^c \otimes \sigma_0^\vee),$$
$$\mathcal{M}(\tilde{w}', s) = \mathcal{M}(\tilde{w}', \tau_s \chi_V \otimes \pi_0)$$

for the unnormalized intertwining operators, which are defined by the integrals

$$\mathcal{M}(\tilde{w}, s)\Phi_s(h) = \int_{U_P} \Phi_s(\tilde{w}^{-1}uh)\,du,$$
$$\mathcal{M}(\tilde{w}', s)\Psi_s(g) = \int_{U_Q} \Psi_s(\tilde{w}'^{-1}ug)\,du$$

for $\Phi_s \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$ and $\Psi_s \in \mathrm{Ind}_Q^{\mathrm{U}(W)}(\tau_s \chi_V \otimes \pi_0)$. By the Howe duality, the diagram

$$\begin{array}{ccc}
\omega \otimes \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee) & \xrightarrow{\mathcal{T}_s} & \mathrm{Ind}_Q^{\mathrm{U}(W)}(\tau_s \chi_V \otimes \pi_0) \\
\Big\downarrow {\scriptstyle 1\otimes\mathcal{M}(\tilde{w},s)} & & \Big\downarrow {\scriptstyle \mathcal{M}(\tilde{w}',s)} \\
\omega \otimes \mathrm{Ind}_P^{\mathrm{U}(V)}(w(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)) & \xrightarrow{\mathcal{T}_{-s}} & \mathrm{Ind}_Q^{\mathrm{U}(W)}(w'(\tau_s \chi_V \otimes \pi_0))
\end{array}$$

commutes up to a scalar. The following proposition determines this constant of proportionality explicitly.

Proposition 8.4 *For $\varphi \in \mathscr{S}$ and $\Phi_s \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$, we have*

$$\begin{aligned}
&\mathcal{M}(\tilde{w}', s)\mathcal{T}_s(\varphi \otimes \Phi_s) \\
&= \Big[\gamma_V^{-1} \cdot \gamma_W \cdot \chi_V((-1)^{n'} \cdot \varepsilon \cdot \kappa_W^{-1}) \cdot \chi_W((-1)^{m'-1} \cdot \kappa_V^{-1}) \cdot (\chi_V^{-n}\chi_W^m)(\delta)\Big]^k \\
&\quad \times \omega_\tau((-1)^{m'+n'-1} \cdot \kappa_V^c \kappa_W^{-1}) \cdot |\kappa_V|^{k(s+\rho_P)} \cdot |\kappa_W|^{-k(s+\rho_Q)} \\
&\quad \times L\Big(s - s_0 + \tfrac{1}{2}, \tau\Big)^{-1} \cdot L\Big(-s - s_0 + \tfrac{1}{2}, (\tau^c)^\vee\Big) \cdot \gamma\Big(-s - s_0 + \tfrac{1}{2}, (\tau^c)^\vee, \psi_E\Big) \\
&\quad \times \mathcal{T}_{-s}(\varphi \otimes \mathcal{M}(\tilde{w}, s)\Phi_s).
\end{aligned}$$

Proof We may assume that $\mathrm{Re}(s) \gg 0$. Let $\check{v} \in \mathscr{V}_{\tau^\vee}$ and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$. Noting that $\det J = 1$, we have by definition

$$\begin{aligned}
&\langle \mathcal{M}(\tilde{w}', s)\mathcal{T}_s(\varphi \otimes \Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle \\
&\quad = \omega_\tau((-1)^{n'} \cdot \kappa_W^{-1}) \cdot \chi_V((-1)^{n'} \cdot \kappa_W^{-1})^k \cdot |\kappa_W|^{-k(s+\rho_Q)} \cdot \omega_{\pi_0}(-1)^k \\
&\qquad \times \langle \mathcal{M}(w_Q, s)\mathcal{T}_s(\varphi \otimes \Phi_s)(g), \tau^\vee(J)\check{v} \otimes \check{v}_0 \rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle \mathcal{T}_{-s}(\varphi \otimes \mathcal{M}(\tilde{w}, s)\Phi_s)(g), \check{v} \otimes \check{v}_0 \rangle \\
&\quad = \omega_\tau((-1)^{m'} \cdot (\kappa_V^c)^{-1}) \cdot \chi_W((-1)^{m'} \cdot \kappa_V)^k \cdot |\kappa_V|^{-k(s+\rho_P)} \cdot \omega_{\sigma_0}(-1)^k \\
&\qquad \times \langle \mathcal{T}_{-s}(\varphi \otimes \mathcal{M}(w_P, s)\Phi_s)(g), \tau^\vee(J)\check{v} \otimes \check{v}_0 \rangle,
\end{aligned}$$

where ω_{π_0} and ω_{σ_0} are the central characters of π_0 and σ_0 respectively. Since $\sigma_0 = \Theta_{\psi,\chi_V,\chi_W,V_0,W_0}(\pi_0)$, we know that

$$\omega_{\sigma_0} = \nu \cdot \omega_{\pi_0},$$

where ν is the character of $\mathrm{Ker}(\mathrm{N}_{E/F})$ defined by

$$\nu(x/x^c) = (\chi_V^{-n_0}\chi_W^{m_0})(x)$$

for $x \in E^\times$. In particular, we have

$$\begin{aligned}
\omega_{\pi_0}(-1) \cdot \omega_{\sigma_0}(-1) &= (\chi_V^{-n_0}\chi_W^{m_0})(\delta) \\
&= (\chi_V^{-n}\chi_W^m)(\delta) \cdot \chi_V(-1)^k \cdot \chi_W(-1)^k.
\end{aligned}$$

Thus it suffices to show that

$$\begin{aligned}
&L\Big(s-s_0+\tfrac{1}{2},\tau\Big)\cdot\mathcal{M}(w_Q,s)\mathcal{T}_s(\varphi\otimes\Phi_s)\\
&=(\chi_V(-\varepsilon)\cdot\gamma_V^{-1}\cdot\gamma_W)^k\cdot\omega_\tau(-1)\\
&\quad\times L\Big(-s-s_0+\tfrac{1}{2},(\tau^c)^\vee\Big)\cdot\gamma\Big(-s-s_0+\tfrac{1}{2},(\tau^c)^\vee,\psi_E\Big)\\
&\quad\times\mathcal{T}_{-s}(\varphi\otimes\mathcal{M}(w_P,s)\Phi_s).
\end{aligned}$$

We have

$$\begin{aligned}
&L\Big(s-s_0+\tfrac{1}{2},\tau\Big)\cdot\langle\mathcal{M}(w_Q,s)\mathcal{T}_s(\varphi\otimes\Phi_s)(g),\check{v}\otimes\check{v}_0\rangle\\
&=L\Big(s-s_0+\tfrac{1}{2},\tau\Big)\cdot\int_{U_Q}\langle\mathcal{T}_s(\varphi\otimes\Phi_s)(w_Q^{-1}ug),\check{v}\otimes\check{v}_0\rangle\,du\\
&=\int_{U_Q}\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\langle\mathcal{T}_{00}(\hat{f}(\varphi)(w_Q^{-1}ugh)\otimes\langle\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,dh\,du\\
&=\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\int_{U_Q}\langle\mathcal{T}_{00}(\hat{f}(\varphi)(w_Q^{-1}ugh)\otimes\langle\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,du\,dh.
\end{aligned}$$

In Lemma 8.6(i) below, we shall show that these integrals are absolutely convergent, so that this manipulation is justified. By Lemma 8.2, we have

$$\begin{aligned}
&L\Big(-s-s_0+\tfrac{1}{2},(\tau^c)^\vee\Big)\cdot\gamma\Big(-s-s_0+\tfrac{1}{2},(\tau^c)^\vee,\psi_E\Big)\\
&\qquad\qquad\times\langle\mathcal{T}_{-s}(\varphi\otimes\mathcal{M}(w_P,s)\Phi_s)(g),\check{v}\otimes\check{v}_0\rangle\\
&=\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\langle\mathcal{T}_{00}(f(\varphi)(gh)\otimes\langle\mathcal{M}(w_P,s)\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,dh\\
&=\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\int_{U_P}\langle\mathcal{T}_{00}(f(\varphi)(gh)\otimes\langle\Phi_s(w_P^{-1}uh),\check{v}\rangle),\check{v}_0\rangle\,du\,dh\\
&=\int_{\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\langle\mathcal{T}_{00}(f(\varphi)(gh)\otimes\langle\Phi_s(w_P^{-1}h),\check{v}\rangle),\check{v}_0\rangle\,dh\\
&=\int_{\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\langle\mathcal{T}_{00}(f(\varphi)(gw_Ph)\otimes\langle\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,dh\\
&=\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\int_{U_P}\langle\mathcal{T}_{00}(f(\varphi)(gw_Puh)\otimes\langle\Phi_s(h),\check{v}\rangle),\check{v}_0\rangle\,du\,dh\\
&=\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\int_{U_P}\\
&\quad\times\langle\mathcal{T}_{00}(f(\varphi)(gw_Pum_P(-1_X)h)\otimes\langle\Phi_s(m_P(-1_X)h),\check{v}\rangle),\check{v}_0\rangle\,du\,dh
\end{aligned}$$

$$= \omega_\tau(-1) \cdot \chi_W(-1)^k \times \int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} \int_{U_P} \times \langle \mathcal{T}_{00}(f(\varphi)(g w_P u m_P(-1_X)h) \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0\rangle \, du \, dh.$$

In Lemma 8.6(ii) below, we shall show that these integrals are absolutely convergent, so that this manipulation is justified. Thus it remains to show that

$$\begin{aligned} &\chi_V(-\varepsilon)^k \cdot \gamma_V^k \cdot \int_{U_Q} \hat{f}(\varphi)(w_Q^{-1}u)\, du \\ &\quad = \chi_W(-1)^k \cdot \gamma_W^k \cdot \int_{U_P} f(\varphi)(w_P u m_P(-1_X))\, du. \end{aligned} \tag{8.3}$$

We may assume that $\varphi = \varphi' \otimes \varphi_0$, where $\varphi' \in \mathscr{S}(V \otimes Y^*)$ and $\varphi_0 \in \mathscr{S}_0$. We have $w_Q^{-1} = m_Q(-\varepsilon 1_Y) \cdot w_Q$ and

$$\begin{aligned} &\hat{\mathfrak{f}}(\varphi)(w_Q^{-1}) \\ &= \int_{X\otimes Y^*} (\omega(w_Q^{-1})\varphi)\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \psi(\varepsilon\langle x, e^*\rangle)\, dx \\ &= \chi_V(-\varepsilon)^k \cdot \int_{X\otimes Y^*} (\omega(w_Q)\varphi)\begin{pmatrix} -\varepsilon x \\ 0 \\ 0 \end{pmatrix} \psi(\varepsilon\langle x, e^*\rangle)\, dx \\ &= \chi_V(-\varepsilon)^k \cdot \int_{X\otimes Y^*} (\omega(w_Q)\varphi)\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \psi(-\langle x, e^*\rangle)\, dx \\ &= \chi_V(-\varepsilon)^k \cdot \gamma_V^{-k} \\ &\quad \times \int_{X\otimes Y^*} \left(\int_{X\otimes Y^*} \int_{V_0\otimes Y^*} \int_{X^*\otimes Y^*} \varphi\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \psi(-\langle I_Y y_3, x\rangle)\, dy_3\, dy_2\, dy_1 \right) \\ &\qquad \times \psi(-\langle x, e^*\rangle)\, dx \\ &= \chi_V(-\varepsilon)^k \cdot \gamma_V^{-k} \\ &\quad \times \int_{X\otimes Y^*} \left(\int_{X\otimes Y^*} \int_{V_0\otimes Y^*} \int_{X^*\otimes Y^*} \varphi\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \psi(\langle x, I_Y y_3\rangle)\, dy_3\, dy_2\, dy_1 \right) \\ &\qquad \times \psi(-\langle x, e^*\rangle)\, dx \\ &= \chi_V(-\varepsilon)^k \cdot \gamma_V^{-k} \cdot \int_{X\otimes Y^*} \int_{V_0\otimes Y^*} \varphi\begin{pmatrix} y_1 \\ y_2 \\ I_Y^{-1} e^* \end{pmatrix} dy_2\, dy_1. \end{aligned}$$

Hence, noting that $I_X^{-1} e = I_Y^{-1} e^* = e^{**}$, we have

$$\chi_V(-\varepsilon)^k \cdot \gamma_V^k \cdot \int_{\mathrm{Herm}(Y^*,Y)} \hat{\mathfrak{f}}(\varphi)(w_Q^{-1} u_Q(c))\, dc$$
$$= \int_{\mathrm{Herm}(Y^*,Y)} \left(\int_{X\otimes Y^*} \int_{V_0\otimes Y^*} \varphi\begin{pmatrix} y_1 \\ y_2 \\ e^{**} \end{pmatrix} \psi\left(\langle c y_1, e^{**}\rangle + \tfrac{1}{2}\langle c y_2, y_2\rangle \right) dy_2\, dy_1 \right) dc.$$

We change the variables

$$\begin{aligned} y_1 &= x_1 e^{**} \in X \otimes Y^*, & x_1 &\in \mathrm{Hom}(X^*, X), \\ y_2 &= x_2 e^{**} \in V_0 \otimes Y^*, & x_2 &\in \mathrm{Hom}(X^*, V_0). \end{aligned}$$

Then the inner integral is equal to

$$\begin{aligned} &\int_{\mathrm{Hom}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi\begin{pmatrix} x_1 e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix} \\ &\qquad \times \psi\left(\langle c x_1 e^{**}, e^{**}\rangle + \frac{1}{2}\langle c x_2 e^{**}, x_2 e^{**}\rangle \right) dx_2\, dx_1 \\ &= \int_{\mathrm{Hom}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi\begin{pmatrix} x_1 e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix} \\ &\qquad \times \psi\left(-\langle x_1 e^{**}, c e^{**}\rangle - \tfrac{1}{2}\langle x_2^* x_2 e^{**}, c e^{**}\rangle \right) dx_2\, dx_1 \\ &= \int_{\mathrm{Hom}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi\begin{pmatrix} \left(x_1 - \tfrac{1}{2} x_2^* x_2\right) e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix} \psi(-\langle x_1 e^{**}, c e^{**}\rangle)\, dx_2\, dx_1. \end{aligned}$$

By Lemma 7.1, the integral over $c \in \mathrm{Herm}(Y^*, Y)$ of this integral is equal to

$$\int_{\mathrm{Herm}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi\begin{pmatrix} \left(c - \tfrac{1}{2} x_2^* x_2\right) e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix} dx_2\, dc.$$

Hence the left-hand side of (8.3) is equal to

$$\chi_V(-\varepsilon)^k \cdot \gamma_V^k \cdot \int_{\mathrm{Hom}(W_0,Y)} \int_{\mathrm{Herm}(Y^*,Y)} \hat{f}(\varphi)(w_Q^{-1} u_Q(c) u_Q(b))\, dc\, db$$
$$= \int_{\mathrm{Hom}(W_0,Y)} \int_{\mathrm{Herm}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \times \mathrm{ev}\Big((\omega(u_Q(b))\varphi)\begin{pmatrix} \big(c - \frac{1}{2}x_2^* x_2\big) e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix}\Big)\, dx_2\, dc\, db$$
$$= \int_{\mathrm{Hom}(W_0,Y)} \int_{\mathrm{Herm}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi' \begin{pmatrix} \big(c - \frac{1}{2}x_2^* x_2\big) e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix}$$
$$\times \psi\Big(\tfrac{1}{2}\langle b^* e^{**}, b^*(c - \tfrac{1}{2}x_2^* x_2) e^{**}\rangle\Big) \rho_{00}((b^* x_2 e^{**}, 0)) \varphi_0(b^* e^{**})\, dx_2\, dc\, db$$
$$= \int_{\mathrm{Hom}(W_0,Y)} \int_{\mathrm{Herm}(X^*,X)} \int_{\mathrm{Hom}(X^*,V_0)} \varphi' \begin{pmatrix} \big(c - \frac{1}{2}x_2^* x_2\big) e^{**} \\ x_2 e^{**} \\ e^{**} \end{pmatrix}$$
$$\times \psi\Big(-\tfrac{1}{2}\langle c b^* e^{**}, b^* e^{**}\rangle\Big) \rho_{00}((x_2 b^* e^{**}, 0)) \varphi_0(b^* e^{**})\, dx_2\, dc\, db.$$

Note that $\langle b^* e^{**}, b^* x_2^* x_2 e^{**}\rangle = \langle b b^* e^{**}, x_2^* x_2 e^{**}\rangle = 0$.

On the other hand, the right-hand side of (8.3) is equal to the product of $\chi_W(-1)^k \cdot \gamma_W^k$ and

$$\int_{\mathrm{Hom}(V_0,X)} \int_{\mathrm{Herm}(X^*,X)} f(\varphi)(w_P u_P(c') u_P(b') m_P(-1_X))\, dc'\, db'$$
$$= \int_{\mathrm{Hom}(V_0,X)} \int_{\mathrm{Herm}(X^*,X)} \varphi' \begin{pmatrix} -\big(c' + \frac{1}{2}b' b'^*\big) e^{**} \\ -b'^* e^{**} \\ e^{**} \end{pmatrix}$$
$$\times \mathrm{ev}(\omega_0(w_P u_P(c') u_P(b') m_P(-1_X))\varphi_0)\, dc'\, db'.$$

We have

$$\mathrm{ev}(\omega_0(w_P u_P(c') u_P(b') m_P(-1_X))\varphi_0)$$
$$= \gamma_W^{-k} \cdot \int_{X^* \otimes W_0} (\omega_0(u_P(c') u_P(b') m_P(-1_X))\varphi_0)(y)\, dy$$
$$= \gamma_W^{-k} \cdot \int_{X^* \otimes W_0} \psi\Big(\tfrac{1}{2}\langle c' y, y\rangle\Big) (\omega_0(u_P(b') m_P(-1_X))\varphi_0)(y)\, dy$$

$$= \gamma_W^{-k} \cdot \int_{X^* \otimes W_0} \psi\left(\tfrac{1}{2}\langle c'y, y\rangle\right) \rho_{00}((b'^* y, 0))(\omega_0(m_P(-1_X))\varphi_0)(y)\, dy$$
$$= \chi_W(-1)^k \cdot \gamma_W^{-k} \cdot \int_{X^* \otimes W_0} \psi\left(\tfrac{1}{2}\langle c'y, y\rangle\right) \rho_{00}((b'^* y, 0))\varphi_0(-y)\, dy.$$

Changing the variables

$$\begin{aligned} b' &= -x_2^* \in \mathrm{Hom}(V_0, X), & x_2 &\in \mathrm{Hom}(X^*, V_0), \\ c' &= -c \in \mathrm{Herm}(X^*, X), & c &\in \mathrm{Herm}(X^*, X), \\ y &= -b^* e^{**} \in X^* \otimes W_0, & b &\in \mathrm{Hom}(W_0, Y), \end{aligned}$$

we see that the equality (8.3) holds. This completes the proof. □

Let ϕ_τ, ϕ_0, and ϕ_0' be the L-parameters of τ, π_0, and σ_0 respectively. As a consequence of Proposition 8.4, we deduce:

Corollary 8.5 *For $\varphi \in \mathscr{S}$ and $\Phi_s \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$, we have*

$$\begin{aligned} &\mathcal{R}(w', \tau_s \chi_V \otimes \pi_0) \mathcal{T}_s(\varphi \otimes \Phi_s) \\ &\quad = \alpha \cdot \beta(s) \cdot \mathcal{T}_{-s}(\varphi \otimes \mathcal{R}(w, \tau_s^c \chi_W^c \otimes \sigma_0^\vee)\Phi_s), \end{aligned}$$

where

$$\begin{aligned} \alpha = &\left[\gamma_V^{-1} \cdot \gamma_W \cdot \chi_V((-1)^{n'} \cdot \varepsilon \cdot \kappa_W^{-1}) \cdot \chi_W((-1)^{m'-1} \cdot \kappa_V^{-1}) \cdot (\chi_V^{-n} \chi_W^m)(\delta)\right]^k \\ &\times \omega_\tau((-1)^{m'+n'-1} \cdot \kappa_V^c \kappa_W^{-1}) \cdot \lambda(w, \psi) \cdot \lambda(w', \psi)^{-1} \end{aligned}$$

and

$$\begin{aligned} \beta(s) = &L\left(s - s_0 + \tfrac{1}{2}, \phi_\tau\right)^{-1} \cdot L\left(-s - s_0 + \tfrac{1}{2}, (\phi_\tau^c)^\vee\right) \\ &\times \gamma\left(-s - s_0 + \tfrac{1}{2}, (\phi_\tau^c)^\vee, \psi_E\right) \cdot |\kappa_V \kappa_W^{-1}|^{ks} \\ &\times \gamma(s, \phi_\tau^c \otimes \phi_0' \otimes \chi_W^c, \psi_E)^{-1} \cdot \gamma(s, \phi_\tau \otimes \phi_0^\vee \otimes \chi_V, \psi_E). \end{aligned}$$

Proof The corollary immediately follows from Proposition 8.4 and the following facts:

- $\gamma(s, \mathrm{As}^+ \circ \phi_{\tau^c}, \psi) = \gamma(s, \mathrm{As}^+ \circ \phi_\tau, \psi)$;
- for any conjugate self-dual character χ of $E^\times$,

$$\gamma(s, \mathrm{As}^+ \circ \phi_{\tau\chi}, \psi) = \begin{cases} \gamma(s, \mathrm{As}^+ \circ \phi_\tau, \psi) & \text{if } \chi|_{F^\times} = \mathbb{1}_{F^\times}; \\ \gamma(s, \mathrm{As}^- \circ \phi_\tau, \psi) & \text{if } \chi|_{F^\times} = \omega_{E/F}. \end{cases}$$

□

8.3 Convergence of integrals

To finish the proof of Proposition 8.4, it remains to show the following convergence of the integrals.

Lemma 8.6 *Let* $\varphi \in \mathscr{S}$, $\Phi_s \in \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau_s^c \chi_W^c \otimes \sigma_0^\vee)$, $\check{v} \in \mathscr{V}_{\tau^\vee}$, *and* $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$. *Assume that* $\mathrm{Re}(s) \gg 0$.

(i) *The integral*

$$\int_{U_Q} \int_{U_P \mathrm{U}(V_0) \backslash \mathrm{U}(V)} \langle \mathcal{T}_{00}(\hat{f}(\varphi)(w_Q^{-1} u h) \otimes \langle \Phi_s(h), \check{v} \rangle), \check{v}_0 \rangle \, dh \, du \quad (8.4)$$

is absolutely convergent.

(ii) *The integral*

$$\int_{U_P \mathrm{U}(V_0) \backslash \mathrm{U}(V)} \int_{U_P} \langle \mathcal{T}_{00}(f(\varphi)(h) \otimes \langle \Phi_s(w_P^{-1} u h), \check{v} \rangle), \check{v}_0 \rangle \, du \, dh \quad (8.5)$$

is absolutely convergent.

Proof Put $t = \mathrm{Re}(s) \gg 0$. Fix a special maximal compact subgroup K of $\mathrm{U}(V)$. We may assume that

- $\varphi = \varphi' \otimes \varphi_0$ for some $\varphi' \in \mathscr{S}(V \otimes Y^*)$ and $\varphi_0 \in \mathscr{S}_0$;
- $\Phi_s|_K$ is independent of s;
- Φ_s is K_0-fixed for some open compact subgroup K_0 of K;
- $\mathrm{supp}\, \Phi_s = P k_0 K_0$ for some $k_0 \in K$;
- $\Phi_s(k_0)$ is a pure tensor in $\mathscr{V}_\tau \otimes \mathscr{V}_{\sigma_0^\vee}$.

In particular, there exist maps $v : K \to \mathscr{V}_\tau$ and $v_0 : K \to \mathscr{V}_{\sigma_0^\vee}$ such that

$$\Phi_s(k) = v(k) \otimes v_0(k)$$

for all $k \in K$.

Recall that τ, π_0, and σ_0 are tempered and hence unitarizable. We can choose invariant Hilbert space norms $\|\cdot\|$ on $\mathscr{V}_\tau$ and $\mathscr{V}_{\tau^\vee}$ so that

$$|\langle v, \check{v} \rangle| \le \|v\| \|\check{v}\|$$

for all $v \in \mathscr{V}_\tau$ and $\check{v} \in \mathscr{V}_{\tau^\vee}$. Similarly, we choose invariant Hilbert space norms on $\mathscr{V}_{\pi_0}$, $\mathscr{V}_{\sigma_0}$, and so on. We may regard $\mathcal{T}_{00}$ as a $\mathrm{U}(V_0) \times \mathrm{U}(W_0)$-equivariant map $\mathcal{T}_{00} : \mathscr{S}_{00} \to \mathscr{V}_{\sigma_0} \otimes \mathscr{V}_{\pi_0}$, i.e.

$$\langle \mathcal{T}_{00}(\varphi_{00}), v_0 \otimes \check{v}_0 \rangle = \langle \mathcal{T}_{00}(\varphi_{00} \otimes v_0), \check{v}_0 \rangle$$

for $\varphi_{00} \in \mathscr{S}_{00}$, $v_0 \in \mathscr{V}_{\sigma_0^\vee}$, and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$. Then we have

$$\|\mathcal{T}_{00}(\omega_{00}(g_0h_0)\varphi_{00})\| = \|\mathcal{T}_{00}(\varphi_{00})\|$$

for $g_0 \in \mathrm{U}(W_0)$ and $h_0 \in \mathrm{U}(V_0)$, and

$$|\langle \mathcal{T}_{00}(\varphi_{00} \otimes v_0), \check{v}_0\rangle| \le \|\mathcal{T}_{00}(\varphi_{00})\| \|v_0\| \|\check{v}_0\|.$$

Fix $\check{v} \in \mathscr{V}_{\tau^\vee}$ and $\check{v}_0 \in \mathscr{V}_{\pi_0^\vee}$, and put

$$C = \|\check{v}\| \|\check{v}_0\| \max_{k \in K} \|v(k)\| \|v_0(k)\|.$$

Let Ψ_t be the K-fixed element in $\mathrm{Ind}_P^{\mathrm{U}(V)}(|\det|^t \otimes \mathbb{1}_{\mathrm{U}(V_0)})$ such that $\Psi_t(1) = 1$. Let ℓ denote the representation of $\mathrm{U}(V)$ on $\mathscr{S}(V \otimes Y^*)$ defined by $(\ell(h)\varphi')(x) = \varphi'(h^{-1}x)$. Recall that $\mathrm{ev} : \mathscr{S}_0 \to \mathscr{S}_{00}$ is the evaluation at 0.

First, we prove the absolute convergence of (8.4). We have

$$\hat{f}(\varphi)(h) = \phi(\ell(h)\varphi')(e^*) \cdot \mathrm{ev}(\omega_0(h)\varphi_0),$$

where $\phi : \mathscr{S}(V \otimes Y^*) \to \mathscr{S}(X^* \otimes Y)$ is defined by

$$\phi(\varphi')(y) = \int_{X \otimes Y^*} \varphi'\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \psi(\varepsilon\langle x, y\rangle)\, dx.$$

Put

$$\begin{aligned} \hat{\xi}_s(g,h) &= \langle \mathcal{T}_{00}(\hat{f}(\varphi)(gh) \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0\rangle \\ &= \chi_W^c(\det a)|\det a|^{s+\rho_P} \langle \tau^c(a)v(k), \check{v}\rangle \\ &\quad \times \langle \mathcal{T}_{00}(\hat{f}(\varphi)(gh) \otimes \sigma_0^\vee(h_0)v_0(k)), \check{v}_0\rangle \end{aligned}$$

for $g \in \mathrm{U}(W)$, $h = um_P(a)h_0k \in \mathrm{U}(V)$, $u \in U_P$, $a \in \mathrm{GL}(X)$, $h_0 \in \mathrm{U}(V_0)$, and $k \in K$. Then we have

$$\begin{aligned} |\hat{\xi}_s(g,h)| &\le |\det a|^{t+\rho_P} \|v(k)\| \|\check{v}\| \cdot \|\mathcal{T}_{00}(\hat{f}(\varphi)(gh))\| \|v_0(k)\| \|\check{v}_0\| \\ &\le C \cdot \Psi_t(h) \cdot \|\mathcal{T}_{00}(\hat{f}(\varphi)(gh))\| \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{T}_{00}(\hat{f}(\varphi)(h))\| &= \|\mathcal{T}_{00}(\hat{f}(\varphi)(m_P(a)k))\| \\ &= |\det a|^{k+n_0/2} |\phi(\ell(k)\varphi')(a^*e^*)| \cdot \|\mathcal{T}_{00}(\mathrm{ev}(\omega_0(k)\varphi_0))\|. \end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} |\hat{\xi}_s(g,h)|\, dh \\
&\quad \le C \cdot \int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} \Psi_t(h) \|\mathcal{T}_{00}(\hat{f}(\varphi)(gh))\|\, dh \\
&\quad = C \cdot \int_{\mathrm{GL}(X)} \int_K |\det a|^{t-\rho_P} \|\mathcal{T}_{00}(\hat{f}(\varphi)(g m_P(a) k))\|\, dk\, da \\
&\quad < \infty
\end{aligned}$$

since the last integral is the zeta integral of Godement–Jacquet associated to the trivial representation of $\mathrm{GL}(X)$. Put

$$\hat{\Xi}_t(g) = C \cdot \int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} \Psi_t(h) \|\mathcal{T}_{00}(\hat{f}(\varphi)(gh))\|\, dh.$$

Then we have

$$\hat{\Xi}_t(u m_Q(a) g_0 g) = |\det a|^{t+\rho_Q}\, \hat{\Xi}_t(g)$$

for $u \in U_Q$, $a \in \mathrm{GL}(Y)$, $g_0 \in \mathrm{U}(W_0)$, and $g \in \mathrm{U}(W)$, i.e. $\hat{\Xi}_t \in \mathrm{Ind}_Q^{\mathrm{U}(W)}(|\det|^t \otimes \mathbb{1}_{\mathrm{U}(W_0)})$. Hence we have

$$\int_{U_Q} \int_{U_P \mathrm{U}(V_0)\backslash \mathrm{U}(V)} |\hat{\xi}_s(w_Q^{-1} u, h)|\, dh\, du \le \int_{U_Q} \hat{\Xi}_t(w_Q^{-1} u)\, du < \infty.$$

Next, we prove the absolute convergence of (8.5). We have

$$f(\varphi)(h) = \hat{\phi}(\ell(h)\varphi')(e) \cdot \mathrm{ev}(\omega_0(h)\varphi_0),$$

where $\hat{\phi} : \mathscr{S}(V \otimes Y^*) \to \mathscr{S}(X \otimes Y^*)$ is defined by

$$\hat{\phi}(\varphi')(x) = \varphi' \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}.$$

Put

$$\begin{aligned}
\xi_s(h,h') &= \langle \mathcal{T}_{00}(f(\varphi)(h') \otimes \langle \Phi_s(h), \check{v}\rangle), \check{v}_0\rangle \\
&= \chi_W^c(\det a) |\det a|^{s+\rho_P} \langle \tau^c(a) v(k), \check{v}\rangle \\
&\qquad \times \langle \mathcal{T}_{00}(f(\varphi)(h') \otimes \sigma_0^\vee(h_0) v_0(k)), \check{v}_0\rangle
\end{aligned}$$

for $h = um_P(a)h_0k, h' \in \mathrm{U}(V)$, $u \in U_P$, $a \in \mathrm{GL}(X)$, $h_0 \in \mathrm{U}(V_0)$, and $k \in K$. Then we have

$$\begin{aligned}|\xi_s(h,h')| &\le |\det a|^{t+\rho_P}\|v(k)\|\|\check{v}\|\cdot\|\mathcal{T}_{00}(f(\varphi)(h'))\|\|v_0(k)\|\|\check{v}_0\|\\ &\le C\cdot\Psi_t(h)\cdot\|\mathcal{T}_{00}(f(\varphi)(h'))\|.\end{aligned}$$

Hence we have

$$\int_{U_P}|\xi_s(w_P^{-1}uh,h')|\,du \le C\cdot\|\mathcal{T}_{00}(f(\varphi)(h'))\|\cdot\int_{U_P}\Psi_t(w_P^{-1}uh)\,du<\infty.$$

Put

$$\Xi_t(h) = C\cdot\|\mathcal{T}_{00}(f(\varphi)(h))\|\cdot\mathcal{M}(w_P,t)\Psi_t(h),$$

where

$$\mathcal{M}(w_P,t)\Psi_t(h)=\int_{U_P}\Psi_t(w_P^{-1}uh)\,du.$$

Then we have

$$\begin{aligned}\Xi_t(um_P(a)h_0h) = C\cdot|\det a|^{-t+\rho_P+n_0/2}&|\hat{\phi}(\ell(h)\varphi')(a^{-1}e)|\\ &\times\|\mathcal{T}_{00}(\mathrm{ev}(\omega_0(h)\varphi_0))\|\cdot\mathcal{M}(w_P,t)\Psi_t(h)\end{aligned}$$

for $u \in U_P$, $a \in \mathrm{GL}(X)$, $h_0 \in \mathrm{U}(V_0)$, and $h \in \mathrm{U}(V)$. Hence, putting

$$C' = C\cdot\max_{k\in K}\|\mathcal{T}_{00}(\mathrm{ev}(\omega_0(k)\varphi_0))\|\cdot\mathcal{M}(w_P,t)\Psi_t(1),$$

we have

$$\begin{aligned}&\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\int_{U_P}|\xi_s(w_P^{-1}uh,h)|\,du\,dh\\ &\quad\le\int_{U_P\mathrm{U}(V_0)\backslash\mathrm{U}(V)}\Xi_t(h)\,dh\\ &\quad\le C'\cdot\int_{\mathrm{GL}(X)}\int_K|\det a|^{-t-\rho_P+n_0/2}|\hat{\phi}(\ell(k)\varphi')(a^{-1}e)|\,dk\,da\\ &\quad<\infty\end{aligned}$$

since the last integral is the zeta integral of Godement–Jacquet associated to the trivial representation of $\mathrm{GL}(X)$. □

8.4 Completion of the proof

Now assume that $\varepsilon = +1$ and $m = n+1$. Let ϕ be a tempered but non-square-integrable L-parameter for $\mathrm{U}(W_n^{\pm})$. Since ϕ is not square-integrable, we can write

$$\phi = (\phi_\tau \otimes \chi_V) \oplus \phi_0 \oplus ((\phi_\tau \otimes \chi_V)^c)^\vee$$

for some irreducible (unitary) square-integrable representation τ of $\mathrm{GL}_k(E)$ and tempered L-parameter ϕ_0 for $\mathrm{U}(W_{n_0}^{\pm})$, where k is a positive integer and $n_0 = n - 2k$. Fix $\epsilon' = \pm 1$, and set $W = W_n^{\epsilon'}$ and $W_0 = W_{n_0}^{\epsilon'}$. Let $\pi = \pi(\eta) \in \Pi_\phi$ be an irreducible tempered representation of $\mathrm{U}(W)$ with associated character $\eta \in \mathrm{Irr}(S_\phi)$. Then π is an irreducible constituent of $\mathrm{Ind}_Q^{\mathrm{U}(W)}(\tau\chi_V \otimes \pi_0)$ for some irreducible tempered representation $\pi_0 = \pi_0(\eta_0) \in \Pi_{\phi_0}$ of $\mathrm{U}(W_0)$ with associated character $\eta_0 \in \mathrm{Irr}(S_{\phi_0})$ such that

$$\eta|_{S_{\phi_0}} = \eta_0.$$

Fix $\epsilon = \pm 1$, and set $V = V_{n+1}^{\epsilon}$ and $V_0 = V_{n_0+1}^{\epsilon}$. Suppose that $\sigma := \Theta_{\psi,V,W}(\pi) \neq 0$. By the argument as in [19, pp. 1674–1676], we see that $\sigma_0 := \Theta_{\psi,V_0,W_0}(\pi_0) \neq 0$ and σ is an irreducible constituent of $\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau\chi_W \otimes \sigma_0)$. This implies that $\sigma^\vee$ is an irreducible constituent of $\mathrm{Ind}_P^{\mathrm{U}(V)}(\tau^c\chi_W^c \otimes \sigma_0^\vee)$. By Theorem 4.4, $\sigma = \sigma(\eta') \in \Pi_{\phi'}$ and $\sigma_0 = \sigma_0(\eta_0') \in \Pi_{\phi_0'}$ are irreducible tempered representations of $\mathrm{U}(V)$ and $\mathrm{U}(V_0)$ respectively, with L-parameters

$$\phi' = (\phi \otimes \chi_V^{-1}\chi_W) \oplus \chi_W \quad \text{and} \quad \phi_0' = (\phi_0 \otimes \chi_V^{-1}\chi_W) \oplus \chi_W,$$

and associated characters $\eta' \in \mathrm{Irr}(S_{\phi'})$ and $\eta_0' \in \mathrm{Irr}(S_{\phi_0'})$ such that

$$\eta'|_{S_{\phi_0'}} = \eta_0'.$$

We need to show that $\eta'|_{S_\phi} = \eta$.

Consider a commutative diagram

$$\begin{array}{ccc} S_\phi & \longrightarrow & S_{\phi'} \\ \uparrow & & \uparrow \\ S_{\phi_0} & \longrightarrow & S_{\phi_0'} \end{array}$$

of natural embeddings. Since $n_0 < n$, we know that $(P2)_{n_0}$ holds by assumption, so that

$$\eta'_0|_{S_{\phi_0}} = \eta_0.$$

Hence, we conclude that

$$\eta'|_{S_{\phi_0}} = (\eta'|_{S_{\phi'_0}})|_{S_{\phi_0}} = \eta'_0|_{S_{\phi_0}} = \eta_0 = \eta|_{S_{\phi_0}}.$$

In particular, if $S_{\phi_0} = S_\phi$, then $\eta'|_{S_\phi} = \eta$ as desired.

Finally, we assume that $S_{\phi_0} \neq S_\phi$, which is the case if and only if ϕ_τ is conjugate orthogonal and $\phi_\tau \otimes \chi_V$ is not contained in ϕ_0. Then the component group S_ϕ is of the form

$$S_\phi = S_{\phi_0} \times (\mathbb{Z}/2\mathbb{Z})a_1,$$

where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ arises from the summand $\phi_\tau \otimes \chi_V$ in ϕ. Since we already know that $\eta'|_{S_{\phi_0}} = \eta|_{S_{\phi_0}}$, it suffices to show that $\eta'(a_1) = \eta(a_1)$. To see this, we recall the $\mathrm{U}(V) \times \mathrm{U}(W)$-equivariant map

$$\mathcal{T}_0 : \omega \otimes \mathrm{Ind}_P^{\mathrm{U}(V)}(\tau^c \chi_W^c \otimes \sigma_0^\vee) \longrightarrow \mathrm{Ind}_Q^{\mathrm{U}(W)}(\tau \chi_V \otimes \pi_0).$$

Since $\mathcal{T}_0(\varphi \otimes \Phi) \in \pi$ for $\varphi \in \mathscr{S}$ and $\Phi \in \sigma^\vee$, it follows by (2.1), Lemma 8.3, and Corollary 8.5 that

$$\epsilon(W)^k \cdot \eta(a_1) = \alpha \cdot \beta(0) \cdot \epsilon(V)^k \cdot \eta_{\sigma^\vee}(a_1),$$

where α and $\beta(s)$ are as in Corollary 8.5, and $\eta_{\sigma^\vee} \in \mathrm{Irr}(S_{(\phi')^\vee})$ is the irreducible character associated to $\sigma^\vee$. But we know that

$$\eta_{\sigma^\vee}(a_1) = \eta'(a_1) \times \begin{cases} 1 & \text{if } n \text{ is even;} \\ \omega_{E/F}(-1)^k & \text{if } n \text{ is odd.} \end{cases}$$

Thus it remains to show that

$$\epsilon(V)^k \cdot \epsilon(W)^k \cdot \alpha \cdot \beta(0) = \begin{cases} 1 & \text{if } n \text{ is even;} \\ \omega_{E/F}(-1)^k & \text{if } n \text{ is odd.} \end{cases}$$

First, we compute $\epsilon(V)^k \cdot \epsilon(W)^k \cdot \alpha$ when n is even. In this case, we see that $\gamma_V = \epsilon(V) \cdot \lambda(E/F, \psi)$ and $\gamma_W = \epsilon(W) \cdot \chi_W(\delta)^{-1}$. Hence $\epsilon(V)^k \cdot \epsilon(W)^k \cdot \alpha$ is equal to

$$\Big[\lambda(E/F,\psi)^{-1}\cdot\chi_W(\delta)^{-1}\cdot\chi_V(-1)^{n'}\cdot\chi_W(-1)^{n'}\cdot(\chi_V^{-n}\chi_W^{n+1})(\delta)\Big]^k$$
$$\times\,\omega_\tau(-1)^{2n'}\cdot\lambda(E/F,\psi)^{(k+1)k/2}\cdot\lambda(E/F,\psi)^{-(k-1)k/2}$$
$$=1.$$

Next, we compute $\epsilon(V)^k\cdot\epsilon(W)^k\cdot\alpha$ when n is odd. In this case, we see that $\gamma_V=\epsilon(V)$ and $\gamma_W=\epsilon(W)\cdot\chi_W(\delta)^{-1}\cdot\lambda(E/F,\psi)$. Hence $\epsilon(V)^k\cdot\epsilon(W)^k\cdot\alpha$ is equal to

$$\Big[\chi_W(\delta)^{-1}\cdot\lambda(E/F,\psi)\cdot\chi_V((-1)^{n'-1}\cdot\delta^{-1})\cdot\chi_W((-1)^{n'-1}\cdot\delta^{-1})\cdot(\chi_V^{-n}\chi_W^{n+1})(\delta)\Big]^k$$
$$\times\,\omega_\tau(-1)^{2n'-1}\cdot\lambda(E/F,\psi)^{(k-1)k/2}\cdot\lambda(E/F,\psi)^{-(k+1)k/2}$$
$$=\omega_{E/F}(-1)^k\cdot\omega_\tau(-1)$$
$$=\omega_{E/F}(-1)^k,$$

where the last equality follows because $\omega_\tau|_{F^\times}=\mathbb{1}_{F^\times}$.

Finally, we compute $\beta(0)$. Noting that $s_0=\frac{1}{2}$, $(\phi_\tau^c)^\vee=\phi_\tau$, and $\phi_0'=(\phi_0\otimes\chi_V^{-1}\chi_W)\oplus\chi_W$, we see that

$$\begin{aligned}\beta(s)&=L(s,\phi_\tau)^{-1}\cdot L(-s,\phi_\tau)\cdot\gamma(-s,\phi_\tau,\psi_E)\cdot\gamma(s,\phi_\tau,\psi_E)^{-1}\\&=\frac{\epsilon(-s,\phi_\tau,\psi_E)}{\epsilon(s,\phi_\tau,\psi_E)}\cdot\frac{L(1+s,\phi_\tau^\vee)}{L(1-s,\phi_\tau^\vee)}.\end{aligned}$$

Since τ is square-integrable, $L(s,\phi_\tau^\vee)$ is holomorphic and nonzero at $s=1$, and hence

$$\beta(0)=1.$$

Thus, we have shown the desired formula for $\epsilon(V)^k\cdot\epsilon(W)^k\cdot\alpha\cdot\beta(0)$ and completed the proof of Theorem 6.1.

Remark 8.7 Using Theorem 4.1 (instead of Theorem 4.4) and the above argument, one can also prove the analog of Theorem 6.1 for (P1). Indeed, this can be reduced to the computation of $\epsilon(V)^k\cdot\epsilon(W)^k\cdot\alpha\cdot\beta(0)$ when $\epsilon=+1$, $m=n$, and ϕ_τ is conjugate symplectic, in which case one sees that

$$\epsilon(V)^k\cdot\epsilon(W)^k\cdot\alpha=\begin{cases}\omega_\tau(\delta)\cdot\omega_{E/F}(-1)^k & \text{if } n \text{ is even;}\\ \omega_\tau(\delta) & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\beta(0)=\epsilon\left(\frac{1}{2},\phi_\tau,\psi_E\right)$$

as desired.

9 Generic case

So far, we have verified the Fourier–Jacobi case (FJ) of the Gross–Prasad conjecture for tempered L-parameters for $\mathrm{U}(W_n) \times \mathrm{U}(W_n)$. As in the proof of [15, Theorem 19.1], this implies (FJ) for tempered L-parameters for $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+2k})$ with $k > 0$. In this section, we extend (FJ) to the case of generic L-parameters.

9.1 Generic L-parameters

Let V be an n-dimensional ε-Hermitian space. Recall that an L-parameter ϕ for $\mathrm{U}(V)$ is generic if, by definition, its associated L-packet Π_ϕ contains generic representations (i.e. those which possess some Whittaker models). In Proposition B.1 below, we shall show that ϕ is generic if and only if its adjoint L-factor $L(s, \mathrm{Ad} \circ \phi) = L(s, \mathrm{As}^{(-1)^n} \circ \phi)$ is holomorphic at $s = 1$.

Let ϕ be an L-parameter for $\mathrm{U}(V)$, so that we may write

$$\phi = \rho \oplus \phi_0 \oplus (\rho^c)^\vee \quad \text{with} \quad \rho = \bigoplus_{i=1}^{r} \rho_i |\cdot|^{s_i},$$

where

- ρ_i is a k_i-dimensional tempered representation of WD_E,
- s_i is a real number such that $s_1 > \cdots > s_r > 0$,
- ϕ_0 is a tempered L-parameter for $\mathrm{U}(V_0)$, where V_0 is the ε-Hermitian space of dimension $n - 2(k_1 + \cdots + k_r)$ such that $\epsilon(V_0) = \epsilon(V)$.

As mentioned in Sect. 2.5, by the construction of the local Langlands correspondence, the representations in the Vogan L-packet Π_ϕ are given by the unique irreducible quotient of the standard module

$$\mathrm{Ind}\left(\left(\bigotimes_{i=1}^{r} \tau_i |\cdot|^{s_i}\right) \otimes \pi_0\right) \tag{9.1}$$

for $\pi_0 \in \Pi_{\phi_0}$, where Ind is the appropriate parabolic induction and τ_i is the irreducible tempered representation of $\mathrm{GL}_{k_i}(E)$ associated to ρ_i. If ϕ is generic, then we have the following result of Heiermann [27], which extends a result of Mœglin–Waldspurger [42, Corollaire 2.14] for special orthogonal groups and symplectic groups.

Proposition 9.1 *Let ϕ be a generic L-parameter for $\mathrm{U}(V)$. Then the standard modules as in (9.1) are all irreducible, so that the L-packet Π_ϕ consists of standard modules.*

9.2 Local theta correspondence

Proposition 9.1 has consequences for the local theta correspondence. Let V be an m-dimensional Hermitian space and W an n-dimensional skew-Hermitian space. Consider the theta correspondence for $\mathrm{U}(V) \times \mathrm{U}(W)$ relative to a pair of characters (χ_V, χ_W). Let ϕ be an L-parameter for $\mathrm{U}(W)$ and π a representation of $\mathrm{U}(W)$ in Π_ϕ. If $m = n$, then by Theorem 4.1, we have $\theta_{\psi,V,W}(\pi) \in \Pi_{\theta(\phi)}$ (if nonzero) with

$$\theta(\phi) = \phi \otimes \chi_V^{-1} \chi_W,$$

so that $L(s, \mathrm{Ad} \circ \theta(\phi)) = L(s, \mathrm{Ad} \circ \phi)$. Thus $\theta(\phi)$ is generic if and only if ϕ is. On the other hand, if $m = n + 1$, then by Theorem 4.4, we have $\theta_{\psi,V,W}(\pi) \in \Pi_{\theta(\phi)}$ (if nonzero) with

$$\theta(\phi) = (\phi \otimes \chi_V^{-1} \chi_W) \oplus \chi_W.$$

In this case, it is possible that $\theta(\phi)$ is nongeneric even if ϕ is. More precisely, since

$$L(s, \mathrm{Ad} \circ \theta(\phi)) = L(s, \mathrm{Ad} \circ \phi) \cdot L(s, \phi \otimes \chi_V^{-1}) \cdot L(s, \omega_{E/F}),$$

$\theta(\phi)$ is generic if and only if ϕ is generic and does not contain $\chi_V |\cdot|^{\pm\frac{k+1}{2}} \boxtimes \mathrm{Sym}^{k-1}$ for any positive integer k, where Sym^{k-1} is the unique k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Hence we see that for all but finitely many choices of χ_V (depending on ϕ), $\theta(\phi)$ is generic if ϕ is.

Proposition 9.2 *Let ϕ be an L-parameter for $\mathrm{U}(W)$ and π a representation of $\mathrm{U}(W)$ in Π_ϕ. Then we have:*

(i) *Assume that $m = n$. If ϕ is generic (so that $\theta(\phi)$ is also generic), then*

$$\Theta_{\psi,V,W}(\pi) = \theta_{\psi,V,W}(\pi).$$

(ii) *Assume that $m = n + 1$. If ϕ is generic and does not contain $\chi_V |\cdot|^{\pm\frac{k+1}{2}} \boxtimes \mathrm{Sym}^{k-1}$ for any positive integer k (so that $\theta(\phi)$ is also generic), then*

$$\Theta_{\psi,V,W}(\pi) = \theta_{\psi,V,W}(\pi).$$

Proof We shall give the proof of (ii) since the proof of (i) is similar. We may assume that $\Theta_{\psi,V,W}(\pi) \neq 0$. If ϕ is tempered, then $\Theta_{\psi,V,W}(\pi)$ is irreducible and tempered by [17, Proposition C.4(i)]. In general, by Proposition 9.1, π is a standard module of the form

$$\mathrm{Ind}\left(\left(\bigotimes_{i=1}^{r} \tau_i |\cdot|^{s_i}\right) \otimes \pi_0\right)$$

as in (9.1). Then by [17, Proposition C.4(ii)], $\Theta_{\psi,V,W}(\pi)$ is a quotient of the standard module

$$\mathrm{Ind}\left(\left(\bigotimes_{i=1}^{r} \tau_i \chi_V^{-1} \chi_W |\cdot|^{s_i}\right) \otimes \Theta_{\psi,V_0,W_0}(\pi_0)\right).$$

Since $\theta(\phi)$ is generic as well, Proposition 9.1 implies that this standard module is irreducible, so that $\Theta_{\psi,V,W}(\pi)$ is irreducible. □

9.3 (B) for generic L-parameters

For special orthogonal groups, Mœglin–Waldspurger [42] extended the Bessel case (B) of the Gross–Prasad conjecture from tempered L-parameters to generic L-parameters. We carry out the analogous extension for unitary groups.

Proposition 9.3 *The statement (B) holds for all generic L-parameters for* $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+2k+1})$.

To prove Proposition 9.3, we adapt the proof of Mœglin–Waldspurger [42] to the case of unitary groups. For any (not necessarily irreducible) smooth representations π and π' of $\mathrm{U}(V_n)$ and $\mathrm{U}(V_{n+2k+1})$ respectively, we write $m(\pi,\pi')$ or $m(\pi',\pi)$ for

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \pi', \nu)$$

with the subgroup H of $\mathrm{U}(V_n) \times \mathrm{U}(V_{n+2k+1})$ and the character ν of H as in [15, Sect. 12]. Then as explained in [42, Sect. 3], Proposition 9.3 follows from (B) for all tempered L-parameters (which was proved by Beuzart-Plessis [4–6]), together with Proposition 9.1 and the following proposition:

Proposition 9.4 *Let* $\pi = \mathrm{Ind}((\bigotimes_{i=1}^{r} \tau_i|\cdot|^{s_i}) \otimes \pi_0)$ *be a smooth representation of* $\mathrm{U}(V_n)$, *where*

- τ_i *is an irreducible tempered representation of* $\mathrm{GL}_{k_i}(E)$,
- s_i *is a real number such that* $s_1 \geq \cdots \geq s_r \geq 0$,

- π_0 *is an irreducible tempered representation of* $\mathrm{U}(V_{n-2(k_1+\cdots+k_r)})$.

Likewise, let $\pi' = \mathrm{Ind}((\bigotimes_{j=1}^{r'} \tau'_j|\cdot|^{s'_j}) \otimes \pi'_0)$ *be a smooth representation of* $\mathrm{U}(V_{n+2k+1})$ *with analogous data* τ'_j, k'_j, s'_j, π'_0. *Then we have*

$$m(\pi, \pi') = m(\pi_0, \pi'_0).$$

Proof Since the proof is similar to that of [42, Proposition 1.3], we shall only give a sketch of the proof. First, we prove that $m(\pi, \pi') \leq m(\pi_0, \pi'_0)$.

(i) Let $\sigma = \mathrm{Ind}(\tau_0|\cdot|^{s_0} \otimes \sigma_0)$ be a smooth representation of $\mathrm{U}(V_{n+1})$, where
- τ_0 is an irreducible (unitary) square-integrable representation of $\mathrm{GL}_{k_0}(E)$,
- s_0 is a real number,
- σ_0 is a smooth representation of $\mathrm{U}(V_{n-2k_0+1})$ of finite length.

Assume that $s_0 \geq s_1$ (which is interpreted as $s_0 \geq 0$ when $r = 0$). Then as in [42, Lemme 1.4], we have

$$m(\pi, \sigma) \leq m(\pi, \sigma_0).$$

(ii) Let σ be as in (i). Assume that
- τ_0 is supercuspidal;
- if a representation $\tau_\sharp \otimes \pi_\sharp$ with
  - an irreducible smooth representation $\tau_\sharp$ of a general linear group;
  - an irreducible smooth representation $\pi_\sharp$ of a general linear group or a unitary group

  intervenes in a Jacquet module of $\tau_i^\vee$, τ_i^c, or $\pi_0^\vee$ as a subquotient, then $\tau_0|\cdot|^s$ does not intervene in the supercuspidal support of $\tau_\sharp$ for any $s \in \mathbb{R}$.

Then by [15, Theorem 15.1] (see also [42, Lemme 1.5]), we have

$$m(\pi, \sigma) = m(\pi, \sigma_0).$$

(iii) To prove $m(\pi, \pi') \leq m(\pi_0, \pi'_0)$ in general, we may assume that τ_i, τ'_j are square-integrable for all i, j. As in [42, Sect. 1.6], we argue by induction on

$$l := \sum_{\substack{1\leq i\leq r \\ s_i \neq 0}} k_i + \sum_{\substack{1\leq j\leq r' \\ s'_j \neq 0}} k'_j.$$

If $l = 0$, then it follows by [6, Sects. 14–15] combined with (ii) that $m(\pi, \pi') = m(\pi_0, \pi'_0)$. Suppose that $l \neq 0$.

(a) If $k = 0$ and $s'_1 \geq s_1$ (in particular $r' \geq 1$), then by (i), we have $m(\pi, \pi') \leq m(\pi, \pi'')$, where $\pi'' = \mathrm{Ind}((\bigotimes_{j=2}^{r'} \tau'_j |\cdot|^{s'_j}) \otimes \pi'_0)$. By induction hypothesis, we have $m(\pi, \pi'') \leq m(\pi_0, \pi'_0)$.

(b) If $s_1 \geq s'_1$ (in particular $r \geq 1$), then we can reduce to (a) by using (ii).

(c) If $s'_1 \geq s_1$ (in particular $r' \geq 1$), then we can reduce to (b) by using (ii).

This proves the assertion (see [42, Sect. 1.6] for details).

Next, we prove that $m(\pi, \pi') \geq m(\pi_0, \pi'_0)$. By (ii), we may assume that $k = 0$. If $m(\pi_0, \pi'_0) = 0$, then there is nothing to prove. If $m(\pi_0, \pi'_0) \neq 0$, then by [1], [15, Corollary 15.3], it suffices to show that

$$m(\pi, \pi') \geq 1.$$

Put

$$\pi_z = \mathrm{Ind}\left(\left(\bigotimes_{i=1}^{r} \tau_i |\cdot|^{z_i}\right) \otimes \pi_0\right)$$

and

$$\pi'_{z'} = \mathrm{Ind}\left(\left(\bigotimes_{j=1}^{r'} \tau'_j |\cdot|^{z'_j}\right) \otimes \pi'_0\right)$$

for $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ and $z' = (z'_1, \ldots, z'_{r'}) \in \mathbb{C}^{r'}$. As in [42, Lemme 1.7], we can define a $\Delta(\mathrm{U}(V_n) \times \mathrm{U}(V_n))$-equivariant map

$$\mathcal{L}_{z,z'} : \pi_z \otimes (\pi_z)^\vee \otimes \pi'_{z'} \otimes (\pi'_{z'})^\vee \longrightarrow \mathbb{C}$$

by (meromorphic continuation of) an integral of matrix coefficients, which is absolutely convergent for (z, z') near $(\sqrt{-1}\mathbb{R})^r \times (\sqrt{-1}\mathbb{R})^{r'}$. Since $m(\pi_0, \pi'_0) \neq 0$, it follows by [6, Théorème 14.3.1, Proposition 15.2.1, Proposition 15.3.1] that the map $(z, z') \mapsto \mathcal{L}_{z,z'}$ is not identically zero. In particular, the leading term of $\mathcal{L}_{z,z'}$ at $z = (s_1, \ldots, s_r)$ and $z' = (s'_1, \ldots, s'_{r'})$ is nonzero and hence $m(\pi, \pi') \geq 1$ (see [42, Sect. 1.8] for details). This completes the proof. ◻

9.4 (FJ) for generic L-parameters

In view of Propositions 9.2 and 9.3, one may repeat the see-saw argument in Sect. 5 for generic L-parameters, using (P1) and (P2) (which were shown for all L-parameters) to prove:

Proposition 9.5 *The statement (FJ) holds for all generic L-parameters for* $\mathrm{U}(W_n) \times \mathrm{U}(W_n)$.

Here, in repeating the see-saw argument, one may choose a character χ_V so that the condition of Proposition 9.2(ii) holds. Finally, Proposition 9.5 together with [15, Theorem 19.1] implies:

Corollary 9.6 *The statement (FJ) holds for all generic L-parameters for* $\mathrm{U}(W_n) \times \mathrm{U}(W_{n+2k})$.

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Appendix A: Addendum to [17]

In this appendix, we elaborate on some results of [17, Appendix C] which are used in the proof of Theorem 4.4. In particular,

- we fill in some missing details in the proof of [17, Proposition C.1(ii)] and streamline its proof by exploiting the recently established Howe duality conjecture [20,21];
- we extend some results of Muić [45, Lemma 4.2 and Theorem 5.1(i)] (used in the proof of [17, Proposition C.1(ii)]), which were written only for symplectic-orthogonal dual pairs, to cover all dual pairs considered in [17], streamlining some of his proofs in the process.

A.1 The issues

Let us be more precise. We freely use the notation of [17, Sect. C.1].

Let π be an irreducible square-integrable representation of $G(W)$ such that

$$\sigma_0 := \Theta_{\tilde{V},W,\boldsymbol{\chi},\psi}(\pi) \neq 0.$$

By the bullet point on [17, p. 645], together with the Howe duality, σ_0 is irreducible and square-integrable. Then we showed that

(i) any irreducible subquotient of $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is tempered in the first bullet point on [17, p. 646];

(ii) $\sigma := \theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is an irreducible constituent of $I_{Q(Y_1)}^{H(V)}(\chi_W \otimes \sigma_0)$ in the the second bullet point on [17, p. 646],

and claimed that

(iii) any irreducible subquotient of $\Theta_{V,W,\chi,\psi}(\pi)$ is not square-integrable in the third bullet point on [17, p. 646];
(iv) any irreducible subquotient of $\Theta_{V,W,\chi,\psi}(\pi)$ is a subrepresentation of $I_{Q(Y_1)}^{H(V)}(\chi_W \otimes \sigma_0')$ for some irreducible smooth representation σ_0' of $H(\tilde{V})$ in the fourth bullet point on [17, p. 646].

However, in the third and fourth bullet points on [17, p. 646], we have used results of Muić [45, Lemma 4.2 and Theorem 5.1(i)], which were written only for symplectic-orthogonal dual pairs. Moreover, we have not given the proof of (iv): we have simply asserted that it is true as if it is obvious (which it is not). Thus, we need to give the details of the proof of (iii) and (iv), as well as that of the results of Muić for all dual pairs considered in [17].

A.2 Proof of (iii)

First, we address (iii). Our original argument in [17] used [45, Lemma 4.2 and Theorem 5.1(i)], which we state and prove in Lemma A.1 and Corollary A.5 below. Here, we give a more streamlined argument using the recently established Howe duality conjecture [20,21].

Let σ' be an irreducible subquotient of $\Theta_{V,W,\chi,\psi}(\pi)$. Suppose that σ' is square-integrable. Since $\Theta_{V,W,\chi,\psi}(\pi)$ is of finite length and tempered by (i), it follows by [60, Corollaire III.7.2] that σ' is in fact a quotient of $\Theta_{V,W,\chi,\psi}(\pi)$. Hence we must have $\sigma' \cong \sigma$ by the Howe duality. But σ is not square-integrable by (ii), which is a contradiction. This completes the proof of (iii).

A.3 Proof of [45, Lemma 4.2]

For the proof of (iv), we will need the following result of Muić [45, Lemma 4.2].

Lemma A.1 (Muić) *Let $G(W) \times H(V)$ be an arbitrary reductive dual pair as in* [17, Sect. 3]. *Let π be an irreducible smooth representation of $G(W)$. Then all irreducible subquotients of $\Theta_{V,W,\chi,\psi}(\pi)$ have the same supercuspidal support.*

Proof We may assume that $\Theta_{V,W,\chi,\psi}(\pi) \neq 0$. Since $\Theta_{V,W,\chi,\psi}(\pi)$ is of finite length, it follows by the theory of the Bernstein center [3] that

$$\Theta_{V,W,\chi,\psi}(\pi) = \sigma_1 \oplus \cdots \oplus \sigma_r$$

for some smooth representations σ_i of $H(V)$ of finite length such that

- for each i, all irreducible subquotients of σ_i have the same supercuspidal support, say, $\operatorname{supp} \sigma_i$;
- if $i \neq j$, then $\operatorname{supp} \sigma_i \neq \operatorname{supp} \sigma_j$.

Of course, if we were willing to appeal to the Howe duality, then it would follow immediately that $r = 1$, so that the lemma is proved. However, we may appeal to an older result of Kudla. Namely, Kudla's supercuspidal support theorem [36] (see also [17, Proposition 5.2] and the references therein) says that the supercuspidal support of $\theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$ is determined by that of π. Hence we must have $r = 1$. $\square$

A.4 Plancherel measures

To prove (iv), we will also need the following property of Plancherel measures. We freely use the convention of [17, Appendix B].

Lemma A.2 *Let $G(W)$ be an arbitrary classical group as in* [17, Sect. 2]. *Let π be an irreducible tempered representation of $G(W)$ such that*

$$\pi \subset I_P^{G(W)}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0),$$

where P is a parabolic subgroup of $G(W)$ with Levi component $\mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E) \times G(W_0)$, τ_i is an irreducible (unitary) square-integrable representation of $\mathrm{GL}_{k_i}(E)$, and π_0 is an irreducible square-integrable representation of $G(W_0)$. Let τ be an irreducible (unitary) square-integrable representation of $\mathrm{GL}_k(E)$ and put

$$\mathcal{I}(\tau) = \{i \mid \tau_i \cong \tau\}.$$

Then we have

$$\operatorname*{ord}_{s=0} \mu(\tau_s \otimes \pi) = 2 \cdot \#\mathcal{I}(\tau) + 2 \cdot \#\mathcal{I}((\tau^c)^\vee) + \operatorname*{ord}_{s=0} \mu(\tau_s \otimes \pi_0).$$

Moreover, we have

$$\operatorname*{ord}_{s=0} \mu(\tau_s \otimes \pi_0) = \begin{cases} 0 \text{ or } 2 & \text{if } (\tau^c)^\vee \cong \tau; \\ 0 & \text{if } (\tau^c)^\vee \ncong \tau. \end{cases}$$

Proof By the multiplicativity of Plancherel measures (see [17, Sect. B.5]), we have

$$\mu(\tau_s \otimes \pi) = \left(\prod_{i=1}^{r} \mu(\tau_s \otimes \tau_i) \cdot \mu(\tau_s \otimes (\tau_i^c)^\vee) \right) \cdot \mu(\tau_s \otimes \pi_0).$$

For any irreducible (unitary) square-integrable representation τ' of $\mathrm{GL}_{k'}(E)$, we have

$$\mu(\tau_s \otimes \tau') = \gamma(s, \tau \times (\tau')^\vee, \psi_E) \cdot \gamma(-s, \tau^\vee \times \tau', \bar{\psi}_E)$$

and hence

$$\operatorname*{ord}_{s=0} \mu(\tau_s \otimes \tau') = \begin{cases} 2 & \text{if } \tau \cong \tau'; \\ 0 & \text{if } \tau \ncong \tau', \end{cases}$$

which reflects the triviality of R-groups for general linear groups. This proves the first assertion. The second assertion follows from [60, Corollaire IV.1.2] if $(\tau^c)^\vee \cong \tau$ and [60, Proposition IV.2.2] if $(\tau^c)^\vee \ncong \tau$. □

A.5 Proof of (iv)

Now we prove (iv). Let σ' be an irreducible subquotient of $\Theta_{V,W,\boldsymbol{\chi},\psi}(\pi)$. By (i) and (iii), we have

$$\sigma' \subset I_Q^{H(V)}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \sigma_0')$$

for some $r \geq 1$ and irreducible square-integrable representations τ_i and σ_0' of $\mathrm{GL}_{k_i}(E)$ and $H(V_0)$ respectively, where Q is a parabolic subgroup of $H(V)$ with Levi component $\mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E) \times H(V_0)$. We need to show that $\tau_i = \chi_W$ for some i.

By Lemma A.1 and the multiplicativity of Plancherel measures, we have

$$\mu((\chi_W)_s \otimes \sigma') = \mu((\chi_W)_s \otimes \sigma).$$

By (ii) and Lemma A.2, the right-hand side has a zero at $s = 0$ of order at least 4. Hence, by Lemma A.2 again, we must have $\tau_i = \chi_W$ for some i. This completes the proof of (iv).

Remark A.1 In the proof of (iii) and (iv), we have used some results of Waldspurger [60], which were written only for connected reductive linear algebraic groups. However, it is straightforward to extend them to the cases of (disconnected) orthogonal groups and (nonlinear) metaplectic groups.

A.6 Proof of [45, Theorem 5.1(i)]

As we noted above, we have used [45, Theorem 5.1(i)] besides [45, Lemma 4.2] in our original argument in [17]. Although it is not necessary for the proof of (iii) and (iv) (because of the use of the Howe duality), we shall give a proof here.

In fact, we prove the following more general result by refining the argument in the proof of (iv).

Lemma A.4 *Let $G(W)$ be an arbitrary classical group as in* [17, Sect. 2]. *Let π be an irreducible tempered representation of $G(W)$ such that*

$$\pi \subset I_P^{G(W)}(\tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0),$$

where P is a parabolic subgroup of $G(W)$ with Levi component $\mathrm{GL}_{k_1}(E) \times \cdots \times \mathrm{GL}_{k_r}(E) \times G(W_0)$, τ_i is an irreducible (unitary) square-integrable representation of $\mathrm{GL}_{k_i}(E)$, and π_0 is an irreducible square-integrable representation of $G(W_0)$. Likewise, let π' be an irreducible tempered representation of $G(W)$ such that

$$\pi' \subset I_{P'}^{G(W)}(\tau_1' \otimes \cdots \otimes \tau_{r'}' \otimes \pi_0')$$

with analogous data P', r', τ_i', π_0'. Assume that

$$\mu(\tau_s \otimes \pi) = \mu(\tau_s \otimes \pi')$$

for all irreducible (unitary) square-integrable representations τ of $\mathrm{GL}_k(E)$ for all $k \geq 1$. Then we have $r = r'$ and

$$\{\tau_1, \ldots, \tau_r, (\tau_1^c)^\vee, \ldots, (\tau_r^c)^\vee\} = \{\tau_1', \ldots, \tau_r', ((\tau_1')^c)^\vee, \ldots, ((\tau_r')^c)^\vee\}$$

as multi-sets. Moreover, we have

$$\mu(\tau_s \otimes \pi_0) = \mu(\tau_s \otimes \pi_0')$$

for all irreducible (unitary) square-integrable representations τ of $\mathrm{GL}_k(E)$ for all $k \geq 1$.

Proof Note that the second assertion is an immediate consequence of the first assertion and the multiplicativity of Plancherel measures. To prove the first assertion, it suffices to show that

$$\#\mathcal{I}(\tau) + \#\mathcal{I}((\tau^c)^\vee) = \#\mathcal{I}'(\tau) + \#\mathcal{I}'((\tau^c)^\vee) \tag{A.1}$$

for any irreducible (unitary) square-integrable representation τ of $\mathrm{GL}_k(E)$, where $\mathcal{I}(\tau) = \{i \mid \tau_i \cong \tau\}$ and $\mathcal{I}'(\tau) = \{i \mid \tau_i' \cong \tau\}$. If $(\tau^c)^\vee \cong \tau$, then by Lemma A.2, we have

$$4 \cdot \#\mathcal{I}(\tau) + \alpha = 4 \cdot \#\mathcal{I}'(\tau) + \alpha'$$

for some $0 \leq \alpha, \alpha' \leq 2$. This forces $\#\mathcal{I}(\tau) = \#\mathcal{I}'(\tau)$, so that (A.1) holds. If $(\tau^c)^\vee \not\cong \tau$, then (A.1) is a direct consequence of Lemma A.2. This completes the proof. □

The following corollary (which is [45, Theorem 5.1(i)]) is now immediate:

Corollary A.5 (Muić) *Suppose that π and π' are irreducible tempered representations of $G(W)$ which have the same supercuspidal support. If π is square-integrable, then so is π'.*

Proof If π and π' have the same supercuspidal support, then the multiplicativity of Plancherel measures implies that

$$\mu(\tau_s \otimes \pi) = \mu(\tau_s \otimes \pi')$$

for all irreducible (unitary) square-integrable representations τ of $\mathrm{GL}_k(E)$ for all $k \geq 1$. The assertion then follows from Lemma A.4. □

A.7 Some variant

Finally, admitting the local Langlands correspondence, we shall state a variant of Lemma A.4 in terms of L-parameters.

Let $G(W)$ be an arbitrary classical group as in [17, Sect. 2]. To each irreducible tempered representation π of $G(W)$, the local Langlands correspondence assigns an L-parameter ϕ, which we regard as a semisimple representation of WD_E as described in [15, Sect. 8]. Moreover, for any irreducible tempered representation τ of $\mathrm{GL}_k(E)$ with associated L-parameter ϕ_τ, Langlands' conjecture on Plancherel measures [38, Appendix II] says that

$$\begin{aligned}\mu(\tau_s \otimes \pi) &= \gamma(s, \phi_\tau \otimes \phi^\vee, \psi_E) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi, \bar{\psi}_E) \\ &\quad \times \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R \circ \phi_\tau^\vee, \bar{\psi}),\end{aligned} \tag{A.2}$$

where

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } G(W) \text{ is odd orthogonal or metaplectic;} \\ \wedge^2 & \text{if } G(W) \text{ is even orthogonal or symplectic;} \\ \mathrm{As}^+ & \text{if } G(W) \text{ is even unitary;} \\ \mathrm{As}^- & \text{if } G(W) \text{ is odd unitary.} \end{cases}$$

In fact, (A.2) immediately follows from [2, Proposition 2.3.1], [44, Proposition 3.3.1], [33, Lemma 2.2.3] (together with induction in stages) for classical groups considered there. (See also §7.3 in the case of unitary groups.) In other words, recalling the definitions of of Plancherel measures and normalized intertwining operators, we see that (A.2) is a consequence of a property of normalized intertwining operators. Also, in the case of metaplectic groups, (A.2) follows from the case of odd orthogonal groups combined with [19, Proposition 10.1].

Lemma A.6 *Let π and π' be irreducible tempered representations of $G(W)$ with associated L-parameters ϕ and ϕ' respectively. Assume that*

$$\mu(\tau_s \otimes \pi) = \mu(\tau_s \otimes \pi')$$

for all irreducible (unitary) square-integrable representations τ of $\mathrm{GL}_k(E)$ for all $k \geq 1$. Then we have

$$\phi = \phi'.$$

Proof For any irreducible (unitary) square-integrable representation τ of $\mathrm{GL}_k(E)$ with associated L-parameter ϕ_τ, we have

$$\gamma(s, \phi_\tau \otimes \phi^\vee, \psi_E) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi, \bar{\psi}_E)$$
$$= \gamma(s, \phi_\tau \otimes (\phi')^\vee, \psi_E) \cdot \gamma(-s, \phi_\tau^\vee \otimes \phi', \bar{\psi}_E)$$

by assumption and (A.2). Comparing the orders of zero at $s = 0$, we see that the multiplicities of ϕ_τ in ϕ and ϕ' are equal (see also [19, Lemma 12.3]). This completes the proof. □

A.8 Erratum to [17]

On this occasion, we also correct some typos in [17].

- Lemma C.2: $\mathrm{Isom}(Y_a', X_a)$ should be read as the set of invertible conjugate linear maps from Y_a' to X_a.
- Bottom of p. 650: Asai should be read as As^+ (resp. As^-) if $G(W^\bullet)$ is even unitary (resp. odd unitary).

Appendix B: Generic L-packets and adjoint L-factors

In this appendix, we prove a conjecture of Gross–Prasad and Rallis [23, Conjecture 2.6] under a certain working hypothesis.

B.1 Notation

Let G be a connected reductive algebraic group defined and quasi-split over F. Fix a Borel subgroup B of G over F and a maximal torus T in B over F. Let

N be the unipotent radical of B, so that $B = TN$. If P is a parabolic subgroup of G over F, we say that P is standard (relative to B) if $P \supset B$. If P is a standard parabolic subgroup of G over F, then we have a Levi decomposition $P = MU$, where M is the unique Levi component of P such that $M \supset T$ and U is the unipotent radical of P. We call M a standard Levi subgroup of G. Let $W^M = \mathrm{Norm}_M(T)/T$ be the Weyl group of M and w_0^M the longest element in W^M. Put

$$\mathfrak{a}_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}),$$

where $\mathrm{Rat}(M)$ is the group of algebraic characters of M defined over F. We write $\langle \cdot, \cdot \rangle : \mathfrak{a}_M^* \times \mathfrak{a}_M \to \mathbb{R}$ for the natural pairing. Let $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}_M^*$. Let A_M be the split component of the center of M and $\Sigma(P)$ the set of reduced roots of A_M in P. We may regard $\Sigma(P)$ as a subset of $\mathfrak{a}_M^* \cong \mathrm{Rat}(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in \Sigma(P)$, let $\alpha^\vee \in \mathfrak{a}_M$ denote its corresponding coroot. Put

$$(\mathfrak{a}_M^*)^+ = \{\lambda \in \mathfrak{a}_M^* \mid \langle \lambda, \alpha^\vee \rangle > 0 \quad \text{for all } \alpha \in \Sigma(P)\}.$$

We define a homomorphism $H_M : M \to \mathfrak{a}_M$ by requiring that

$$|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle}$$

for all $\chi \in \mathrm{Rat}(M)$ and $m \in M$, where q is the cardinality of the residue field of F.

Let π be an irreducible smooth representation of M. For $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, we define a representation π_λ of M by $\pi_\lambda(m) = q^{-\langle \lambda, H_M(m) \rangle} \pi(m)$. We write

$$I_P^G(\pi_\lambda) := \mathrm{Ind}_P^G(\pi_\lambda)$$

for the induced representation of G. If π is tempered and $\mathrm{Re}(\lambda) \in (\mathfrak{a}_M^*)^+$, then $I_P^G(\pi_\lambda)$ has a unique irreducible quotient $J_P^G(\pi_\lambda)$.

Let $\widehat{M}$ be the dual group of M and ${}^L M = \widehat{M} \rtimes W_F$ the L-group of M. Let $Z(\widehat{M})$ be the center of $\widehat{M}$. We write $\iota_M : {}^L M \hookrightarrow {}^L G$ for the natural embedding. If $\phi : WD_F \to {}^L M$ is an L-parameter, we say that ϕ is tempered if the projection of $\phi(W_F)$ to $\widehat{M}$ is bounded. For $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, we define an L-parameter $\phi_\lambda : WD_F \to {}^L M$ by $\phi_\lambda = a_\lambda \cdot \phi$, where $a_\lambda \in Z^1(W_F, Z(\widehat{M}))$ is a 1-cocycle which determines the character $m \mapsto q^{-\langle \lambda, H_M(m) \rangle}$ of M.

B.2 Hypothesis

In this appendix, we admit the local Langlands correspondence for any standard Levi subgroup M of G:

$$\mathrm{Irr}(M) = \bigsqcup_{\phi} \Pi_{\phi},$$

where the disjoint union on the right-hand side runs over all equivalence classes of L-parameters ϕ for M and Π_{ϕ} is a finite set of representations of M, the so-called L-packet. More precisely, we will use the following properties of the local Langlands correspondence:

(i) $\pi \in \Pi_{\phi}$ is tempered if and only if ϕ is tempered.
(ii) $\Pi_{\phi_{\lambda}} = \{\pi_{\lambda} \mid \pi \in \Pi_{\phi}\}$ for $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$.
(iii) If ϕ is an L-parameter for G, then replacing ϕ by its $\widehat{G}$-conjugate if necessary, we can write

$$\phi = \iota_M \circ (\phi_M)_{\lambda_0},$$

where
- M is a standard Levi subgroup of G,
- ϕ_M is a tempered L-parameter for M,
- $\lambda_0 \in (\mathfrak{a}^*_M)^+$.

Then we have

$$\Pi_{\phi} = \{J^G_P(\pi_{\lambda_0}) \mid \pi \in \Pi_{\phi_M}\},$$

where P is the standard parabolic subgroup of G with Levi component M. Note that $\pi \in \Pi_{\phi_M}$ is tempered by (i) and π_{λ_0} has L-parameter $(\phi_M)_{\lambda_0}$ by (ii).
(iv) If ϕ is a tempered L-parameter for M, then for any generic character ψ_{N_M} of $N_M := N \cap M$, Π_{ϕ} contains a (N_M, ψ_{N_M})-generic representation π of M (see [53, Conjecture 9.4]). Moreover, we have

$$\gamma^{\mathrm{Sh}}(s, \pi_{\lambda}, r_M, \psi) = \gamma(s, r_M \circ \phi_{\lambda}, \psi),$$

where the left-hand side is Shahidi's γ-factor [53] and r_M is the adjoint representation of ${}^L M$ on $\mathrm{Lie}({}^L U)$. In fact, we only need the equality up to an invertible function.

The above hypothesis is known to hold for general linear groups by [26,29,51] and for classical groups by [2,44].

B.3 A conjecture of Gross–Prasad and Rallis

If ϕ is an L-parameter for G, we say that ϕ is generic if its associated L-packet Π_ϕ contains a (N, ψ_N)-generic representation of G for some generic character ψ_N of N.

Proposition B.1 *Let ϕ be an L-parameter for G. Then, under the hypothesis in Sect.* B.2 *, ϕ is generic if and only if $L(s, \mathrm{Ad} \circ \phi)$ is holomorphic at $s = 1$. Here,* Ad *is the adjoint representation of ${}^L G$ on its Lie algebra* $\mathrm{Lie}({}^L G)$.

B.4 Proof of Proposition B.1

Fix an L-parameter ϕ for G and write $\phi = \iota_M \circ (\phi_M)_{\lambda_0}$ as in (iii). Then by (iii), ϕ is generic if and only if $J_P^G(\pi_{\lambda_0})$ is (N, ψ_N)-generic for some $\pi \in \Pi_{\phi_M}$ and some generic character ψ_N of N, in which case π is necessarily $(N_M, \psi_N|_{N_M})$-generic by a result of Rodier [50], [7, Corollary1.7]. Here, we have also used the fact that for any element w in W^G, there exists a representative $\tilde{w}$ of w (depending on ψ_N) such that ψ_N is compatible with $\tilde{w}$ (see [54, Sect. 2], [12, Sect. 1.2]). Now we invoke the following result of Heiermann–Muić [28, Proposition 1.3].

Lemma B.2 *Let ψ_N be a generic character of N and π an irreducible tempered $(N_M, \psi_N|_{N_M})$-generic representation of M. Then $J_P^G(\pi_{\lambda_0})$ is (N, ψ_N)-generic if and only if $\gamma^{\mathrm{Sh}}(0, \pi_\lambda, r_M, \psi)$ is holomorphic at $\lambda = \lambda_0$.*

Proof Since the assertion in [28, Proposition 1.3] is slightly different, we include a proof for the convenience of the reader. We realize the representation $I_P^G(\pi_\lambda)$ by using the unique (up to a scalar) Whittaker functional on π with respect to $(N_M, \psi_N|_{N_M})$. Then we can define a Whittaker functional

$$\Lambda(\pi_\lambda) : I_P^G(\pi_\lambda) \longrightarrow \mathbb{C}$$

with respect to (N, ψ_N) by (holomorphic continuation of) the Jacquet integral (see [52, Proposition 3.1]). By [50], [7, Corollary 1.7], $\Lambda(\pi_\lambda)$ is a basis of $\mathrm{Hom}_N(I_P^G(\pi_\lambda), \psi_N)$ for all $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. Put $w = w_0^G w_0^M$ and choose its representative $\tilde{w}$ so that ψ_N is compatible with $\tilde{w}$. As in Sect. 7.3, we can define an unnormalized intertwining operator

$$\mathcal{M}(\tilde{w}, \pi_\lambda) : I_P^G(\pi_\lambda) \longrightarrow I_{w(P)}^G(w(\pi_\lambda))$$

by (meromorphic continuation of) an integral which is absolutely convergent for $\mathrm{Re}(\lambda) \in (\mathfrak{a}^*_M)^+$ (see [60, Proposition IV.2.1]), where $w(P)$ is the standard parabolic subgroup of G with Levi component wMw^{-1}. Then we have

$$\Lambda(\pi_\lambda) = C(\tilde{w}, \pi_\lambda) \cdot \Lambda(w(\pi_\lambda)) \circ \mathcal{M}(\tilde{w}, \pi_\lambda) \tag{B.1}$$

for some meromorphic function $C(\tilde{w}, \pi_\lambda)$, the so-called local coefficient. Here, $C(\tilde{w}, \pi_\lambda)$ depends on the choice of Haar measures in the definitions of $\Lambda(\pi_\lambda)$, $\Lambda(w(\pi_\lambda))$, $\mathcal{M}(\tilde{w}, \pi_\lambda)$, but we ignore the normalization of Haar measures since it does not affect the proof. Since $J_P^G(\pi_{\lambda_0})$ is isomorphic to the image of $\mathcal{M}(\tilde{w}, \pi_{\lambda_0})$ and the functor $\mathrm{Hom}_N(\,\cdot\,, \psi_N)$ is exact, $J_P^G(\pi_{\lambda_0})$ is (N, ψ_N)-generic if and only if the restriction of $\Lambda(w(\pi_{\lambda_0}))$ to the image of $\mathcal{M}(\tilde{w}, \pi_{\lambda_0})$ is nonzero. By (B.1), this condition is equivalent to the holomorphy of $C(\tilde{w}, \pi_\lambda)$ at $\lambda = \lambda_0$. On the other hand, by the definition of Shahidi's γ-factor, we have

$$C(\tilde{w}, \pi_\lambda) = \gamma^{\mathrm{Sh}}(0, \pi_\lambda, r_M, \psi)$$

up to an invertible function. (Note that the convention in [53] is different from ours: the homomorphism H_M is normalized so that $|\chi(m)|_F = q^{\langle \chi, H_M(m) \rangle}$ in [53]. This is why we have $\gamma^{\mathrm{Sh}}(0, \pi_\lambda, r_M, \psi)$ on the right-hand side rather than $\gamma^{\mathrm{Sh}}(0, \pi_\lambda, r_M^\vee, \bar{\psi})$.) This completes the proof. □

Now it follows by Lemma B.2 combined with (iv) that ϕ is generic if and only if

$$\frac{L(1, r_M^\vee \circ (\phi_M)_\lambda)}{L(0, r_M \circ (\phi_M)_\lambda)} \tag{B.2}$$

is holomorphic at $\lambda = \lambda_0$. We consider the analytic property of (B.2). For $\alpha \in \Sigma(P)$, let A_α be the identity component of $\mathrm{Ker}(\alpha)$, M_α the centralizer of A_α in G, and U_α the root subgroup associated to α. Then M_α is a Levi subgroup of G (but not necessarily a Levi component of a standard parabolic subgroup of G) and MU_α is a maximal parabolic subgroup of M_α. We may regard $\mathfrak{a}_{M_\alpha}$ as a subspace of $\mathfrak{a}_M$. Put

$$(\mathfrak{a}_M^{M_\alpha})^* = \{\lambda \in \mathfrak{a}_M^* \mid \langle \lambda, H \rangle = 0 \quad \text{for all } H \in \mathfrak{a}_{M_\alpha}\}.$$

For $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, let λ^{M_α} denote its orthogonal projection to $(\mathfrak{a}_M^{M_\alpha})^* \otimes_{\mathbb{R}} \mathbb{C}$. We can write

$$\lambda^{M_\alpha} = s_\alpha(\lambda) \cdot \varpi_\alpha$$

for some $s_\alpha(\lambda) \in \mathbb{C}$, where $\varpi_\alpha \in (\mathfrak{a}_M^{M_\alpha})^*$ is the unique element such that $\langle \varpi_\alpha, \alpha^\vee \rangle = 1$. Then we have

$$(\mathrm{B.2}) = \prod_{\alpha \in \Sigma(P)} \frac{L(1 - s_\alpha(\lambda), r_\alpha^\vee \circ \phi_M)}{L(s_\alpha(\lambda), r_\alpha \circ \phi_M)},$$

where r_α is the adjoint representation of ${}^L M$ on $\mathrm{Lie}({}^L U_\alpha)$. Note that $L(s, r_\alpha \circ \phi_M)$ is holomorphic and nonzero for $\mathrm{Re}(s) > 0$ since ϕ_M is tempered. Since $\lambda_0 \in (\mathfrak{a}_M^*)^+$, $s_\alpha(\lambda_0)$ is a positive real number for all $\alpha \in \Sigma(P)$. Hence (B.2) is holomorphic at $\lambda = \lambda_0$ if and only if

$$\prod_{\alpha \in \Sigma(P)} L(1 - s_\alpha(\lambda) - s_\alpha(\lambda_0), r_\alpha^\vee \circ \phi_M)$$

is holomorphic at $\lambda = 0$. Since the L-factors have no zeros, this condition is equivalent to the holomorphy of $L(s - s_\alpha(\lambda_0), r_\alpha^\vee \circ \phi_M)$ at $s = 1$ for all $\alpha \in \Sigma(P)$, which in turn is equivalent to the holomorphy of

$$L(s, r_M^\vee \circ (\phi_M)_{\lambda_0}) = \prod_{\alpha \in \Sigma(P)} L(s - s_\alpha(\lambda_0), r_\alpha^\vee \circ \phi_M)$$

at $s = 1$. Thus, we have shown that ϕ is generic if and only if $L(s, r_M^\vee \circ (\phi_M)_{\lambda_0})$ is holomorphic at $s = 1$.

On the other hand, we have

$$L(s, \mathrm{Ad} \circ \phi) = L(s, r_M \circ (\phi_M)_{\lambda_0}) \cdot L(s, \mathrm{Ad}_M \circ (\phi_M)_{\lambda_0}) \cdot L(s, r_M^\vee \circ (\phi_M)_{\lambda_0}),$$

where Ad_M is the adjoint representation of ${}^L M$ on $\mathrm{Lie}({}^L M)$. Since ϕ_M is tempered and $s_\alpha(\lambda_0) > 0$ for all $\alpha \in \Sigma(P)$,

$$L(s, r_M \circ (\phi_M)_{\lambda_0}) = \prod_{\alpha \in \Sigma(P)} L(s + s_\alpha(\lambda_0), r_\alpha \circ \phi_M)$$

and $L(s, \mathrm{Ad}_M \circ (\phi_M)_{\lambda_0}) = L(s, \mathrm{Ad}_M \circ \phi_M)$ are holomorphic and nonzero for $\mathrm{Re}(s) > 0$. Hence $L(s, \mathrm{Ad} \circ \phi)$ is holomorphic at $s = 1$ if and only if $L(s, r_M^\vee \circ (\phi_M)_{\lambda_0})$ is holomorphic at $s = 1$. This completes the proof of Proposition B.1.

Remark B.3 If G is a classical group, then one has the following variant of Proposition B.1 which does not rely on the local Langlands correspondence. Fix a generic character ψ_N of N. If π is an irreducible (N, ψ_N)-generic representation of G, let Π be its functorial lift to the general linear group established in [10,11,13,34,35] (see [11, Definition 7.1] for the precise definition in the case when G is split over F). Put

$$L^{\mathrm{Sh}}(s, \pi, \mathrm{Ad}) := L^{\mathrm{Sh}}(s, \Pi, R),$$

where the right-hand side is Shahidi's L-factor [53] and

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } G \text{ is odd special orthogonal;} \\ \wedge^2 & \text{if } G \text{ is even special orthogonal or symplectic;} \\ \mathrm{As}^+ & \text{if } G \text{ is even unitary;} \\ \mathrm{As}^- & \text{if } G \text{ is odd unitary.} \end{cases}$$

If π is tempered, then so is Π (see [11, Proposition 7.4] when G is split over F and [35, Proposition 8.6] when G is even unitary) and hence $L^{\mathrm{Sh}}(s, \pi, \mathrm{Ad})$ is holomorphic and nonzero for $\mathrm{Re}(s) > 0$ (see [53, Proposition 7.2]). If we admit the local Langlands correspondence, then by [30], we have $L^{\mathrm{Sh}}(s, \pi, \mathrm{Ad}) = L(s, \mathrm{Ad} \circ \phi)$, where ϕ is the L-parameter of π.

Now let P be a standard parabolic subgroup of G with Levi component M and π an irreducible tempered $(N_M, \psi_N|_{N_M})$-generic representation of M. For any $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, one has the L-factor $L^{\mathrm{Sh}}(s, I_P^G(\pi_\lambda), \mathrm{Ad})$ as above since the set of λ such that $I_P^G(\pi_\lambda)$ is irreducible and (N, ψ_N)-generic is Zariski dense in $\mathfrak{a}^*_{M,\mathbb{C}}$. Then by the above argument (together with the multiplicativity), one can show that for $\lambda_0 \in (\mathfrak{a}^*_M)^+$, $J_P^G(\pi_{\lambda_0})$ is (N, ψ_N)-generic if and only if $L^{\mathrm{Sh}}(s, I_P^G(\pi_{\lambda_0}), \mathrm{Ad})$ is holomorphic at $s = 1$.

References

1. Aizenbud, A., Gourevitch, D., Rallis, S., Schiffmann, G.: Multiplicity one theorems. Ann. Math. **172**, 1407–1434 (2010)
2. Arthur, J.: The endoscopic classification of representations: orthogonal and symplectic groups. In: Colloquium Publications, vol. 61. American Mathematical Society, New York (2013)
3. Bernstein, J.N.: Le "centre" de Bernstein, Travaux en Cours. In: Representations of Reductive Groups over a Local Field, pp. 1–32, Hermann (1984)
4. Beuzart-Plessis, R.: Expression d'un facteur epsilon de paire par une formule intégrale. Can. J. Math. **66**, 993–1049 (2014)
5. Beuzart-Plessis, R.: Endoscopie et conjecture raffinée de Gan–Gross–Prasad pour les groupes unitaires. Compos. Math. **151**, 1309–1371 (2015)
6. Beuzart-Plessis, R.: La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes unitaires. Mém. Soc. Math. Fr. (to appear)
7. Casselman, W., Shalika, J.: The unramified principal series of p-adic groups. II. The Whittaker function. Compos. Math. **41**, 207–231 (1980)
8. Chaudouard, P.-H., Laumon, G.: Le lemme fondamental pondéré. II. Énoncés cohomologiques. Ann. Math. **176**, 1647–1781 (2012)
9. Clozel, L.: On the cohomology of Kottwitz's arithmetic varieties. Duke Math. J. **72**, 757–795 (1993)
10. Cogdell, J.W., Kim, H.H., Piatetski-Shapiro, I.I., Shahidi, F.: On lifting from classical groups to GL_N. Publ. Math. Inst. Hautes Études Sci. **93**, 5–30 (2001)
11. Cogdell, J.W., Kim, H.H., Piatetski-Shapiro, I.I., Shahidi, F.: Functoriality for the classical groups. Publ. Math. Inst. Hautes Études Sci. **99**, 163–233 (2004)
12. Cogdell, J.W., Piatetski-Shapiro, I.I., Shahidi, F.: Partial Bessel functions for quasi-split groups. Automorphic representations, L-functions and applications: progress and

prospects. In: Ohio State University Mathematical Research Institute Publication, vol. 11, pp. 95–128. de Gruyter, New York (2005)
13. Cogdell, J.W., Piatetski-Shapiro, I.I., Shahidi, F.: Functoriality for the quasisplit classical groups. On certain L-functions. Clay Math. Proc. **13**, 117–140 (2011) (American Mathematical Society)
14. Deligne, P.: Les constantes des équations fonctionnelles des fonctions L. Modular functions of one variable, II. Lect. Notes Math. **349**, 501–597 (1973) (Springer)
15. Gan, W.T., Gross, B.H., Prasad, D.: Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups. Astérisque **346**, 1–109 (2012)
16. Gan, W.T., Gross, B.H., Prasad, D.: Restrictions of representations of classical groups: examples. Astérisque **346**, 111–170 (2012)
17. Gan, W.T., Ichino, A.: Formal degrees and local theta correspondence. Invent. Math. **195**, 509–672 (2014)
18. Gan, W.T., Qiu, Y., Takeda, S.: The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula. Invent. Math. **198**, 739–831 (2014)
19. Gan, W.T., Savin, G.: Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence. Compos. Math. **148**, 1655–1694 (2012)
20. Gan, W.T., Takeda, S.: On the Howe duality conjecture in classical theta correspondence. In: Contemporary Mathematics. Advances in the Theory of Automorphic Forms and Their L-Functions, vol. 664, pp. 105–117. American Mathematical Society, Province, RI (2016)
21. Gan, W.T., Takeda, S.: A proof of the Howe duality conjecture. J. Am. Math. Soc. **29**, 473–493 (2016)
22. Godement, R., Jacquet, H.: Zeta functions of simple algebras. Lect. Notes Math. **260** (1972) (Springer)
23. Gross, B.H., Prasad, D.: On the decomposition of a representation of SO_n when restricted to SO_{n-1}. Can. J. Math. **44**, 974–1002 (1992)
24. Gross, B.H., Prasad, D.: On irreducible representations of $SO_{2n+1} \times SO_{2m}$. Can. J. Math. **46**, 930–950 (1994)
25. Harris, M., Kudla, S.S., Sweet Jr., W.J.: Theta dichotomy for unitary groups. J. Am. Math. Soc. **9**, 941–1004 (1996)
26. Harris, M., Taylor, R.: The geometry and cohomology of some simple Shimura varieties. Ann. Math. Stud. **151** (2001) (Princeton University Press)
27. Heiermann, V.: A note on standard modules and Vogan L-packets. Manuscr. Math. (to appear)
28. Heiermann, V., Muić, G.: On the standard modules conjecture. Math. Z. **255**, 847–853 (2007)
29. Henniart, G.: Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. Invent. Math. **139**, 439–455 (2000)
30. Henniart, G.: Correspondance de Langlands et fonctions L des carrés extérieur et symétrique. Int. Math. Res. Not., 633–673 (2010)
31. Ichino, A.: On the local theta correspondence and R-groups. Compos. Math. **140**, 301–316 (2004)
32. Kaletha, T.: Genericity and contragredience in the local Langlands correspondence. Algebra Number Theory **7**, 2447–2474 (2013)
33. Kaletha, T., Mínguez, A., Shin, S.W., White, P.-J.: Endoscopic classification of representations: inner forms of unitary groups. arXiv:1409.3731
34. Kim, H.H., Krishnamurthy, M.: Base change lift for odd unitary groups. Functional analysis VIII. Various Publ. Ser. (Aarhus) **47**, 116–125 (2004) (Aarhus University)
35. Kim, H.H., Krishnamurthy, M.: Stable base change lift from unitary groups to GL_n. Int. Math. Res. Pap., 1–52 (2005)
36. Kudla, S.S.: On the local theta-correspondence. Invent. Math. **83**, 229–255 (1986)

37. Kudla, S.S.: Splitting metaplectic covers of dual reductive pairs. Israel J. Math. **87**, 361–401 (1994)
38. Langlands, R.P.: On the functional equations satisfied by Eisenstein series. In: Lecture Notes in Mathematics, vol. 544. Springer, New York (1976)
39. Langlands, R.P., Shelstad, D.: On the definition of transfer factors. Math. Ann. **278**, 219–271 (1987)
40. Lapid, E.M., Rallis, S.: On the local factors of representations of classical groups. Automorphic representations, L-functions and applications: progress and prospects. In: Ohio State University Mathematical Research Institute Publication, vol. 11, pp. 309–359. de Gruyter, New York (2005)
41. Mœglin, C.: Conjecture d'Adams pour la correspondance de Howe et filtration de Kudla. Arithmetic geometry and automorphic forms. Adv. Lect. Math. **19**, 445–503 (2011) (International Press)
42. Mœglin, C., Waldspurger, J.-L.: La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux: le cas général. Astérisque **347**, 167–216 (2012)
43. Mœglin, C., Waldspurger, J.-L.: Stabilisation de la Formule des Traces Tordue. Progress in Mathematics, vol. 316/317. Birkhäuser, Basel (**to appear**)
44. Mok, C.P.: Endoscopic classification of representations of quasi-split unitary groups. Mem. Am. Math. Soc. **235**, 1108 (2015)
45. Muić, G.: On the structure of theta lifts of discrete series for dual pairs $(\mathrm{Sp}(n), \mathrm{O}(V))$. Israel J. Math. **164**, 87–124 (2008)
46. Piatetski-Shapiro, I.I., Rallis, S.: L-functions for the classical groups. Explicit constructions of automorphic L-functions. In: Lecture Notes in Mathematics, vol. 1254, pp. 1–52. Springer, New York (1987)
47. Prasad, D.: On the local Howe duality correspondence. Int. Math. Res. Not., 279–287 (1993)
48. Prasad, D.: Theta correspondence for unitary groups. Pac. J. Math. **194**, 427–438 (2000)
49. Ranga Rao, R.: On some explicit formulas in the theory of Weil representation. Pac. J. Math. **157**, 335–371 (1993)
50. Rodier, F.: Whittaker models for admissible representations of reductive p-adic split groups. Harmonic analysis on homogeneous spaces. In: Proceedings of the Symposium on Pure Mathematics, vol. 26, pp. 425–430. American Mathematical Society, New York (1973)
51. Scholze, P.: The local Langlands correspondence for GL_n over p-adic fields. Invent. Math. **192**, 663–715 (2013)
52. Shahidi, F.: On certain L-functions. Am. J. Math. **103**, 297–355 (1981)
53. Shahidi, F.: A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. Ann. Math. **132**, 273–330 (1990)
54. Shahidi, F.: Local coefficients as Mellin transforms of Bessel functions: towards a general stability. Int. Math. Res. Not., 2075–2119 (2002)
55. Shin, S.W.: Automorphic Plancherel density theorem. Israel J. Math. **192**, 83–120 (2012)
56. Sun, B.: Multiplicity one theorems for Fourier-Jacobi models. Am. J. Math. **134**, 1655–1678 (2012)
57. Sun, B., Zhu, C.-B.: Conservation relations for local theta correspondence. J. Am. Math. Soc. **28**, 939–983 (2015)
58. Vogan Jr, D.A.: The local Langlands conjecture. Representation theory of groups and algebras. Contemp. Math. **145**, 305–379 (1993) (American Mathematical Society)
59. Waldspurger, J.-L.: Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I. In: Israel Mathematical Conference Proceedings, vol. 2, pp. 267–324. Weizmann, Rehovot (1990)
60. Waldspurger, J.-L.: La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra). J. Inst. Math. Jussieu **2**, 235–333 (2003)

61. Waldspurger, J.-L.: Une formule intégrale reliée à la conjecture locale de Gross–Prasad. Compos. Math. **146**, 1180–1290 (2010)
62. Waldspurger, J.-L.: Une formule intégrale reliée à la conjecture locale de Gross–Prasad, 2ème partie: extension aux représentations tempérées. Astérisque **346**, 171–312 (2012)
63. Waldspurger, J.-L.: Calcul d'une valeur d'un facteur ϵ par une formule intégrale. Astérisque **347**, 1–102 (2012)
64. Waldspurger, J.-L.: La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux. Astérisque **347**, 103–165 (2012)
65. Wallach, N.R.: On the constant term of a square integrable automorphic form. Operator algebras and group representations, vol. II. In: Monographic Studies in Mathematics, vol. 18, pp. 227–237. Pitman, New York (1984)