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Inflations of self-affine tilings are integral algebraic Perron

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Abstract We prove that any expanding linear map $\phi : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ that is the inflation map in an inflation-substitution process generating a self-affine tiling of $\mathbb{R}^{\mathrm{d}}$ is integral algebraic and Perron. This means that ϕ is linearly conjugate to a restriction of an integer matrix to a subspace $\mathcal{E}$ satisfying a maximal growth condition that generalizes the characterization of Perron numbers as numbers that are larger than the moduli of their algebraic conjugates. The case of diagonalizable ϕ has been previously resolved by Richard Kenyon and Boris Solomyak, and it is rooted in Thurston's idea of lifting the tiling from the *physical space* $\mathbb{R}^{\mathrm{d}}$ to a higher dimensional *mathematical space* where the tiles (their control points) sit on a lattice. The main novelty of our approach is in lifting the inflation-substitution process to the mathematical space and constructing a certain vector valued cocycle defined over the translation induced $\mathbb{R}^{\mathrm{d}}$-action on the tiling space. The subspace $\mathcal{E}$ is obtained then by ergodic averaging of the cocycle. More broadly, we assemble a powerful framework for studying self-affine tiling spaces.

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1 Introduction

Prologue

Aperiodic repetitive tilings of the Cartesian space $\mathbb{R}^{\mathrm{d}}$ (with the dimension $\mathrm{d} \geq 1$) are both mathematically interesting and practically relevant as idealizations of physical quasi-crystals. The ones that are of central interest are fractal in the sense that they enjoy the additional property of self-similarity or ϕ-self-affinity where ϕ is some linear expanding transformation of $\mathbb{R}^{\mathrm{d}}$. Self-affine quasi-crystals have been both synthesized and found in nature [2]. Since self-affinity is a very prominent feature and a powerful theoretical tool, the first step toward classifying all tilings is understanding the ϕ that can arise in the self-affine case. This program was put in motion by William Thurston [26] and continued by Richard Kenyon in his PhD thesis [9] and Richard Kenyon and Boris Solomyak in [10]. Building on this work, we propose a condition that should characterize the tiling-borne ϕ among all linear transformations. In this work we prove the necessity of this condition.

The result

We consider tilings $\mathcal{T}$ of $\mathbb{R}^{\mathrm{d}}$, i.e., coverings of $\mathbb{R}^{\mathrm{d}}$ by sets (*tiles*), each a closure of its interior, such that the interiors are pairwise disjoint. The number of tile types is to be finite, meaning that each tile is a translate of one of finitely many *prototiles* $T_1, \ldots, T_m$ ($m \in \mathbb{N}$). A translate of T_i by $p \in \mathbb{R}^{\mathrm{d}}$ will be denoted[1] by $T_i @ p$. Finite subcollections of $\mathcal{T}$ are called *patches*, and we postulate *finite local complexity* of $\mathcal{T}$, i.e., for any given $R > 0$, up to translation, there are finitely many patches whose diameter does not exceed R. We also ask that $\mathcal{T}$ satisfies a kind of almost periodicity hypothesis called *repetitivity*, i.e., for any fixed finite patch, there is $R' > 0$ such that a translated copy of that patch can be found inside any ball of radius R' in $\mathbb{R}^{\mathrm{d}}$. (We do not assume that the tiling is aperiodic; although, this is certainly the most interesting case).

Given a linear $\phi : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ that is expanding (i.e., all its eigenvalues λ satisfy $|\lambda| > 1$), a *ϕ-self-affine* tiling is a tiling $\mathcal{T}$ for which one can find a *substitution rule* that recovers $\mathcal{T}$ when applied to the inflated tiling $\phi\mathcal{T}$ (the image of $\mathcal{T}$ under ϕ). To express this precisely, the inflation-substitution process is formalized as the so called *inflation-substitution map* Φ. Besides ϕ, Φ is determined by finite sets $D_{i,j} \subset \mathbb{R}^{\mathrm{d}}$ indexed by $i, j \in \{1, \ldots, m\}$. ($D_{i,j}$ are called *digit sets*.) Φ applied to a tile $T_j @ p$ is a finite collection of tiles given as follows

$$\Phi(T_j @ p) := \{T_i @ (\phi p + d) : \ d \in D_{i,j}, 1 \leq i \leq m\}. \tag{1.1}$$

[1] This is more compact than the standard notation $T_i + p$.

Crucially, $\Phi(T_j@p)$ is required to tile the subset $\phi(T_j@p)$ (i.e., the tiles in $\Phi(T_j@p)$ cover $\phi(T_j@p)$ and have pairwise disjoint interiors). Now, Φ can be applied to $\mathcal{T}$ tile-by-tile to render a tiling of $\mathbb{R}^{\mathrm{d}}$ denoted by $\Phi(\mathcal{T})$. $\mathcal{T}$ is *ϕ-self-affine* iff $\Phi(\mathcal{T}) = \mathcal{T}$ (for some choice of the sets $D_{i,j}$).

Note that $a_{ij} := \#D_{i,j} \geq 0$ is the number of tiles of type i going into substitution of the (inflated) tile of type j. The so called *substitution matrix* $A := (a_{ij})$ is assumed to be primitive (i.e., some power of it has positive entries). Pictures of tilings abound in literature and can be found on the Internet; see `tilings.math.uni-bielefeld.de` for a nicely curated collection.

Theorem 1.1 *If $\phi : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ is an expanding linear map for which there exists a ϕ-self-affine tiling $\mathcal{T}$ then*

- `(IA)` *ϕ is integral algebraic;*
- `(P)` *ϕ is Perron.*

Even before defining `(IA)` and `(P)`, we should note that, under the additional hypothesis that ϕ is diagonalizable, this theorem was first proposed in [9] and then completely proven by Kenyon and Solomyak in [10]. This is following Thurston's original treatment of the two-dimensional conformal ϕ in [26] and predated by Lind's [17] from which the one-dimensional version can be readily extracted. It is also expected that the implication can be reversed, making the theorem sharp. At the very least this should be so at the expense of passing to an iterate of ϕ:

Conjecture 1.2 *Given an expanding ϕ that is integral algebraic and Perron, there is $n \in \mathbb{N}$ such that a ϕ^n-self-affine tiling exists.*

The one- and two-dimensional conformal cases are resolved in [8, 17].

We define `(IA)` and `(P)` now. `(IA)` simply means that every eigenvalue of ϕ is an algebraic integer. In the diagonalizable case, `(P)` is also a condition on the eigenvalues of ϕ and stipulates that, whenever λ is an eigenvalue and k is its multiplicity and γ is an algebraic conjugate of λ (over the field of rationals $\mathbb{Q}$), then either $|\gamma| < |\lambda|$ or γ is also an eigenvalue and has multiplicity at least k. To formulate `(P)` in the general case, one has to replace the eigenvalues by the *Jordan blocks* appearing in the *Jordan decomposition of ϕ*. To be precise, given $r \geq 1$ and $\lambda \in \mathbb{C}$ the (λ, r)-*Jordan block* is the $r \times r$ block matrix

$$J_{\lambda,r} = \begin{bmatrix} \Lambda_\lambda & I_\lambda & & \\ & \Lambda_\lambda & I_\lambda & \\ & & \cdots & \\ & & & \Lambda_\lambda \end{bmatrix} \tag{1.2}$$

where $\Lambda_\lambda = \lambda$ and $I_\lambda = 1$ when λ is real and $\Lambda_\lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ and $I_\lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when λ lambda is non-real and $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. We refer to λ and r as the *eigenvalue* and the *rank of the block* (although, for non-real λ the real rank is $2r$). The real Jordan canonical form theorem asserts that any real linear transformation ϕ is linearly conjugate[2] to a direct sum of Jordan blocks (acting by multiplication on the column vectors). We note that a (λ, r)-Jordan block is linearly conjugate to a (λ', r')-Jordan block iff $r' = r$ and λ' coincides with λ up to complex conjugation, i.e., $\lambda' = \lambda$ or $\lambda' = \overline{\lambda}$, which we will denote by $\lambda' \equiv \lambda$. Therefore we shall consider the pairs (λ, r) and $(\overline{\lambda}, r)$ as equivalent, denote the equivalence class by $[\lambda, r]$, and refer to it as the *datum* of the Jordan block.

Although the linear conjugacy in the real Jordan decomposition theorem is not unique, it uniquely determines the *Jordan spectrum of* ϕ, by which we understand the list of datums of all Jordan blocks in the direct sum. Here the *list* refers to a set with repetitions, i.e., a function with values in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ equal to 0 for all but finitely many of its arguments $[\lambda, r]$. This function is referred to as the *multiplicity* of ϕ and its value on $[\lambda, r]$ is denoted by $\text{mult}_\phi(\lambda, r)$. For instance, the multiplicity function given by $\text{mult}_\phi(\sqrt{3}, 2) = 1$, $\text{mult}_\phi(\sqrt{2}, 1) = 2$, and zero otherwise, corresponds to the list $\{[\sqrt{3}, 2], [\sqrt{2}, 1], [\sqrt{2}, 1]\}$, also expressed as $\{[\sqrt{3}, 2]^1, [\sqrt{2}, 1]^2\}$.

Given a pair of complex numbers λ and γ, we say that γ *dominates* λ, denoted γ*dominates*λ, iff λ and γ are algebraically conjugate and $|\gamma| > |\lambda|$. We say that γ *weakly-dominates* λ, denoted $\gamma \succeq \lambda$, iff λ and γ are algebraically conjugate and $|\gamma| \geq |\lambda|$ with $\gamma \not\equiv \lambda$.

Definition 1.3 A linear map ϕ satisfies (P) (is called *Perron*) iff whenever $[\lambda, r]$ is in the spectrum of ϕ with multiplicity k and γ is algebraically conjugate to λ then either $|\gamma| < |\lambda|$ or $[\gamma, r]$ is also in the spectrum of ϕ with multiplicity at least k. Equivalently, the multiplicity function satisfies

$$\gamma \succeq \lambda \implies \text{mult}_\phi(\gamma, \cdot) \geq \text{mult}_\phi(\lambda, \cdot). \tag{1.3}$$

For diagonalizable ϕ, the only possible value of the rank is $r = 1$ and this definition coincides with the one in [10].

Let us look at an example. We shall refer to ϕ as *realized* iff there exists a ϕ-self-affine tiling (with the properties stated in the first paragraph of this introduction).

[2] Maps A and B are *linearly conjugate* iff $A = L^{-1} \circ B \circ L$ for some linear isomorphism L.

Example 1 Kenyon and Solomyak conjectured that the following ϕ is not realized:

$$\phi = \begin{bmatrix} 3+\sqrt{2} & 1 & 0 \\ 0 & 3+\sqrt{2} & 0 \\ 0 & 0 & 3-\sqrt{2} \end{bmatrix}. \tag{1.4}$$

This ϕ is already presented in its real Jordan canonical form. The minimal monic polynomial for $3 \pm \sqrt{2}$ is $p(z) = z^2 - 6z + 7$, so the two roots $3 \pm \sqrt{2}$ form a full conjugacy class of algebraic integers. The Jordan spectrum of ϕ is $\{[3+\sqrt{2}, 2], [3-\sqrt{2}, 1]\}$. It is not Perron because it does not contain $[3+\sqrt{2}, 1]$ and violates (1.3): $\text{mult}_\phi(3+\sqrt{2}, 1) = 0 < 1 = \text{mult}_\phi(3-\sqrt{2}, 1)$ but $3+\sqrt{2} \succ 3-\sqrt{2}$. By our theorem, ϕ is not realized. Note how Perronness failed even though ϕ has an eigenspace associated to $3+\sqrt{2}$. This eigenspace did not yield a $(3+\sqrt{2}, 1)$-Jordan block because it is not invariantly complemented, i.e., it is not a direct summand in a splitting of $\mathbb{R}^3$ into ϕ-invariant subspaces.

Our next task is to explain what it takes to prove the theorem and further elucidate the meaning of `(IA)` and `(P)`. To start, we recast `(IA)`. Given a real vector space $\mathcal{V}$ and a lattice $\mathcal{J}$ (i.e., a discrete co-compact subgroup of $\mathcal{V}$), we call a linear transformation $M : \mathcal{V} \to \mathcal{V}$ *integral* iff it maps $\mathcal{J}$ to itself. (Using a basis of the lattice $\mathcal{J}$, M can be represented by a matrix with integer entries.) If the lattice is not specified for $M : \mathcal{V} \to \mathcal{V}$, we mean that such a lattice exists.

Proposition 1.4 *Suppose $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a linear transformation. ϕ is integral algebraic (satisfies* `(IA)`*) iff there is a real vector space $\mathcal{V}$ with a lattice $\mathcal{J} \subset \mathcal{V}$ and an integral linear transformation $M : \mathcal{V} \to \mathcal{V}$ for which an M-invariant splitting into subspaces $\mathcal{V} = \mathcal{E} \oplus \mathcal{K}$ exists so that the restriction $M|_\mathcal{E}$ is linearly conjugate to ϕ.*

Proof (cf. Proposition 3.3.) ($\Leftarrow$): An eigenvalue of ϕ is an eigenvalue of M and thus a root of the characteristic polynomial of M, which is monic with integer coefficients.

($\Rightarrow$): By the Jordan theorem, ϕ is linearly conjugate to a direct sum of Jordan blocks. Each block is a (λ, r)-Jordan block for some eigenvalue λ, which is an algebraic integer per `(IA)`. Taking p to be the minimal polynomial of λ, such block is linearly conjugated to the companion matrix of the r-th power p^r restricted to an invariantly complemented subspace. Taking the direct sum of such companion matrices yields an integral M that restricts to ϕ along invariant and invariantly complemented subspace. □

The proof of Theorem 1.1 begins with constructing $\mathcal{V}$, $\mathcal{J}$, M, $\mathcal{E}$, $\mathcal{K}$ as in Proposition 1.4. This secures `(IA)`. Much more work is necessary to establish `(P)`.

It is hard to convey the main ideas without a considerable buildup, but let us comment that the algebraic framework of Thurston–Kenyon–Solomayk theory [9,10,26] only employs $\mathcal{V}$, $\mathcal{J}$, M, $\mathcal{K}$ and offsets the absence of $\mathcal{E}$ by the fact that, in the diagonalizable case, any invariant subspace is invariantly complemented. The linear space $\mathcal{V}$ is what we call[3] the *mathematical space*, into which the original tiling in the *physical space* $\mathbb{R}^{\mathtt{d}}$ can be lifted and where the tiles (their *control points*) sit on a lattice (Sect. 4). The action of ϕ on the lattice gives rise to M, and $\mathcal{K}$ is the kernel of the projection $\pi_{\mathbb{R}^{\mathtt{d}}} : \mathcal{V} \to \mathbb{R}^{\mathtt{d}}$. The most apparent conceptual novelty of our work is a construction (Sect. 11) of a natural $\mathcal{E}$ that facilitates the invariant splitting $\mathcal{V} = \mathcal{E} \oplus \mathcal{K}$ by complementing the space $\mathcal{K}$. The key idea is lifting (Sect. 6) of the inflation-substitution map to the level of $\mathcal{V}$ and its exploitation (Sect. 7) in a construction of a certain new vector valued cocycle (Sect. 8) over the minimal uniquely ergodic $\mathbb{R}^{\mathtt{d}}$-action on the tiling space of the tiling, also called the hull of the tiling (Sect. 5). The subspace $\mathcal{E}$ is the asymptotic direction of the cocycle computed by ergodic averaging (Sect. 11). The proof of `(P)` rests on the interaction of the non-linear part of the cocycle with the algebraic action induced by M (Sects. 12 and 14).

This synopsis neglects the role of the return vectors for the $\mathbb{R}^{\mathtt{d}}$-action in both pinning down the cocycle (Sects. 9 and 10) and constructing certain crucial filtrations of linear subspaces of $\mathcal{V}$ (Sects. 13, and 15, 16, 17). Here we borrowed many ideas from [10], but adapting to the non-diagonalizable context and ensuring that different parts mesh together offered many challenges. The net result is a powerful general framework for studying self-affine tiling spaces.

To clearly articulate the ideas, we proceed in short modular sections, each introducing a different aspects of the framework. (In Sect. 14, the proof of the theorem is then a short affair.) Despite the length of the buildup, depending on taste, our techniques may be in some key aspects simpler than those in [10]. Chiefly, our use of rather elementary ergodic cocycle averaging (Sect. 11) subsumes the impressive analytical part in [10] resting on the trio of: Rademacher's Theorem on a.e. differentiability, Egorov's Theorem, and Lebesgue–Vitali Density Theorem. What we have to offer may then appeal even to a reader who only cares about the diagonalizable case in the physically relevant dimension $\mathtt{d} = 3$.

Before starting in earnest, we turn to some basic algebraic preliminaries. We also give a geometric interpretation of the Perron property via growth maximization, which speaks to our intuition much better than the original algebraic formulation.

[3] Another terms for $\mathbb{R}^{\mathtt{d}}$ and $\mathcal{V}$ are the *internal space* and the *embedding space*, see e.g. [20].

2 Algebraic preliminaries

We collect some basic notions from the theory of Jordan and rational canonical forms (see e.g. [4]). The goal is to clearly articulate what we need for the proof and avoid the pitfalls stemming from the lack of uniqueness of the Jordan decomposition and its quirkiness under passage to an invariant subspace (see e.g. [3,5]).

Given any real linear transformation $M : \mathcal{V} \to \mathcal{V}$, the space $\mathcal{V}$ can be presented as a direct sum of *Jordan spaces*. To be precise, we call an $\mathbb{R}$-subspace of $\mathcal{V}$ a (λ, r)-*Jordan space* if it is M-invariant, is complemented by an M-invariant subspace, and M restricted to it is linearly conjugate to the (λ, r)-Jordan block. Any decomposition of $\mathcal{V}$ into such spaces is called a *Jordan decomposition* for M. The number of (λ, r)-Jordan spaces equals $\text{mult}_M(\lambda, r)$. Jordan decomposition is generally not unique; however, for any eigenvalue λ, the direct sum of all (λ, r)-Jordan spaces ($r \in \mathbb{N}$) is determined by M and coincides with the *generalized eigenspace* associated to λ, denoted by $\mathcal{V}_\lambda$.

If we have an invariant splitting $\mathcal{V} = \mathcal{E} \oplus \mathcal{K}$ then Jordan decompositions for the restrictions $M|_\mathcal{E}$ and $M|_\mathcal{K}$ add to form a Jordan decomposition for M, so the multiplicity functions satisfy

$$\text{mult}_M = \text{mult}_{M|_\mathcal{E}} + \text{mult}_{M|_\mathcal{K}}. \tag{2.1}$$

In particular, for any invariantly complemented subspace $\mathcal{E} \subset \mathcal{V}$, we have[4]

$$\text{mult}_{M|_\mathcal{E}} \leq \text{mult}_M. \tag{2.2}$$

Now, suppose additionally that $M : \mathcal{V} \to \mathcal{V}$ is integral by virtue of preserving a lattice $\mathcal{J} \subset \mathcal{V}$. Let $\mathcal{V}_\mathbb{Q}$ be the *rational part* of $\mathcal{V}$ consisting of all linear combinations of the vectors in $\mathcal{J}$ with rational coefficients. We call a real linear subspace $\mathcal{U} \subset \mathcal{V}$ *rational* iff it is the real linear span of a rational linear subspace of $\mathcal{V}_\mathbb{Q}$, i.e., $\mathcal{U} = \text{span}_\mathbb{R}(\mathcal{U} \cap \mathcal{V}_\mathbb{Q})$. Considered as a $\mathbb{Q}$-linear transformation, $M : \mathcal{V}_\mathbb{Q} \to \mathcal{V}_\mathbb{Q}$ has a rational canonical form and an accompanying decomposition into rational invariant subspaces called *cyclic spaces*. Specifically, the characteristic and the minimal polynomials of M are of the form

$$p_M^{\text{char}}(z) = (-1)^N p_1(z)^{n_1} \dots p_k(z)^{n_k} \quad \text{and} \quad p_M^{\min}(z) = p_1(z)^{m_1} \dots p_k(z)^{m_k}$$

[4] This may fail for invariant $\mathcal{E}$ that are not invariantly complemented. It can even be that no Jordan decomposition for $M|_\mathcal{E}$ is obtained by restricting a decomposition for M.

where $N := \dim \mathcal{V}$, the p_i are distinct monic polynomials[5] that are irreducible over $\mathbb{Q}$, and the m_i, n_i are positive integers with $1 \le m_i \le n_i$. Set $d_i := \deg(p_i)$. Each factor p_i has associated to it a subspace, sometimes called a *primary subspace* of M, given by

$$\mathcal{V}_{p_i} := \left\{v \in \mathcal{V} :\ \exists_{r\in\mathbb{N}}\ p_i(M)^r v = 0\right\} = \left\{v \in \mathcal{V} :\ p_i(M)^{m_i} v = 0\right\},$$

which is rational and of dimension $\dim(\mathcal{V}_{p_i}) = n_i d_i$. In particular, the restriction $M|_{\mathcal{V}_{p_i}} : \mathcal{V}_{p_i} \to \mathcal{V}_{p_i}$ is integral (by virtue of preserving the lattice $\mathcal{V}_{p_i} \cap \mathcal{J}$). We have $\mathcal{V} = \bigoplus_i \mathcal{V}_{p_i}$ and each $\mathcal{V}_{p_i}$, in turn, can be decomposed into a direct sum of *cyclic subspaces*, that is subspaces that have a basis of the form $\{v, Mv, \ldots, M^{s-1}v\}$ for some $v \in \mathcal{V}_{\mathbb{Q}}$. The action of M on each such subspace $\mathcal{C}$ is $\mathbb{Q}$-linearly conjugate to that of the companion matrix of p_i^r for a certain $1 \le r \le m_i$ and the dimension of $\mathcal{C}$ is $s = rd_i$. (Unlike the $\mathcal{V}_{p_i}$, the cyclic subspaces are not uniquely determined, only their number and dimensions are unique.) The Jordan spectrum of $M|_{\mathcal{C}}$ is of the form $\{[\gamma_1, r], \ldots, [\gamma_d, r]\}$ where $\{\gamma_1, \ldots, \gamma_d\}$ are the roots of p_i (thus a complete algebraic conjugacy class) with the caveat that we include only one root from each pair of complex conjugates. (So $d = d_i$ only if all the roots are real.) In particular, $\mathrm{mult}_{M|_{\mathcal{C}}}(\cdot, r)$ is constant across $\{\gamma_1, \ldots, \gamma_d\}$. By taking direct sums over cyclic spaces (and invoking (2.1)), we get

$$\lambda \text{ and } \lambda' \text{ are algebraically conjugated} \quad \Longrightarrow \quad \mathrm{mult}_M(\lambda, \cdot) = \mathrm{mult}_M(\lambda', \cdot). \tag{2.3}$$

The implication (2.3) is also true when $M : \mathcal{V} \to \mathcal{V}$ is only rational (i.e., nM is integral for some $n \in \mathbb{N}$).

Example 2 Still using the monic irreducible polynomial $p(z) = z^2 - 6z + 7$ from Example 1, the matrices

$$B := \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C := \begin{bmatrix} B & I \\ 0 & B \end{bmatrix}. \tag{2.4}$$

have $p_B^{\mathrm{char}}(z) = p_B^{\min}(z) = p(z)$ and $p_C^{\mathrm{char}}(z) = p_C^{\min}(z) = p(z)^2$. The 6×6 matrix

$$M := C \oplus B$$

has $p_M^{\min}(z) = p(z)^2$ and $p_M^{\mathrm{char}}(z) = p(z)^3$. There is only one primary space, $\mathcal{V}_1 = \mathcal{V} = \{v :\ p(M)^2 v = 0\} = \mathbb{R}^6$. One sees that B and C are $\mathbb{Q}$-linearly conjugate to the companion matrices of $p(z)$ and $p(z)^2$, respectively.

[5] Along with p_M^{char}, the p_i also have integer coefficients (by Gauss' Lemma).

In particular, using the standard unit vectors e_i, a splitting into cyclic subspaces can be taken as

$$\mathbb{R}^6 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3, e_4\} \oplus \mathrm{span}_{\mathbb{R}}\{e_5, e_6\}. \tag{2.5}$$

B and C are $\mathbb{R}$-linearly conjugate to their Jordan forms:

$$B_J := \begin{bmatrix} 3+\sqrt{2} & 0 \\ 0 & 3-\sqrt{2} \end{bmatrix} \quad \text{and}$$

$$C_J := \begin{bmatrix} 3+\sqrt{2} & 1 & 0 & 0 \\ 0 & 3+\sqrt{2} & 0 & 0 \\ 0 & 0 & 3-\sqrt{2} & 1 \\ 0 & 0 & 0 & 3-\sqrt{2} \end{bmatrix},$$

and the Jordan canonical form of M is $M_J = C_J \oplus B_J$. The Jordan spaces of M come in two groups of two, each group decomposing a different cyclic space. Expressed in the $\mathbb{R}^6$ acted on by M_J, they are $\mathrm{span}_{\mathbb{R}}(e_1, e_2, e_3, e_4) = \mathrm{span}_{\mathbb{R}}(e_1, e_2) \oplus \mathrm{span}_{\mathbb{R}}(e_3, e_4)$ and $\mathrm{span}_{\mathbb{R}}(e_5, e_6) = \mathrm{span}_{\mathbb{R}}(e_5) \oplus \mathrm{span}_{\mathbb{R}}(e_6)$. Moreover, the direct sums of Jordan spaces corresponding to the roots $3+\sqrt{2}$ and $3-\sqrt{2}$ are the two generalized eigenspaces of M, $\mathcal{V}_{3+\sqrt{2}} = \mathrm{span}_{\mathbb{R}}(e_1, e_2, e_5)$ and $\mathcal{V}_{3-\sqrt{2}} = \mathrm{span}_{\mathbb{R}}(e_3, e_4, e_6)$, correspondingly. The Jordan datum of M (stratified by rank) is

$$\text{rank } 2 : [3+\sqrt{2}, 2] \quad [3-\sqrt{2}, 2]\,; \quad \text{rank } 1 : [3+\sqrt{2}, 1] \quad [3-\sqrt{2}, 1]$$

3 Perronness as growth semi-maximality

This section is not strictly needed for the proof of the theorem but it clarifies the geometric meaning of `(P)` and contains some further discussion and examples of (Perron) Jordan spectra.

The *growth* of an M-invariant subspace is the modulus of the determinant of the restriction of M to that subspace. (For $\mathcal{V}_{3+\sqrt{2}}$ in Example 2, it is $(3+\sqrt{2})^3$.)

Definition 3.1 *(cf.* [10]*)* Suppose that $M : \mathcal{V} \to \mathcal{V}$ is a linear transformation. An invariantly complemented M-invariant $\mathbb{R}$-subspace $\mathcal{E} \subset \mathcal{V}$ is *growth maximal* iff the growth of $\mathcal{E}$ is strictly larger than the growth of any other invariantly complemented M-invariant subspace $\mathcal{E}' \subset \mathcal{V}$ of the same dimension as $\mathcal{E}$.

To see that there is a connection between Perron property `(P)` and growth maximizing consider a class of particularly simple examples.

Example 3 Suppose that $p(z)$ is an integral monic polynomial that is irreducible over $\mathbb{Q}$ (e.g., the $p(z)$ in Example 1). From each complex conjugacy class $\gamma, \overline{\gamma}$ of the roots of $p(z)$ select one root to form a set $\{\gamma_1, \ldots, \gamma_d\}$, and index it so that $|\gamma_i| \geq |\gamma_{i+1}|$ $(1 \leq i < d)$. Let M be the companion matrix of $p(z)^r$. By our discussion of the rational canonical form, the Jordan spectrum of M is exactly $\{[\gamma_1, r], \ldots, [\gamma_d, r]\}$. In fact, the Jordan decomposition is unique and the generalized real eigenspace E_{γ_i} of γ_i is the sole (γ_i, r)-Jordan space. Any invariantly complemented subspace $\mathcal{E}$ is a direct sum of a subcollection of the E_{γ_i}. It is easy to see that $\mathcal{E}$ is of maximal growth iff $\mathcal{E} = \bigoplus_{i=1,\ldots,d'} E_{\gamma_i}$ where either $d' < d$ and $|\gamma_{d'}| > |\gamma_{d'+1}|$ or $d' = d$ (so $\mathcal{E}$ is the full space). Such $\mathcal{E}$ is a *dominating* subspace for M in the sense that each eigenvalue γ_i of $M|_{\mathcal{E}}$ dominates any γ_j that is not an eigenvalue of $M|_{\mathcal{E}}$.

It is asserted in [10] that a diagonalizable ϕ satisfies `(IA)` and `(P)` iff ϕ is linearly conjugate to the restriction of some integral M to a growth maximal subspace $\mathcal{E}$. This is very close to but not quite true. We give an example of an offending ϕ and show that it can be realized by a tiling.

Counter-Example 4 Take a 4×4 unimodular integral A with simple eigenvalues such that $|\lambda_1| > |\lambda_2| > |\lambda_3| > 1 > |\lambda_4|$ and let $M := A \oplus A \oplus A$. The three expanding eigenspaces of A yield nine 1-dimensional M-invariant subspaces indexed by the eigenvalues arranged in a 3×3 array. Denote by the $\mathcal{E}$ and $\mathcal{E}'$ the direct sums of the eigenspaces of the five boxed eigenvalues, as depicted:

$$(\mathcal{E}) \quad \text{rank } 1: \quad \begin{array}{l} \boxed{\lambda_1\lambda_2\lambda_3} \\ \boxed{\lambda_1}\,\lambda_2\lambda_3 \\ \boxed{\lambda_1}\,\lambda_2\lambda_3 \end{array} \qquad \text{and} \qquad (\mathcal{E}') \quad \text{rank } 1: \quad \begin{array}{l} \boxed{\lambda_1\lambda_2}\,\lambda_3 \\ \boxed{\lambda_1\lambda_2}\,\lambda_3 \\ \boxed{\lambda_1}\,\lambda_2\lambda_3 \end{array}$$

(Above, “rank 1” indicates that each λ_i contributes $[\lambda_i, 1]$ to the Jordan spectrum.) Note that $\mathcal{E}$ is not growth maximal since its growth is strictly less than that of $\mathcal{E}'$: $|\lambda_1^3\lambda_2\lambda_3| < |\lambda_1^3\lambda_2^2|$. Crucially, $M|_{\mathcal{E}}$ is Perron and it is possible to make a ϕ-self-affine tiling with ϕ that is linearly conjugate to $M|_{\mathcal{E}}$. Indeed, by using any Markov partition ([23], see also [19]) for the toral automorphism induced by A, one can produce[6] a three-dimensional $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$-self-affine tiling $\mathcal{T}_{123}$. There is also a one-dimensional λ_1-self-affine tiling $\mathcal{T}_1$ (by [17]). The product tiling[7] $\mathcal{T}_{123} \times \mathcal{T}_1 \times \mathcal{T}_1$ is as desired.

Note that the $\mathcal{E}$ in the counter-example is a direct sum of three growth maximal subspaces (corresponding to the three rows) and thus is growth semi-maximal in the following sense.

Definition 3.2 Suppose that $M : \mathcal{V} \to \mathcal{V}$ is an integral (or merely rational) linear transformation. An invariantly complemented M-invariant $\mathbb{R}$-subspace

[6] By intersecting a leaf of the unstable foliation with the Markov boxes, cf. Section 3 of [11].

[7] Its tiles are $T \times T' \times T''$ where T is a tile of $\mathcal{T}_{123}$ and T' and T'' are tiles of $\mathcal{T}_1$.

$\mathcal{E} \subset \mathcal{V}$ is *growth semi-maximal* iff there is a direct sum splitting $\mathcal{V} = \bigoplus_j \mathcal{U}_j$ into rational M-invariant subspaces such that (for every j) $\mathcal{E}_j := \mathcal{U}_j \cap \mathcal{E}$ is invariantly complemented in $\mathcal{U}_j$ by some $\mathcal{K}_j$, $\mathcal{E} = \bigoplus_j \mathcal{E}_j$, and $\mathcal{E}_j \subset \mathcal{U}_j$ is growth maximal (for the restriction $M|_{\mathcal{U}_j}$).

Proposition 3.3 *A linear transformation $\phi : \mathbb{R}^{\mathrm{d}} \to \mathbb{R}^{\mathrm{d}}$ is integral algebraic Perron (i.e., satisfies* `(IA)` *and* `(P)`*) iff there is an integral linear transformation $M : \mathcal{V} \to \mathcal{V}$ such that $M|_{\mathcal{E}}$ is linearly conjugate to ϕ and $\mathcal{E} \subset \mathcal{V}$ is growth semi-maximal.*

Proof of $\Leftarrow$: Suppose that ϕ is linearly conjugate to $M|_{\mathcal{E}}$ where $\mathcal{E}$ is growth semi-maximal and $\mathcal{U}_j$ and $\mathcal{E}_j$ are as in Definition 3.2. By (2.1), $\mathrm{mult}_\phi = \sum_j \mathrm{mult}_{M|_{\mathcal{E}_j}}$, so it suffices to show that each $M|_{\mathcal{E}_j}$ is Perron. (This is because the inequalities in (1.3) can be added side-by-side.) Suppose then that $M|_{\mathcal{E}_j}$ is not Perron for some j and there is γ weakly-dominating λ (so $|\gamma|/|\lambda| \geq 1$) such that

$$m_\gamma := \mathrm{mult}_{M|_{\mathcal{E}_j}}(\gamma, r) < \mathrm{mult}_{M|_{\mathcal{E}_j}}(\lambda, r) =: m_\lambda.$$

Since $\mathcal{U}_j$ is rational and λ and γ are algebraically conjugate, we have (by (2.3) and (2.2)) $m := \mathrm{mult}_{M|_{\mathcal{U}_j}}(\gamma, r) = \mathrm{mult}_{M|_{\mathcal{U}_j}}(\lambda, r) \geq \max\{m_\gamma, m_\lambda\}$. This means that, if we fix a Jordan decomposition for $M|_{\mathcal{U}_j}$ obtained by taking the direct sum of decompositions for $M|_{\mathcal{E}_j}$ and $M|_{\mathcal{K}_j}$, then $\mathcal{E}_j$ contains a direct sum of m_γ (γ, r)-Jordan spaces and m_λ (λ, r)-Jordan spaces. One can create $\mathcal{E}'_j$ by replacing $\Delta := m_\lambda - m_\gamma > 0$ of those (λ, r)-Jordan spaces by some (γ, r)-Jordan spaces of M not already in $\mathcal{E}_j$ (i.e., Jordan spaces for $M|_{\mathcal{K}_j}$). Such $\mathcal{E}'_j$ has the same dimension as $\mathcal{E}_j$, is invariantly complemented, and

$$\mathrm{growth}(\mathcal{E}'_j) = \mathrm{growth}(\mathcal{E}_j)\left(\frac{|\gamma|}{|\lambda|}\right)^{r\Delta} \geq \mathrm{growth}(\mathcal{E}_j).$$

This contradicts the growth maximality of $\mathcal{E}_j \subset \mathcal{U}_j$. □

Proof of $\Rightarrow$: Suppose that ϕ is integral algebraic Perron. We will perform a more controlled version of the construction of M than the one outlined in the proof of Proposition 1.4.

$\mathbb{R}^{\mathrm{d}}$ is the direct sum of generalized eigenspaces $E_\lambda \subset \mathbb{R}^{\mathrm{d}}$ associated to the eigenvalues λ of ϕ. The set of eigenvalues of ϕ can be split into algebraic conjugacy classes, each made of roots of an irreducible monic polynomial with integer coefficients. Suppose the polynomials are $p_1, \ldots, p_k$ so that the algebraic conjugacy classes are indexed by $i = 1, \ldots, k$. Fix one such i, and consider the restriction of ϕ to the direct sum E_{p_i} of the generalized eigenspaces associated to the λ within the i-th class,

$$E_{p_i} := \bigoplus_{p_i(\lambda)=0} E_\lambda.$$

(As before, only one is used of each pair of complex conjugates λ and $\overline{\lambda}$.) Pick a Jordan decomposition for $\phi|_{E_{p_i}}$ to present E_{p_i} as a direct sum of Jordan spaces. For $r \in \mathbb{N}$, let

$$m_{i,r} := \max\left\{\mathrm{mult}_{\phi|_{E_{p_i}}}(\lambda, r) : \ p_i(\lambda) = 0\right\}.$$

Naturally, we only need to consider ranks r for which $m_{i,r} > 0$, i.e., $\phi|_{E_{p_i}}$ has datum $[\lambda, r]$ in its Jordan spectrum for some root λ of p_i.

Fix for a moment one such r and consider the linear transformation

$$M_{i,r} = \bigoplus_{1\le j\le m_{i,r}} M_{i,r,j} \tag{3.1}$$

where $M_{i,r,j}$ is a copy of the companion matrix of p_i^r (and thus has Jordan spectrum as described in Example 3). The Jordan spectrum of $M_{i,r}$ can be arranged then into a $m_{i,r} \times d$ array

$$\begin{array}{ccc} [\gamma_1, r] & \dots & [\gamma_d, r] \\ [\gamma_1, r] & \dots & [\gamma_d, r] \\ \dots & \dots & \dots \\ [\gamma_1, r] & \dots & [\gamma_d, r] \end{array}$$

where each datum has associated a Jordan space of $M_{i,r}$. Since $\mathrm{mult}_{\phi|_{E_{p_i}}}(\cdot, r) \le m_{i,r}$, the rank r datums in Jordan spectrum of $\phi|_{E_{p_i}}$ can be arranged in the array as subcolumns starting at the top. Crucially, Perron property `(P)` ensures that the subcolumns are of non-increasing length and create an (inverted) staircase with each row of the form $[\gamma_1, r], \dots, [\gamma_{d'}, r]$ where d' is as in Example 3. (See Examples 4 and 5 for example staircases.) As a result, the direct sum $\mathcal{E}_{i,r,j}$ of the Jordan spaces associated to the jth row of the staircase and the sum $\mathcal{K}_{i,r,j}$ of the Jordan spaces associated to the part of the jth row outside the staircase form a rational subspace $U_{i,r,j} := \mathcal{E}_{i,r,j} \oplus \mathcal{K}_{i,r,j}$ in which $\mathcal{E}_{i,r,j}$ is growth maximal.

Taking

$$M := \bigoplus_{i,r,j} M_{i,r,j} \quad \text{and} \quad \mathcal{E} := \bigoplus_{i,r,j} \mathcal{E}_{i,r,j} \quad \text{and} \quad \mathcal{K} := \bigoplus_{i,r,j} \mathcal{K}_{i,r,j}$$

we see that ϕ is linearly conjugate to $M|_{\mathcal{E}}$ and $\mathcal{E}$ is growth semi-maximal. (Here $\mathcal{V} = \mathbb{R}^N$ and $\mathcal{J} = \mathbb{Z}^N$ for some $N \in \mathbb{N}$.) □

The construction of M in the proof above includes a way of organizing the Jordan spectrum of ϕ into a union of sets (as opposed to lists), each set made of $[\lambda, r]$ where r is fixed and λ ranges over a subset of the full conjugacy class of algebraic integers. The Peronness of ϕ is manifested by each λ in the set dominating the λ' in the complement of the set. We revisit Examples 1 and 2 in this light.

Example 5 Still taking ϕ given by (1.4), Proposition 1.4 is exemplified by ϕ coinciding with the restriction of M_J to $\mathcal{E} = \text{span}_{\mathbb{R}}(e_1, e_2, e_6)$. (The restriction of C_J to $\text{span}_{\mathbb{R}}(e_1, e_2, e_3)$ would not do because this subspace is not invariantly complemented.) That ϕ is not Perron is due to its Jordan spectrum (boxed below), treated as a subset of the spectrum of M, skipping a dominating datum:

$$\begin{array}{lll} \text{rank } 2: & \boxed{[3+\sqrt{2}, 2]} & [3-\sqrt{2}, 2]\,; \\ \text{rank } 1: & [3+\sqrt{2}, 1] & \boxed{[3-\sqrt{2}, 1]} \end{array}$$

or, expressed more succinctly:

$$\text{rank } 2: \quad \boxed{3+\sqrt{2}} \quad 3-\sqrt{2} \qquad \text{rank } 1: \quad 3+\sqrt{2} \quad \boxed{3-\sqrt{2}}$$

To give a positive example, consider a bigger matrix

$$M_{\text{big}} = C \oplus C \oplus C \oplus B \oplus B \tag{3.2}$$

and let ϕ be the restriction of M_{big} to an invariantly complemented subspace $\mathcal{E}$ that is the direct sum of Jordan spaces of the boxed elements of the Jordan spectrum of M_{big}:

$$\text{rank } 2: \quad \begin{array}{cc} \boxed{3+\sqrt{2}} & \boxed{3-\sqrt{2}} \\ \boxed{3+\sqrt{2}} & 3-\sqrt{2} \\ \boxed{3+\sqrt{2}} & 3-\sqrt{2} \end{array} \qquad \text{rank } 1: \quad \begin{array}{cc} \boxed{3+\sqrt{2}} & \boxed{3-\sqrt{2}} \\ \boxed{3+\sqrt{2}} & 3-\sqrt{2} \end{array}$$

To be more specific about $\mathcal{E}$ and verify its semi-maximality, take $\mathcal{U}_j \subset \mathbb{R}^{16}$ that are the subspaces corresponding to the summands in (3.2): $\mathcal{U}_1 := \text{span}_{\mathbb{R}}(e_1, \ldots, e_4)$, $\mathcal{U}_2 := \text{span}_{\mathbb{R}}(e_5, \ldots, e_8), \ldots, \mathcal{U}_5 := \text{span}_{\mathbb{R}}(e_{15}, e_{16})$. Associate to each $\mathcal{U}_j$ a different row (within an appropriate rank) in the diagram above. Taking the sum of Jordan spaces of the boxed datums in the row of $\mathcal{U}_j$ yields a growth maximal subspace of $\mathcal{E}_j \subset \mathcal{U}_j$. Thus $\mathcal{E} := \bigoplus_{j=1}^{5} \mathcal{E}_j$ is growth semi-maximal in $\mathbb{R}^{16}$ with $\dim(\mathcal{E}) = 4+2+2+2+1 = 11$. (Moving a box in the right column of either rank shows that $\mathcal{E}$ is not growth maximal.)

Of course, it is another matter to actually construct a tiling realizing this ϕ.

4 Control points and lattice

The rest of this paper is devoted to the proof of Theorem 1.1 (save for the appendices). We therefore fix a ϕ-self-affine tiling $\mathcal{T}$ satisfying the hypotheses listed in the introduction. The first step is a construction of a lattice $\mathcal{J}$ and an integral linear transformation M preserving that lattice. This goes back to Thurston [26].

Think of the tile types as having different colors indexed by $i = 1, \ldots, m$. By marking a point in each prototile, one can associate to the tiling a set of colored points in $\mathbb{R}^{\mathrm{d}}$. (Each tile in $\mathcal{T}$ contributes a point.) As a consequence of repetitivity and finite local complexity, such a set of colored points is a *Delone multiset*. To be precise, following [12,13,15], by a *Delone multiset* in $\mathbb{R}^{\mathrm{d}}$ we understand an m-tuple $\boldsymbol{\Lambda} = (\Lambda_1, \ldots, \Lambda_m)$ where the individual sets Λ_i as well as their union $\bigcup_i \Lambda_i$ are subsets of $\mathbb{R}^{\mathrm{d}}$ with the *Delone property*, i.e., they are uniformly discrete and relatively dense. We use the notation $i@p$ to denote a point of color i at $p \in \mathbb{R}^{\mathrm{d}}$, allowing us to write

$$\boldsymbol{\Lambda} = \{i@p : \ p \in \Lambda_i, \ i = 1, \ldots, m\}.$$

E.g., $\{1@0\}$ is just a convenient notation for the multiset $(\{0\}, \emptyset, \ldots, \emptyset)$. Note that, with the marking of prototiles fixed, any tiling of $\mathbb{R}^{\mathrm{d}}$ using these prototiles can be recovered from its Delone multiset. This allows one to phrase the theory of such tilings entirely in terms of Delone multisets.

Thurston's original insight was that the ϕ-self-affinity affords a choice of markings so that $\boldsymbol{\Lambda}$ maps under ϕ into itself when treated as an ordinary set (with the colors forgotten), i.e.,

$$\phi\left(\bigcup_i \Lambda_i\right) \subset \bigcup_i \Lambda_i. \tag{4.1}$$

The points of such a special $\boldsymbol{\Lambda}$ are called *control points* for the tiling and their construction bears repeating. For each $j \in \{1, \ldots, m\}$, use primitivity of A to pick $i \in \{1, \ldots, m\}$ such that $D_{i,j} \neq \emptyset$ and select $d \in D_{i,j}$. Denote $\sigma(j) := i$ and $d_{\sigma(j),j} := d$. The control point c of tile $T_j@p$ of $\mathcal{T}$ is obtained by iterating σ:

$$c := p + \phi^{-1} d_{\sigma(j),j} + \phi^{-2} d_{\sigma^2(j),\sigma(j)} + \phi^{-3} d_{\sigma^3(j),\sigma^2(j)} + \cdots . \tag{4.2}$$

To show (4.1), let us see that ϕc is again a control point of a tile of $\mathcal{T}$. Indeed, taking j, i, d as above, the tile $T_i@\phi p + d$ belongs to $\Phi(T_j@p)$ and thus is a tile of $\mathcal{T}$. By another instance of (4.2), the control point of $T_i@\phi p + d$ is

$$\begin{aligned}&\phi p + d + \phi^{-1} d_{\sigma(i),i} + \phi^{-2} d_{\sigma^2(i),\sigma(i)} + \cdots \\ &\quad = \phi\left(p + \phi^{-1} d_{\sigma(j),j} + \phi^{-2} d_{\sigma^2(j),\sigma(j)} + \cdots\right) = \phi c.\end{aligned}$$

To construct a lattice, let $\mathcal{J}$ be the $\mathbb{Z}$-module generated by the control points,

$$\mathcal{J} := \left\{ \sum_k a_k v_k \,:\, a_k \in \mathbb{Z},\ v_k \in \bigcup_{i=1}^{m} \Lambda_i \right\}. \tag{4.3}$$

It is a simple yet pivotal consequence of the finite local complexity hypothesis that there is $R > 0$ such that the control points c and the differences of control points $c - c'$ satisfying $|c|, |c - c'| < R$ already generate $\mathcal{J}$. In particular, $\mathcal{J}$ is a free module of a finite rank:

Proposition 4.1 (Thurston) $N := \operatorname{rank}(\mathcal{J}) < \infty$.

Let $\mathcal{V}$ be the real linear space obtained by extending scalars in $\mathcal{J}$ from $\mathbb{Z}$ to $\mathbb{R}$, that is, $\mathcal{V}$ is the tensor product of $\mathbb{Z}$-modules

$$\mathcal{V} := \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{J}. \tag{4.4}$$

To paraphrase, $\mathcal{V}$ is obtained by taking the linear space of all formal sums $\sum_k a_k v_k$ where $a_k \in \mathbb{R}$ and $v_k \in \mathcal{J}$ and factoring by the subspace generated by the formal sums $\sum_k a_k v_k$ with $a_k \in \mathbb{Z}$ such that $\sum_k a_k v_k = 0$ when evaluated in $\mathbb{R}^{\mathrm{d}}$. (The notation for the coset of the formal sum $\sum_k a_k v_k$ is $\sum_k a_k \otimes v_k$.) $\mathcal{V}$ is finite dimensional of dimension $N \in \mathbb{N}$ and the natural map $v \mapsto 1 \otimes v$ identifies $\mathcal{J}$ with a discrete lattice $\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{J} = 1 \otimes_{\mathbb{Z}} \mathcal{J}$ in $\mathcal{V}$. When convenient, we use this identification of $\mathcal{J}$ and $1 \otimes_{\mathbb{Z}} \mathcal{J}$ without further notice. Unlike in [10], we prefer not to work in terms of any specific integral basis[8] of $\mathcal{J}$, but had we chosen such a basis $v_1, \ldots, v_N$, we would have

$$\mathcal{J} = \left\{ \sum_{k=1}^{N} a_k v_k \,:\, a_k \in \mathbb{Z} \right\} \quad \text{and} \quad \mathcal{V} = \left\{ \sum_{k=1}^{N} a_k (1 \otimes v_k) \,:\, a_k \in \mathbb{R} \right\} \tag{4.5}$$

allowing explicit identification of $\mathcal{J}$ and $\mathcal{V}$ with $\mathbb{Z}^N$ and $\mathbb{R}^N$, respectively.

We refer to the $\mathbb{R}^{\mathrm{d}}$ containing Λ as the *physical space* of the tiling $\mathcal{T}$ and to the $\mathcal{V}$ as the *mathematical space* of $\mathcal{T}$.

The embedding $\mathcal{J} \to \mathcal{V}$ has a one-sided inverse $\pi_{\mathbb{R}^{\mathrm{d}}} : \mathcal{V} \to \mathbb{R}^{\mathrm{d}}$ given on the simple tensors by

$$\pi_{\mathbb{R}^{\mathrm{d}}}(1 \otimes v) := v, \tag{4.6}$$

[8] I.e. generators of the free abelian group $\mathcal{J}$.

which is to say that

$$\pi_{\mathbb{R}^{\mathrm{d}}}\left(\sum_k a_k \otimes v_k\right) = \pi_{\mathbb{R}^{\mathrm{d}}}\left(\sum_k a_k(1 \otimes v_k)\right) := \sum_k a_k v_k. \tag{4.7}$$

Let $\mathcal{K}$ be its kernel

$$\mathcal{K} = \left\{x \in \mathcal{V} :\ \pi_{\mathbb{R}^{\mathrm{d}}}(x) = 0\right\}. \tag{4.8}$$

Since all elements of $1 \otimes_{\mathbb{Z}} \mathcal{J}$ are simple tensors, $\pi_{\mathbb{R}^{\mathrm{d}}}$ restricted to $1 \otimes_{\mathbb{Z}} \mathcal{J} \to \mathcal{J}$ is injective and

$$\mathcal{K} \cap (1 \otimes_{\mathbb{Z}} \mathcal{J}) = \{0\}. \tag{4.9}$$

Due to (4.1), we have $\phi(\mathcal{J}) \subset \mathcal{J}$ and the map $\phi : \mathcal{J} \to \mathcal{J}$ induces a linear transformation $M : \mathcal{V} \to \mathcal{V}$ given on the simple tensors by

$$M(1 \otimes v) := 1 \otimes (\phi v). \tag{4.10}$$

By construction, M factors to ϕ via $\pi_{\mathbb{R}^{\mathrm{d}}}$,

$$\pi_{\mathbb{R}^{\mathrm{d}}} \circ M = \phi \circ \pi_{\mathbb{R}^{\mathrm{d}}}. \tag{4.11}$$

It follows that the eigenvalues of ϕ are among the eigenvalues of M, so ϕ is integral algebraic, satisfies `(IA)`. However, we will have to wait until Sect. 11 before we identify a linear subspace $\mathcal{E} \subset \mathcal{V}$ so that $\mathcal{V} = \mathcal{E} \oplus \mathcal{K}$ is an M-invariant splitting with $M|_{\mathcal{E}}$ linearly conjugate to ϕ (as stipulated in Proposition 1.4). Existence of such $\mathcal{E}$ is automatic in the diagonalizable case.

5 Substitution Delone set and hull

Compared to [10], our argument will explicitly invoke the hull of the tiling and make an extensive use of the self-map of that hull induced by the inflation-substitution map Φ. Again, we phrase everything in terms of the Delone multiset $\mathbf{\Lambda}$ of the control points.

The hull $\mathcal{X}$ (going back at least to [22]) is typically defined as the completion of the set of all translates $\{\mathbf{\Lambda} + t :\ t \in \mathbb{R}^{\mathrm{d}}\}$ metrized so that two multisets are close if they agree up to a small translation on a large ball about the origin (see e.g. [14,21]). Constructing a suitable metric is not without pitfalls and we like the approach using the spherical metric $\mathbf{d}$ on $\mathbb{R}^{\mathrm{d}}$. This is the metric induced on $\mathbb{R}^{\mathrm{d}}$ identified with the unit sphere $\mathbb{S}^{\mathrm{d}} \subset \mathbb{R}^{\mathrm{d}+1}$ (sans a point) via the usual stereographic projection. Actually, any other metric on $\mathbb{R}^{\mathrm{d}}$ that induces the standard topology and with completion that is a one point compactification of $\mathbb{R}^{\mathrm{d}}$ would do, e.g., $\mathbf{d}(x, y) := \min\{|x - y|, 1/(1 + |x|) + 1/(1 + |y|)\}$ (see

[18]). Taking $\mathbf{d}_{\mathrm{H}}$ to be the Hausdorff metric induced by $\mathbf{d}$ on the space of all closed subsets of $\mathbb{R}^{\mathrm{d}}$, the distance between two closed multisets is

$$\mathbf{d}_D((\Lambda_i), (\Lambda_i')) := \max_i \mathbf{d}_{\mathrm{H}}(\Lambda_i, \Lambda_i'). \tag{5.1}$$

The topology induced by $\mathbf{d}_D$ is basically that of uniform convergence on compact subsets (of $\mathbb{R}^{\mathrm{d}}$). The hull can be then defined as the following closure in the space of all m-tuples of closed subsets of $\mathbb{R}^{\mathrm{d}}$

$$\mathcal{X} := \mathrm{cl}\{\mathbf{\Lambda} + t : \ t \in \mathbb{R}^{\mathrm{d}}\}. \tag{5.2}$$

(This closure would be the same had we used the standard metric given in [14]; see Appendix B, Proposition 19.1.)

The important point for us is that $\mathcal{X}$ is a compact (Hausdorff) space and that translating Delone multisets induces a continuous $\mathbb{R}^{\mathrm{d}}$-action on $\mathcal{X}$ which (by virtue of repetitivity) is *minimal*, i.e., for every $\mathbf{x} \in \mathcal{X}$, its orbit $\{\mathbf{x}+t : t \in \mathbb{R}^{\mathrm{d}}\}$ is dense in $\mathcal{X}$. Furthermore, we shall use that this action is *uniquely ergodic* (i.e., it has only one invariant Borel probability measure). This is shown in [25]; see also [11,21].

Now, let us turn to the inflation-substitution on multisets. If we replace tiles by their control points, (1.1) dictates that a single (colored) point in $\mathbb{R}^{\mathrm{d}}$ transforms into a finite multiset as follows

$$\Phi(j@p) := \left\{i@(\phi p + d) : d \in D_{i,j}\right\} = i@\phi p + D_{i,j}. \tag{5.3}$$

(We overloaded the notation Φ.) We can apply the above formula point by point to obtain a version of the *inflation-substitution* acting on an arbitrary multiset $\mathbf{\Lambda}'$ in $\mathbb{R}^{\mathrm{d}}$; it reads

$$\Phi(\mathbf{\Lambda}') = \left(\bigcup_{j=1}^{m} \phi\Lambda_j' + D_{i,j}\right)_{i=1}^{m}. \tag{5.4}$$

From $\Phi(\mathcal{T}) = \mathcal{T}$, we see that $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$, which is to say that

$$\Lambda_i = \bigcup_{j=1}^{m} \phi\Lambda_j + D_{i,j} \quad (i = 1, \ldots, m). \tag{5.5}$$

This makes $\mathbf{\Lambda}$ a *substitution Delone multiset* in the sense of [16], which is a slight narrowing of the original definition proposed in [13][9].

[9] Lagarias and Wang allow sets with multiplicity for the sake of dealing with *multiple tilings* (*over-tilings*), a level of generality we do not need here.

Note that $\Phi(\mathbf{\Lambda} + t) = \mathbf{\Lambda} + \phi t$ for $t \in \mathbb{R}^{\mathrm{d}}$, rendering a frequently used identity

$$\Phi(\mathbf{x} + t) = \Phi(\mathbf{x}) + \phi t \quad (t \in \mathbb{R}^{\mathrm{d}},\ \mathbf{x} \in \mathcal{X}). \tag{5.6}$$

In particular, Φ maps the $\mathbb{R}^{\mathrm{d}}$-orbit of $\mathbf{\Lambda}$ onto itself and, restricted to $\mathcal{X}$, is a surjective self-map of $\mathcal{X}$. Under the additional hypothesis that $\mathbf{\Lambda}$ is *aperiodic* (i.e., $\mathbf{x} + t \neq \mathbf{x}$ for all $t \in \mathbb{R}^{\mathrm{d}} \backslash \{0\}$ and $\mathbf{x} \in \mathcal{X}$), Φ is known to be a self-homeomorphism of $\mathcal{X}$ by the *recognizability theorem* in [24] (see also [11]). Although we do not assume aperiodicity in Theorem 1.1, with other applications in mind, we will pay special attention to the aperiodic case where it differs from the general case.

6 Lifting

As we already indicated [10] rests on *lifting (un-projecting)* the multiset $\mathbf{\Lambda}$ in the *physical space* $\mathbb{R}^{\mathrm{d}}$ to a multiset $\tilde{\mathbf{\Lambda}}$ in the *mathematical space* $\mathcal{V} \simeq \mathbb{R}^N$. (This idea is already present in earlier works, see e.g. [12] and the references therein.) We propose a natural addition to this scheme whereby the inflation-substitution map Φ is *lifted* to a map $\tilde{\Phi}$ that acts on multisets living in $\mathcal{V}$ and fixes a lifted multiset $\tilde{\mathbf{\Lambda}}$. As we did before in $\mathbb{R}^{\mathrm{d}}$, the space of all closed multisets in $\mathcal{V}$ is taken with the topology of uniform convergence on compact subsets (as induced by the analogue of the metric $\mathbf{d}_D$ given in (5.1)).

For $v \in \mathbb{R}^{\mathrm{d}}$, by a *lift* of v we understand any $\tilde{v} \in \mathcal{V}$ that projects to v along $\mathcal{K}$, $\pi_{\mathbb{R}^{\mathrm{d}}}(\tilde{v}) = v$. Note that, for any $v \in \mathbb{R}^{\mathrm{d}}$, no two lifts differ by a non-zero vector in $1 \otimes_{\mathbb{Z}} \mathcal{J}$ because $\pi_{\mathbb{R}^{\mathrm{d}}} : 1 \otimes_{\mathbb{Z}} \mathcal{J} \to \mathcal{J}$ is injective (by (4.9)). If $v \in \mathcal{J} \subset \mathbb{R}^{\mathrm{d}}$, the *canonical lift* of v is the $\tilde{v}$ given by the natural identification of $\mathcal{J}$ and $1 \otimes_{\mathbb{Z}} \mathcal{J} \subset \mathcal{V}$,

$$\tilde{v}_{\text{canonical}} = 1 \otimes v \in \mathcal{V}. \tag{6.1}$$

Collecting the canonical lifts for each $i @ p \in \mathbf{\Lambda}$ yields the multiset

$$\tilde{\mathbf{\Lambda}} := \{i @ \tilde{p}_{\text{canonical}} :\ i @ p \in \mathbf{\Lambda}\}, \tag{6.2}$$

to which we refer as the *canonical lift* of $\mathbf{\Lambda}$.

We should record a well known (see e.g. Lemma 3.2 in [10]) consequence of the finite local complexity of $\mathbf{\Lambda}$: for any norms on $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{V}$, the canonical lifting is L-Lipschitz on $\mathbf{\Lambda}$ for some $L > 0$,

$$|1 \otimes v - 1 \otimes v'| \leq L|v - v'| \quad (i @ v, i' @ v' \in \mathbf{\Lambda}). \tag{6.3}$$

(L depends on the choice of the norms in $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{V}$, which we will make only in the next section.) Furthermore, for any $t \in \mathbb{R}^{\mathrm{d}}$, upon choosing its lift $\tilde{t} \in \mathcal{V}$, the multiset $\tilde{\mathbf{x}} = \tilde{\mathbf{\Lambda}} + \tilde{t}$ not only projects to $\mathbf{x}$ via $\pi_{\mathbb{R}^d}$, $\pi_{\mathbb{R}^d}(\tilde{\mathbf{x}}) = \mathbf{x}$, but also enjoys the following *lattice displacement property*:

$$\{\tilde{p} - \tilde{q} \,:\, i@\tilde{p},\, j@\tilde{q} \in \tilde{\mathbf{x}}\} \subset 1 \otimes_{\mathbb{Z}} \mathcal{J}. \tag{6.4}$$

Fact 6.1 *For any Delone multiset* $\mathbf{x}$ *in the hull* $\mathcal{X}$*, there is a multiset* $\tilde{\mathbf{x}}$ *in* $\mathcal{V}$ *so that* $\pi_{\mathbb{R}^d}(\tilde{\mathbf{x}}) = \mathbf{x}$ *and (6.4) holds.*

Proof All one has to observe is that, because $\mathbf{x} \in \mathcal{X}$ is a limit of translates of $\mathbf{\Lambda}$, it satisfies $\{p - q \in \mathcal{J} \,:\, i@p, j@q \in \mathbf{x}\}$. (To see this use that $i@(p - t_k + \tau)$, $j@(q - t_k + \tau)$ belong to $\mathbf{\Lambda}$ for some $\tau \in \mathbb{R}^d$ if $\mathbf{\Lambda} + t_k$ is sufficiently close to $\mathbf{x}$. See (i) of Proposition 19.1.) Thus, upon fixing any lift $\tilde{q}$ of q, the $\tilde{\mathbf{x}}$ satisfying (6.4) is constructed by lifting each $i@p \in \mathbf{x}$ to $i@\tilde{p}$ where $\tilde{p} := \tilde{q} + \widetilde{(p - q)}_{\text{canonical}}$. □

We shall refer to $\tilde{\mathbf{x}}$ satisfying the assertion of Fact 6.1 simply as *lifts* of $\mathbf{x}$. (So (6.4) is tacitly assumed.) Note that any two lifts $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ of $\mathbf{x}$ must coincide up to a <u>common</u> translation along $\mathcal{K}$, i.e., $\tilde{\mathbf{x}}_2 = \tilde{\mathbf{x}}_1 + s$ for some $s \in \mathcal{K}$.

Fact 6.2

$$D_{i,j} \subset \mathcal{J} \quad (\forall i, j). \tag{6.5}$$

Proof Let $d \in D_{i,j}$. Take some $j@p \in \mathbf{\Lambda}$, i.e., $p \in \mathbf{\Lambda}_j$. Then $i@(\phi p + d) \in \mathbf{\Lambda}$ by $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$ and the definition of Φ. Thus $\phi p \in \phi \Lambda_j \subset \phi \mathcal{J} \subset \mathcal{J}$ and $\phi p + d \in \Lambda_i \subset \mathcal{J}$ so that $d = (\phi p + d) - \phi p$ is in $\mathcal{J}$ as well. □

Fact 6.2 allows using canonical lifting to form *lifted digit sets*

$$\tilde{D}_{i,j} := \{1 \otimes d \,:\, d \in D_{i,j}\} \subset 1 \otimes_{\mathbb{Z}} \mathcal{J} \tag{6.6}$$

Clearly, $\pi_{\mathbb{R}^d}(\tilde{D}_{i,j}) = D_{i,j}$ and any two points in $\bigcup_{i,j} \tilde{D}_{i,j}$ differ by a vector in the lattice $1 \otimes_{\mathbb{Z}} \mathcal{J}$. Let $\tilde{\Phi}$ be the map on multisets in $\mathcal{V}$ given on individual (colored) points by

$$\tilde{\Phi}(j@\tilde{p}) = \{i@(M\tilde{p} + \tilde{d}) \,:\, \tilde{d} \in \tilde{D}_{i,j}\}. \tag{6.7}$$

By comparing this with (5.3), we see that

$$\pi_{\mathbb{R}^d} \circ \tilde{\Phi} = \Phi \circ \pi_{\mathbb{R}^d} \tag{6.8}$$

where the projection $\pi_{\mathbb{R}^d} : \mathcal{V} \to \mathbb{R}^d$ is acting on multisets point-by-point. [10]

[10] To avoid notational glut, we shall frequently overload the notation if its meaning is clear from the context. In particular, we use the same letter for a map on points and the induced mapping on multisets.

Because $\tilde{\Phi}(\tilde{\mathbf{\Lambda}})$ is a lift of $\Phi(\mathbf{\Lambda}) = \mathbf{\Lambda}$ and $\tilde{\Phi}(\tilde{\mathbf{\Lambda}})$ lives in $1 \otimes_{\mathbb{Z}} \mathcal{J}$ (since $M\tilde{\mathbf{\Lambda}}$ and the $\tilde{D}_{i,j}$ do), $\tilde{\Phi}(\tilde{\mathbf{\Lambda}})$ is the canonical lift of $\mathbf{\Lambda}$ and thus coincides with $\tilde{\mathbf{\Lambda}}$,

$$\tilde{\Phi}(\tilde{\mathbf{\Lambda}}) = \tilde{\mathbf{\Lambda}}. \tag{6.9}$$

7 Stable/unstable positioning

We saw (Fact 6.1) that any multiset $\mathbf{x} \in \mathcal{X}$ can be lifted to a multiset $\tilde{\mathbf{x}}$ in $\mathcal{V}$ but $\tilde{\mathbf{x}}$ was only determined up to a translation along $\mathcal{K}$. The *lifted inflation-substitution map* $\tilde{\Phi}$ will allow us to go some way towards removing this ambiguity.

Here we come to a point in our exposition where a slightly different treatment is due depending on whether the inflation-substitution induced map $\Phi : \mathcal{X} \to \mathcal{X}$ is continuously invertible or not. For the proof of Theorem 1.1, we do not need invertibility as all the necessary structure can be put in place by using only the forward iterates $(\Phi^n)_{n=0}^{\infty}$. However, a much more complete picture arises when positive and negative iterates $(\Phi^n)_{n=-\infty}^{\infty}$ are available. As we already mentioned, Φ is invertible (a homeomorphism) under the aperiodicity hypothesis. Because aperiodic tilings are of prime interest and the arguments in the non-invertible case amount to replacing $\mathbb{Z}$ by $\mathbb{N}_0 := \{k \in \mathbb{Z} : k \geq 0\}$ in the invertible case, we provide proofs for the latter and only occasionally comment on how the non-invertible case differs.

To start, let

$$\mathcal{V} = \mathcal{V}^s \oplus \mathcal{V}^u \oplus \mathcal{V}^c \oplus \mathcal{V}^0 \tag{7.1}$$

be the splitting into the *stable, unstable, central*, and *eventual kernel* subspaces associated to the linear endomorphism M, as given by the direct sums of the (real) generalized eigenspaces with the eigenvalues λ that satisfy, respectively, $|\lambda| \in (0, 1)$, $|\lambda| \in (1, \infty)$, $|\lambda| = 1$, and $|\lambda| = 0$. For $v \in \mathcal{V}$, its splitting per (7.1) is expressed as $v = v^s + v^u + v^c + v^0$. We shall also write M_u for the restriction $M|_{\mathcal{V}^u} : \mathcal{V}^u \to \mathcal{V}^u$, and give the analogous meaning to M_s and M_c. Additionally, $M_{su} := M_s \oplus M_u$, etc. For instance, M_{suc} stands for the automorphism of $\mathcal{V}^{suc} := \mathcal{V}^s \oplus \mathcal{V}^u \oplus \mathcal{V}^c$ induced by M.

For geometric considerations we will have to use norms on $\mathcal{V}$ and $\mathbb{R}^{\mathrm{d}}$, which we choose as follows. We fix any norm $|\cdot|_{c0}$ on $\mathcal{V}^{c0}$. On $\mathcal{V}^s$ and $\mathcal{V}^u$ we choose *adapted norms* $|\cdot|_s$ and $|\cdot|_u$, i.e., norms making the operator norms of M_s and M_u^{-1} less than one, $\|M_s\|$, $\|M_u^{-1}\| < 1$. The norm on $\mathcal{V}$ is then taken as $|v| := |v^{c0}|_{c0} + |v^s|_s + |v^u|_u$. On $\mathbb{R}^{\mathrm{d}} \simeq \mathcal{V}/\mathcal{K}$ we take the quotient norm, simply denoted by $|\cdot|$ since there will be no risk of confusion. Observe that (since $V^{sc0} \subset \mathcal{K}$) this norm is adapted to ϕ, i.e., $\|\phi^{-1}\| < 1$. This means that ϕ expands by at least $\lambda_\phi := \|\phi^{-1}\|^{-1} > 1$; namely, $|\phi t| > \lambda_\phi |t|$ for all $t \in \mathbb{R}^{\mathrm{d}} \backslash \{0\}$. Also $|\pi_{\mathbb{R}^{\mathrm{d}}}(v)| \leq |v|$ for any $v \in \mathcal{V}$.

Consider a Delone multiset $\mathbf{x} \in \mathcal{X}$. Assuming Φ is invertible, let $(\mathbf{x}_n)_{n=-\infty}^{\infty}$ be the bi-infinite Φ-orbit of $\mathbf{x}$ in $\mathcal{X}$, i.e.,

$$\mathbf{x}_0 = \mathbf{x} \quad \text{and} \quad \Phi(\mathbf{x}_n) = \mathbf{x}_{n+1} \quad (n \in \mathbb{Z}). \tag{7.2}$$

Let $\tilde{\mathbf{x}}_n$ be a lift of $\mathbf{x}_n$. Because $\tilde{\Phi}(\tilde{\mathbf{x}}_n)$ is a lift of $\Phi(\mathbf{x}_n) = \mathbf{x}_{n+1}$, there are $t_n \in \mathcal{K}$ such that

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n) = \tilde{\mathbf{x}}_{n+1} + t_{n+1} \quad (n \in \mathbb{Z}). \tag{7.3}$$

Furthermore, for any choice of $\tau_n \in \mathcal{K}$, we have

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n + \tau_n) = \tilde{\mathbf{x}}_{n+1} + M\tau_n + t_{n+1} \quad (n \in \mathbb{Z}). \tag{7.4}$$

By determining $\tau_n^u \in \mathcal{V}^u$ and $\tau_n^s \in \mathcal{V}^s$ from non-singular linear systems

$$(M_u - I)\tau_n^u = -t_{n+1}^u \quad \text{and} \quad (M_s - I)\tau_n^s = -t_{n+1}^s, \tag{7.5}$$

then picking $\tau_n^{c0} \in \mathcal{V}^{c0}$ arbitrarily and setting $\tau_n := \tau_n^s + \tau_n^u + \tau_n^{c0}$, we can replace $\tilde{\mathbf{x}}_n$ by $\tilde{\mathbf{x}}_n + \tau_n$ to secure

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n)^{su} = \tilde{\mathbf{x}}_{n+1}^{su} \quad (n \in \mathbb{Z}). \tag{7.6}$$

Here the superscript indicates the $\mathcal{V}^{su}$ component, so this means $\tilde{\Phi}(\tilde{\mathbf{x}}_n) = \tilde{\mathbf{x}}_{n+1} \pmod{\mathcal{V}^{c0}}$. We shall refer to the sequences $(\tilde{\mathbf{x}}_n)_{n=-\infty}^{\infty}$ satisfying (7.6) as $\tilde{\Phi}^{su}$-*orbits*. When $\pi_{\mathbb{R}^d}(\tilde{\mathbf{x}}_n) = \mathbf{x}_n$ for all n — as is the case above — the $\tilde{\Phi}^{su}$-orbit $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ is called a *lift* of the Φ-orbit $(\mathbf{x}_n)_{-\infty}^{\infty}$. Note that the lift $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ of $(\mathbf{x}_n)_{-\infty}^{\infty}$ is not unique but all other lifts are (mod $\mathcal{V}^{c0}$) of the from

$$(\tilde{\mathbf{x}}_n + M_{su}^n \tau)_{-\infty}^{\infty} \quad \text{where } \tau \in \mathcal{K} \cap \mathcal{V}^{su}. \tag{7.7}$$

Diverting attention for a moment to the case when Φ is not invertible, (7.6) is replaced by

$$\tilde{\Phi}(\tilde{\mathbf{x}}_n)^u = \tilde{\mathbf{x}}_{n+1}^u \quad (n \in \mathbb{N}_0), \tag{7.8}$$

we speak of $(\tilde{\mathbf{x}}_n)_{n=0}^{\infty}$ as $\tilde{\Phi}^u$-*orbits*, and all other lifts are as in (7.7) but are indexed by $n \geq 0$, considered mod $\mathcal{V}^{sc0}$, and with the free parameter $\tau \in \mathcal{K} \cap \mathcal{V}^u$.

Our next goal is to find, for each Φ-orbit, a distinguished parameter τ (and thus a distinguished lift) by means of the following central definition. (Below, the distance from the origin to a multiset $\mathbf{y}$ is $\operatorname{dist}(0, \mathbf{y}) := \min\{|p| : i@p \in \mathbf{y}\}$.)

Definition 7.1 A sequence $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ of multisets in $\mathcal{V}$ is *well su-positioned* iff

$$\sup_{n\in\mathbb{Z}} \operatorname{dist}\left(0^{su}, \tilde{\mathbf{x}}_n^{su}\right) < +\infty.$$

A multiset $\tilde{\mathbf{x}}$ in $\mathcal{V}$ is *well su-positioned* iff $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0$ where $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ is a well su-positioned $\tilde{\Phi}^{su}$-orbit. Likewise, $(\tilde{\mathbf{x}}_n)_0^{\infty}$ is *well u-positioned* iff $\sup_{n\in\mathbb{N}_0} \operatorname{dist}\left(0^u, \tilde{\mathbf{x}}_n^u\right) < +\infty$; and $\tilde{\mathbf{x}}$ is *well u-positioned* iff $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0$ where $(\tilde{\mathbf{x}}_n)_0^{\infty}$ is a well u-positioned $\tilde{\Phi}^u$-orbit.

To rephrase, an orbit $(\tilde{\mathbf{x}}_n)$ is *well su/u-positioned* iff each multiset $\tilde{\mathbf{x}}_n$ has some points within certain fixed distance from the origin when projected into the appropriate space ($\mathcal{V}^{su}$ or $\mathcal{V}^u$).

Lemma 7.2 *For any bi-infinite Φ-orbit $(\mathbf{x}_n)_{-\infty}^{\infty}$, there exists a well su-positioned lift $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ and such a lift is unique up to a translation along $\mathcal{V}^{c0}$. Moreover, $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ depends continuously on $(\mathbf{x}_n)_{-\infty}^{\infty}$ (when using the product topologies on sequences of multisets). The analogous statement also holds for well u-positioned lifts of one-sided orbits $(\mathbf{x}_n)_0^{\infty}$ of non-invertible Φ (but these lifts are only unique up to a translation along $\mathcal{V}^{sc0}$).*

The proof of the lemma (below) hinges on basic facts about non-stationary bounded additive perturbations of a hyperbolic map, M_{su} or M_u. In the argument we use the *central R-patch* of $\mathbf{x} \in \mathcal{X}$, denoted $\mathbf{x}|_R$, defined by $\mathbf{x}|_R := \{j@p \in \mathbf{x} : \ |p| < R\}$. Likewise, for a multiset $\tilde{\mathbf{x}}$ in $\mathcal{V}$, its *central R-patch* is $\tilde{\mathbf{x}}|_R := \{j@\tilde{p} \in \tilde{\mathbf{x}} : |\pi_{\mathbb{R}^d}(\tilde{p})| < R\}$. (In Sect. 9, to streamline arguments, we will introduce a slightly different definition of the *central R-patch* that refers back to the tiling.)

Proof of Lemma 7.2 We deal with the harder invertible case and start with the existence. Set $C_\Phi := \sup\{|\tilde{d}| : \ \tilde{d} \in \bigcup_{i,j} \tilde{D}_{i,j}\}$. Take $R_\Phi > 0$ large enough that the central R_Φ-patch of any $\mathbf{x} \in \mathcal{X}$ is non-empty and *maps over* the central R_Φ-patch of $\Phi(\mathbf{x})$, i.e., if $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $\mathbf{y} = \Phi(\mathbf{x})$ then there exist $j@q \in \mathbf{y}$ with $|q| < R_\Phi$ and, for each such $j@q$, one can find $i@p \in \mathbf{x}$ with $|p| < R_\Phi$ so that $\Phi(i@p)$ contains $j@q$. (It suffices that $(\|\phi^{-1}\|^{-1} - 1)R_\Phi > C_\Phi$.)

We fix a Φ-orbit $(\mathbf{x}_n)_{-\infty}^{\infty}$ and its lift $(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$. By the choice of R_Φ, for any $N \in \mathbb{N}$, there is an infinite sequence of points $i_N@p_N \in \mathbf{x}_N$, $i_{N-1}@p_{N-1} \in \mathbf{x}_{N-1}, \ldots$, called a selection, such that, for all $n \le N$, we have $|p_n| < R_\Phi$ and $\Phi(i_{n-1}@p_{n-1})$ contains $i_n@p_n$. By taking a subsequential limit of such selections with $N \to \infty$, we get $i_n@p_n \in \mathbf{x}_n$ for all $n \in \mathbb{Z}$ such that $|p_n| \le R_\Phi$ and $\Phi(i_n@p_n)$ contains $i_{n+1}@p_{n+1}$. (To take the limit, artificially extend each selection $(i_n@p_n)_{n\le N}$ to a bi-infinite sequence $(i_n@p_n)_{n\in\mathbb{Z}}$ where, say, $p_n = 0$ and $i_n = 1$ for $n > N$, and then use the product topology on $(\{1,\ldots,m\} \times \{p : |p| \le R_\Phi\})^{\mathbb{Z}}$.) Keeping in mind that

$(\tilde{\mathbf{x}}_n)_{-\infty}^{\infty}$ is a $\tilde{\Phi}^{su}$-orbit, the corresponding lifts $i_n@\tilde{p}_n \in \tilde{\mathbf{x}}_n$ are such that $|\pi_{\mathbb{R}^d}(\tilde{p}_n)| = |p_n| \le R_\Phi$ and $\tilde{\Phi}(i_n@\tilde{p}_n)$ contains $i_{n+1}@\tilde{p}_{n+1}$ for all $n \in \mathbb{Z}$. In particular, recalling the form of $\tilde{\Phi}$ (given by (5.4)), there are $\tilde{d}_n^{su} \in \mathcal{V}^{su}$ with $\sup_{n\in\mathbb{Z}} |\tilde{d}_n^{su}| \le C_\Phi < \infty$ such that

$$\tilde{p}_{n+1}^{su} = M_{su}\tilde{p}_n^{su} + \tilde{d}_n^{su} \quad (n \in \mathbb{Z}).$$

This is to say that the bi-infinite sequence $(s_n)_{-\infty}^{\infty} := (\tilde{p}_n^{su})_{-\infty}^{\infty}$ in $\mathcal{V}^{su}$ satisfies the following recurrence equation

$$s_{n+1} = M_{su}s_n + \tilde{d}_n^{su} \quad (n \in \mathbb{Z}). \tag{7.9}$$

Crucially, due to the hyperbolicity of M_{su}, (7.9) has a unique solution $(s_n^*)_{-\infty}^{\infty} \subset \mathcal{V}^{su}$ that is bounded, i.e., $\sup_{n\in\mathbb{Z}} |s_n^*| < \infty$. (Explicitly, $s_n^* = s_n^s + s_n^u$ with $s_n^s := \sum_{k=0}^{\infty} M_s^k \tilde{d}_{n-k-1}^s$ and $s_n^u := \sum_{k=0}^{\infty} -M_u^{-k-1}\tilde{d}_{n+k}^u$, geometrically convergent series.) Note that there is $C'_\Phi > 0$ depending on M_{su} and C_Φ only, such that $\sup_{n\in\mathbb{Z}} |s_n| < C'_\Phi$, whenever this sup is finite. Because the difference of any two solutions to (7.9) is just a bi-infinite orbit of M_{su}, there is $\tau \in \mathcal{V}^{su}$ such that $s_n^* := \tilde{p}_n^{su} + M_{su}^n\tau$. As both $p_n = \pi_{\mathbb{R}^d}(\tilde{p}_n)$ and $\pi_{\mathbb{R}^d}(s_n^*)$ form bounded sequences in $\mathbb{R}^d$, the same is true of $\pi_{\mathbb{R}^d}(M_{su}^n\tau) = \phi^n\pi_{\mathbb{R}^d}(\tau)$, so that $\pi_{\mathbb{R}^d}(\tau) = 0$ and thus $\tau \in \mathcal{K}$. This means that $(\tilde{\mathbf{x}}_n + M_{su}^n\tau)_{-\infty}^{\infty}$ is a lift of $(\mathbf{x}_n)_{-\infty}^{\infty}$. This lift is well su-positioned because $i_n@s_n^* = i_n@(\tilde{p}_n^{su} + M_{su}^n\tau)$ are within C'_Φ from the origin in $\mathcal{V}^{su}$ so that $\mathrm{dist}(0^{su}, (\tilde{\mathbf{x}}_n + M_{su}^n\tau)^{su}) \le C'_\Phi$ for all $n \in \mathbb{Z}$.

Now that the existence is established, the uniqueness of a well su-positioned lift is an easy exercise (based on the uniqueness of a bounded solution (s_n^*) above).

As for the continuity of the su-well positioned lifting, one can argue as follows. Since we are using the product topology on sequences, two orbits $(\mathbf{x}_n)_{-\infty}^{\infty}$ and $(\mathbf{x}'_n)_{-\infty}^{\infty}$ are near iff $\mathbf{d}_D(\mathbf{x}_n, \mathbf{x}'_n)$ is small whenever $-N \le n \le N$ for some large $N \in \mathbb{N}$. This means (via Proposition 19.1) that the central R-patches of $\mathbf{x}_n$ and $\mathbf{x}'_n$ coincide up to a translation by no more than $1/R$ for some large $R > R_\Phi$. Denoting those patches by $\mathbf{x}'_n|_R$ and $\mathbf{x}_n|_R$, we can express this as $\mathbf{x}'_n|_R = \mathbf{x}_n|_R + \kappa_n$ where $|\kappa_n| < 1/R$. (Note that $\kappa_{n+1} = \phi\kappa_n$ because $\mathbf{x}'_{n+1}|_R = \mathbf{x}_{n+1}|_R + \phi\kappa_n = \mathbf{x}_{n+1}|_R + \kappa_{n+1}$.)

It suffices to show that, given any $N_1 \in \mathbb{N}$ and $R > R_\Phi$, once N is large enough we have the analogous proximity of the well su-positioned lifts, i.e., $\tilde{\mathbf{x}}_n'^{su}|_R = \tilde{\mathbf{x}}_n^{su}|_R + \tilde{\kappa}_n$ for some $\tilde{\kappa}_n \in \mathcal{V}^{su}$ with $|\tilde{\kappa}_n| < 1/R$ and all $-N_1 \le n \le N_1$.

We certainly have $\tilde{\mathbf{x}}_n'^{su}|_R = \tilde{\mathbf{x}}_n^{su}|_R + \tilde{\kappa}_n$ where $\tilde{\kappa}_n \in \mathcal{V}^{su}$ are some lifts of the κ_n, $\pi_{\mathbb{R}^d}(\tilde{\kappa}_n) = \kappa_n$. Additionally, by the su-well positioning, $\mathrm{dist}(0^{su}, \tilde{\mathbf{x}}_n^{su})$

and $\text{dist}(0^{su}, \tilde{\mathbf{x}}_n'^{su})$ are uniformly bounded (by C'_Φ) and thus so are the $|\tilde{\kappa}_n|$; precisely, $|\tilde{\kappa}_n| \leq C := 2C'_\Phi + 2R + 2/R < \infty$. (All that for $-N \leq n \leq N$.)

Finally, $\tilde{\mathbf{x}}^{su}_{n+1}|_R$ is contained in $\tilde{\Phi}_{su}(\tilde{\mathbf{x}}^{su}_n|_R)$ and $\tilde{\mathbf{x}}^{su}_{n+1}|_R + \tilde{\kappa}_{n+1} = \tilde{\mathbf{x}}'^{su}_{n+1}|_R$ is contained in $\tilde{\Phi}_{su}(\tilde{\mathbf{x}}'^{su}_n|_R) = \tilde{\Phi}_{su}(\tilde{\mathbf{x}}^{su}_n|_R + \tilde{\kappa}_n) = \tilde{\Phi}_{su}(\tilde{\mathbf{x}}^{su}_n|_R) + M_{su}\tilde{\kappa}_n$. Hence, we must have $\tilde{\kappa}_{n+1} = M_{su}\tilde{\kappa}_n$ for $-N \leq n \leq N$. (We used that $\tilde{\mathbf{x}}^{su}_{n+1}|_R$ is non-empty and $\pi_{\mathbb{R}^d}(\tilde{\kappa}_{n+1} - M_{su}\tilde{\kappa}_n) = \kappa_{n+1} - \phi\kappa_n = 0$.) This together with $|\tilde{\kappa}_n| \leq C$, implies that $|\tilde{\kappa}_n| \leq 1/R$ for all $-N_1 \leq n \leq N_1$ provided N was selected large enough for the given N_1 and R. □

Definition 7.3 The *well u-positioned lift* of $\mathbf{x} \in \mathcal{X}$ is the multiset $\tilde{\mathbf{x}} := \tilde{\mathbf{x}}_0$ in $\mathcal{V}^u$ where $(\tilde{\mathbf{x}}_n)_0^\infty$ is the unique well u-positioned lift of the orbit $(\Phi^n(\mathbf{x}))_0^\infty$, as given by Lemma 7.2. When Φ-is invertible, the *well su-positioned lift* of $\mathbf{x} \in \mathcal{X}$ is the multiset $\tilde{\mathbf{x}} := \tilde{\mathbf{x}}_0$ in $\mathcal{V}^{su}$ where $(\tilde{\mathbf{x}}_n)_{-\infty}^\infty$ is the unique well su-positioned lift of the orbit $(\Phi^n(\mathbf{x}))_{-\infty}^\infty$, as given by Lemma 7.2.

Although we dropped the su or u superscript, the $\tilde{\mathbf{x}}$ stands for the equivalence class of multisets in $\mathcal{V}$ modulo the $\mathcal{V}^{c0}$-translations or $\mathcal{V}^{sc0}$-translations (depending on invertibility of Φ).[11] The *su-lifted version* of the hull $\mathcal{X}$ is the set of all such equivalence classes,

$$\tilde{\mathcal{X}}^{su} := \{\tilde{\mathbf{x}} : \ \tilde{\mathbf{x}} \text{ is a well su-positioned lift of } \mathbf{x} \in \mathcal{X}\}, \tag{7.10}$$

with the analogous definition for $\tilde{\mathcal{X}}^u$. Whenever it does not matter which one of $\tilde{\mathcal{X}}^{su}$ or $\tilde{\mathcal{X}}^u$ we speak off, we shall drop the superscript and write $\tilde{\mathcal{X}}$.

The elements $\tilde{\mathbf{x}}$ of $\tilde{\mathcal{X}}$ are not Delone multisets in $\mathcal{V}^{su/u}$ but the points of each $\tilde{\mathbf{x}}$ are in a bijective correspondence with those of $\mathbf{x} := \pi_{\mathbb{R}^d}(\tilde{\mathbf{x}})$, and the point map $\mathbf{x} \to \tilde{\mathbf{x}}$ is L-Lipschitz with a uniform constant L secured by (6.3). Stated loosely, an element in $\tilde{\mathcal{X}}$ is an L-Lipschitz embedding of the corresponding Delone multi-set in $\mathcal{X}$, and the image of this embedding comes within a uniformly bounded distance to the origin. By a standard argument, one proves that $\tilde{\mathcal{X}}$ is compact (with the topology on closed subsets of $\mathcal{V}^{su/u}$ induced by the metric analogous to (5.1)). The well positioned lifting map $\mathcal{X} \ni \mathbf{x} \mapsto \tilde{\mathbf{x}} \in \tilde{\mathcal{X}}$ is then a homeomorphism since it is surjective, continuous (by Lemma 7.2), and manifestly injective. Moreover, $\tilde{\Phi}$ maps $\tilde{\mathcal{X}}$ to itself because the $\tilde{\mathbf{x}}_1$ in the well lifted orbit $(\tilde{\mathbf{x}}_n)$ is a well positioned lift of $\mathbf{x}_1 = \Phi(\mathbf{x}_0)$. From (6.8), we have

$$\pi_{\mathbb{R}^d} \circ \tilde{\Phi}|_{\tilde{\mathcal{X}}} = \Phi|_{\mathcal{X}} \circ \pi_{\mathbb{R}^d}. \tag{7.11}$$

Since $\mathcal{X} \ni \mathbf{x} \mapsto \tilde{\mathbf{x}} \in \tilde{\mathcal{X}}$ is the inverse of $\pi_{\mathbb{R}^d}|_{\tilde{\mathcal{X}}} : \tilde{\mathcal{X}} \to \mathcal{X}$, the above commutation gives

$$\tilde{\Phi}(\tilde{\mathbf{x}}) = \widetilde{\Phi(\mathbf{x})} \quad (\forall \mathbf{x} \in \mathcal{X}). \tag{7.12}$$

[11] Little can be done to take away this $\mathcal{V}^{c0}$-ambiguity in the general setting we consider here.

This is to say that the operation of well positioned lifting homeomorphically conjugates the two continuous maps $\Phi : \mathcal{X} \to \mathcal{X}$ and $\tilde{\Phi} : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$.

8 Positioning cocycle

In this section, we lift the $\mathbb{R}^{\mathrm{d}}$-action on $\mathcal{X}$ to an action on the lifted hull $\tilde{\mathcal{X}}$, where $\tilde{\mathcal{X}} := \tilde{\mathcal{X}}^{su}$ in the invertible case and $\tilde{\mathcal{X}} := \tilde{\mathcal{X}}^{u}$ in the non-invertible case. This will generate a $\mathcal{V}^{su}$- or $\mathcal{V}^{u}$-valued cocycle, denoted α_{su} and α_u, respectively. In the invertible case, α_u will coincide with the unstable component of α_{su} in the splitting $\alpha_{su} = \alpha_s \oplus \alpha_u$ (induced by $\mathcal{V}^{su} = \mathcal{V}^s \oplus \mathcal{V}^u$). Again, although Theorem 1.1 only requires α_u, we record some basic properties of α_{su}.

Let $\mathbf{x} \in \mathcal{X}$ and $\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}^{su}$ be its well su-positioned lift. For $t \in \mathbb{R}^{\mathrm{d}}$, the unique well su-positioned lift $(\mathbf{x} + t)\tilde{}$ of $\mathbf{x} + t$ is given by

$$(\mathbf{x} + t)\tilde{} = \tilde{\mathbf{x}} + \alpha_{su}(\mathbf{x}, t) \tag{8.1}$$

for some unique vector $\alpha_{su}(\mathbf{x}, t) \in \mathcal{V}^{su}$ that is a lift of t; in particular,

$$\pi_{\mathbb{R}^{\mathrm{d}}} \circ \alpha_{su}(\mathbf{x}, t) = t. \tag{8.2}$$

The equality (8.1) asserts that the well su-positioned lifting conjugates the $\mathbb{R}^{\mathrm{d}}$-action on $\mathcal{X}$ to a $\mathbb{R}^{\mathrm{d}}$-action on $\tilde{\mathcal{X}}^{su}$ whereby t acts on $\tilde{\mathbf{x}} \in \tilde{\mathcal{X}}^{su}$ by translating it by $\alpha_{su}(\mathbf{x}, t)$. In particular, α_{su} is a cocycle over the $\mathbb{R}^{\mathrm{d}}$-action on $\mathcal{X}$, i.e.,

$$\alpha_{su}(\mathbf{x}, t + s) = \alpha_{su}(\mathbf{x}, t) + \alpha_{su}(\mathbf{x} + t, s) \quad (t, s \in \mathbb{R}^{\mathrm{d}}, \mathbf{x} \in \mathcal{X}). \tag{8.3}$$

From (7.12), we get the *self-affinity* of the cocycle:

$$M_{su}\alpha_{su}(\mathbf{x}, t) = \alpha_{su}(\Phi(\mathbf{x}), \phi t) \quad (\mathbf{x} \in \mathcal{X},\ t \in \mathbb{R}^{\mathrm{d}}). \tag{8.4}$$

All of the above can be repeated with su replaced by u to get a $\mathcal{V}^u$-valued cocycle α_u. (Note that, even if $\mathbf{x}$ has a period, i.e., $\mathbf{x}+t = \mathbf{x}$ for some non-zero t, $\alpha_{su}(\mathbf{x}, t)$ is still uniquely defined because we insist that it is a lift of t.)

By the continuity of the well positioning operation (Lemma 7.2) and the $\mathbb{R}^{\mathrm{d}}$-action on $\mathcal{X}$, both α_{su} and α_u are continuous functions of their arguments. In fact, much more is true about the regularity of the cocycles.

Proposition 8.1 (cocycle regularity)

(i) *The map $(\mathbf{x}, t) \mapsto \alpha_u(\mathbf{x}, t)$ is Hölder in $t \in \mathbb{R}^{\mathrm{d}}$ (with the Hölder constant independent of $\mathbf{x} \in \mathcal{X}$) and transversally locally constant as a function of $\mathbf{x} \in \mathcal{X}$.*

(ii) *Assuming that* Φ *is invertible, the map* $(\mathbf{x}, t) \mapsto \alpha_s(\mathbf{x}, t)$ *is locally constant as a function of* $t \in \mathbb{R}^{\mathrm{d}}$ *and satisfies*

$$\forall_{L>0}\exists_{C>0,\epsilon>0} \quad \mathbf{x}|_R = \mathbf{y}|_R, \ |t| < L \\ \implies |\alpha_s(\mathbf{x}, t) - \alpha_s(\mathbf{y}, t)| \leq CR^{-\epsilon} \quad (\forall \mathbf{x}, \mathbf{y} \in \mathcal{X}, \ R > 0).$$

(which can be thought of as Hölder continuity of α_s *as a function of* $\mathbf{x} \in \mathcal{X}$*).*

We should indicate that $\mathbf{x}|_R = \mathbf{y}|_R$ in (ii) refers to the coincidence of central R-patches in the stronger sense defined in the next section. In any case, (ii) is not going to be used in the proof of Theorem 1.1 and we included it just to round the picture and bring out the $s \leftrightarrow u$ duality. ((ii) will be shown in Appendix A) On the other hand, the local transversal constancy of $\alpha_u(\mathbf{x}, t)$ as a function of $\mathbf{x}$ will be important for Theorem 1.1 and we formalize and prove it in Proposition 9.2 below.

9 Local transversal constancy

In this section, we further investigate the *well u-positioning*. (Accordingly, we do not assume that Φ is invertible and only use the forward iterates of $\tilde{\Phi} : \tilde{\mathcal{X}}^u \to \tilde{\mathcal{X}}^u$.) Our main objective is Proposition 9.2 (below) showing that, for any given $t \in \mathbb{R}^{\mathrm{d}}$, $\alpha_u(\mathbf{x}, t)$ is fully determined by a finite patch of $\mathbf{x}$ around the origin. This is the intended precise meaning of the *transversal local constancy* in Proposition 8.1.

Recall that we use a norm $|\cdot|$ in $\mathbb{R}^{\mathrm{d}}$ *adapted* to ϕ, i.e., $|\phi t| > \lambda_\phi |t|$ where $\lambda_\phi > 1$ is independent of $t \in \mathbb{R}^{\mathrm{d}} \setminus \{0\}$. Given $R > 0$ and $\mathbf{x} \in \mathcal{X}$, let us (re)define $\mathbf{x}|_R$ as the sub-multiset of $\mathbf{x}$ — still referred to as the *central R-patch* — made of the control points whose tiles intersect $B_R(0)$, the R-ball about the origin. This is a slight departure from our previous usage of the term *central R-patch* (in the proof of Lemma 7.2), which comprised the set of all points of $\mathbf{x}$ within distance R from the origin. The purpose of modifying the definition is in securing that $\Phi(\mathbf{x})|_{\lambda_\phi R}$ is a sub-multiset of $\Phi(\mathbf{x}|_R)$ for all $R > 0$, a convenient property that would otherwise hold only for sufficiently large R. (We dread introducing a "length scale" in our formulations and also try to parallel [10].)

As before, collecting the points of a multiset $\tilde{\mathbf{x}}$ in $\mathcal{V}$ that project (via $\pi_{\mathbb{R}^{\mathrm{d}}} : \mathcal{V} \to \mathbb{R}^{\mathrm{d}}$) into the points of $\mathbf{x}|_R$ yields the *central R-patch* of $\tilde{\mathbf{x}}$, denoted by $\tilde{\mathbf{x}}|_R$. In particular, if $\tilde{\mathbf{x}}$ is the well u-positioned lift of $\mathbf{x}$, we shall denote this patch by $\tilde{\mathbf{x}}|_R^u$, with the slightly idiosyncratic superscript u reminding us that this multiset is only well defined up to translation along $\mathcal{V}^{sc0}$. With these definitions, $\tilde{\Phi}(\tilde{\mathbf{x}})|_{\lambda_\phi R}^u$ is a sub-multiset of $\tilde{\Phi}(\tilde{\mathbf{x}}|_R^u)$.

We start with a simple yet crucial observation that the well u-positioning depends only on how the mult-set looks near the origin:

Fact 9.1 *For* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ *and* $R > 0$, $\mathbf{x}|_R = \mathbf{y}|_R \implies \tilde{\mathbf{x}}|_R^u = \tilde{\mathbf{y}}|_R^u$.

Proof Let $(\tilde{\mathbf{x}}_n)_0^\infty$ and $(\tilde{\mathbf{y}}_n)_0^\infty$ be the forward $\tilde{\Phi}^u$-orbits of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in $\tilde{\mathcal{X}}^u$. That means (recall (7.8)) that $\tilde{\Phi}(\tilde{\mathbf{x}}_n)^u = \tilde{\mathbf{x}}_{n+1}^u$ and $\tilde{\Phi}(\tilde{\mathbf{y}}_n)^u = \tilde{\mathbf{y}}_{n+1}^u$ for $n \geq 0$, From $\mathbf{x}|_R = \mathbf{y}|_R$, we have that $\tilde{\mathbf{y}}|_R^u = \tilde{\mathbf{x}}|_R^u + v^u$ for some $v \in \mathcal{K}$. Hence $\tilde{\mathbf{y}}_n|_{\lambda_\phi^n R}^u = \tilde{\mathbf{x}}_n|_{\lambda_\phi^n R}^u + M_u^n v^u$ for $n \geq 0$. Because of the well u-positioning, $\sup_{n \geq 0} |M_u^n v^u| < +\infty$. This forces v^u to be zero, so the fact follows by taking $n = 0$ in the last equality. □

Proposition 9.2 (local transversal constancy) *Suppose that, for some* $\mathbf{x}, \mathbf{y} \in \mathcal{X}$*,* $s \in \mathbb{R}^{\mathrm{d}}$*, and* $R, R' > 0$*, we have that* $\mathbf{x}|_R = \mathbf{y}|_R$ *and* $\mathbf{x} + s|_{R'} = \mathbf{y} + s|_{R'}$*, then*

$$\alpha_u(\mathbf{y}, s) = \alpha_u(\mathbf{x}, s). \tag{9.1}$$

The "local transversal" alludes here to the set of all $\mathbf{y} \in \mathcal{X}$ that satisfy $\mathbf{y}|_R = \mathbf{x}|_R$ for some fixed $\mathbf{x} \in \mathcal{X}$ and $R > 0$. Although we shall not rely on this, keep in mind that a neighborhood of any $\mathbf{x} \in \mathcal{X}$ has a *product structure*: it is homeomorphic to the product (a local transversal) × (an open set in $\mathbb{R}^{\mathrm{d}}$) ([1,6,7]).[12] With that in mind, note that $\mathbf{x}|_R = \mathbf{y}|_R$ already implies $\mathbf{x} + s|_{R'} = \mathbf{y} + s|_{R'}$ so that (9.1) holds and expresses constancy of the function $\mathbf{x} \mapsto \alpha_u(\mathbf{x}, s)$ on the local transversal.

Proof of Proposition 9.2 Fix some well u-positioned lifts $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ of $\mathbf{x}$ and $\mathbf{y}$ and some well u-positioned lifts $(\mathbf{x}+s)\tilde{}$ and $(\mathbf{y}+s)\tilde{}$ of $\mathbf{x}+s$ and $\mathbf{y}+s$. At this point, it is best to think of these as concrete multisets in $\mathcal{V}$ serving as representatives of their respective equivalence classes mod $\mathcal{V}^{sc0}$. Then $(\mathbf{x} + s)\tilde{} = \tilde{\mathbf{x}} + a$ and $(\mathbf{y} + s)\tilde{} = \tilde{\mathbf{y}} + b$ where $\alpha_u(\mathbf{x}, s) = a^u$ and $\alpha_u(\mathbf{y}, s) = b^u$. Our goal is to show equality of these $\mathcal{V}^u$-components, $a^u = b^u$.

Note that $(\mathbf{x} + s)\tilde{}|_{R'} - a$ is a patch of $\tilde{\mathbf{x}}$; let us denote it by $\tilde{\mathbf{x}}|_{B_{R'}(-s)}$ (as it projects to a patch of $\mathbf{x}$ centered at $-s$ and of radius R'). Likewise, $(\mathbf{y}+s)\tilde{}|_{R'} - b$ is a patch of $\tilde{\mathbf{y}}$, which we denote by $\tilde{\mathbf{y}}|_{B_{R'}(-s)}$.

By Fact 9.1, $\tilde{\mathbf{y}}|_R^u = \tilde{\mathbf{x}}|_R^u$, so there is a vector $\delta \in \mathcal{K}$ with $\delta^u = 0$ such that $\tilde{\mathbf{y}}|_R = \tilde{\mathbf{x}}|_R + \delta$. Likewise, there is a vector $\delta' \in \mathcal{K}$ with $\delta'^u = 0$ such that $(\mathbf{y}+s)\tilde{}|_{R'} = (\mathbf{x}+s)\tilde{}|_{R'} + \delta'$ or, equivalently, $\tilde{\mathbf{y}}|_{B_{R'}(-s)} + b = \tilde{\mathbf{x}}|_{B_{R'}(-s)} + a + \delta'$.

Picture $\tilde{\mathbf{x}} + \delta$ and $\tilde{\mathbf{y}}$: $\tilde{\mathbf{x}} + \delta$ agrees with $\tilde{\mathbf{y}}$ on a patch centered over 0 and differs from $\tilde{\mathbf{y}}$ by the translation by $b - a - \delta' - \delta$ on a patch centered over $-s$. Due to the lattice displacement property (6.4) of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$, the cumulative translation given by $b - a - \delta' - \delta$ is a lattice vector, it belongs to $1 \otimes_{\mathbb{Z}} \mathcal{J}$. On the other hand,

[12] Existence of local product structure in a general setting is given in [11].

$$\begin{aligned}\pi_{\mathbb{R}^d}(b-a-\delta'-\delta) &= \pi_{\mathbb{R}^d}(b)-\pi_{\mathbb{R}^d}(a)-\pi_{\mathbb{R}^d}(\delta')-\pi_{\mathbb{R}^d}(\delta)\\ &= s-s-0-0=0\end{aligned}$$

and thus $b-a-\delta'-\delta=0$ by the injectivity of $\pi_{\mathbb{R}^d}$ on $1\otimes_{\mathbb{Z}}\mathcal{J}$ (recall (4.9)). By taking the unstable components, we get $b^u-a^u=\delta'^u+\delta^u=0+0=0$, as desired. □

10 Return vectors

The vectors t for which $\mathbf{x}+t|_R=\mathbf{x}|_R$ for some $\mathbf{x}\in\mathcal{X}$ and $R>0$ are called the R-*return vectors* for $\mathbf{x}$, and their totality is denoted by

$$\mathcal{R}_{R,\mathbf{x}} := \left\{t\in\mathbb{R}^d:\ \mathbf{x}+t|_R=\mathbf{x}|_R\right\}. \tag{10.1}$$

Note that t is a *return vector* for $\mathbf{x}$, i.e. $t\in\mathcal{R}_{\mathbf{x}}:=\bigcup_{R>0}\mathcal{R}_{\mathbf{x},R}$, iff the tilings corresponding to $\mathbf{x}$ and $\mathbf{x}+t$ coincide on the tiles containing some open neighborhood of 0. Because every patch of $\mathbf{x}\in\mathcal{X}$ is a patch of $\mathbf{\Lambda}$, it is also easy to see that the totality of return vectors coincides with the subset of $\mathcal{J}$ consisting of translations between tiles of the same type (as considered in [10]), i.e.,

$$\Theta := \bigcup_{\mathbf{x}\in\mathcal{X},\ R>0}\mathcal{R}_{R,\mathbf{x}}=\{p-q:\ i@p, i@q\in\mathbf{\Lambda},\ i=1,\dots,m\}. \tag{10.2}$$

Observe that $\Phi(\mathbf{\Lambda})=\mathbf{\Lambda}$ readily yields

$$\phi(\Theta)\subset\Theta. \tag{10.3}$$

The nice thing is that $\alpha_u(\mathbf{x},\cdot)$ on $\mathcal{R}_{\mathbf{x}}$ is given by the canonical lifting:

Proposition 10.1 *For* $\mathbf{x}\in\mathcal{X}$ *and* $R>0$*, we have*

$$\mathbf{x}+t|_R=\mathbf{x}|_R\implies t\in\mathcal{J}\ \text{ and }\ \alpha_u(\mathbf{x},t)=(1\otimes t)^u.$$

Proof By (10.2), $t\in\Theta\subset\mathcal{J}$. The canonical lift of t is $\tilde{t}:=1\otimes t$. Let $\tilde{\mathbf{x}}$ be the well u-positioned lift of $\mathbf{x}$, and think of it as a fixed multiset in $\mathcal{V}$ (representing its equivalence class modulo $\mathcal{V}^{sc0}$). By the lattice displacement property of $\tilde{\mathbf{x}}$ (see (6.4)), the differences between points in the two lifted patches $\tilde{\mathbf{x}}+\tilde{t}|_R$ and $\tilde{\mathbf{x}}|_R$ are in the lattice $1\otimes_{\mathbb{Z}}\mathcal{J}$. Since after projecting via $\pi_{\mathbb{R}^d}$ we have $\mathbf{x}+t|_R=\mathbf{x}|_R$, the injectivity of $\pi_{\mathbb{R}^d}$ on $1\otimes_{\mathbb{Z}}\mathcal{J}$ gives $\tilde{\mathbf{x}}+\tilde{t}|_R=\tilde{\mathbf{x}}|_R$.

At the same time, Fact 9.1 gives $\widetilde{(\mathbf{x}+t)}|_R^u=\tilde{\mathbf{x}}|_R^u$. Via the definition of α_u, we can get then $(\tilde{\mathbf{x}}+\alpha_u(\mathbf{x},t))|_R^u=\tilde{\mathbf{x}}|_R^u=(\tilde{\mathbf{x}}+\tilde{t})|_R^u$. Thus $\alpha_u(\mathbf{x},t)=\tilde{t}^u$ by the uniqueness of well u-positioning. (We also used that the difference $\tilde{t}-\alpha_u(\mathbf{x},t)$ belongs to $\mathcal{K}$.) □

Proposition 10.2 (local linearity) *Let $t, s \in \mathbb{R}^{\mathrm{d}}$ and $\mathbf{x} \in \mathcal{X}$. Suppose that t is a return vector for both $\mathbf{x}$ and $\mathbf{x} + s$ (i.e. $t \in \mathcal{R}_{\mathbf{x}} \cap \mathcal{R}_{\mathbf{x}+s}$), then*

$$\alpha_u(\mathbf{x} + t, s) = \alpha_u(\mathbf{x}, s) \tag{10.4}$$

and

$$\alpha_u(\mathbf{x}, t + s) = \alpha_u(\mathbf{x}, t) + \alpha_u(\mathbf{x}, s). \tag{10.5}$$

We note that $t \in \mathcal{R}_{\mathbf{x}}$ is also return for $\mathbf{x} + s$ when s is sufficiently small, so (10.5) ensures that α_u is *locally linear* near t. In Sect. 16, this is elevated to true (global) linearity in certain circumstances.

Proof of Proposition 10.2 The hypothesis that t is a return vector for both $\mathbf{x}$ and $\mathbf{x} + s$ means that $\mathbf{x} + t|_R = \mathbf{x}|_R$ and $\mathbf{x} + s + t|_{R'} = \mathbf{x} + s|_{R'}$ for some $R, R' > 0$. Hence, Proposition 9.2 gives (10.4). Then (10.5) follows by combining (10.4) with the cocycle property: $\alpha_u(\mathbf{x}, t + s) = \alpha_u(\mathbf{x}, t) + \alpha_u(\mathbf{x} + t, s)$ (cf. (8.3)). □

11 Ergodic embedding $A : \mathbb{R}^{\mathrm{d}} \to \mathcal{E}$

Finally, we arrive at the juncture where the viewpoint emphasizing the hull $\mathcal{X}$ (over the individual multiset $\mathbf{\Lambda}$) and the introduction of the cocycle α_u pays off. We use ergodic averaging for the $\mathbb{R}^{\mathrm{d}}$ action on $\mathcal{X}$ to linearize $t \mapsto \alpha_u(\mathbf{x}, t)$ at the *large scale*, i.e., when $|t| \approx \infty$. This is a major departure from the local linearization (when $|t| \approx 0$) employed by the pivotal Lemma 3.7 in [10] (and effected by the elegant tool of a.e. differentiation of Lipschitz functions). One advantage of our approach is that it readily gives an M-invariant subspace complementing $\mathcal{K}$. (Because we only work with α_u, we do not need to assume that Φ is invertible.)

As we already noted, the translation action on $\mathcal{X}$ is not only minimal but also uniquely ergodic. This allows invoking Lemma 4.1 in [11] for the cocycle α_u (which is continuous by Lemma 7.2), to see that α_u is asymptotically linear in the following sense. There is a linear transformation $A : \mathbb{R}^{\mathrm{d}} \to \mathcal{V}^u$ such that

$$\alpha_u(\mathbf{x}, t) = At + \mathrm{Error}(\mathbf{x}, t) \tag{11.1}$$

where

$$\lim_{R\to\infty} \sup_{\mathbf{x}\in\mathcal{X}, |t|=R} \frac{|\mathrm{Error}(\mathbf{x}, t)|}{|t|} = 0. \tag{11.2}$$

From (8.2), $\pi_{\mathbb{R}^{\mathrm{d}}} \circ At = t$ (for $t \in \mathbb{R}^{\mathrm{d}}$), so A is a linear embedding and a right inverse of $\pi_{\mathbb{R}^{\mathrm{d}}}$:

$$\pi_{\mathbb{R}^{\mathrm{d}}} \circ A = \mathrm{Id}_{\mathbb{R}^{\mathrm{d}}}. \tag{11.3}$$

We shall refer to A as the *ergodic embedding* of the physical space $\mathbb{R}^{\mathrm{d}}$ into the mathematical space $\mathcal{V}$. Let $\mathcal{E}$ be the range of A,

$$\mathcal{E} := \{At : t \in \mathbb{R}^{\mathrm{d}}\}. \tag{11.4}$$

From (8.4), we get that A intertwines ϕ and M:

$$M \circ A = A \circ \phi, \tag{11.5}$$

nicely complementing (4.11), which read $\pi_{\mathbb{R}^{\mathrm{d}}} \circ M = \phi \circ \pi_{\mathbb{R}^{\mathrm{d}}}$. In particular, $\mathcal{E}$ is an M_u-invariant subspace of $\mathcal{V}^u$ and thus also an M-invariant subspace of $\mathcal{V}$. Also from (11.3) and (11.5), we have an M-invariant splitting

$$\mathcal{V} = \mathcal{E} \oplus \mathcal{K} \tag{11.6}$$

with the restriction $M|_{\mathcal{E}}$ linearly conjugated to ϕ. When convenient, without further mention, we will use the linear isomorphism $A : \mathbb{R}^{\mathrm{d}} \to \mathcal{E}$ to identify the two spaces $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{E}$ and blur the distinction between ϕ and $M|_{\mathcal{E}}$.

12 Lyapunov shyness

We turn attention to the non-linear features of the cocycle α_u.

Let

$$\mathbb{R}^{\mathrm{d}} = \bigoplus_{\lambda} E_\lambda \quad \text{and} \quad \mathcal{V} = \bigoplus_{\gamma} \mathcal{V}_\gamma \tag{12.1}$$

be the decompositions of $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{V}$ into the generalized real eigenspaces (indexed by eigenvalues) for ϕ and M, respectively. Using $\pi_\lambda : \mathcal{V} \to \mathcal{V}_\lambda$ for the associated projections, we have the *components* of A given by $A_{\gamma,\lambda} := \pi_\gamma \circ A|_{E_\lambda} : E_\lambda \to \mathcal{V}_\gamma$ Note that, if $\gamma \not\equiv \lambda$ (i.e., $\gamma \notin \{\lambda, \overline{\lambda}\}$), then

$$A_{\gamma,\lambda} = 0, \tag{12.2}$$

as otherwise $A_{\gamma,\lambda}$ would non-trivially intertwine $\phi|_{E_\lambda}$ and $M|_{\mathcal{V}_\gamma}$, that is $A_{\gamma,\lambda} \circ \phi|_{E_\lambda} = M|_{\mathcal{V}_\gamma} \circ A_{\gamma,\lambda}$ (cf. (11.5)). (This is impossible unless $\gamma = \lambda$ or $\gamma = \overline{\lambda}$.) Therefore, the embedding $A : \mathbb{R}^{\mathrm{d}} \to \mathcal{V}$ decomposes into a direct sum of the embeddings $A_{\lambda,\lambda} : E_\lambda \to \mathcal{V}_\lambda$,

$$A = \bigoplus_{\lambda} A_{\lambda,\lambda}. \tag{12.3}$$

Such simple direct sum structure is not true for the unstable cocycle, which typically deviates from its (asymptotic) linearization A. Indeed, its components

$$\alpha_{\gamma,\lambda}(\mathbf{x},\cdot) := \pi_\gamma \circ \alpha_u(\mathbf{x},\cdot)|_{E_\lambda} : E_\lambda \to \mathcal{V}_\gamma \quad (\mathbf{x} \in \mathcal{X},\ |\gamma| > 1)$$

may satisfy $\alpha_{\gamma,\lambda} \neq 0$ even if $\lambda \not\equiv \gamma$. In such case we say that λ *pollutes* γ.

Lemma 12.1 (Shyness) *For any* $\mathbf{x} \in \mathcal{X}$*, if* γ *is an eigenvalue of* M_u *and* λ *is an eigenvalue of* ϕ *and* $|\gamma| \geq |\lambda|$*, then* $\alpha_{\gamma,\lambda}(\mathbf{x},\cdot)$ *is linear and coincides with* $A_{\gamma,\lambda}$*. In particular,* $\alpha_{\gamma,\lambda}(\mathbf{x},\cdot) = 0$ *if additionally* $\lambda \not\equiv \gamma$.

This lemma conveys the same general idea as Lemma 3.7 in [10] and will go to the heart of Perronness `(P)` by ensuring that λ is not polluting the γ that weakly-dominate λ, denoted $\gamma \succeq \lambda$.

Proof of Lemma 12.1 Taking λ and γ as in the lemma, we have to show that $\beta(\mathbf{x},\cdot)|_{E_\lambda} = 0$ ($\mathbf{x} \in \mathcal{X}$) where β is the difference cocycle

$$\beta(\mathbf{x},t) := \pi_\gamma\,(\alpha_u(\mathbf{x},t) - At) \quad (t \in \mathbb{R}^{\mathrm{d}},\ \mathbf{x} \in \mathcal{X}). \tag{12.4}$$

All we shall need is that β takes values in $\mathcal{V}_\gamma$ and intertwines ϕ and M (see (8.4) and (11.5)):

$$\beta(\Phi\mathbf{x},\phi t) = M\beta(\mathbf{x},t) \quad (t \in E_\lambda,\ \mathbf{x} \in \mathcal{X}). \tag{12.5}$$

Note that if $E', E'' \subset \mathbb{R}^{\mathrm{d}}$ are two subspaces such that $\beta(\mathbf{x},\cdot)|_{E'} = 0$ and $\beta(\mathbf{x},\cdot)|_{E''} = 0$ for all $\mathbf{x} \in \mathcal{X}$ then $\beta(\mathbf{x},\cdot)|_{E'+E''} = 0$ for all $\mathbf{x} \in \mathcal{X}$. Indeed, for $v' \in E'$ and $v'' \in E''$, the cocycle property (8.3) gives

$$\beta(\mathbf{x},v'+v'') = \beta(\mathbf{x},v') + \beta(\mathbf{x}+v',v'') = 0 + 0 \quad (\mathbf{x} \in \mathcal{X}).$$

Thus, if we choose a Jordan decomposition for $\phi|_{E_\lambda}$ and represent E_λ as a direct sum of Jordan spaces, it suffices to show that the cocycle β vanishes on an arbitrary Jordan space. Let then $\tilde{E}_\lambda \subset E_\lambda$ be a Jordan subspace. In a familiar way, $\tilde{E}_\lambda$ further stratifies into invariant (but not invariantly complemented) subspaces:[13]

$$\{0\} = \tilde{E}_\lambda^{(0)} \subset \tilde{E}_\lambda^{(1)} \subset \tilde{E}_\lambda^{(2)} \subset \tilde{E}_\lambda^{(3)} \subset \cdots \subset \tilde{E}_\lambda^{(r)} = \tilde{E}_\lambda \tag{12.6}$$

where the ϕ-induced action on the quotient $\tilde{E}_\lambda^{(j+1)}/\tilde{E}_\lambda^{(j)}$ is simply that of $(\lambda, 1)$-Jordan block. (It multiplies by λ for real λ and rotates and scales by $|\lambda|$ for complex λ.)

We shall show that $\beta(\mathbf{x},\cdot)|_{\tilde{E}_\lambda^{(j)}} = 0$ for $j = 0, 1, \ldots, r$ by induction on j. For $j = 0$, $\beta(\cdot, 0) = 0$ from the definition (8.1). To make the induction step, assume that $\beta(\mathbf{x},\cdot)|_{\tilde{E}_\lambda^{(j)}} = 0$ for all $\mathbf{x} \in \mathcal{X}$ and some $j < r$. By the cocycle

[13] For real λ, $\tilde{E}_\lambda^{(j)} = \{v \in E_\lambda :\ (\phi - \lambda I)^j v = 0\}$.

property, $\beta(\mathbf{x}, \cdot)$ is constant on the $\tilde{E}_\lambda^{(j)}$ cosets in $\tilde{E}_\lambda^{(j+1)}$, i.e., for $v \in \tilde{E}_\lambda^{(j+1)}$ and $w \in \tilde{E}_\lambda^{(j)}$, we have

$$\beta(\mathbf{x}, v + w) = \beta(\mathbf{x}, v) + \beta(\mathbf{x} + v, w) = \beta(\mathbf{x}, v) + 0 \quad (\mathbf{x} \in \mathcal{X}). \tag{12.7}$$

Suppose now that β does not vanish on $\tilde{E}_\lambda^{(j+1)}$, i.e., there is $\mathbf{x} \in \mathcal{X}$ and $v \in \tilde{E}_\lambda^{(j+1)} \backslash \tilde{E}_\lambda^{(j)}$ with $u := \beta(\mathbf{x}, v) \in \mathcal{V}_\gamma \backslash \{0\}$. Because the ϕ-induced action on the quotient $\tilde{E}_\lambda^{(j+1)} / \tilde{E}_\lambda^{(j)}$ is that of a $(\lambda, 1)$-Jordan block, there is $C > 0$ such that, for each $n \in \mathbb{N}$, we can pick $v_n \in \tilde{E}_\lambda^{(j+1)}$ with $v_n - v \in \tilde{E}_\lambda^{(j)}$ and $|\phi^n v_n| \le C|\lambda|^n$. On the other hand, the fact that $u \in \mathcal{V}_\gamma \backslash \{0\}$ ensures that, after increasing $C > 0$ if necessary (to suit u), we have $|M^n u| \ge C^{-1} |\gamma|^n$ for all $n \in \mathbb{N}$.

Hence, first using (12.7) and (12.5), the hypothesis $|\gamma| \ge |\lambda|$ allows us to write

$$\begin{aligned} \frac{|\beta(\Phi^n \mathbf{x}, \phi^n v_n)|}{|\phi^n v_n|} &= \frac{|\beta(\Phi^n \mathbf{x}, \phi^n v)|}{|\phi^n v_n|} \\ &= \frac{|M^n \beta(\mathbf{x}, v)|}{|\phi^n v_n|} = \frac{|M^n u|}{|\phi^n v_n|} \ge \frac{C^{-1} |\gamma|^n}{C |\lambda|^n} \ge C^{-2} > 0. \end{aligned}$$

This contradicts the fact that $\lim_{n \to \infty} \frac{|\beta(\Phi^n \mathbf{x}, \phi^n v_n)|}{|\phi^n v_n|} = 0$ by the ergodic averaging (11.2) and the fact that $|\phi^n v_n| \to \infty$ (since $|\phi^n v_n| \ge \left| \phi^n v \pmod{\tilde{E}_\lambda^{(j)}} \right| \sim |\lambda|^n \to \infty$, where the middle norm is the quotient norm on $\tilde{E}_\lambda^{(j+1)} / \tilde{E}_\lambda^{(j)}$.) □

13 Subspace thickening: summary

Completing the build-up is a construction associating to any ϕ-invariant subspace $E \subset \mathbb{R}^{\mathrm{d}}$ a possibly larger subspace $\overline{W}_E \subset \mathbb{R}^{\mathrm{d}}$ and a rational subspace $\overline{\mathcal{W}}_E \subset \mathcal{V}$. The role of the pair $(\overline{W}_E, \overline{\mathcal{W}}_E)$ is to serve as the smallest replacement for $(\mathbb{R}^{\mathrm{d}}, \mathcal{V})$ allowing self-contained study of the unstable cocycle restricted to E. The idea is to apply repeatedly (to saturation) a version of the construction used in [10] for E equal to an eigenspace of ϕ. The complete story is a bit involved so we only summarize it now, relegating most arguments to the three sections following the proof of the theorem.

Fix an arbitrary ϕ-invariant linear subspace $E \subset \mathbb{R}^{\mathrm{d}}$. Here is the basic construction. First, for $\epsilon > 0$, consider the subspaces of $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{V}$ generated by the vectors in Θ (defined in (10.2)) that are ϵ-close to E:

$$\begin{aligned} W_\epsilon &:= \operatorname{span}_{\mathbb{R}} \{v : \ v \in \Theta \cap B_\epsilon(E)\} \quad \text{and} \\ \mathcal{W}_\epsilon &:= \operatorname{span}_{\mathbb{R}} \{1 \otimes v : \ v \in \Theta \cap B_\epsilon(E)\} \end{aligned}$$

where $B_\epsilon(E)$ is the ϵ-neighborhood of E in $\mathbb{R}^{\mathrm{d}}$. Second, set

$$W_E := \bigcap_{\epsilon>0} W_\epsilon \subset \mathbb{R}^{\mathrm{d}} \quad \text{and} \quad \mathcal{W}_E := \bigcap_{\epsilon>0} \mathcal{W}_\epsilon \subset \mathcal{V}. \tag{13.1}$$

Of course, there is $\epsilon_0 > 0$ such that W_ϵ and $\mathcal{W}_\epsilon$ stabilize: $W_\epsilon = W_E$ and $\mathcal{W}_\epsilon = \mathcal{W}_E$ for $\epsilon \in (0, \epsilon_0]$. We call W_E the *thickening* of E. Note that for $E = 0$ we get $W_E = 0$ and $\mathcal{W}_E = 0$ because 0 is isolated in Θ.

Example 6 The simplest examples of non-trivial $W_E \subsetneq \mathbb{R}^{\mathrm{d}}$ come from the product tilings. Let $\mathcal{T}_i$ be tilings of $\mathbb{R}^{\mathrm{d}_i}$ $(i = 1, 2)$ and $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ be the product tiling of $\mathbb{R}^{\mathrm{d}_1} \times \mathbb{R}^{\mathrm{d}_2}$. The projection of Θ into $\mathbb{R}^{\mathrm{d}_1} \times 0$ equals $\Theta_1 \times 0$. Hence, for any $E \subset \mathbb{R}^{\mathrm{d}_1} \times 0$, we have $W_E \subset \mathbb{R}^{\mathrm{d}_1} \times 0$. In particular, $W_{\mathbb{R}^{\mathrm{d}_1} \times 0} = \mathbb{R}^{\mathrm{d}_1} \times 0$.

W_E and $\mathcal{W}_E$ inherit some basic properties from $\mathbb{R}^{\mathrm{d}}$ and $\mathcal{V}$.

Proposition 13.1 (Thickening) *We have*

(i) *W_E is ϕ-invariant, and $\mathcal{W}_E$ is a rational M-invariant subspace of $\mathcal{V}$;*
(ii) *$\pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E} : \mathcal{W}_E \to W_E$ factors $M|_{\mathcal{W}_E}$ onto $\phi|_{W_E}$, i.e., it is surjective and*

$$\pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E} \circ M|_{\mathcal{W}_E} = \phi|_{W_E} \circ \pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E};$$

(iii) *$E \subset W_E$;*
(iv) *$\mathcal{W}_E^u \subset \mathrm{span}_{\mathbb{R}}\{\alpha_u(\mathbf{x}, t) \in \mathcal{W}_E^u : t \in W_E,\ \mathbf{x} \in \mathcal{X}\}$.*

Proof (i) W_E is ϕ-invariant because so are E and Θ (see (10.3)). The M-invariance of $\mathcal{W}_E$ follows then via $M(1 \otimes v) = 1 \otimes \phi v$ (see (4.10)). $\mathcal{W}_E$ is rational because $\Theta \subset \mathcal{J}$, so the generators $1 \otimes v$ of $\mathcal{W}_E$ are in the lattice $1 \otimes_{\mathbb{Z}} \mathcal{J}$.

(ii) $\mathcal{W}_E$ is spanned by canonical lifts $1 \otimes v$ (cf. (6.1)) of the vectors $v = \pi_{\mathbb{R}^{\mathrm{d}}}(1 \otimes v)$ spanning W_E. Hence, $\pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E} : \mathcal{W}_E \to W_E$ is surjective. That $\pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E} \circ M|_{\mathcal{W}_E} = \phi|_{W_E} \circ \pi_{\mathbb{R}^{\mathrm{d}}}|_{\mathcal{W}_E}$ is due to (4.11).

(vi) $\mathcal{W}_E^u$ is spanned by vectors $(1 \otimes v)^u$ with $v \in \Theta \cap W_E$. Any $v \in \Theta$ is a return vector for some $\mathbf{x} \in \mathcal{X}$ (see (10.2)). By Proposition 10.1, $\alpha_u(\mathbf{x}, v) = (1 \otimes v)^u$ (for such $\mathbf{x}$), proving the inclusion.

(iii) This is more involved and will be shown as Lemma 15.2 in Sect. 15. □

W_E is not sufficiently well behaved for our purposes. To remedy this we shall iterate to saturation the operation associating W_E to E. Setting $W^0 := E$ and $W^{j+1} := W_{W^j}$ for $j = 0, 1, \ldots$ generates a stabilizing filtration

$$E =: W^0 \subset W^1 \subset \cdots \subset W^k = W^{k+1} =: \overline{W}_E. \tag{13.2}$$

(Here the inclusions are due to (iii) in Proposition 13.1.) The end space $\overline{W}_E$ of the filtration, which we call the *saturated thickening* of E, has the property that it contains all nearby elements of Θ; precisely, there is $\epsilon = \epsilon_E > 0$ such that

$$\overline{W}_E = \mathrm{span}_{\mathbb{R}} \left\{ v : \; v \in \Theta \cap B_\epsilon(\overline{W}_E) \right\}. \tag{13.3}$$

We call subspaces of $\mathbb{R}^{\mathrm{d}}$ satisfying (13.3) *saturated*. The counterpart of $\overline{W}_E$ in the mathematical space is

$$\overline{\mathcal{W}}_E := \mathrm{span}_{\mathbb{R}} \left\{ 1 \otimes v : \; v \in \Theta \cap \overline{W}_E \right\}. \tag{13.4}$$

So why did we saturate thickening? $\overline{W}_E$ and $\overline{\mathcal{W}}_E$ have a longer list of good properties:

Proposition 13.2 (Saturated thickening) *We have*

(i) $\overline{W}_E$ *is* ϕ*-invariant, and* $\overline{\mathcal{W}}_E$ *is a rational* M*-invariant subspace of* $\mathcal{V}$*;*
(ii) $\pi_{\mathbb{R}^{\mathrm{d}}}|_{\overline{\mathcal{W}}_E} : \overline{\mathcal{W}}_E \to \overline{W}_E$ *factors* $M|_{\overline{\mathcal{W}}_E}$ *onto* $\phi|_{\overline{W}_E}$*;*
(iii) $E \subset \overline{W}_E$*;*
(iv) $\overline{\mathcal{W}}^u_E = \mathrm{span}_{\mathbb{R}}\{\alpha_u(\mathbf{x}, t) : \; t \in \overline{W}_E, \; \mathbf{x} \in \mathcal{X}\}$*;*
(v) $A(\overline{W}_E) = \mathcal{E} \cap \overline{\mathcal{W}}_E$*;*
(vi) $\overline{\mathcal{W}}_E = (\mathcal{E} \cap \overline{\mathcal{W}}_E) \oplus (\mathcal{K} \cap \overline{\mathcal{W}}_E)$.

The parts (i), (ii), (iii) follow directly from Proposition 13.2. In (iv) there is however a substantial gain as we go from Propositions 13.1 to 13.2: the inclusion turns into an equality. ($\mathcal{W}_E$ was not quite big enough to contain the restricted cocycle $\alpha_u|_E$ and $\overline{\mathcal{W}}_E$ is just right.) The proofs of (iv), (v), (vi) are given in Sect. 17.

Finally, we turn to the most remarkable connection between E and $\overline{W}_E$. Given an unstable eigenvalue γ of M (i.e. $|\gamma| > 1$), denote by α_γ and A_γ the $\mathcal{V}_\gamma$ components of the unstable cocycle α_u and its (asymptotic) linearization A,

$$\alpha_\gamma(\mathbf{x}, t) := \pi_\gamma \circ \alpha_u(\mathbf{x}, t) \quad \text{and} \quad A_\gamma := \pi_\gamma \circ A \quad (\mathbf{x} \in \mathcal{X}, \; t \in \mathbb{R}^{\mathrm{d}}). \tag{13.5}$$

Lemma 13.3 (Linearity is thick) *For any unstable eigenvalue* γ *of* M*, if* $\alpha_\gamma(\mathbf{x}, \cdot)|_E = A_\gamma|_E$ *for all* $\mathbf{x} \in \mathcal{X}$*, then* $\alpha_\gamma(\mathbf{x}, \cdot)|_{\overline{W}_E} = A_\gamma|_{\overline{W}_E}$ *for all* $\mathbf{x} \in \mathcal{X}$.

This means that if α_γ is linear on E then it is also linear on its saturated thickening $\overline{W}_E$. The basic mechanism behind this phenomenon is already present in Lemma 3.8 in [10]. Section 16 is devoted to the proof.

To keep track of our liabilities, beside Lemma 13.3, still needing a proof are (iii) of Proposition 13.1 (Sect. 15) and (iv), (v), (vi) of Proposition13.2 (Sect. 17). But we finish off the theorem first.

14 Proof of Theorem

We have known that ϕ satisfies `(IA)` since (4.11). It remains to show `(P)`, i.e., we have to see that $\mathrm{mult}_\phi(\gamma,\cdot) \geq \mathrm{mult}_\phi(\lambda,\cdot)$ whenever γ weakly-dominates λ (denoted $\gamma \succeq \lambda$) and λ is an eigenvalue of ϕ. We then fix λ and restrict attention to its algebraic conjugacy class $[\lambda] := \{\gamma_1, \dots, \gamma_d\}$, ordered so that the moduli $|\gamma_j|$ are non-increasing and with only one member of each complex conjugate pair $\gamma, \overline{\gamma}$ listed (cf. Example 3 in Sect. 3). We also fix γ so that $\gamma \succeq \lambda$. Since $\lambda, \gamma \in [\lambda]$, there are j, j' so that $\lambda \equiv \gamma_j$ and $\gamma \equiv \gamma_{j'}$. From $\gamma \succeq \lambda$ we have $|\gamma| \geq |\lambda|$, so $j > j'$, possibly after swapping the indices j, j' (when $|\gamma_j| = |\gamma_{j'}|$). It suffices then to show

$$\mathrm{mult}_\phi(\gamma_i,\cdot) \geq \mathrm{mult}_\phi(\gamma_{i+1},\cdot) \quad (i = 1, \dots, d-1). \tag{14.1}$$

Recalling the decompositions (12.1) into generalized eigenspaces and the saturated thickening construction (Sect. 13), we consider

$$E_{[\lambda]} := \bigoplus_{\lambda' \in [\lambda]} E_{\lambda'} \quad \text{and} \quad \overline{W}_{[\lambda]} := \overline{W}_{E_{[\lambda]}} \quad \text{and} \quad \overline{\mathcal{W}}_{[\lambda]} := \overline{\mathcal{W}}_{E_{[\lambda]}}.$$

These three spaces are further filtrated as follows

$$\begin{aligned} &E_{[\lambda]} = E'_1 \supset \cdots \supset E'_d \ \text{ and} \\ &\overline{W}_{[\lambda]} = \overline{W}_1 \supset \cdots \supset \overline{W}_d \ \text{ and } \ \overline{\mathcal{W}}_{[\lambda]} = \overline{\mathcal{W}}_1 \supset \cdots \supset \overline{\mathcal{W}}_d \end{aligned}$$

where

$$E'_i := \bigoplus_{j \geq i} E_{\gamma_j} \quad \text{and} \quad \overline{W}_i := \overline{W}_{E'_i} \quad \text{and} \quad \overline{\mathcal{W}}_i := \overline{\mathcal{W}}_{E'_i} \quad (i = 1, \dots, d).$$

We adopt here the convention that $E_{\gamma_j} = 0$ if γ_j is not an eigenvalue of ϕ. For instance, when $|\gamma_j| \leq 1$ we have $E'_j = \overline{W}_j = 0$ and $\overline{\mathcal{W}}_j = 0$. We may well **assume** then that $|\gamma_i|, |\gamma_{i+1}| > 1$ (as otherwise (14.1) is trivially satisfied).[14]

The generalized eigenspaces of $\phi|_{\overline{W}_i}$ and $M|_{\overline{\mathcal{W}}_i}$ associated to γ_j are, respectively,

$$\overline{W}_{i,\gamma_j} := \overline{W}_i \cap E_{\gamma_j} \quad \text{and} \quad \overline{\mathcal{W}}_{i,\gamma_j} := \overline{\mathcal{W}}_i \cap \mathcal{V}_{\gamma_j}.$$

By construction $\overline{W}_{i,\gamma_j} = E_{\gamma_j}$ for $j \geq i$. (For $j < i$, merely $\overline{W}_{i,\gamma_j} \subset E_{\gamma_j}$.)

Recall (from Sect. 12) that the components $A_{\gamma_j,\gamma_i} := \pi_{\gamma_j} \circ A|_{E_{\gamma_i}} : E_{\gamma_i} \to \mathcal{V}_{\gamma_j}$ are zero if $i \neq j$, and the $A_{\gamma_i,\gamma_i} = A|_{E_{\gamma_i}}$ are embeddings. (Also remember

[14] A posteriori, $E_{\gamma_j} \neq 0$ exactly for $j = 1, \dots, d'$ for some $d' \leq d$.

the notation $A_{\gamma_i} := \pi_{\gamma_i} \circ A$.) The range of A_{γ_i,γ_i} actually sits in $\overline{\mathcal{W}}_{i,\gamma_i} \subset \mathcal{V}_{\gamma_i}$ because, by (v) of Proposition 13.2, $A(\overline{W}_i) \subset \overline{\mathcal{W}}_i$ and so also

$$A(E_{\gamma_i}) = A(\overline{W}_{i,\gamma_i}) \subset \overline{\mathcal{W}}_{i,\gamma_i}. \tag{14.2}$$

Lemma 14.1

$$A|_{E_{\gamma_i}} = A_{\gamma_i,\gamma_i} : E_{\gamma_i} \to \overline{\mathcal{W}}_{i,\gamma_i}$$

is a linear isomorphism. It linearly conjugates $\phi|_{E_{\gamma_i}}$ *and* $M|_{\overline{\mathcal{W}}_{i,\gamma_i}}$, *i.e.,*

$$A_{\gamma_i,\gamma_i} \circ \phi|_{E_{\gamma_i}} = M|_{\overline{\mathcal{W}}_{i,\gamma_i}} \circ A_{\gamma_i,\gamma_i}.$$

Proof of Lemma 14.1 We already know that $A|_{E_{\gamma_i}} = A_{\gamma_i,\gamma_i} : E_{\gamma_i} \to \overline{\mathcal{W}}_{i,\gamma_i}$ is injective so we only have to establish its surjectivity to see that it is an isomorphism. Then the linear conjugacy is automatic by taking the γ_i component in the old $A \circ \phi = M \circ A$ (see (11.5)).

By shyness (Lemma 12.1), we have

$$\alpha_{\gamma_i,\gamma_i}(\mathbf{x}, \cdot) := \alpha_{\gamma_i}(\mathbf{x}, \cdot)|_{E_{\gamma_i}} = A_{\gamma_i,\gamma_i} \quad (\mathbf{x} \in \mathcal{X}). \tag{14.3}$$

(This includes the possibility that $A_{\gamma_i,\gamma_i} = 0$, in which case also $\alpha_{\gamma_i,\gamma_i} = 0$.) Now, (iv) of Proposition 13.2 gives $\mathrm{span}_{\mathbb{R}}\{\alpha_u(\mathbf{x}, t) : t \in \overline{W}_i, \ \mathbf{x} \in \mathcal{X}\} = \overline{\mathcal{W}}_i^u$ and, after applying π_{γ_i} to both sides (since $\overline{\mathcal{W}}_{i,\gamma_i} \subset \overline{\mathcal{W}}_i^u$), we get

$$\mathrm{span}_{\mathbb{R}} \left\{\alpha_{\gamma_i}(\mathbf{x}, t) : \ t \in \overline{W}_i, \ \mathbf{x} \in \mathcal{X}\right\} = \overline{\mathcal{W}}_{i,\gamma_i}. \tag{14.4}$$

The surjectivity would be shown if we could justify writing

$$\begin{aligned} A_{\gamma_i,\gamma_i}(E_{\gamma_i}) &= \mathrm{span}_{\mathbb{R}} \left\{\alpha_{\gamma_i}(\mathbf{x}, t_i) : \ t_i \in E_{\gamma_i}, \ \mathbf{x} \in \mathcal{X}\right\} \\ &= \mathrm{span}_{\mathbb{R}} \left\{\alpha_{\gamma_i}(\mathbf{x}, t) : \ t \in \overline{W}_i, \ \mathbf{x} \in \mathcal{X}\right\} \\ &= \overline{\mathcal{W}}_{i,\gamma_i}. \end{aligned}$$

The first and third equalities are by (14.3) and (14.4), so only the second has to be justified. Consider then any $t \in \overline{W}_i$. Much like all of $\mathbb{R}^{\mathrm{d}}$ in (12.1), $\overline{W}_i$ is a direct sum of the generalized eigenspaces of $\phi|_{\overline{W}_i}$ and we can decompose t accordingly

$$t = t_i + \sum_{\lambda' \neq \gamma_i} t_{\lambda'}$$

where $t_i \in \overline{W}_{i,\gamma_i} = E_{\gamma_i}$ and $t_{\lambda'} \in \overline{W}_{i,\lambda'} := \overline{W}_i \cap E_{\lambda'}$. (Note that the eigenvalue λ' may or may not be algebraically conjugate to γ_i, and it is possible that $t_i = 0$ if γ_i is not an eigenvalue of ϕ.) We need to show

$$\alpha_{\gamma_i}(\mathbf{x}, t) = \alpha_{\gamma_i}(\mathbf{x}, t_i). \tag{14.5}$$

By using the cocycle property (8.3), we can write

$$\alpha_{\gamma_i}(\mathbf{x}, t) = \alpha_{\gamma_i}(\mathbf{x}, t_i) + \sum_{\lambda' \not\equiv \gamma_i} \alpha_{\gamma_i}(\mathbf{x}_{\lambda'}, t_{\lambda'})$$

for suitable $\mathbf{x}_{\lambda'} \in \mathcal{X}$. The task is to see that each term under the sum is zero.

We consider two cases. The first case is when $|\lambda'| \leq |\gamma_i|$. It is dispatched by observing that $\alpha_{\gamma_i}(\mathbf{x}, \cdot)|_{E_{\lambda'}} = 0$ (for all $\mathbf{x} \in \mathcal{X}$) by shyness (Lemma 12.1).

The second case is when $|\lambda'| \geq |\gamma_i|$. The idea is to apply the first case to see that α_{γ_i} is linear on E'_i. Precisely, for any $s \in E'_i$ and $\mathbf{x} \in \mathcal{X}$, we again use the cocycle property to write $\alpha_{\gamma_i}(\mathbf{x}, s) = \sum_{j \geq i} \alpha_{\gamma_i}(\mathbf{x}_j, s_j)$ where $s_j \in E_{\gamma_j}$ and $\mathbf{x}_j \in \mathcal{X}$. By the first case, $\alpha_{\gamma_i}(\mathbf{x}_j, s_j) = 0$ for $j > i$ (since $|\gamma_j| \leq |\gamma_i|$), so $\alpha_{\gamma_i}(\mathbf{x}, s) = \alpha_{\gamma_i}(\mathbf{x}_i, s_i) = A_{\gamma_i} s_i$ (via (14.3)). This shows that $\alpha_{\gamma_i}|_{E'_i} = A_{\gamma_i}|_{E'_i}$, and we can invoke thickness of linearity (Lemma 13.3) to get that $\alpha_{\gamma_i}|_{\overline{W}_i} = A_{\gamma_i}|_{\overline{W}_i}$. In particular, $\alpha_{\gamma_i}(\mathbf{x}_{\lambda'}, t_{\lambda'}) = A_{\gamma_i} t_{\lambda'} = A_{\gamma_i,\lambda'} t_{\lambda'} = 0$, where $A_{\gamma_i,\lambda'} = 0$ due to $\lambda' \not\equiv \gamma_i$. We established the desired equality (14.5). □

Let us record the linearity of α_{γ_i} on $\overline{W}_i$ (extracted from the paragraph above):

Corollary 14.2 *For any eigenvalue γ_i of ϕ and all $\mathbf{x} \in \mathcal{X}$, we have*

$$\alpha_{\gamma_i}(\mathbf{x}, \cdot)|_{\overline{W}_i} = A_{\gamma_i}|_{\overline{W}_i}. \tag{14.6}$$

We could say that, by restricting to $\overline{W}_i$, γ_i is freed of any pollution from other eigenvalues. More pertinently, the lemma cleaves apart the part of the Jordan spectrum of ϕ corresponding to γ_i and equates it with the analogous part of the Jordan spectrum of some rational map (namely $M|_{\overline{\mathcal{W}}_i}$), as follows:

Corollary 14.3 *$mult_\phi(\gamma_i, \cdot) = mult_{M|_{\overline{\mathcal{W}}_i}}(\gamma_i, \cdot) \quad (i = 1, \ldots, d)$*

Proof If $E_{\gamma_i} = 0$ then $\overline{\mathcal{W}}_{i,\gamma_i} = 0$ by the lemma and there is nothing to prove. Assume then that $E_{\gamma_i} \neq 0$. We have

$$\begin{aligned} \text{mult}_\phi(\gamma_i, \cdot) &= \text{mult}_{\phi|_{E_{\gamma_i}}}(\gamma_i, \cdot) \\ &= \text{mult}_{M|_{\overline{\mathcal{W}}_{i,\gamma_i}}}(\gamma_i, \cdot) \\ &= \text{mult}_{M|_{\overline{\mathcal{W}}_i}}(\gamma_i, \cdot) \end{aligned} \tag{14.7}$$

where the first equality is because E_{γ_i} is the generalized eigenspace of ϕ corresponding to γ_i, the second is due to $\phi|_{E_{\gamma_i}}$ being linearly conjugated to $M|_{\overline{\mathcal{W}}_{i,\gamma_i}}$ by the lemma, and the third owes to $\overline{\mathcal{W}}_{i,\gamma_i}$ being the generalized eigenspace of $M|_{\overline{\mathcal{W}}_i}$ corresponding to γ_i. □

Proof of (14.1), concluding the proof of Theorem 1.1 We have

$$\begin{aligned} \operatorname{mult}_\phi(\gamma_{i+1},\cdot) &= \operatorname{mult}_{\phi|_{\overline{W}_i}}(\gamma_{i+1},\cdot) \\ &= \operatorname{mult}_{M|_{\overline{\mathcal{W}}_i\cap\mathcal{E}}}(\gamma_{i+1},\cdot) \\ &\le \operatorname{mult}_{M|_{\overline{\mathcal{W}}_i}}(\gamma_{i+1},\cdot) \\ &= \operatorname{mult}_{M|_{\overline{\mathcal{W}}_i}}(\gamma_i,\cdot) \\ &= \operatorname{mult}_\phi(\gamma_i,\cdot). \end{aligned} \tag{14.8}$$

Here the first equality is due to $E_{\gamma_{i+1}} \subset E_i' \subset \overline{W}_i$ (where the first inclusion is by construction of E_i' and the second inclusion is (iii) of Proposition 13.2). The second equality is because $\phi|_{\overline{W}_i}$ is linearly conjugate to $M|_{\overline{\mathcal{W}}_i\cap\mathcal{E}}$, as follows from (v) of Proposition 13.2 and $A\circ\phi = M\circ A$. The inequality follows (via (2.2)) from $\mathcal{E}\cap\overline{\mathcal{W}}_i$ being invariantly complemented in $\overline{\mathcal{W}}_i$, as secured by (vi) of Proposition 13.2. The second to last equality is because $M|_{\overline{\mathcal{W}}_i}$ is rational and γ_{i+1} is algebraically conjugate to γ_i (so we can invoke (2.3)). The last equality is from Corollary 14.3. □

We are done except for the details of the thickening theory.

15 Subspace thickening: fundamentals of W_E

This is the first of three sections establishing the properties of the thickening procedure summarized in Sect. 13. We fix an arbitrary ϕ-invariant subspace $E \subset \mathbb{R}^{\mathrm{d}}$ and drop the subscript E from the notation, so $W := W_E$ and $\mathcal{W} := \mathcal{W}_E$. Our immediate goal is to complete the proof of Proposition 13.1 by showing (iii) (to the effect that $E \subset W_E$, see Lemma 15.2 below). But we also give a different definition of W and $\mathcal{W}$, which jibes better with α_u and is used in Sects. 16 and 17. Indeed, diverging from (15.1) (and [10]), we are compelled to start with an a priori $\mathbf{x}$ dependent variant of W built from the set $\mathcal{R}_{\mathbf{x},R}-\mathcal{R}_{\mathbf{x},R} := \{t_1-t_2 : t_1, t_2 \in \mathcal{R}_{\mathbf{x},R}\}$ of differences of R-return vectors for $\mathbf{x}$ (as introduced in Sect. 10). For $\mathbf{x}\in\mathcal{X}$ and $R, \epsilon > 0$, we have the subspace of $\mathbb{R}^{\mathrm{d}}$ generated by such differences that are ϵ-close to E:

$$W_{\mathbf{x},R,\epsilon} := \operatorname{span}_{\mathbb{R}}\left\{v\in\mathbb{R}^{\mathrm{d}} : v\in(\mathcal{R}_{\mathbf{x},R}-\mathcal{R}_{\mathbf{x},R})\cap B_\epsilon(E)\right\}\subset\mathbb{R}^{\mathrm{d}}. \tag{15.1}$$

It is only natural to take the limit with $\epsilon \to 0$ and $R \to \infty$, i.e., we define

$$W_{\mathbf{x},R} := \bigcap_{\epsilon>0} W_{\mathbf{x},R,\epsilon} \quad \text{and} \quad W_{\mathbf{x}} := \bigcap_{R>0} W_{\mathbf{x},R}. \tag{15.2}$$

As before, there is $\epsilon_{\mathbf{x},R} > 0$ such that $W_{\mathbf{x},R,\epsilon}$ stabilizes so that $W_{\mathbf{x},R} = W_{\mathbf{x},R,\epsilon}$ provided $\epsilon \in (0, \epsilon_{\mathbf{x},R}]$, and there is $R_{\mathbf{x}} > 0$ such that $W_{\mathbf{x}} = W_{\mathbf{x},R}$ provided $R \in [R_{\mathbf{x}}, \infty)$.

To come back to W, we link Θ with the differences of return vectors and prove the following. (Below the norm in $\mathbb{R}^{\mathrm{d}}$ is the ϕ-adapted norm, used in Sects. 7 and 9, and $\|\phi\|$ is the associated operator norm of ϕ.)

Proposition 15.1 (i) *$W_{\mathbf{x}} = W$ for all $\mathbf{x} \in \mathcal{X}$;*
(ii) *For any $R > 0$, there is $n = n(R) \geq 0$ so that*

$$\begin{aligned} &v \in \Theta \cap B_{\epsilon}(E) \\ &\quad \Longrightarrow \phi^n v \in (\mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}) \cap B_{\|\phi\|^n \epsilon}(E) \quad (\forall \mathbf{x} \in \mathcal{X},\ \epsilon > 0). \end{aligned}$$

Proof of Proposition 15.1 (ii) Take $R > 0$ and $\mathbf{x} \in \mathcal{X}$ and $\epsilon > 0$. Use primitivity to pick $n = n(R) \geq 0$ so that, for every i, $\Phi^n(i @ 0)$ contains a translated copy of any R-patch in $\mathbf{x}$. Take $\mathbf{x}' \in \mathcal{X}$ with $\Phi^n(\mathbf{x}') = \mathbf{x}$. By repetitivity, a vector $v \in \Theta \cap B_{\epsilon}(E)$ can be written as $v = s_2 - s_1$ where $i @ s_i \in \mathbf{x}'$ $(i = 1, 2)$. After applying Φ^n and locating translated copies of $\mathbf{x}|_R$ in $\Phi^n(i @ s_i)$, we see that $\mathbf{x}|_R = \mathbf{x} - t_1|_R = \mathbf{x} - t_2|_R$ where $t_2 - t_1 = \phi^n v \in B_{\|\phi\|^n \epsilon}(E)$. Of course, also $\phi^n v = t_2 - t_1 \in \mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}$ by construction.

(i) Fix $\mathbf{x} \in \mathcal{X}$. The inclusion $W \supset W_{\mathbf{x}}$ is immediate since $\mathcal{R}_{\mathbf{x}} - \mathcal{R}_{\mathbf{x}} \subset \Theta$. We show $W \subset W_{\mathbf{x}}$. Take any $R > 0$, and set $\epsilon > 0$ small enough that

$$\begin{aligned} W &:= \operatorname{span}_{\mathbb{R}} (\Theta \cap B_{\epsilon}(E)) \quad \text{and} \\ W_{\mathbf{x},R} &= \operatorname{span}_{\mathbb{R}} \left((\mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}) \cap B_{\|\phi\|^n \epsilon}(E) \right). \end{aligned}$$

Taking $n = n(R)$, (ii) yields $\phi^n W \subset W_{\mathbf{x},R}$. However, $\phi^n W = W$ because $\phi W \subset W$ and ϕ is a linear isomorphism. This shows $W \subset W_{\mathbf{x},R}$. Intersecting over all $R > 0$ gives $W \subset W_{\mathbf{x}}$. □

Lemma 15.2 *$E \subset W$.*

The proof borrows a compactness argument from Lemma 3.9 in [10]. It also reveals why we defined $W_{\mathbf{x}}$ by using the differences of the return vectors and not the vectors themselves.

Proof of Lemma 15.2 Since $W = W_{\mathbf{x}}$, we show that $E \subset W_{\mathbf{x}}$. Fix $\mathbf{x} \in \mathcal{X}$ and $u \in E$. Take an arbitrary $R > 0$ and set $\epsilon := \epsilon_{\mathbf{x},R} > 0$. By repetitivity, there is $R_0 = R_0(R) > 0$ such that any patch of $\mathbf{x}$ of radius larger than R_0 contains a translated copy of the central R-patch of $\mathbf{x}$. Specifically, there are $t_j \in \mathbb{R}^{\mathrm{d}}$ with $|t_j - ju| < R_0$ such that $\mathbf{x} - t_j|_R = \mathbf{x}|_R$ (i.e. $-t_j \in \mathcal{R}_{\mathbf{x},R}$) for all $j \in \mathbb{N}$. By this construction $t_j \in B_{R_0}(E)$, so the projections $s_j \in \mathbb{R}^{\mathrm{d}}$ of the t_j into some fixed subspace of $\mathbb{R}^{\mathrm{d}}$ complementary to E form a bounded sequence (s_j). Thus there is $s \in \mathbb{R}^{\mathrm{d}}$ and a sequence of pairs $(j_k, j'_k) \in \mathbb{N} \times \mathbb{N}$ such that $j_k - j'_k \to \infty$ and $s_{j_k} \to s$ and $s_{j'_k} \to s$. For all large k, we have then

$$t_{j_k} - t_{j'_k} \in B_\epsilon(E).$$

By the choice of ϵ, $(t_{j_k} - t_{j'_k}) \in W_{\mathbf{x},R}$ and thus also $w_k := \frac{1}{j_k - j'_k}(t_{j_k} - t_{j'_k}) \in W_{\mathbf{x},R}$. It remains to observe that

$$|w_k - u| \leq \frac{|(t_{j_k} - t_{j'_k}) - (j_k - j'_k)u|}{j_k - j'_k} \leq \frac{2R_0}{j_k - j'_k} \to 0,$$

which secures $u \in W_{\mathbf{x},R}$ because $W_{\mathbf{x},R}$ is closed. By arbitrariness of $u \in E$ and $R > 0$, $E \subset W_{\mathbf{x}}$. □

16 Subspace thickening: linearity is thick

This section is devoted to the proof of Lemma 13.3 about linearity of the cocycle on E automatically extending to the saturated thickening $\overline{W}_E$. Observe that it is enough to show linearity on the thickening W_E (see (iii) of Lemma 16.1 below), as then one can iterate the thickening to get to $\overline{W}_E$.

As before, set $W := W_E$ and $\mathcal{W} := \mathcal{W}_E$. Recall that $A_\gamma := \pi_\gamma \circ A$. Additionally, $\pi^\perp : t \mapsto t^\perp$ is a projection from $\mathbb{R}^{\mathrm{d}}$ along E and onto a subspace $E^\perp \subset \mathbb{R}^{\mathrm{d}}$ complementary to E. Also, $\pi_\parallel := \mathrm{Id} - \pi^\perp$ and $t^\parallel := \pi_\parallel(t)$. We do not require that $E^\perp$ is ϕ-invariant but it will be convenient to assume that $\pi^\perp$ is a weak contraction, i.e., with some overloading of the $|\cdot|$ notation, $|t^\perp| \leq |t|$ for all $t \in \mathbb{R}^{\mathrm{d}}$. (This can be achieved by assuming that the ϕ-adapted norm $|\cdot|$ on $\mathbb{R}^{\mathrm{d}}$ had been generated from a suitable inner product and by taking $E^\perp$ to be the orthogonal complement of E).

Lemma 16.1 *Suppose that γ is an unstable eigenvalue of M (i.e. $|\gamma| > 1$) and $\alpha_\gamma(\mathbf{x}, \cdot)|_E = A_\gamma|_E$ for all $\mathbf{x} \in \mathcal{X}$. Then, for all $\mathbf{x} \in \mathcal{X}$, we have*

(i) $\alpha_\gamma(\mathbf{x}, t) = \alpha_\gamma(\mathbf{x}, t^\perp) + A_\gamma t^\parallel$ *(for all $t \in \mathbb{R}^{\mathrm{d}}$)*;
(ii) *If $a \in \mathbb{R}^{\mathrm{d}}$ and $t_1, t_2 \in \mathcal{R}_{\mathbf{x},R}$ with $|a^\perp| < R$ and $|(t_1 - t_2 + a)^\perp| < R$ then*

$$\alpha_\gamma(\mathbf{x} + a^\perp, t_1 - t_2) = \pi_\gamma(1 \otimes (t_1 - t_2));$$

(iii) $\alpha_\gamma(\mathbf{x}, \cdot)|_{W_{\mathbf{x}}} = A_\gamma|_{W_{\mathbf{x}}}$.

Proof of Lemma 16.1 (i) Just use our hypothesis and the cocycle property:

$$\alpha_\gamma(\mathbf{x}, t) = \alpha_\gamma(\mathbf{x}, t^\parallel + t^\perp) = \alpha_\gamma(\mathbf{x}, t^\perp) + \alpha_\gamma(\mathbf{x} + t^\perp, t^\parallel) = \alpha_\gamma(\mathbf{x}, t^\perp) + A_\gamma t^\parallel.$$

(ii) Note that, in terms of the cocycle $\beta(\mathbf{x}, t) := \alpha_\gamma(\mathbf{x}, t) - A_\gamma t^\parallel$, (i) amounts to

$$\beta(\mathbf{x}, t) = \beta(\mathbf{x}, t^\perp) \quad (\forall \mathbf{x} \in \mathcal{X},\ t \in \mathbb{R}^{\mathrm{d}}). \tag{16.1}$$

Take $s := (t_1 - t_2)^\perp$. From $t_2 \in \mathcal{R}_{\mathbf{x},R}$, $\mathbf{x}|_R = \mathbf{x} + t_2|_R$ so also $\mathbf{x} + s + a^\perp|_{R'} = \mathbf{x} + s + a^\perp + t_2|_{R'}$ for $R' := R - |s + a^\perp| > 0$. That is t_2 is a return vector for $\mathbf{x} + a^\perp + s$. Hence, $\alpha_\gamma(\mathbf{x} + a^\perp + s, t_2) = \pi_\gamma(1 \otimes t_2)$ by Proposition 10.1. Likewise, $\alpha_\gamma(\mathbf{x} + a^\perp, t_1) = \pi_\gamma(1 \otimes t_1)$. Using (i), the cocycle property, and (i) again yields then

$$\begin{aligned}
\alpha_\gamma(\mathbf{x} + a^\perp, t_1 - t_2) &= \beta(\mathbf{x} + a^\perp, t_1 - t_2) + A_\gamma(t_1 - t_2)^\parallel \\
&= \beta(\mathbf{x} + a^\perp, s) + A_\gamma(t_1 - t_2)^\parallel \\
&= \beta(\mathbf{x} + a^\perp, t_2 + s) - \beta(\mathbf{x} + a^\perp + s, t_2) + A_\gamma(t_1 - t_2)^\parallel \\
&= \beta(\mathbf{x} + a^\perp, t_1) - \beta(\mathbf{x} + a^\perp + s, t_2) + A_\gamma(t_1 - t_2)^\parallel \\
&= \alpha_\gamma(\mathbf{x} + a^\perp, t_1) - \alpha_\gamma(\mathbf{x} + a^\perp, t_2) \\
&= \pi_\gamma(1 \otimes t_1) - \pi_\gamma(1 \otimes t_2) \\
&= \pi_\gamma(1 \otimes (t_1 - t_2)).
\end{aligned} \tag{16.2}$$

(iii) As we said before, the core mechanism comes from Lemma 3.8 in [10].

It suffices to show linearity on $W_{\mathbf{x}}$ for the cocycle β, as then $\alpha_\gamma|_{W_{\mathbf{x}}}$ is also linear and thus equal to its linearization $A_\gamma|_{W_{\mathbf{x}}}$. The cocycle β is constant on the cosets of E by (16.1). Fix $\mathbf{x} \in \mathcal{X}$ and $R > 0$. **Provided** $|a^\perp|, |v^\perp|, |a^\perp + v^\perp| < R$ with $a \in \mathbb{R}^{\mathrm{d}}$ and $v \in \mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}$, (ii) gives $\alpha_\gamma(\mathbf{x} + a^\perp, v) = \pi_\gamma(1 \otimes v) = \alpha_\gamma(\mathbf{x}, v)$ so that

$$\beta(\mathbf{x} + a^\perp, v^\perp) = \alpha_\gamma(\mathbf{x} + a^\perp, v) - A_\gamma v^\parallel = \alpha_\gamma(\mathbf{x}, v) - A_\gamma v^\parallel = \beta(\mathbf{x}, v^\perp), \tag{16.3}$$

and we can use the cocycle property to write

$$\begin{aligned}
\beta(\mathbf{x}, v^\perp + a^\perp) &= \beta(\mathbf{x}, a^\perp) + \beta(\mathbf{x} + a^\perp, v^\perp) \\
&= \beta(\mathbf{x}, a^\perp) + \beta(\mathbf{x}, v^\perp).
\end{aligned} \tag{16.4}$$

By replacing $a^\perp$ with $v_1^\perp + a^\perp$ where $v_1 \in \mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}$, two applications of (16.4) yield

$$\begin{aligned}
\beta(\mathbf{x}, v^\perp + v_1^\perp + a^\perp) &= \beta(\mathbf{x}, v^\perp) + \beta(\mathbf{x}, v_1^\perp + a^\perp) \\
&= \beta(\mathbf{x}, v^\perp) + \beta(\mathbf{x}, v_1^\perp) + \beta(\mathbf{x}, a^\perp)
\end{aligned}$$

provided $|a^\perp + v_1^\perp|, |v^\perp|, |a^\perp + v_1^\perp + v^\perp| < R$ and $|a^\perp|, |v_1^\perp|, |a^\perp + v_1^\perp| < R$, which two conditions simplify to $|a^\perp|, |v_1^\perp|, |v^\perp| < R$ and $|v_1^\perp + a^\perp|, |v^\perp + v_1^\perp + a^\perp| < R$.

Continuing in this way, after the total of $r + 1$ applications of (16.4) (then setting $a = 0$ and using $\beta(\mathbf{x}, 0) = 0$), results in

$$\beta\left(\mathbf{x}, \sum_{i=0}^{r} v_i^\perp\right) = \sum_{i=0}^{r} \beta(\mathbf{x}, v_i^\perp) \tag{16.5}$$

provided $v_i \in \mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}$ $(i = 0, \ldots, r, r \in \mathbb{N})$ and the following conditions hold

$$|v_i^\perp| < R \quad \text{for } i = 1, \ldots, r \tag{16.6}$$

and

$$|v_j^\perp + \cdots + v_r^\perp| < R \quad \text{for } j = 1, \ldots, r. \tag{16.7}$$

It remains to use an approximation argument to obtain a true $\mathbb{R}$-linearity from (16.5). Fix then an arbitrary $\epsilon > 0$. For small $\epsilon' > 0$, take $w_1, \ldots, w_m$ in $(\mathcal{R}_{\mathbf{x},R} - \mathcal{R}_{\mathbf{x},R}) \cap B_{\epsilon'}(E)$ that form a basis of the linear space $W_{\mathbf{x}}$. Consider the linear combinations of the form $\sum_{i=0}^{r} v_i^\perp$ with each v_i equal to some w_k and satisfying the version of (16.6) and (16.7) with R **replaced** by $R/2$. Such linear combinations are ϵ-dense in $B_{R/4}(E) \cap W_{\mathbf{x}} \cap E^\perp$ provided $\epsilon' > 0$ is small enough.

Therefore, for any $u_1, u_2 \in W_{\mathbf{x}} \cap B_{R/4}(E)$, we can first approximate each of $u_1^\perp$ and $u_2^\perp$ by a linear combination as above and then add the two approximations to approximate $u_1^\perp + u_2^\perp$ by a linear combination that satisfies conditions (16.6) and (16.7). Invoking (16.5) three times gives then

$$\beta(\mathbf{x}, u_1^\perp + u_2^\perp) = \beta(\mathbf{x}, u_1^\perp) + \beta(\mathbf{x}, u_2^\perp) + \text{Error} \tag{16.8}$$

where Error converges to zero with $\epsilon \to 0$ by the continuity of $\beta(\mathbf{x}, \cdot)$. After passing to the limit and using the arbitrariness of $R > 0$, we arrive at an identity that holds for all $u_1, u_2 \in W_{\mathbf{x}}$:

$$\beta(\mathbf{x}, u_1^\perp + u_2^\perp) = \beta(\mathbf{x}, u_1^\perp) + \beta(\mathbf{x}, u_2^\perp) \quad (\forall u_1, u_2 \in W_{\mathbf{x}}). \tag{16.9}$$

This implies that $\beta(\mathbf{x}, \cdot)$ is linear on $E^\perp$ and thus on all of $W_{\mathbf{x}}$ (via (16.1)). □

17 Subspace thickening: $\overline{W}_E$ vis-à-vis $\overline{\mathcal{W}}_E$

We turn attention to the saturated thickening in order to complete the proof of Proposition 13.2 by showing (iv, v, vi), which are Lemma 17.1 and (b) and (c) of Corollary 17.3, respectively. Our considerations apply to any ϕ-invariant $\overline{W} \subset \mathbb{R}^{\mathrm{d}}$ and $\overline{\mathcal{W}} \subset \mathcal{V}$ such that the saturation property holds

$$\exists_{\epsilon>0} \quad \overline{W} = \operatorname{span}_{\mathbb{R}} \left\{ v : \ v \in \Theta \cap B_\epsilon(\overline{W}) \right\} \tag{17.1}$$

and

$$\overline{\mathcal{W}} = \operatorname{span}_{\mathbb{R}} \left\{ 1 \otimes v : \ v \in \Theta \cap \overline{W} \right\}. \tag{17.2}$$

(This is the case for $\overline{W} := \overline{W}_E$ and $\overline{\mathcal{W}} := \overline{\mathcal{W}}_E$ by (13.3) and (13.4).)

Lemma 17.1

$$\overline{\mathcal{W}}^u = \operatorname{span}_{\mathbb{R}} \left\{ \alpha_u(\mathbf{x}, s) : \ s \in \overline{W}, \ \mathbf{x} \in \mathcal{X} \right\}.$$

To prove the lemma, we need the following general fact.

Fact 17.2 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function such that*

$$\forall_{t,s \in \mathbb{R}^n} \exists_{m \in \mathbb{N}} \ f(t + ms) = f(t). \tag{17.3}$$

Then f is constant.

Proof of Fact 17.2 A non-constant continuous f would have an uncountable family of disjoint level sets $\{f^{-1}(z)\}_{z \in f(\mathbb{R}^n)}$. This is impossible if the interiors $\operatorname{int}(f^{-1}(z))$ are all non-empty, which we show now. For $z \in f(\mathbb{R}^n)$, fix $t \in f^{-1}(z)$ and consider the translated level set $L_t := f^{-1}(f(t)) - t = \{s \in \mathbb{R}^n : \ f(t+s) = f(t)\}$. For any $s \in \mathbb{R}^n$, $ms \in L_t$ for some $m \in \mathbb{N}$, i.e., $\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} \frac{1}{m} L_t$. Thus $\operatorname{int}(L_t) \neq \emptyset$ by Baire's Theorem. □

Proof of Lemma 17.1 The saturation property amounts to $W_{\overline{W}} = \overline{W}$ and $\overline{\mathcal{W}} = \mathcal{W}_{\overline{W}}$. Thus (iv) of Proposition 13.1 with $E = \overline{W}$ gives the inclusion

$$\overline{\mathcal{W}}^u = \mathcal{W}^u_{\overline{W}} \subset \operatorname{span}_{\mathbb{R}} \{\alpha_u(\mathbf{x}, s) : \ s \in W_{\overline{W}} = \overline{W}, \ \mathbf{x} \in \mathcal{X}\}.$$

The harder part is the inclusion $\operatorname{span}_{\mathbb{R}} \{\alpha_u(\mathbf{x}, s) : \ s \in \overline{W}, \ \mathbf{x} \in \mathcal{X}\} \subset \overline{\mathcal{W}}^u$. First we claim that it suffices to show the following

$$\exists_{m_0} \forall_{\mathbf{x} \in \mathcal{X}, s \in \overline{W}} \exists_{m \in \{1, \dots, m_0\}} \ \alpha_u(\mathbf{x}, ms) \in \overline{\mathcal{W}}. \tag{17.4}$$

Indeed, fix $\mathbf{x} \in \mathcal{X}$ and consider the function $f(t) := \xi \circ \alpha_u(\mathbf{x}, t)$ where $t \in \overline{W} \simeq \mathbb{R}^n$ (for some n) and ξ is an arbitrary linear functional vanishing on $\overline{\mathcal{W}}$. Note that (17.4) secures the hypothesis (17.3) because the cocycle property (8.3) gives

$$f(t + ms) = f(t) + \xi \circ \alpha_u(\mathbf{x} + t, ms).$$

Therefore, by Fact 17.2, f is constant equal to $f(0) = 0$. By arbitrariness of ξ, $\alpha_u(\mathbf{x}, t) \in \overline{\mathcal{W}}$.

It remains to prove (17.4). Fix $s \in \overline{W}$ and $\mathbf{x} \in \mathcal{X}$ and $\epsilon > 0$ as in (17.1). Take any $R > 0$. By repetitivity, there is $L > 0$ so that, for any $n, j \in \mathbb{N}$, we can find a return vector $t_{n,j} \in \mathcal{R}_{\Phi^n \mathbf{x}, R}$ with $|j\phi^n s - t_{n,j}| < L$. Since the distance of $t_{n,j}$ from $\overline{W}$ is uniformly bounded (by L), there is $m_0 \in \mathbb{N}$ that depends only on ϵ and L so that the set $\{t_{n,1}, \dots, t_{n,m_0}\}$ contains two distinct points, denote them t_{j_n} and $t_{j_n'}$, such that $t_{j_n} - t_{j_n'} \in B_\epsilon(\overline{W})$. In particular, $t_{j_n} - t_{j_n'}$ are in $\overline{W}$ by the saturation property (17.1).

Now, along a subsequence of $n \to \infty$, the difference $j_n - j_n'$ is constant and equal to some $m \in \{1, \dots, m_0\}$. Because

$$|m\phi^n s - (t_{j_n} - t_{j_n'})| = |(j_n - j_n')\phi^n s - (t_{j_n} - t_{j_n'})| < 2L$$

there is a constant $C = C(L) > 0$ so that, uniformly in $\mathbf{x}$ and n,

$$|\alpha_u(\Phi^n \mathbf{x}, m\phi^n s) - \alpha_u(\Phi^n \mathbf{x}, t_{j_n} - t_{j_n'})| \le C.$$

Hence, by using that M_u^{-1} is a contracting map of $\overline{\mathcal{W}}^u$ onto itself, we can take the limit (along the subsequence) as follows

$$\begin{aligned} \alpha_u(\mathbf{x}, ms) &= M_u^{-n} \alpha_u(\Phi^n \mathbf{x}, m\phi^n s) \\ &= \lim_{n\to\infty} M_u^{-n} \alpha_u(\Phi^n \mathbf{x}, t_{j_n} - t_{j_n'}) \\ &= \lim_{n\to\infty} M_u^{-n} (1 \otimes (t_{j_n} - t_{j_n'}))^u. \end{aligned} \tag{17.5}$$

Since $(1 \otimes (t_{j_n} - t_{j_n'}))^u \in \overline{\mathcal{W}}^u$, this gives $\alpha_u(\mathbf{x}, ms) \in \overline{\mathcal{W}}^u$. □

Corollary 17.3 *We have*

(a) $A(\overline{W}) \subset \overline{\mathcal{W}}$,
(b) $\overline{\mathcal{W}} = (\mathcal{E} \cap \overline{\mathcal{W}}) \oplus (\mathcal{K} \cap \overline{\mathcal{W}})$,
(c) $A(\overline{W}) = \mathcal{E} \cap \overline{\mathcal{W}}$.

Proof (a) Fix $\mathbf{x} \in \mathcal{X}$. For any $v \in \overline{W}$, the definition (11.1) of A gives $Av = \lim_{t\to\infty} \frac{1}{t} \alpha_u(\mathbf{x}, tv)$ so $Av \in \overline{\mathcal{W}}$ since $\alpha_u(\mathbf{x}, tv) \in \overline{\mathcal{W}}$ by the lemma.

(b) The inclusion "$\supset$" is clear. To show "$\subset$", take $w \in \overline{\mathcal{W}}$. By (11.6), $w = e + k$ where $e \in \mathcal{E}$ and $k \in \mathcal{K}$. Since $\pi_{\mathbb{R}^d}(\overline{\mathcal{W}}) = \overline{W}$ (by (13.4)), we have $s := \pi_{\mathbb{R}^d}(e) = \pi_{\mathbb{R}^d}(w) \in \overline{W}$, and $e = As$ (because $A \circ \pi_{\mathbb{R}^d}|_{\mathcal{E}} = \mathrm{Id}_{\mathcal{E}}$). In particular, $e \in A(\overline{W}) \subset \overline{\mathcal{W}}$. Then also $k = w - e \in \overline{\mathcal{W}}$. Therefore $e \in \mathcal{E} \cap \overline{\mathcal{W}}$ and $k \in \mathcal{K} \cap \overline{\mathcal{W}}$, so w belongs to the direct sum in (b).

(c) The inclusion "$\subset$" follows from (a) (and $A(\mathbb{R}^d) \subset \mathcal{E}$). For "$\supset$", from the proof above, any $e \in \mathcal{E} \cap \overline{\mathcal{W}}$ is of the form $e = As$ for $s := \pi_{\mathbb{R}^d}(e) \in \overline{W}$. □

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Appendix A: Cocycle regularity

In this appendix we complete the proof of Proposition 8.1 on regularity of the cocycles $\alpha_{u/su}$. (Recall that the continuity of $\alpha_{u/su}$ follows from Lemma 7.2.)

Proof of Proposition 8.1 (i) Recall that the local transversal constancy in (i) has been already defined and demonstrated in Sect. 9 (see Proposition 9.2). What remains to be shown is the Hölder continuity of $\alpha_u(\mathbf{x}, t)$ in t. This readily follows from the self-affinity (8.4) by a standard argument already exploited in [10], which we include for the reader's convenience.

Consider two different $t, t' \in \mathbb{R}^{\mathrm{d}}$. Since we are using the adapted norm on $\mathbb{R}^{\mathrm{d}}$, we have $|\phi^n t - \phi^n t'| \geq \lambda_\phi^n |t - t'|$ (where $\lambda_\phi > 1$). There is then the smallest $n \in \mathbb{N}$ so that $|\phi^n t - \phi^n t'| \geq 1$. Note that $n - 1 \leq -\ln|t - t'|/\ln\lambda_\phi$. Because $|\phi^{n-1} t - \phi^{n-1} t'| < 1$ also $|\phi^n t - \phi^n t'| < \|\phi\|$, so the uniform continuity of α_u on a compact set implies that

$$\left|\alpha_u(\Phi^n \mathbf{x}, \phi^n t) - \alpha_u(\Phi^n \mathbf{x}, \phi^n t')\right| = \left|\alpha_u(\Phi^n \mathbf{x} + \phi^n t', \phi^n t - \phi^n t')\right| \leq C \tag{18.1}$$

where $C > 0$ is a constant that is independent of $\mathbf{x}, t, t', n$.

By using (8.4) and (18.1) and the upper bound for n, we get

$$\begin{aligned} |\alpha_u(\mathbf{x}, t) - \alpha_u(\mathbf{x}, t')| &= \left|M_u^{-n}(\alpha_u(\Phi^n \mathbf{x}, \phi^n t) - \alpha_u(\Phi^n \mathbf{x}, \phi^n t'))\right| \\ &\leq \|M_u^{-1}\|^n C \\ &\leq C_1 \exp\left((-\ln|t - t'|/\ln\lambda_\phi)\ln\|M_u^{-1}\|\right) \\ &= C_1 |t - t'|^{-\ln\|M_u^{-1}\|/\ln\lambda_\phi} \end{aligned}$$

This proves that $\alpha_u(\mathbf{x}, t)$ is $-\ln\|M_u^{-1}\|/\ln\lambda_\phi$-Hölder in t with the constant $C_1 := \|M_u^{-1}\| C$ independent of $\mathbf{x} \in \mathcal{X}$.

(ii) Fix $\lambda > \|\phi\|$. We have to invoke the recognizability theorem in [24] asserting not only that Φ is invertible but also that a central patch of $\Phi^{-1}(\mathbf{x})$ is determined by a (perhaps large) central patch of $\mathbf{x}$. By a simple argument (using (5.6)), there is then $R_0 > 0$ such that

$$\forall_{R > R_0} \quad \mathbf{x}|_R = \mathbf{y}|_R \quad \Longrightarrow \quad \Phi^{-1}(\mathbf{x})|_{\lambda^{-1}R} = \Phi^{-1}(\mathbf{y})|_{\lambda^{-1}R} \quad (\forall \mathbf{x}, \mathbf{y} \in \mathcal{X}). \tag{18.2}$$

Fix any $L > 0$. Suppose that $\mathbf{x}|_R = \mathbf{y}|_R$. Take the maximal $n \geq 0$ such that $\lambda^{-n} R \geq R_0$. Then $\lambda^{-n-1} R \leq R_0$ so $n + 1 \geq (\ln R - \ln R_0)/\ln\lambda$. Also, setting $\mathbf{x}_{-n} := \Phi^{-n}\mathbf{x}$ and $\mathbf{y}_{-n} := \Phi^{-n}\mathbf{y}$, the continuity of α_s implies that

$$|\alpha_s(\mathbf{x}_{-n}, s) - \alpha_s(\mathbf{y}_{-n}, s)| \leq C_2 \qquad (\forall |s| < L) \tag{18.3}$$

where $C_2 > 0$ is a constant that is independent of $\mathbf{x}, \mathbf{y}, n$. Hence, as long as $|t| < L$ (and thus also $|\phi^{-n}t| < L$), putting together (18.3) and the lower bound on $n+1$ gives

$$\begin{aligned}|\alpha_s(\mathbf{x}, t) - \alpha_s(\mathbf{y}, t)| &= \left|M_s^n \alpha_s(\mathbf{x}_{-n}, \phi^{-n}t) - M_s^n \alpha_s(\mathbf{y}_{-n}, \phi^{-n}t)\right| \\ &\leq \|M_s\|^n C_2 \\ &\leq C_3 \exp\left(\frac{\ln R - \ln R_0}{\ln \lambda} \ln \|M_s\|\right) \\ &\leq C_3\, R^{\ln \|M_s\| / \ln \lambda}\end{aligned}$$

where the constant $C_3 := C_2 \|M_s\|^{-1} > 0$ is independent of $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. □

Remark 18.1 As for any continuous cocycle, there is $C > 0$ so that

$$|\alpha_u(\mathbf{x}, t)| \leq C + C|t| \quad (\forall\, t \in \mathbb{R}^{\mathrm{d}}, \mathbf{x} \in \mathcal{X}).$$

Via the cocycle property, this gives that there is $L > 0$ such that, for any $\mathbf{x} \in \mathcal{X}$, $\alpha_u(\mathbf{x}, t)$ is L-Lipschitz at large scales, i.e.,

$$|\alpha_u(\mathbf{x}, t) - \alpha_u(\mathbf{x}', t')| \leq L|t - t'| + L \quad (\forall\, t, t' \in \mathbb{R}^{\mathrm{d}},\ \mathbf{x}, \mathbf{x}' \in \mathcal{X}).$$

The same can be said about α_s when Φ is invertible.

Appendix B: The hull metric

Recall that $\mathbf{d}_{\mathrm{H}}$, given by (5.1), is the Hausdorff metric on the space of all Delone multisets induced by the spherical metric $\mathbf{d}$ on $\mathbb{R}^{\mathrm{d}}$. The following shows that, owing to the finite local complexity of $\mathbf{\Lambda}$, two Delone multisets $\mathbf{x}, \mathbf{x}'$ in $\mathcal{X}$ are close in the $\mathbf{d}_{\mathrm{H}}$-sense iff they coincide, up to a small translation, on a big ball $B_R(0)$ around the origin. (In our notation, this coincidence is expressed as $\mathbf{x} + \tau|_R = \mathbf{x}' + \tau'|_R$.)

Proposition 19.1 (i) *For any sufficiently large $R > 0$ there is $\delta > 0$ such that*

$$\mathbf{d}_D(\mathbf{x}, \mathbf{x}') < \delta \implies \exists_{\tau, \tau' \in B_{1/R}(0)} \quad \mathbf{x} + \tau|_R = \mathbf{x}' + \tau'|_R \qquad (\forall \mathbf{x}, \mathbf{x}' \in \mathcal{X}).$$

(ii) *For any $\delta > 0$ there is $R > 0$ such that*

$$\exists_{\tau, \tau' \in B_{1/R}(0)} \quad \mathbf{x} + \tau|_R = \mathbf{x}' + \tau'|_R \implies \mathbf{d}_D(\mathbf{x}, \mathbf{x}') < \delta \qquad (\forall \mathbf{x}, \mathbf{x}' \in \mathcal{X}).$$

We note that, the infimum $\inf 1/R$ where R is as in the proposition is used to express the distance between $\mathbf{x}$ and $\mathbf{x}'$; in fact, the quantity $\min\{\inf 1/R, 1/\sqrt{2}\}$ is taken as a metric on Delone multisets in [14]. As a result, the completions of the set of all translates of $\boldsymbol{\Lambda}$ with respect to this metric or our $\mathbf{d}_D$ coincide.

Proof of Proposition 19.1 (i) Let R_D be such that each component set Λ_i of $\boldsymbol{\Lambda}$ is R_D-dense in $\mathbb{R}^{\mathrm{d}}$. To avoid dealing with empty sets we will assume $R > R_D$. First, consider the collection of all patches of the form $\mathbf{x}|_{R+1/R}$ where $\mathbf{x} \in \mathcal{X}$. By the finite local complexity hypothesis, the number of translation equivalence classes of such patches is finite. Let $\mathcal{P}$ be a finite subcollection containing one patch from each class. There is $\delta > 0$ so that no two patches in $\mathcal{P}$ can be translated to be brought closer in the $\mathbf{d}_D$-distance than δ.

If the proposition were to fail, there would be $R > R_D$ and $\mathbf{x}_n, \mathbf{x}'_n \in \mathcal{X}$ such that $\lim \mathbf{d}_D(\mathbf{x}_n, \mathbf{x}'_n) = 0$ and $\mathbf{x}_n + \tau_n|_R \neq \mathbf{x}'_n + \tau'_n|_R$ no matter what $|\tau_n|, |\tau'_n| < 1/R$ are taken. By passing to a subsequence, we can assume that $\mathbf{x}_n|_{R+1/R} = P + u_n$ and $\mathbf{x}'_n|_{R+1/R} = Q + v_n$ for some fixed $P, Q \in \mathcal{P}$ and $u_n, v_n \in \mathbb{R}^{\mathrm{d}}$. As soon as $\mathbf{d}_D(\mathbf{x}_n, \mathbf{x}'_n) < \delta$, we see that the choice of $\delta > 0$ forces $P = Q$. Also, $\lim \mathbf{d}_D(\mathbf{x}_n, \mathbf{x}'_n) = 0$ readily gives $\tau_n := v_n - u_n \to 0$. For n so large that $|\tau_n| = |u_n - v_n| < 1/R$, we get then

$$\mathbf{x}_n + \tau_n|_R = P + u_n + \tau_n|_R = P + v_n|_R = \mathbf{x}'_n|_R,$$

a contradiction.

(ii) This is a simple consequence of the continuity of the addition on $\mathbb{R}^{\mathrm{d}}$ and the property of the spherical metric that, for large enough $R > 0$ (and all $p, q \in \mathbb{R}^{\mathrm{d}}$), we have $|p|, |q| > R \implies \mathbf{d}(p, q) < \delta$. □

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