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Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces

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Abstract For the d-dimensional incompressible Euler equation, the standard energy method gives local wellposedness for initial velocity in Sobolev space $H^s(\mathbb{R}^d)$, $s > s_c := d/2 + 1$. The borderline case $s = s_c$ was a folklore open problem. In this paper we consider the physical dimension $d = 2$ and show that if we perturb any given smooth initial data in H^{s_c} norm, then the corresponding solution can have infinite H^{s_c} norm instantaneously at $t > 0$. In a companion paper [1] we settle the 3D and more general cases. The constructed solutions are unique and even C^∞-smooth in some cases. To prove these results we introduce a new strategy: *large Lagrangian deformation induces critical norm inflation*. As an application we also settle several closely related open problems.

1 Introduction

The d-dimensional incompressible Euler equation takes the form

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ \nabla \cdot u = 0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

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where $u = u(t,x) = (u_1(t,x), \ldots, u_d(t,x)) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ denotes the velocity of the fluid and $p = p(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is the pressure. The second equation $\nabla \cdot u = 0$ in (1.1) is usually called the incompressibility (divergence-free) condition. By taking the divergence on both sides of the first equation in (1.1), one can recover the pressure from the quadratic term in velocity by inverting the Laplacian in suitable functional spaces. Another way to eliminate the pressure is to use the vorticity formulation. For this we will discuss separately the 2D and 3D case. In 2D, introduce the scalar-valued vorticity function

$$\omega = -\partial_2 u_1 + \partial_1 u_2 = \nabla^\perp \cdot u, \quad \nabla^\perp := (-\partial_2, \partial_1).$$

By taking $\nabla^\perp \cdot$ on both sides of (1.1), we have the equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi), \quad \Delta \psi = \omega, \\ \omega\big|_{t=0} = \omega_0. \end{cases} \tag{1.2}$$

Under some suitable regularity assumptions, the second equations in (1.2) can be written as a single equation

$$u = \Delta^{-1} \nabla^\perp \omega, \tag{1.3}$$

which is the usual Biot–Savart law. Alternatively one can express (1.3) as a convolution integral

$$u = K * \omega, \quad K(x) = \frac{1}{2\pi} \cdot \frac{x^\perp}{|x|^2}, \quad x^\perp := (-x_2, x_1).$$

We can then rewrite (1.2) more compactly as

$$\begin{cases} \partial_t \omega + (\Delta^{-1} \nabla^\perp \omega \cdot \nabla) \omega = 0, \\ \omega\Big|_{t=0} = \omega_0. \end{cases} \tag{1.4}$$

We shall frequently refer to (1.4) as the usual 2D Euler equation in vorticity formulation. Note that (1.4) is a transport equation which preserves all L^p, $1 \le p \le \infty$ norm of the vorticity ω. In the 3D case the vorticity is vector-valued and given by

$$\omega = \operatorname{curl} u = \nabla \times u.$$

The 3D Euler equation in vorticity formulation has the form

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u, \\ u = -\Delta^{-1} \nabla \times \omega, \\ \omega\Big|_{t=0} = \omega_0. \end{cases}$$

Note that the second equation above is just the Biot–Savart law in 3D. The expression $(\omega \cdot \nabla)u$ is often referred to as the vorticity stretching term. It is one of the main sources of difficulties in the wellposedness theory of 3D Euler.

There is by now an extensive literature on the wellposedness theory for Euler equations. We shall only mention a few and refer to Majda-Bertozzi [23] and Constantin [8] for more extensive references. The papers of Lichtenstein [21] and Gunther [15] started the subject of local wellposedness in Hölder spaces $C^{k,\alpha}$ ($k \geq 1$, $0 < \alpha < 1$). In [29] Wolibner obtained global solvability of classical (belonging to Hölder class) solutions for 2D Euler (see Chemin [6] for a modern exposition). In [12] Ebin and Marsden proved the short time existence, uniqueness, regularity, and continuous dependence on initial conditions for solutions of the Euler equation on general compact manifolds (possibly with C^∞ boundary). Their method is to topologize the space of diffeomorphisms by Sobolev H^s, $s > d/2 + 1$ norms and then solve the geodesic equation using contractions. In [5] Bourguignon and Brezis generalized H^s to the case of $W^{s,p}$ for $s > d/p + 1$. In [18] Kato proved local wellposedness of d-dimensional Euler in $C_t^0 H_x^m$ for initial velocity $u_0 \in H^m(\mathbb{R}^d)$ with integer $m > d/2 + 1$. Later Kato and Ponce [20] proved wellposedness in the general Sobolev space $W^{s,p}(\mathbb{R}^d) = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^d)$ with real $s > d/p + 1$ and $1 < p < \infty$. The key argument in [20] is the following commutator estimate[1] for the operator $J^s = (1 - \Delta)^{s/2}$:

$$\|J^s(fg) - f J^s g\|_p \lesssim_{d,s,p} \|Df\|_\infty \|J^{s-1} g\|_p + \|J^s f\|_p \|g\|_\infty, \quad 1 < p < \infty,\ s \geq 0. \tag{1.5}$$

To extend the local solutions globally in time, one can use the Beale-Kato-Majda criterion [4] which asserts that (here $s > d/2 + 1$)

$$\limsup_{t \to T^*} \|u(t, \cdot)\|_{H^s(\mathbb{R}^d)} = +\infty,$$

[1] The L^∞ end-point Kato-Ponce inequality (conjectured in [14]) and several new Kato-Ponce type inequalities are proved in recent [2].

if and only if

$$\limsup_{t\to T^*} \int_0^t \|\omega(s,\cdot)\|_{L^\infty(\mathbb{R}^d)} ds = +\infty.$$

By using this criterion and conservation of $\|\omega\|_\infty$ in 2D, one can immediately deduce the global existence of Kato's solutions in dimension two. In [27] (see also [28]) Vishik considered the borderline case $s = d/p + 1$ and obtained global solvability for the 2D Euler in Besov space $B^{2/p+1}_{p,1}$ with $1 < p < \infty$. In [7] Chae proved local existence and uniqueness of solutions to d-dimensional Euler in critical Besov space (for velocity) $B^{d/p+1}_{p,1}(\mathbb{R}^d)$ with $1 < p < \infty$. The local wellposedness in $B^1_{\infty,1}(\mathbb{R}^d)$, $d \geq 2$ was settled by Pak and Park in [25]. Roughly speaking, all the aforementioned local wellposedness results rely on finding a certain Banach space X with the norm $\|\cdot\|_X$ such that (take $f = \nabla \times u$ and $X = B^{d/p}_{p,1}$ for example)

(1) If $f \in X$, then $\|f\|_{L^\infty} + \|\mathcal{R}_{ij} f\|_{L^\infty} \lesssim \|f\|_X$ ($\mathcal{R}_{ij}$ is the Riesz transform);
(2) Some version of a commutator estimate similar to (1.5) holds in X.

The above are essentially minimal conditions needed to close the energy estimates. On the other hand, this type of scheme completely breaks down for the natural borderline Sobolev spaces such as $H^{d/2+1}$ (in terms of vorticity we have $X = H^{d/2}$) since both conditions will be violated. In [9,11], well-posedness in critical $H^{d/2+1}$ spaces were proved for some logarithmically regularized Euler equations. In [26], Takada constructed[2] several counterexamples of Kato-Ponce-type commutator estimates in critical Besov $B^{d/p+1}_{p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin spaces $F^{d/p+1}_{p,q}(\mathbb{R}^d)$ for various exponents p and q (For Besov: $1 \leq p \leq \infty$, $1 < q \leq \infty$; For Triebel-Lizorkin: $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$). It should be noted that the vector fields used in his counterexamples are divergence-free. In light of these considerations, a well-known long standing open problem was the following

Conjecture 1.1 *The Euler equation (1.1) is ill-posed for a class of initial data in $H^{d/2+1}(\mathbb{R}^d)$.*

Of course one can state analogous versions of Conjecture 1.1 in similar Sobolev spaces $W^{d/p+1,p}$ or other Besov or Triebel-Lizorkin type spaces with various boundary conditions. A rather delicate matter is to give a precise (and satisfactory) formulation of the ill-posedness statement in Conjecture 1.1. The formulation and the proof of such a statement requires a deep understanding of how the critical space topology changes under the Euler dynamics.

[2] Counter examples for the case $s < d/p + 1$ was also considered therein.

To begin, one can consider explicit solutions to (1.1). In [13], DiPerna and Majda introduced the following shear flow (in their study of measure-valued solutions for 3D Euler):

$$u(t,x) = (f(x_2), 0, g(x_1 - tf(x_2))), \quad x = (x_1, x_2, x_3),$$

where f and g are given single variable functions. This explicit flow (sometimes called "2+1/2"-dimensional flow) solves (1.1) with pressure $p = 0$. DiPerna and Lions used the above flow (see e.g. p152 of [22]) to show that for every $1 \le p < \infty$, $T > 0$, $M > 0$, there exists a smooth shear flow for which $\|u(0)\|_{W^{1,p}(\mathbb{T}^3)} = 1$ and $\|u(T)\|_{W^{1,p}(\mathbb{T}^3)} > M$. Recently Bardos and Titi [3] revisited this example and constructed a weak solution which initially lies in C^α but does not belong to any C^β for any $t > 0$ and $1 > \beta > \alpha^2$. By similar arguments one can also deduce ill-posedness in $F^1_{\infty,2}$ and $B^1_{\infty,\infty}$ (see Remark 1 therein). In [24], Misiołek and Yoneda considered the logarithmic Lipschitz space $\mathrm{LL}_\alpha(\mathbb{R}^d)$ consisting of continuous functions such that

$$\|f\|_{\mathrm{LL}_\alpha} = \|f\|_\infty + \sup_{0<|x-y|<\frac{1}{2}} \frac{|f(x)-f(y)|}{|x-y||\log|x-y||^\alpha} < \infty.$$

They used the above shear-flow example to generate ill-posedness of 3D Euler in LL_α for any $0 < \alpha \le 1$. In connection with Conjecture 1.1, a related issue is the dependence of the solution operator on the underlying topology. In [19], to describe the sharpness of the continuous dependence on initial data in his well-posedness result, Kato showed that (see Example 5.2 therein) the solution operator for the Burgers equation is not Hölder continuous in $H^s(\mathbb{R})$, $s \ge 2$ norm for any prescribed Hölder exponent. In [16] Himonas and Misiołek proved that for the Euler equation the data-to-solution map is not uniform continuous in $H^s(\Omega)$ topology where $s \in \mathbb{R}$ if $\Omega = \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ and $s > 0$ if $\Omega = \mathbb{R}^d$. Very recently Inci [17] strengthened this result and showed for any $T > 0$ that the solution map $u(0) \to u(T)$ is nowhere locally uniformly continuous for $H^s(\mathbb{R}^n)$, $s > n/2+1$. In [10], Cheskidov and Shvydkoy proved ill-posedness of d-dimensional Euler in Besov spaces $B^s_{r,\infty}(\mathbb{T}^d)$ where $s > 0$ if $r > 2$ and $s > d(\frac{2}{r}-1)$ if $1 \le r \le 2$. However, as was pointed out by the aforementioned authors, the above works do not address the borderline Sobolev space $H^{d/2+1}$ or similar critical spaces which was an outstanding open problem.

The purpose of this work and the companion [1] is to completely settle the borderline case $H^{d/2+1}$ (Conjecture 1.1) and several other related open problems. Roughly speaking, we prove the following

Theorem *Let the dimension $d = 2, 3$. The Euler equation (1.1) is ill-posed in the Sobolev space $W^{d/p+1,p}$ for any $1 \le p < \infty$ or the Besov space $B^{d/p+1}_{p,q}$ for any $1 \le p < \infty$, $1 < q \le \infty$.*

As a matter of fact, we shall show that in the borderline case, ill-posedness holds in the strongest sense. Namely for *any* given smooth initial data, we shall find special perturbations which can be made arbitrarily small in the critical Sobolev norm, such that the corresponding perturbed solution is unique (in other functional spaces) but loses borderline Sobolev regularity instantaneously in time. Our analysis shows that in some sense the ill-posedness happens in a very generic way. In particular, it is "dense" in the $H^{d/2+1}$ (and similarly for other critical spaces) topology.

In order to expound the main ideas without clouding it by technicalities, we only treat the 2D case in this paper. The companion paper [1] is devoted to the (more technical) strong ill-posedness results in borderline Besov and Sobolev spaces for 3D. The prominent difficulty in extending the analysis of this paper to 3D is the vorticity stretching effect (e.g. it renders the L^∞-norm of vorticity hard to control) and certain nonlocal obstructions. We defer the discussion of these technical aspects to the end of this introduction.

We now state more precisely the main results. The first result is for 2D Euler with non-compactly supported data. A special feature is that our constructed solutions are C^∞-smooth which are classical solutions.

Theorem 1.2 (2D non-compact case) *For any given* $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ *and any* $\epsilon > 0$*, we can find a* C^∞ *perturbation* $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ *such that the following hold true:*

(1) $\|\omega_0^{(p)}\|_{\dot{H}^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \epsilon.$

(2) *Let* $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$*. The initial velocity* $u_0 = \Delta^{-1}\nabla^\perp\omega_0$ *has regularity* $u_0 \in H^2(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$*. The gradient of* u_0 *has no decay at infinity in the sense that*

$$\|\nabla u_0\|_{L^\infty(\mathbb{R}^2)} = +\infty.$$

(3) *There exists a unique classical solution* $\omega = \omega(t)$ *to the 2D Euler equation (in vorticity form)*

$$\begin{cases} \partial_t\omega + (\Delta^{-1}\nabla^\perp\omega \cdot \nabla)\omega = 0, & 0 < t \le 1,\ x \in \mathbb{R}^2, \\ \omega\Big|_{t=0} = \omega_0, \end{cases}$$

satisfying

$$\max_{0\le t\le 1}\Big(\|\omega(t,\cdot)\|_{L^1} + \|\omega(t,\cdot)\|_{L^\infty} + \|\omega(t,\cdot)\|_{\dot{H}^{-1}}\Big) < \infty.$$

Here $\omega(t) \in C^\infty$*,* $u(t) = \Delta^{-1}\nabla^\perp\omega(t) \in C^\infty \cap L^2 \cap L^\infty$ *for each* $0 \le t \le 1$.

(4) *For any* $0 < t_0 \le 1$, *we have*

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{H}^1} = +\infty. \tag{1.6}$$

Remark 1.3 The $\dot{H}^{-1}$ assumption on the vorticity data $\omega_0^{(g)}$ can be removed. We include it here simply to stress that the perturbed solution can inherit $\dot{H}^{-1}$ regularity which is natural since the corresponding velocity will be in L^2. Of course one can also state similar results for $\omega_0^{(g)} \in H^s$ with $s > 1$ or some other subcritical functional spaces.

Remark 1.4 In [19] Kato introduced the uniformly local Sobolev spaces $L^p_{ul}(\mathbb{R}^d)$ [see (2.3)] and $H^s_{ul}(\mathbb{R}^d)$. These spaces contain $H^s(\mathbb{R}^d)$ and the periodic space $H^s(\mathbb{T}^d)$. The statement (1.6) in Theorem 1.2 can be improved to

$$\text{ess-sup}_{0<t\le t_0} \|\nabla\omega(t,\cdot)\|_{L^2_{ul}(\mathbb{R}^2)} = +\infty.$$

Similar results also hold for Theorem 1.5 below.

Our next result is for the compactly supported data for the 2D Euler equation. Note that this result carries over (with simple changes) to the periodic case as well. For simplicity we shall consider vorticity functions having one-fold symmetry. For example, we shall say $g = g(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ is odd in x_2 if

$$g(x_1, -x_2) = -g(x_1, x_2), \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

It is not difficult to check that the one-fold odd symmetry (of the vorticity function) is preserved by the Euler flow.

Theorem 1.5 (2D compact case) *Let* $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ *be any given vorticity function which is odd in* x_2.[3] *For any such* $\omega_0^{(g)}$ *and any* $\epsilon > 0$, *we can find a perturbation* $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ *such that the following hold true:*

(1) $\omega_0^{(p)}$ *is compactly supported (in a ball of radius* ≤ 1*), continuous and*

$$\|\omega_0^{(p)}\|_{\dot{H}^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \epsilon.$$

(2) *Let* $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. *Corresponding to* ω_0 *there exists a unique time-global solution* $\omega = \omega(t)$ *to the Euler equation satisfying* $\omega(t) \in L^\infty \cap \dot{H}^{-1}$. *Furthermore* $\omega \in C_t^0 C_x^0$ *and*[4] $u = \Delta^{-1}\nabla^\perp\omega \in C_t^0 L_x^2 \cap C_t^0 C_x^\alpha$ *for any* $0 < \alpha < 1$.

[3] Similar results also hold for vorticity functions which are odd in x_1, or odd in both x_1 and x_2.

[4] Actually it is easy to show that u is log-Lipschitz.

(3) *$\omega(t)$ has additional local regularity in the following sense: there exists $x_* \in \mathbb{R}^2$ such that for any $x \neq x_*$, there exists a neighborhood $N_x \ni x$, $t_x > 0$ such that $\omega(t, \cdot) \in C^\infty(N_x)$ for any $0 \le t \le t_x$.*
(4) *For any $0 < t_0 \le 1$, we have*

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{H}^1} = +\infty.$$

More precisely, there exist $0 < t_n^1 < t_n^2 < \frac{1}{n}$, open precompact sets Ω_n, $n = 1, 2, 3, \ldots$ such that $\omega(t) \in C^\infty(\Omega_n)$ for all $0 \le t \le t_n^2$, and

$$\|\nabla\omega(t,\cdot)\|_{L^2(\Omega_n)} > n, \quad \forall\, t \in [t_n^1, t_n^2].$$

Remark 1.6 In [30] Yudovich proved the existence and uniqueness of weak solutions to 2D Euler in bounded domains for L^∞ vorticity data. In our construction, since we have uniform in time L^∞ control of the vorticity ω in 2D, the uniqueness of the constructed solution is not an issue and we shall not discuss this point further in this work.

In the rest of this introduction, we give a brief overview of the proofs of Theorems 1.2 and 1.5. The overall scheme consists of three steps. The first two steps are devoted to local constructions. The last step is a global patching argument. Some additional technical points needed to treat the 3D case in [1] will be clarified at the end.

Step 1. Creation of large Lagrangian deformation. Define the flow map associated to (1.1) as $\phi = \phi(t, x)$ which solves

$$\begin{cases} \partial_t\phi(t,x) = u(t,\phi(t,x)),\\ \phi(0,x) = x. \end{cases}$$

For any $0 < T \ll 1$, $B(x_0, \delta) \subset \mathbb{R}^2$ with $x_0 \in \mathbb{R}^2$ arbitrary and $\delta \ll 1$, we choose initial (vorticity) data $\omega_a^{(0)}$ with $\mathrm{supp}(\omega_a^{(0)}) \subset B(x_0, \delta)$ such that

$$\|\omega_a^{(0)}\|_{L^1} + \|\omega_a^{(0)}\|_{L^\infty} + \|\omega_a^{(0)}\|_{H^1} \ll 1,$$

and

$$\sup_{0<t\le T} \|D\phi_a(t,\cdot)\|_{L_x^\infty(B(x_0,\delta))} \gg 1.$$

Here ϕ_a is the flow map associated with the velocity $u = u_a$ which solves (1.1) with $\omega_a^{(0)}$ as vorticity initial data. By translation invariance of Euler it suffices to consider the case $x_0 = 0$. In our construction we restrict to some special flows which have odd symmetry and admit the origin as a stagnation point. We prove that the deformation matrix Du remains essentially hyperbolic near the spatial origin in the short time interval considered (cf. Propositions 3.4 and 3.5).

Step 2. Local inflation of the critical norm. As was already mentioned, the critical norm for the vorticity is H^1. The solution constructed in Step 1 does not necessarily obey $\sup_{0<t\le T} \|\nabla\omega_a(t)\|_2 \gg 1$. We then perturb the initial data $\omega_a^{(0)}$ and take

$$\omega_b^{(0)} = \omega_a^{(0)} + \frac{1}{k}\sin(kf(x))g(x),$$

where k is a very large parameter. The function g is smooth and has $o(1)$ L^2 norm.[5] The function $f(x)$ and the support of g will be chosen depending on the exact location of the maximum of $\|D\phi_a(t,\cdot)\|_\infty$. Of course since the initial data is altered, the corresponding characteristic line (flow map) is changed as well. For this we run a perturbation argument in $W^{1,4}$ so that $\|D\phi_b(t,\cdot) - D\phi_a(t,\cdot)\|_\infty \ll 1$. The same argument is used to show that in the main order the H^1 norm of the solution corresponding to $\omega_b^{(0)}$ is inflated through the Lagrangian deformation matrix $D\phi_a$. The technical details are elaborated in Proposition 4.2.

Step 3. Gluing of patch solutions. The construction in previous two steps can be repeated in infinitely many small patches which stay away from each other initially. To glue these solutions together we need to differentiate two situations. In the case of Theorem 1.2, we exploit the unboundedness nature of $\mathbb{R}^2$ and add each patches sequentially. Each time a new patch is added, we choose the distance between it and the old patches sufficiently large such that their interaction is very small. The key properties exploited here are the finite transport speed of the Euler flow and spatial decay of the Riesz kernel. In the case of Theorem 1.5, we need to deal with compactly supported data. This forces us to analyze in detail the interactions of the patches since the patches can become infinitely close to each other. For each $n \ge 2$, define $\omega_{\le n-1}$ the existing patch and ω_n the current (to be added) patch. It turns out that there exists a patch time T_n such that for $0 \le t \le T_n$, the patch ω_n has disjoint support from $\omega_{\le n-1}$, and obeys the dynamics

$$\partial_t\omega_n + \Delta^{-1}\nabla^\perp\omega_{\le n-1}\cdot\nabla\omega_n + \Delta^{-1}\nabla^\perp\omega_n\cdot\nabla\omega_n = 0.$$

[5] In the actual perturbation argument, we need to divide it by a suitable power of $\|D\phi\|_\infty$.

By a suitable re-definition of the patch center and change of variable, we find that $\tilde{\omega}_n$ (which is ω_n expressed in the new variable) satisfies the equation

$$\partial_t \tilde{\omega}_n + \Delta^{-1}\nabla^{\perp}\tilde{\omega}_n \cdot \nabla\tilde{\omega}_n + b(t)\begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} \cdot \nabla\tilde{\omega}_n + r(t,y)\cdot\nabla\tilde{\omega}_n = 0,$$

where $b(t) = O(1)$ and $|r(t,y)| \lesssim |y|^2$. We then choose initial data for ω_n such that within patch time $0 < t \le T_n$ the critical norm of ω_n inflates rapidly. As we take $n \to \infty$, the patch time $T_n \to 0$ and ω_n becomes more and more localized. Note that the whole solution (consisting of all patches ω_n) is actually a time-global solution. During interaction time T_n the patch ω_n produces the desired norm inflation since it stays disjoint from all the other patches. The details of the perturbation analysis can be found in Lemma 6.4 (and some related lemmas in Sect. 6).

The 3D case. As was already mentioned, in [1] we settle the 3D case which is technically more involved. To put things into perspective, we briefly explain the main difficulties therein and how to overcome them. Compared with the 2D case, the first difficulty in 3D is the lack of L^p conservation of the vorticity. It is deeply connected with the vorticity stretching term $(\omega\cdot\nabla)u$. To simplify the analysis we take the axisymmetric flow without swirl as the basic building block for the whole construction. The vorticity equation in the axisymmetric case takes the form

$$\partial_t\left(\frac{\omega}{r}\right) + (u\cdot\nabla)\left(\frac{\omega}{r}\right) = 0, \quad r = \sqrt{x_1^2 + x_2^2},\ x = (x_1, x_2, z).$$

Owing to the denominator r, the solution formula for ω then acquires an additional metric factor (compared with 2D) which represents the vorticity stretching effect in the axisymmetric setting. A lot of analysis goes into controlling the metric factor by the large Lagrangian deformation matrix and producing the desired $H^{3/2}$ norm inflation of vorticity. In our construction the patch solutions which are made of asymmetric without swirl flows typically carry infinite $\|\omega/r\|_{L^{3,1}}$ norm (when summing all the patches together). To glue these solutions together in the 3D compactly supported case, we need to run a new perturbation argument which allows to add each new patch ω_n with sufficiently small $\|\omega_n\|_\infty$ norm (over the whole lifespan) such that the effect of the large $\|\omega_n/r\|_{L^{3,1}}$ becomes negligible. All in all, the constructed patch solutions converge in the C^0 metric after building several auxiliary lemmas.

We have roughly described the whole strategy of the proof although some technical points could not be elucidated or even mentioned in this short introduction. In some sense our approach is a hybrid of the Lagrangian point of view and the Eulerian one, using in an essential way several features of the Euler dynamics: finite speed propagation and weak interaction between well-separated "patch" solutions. The rest of this paper is organized as follows. In Sect. 2 we set up some basic notations and preliminaries. In Sect. 3 we describe in detail the first part of the local construction for the 2D case. Sect. 4 is devoted to the perturbation argument needed for the 2D local construction step. In Sects. 5 and 6 we treat the 2D noncompact case and compactly supported case separately.

2 Notation and preliminaries

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some harmless constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim_{Z_1, Z_2, \ldots, Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $(Z_1, \ldots, Z_k)$. Similarly we define $\gtrsim_{Z_1,\ldots,Z_k}$ and $\sim_{Z_1,\ldots,Z_k}$.

We shall denote by $X+$ any quantity of the form $X + \epsilon$ for any $\epsilon > 0$. For example we shall write

$$Y \lesssim 2^{X+} \tag{2.1}$$

if $Y \lesssim_\epsilon 2^{X+\epsilon}$ for any $\epsilon > 0$. The notation $X-$ is similarly defined.

For any center $x_0 \in \mathbb{R}^d$ and radius $R > 0$, we use $B(x_0, R) := \{x \in \mathbb{R}^d : |x - x_0| < R\}$ to denote the open Euclidean ball. More generally for any set $A \subset \mathbb{R}^d$, we denote

$$B(A, R) := \{y \in \mathbb{R}^d : \ |y - x| < R \text{ for some } x \in A\}. \tag{2.2}$$

For any two sets $A_1, A_2 \subset \mathbb{R}^d$, we define

$$d(A_1, A_2) = \operatorname{dist}(A_1, A_2) = \inf\{|x - y| : \ x \in A_1, \ y \in A_2\}.$$

For any f on $\mathbb{R}^d$, we denote the Fourier transform of f as

$$(\mathcal{F} f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx.$$

The inverse Fourier transform of any g is given by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} g(\xi)e^{ix\cdot\xi}\,d\xi.$$

For any $1 \le p \le \infty$ we use $\|f\|_p$, $\|f\|_{L^p(\mathbb{R}^d)}$, or $\|f\|_{L^p_x(\mathbb{R}^d)}$ to denote the usual Lebesgue norm on $\mathbb{R}^d$. The Sobolev space $H^1(\mathbb{R}^d)$ is defined in the usual way as the completion of C_c^∞ functions under the norm $\|f\|_{H^1} = \|f\|_2 + \|\nabla f\|_2$. For any $s \in \mathbb{R}$, we define the homogeneous Sobolev norm of a tempered distribution $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

We use the Fourier transform to define the fractional differentiation operators $|\nabla|^s$ by the formula

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s\hat{f}(\xi).$$

For any integer $n \ge 0$ and any open set $U \subset \mathbb{R}^d$, we use the notation $C^n(U)$ to denote functions on U whose n^{th} derivatives are all continuous.

For any $1 \le p < \infty$, we denote by $L^p_{ul}(\mathbb{R}^d)$ the Banach space endowed with the norm

$$\|u\|_{L^p_{ul}(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d}\left(\int_{|y-x|<1} |u(y)|^p dy\right)^{\frac{1}{p}}. \tag{2.3}$$

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be not identically zero. The condition $u \in L^p_{ul}$ is equivalent to

$$\sup_{x\in\mathbb{R}^d} \|\phi(\cdot - x)u(\cdot)\|_{L^p(\mathbb{R}^d)} < \infty.$$

For any $s \in \mathbb{R}$ and any function $u \in H^s_{loc}(\mathbb{R}^d)$, one can define

$$\|u\|_{H^s_{ul}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|\phi(\cdot - x)u(\cdot)\|_{H^s(\mathbb{R}^d)}.$$

We will need to use the Littlewood–Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \le 2$ and equal to one on the ball $|\xi| \le 1$. For any real number $N > 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, define the frequency localized (LP) projection operators:

$$\widehat{P_{\le N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi),$$
$$\widehat{P_{>N} f}(\xi) := [1-\varphi(\xi/N)]\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := [\varphi(\xi/N)-\varphi(2\xi/N)]\hat{f}(\xi).$$

Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever $N > M > 0$ are real numbers. We will usually use these operators when M and N are dyadic numbers. The summation over N or M are understood to be over dyadic numbers. Occasionally for convenience of notation we allow M and N not to be a power of 2.

We recall the following Bernstein estimates: for any $1 \le p \le q \le \infty$, $s \in \mathbb{R}$,

$$\||\nabla|^s P_N f\|_{L_x^p(\mathbb{R}^d)} \sim N^s \|P_N f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_{\le N} f\|_{L_x^q(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{p}-\frac{1}{q})} \|P_{\le N} f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_N f\|_{L_x^q(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{p}-\frac{1}{q})} \|P_N f\|_{L_x^p(\mathbb{R}^d)}.$$

For any $s \in \mathbb{R}$, $1 \le p, q \le \infty$, we define the homogeneous Besov seminorm as

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \left(\sum_{N>0} N^{sq} \|P_N f\|^q_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty; \\ \sup_{N>0} N^s \|P_N f\|_{L^p(\mathbb{R}^d)}, & \text{if } q = \infty. \end{cases}$$

The inhomogeneous Besov norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ is

$$\|f\|_{B^s_{p,q}} = \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}.$$

3 Local construction for 2D case

We begin by describing the choice of initial data for the local construction.

Let $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ be a radial bump function such that $\mathrm{supp}(\varphi_0) \subset B(0,1)$ and $0 \le \varphi_0 \le 1$. Define

$$\eta_0(x_1,x_2) = \sum_{a_1,a_2=\pm 1} a_1 a_2 \cdot \varphi_0\Big(\frac{(x_1-a_1, x_2-a_2)}{2^{-10}}\Big).$$

Clearly by definition η_0 is odd in x_1, x_2, i.e.

$$\eta_0(x_1,x_2) = -\eta_0(-x_1,x_2) = -\eta_0(x_1,-x_2), \quad \forall\, x=(x_1,x_2) \in \mathbb{R}^2.$$

Define for each integer $k \geq 1$,

$$\eta_k(x) = \eta_0(2^k x). \tag{3.1}$$

Obviously,

$$\mathrm{supp}(\eta_k) \subset \bigcup_{a_1,a_2=\pm 1} B\Big((2^{-k}a_1, 2^{-k}a_2),\ 2^{-(k+10)}\Big), \tag{3.2}$$

so that η_k and η_l have disjoint supports for $k \neq l$, and

$$\|\partial^\alpha \eta_k\|_\infty \lesssim_\alpha 2^{k|\alpha|}. \tag{3.3}$$

Take any $A \gg 1$ and define the following one parameter family of functions:

$$h_A(x) = \frac{\sqrt{\log A}}{A} \sum_{A \leq k \leq 2A} \eta_k(x). \tag{3.4}$$

It is easy to check

$$\|h_A\|_1 + \|h_A\|_\infty \lesssim \frac{\sqrt{\log A}}{A}$$

and

$$\|h_A\|_{H^1} \lesssim \frac{\sqrt{\log A}}{\sqrt{A}}.$$

Note that in computing the H^1-norm above, we have a saving of $A^{\frac{1}{2}}$ due to the fact that each composing piece η_k has $O(1)$ H^1-norm and they have disjoint supports.

We begin with a simple interpolation lemma.

Lemma 3.1 *Let $\mathcal{R} = \mathcal{R}_{ij}$ be a Riesz transform on $\mathbb{R}^2$, then*

$$\|\mathcal{R}f\|_\infty \lesssim \|f\|_2^{\frac{1}{2}} \|\nabla f\|_\infty^{\frac{1}{2}}. \tag{3.5}$$

Proof By using the Littlewood-Paley decomposition, splitting into dyadic frequencies and the Bernstein inequality, we have

$$\begin{aligned}\|\mathcal{R}f\|_\infty &\lesssim \sum_N \|P_N f\|_\infty \lesssim \sum_{N<N_0} N\|P_N f\|_2 + \sum_{N>N_0} N^{-1}\|P_N \nabla f\|_\infty \\ &\lesssim N_0\|f\|_2 + N_0^{-1}\|\nabla f\|_\infty.\end{aligned}$$

Choosing $N_0 \in 2^{\mathbb{Z}}$ such that $N_0 \sim \left(\frac{\|\nabla f\|_\infty}{\|f\|_2}\right)^{\frac{1}{2}}$ then yields (3.5). □

The following lemma gives the estimates of Riesz transforms of compositions with Lipschitz maps on $\mathbb{R}^2$ for the functions h_A defined earlier.

Lemma 3.2 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a bi-Lipschitz function satisfying the following conditions:*

(i) $\phi(0) = 0$.
(ii) $\phi = (\phi_1, \phi_2)$ *commutes with the reflection map* $\sigma_2(x_1, x_2) = (x_1, -x_2)$, *i.e.*

$$\phi_1(x_1, -x_2) = \phi_1(x_1, x_2),$$
$$\phi_2(x_1, -x_2) = -\phi_2(x_1, x_2), \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

(iii) *For some integer* $n_0 \geq 1$,

$$\|D\phi\|_\infty \leq 2^{n_0} \quad \textit{and} \ \|D(\phi^{-1})\|_\infty \leq 2^{n_0}. \tag{3.6}$$

Here ϕ^{-1} denotes the inverse map of ϕ. Note that equivalently we can write

$$\|(D\phi)^{-1}\|_\infty \leq 2^{n_0},$$

where $(D\phi)^{-1}$ is the matrix inverse of $D\phi$.

Then with $\omega = h_A$ defined in (3.4), we have

$$\|\mathcal{R}_{11}(\omega \circ \phi)\|_\infty \leq C \cdot n_0 \cdot (\|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}} \cdot 2^{\frac{n_0}{2}} + 1) \cdot \frac{\sqrt{\log A}}{A}, \tag{3.7}$$
$$\|\mathcal{R}_{22}(\omega \circ \phi)\|_\infty \leq C \cdot n_0 \cdot (\|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}} \cdot 2^{\frac{n_0}{2}} + 1) \cdot \frac{\sqrt{\log A}}{A}. \tag{3.8}$$

Here $C > 0$ is an absolute constant. $\mathcal{R}_{11} = \Delta^{-1}\partial_{11}$ and $\mathcal{R}_{22} = \Delta^{-1}\partial_{22}$ are the Riesz transforms.

Remark 3.3 The same result holds if ϕ commutes with the map $\sigma_1(x_1, x_2) = (-x_1, x_2)$. Note also that in the proof below, we only used the oddness in x_2 of h_A defined in (3.4).

Proof of Lemma 3.2 First, note that by assumption (ii) on the map ϕ, the function $\eta_k \circ \phi$ is still odd in x_2. Since $\mathcal{R}_{11}$ is an even operator, it follows that $\mathcal{R}_{11}(\eta_k \circ \phi)(0) = 0$. (More precisely one just recalls that $\mathcal{R}_{11}$ is obtained by convolution with the even kernel $K(x) = p.v. \left(\frac{1}{2\pi} \cdot \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}\right) + \frac{1}{2}\delta(x)$, and $\mathcal{R}_{11}(\eta_k \circ \phi)(0) = \langle \eta_k \circ \phi, K \rangle = 0$.)

Now let $x \in \mathbb{R}^2 \setminus \{0\}$, $|x| \sim 2^{-l}$. We evaluate $\mathcal{R}_{11}(\eta_k \circ \phi)(x)$ by considering 3 cases.

Case 1. $2^k \ll 2^{l-n_0}$. [see (3.6) for the definition of n_0.]

By definition

$$|\mathcal{R}_{11}(\eta_k \circ \phi)(x)| = \left| \int_{\mathbb{R}^2} (\eta_k \circ \phi)(x-y) K(y) dy \right|. \tag{3.9}$$

The integrand in (3.9) vanishes unless $|\phi(x-y)| \sim 2^{-k}$ [see (3.2)]. By (3.6) and $\phi(0) = 0$, we have

$$2^{-k+n_0} \gtrsim |x-y| \gtrsim 2^{-k-n_0} \gg 2^{-l}.$$

Therefore $2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0}$. Since $(\eta_k \circ \phi)(y_1, y_2)$ is odd in the y_2 variable, obviously

$$\int_{2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0}} (\eta_k \circ \phi)(-y) K(y) dy = 0.$$

We then insert the above into (3.9) and compute

$$\begin{aligned}
&|\mathcal{R}_{11}(\eta_k \circ \phi)(x)| \\
&\quad \le \int_{2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0}} |(\eta_k \circ \phi)(x-y) - (\eta_k \circ \phi)(-y)| |K(y)| dy \\
&\quad \le |x| \cdot \|\nabla(\eta_k \circ \phi)\|_\infty \cdot \int_{2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0}} |K(y)| dy \\
&\quad \lesssim 2^{-l} \cdot 2^{n_0} \cdot 2^k \cdot n_0 \lesssim n_0.
\end{aligned}$$

Case 2. $2^k \gg 2^{l+n_0}$.

Again the integrand in (3.9) vanishes unless $|\phi(x-y)| \sim 2^{-k}$ which yields $2^{-k+n_0} \gtrsim |x-y| \gtrsim 2^{-k-n_0}$. Since $2^{-l} \gg 2^{-k+n_0}$ and $|x| \sim 2^{-l}$, we get $|y| \sim 2^{-l}$. Therefore

$$\begin{aligned}
|\mathcal{R}_{11}(\eta_k \circ \phi)(x)| &\le \|K\|_{L^\infty(|y| \sim 2^{-l})} \cdot \|\eta_k \circ \phi\|_1 \\
&\lesssim 4^l \cdot 4^{-k} \cdot 4^{n_0} = 4^{-k+l+n_0} \lesssim 1.
\end{aligned}$$

Case 3. $2^{l-n_0} \lesssim 2^k \lesssim 2^{l+n_0}$.

In this case we use Lemma 3.1. Then by (3.5) and (3.3),

$$\begin{aligned}\|\mathcal{R}_{11}(\eta_k\circ\phi)\|_\infty &\lesssim \|\eta_k\circ\phi\|_2^{\frac{1}{2}}\cdot\|\nabla(\eta_k\circ\phi)\|_\infty^{\frac{1}{2}}\\ &\lesssim \|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}}\cdot\|\eta_k\|_2^{\frac{1}{2}}\cdot\|\nabla\eta_k\|_\infty^{\frac{1}{2}}\cdot 2^{\frac{n_0}{2}}\\ &\lesssim \|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}}\cdot 2^{-\frac{k}{2}}\cdot 2^{\frac{k}{2}}\cdot 2^{\frac{n_0}{2}}\\ &\lesssim \|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}}\cdot 2^{\frac{n_0}{2}}.\end{aligned}$$

Collecting all the estimates, we then obtain

$$\sum_k |\mathcal{R}_{11}(\eta_k\circ\phi)(x)| \lesssim \|\det(D(\phi^{-1}))\|_\infty^{\frac{1}{2}}\cdot 2^{\frac{n_0}{2}}\cdot n_0 + n_0.$$

The bound (3.7) follows from this and the normalizing factor in (3.4). Similarly one can prove (3.8) or just use the identity $\mathcal{R}_{11}+\mathcal{R}_{22}=\mathrm{Id}$. □

We are now ready to describe the details of the local construction: namely the existence of large deformation for well-chosen initial data.

To be more specific, we consider the Euler equation

$$\begin{cases}\partial_t\omega + \left(\Delta^{-1}\nabla^\perp\omega\cdot\nabla\right)\omega = 0, & t>0,\\ \omega\Big|_{t=0} = h_A,\end{cases} \tag{3.10}$$

where h_A is defined in (3.4). Easy to check that ω is odd in both x_1 and x_2. We suppress the dependence of the solution ω on the parameter A for simplicity of notation.

The equation for the (forward) characteristic lines takes the form

$$\begin{cases}\partial_t\phi(t,x) &= (\Delta^{-1}\nabla^\perp\omega)(t,\phi(t,x)),\\ \phi(0,x) &= x\in\mathbb{R}^2.\end{cases} \tag{3.11}$$

It is easy to check that $\phi=\phi(t,x)$ is a symplectic map and $\phi(t,0)\equiv 0$. Due to the special choice of the initial data h_A, the flow associated with (3.10) and (3.11) is hyperbolic near the origin with a large deformation gradient. The following proposition quantifies this fact.

Proposition 3.4 *With the notation in (3.10, 3.11), we have for A sufficiently large,*

$$\max_{0\le t\le t_A}\|(D\phi)(t,\cdot)\|_\infty > M_A, \tag{3.12}$$

where $M_A = \log\log A$ *and* $t_A = 1/\log\log A$.

Proof of Proposition 3.4 We shall argue by contradiction. Assume that

$$\max_{0\le t\le t_A} \|(D\phi)(t,\cdot)\|_\infty \le M_A. \tag{3.13}$$

Since $\det(D\phi) \equiv 1$, it is easy to check that $\|D(\phi^{-1})\|_\infty$ has the same bound.

Now by Lemma 3.2, we have

$$\max_{0\le t\le t_A} \|\mathcal{R}_{11}\omega\|_\infty \lesssim M_A \frac{\sqrt{\log A}}{A},$$
$$\max_{0\le t\le t_A} \|\mathcal{R}_{22}\omega\|_\infty \lesssim M_A \frac{\sqrt{\log A}}{A}. \tag{3.14}$$

Denote $D(t) = D(t,\cdot) = (D\phi)(t,\cdot)$. By (3.11) and (3.14), we have

$$\begin{aligned}\partial_t D(t) &= \begin{pmatrix} -\mathcal{R}_{12}\omega & -\mathcal{R}_{22}\omega \\ \mathcal{R}_{11}\omega & \mathcal{R}_{12}\omega \end{pmatrix} D(t) \\ &=: \begin{pmatrix} -\lambda(t) & 0 \\ 0 & \lambda(t) \end{pmatrix} D(t) + P(t)D(t),\end{aligned} \tag{3.15}$$

where $\lambda(t,x) = (\mathcal{R}_{12}\omega)(t,\phi(t,x))$, and

$$\max_{0\le t\le t_A} \|P(t)\|_\infty \lesssim M_A \frac{\sqrt{\log A}}{A}. \tag{3.16}$$

Integrating (3.15) in time and noting that $D(0) = \mathrm{Id}$, we get

$$D(t) = \begin{pmatrix} e^{-\int_0^t \lambda} & 0 \\ 0 & e^{\int_0^t \lambda} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-\int_\tau^t \lambda} & 0 \\ 0 & e^{\int_\tau^t \lambda} \end{pmatrix} P(\tau)D(\tau)d\tau. \tag{3.17}$$

Now note that $|\int_\tau^t \lambda| \le |\int_0^t \lambda| + |\int_0^\tau \lambda| \le 2\max_{0\le s\le t} |\int_0^s \lambda|$.

By (3.13, 3.16) and (3.17), we have for all $0 \le t \le t_A$,

$$e^{|\int_0^t \lambda|} \le M_A + C_2 \cdot M_A^2 \cdot \frac{\sqrt{\log A}}{A} \cdot \max_{0\le s\le t} \left(e^{2|\int_0^s \lambda|}\right),$$

where $C_2 > 0$ is some absolute constant.

By taking A sufficiently large and a standard continuity argument, we get

$$e^{|\int_0^t \lambda|} \le 2M_A, \quad \forall\, 0 \le t \le t_A. \tag{3.18}$$

Now denote

$$D = \begin{pmatrix} e^{-\alpha} & 0 \\ 0 & e^{\alpha} \end{pmatrix} + \beta, \tag{3.19}$$

where $\alpha(t,x) := \int_0^t \lambda(\tau, x)d\tau$ and

$$|\beta| \le C_2 \cdot M_A^2 \cdot \frac{\sqrt{\log A}}{A} \cdot 4M_A^2.$$

From (3.19) we can get more information on the transport map $\phi = \phi(t,x)$. Indeed for fixed t, using the fact that $\phi(t,0) \equiv 0$, we have

$$\begin{aligned}
\phi(t,x) &= \phi(t,x) - \phi(t,0) \\
&= \int_0^1 \frac{d}{ds}\big(\phi(t,sx)\big)ds \\
&= \left(\int_0^1 (D\phi)(t,sx)ds\right)x \\
&= \left(\left(\int_0^1 e^{-\alpha(t,sx)}ds\right)x_1,\ \left(\int_0^1 e^{\alpha(t,sx)}ds\right)x_2\right) + \tilde{\beta},
\end{aligned}$$

where

$$|\tilde{\beta}| \lesssim M_A^4 \cdot \frac{\sqrt{\log A}}{A} \cdot |x|.$$

Note that by (3.18), for any $0 \le t \le t_A$,

$$\begin{aligned}
\frac{1}{2M_A} &\le \int_0^1 e^{\alpha(t,sx)}ds \le 2M_A, \\
\frac{1}{2M_A} &\le \int_0^1 e^{-\alpha(t,sx)}ds \le 2M_A.
\end{aligned}$$

Since

$$M_A^4 \cdot \frac{\sqrt{\log A}}{A} \ll \frac{1}{M_A},$$

we have if $x_1 > 0$, $x_2 > 0$, and

$$\frac{1}{2} < \frac{x_1}{x_2} < 2,$$

then for $\phi(t,x) = (\phi_1(t,x), \phi_2(t,x))$, $0 \le t \le t_A$,

$$\frac{1}{10M_A^2} < \frac{\phi_1(t,x)}{\phi_2(t,x)} < 10M_A^2. \tag{3.20}$$

By (3.13), we also have

$$|\phi(t,x)| \le M_A |x|. \tag{3.21}$$

These bounds will be needed later.

Now we analyze $\lambda(t,\cdot)$ at $x = 0$ to get a contradiction. We have (recall $\omega(0,x) = h_A(x)$)

$$\begin{aligned}\lambda(t,0) &= (\mathcal{R}_{12}\omega)(t,\phi(t,0)) = (\mathcal{R}_{12}\omega)(t,0)\\ &= -\frac{1}{\pi}\int_{\mathbb{R}^2} \omega(t,x)\frac{x_1x_2}{(x_1^2+x_2^2)^2}dx\\ &= -\frac{1}{\pi}\int_{\mathbb{R}^2} h_A(x)\cdot\frac{\phi_1(t,x)\phi_2(t,x)}{(\phi_1(t,x)^2+\phi_2(t,x)^2)^2}dx. \end{aligned} \tag{3.22}$$

In the last step above we have made a change of variable $x \to \phi(t,x)$ and used the fact $\omega(t,\phi(t,x)) = \omega(0,x) = h_A(x)$.

To continue, let us observe that the maps ϕ_1 and ϕ_2 are sign-preserving, i.e. if $x_1 \ge 0$ (resp. $x_2 \ge 0$) then $\phi_1 \ge 0$ (resp. $\phi_2 \ge 0$). To check this, one can use (3.11) and the fact that ω is odd in x_1 and x_2 to get

$$\begin{aligned}\partial_t\phi_1 &= (-\Delta^{-1}\partial_2\omega)(t,\phi_1,\phi_2) - (-\Delta^{-1}\partial_2\omega)(t,0,\phi_2)\\ &= F(t,\phi_1,\phi_2)\phi_1,\end{aligned}$$

which (by integrating in time) yields that $\text{sign}(\phi_1(t)) = \text{sign}(\phi_1(0)) = \text{sign}(x_1)$.

By using the sign property mentioned above and the parity of our solution, we conclude that the RHS integral of (3.22) is always non-negative and can be restricted to the first quadrant. Hence by (3.20–3.22),we have for all $0 \le t \le t_A$,

$$\begin{aligned}-\frac{\pi}{4}\lambda(t,0) &= \int_{x_1>0,x_2>0} h_A(x)\cdot\frac{\phi_1(t,x)\phi_2(t,x)}{(\phi_1^2(t,x)+\phi_2^2(t,x))^2}dx\\ &= \int_{x_1>0,x_2>0} h_A(x)\cdot\frac{1}{\frac{\phi_1(t,x)}{\phi_2(t,x)}+\frac{\phi_2(t,x)}{\phi_1(t,x)}}\cdot\frac{1}{\phi_1^2(t,x)+\phi_2^2(t,x)}dx\\ &\ge \int_{x_1>0,x_2>0} h_A(x)\cdot\frac{1}{20M_A^2}\cdot\frac{1}{M_A^2}\cdot\frac{1}{|x|^2}dx\end{aligned}$$

$$\begin{aligned}&\gtrsim \frac{1}{M_A^4}\cdot\frac{\sqrt{\log A}}{A}\cdot\sum_{A\le k\le 2A}\int_{x_1>0,x_2>0}\frac{\eta_k(x)}{|x|^2}dx\\&\gtrsim M_A^{-4}\cdot\sqrt{\log A}.\end{aligned}$$

Therefore

$$\left|\int_0^{t_A}\lambda(t,0)dt\right|\gtrsim t_A\cdot M_A^{-4}\cdot\sqrt{\log A}$$

which obviously contradicts (3.18). □

The special initial data h_A in Proposition 3.4 can be generalized to a slightly larger class of functions. Also the proof of Proposition 3.4 can be simplified if we take full advantage of the odd symmetry of the data. The main observation is that by parity $x=0$ is invariant under the flow and $(Du)(t,0)$ is diagonal for all $t>0$. We now state a more general result taking into account all these considerations. The argument below bypasses Lemma 3.2 and is more streamlined and quantitative. In particular the contradiction argument is replaced by a more effective integral (in time) inequality.

Consider

$$\begin{cases}\partial_t\omega+(\Delta^{-1}\nabla^{\perp}\omega\cdot\nabla)\omega=0, & t>0,\\ \omega\Big|_{t=0}=g.\end{cases}$$

Assume $g\in C_c^{\infty}(\mathbb{R}^2)$ satisfies

(i) g is odd in x_1 and x_2, and

$$g(x_1,x_2)\ge 0,\qquad \text{if } x_1\ge 0 \text{ and } x_2\ge 0.$$

(ii)

$$\int_{\mathbb{R}^2}g(x)\frac{x_1x_2}{|x|^4}dx=B>0.$$

Denoting by $\phi=\phi(t,x)$ the (forward) characteristic lines, we have

Proposition 3.5

$$\int_0^t\frac{1}{\|D\phi(s)\|_\infty^4}ds\le\frac{\pi}{4B}\log\left(1+\frac{4B}{\pi}t\right),\quad \forall t\ge 0.$$

In particular,

$$\max_{0\le s\le t}\|D\phi(s)\|_\infty \ge \left(\frac{4B}{\pi}\cdot\frac{t}{\log(1+\frac{4B}{\pi}t)}\right)^{\frac14}, \quad \forall t>0.$$

Proof of Proposition 3.5 By parity, we have $\phi(t,0)\equiv 0$ and

$$(Du)(t,0)=\begin{pmatrix}-\lambda(t) & 0\\ 0 & \lambda(t)\end{pmatrix},$$

where $\lambda(t)=(\mathcal{R}_{12}\omega)(t,0)$. The off-diagonal terms of Du vanish at $x=0$ since $\mathcal{R}_{11}\omega$ and $\mathcal{R}_{22}\omega$ are both odd functions of x_1, x_2. Integrating in time gives

$$(D\phi)(t,0)=\begin{pmatrix}e^{-\int_0^t\lambda(\tau)d\tau} & 0\\ 0 & e^{\int_0^t\lambda(\tau)d\tau}\end{pmatrix}.$$

Write $\phi=(\phi_1,\phi_2)$. By parity it is easy to check $\phi_1(t,0,x_2)\equiv 0$, $\phi_2(t,x_1,0)\equiv 0$ for any $x_1,x_2\in\mathbb{R}$. By this and sign preservation it follows that for any $x_1\ge 0$, $x_2\ge 0$,

$$\begin{aligned}\frac{1}{\|D\phi(t)\|_\infty}\phi_1(t,x_1,x_2)\le x_1\le \phi_1(t,x_1,x_2)\cdot\|D\phi(t)\|_\infty,\\ \frac{1}{\|D\phi(t)\|_\infty}\phi_2(t,x_1,x_2)\le x_2\le \phi_2(t,x_1,x_2)\cdot\|D\phi(t)\|_\infty.\end{aligned}$$

Therefore for any $x_1>0$, $x_2>0$,

$$\begin{aligned}\frac{\phi_1\phi_2}{(\phi_1^2+\phi_2^2)^2}&=\frac{1}{\frac{\phi_1}{\phi_2}+\frac{\phi_2}{\phi_1}}\cdot\frac{1}{\phi_1^2+\phi_2^2}\\ &\ge\frac{1}{\|D\phi\|_\infty^4}\cdot\frac{1}{\frac{x_1}{x_2}+\frac{x_2}{x_1}}\cdot\frac{1}{|x|^2}\\ &=\frac{1}{\|D\phi\|_\infty^4}\cdot\frac{x_1x_2}{|x|^4}.\end{aligned}$$

We compute $\lambda(t)$ as

$$\begin{aligned}-\pi\lambda(t)&=\int_{\mathbb{R}^2}g(x)\frac{\phi_1(t,x)\phi_2(t,x)}{|\phi(t,x)|^4}dx\\ &=4\int_{x_1>0,x_2>0}g(x)\frac{\phi_1(t,x)\phi_2(t,x)}{|\phi(t,x)|^4}dx\end{aligned}$$

$$\geq \frac{4}{\|D\phi(t)\|_\infty^4} \int_{x_1>0, x_2>0} g(x) \frac{x_1 x_2}{|x|^4} dx$$
$$= \frac{B}{\|D\phi(t)\|_\infty^4}.$$

Since

$$\|D\phi(t,\cdot)\|_\infty \geq \|(D\phi)(t,0)\|_\infty \geq \exp\left(-\int_0^t \lambda(s)ds\right),$$

we get

$$\|D\phi(t)\|_\infty \geq \exp\left(\frac{B}{\pi}\int_0^t \frac{1}{\|D\phi(s)\|_\infty^4} ds\right).$$

Equivalently,

$$\frac{d}{dt}\left(\exp\left(\frac{4B}{\pi}\int_0^t \frac{1}{\|D\phi(s)\|_\infty^4} ds\right)\right) \leq \frac{4B}{\pi}, \quad \forall t \geq 0.$$

Integrating in time, we get

$$\int_0^t \frac{1}{\|D\phi(s)\|_\infty^4} ds \leq \frac{\pi}{4B} \log\left(1 + \frac{4B}{\pi} t\right), \quad \forall t \geq 0.$$

□

4 $\dot{H}^1$ norm inflation by large Lagrangian deformation

We begin with a simple ODE perturbation lemma.

Lemma 4.1 *Suppose* $u = u(t,x) : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$, $v = v(t,x) : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ *are given smooth vector fields. Let* ϕ_1, ϕ_2 *solve respectively*

$$\begin{cases} \partial_t \phi_1(t,x) = u(t,\phi_1(t,x)), \\ \phi_1(0,x) = x \in \mathbb{R}^2, \end{cases}$$

and

$$\begin{cases} \partial_t \phi_2(t,x) = u(t,\phi_2(t,x)) + v(t,\phi_2(t,x)), \\ \phi_2(0,x) = x \in \mathbb{R}^2. \end{cases}$$

Then for some constant $C = C(\max_{0\leq t\leq 1} \|D^2 u(t)\|_\infty, \max_{0\leq t\leq 1} \|Du(t)\|_\infty) > 0$, *we have*

$$\max_{0\le t\le 1}\Big(\|\phi_2(t,\cdot)-\phi_1(t,\cdot)\|_\infty+\|(D\phi_2)(t)-(D\phi_1)(t)\|_\infty\Big)$$
$$\le C\cdot\max_{0\le t\le 1}(\|v(t)\|_\infty+\|Dv(t)\|_\infty)\cdot exp(C_1\max_{0\le t\le 1}\|Dv(t)\|_\infty),$$

where $C_1>0$ is an absolute constant.

Proof of Lemma 4.1 This is quite standard. We sketch the details for the sake of completeness.

Set $\eta(t,x)=\phi_2(t,x)-\phi_1(t,x)$. Then

$$\begin{aligned}\partial_t\eta &= u(t,\phi_2)-u(t,\phi_1)+v(t,\phi_2)\\ &= \int_0^1 (Du)(t,\phi_1+(\phi_2-\phi_1)\theta)d\theta\,\eta+v(t,\phi_2).\end{aligned}$$

A Gronwall in time argument then yields

$$\max_{0\le t\le 1}\|\eta(t)\|_\infty\le C\max_{0\le t\le 1}\|v(t)\|_\infty,$$

where the constant $C=C(\max_{0\le t\le 1}\|Du(t)\|_\infty)$.

Now for $D\eta$ note that

$$\begin{aligned}\partial_t(D\eta) &= (Du)(t,\phi_2)D\phi_2-(Du)(t,\phi_1)D\phi_1+(Dv)(t,\phi_2)D\phi_2\\ &= ((Du)(t,\phi_2)-(Du)(t,\phi_1))D\phi_2+(Du)(t,\phi_1)D\eta+(Dv)D\phi_2\\ &= O(\|D^2u\|_\infty\cdot\|\eta\|_\infty\cdot\|D\phi_2\|_\infty)+O(\|Du\|_\infty)D\eta\\ &\quad +O(\|Dv\|_\infty\cdot\|D\phi_2\|_\infty).\end{aligned}$$

It is easy to estimate

$$\max_{0\le t\le 1}\|(D\phi_2)(t,\cdot)\|_\infty\le\exp\Big(\text{const}\cdot\big(\max_{0\le t\le 1}(\|Du(t)\|_\infty+\|Dv(t)\|_\infty)\big)\Big).$$

Hence the desired bound follows from Gronwall. □

The following key proposition shows that large deformation of the transportation map can produce large $\dot H^1$ norm, provided we perturb the initial data judiciously.

Proposition 4.2 (Large deformation induces $\dot H^1$ inflation) *Suppose ω is a smooth solution to the Euler equation*

$$\begin{cases}\partial_t\omega+\Delta^{-1}\nabla^\perp\omega\cdot\nabla\omega=0, & 0<t\le 1,\\ \omega\Big|_{t=0}=\omega_0\end{cases}$$

satisfying the following conditions:

- $\|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty} + \|\omega_0\|_{\dot{H}^{-1}} < \infty.$
- *For some* $z_0 \in \mathbb{R}^2$, $R_0 > 0$, *we have*

$$\operatorname{supp}(\omega(t,\cdot)) \subset B\left(z_0, \frac{1}{2}R_0\right), \quad \forall\, 0 \le t \le 1.$$

- *For some* $0 < t_0 \le 1$ *and some* $M \gg 1$ *(*$M \ge 10^7$ *will suffice), we have*

$$\|(D\phi)(t_0,\cdot)\|_\infty > M, \tag{4.1}$$

where $\phi = \phi(t,x)$ *is the (forward) characteristics:*

$$\begin{cases} \partial_t \phi(t,x) = (\Delta^{-1}\nabla^\perp \omega)(t,\phi(t,x)), \\ \phi(0,x) = x. \end{cases}$$

Then we can find a smooth solution $\tilde{\omega}$ *also solving the Euler equation*

$$\begin{cases} \partial_t \tilde{\omega} + \Delta^{-1}\nabla^\perp \tilde{\omega} \cdot \nabla \tilde{\omega} = 0, \quad 0 < t \le 1 \\ \tilde{\omega}\Big|_{t=0} = \tilde{\omega}_0 \end{cases}$$

such that the following hold:

(1) $\tilde{\omega}_0$ *can be bounded in terms of* ω_0*:*

$$\|\tilde{\omega}_0\|_{L^1} \le 2\|\omega_0\|_{L^1}, \tag{4.2}$$
$$\|\tilde{\omega}_0\|_{L^\infty} \le 2\|\omega_0\|_{L^\infty}, \tag{4.3}$$
$$\|\tilde{\omega}_0\|_{\dot{H}^{-1}} \le 2\|\omega_0\|_{\dot{H}^{-1}}, \tag{4.4}$$
$$\|\tilde{\omega}_0\|_{\dot{H}^1} \le \|\omega_0\|_{\dot{H}^1} + M^{-\frac{1}{2}}. \tag{4.5}$$

(2) *For the same* t_0 *as in* (4.1)*, we have*

$$\|\tilde{\omega}(t_0,\cdot)\|_{\dot{H}^1} > M^{\frac{1}{3}}. \tag{4.6}$$

(3) $\tilde{\omega}$ *is also compactly supported:*

$$\operatorname{supp}(\tilde{\omega}(t)) \subset B(z_0,\ R_0), \quad \forall\, 0 \le t \le 1. \tag{4.7}$$

Proof of Proposition 4.2 To simplify the later computation, we begin with a general derivation. Let $W = W(t, x)$ be a smooth solution to the Euler equation

$$\begin{cases} \partial_t W + \Delta^{-1}\nabla^{\perp} W \cdot \nabla W = 0, \\ W\Big|_{t=0} = f. \end{cases}$$

Denote the associated (forward) characteristics as $\Phi = \Phi(t, x)$ which solves

$$\begin{cases} \partial_t \Phi(t, x) & = (\Delta^{-1}\nabla^{\perp} W)(t, \Phi(t, x)), \\ \Phi(0, x) & = x. \end{cases}$$

Let $\tilde{\Phi}(t, x)$ be the inverse map of $\Phi(t, x)$. Then

$$\tilde{\Phi}(t, \Phi(t, x)) = x.$$

Differentiating the above gives us

$$(D\tilde{\Phi})(t, \Phi(t, x))(D\Phi)(t, x) = Id$$

or

$$(D\tilde{\Phi})(t, \Phi(t, x)) = (D\Phi(t, x))^{-1}, \tag{4.8}$$

where $(D\Phi(t, x))^{-1}$ is the usual matrix inverse.

Since $\Phi(t)$ is a smooth symplectic map with $\Phi(0, x) = x$, we have $\det(D\Phi) = 1$. Denote $\Phi(t, x) = (\Phi_1(t, x), \Phi_2(t, x))$ and recall

$$D\Phi = \begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1} & \frac{\partial \Phi_1}{\partial x_2} \\ \frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_2}{\partial x_2} \end{pmatrix}.$$

Then

$$(D\Phi)^{-1} = \begin{pmatrix} \frac{\partial \Phi_2}{\partial x_2} & -\frac{\partial \Phi_1}{\partial x_2} \\ -\frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_1}{\partial x_1} \end{pmatrix}. \tag{4.9}$$

Since $W(t, x) = f(\tilde{\Phi}(t, x))$, we get

$$\begin{aligned} \int_{\mathbb{R}^2} |(DW)(t, x)|^2 dx &= \int_{\mathbb{R}^2} |(Df)(\tilde{\Phi}(t, x))(D\tilde{\Phi})(t, x)|^2 dx \\ &= \int_{\mathbb{R}^2} |(Df)(x)(D\Phi(t, x))^{-1}|^2 dx, \end{aligned} \tag{4.10}$$

where we have performed a measure-preserving change of variables $x \to \Phi(t, x)$ and used (4.8).

By (4.9), we can then write (4.10) as

$$\|W(t,\cdot)\|_{\dot{H}^1}^2 = \int_{\mathbb{R}^2} |(\nabla f)(x) \cdot (\nabla^\perp \Phi_2)(t,x)|^2 dx + \int_{\mathbb{R}^2} |(\nabla f)(x) \cdot (\nabla^\perp \Phi_1)(t,x)|^2 dx. \tag{4.11}$$

We shall need this formula below.

Now discuss two cases.

`Case 1`: $\|\omega(t_0,\cdot)\|_{\dot{H}^1} > M^{\frac{1}{3}}$. In this case we just set $\tilde{\omega} = \omega$ and no work is needed.

`Case 2`: $\|\omega(t_0,\cdot)\|_{\dot{H}^1} \le M^{\frac{1}{3}}$. It is this case which requires a nontrivial analysis. We shall use a perturbation argument.

By (4.1), we can find x_* such that

$$\|(D\phi)(t_0, x_*)\|_\infty > M.$$

Here for a matrix $A = (a_{ij})$, $\|A\|_\infty := \max |a_{ij}|$.

Denote $\phi(t_0, x) = (\phi_1(t_0, x), \phi_2(t_0, x))$. Without loss of generality, we may assume one of the entries of $(D\phi)(t_0, x_*)$ is at least M, namely

$$\left| \frac{\partial \phi_2}{\partial x_2}(t_0, x_*) \right| > M.$$

By continuity we can find $\delta > 0$ sufficiently small such that $\{x : |x - x_*| \le 2\delta\} \subset B(z_0, \frac{1}{2}R_0)$ and

$$\left| \frac{\partial \phi_2}{\partial x_2}(t_0, x) \right| > M, \quad \forall\, |x - x_*| \le 2\delta. \tag{4.12}$$

Now let $\Phi_0 \in C_c^\infty(\mathbb{R}^2)$ be a radial bump function such that $0 \le \Phi_0(x) \le 1$ for all $x \in \mathbb{R}^2$, $\Phi_0(x) = 1$ for $|x| \le 1$ and $\Phi_0(x) = 0$ for $|x| \ge 2$. Obviously

$$\sqrt{\pi} \le \|\Phi_0\|_2 \le 2\sqrt{\pi}. \tag{4.13}$$

Depending on the location of x_*, we need to shrink $\delta > 0$ slightly further if necessary and define an even function $b \in C_c^\infty(\mathbb{R}^2)$ as follows. If $x_* = (0, 0)$, we just define

$$b(x) = \frac{1}{\delta}\Phi_0\Big(\frac{x}{\delta}\Big).$$

If $x_* = (a_*, 0)$ for some $a_* \neq 0$, then we shrink $\delta > 0$ such that $\delta \ll |a_*|$ and define

$$b(x) = \frac{1}{\delta}\bigg(\Phi_0\Big(\frac{x - x_*}{\delta}\Big) + \Phi_0\Big(\frac{x + x_*}{\delta}\Big)\bigg).$$

The case $x_* = (0, a_*)$ for some $a_* \neq 0$ is similar. Now if $x_* = (a_*, c_*)$ for some $a_* \neq 0$ and $c_* \neq 0$, then we take $\delta \ll \min\{|a_*|, |c_*|\}$ and define

$$b(x) = \frac{1}{\delta}\sum_{\epsilon_1,\epsilon_2=\pm 1}\Phi_0\Big(\frac{x - (\epsilon_1 a_*, \epsilon_2 c_*)}{\delta}\Big).$$

Easy to check that in all cases the function $b(x)$ defined above is even in x_1, x_2, i.e.

$$b(x_1, x_2) = b(-x_1, x_2) = b(x_1, -x_2), \qquad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

Now introduce the perturbation

$$\beta(x) = \frac{1}{10k}\sin(kx_1)\cdot b(x)\cdot\frac{1}{M^{\frac{1}{2}}}, \tag{4.14}$$

and define

$$\tilde{\omega}_0(x) = \omega_0(x) + \beta(x). \tag{4.15}$$

We now show that if the parameter $k > 0$ is taken sufficiently large then the corresponding solution $\tilde{\omega}$ will satisfy all the requirements. In the rest of this proof, to simplify the presentation, we shall use the notation $X = O(\frac{1}{k^\alpha})$ if the quantity X obeys the bound $X \leq C_1 \cdot \frac{1}{k^\alpha}$ and the constant C_1 can depend on $(\omega, M, \Phi_0, \delta, \phi, R_0)$.

We first check (4.2–4.5).

Obviously by (4.14), if k is sufficiently large, then

$$\|\beta\|_{L^1} \leq \frac{1}{k}\cdot\frac{1}{\sqrt{M}}\|b\|_{L^1} \leq \|\omega_0\|_{L^1},$$

Similarly we can take k large such that

$$\|\beta\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}.$$

For the $\dot{H}^{-1}$-norm, note that β is an odd function and $\hat{\beta}(0) = 0$. Thus

$$\begin{aligned}\||\nabla|^{-1}\beta\|_2 &\lesssim \|\widehat{x\beta}\|_\infty + \|\beta\|_2 \\ &= O(k^{-1}) \le \|\omega_0\|_{\dot{H}^{-1}}\end{aligned}$$

if k is taken sufficiently large.

For the $\dot{H}^1$-norm, by (4.13) we have

$$\begin{aligned}\|\nabla\beta\|_{L^2}^2 &\le O\left(\frac{1}{k^2}\right) + \frac{1}{M}\cdot 10^{-2}\int b^2(x)\cos^2 kx_1 dx \\ &\le O\left(\frac{1}{k^2}\right) + \frac{1}{2M}\cdot 10^{-2}\int b^2(x)dx \\ &\le O\left(\frac{1}{k^2}\right) + \frac{1}{2M}\cdot 10^{-2}\cdot 4\cdot 4\pi < \frac{1}{M},\end{aligned}$$

where we again take k sufficiently large. Consequently the bound (4.5) follows. On the other hand, (4.7) follows from the assumption $\mathrm{supp}(\omega(t,\cdot)) \subset B(z_0, \frac{1}{2}R_0)$ for all $0 \le t \le 1$, and the fact that we can take k sufficiently large.

It remains to show (4.6). We shall proceed in several steps.

First we shall show

$$\max_{0\le t\le 1} \|\nabla\tilde{\omega}(t)\|_{L^4} \lesssim 1. \tag{4.16}$$

Here the implied constant is independent of k (but is allowed to depend on other parameters).

By a standard energy estimate, we have

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla\tilde{\omega}(t)\|_4^4\right) &\lesssim \|\mathcal{R}_{ij}\tilde{\omega}(t)\|_\infty \cdot \|\nabla\tilde{\omega}(t)\|_4^4 \\ &\lesssim \log(10 + \|\tilde{\omega}\|_2^2 + \|\nabla\tilde{\omega}\|_4^4)\cdot\|\nabla\tilde{\omega}\|_4^4.\end{aligned}$$

A Gronwall in time argument then yields (4.16) [by (4.14), it is easy to check that the initial data $\tilde{\omega}_0$ satisfies (4.16)].

Set $\eta = \omega - \tilde{\omega}$. Then

$$\partial_t\eta + \Delta^{-1}\nabla^\perp\omega\cdot\nabla\eta + \Delta^{-1}\nabla^\perp\eta\cdot\nabla\tilde{\omega} = 0.$$

Therefore noting that $\mathrm{supp}(\eta(t)) \subset B(z_0, R_0)$ for any $0 \le t \le 1$, we have

$$\frac{d}{dt}(\|\eta\|_2^2) \lesssim \|\Delta^{-1}\nabla^{\perp}\eta\|_4 \cdot \|\nabla\tilde{\omega}\|_4 \cdot \|\eta\|_2$$
$$\lesssim \|\eta\|_2^2 \cdot \|\nabla\tilde{\omega}\|_4.$$

Integrating in time then gives

$$\max_{0\le t\le 1} \|\eta(t)\|_2 = O\left(k^{-1}\right). \tag{4.17}$$

Interpolating the bound (4.17) with (4.16) [note that ω also satisfies the same bound (4.16)], we obtain

$$\max_{0\le t\le 1} \|D\Delta^{-1}\nabla^{\perp}(\tilde{\omega}(t) - \omega(t))\|_\infty + \max_{0\le t\le 1} \|\Delta^{-1}\nabla^{\perp}(\tilde{\omega}(t) - \omega(t))\|_\infty$$
$$= O\left(\frac{1}{k^\alpha}\right), \tag{4.18}$$

where $\alpha > 0$ is some absolute constant.

Denote the forward characteristic lines associated with $\tilde{\omega}$ as $\tilde{\phi}(t, x)$ which solves

$$\begin{cases} \partial_t\tilde{\phi}(t, x) = (\Delta^{-1}\nabla^{\perp}\tilde{\omega})(t, \tilde{\phi}(t, x)), \\ \tilde{\phi}(0, x) = x. \end{cases}$$

By Lemma 4.1 and (4.18), we have

$$\max_{0\le t\le 1}\left(\|\tilde{\phi}(t, \cdot) - \phi(t, \cdot)\|_\infty + \|(D\phi)(t, \cdot) - (D\tilde{\phi})(t, \cdot)\|_\infty\right) = O\left(\frac{1}{k^\alpha}\right).$$

Write $\tilde{\phi}(t, x) = (\tilde{\phi}_1(t, x), \tilde{\phi}_2(t, x))$. By (4.11), we get

$$\|\tilde{\omega}(t_0, \cdot)\|_{\dot{H}^1}^2 \ge \int |(\nabla\tilde{\omega}_0)(x) \cdot \nabla^{\perp}\tilde{\phi}_2(t_0, x)|^2 dx$$
$$\ge \int |\nabla\tilde{\omega}_0(x) \cdot \nabla^{\perp}\phi_2(t_0, x)|^2 dx - O\left(\frac{1}{k^\alpha}\right)$$
$$\ge \frac{1}{2}\int |\nabla\beta(x) \cdot \nabla^{\perp}\phi_2(t_0, x)|^2 dx$$
$$- \int |\nabla\omega_0(x) \cdot \nabla^{\perp}\phi_2(t_0, x)|^2 dx - O\left(\frac{1}{k^\alpha}\right), \tag{4.19}$$

where in the last step we used the simple inequality

$$|a + b|^2 \ge \frac{1}{2}|a|^2 - |b|^2, \qquad \forall\, a, b \in \mathbb{R}^d.$$

Since we are in Case 2, we have $\|\omega(t_0,\cdot)\|_{\dot{H}^1} \le M^{\frac{1}{3}}$. By (4.11), we get

$$\int |\nabla\omega_0(x)\cdot\nabla^{\perp}\phi_2(t_0,x)|^2dx \le \|\omega(t_0,\cdot)\|_{\dot{H}^1}^2 \le M^{\frac{2}{3}}. \tag{4.20}$$

By our choice of the function β and (4.12), we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\beta(x)\cdot\nabla^{\perp}\phi_2(t_0,x)|^2dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^2}\left|\frac{\cos(kx_1)b(x)}{10\sqrt{M}}\cdot\frac{\partial\phi_2}{\partial x_2}(t_0,x)\right|^2dx - O\left(k^{-2}\right) \\ &\ge \frac{1}{2}10^{-2}\cdot M\cdot\int b^2(x)\cos^2(kx_1)dx - O\left(k^{-2}\right) \\ &\ge \frac{\pi}{4}\cdot 10^{-2}\cdot M - O\left(k^{-2}\right). \end{aligned} \tag{4.21}$$

Plugging (4.20) and (4.21) into (4.19), we get

$$\begin{aligned} \|\tilde{\omega}(t_0,\cdot)\|_{\dot{H}^1}^2 &\ge \frac{\pi}{4}10^{-2}M - M^{\frac{2}{3}} - O\left(k^{-2}\right) - O\left(k^{-\alpha}\right) \\ &\ge 0.7\cdot 10^{-2}M - M^{\frac{2}{3}}, \end{aligned}$$

if k is taken sufficiently large. Clearly (4.6) follows. □

5 Local to global: gluing the patches

In this section we prove a general proposition which allows us to glue the local solutions into a global one. We begin with some auxiliary lemmas.

To state the next lemma, we need to fix a sufficiently large constant $A_1 > 1$ such that

$$\|\Delta^{-1}\nabla^{\perp}f\|_{\infty} \le A_1\cdot(\|f\|_1 + \|f\|_{\infty}), \quad \forall f\in L^1(\mathbb{R}^2)\cap L^{\infty}(\mathbb{R}^2). \tag{5.1}$$

Note that A_1 is an absolute constant which does not depend on any parameters.

Lemma 5.1 *Consider the Euler equation on* $\mathbb{R}^2$*:*

$$\begin{cases} \partial_t\omega + \Delta^{-1}\nabla^{\perp}\omega\cdot\nabla\omega = 0, \quad 0 < t \le 1, \\ \omega\Big|_{t=0} = \omega_0 = f + g. \end{cases} \tag{5.2}$$

Assume $f \in H^k \cap L^1$ *for some* $k \geq 2$, $g \in H^2 \cap L^1$ *and*

$$\|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty} \leq C_1 < \infty, \tag{5.3}$$

$$d(\mathrm{supp}(f),\ \mathrm{supp}(g)) \geq 100 A_1 C_1 > 0, \tag{5.4}$$

where A_1 *is the same constant as in* (5.1).

Then for any $0 \leq t \leq 1$, *the following hold true:*

(1) *The solution* $\omega(t)$ *to* (5.2) *can be decomposed as*

$$\omega(t) = \omega_f(t) + \omega_g(t), \tag{5.5}$$

where $\omega_f(0) = f$, $\omega_g(0) = g$, *and* [see (2.2)]

$$\mathrm{supp}(\omega_f(t)) \subset B(\mathrm{supp}(f),\ 2A_1C_1),$$
$$\mathrm{supp}(\omega_g(t)) \subset B(\mathrm{supp}(g), 2A_1C_1).$$

(2) *The Sobolev norm of* $\omega_f(t)$ *can be bounded in terms of* $\|f\|_{H^k}$ *and* C_1 *only:*

$$\max_{0 \leq t \leq 1} \|\omega_f(t)\|_{H^k} \leq C(\|f\|_{H^k}, C_1) < \infty. \tag{5.6}$$

Proof of Lemma 5.1 By (5.3) and (5.1), we have

$$\max_{0 \leq t \leq 1} \|\Delta^{-1}\nabla^{\perp}\omega(t)\|_\infty \leq A_1 C_1.$$

By the transport nature of the equation, the support of the solution $\omega(t)$ is enlarged at most a distance $A_1 C_1$ from its original support in unit time. The decomposition (5.5) follows easily from this observation and (5.4). More precisely, ω_f and ω_g are solutions to the following *linear* equations:

$$\begin{cases} \partial_t \omega_f + (u(t)\cdot\nabla)\omega_f = 0, \\ \omega_f\Big|_{t=0} = f; \end{cases}$$

$$\begin{cases} \partial_t \omega_g + (u(t)\cdot\nabla)\omega_g = 0, \\ \omega_f\Big|_{t=0} = g. \end{cases}$$

Here $u = \Delta^{-1}\nabla^{\perp}\omega$. Note that $\omega_f(t)$ and $\omega_g(t)$ stay well separated for all $0 \leq t \leq 1$:

$$d(\mathrm{supp}(\omega_f(t)),\ \mathrm{supp}(\omega_g(t))) \geq 90 A_1 C_1 > 0. \tag{5.7}$$

To show (5.6), we note that the equation for $\omega_f(t)$ can be rewritten as

$$\partial_t \omega_f + \Delta^{-1}\nabla^{\perp}\omega_f \cdot \nabla\omega_f + \Delta^{-1}\nabla^{\perp}\omega_g \cdot \nabla\omega_f = 0. \tag{5.8}$$

Note that for any multi-index α, we have

$$(\Delta^{-1}\nabla^{\perp}\partial^{\alpha}\omega_g)(x) = \int_{\mathbb{R}^2} K(x-y)(\partial^{\alpha}\omega_g)(y)dy, \tag{5.9}$$

where $K(\cdot)$ is the kernel function corresponding to the operator $\Delta^{-1}\nabla^{\perp}$.

By (5.7), for any $x \in \operatorname{supp}(\omega_f(t))$, $y \in \operatorname{supp}(\omega_g(t))$, we have $|x-y| \geq 90A_1C_1$. Therefore we can introduce a smooth cut-off function χ on the kernel $K(\cdot)$ and rewrite (5.9) as

$$\begin{aligned}(\Delta^{-1}\nabla^{\perp}\partial^{\alpha}\omega_g)(x) &= \int_{\mathbb{R}^2} K(x-y)\chi_{|x-y|\geq 80A_1C_1}(\partial^{\alpha}\omega_g)(y)dy \\ &= \int_{\mathbb{R}^2} (-1)^{|\alpha|}\partial_y^{\alpha}\Big(K(x-y)\chi_{|x-y|\geq 80A_1C_1}\Big)\omega_g(y)dy \\ &= \int_{\mathbb{R}^2} \tilde{K}_{\alpha}(x-y)\omega_g(y)dy, \end{aligned} \tag{5.10}$$

where the modified kernel $\tilde{K}_{\alpha}$ satisfies

$$|\tilde{K}_{\alpha}(z)| \lesssim_{C_1,\alpha} (1+|z|^2)^{-\frac{1}{2}}, \quad \forall z \in \mathbb{R}^2. \tag{5.11}$$

By using L^1 and L^{∞} conservation, we have

$$\|\omega_f(t)\|_{L^1} + \|\omega_f(t)\|_{L^{\infty}} + \|\omega_g(t)\|_{L^1} + \|\omega_g(t)\|_{L^{\infty}} \leq 2C_1. \tag{5.12}$$

Therefore by (5.10–5.12) and the Cauchy-Schwartz inequality, we have

$$\max_{0\leq t\leq 1} \max_{x\in\operatorname{supp}(\omega_f(t))} |(\Delta^{-1}\nabla^{\perp}\partial^{\alpha}\omega_g)(t,x)| \lesssim_{C_1,\alpha} 1. \tag{5.13}$$

The estimate (5.13) shows that the drift term $\Delta^{-1}\nabla^{\perp}\omega_g$ in (5.8) is arbitrarily smooth on the support of ω_f. Therefore the estimate (5.6) follows from the standard energy estimate. For the sake of completeness we sketch the detail here for $k = 2$. By (5.8), we have

$$\begin{aligned}\frac{d}{dt}\Big(\|\Delta\omega_f(t)\|_2^2\Big) \;\leq\; & \left|\int_{\mathbb{R}^2} \Delta(\Delta^{-1}\nabla^{\perp}\omega_f \cdot \nabla\omega_f)\Delta\omega_f dx\right| \\ & + \left|\int_{\mathbb{R}^2} \Delta(\Delta^{-1}\nabla^{\perp}\omega_g \cdot \nabla\omega_f)\Delta\omega_f dx\right|\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^2} |\Delta^{-1}\nabla^{\perp}\partial\omega_f| \cdot |\partial^2\omega_f| \cdot |\Delta\omega_f| dx \\
&\quad + \max_{\substack{x\in \mathrm{supp}(\omega_f(t)) \\ |\alpha|\le 2}} |\Delta^{-1}\nabla^{\perp}\partial^{\alpha}\omega_g(x)| \cdot \|\omega_f(t)\|_{H^2}^2 \\
&\lesssim_{C_1} (1 + \|\mathcal{R}_{ij}\omega_f(t)\|_{\infty}) \cdot \|\omega_f(t)\|_{H^2}^2,
\end{aligned}
$$

where $\mathcal{R}_{ij}$ denotes the Riesz transform. By the usual log interpolation inequality and (5.12), we have

$$
\|\mathcal{R}_{ij}\omega_f\|_{\infty} \lesssim_{C_1} \log\Big(10 + \|\omega_f(t)\|_{H^2}^2\Big).
$$

Therefore

$$
\frac{d}{dt}\Big(\|\omega_f(t)\|_{H^2}^2\Big) \lesssim_{C_1} \log(10 + \|\omega_f(t)\|_{H^2}^2) \cdot \|\omega_f(t)\|_{H^2}^2.
$$

A log Gronwall in time argument then yields (5.6). □

Lemma 5.2 *Let ω and $\tilde{\omega}$ be solutions to the Euler equations*

$$
\begin{cases}
\partial_t\omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = 0, \quad 0 < t \le 1, \\
\omega\Big|_{t=0} = \omega_0 = f + g.
\end{cases}
$$

and

$$
\begin{cases}
\partial_t\tilde{\omega} + \Delta^{-1}\nabla^{\perp}\tilde{\omega} \cdot \nabla\tilde{\omega} = 0, \quad 0 < t \le 1 \\
\tilde{\omega}\Big|_{t=0} = f.
\end{cases}
$$

Assume $f \in H^3 \cap L^1$, $g \in H^2 \cap L^1$ and

$$
\|\omega_0\|_{L^1} + \|\omega_0\|_{L^\infty} \le C_1 < \infty.
$$

Assume also f is compactly supported such that

$$
\mathrm{Leb}(\mathrm{supp}(f)) \le C_2 < \infty. \tag{5.14}
$$

Then for any $\epsilon > 0$, there exists $R_\epsilon = R_\epsilon(\epsilon, \|f\|_{H^3}, C_1, C_2) > 0$ such that if

$$
d(\mathrm{supp}(f), \mathrm{supp}(g)) \ge R_\epsilon > 0,
$$

then for any $0 < t \le 1$, the following hold true:

(1) *$\omega(t)$ has the decomposition*

$$\omega(t) = \omega_f(t) + \omega_g(t), \tag{5.15}$$

where

$$\begin{aligned} &\mathrm{supp}(\omega_f(t)) \subset B(\mathrm{supp}(f), 2A_1C_1); \\ &\mathrm{supp}(\omega_g(t)) \subset B(\mathrm{supp}(g), 2A_1C_1); \\ &d(\mathrm{supp}(\omega_f(t)),\ \mathrm{supp}(\omega_g(t))) \geq 100A_1C_1. \end{aligned} \tag{5.16}$$

Here A_1 is the same constant in (5.1).

(2) *The support of $\tilde{\omega}(t)$ also satisfies*

$$\mathrm{supp}(\tilde{\omega}(t)) \subset B(\mathrm{supp}(f), 2A_1C_1). \tag{5.17}$$

(3) *$\omega_f(t)$ and $\tilde{\omega}(t)$ are close:*

$$\max_{0\leq t\leq 1} \|\omega_f(t) - \tilde{\omega}(t)\|_{H^2} < \epsilon. \tag{5.18}$$

Proof of Lemma 5.2 Note that (5.15) and (5.17) follows directly from Lemma 5.1: we just need to take $R_\epsilon \geq 100A_1C_1$. By Lemma 5.1, we have

$$\begin{aligned} &\max_{0\leq t\leq 1} \|\omega_f(t) - \tilde{\omega}(t)\|_{H^3} \\ &\quad\leq \max_{0\leq t\leq 1} \|\omega_f(t)\|_{H^3} + \max_{0\leq t\leq 1} \|\tilde{\omega}(t)\|_{H^3} \\ &\quad\leq C_3 = C_3(\|f\|_{H^3}, C_1). \end{aligned} \tag{5.19}$$

Set $\eta(t) = \omega_f(t) - \tilde{\omega}(t)$. Then by (5.8), we have

$$\begin{cases} \partial_t\eta + \Delta^{-1}\nabla^\perp\eta \cdot \nabla\omega_f + \Delta^{-1}\nabla^\perp\tilde{\omega} \cdot \nabla\eta + \Delta^{-1}\nabla^\perp\omega_g \cdot \nabla\omega_f = 0, \quad 0 < t \leq 1, \\ \eta(0) = 0. \end{cases}$$

For $x \in \mathrm{supp}(\omega_f(t))$, we have

$$\begin{aligned} |(\Delta^{-1}\nabla^\perp\omega_g)(t,x)| &\lesssim \int_{|x-y|\geq\frac{1}{2}R_\epsilon} \frac{1}{|x-y|}|\omega_g(t,y)|dy \\ &\lesssim R_\epsilon^{-\frac{1}{2}} \cdot \|\omega_g\|_{\frac{4}{3}} \lesssim R_\epsilon^{-\frac{1}{2}} \cdot C_1. \end{aligned}$$

Therefore

$$\frac{d}{dt}\Big(\|\eta(t)\|_2^2\Big) \lesssim \|\Delta^{-1}\nabla^{\perp}\eta\|_3 \cdot \|\eta\|_2 \cdot \|\nabla\omega_f\|_6 + R_\epsilon^{-\frac{1}{2}} \cdot C_1 \cdot \|\nabla\omega_f\|_2 \cdot \|\eta\|_2. \tag{5.20}$$

By Sobolev embedding, (5.14, 5.16, 5.17) and Hölder, we have

$$\begin{aligned}\|\Delta^{-1}\nabla^{\perp}\eta\|_3 &\lesssim \|\eta\|_{\frac{6}{5}} \\ &\lesssim_{C_1,C_2} \|\eta\|_2.\end{aligned}$$

By (5.19) and Sobolev embedding we have

$$\|\nabla\omega_f\|_6 \lesssim C_3.$$

Therefore integrating (5.20) in time, we obtain for some $C_4 = C_4(C_1, C_2, C_3) > 0$ that

$$\max_{0\le t\le 1} \|\eta(t)\|_2 \le R_\epsilon^{-\frac{1}{2}} \cdot C_4. \tag{5.21}$$

The desired estimate (5.18) follows easily from interpolating (5.19, 5.21) and taking R_ϵ sufficiently large. □

Proposition 5.3 (Almost non-interacting patches) *Let $\{\omega_j\}_{j=1}^{\infty}$ be a sequence of functions in $C_c^{\infty}(B(0,1))$ that satisfy the following condition:*

$$\sum_{j=1}^{\infty} \|\omega_j\|_{H^1}^2 + \sum_{j=1}^{\infty} \|\omega_j\|_{L^1} + \sup_j \|\omega_j\|_{L^\infty} \le C_1 < \infty. \tag{5.22}$$

Here we may assume $C_1 > 1$.

Then there exist centers $x_j \in \mathbb{R}^2$ whose mutual distances are sufficiently large (i.e. $|x_j - x_k| \gg 1$ if $j \neq k$) such that the following hold:

(1) *Take the initial data*

$$\omega_0(x) = \sum_{j=1}^{\infty} \omega_j(x - x_j),$$

then $\omega_0 \in L^1 \cap L^\infty \cap H^1 \cap C^\infty$. Furthermore for any $j \neq k$

$$B(x_j, 100A_1C_1) \cap B(x_k, 100A_1C_1) = \varnothing. \tag{5.23}$$

Here A_1 is the same absolute constant as in (5.1).

(2) *With* ω_0 *as initial data, there exists a unique solution* ω *to the Euler equation*

$$\partial_t \omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = 0$$

on the time interval $[0, 1]$ *satisfying* $\omega \in L^1 \cap L^\infty \cap C^\infty$, $u = \Delta^{-1}\nabla^{\perp}\omega \in C^\infty$. *Moreover for any* $0 \le t \le 1$,

$$\operatorname{supp}(\omega(t,\cdot)) \subset \bigcup_{j=1}^{\infty} B(x_j, 3A_1C_1). \tag{5.24}$$

(3) *For any* $\epsilon > 0$, *there exists an integer* J_ϵ *sufficiently large such that if* $j \ge J_\epsilon$, *then*

$$\max_{0\le t\le 1} \|\omega(t,\cdot) - \tilde{\omega}_j(t,\cdot)\|_{H^2(B(x_j,3A_1C_1))} < \epsilon. \tag{5.25}$$

Here $\tilde{\omega}_j$ *is the solution solving the equation*

$$\begin{cases} \partial_t \tilde{\omega}_j + \Delta^{-1}\nabla^{\perp}\tilde{\omega}_j \cdot \nabla\tilde{\omega}_j = 0, \quad 0 < t \le 1, \ x \in \mathbb{R}^2; \\ \tilde{\omega}_j(t=0, x) = \omega_j(x - x_j), \quad x \in \mathbb{R}^2. \end{cases}$$

Proof of Proposition 5.3

`Step 1`. Choice of the centers x_j.

For each ω_j, $j \ge 1$, we choose $R_j = R_j(\|\omega_j\|_{H^3}, C_1) > 0$ corresponding to $f = \omega_j$ and $\epsilon = 2^{-j}$ in Lemma 5.2 [C_1 is the same constant as in (5.22)]. More precisely, if we take

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = 0, \quad 0 < t \le 1, \\ \omega\Big|_{t=0} = \omega_j + g, \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{\omega} + \Delta^{-1}\nabla^{\perp}\tilde{\omega} \cdot \nabla\tilde{\omega} = 0, \\ \tilde{\omega}\Big|_{t=0} = \omega_j, \end{cases}$$

with

$$\|\omega_j + g\|_{L^1} + \|\omega_j + g\|_{L^\infty} \le C_1 < \infty,$$

and

$$d(\operatorname{supp}(\omega_j), \operatorname{supp}(g)) \ge R_j, \tag{5.26}$$

then $\omega(t) = \omega_f(t) + \omega_g(t)$ with

$$\mathrm{supp}(\omega_f(t)) \subset B(0, 1 + 2A_1C_1)$$

and

$$\max_{0\le t\le 1} \|\omega_f(t) - \tilde{\omega}(t)\|_{H^2} < 2^{-j}. \tag{5.27}$$

With the numbers R_j properly defined, we now describe how to choose the centers x_j inductively. First set $x_1 = 0$. For $j \ge 2$, assume $x_1, \ldots, x_{j-1}$ have already been chosen. Let

$$f_{j-1}(x) = \sum_{l=1}^{j-1} \omega_l(x - x_l)$$

and consider the problems

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega = 0, \quad 0 < t \le 1, \\ \omega\Big|_{t=0} = f_{j-1} + g, \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{\omega} + \Delta^{-1}\nabla^\perp \tilde{\omega} \cdot \nabla \tilde{\omega} = 0, \\ \tilde{\omega}\Big|_{t=0} = f_{j-1} \end{cases}$$

with

$$\|f_{j-1} + g\|_{L^1} + \|f_{j-1} + g\|_{L^\infty} \le C_1 < \infty.$$

By Lemma 5.2, we can find $\tilde{R}_j = \tilde{R}_j(\|f_{j-1}\|_{H^3}, C_1) > 0$ such that if

$$d(\mathrm{supp}(f_{j-1}),\ \mathrm{supp}(g)) > \tilde{R}_j, \tag{5.28}$$

then

$$\max_{0\le t\le 1} \|\omega_{f_{j-1}}(t) - \tilde{\omega}(t)\|_{H^2} < 2^{-j}. \tag{5.29}$$

We now choose x_j such that

$$d(\mathrm{supp}(f_{j-1}), x_j) > 2\tilde{R}_j + 2\sum_{l=1}^{j} R_l + 1000A_1C_1 + 10^j. \tag{5.30}$$

By induction it is easy to verify that (5.23) holds.

`Step 2.` Construction of the solution $\omega(t)$ by patching.

Since $\omega_0 \in L^1 \cap L^\infty$, the usual Yudovich theory already gives existence and uniqueness of a weak solution in $L^1 \cap L^\infty$. Here thanks to the special type of initial data we shall give a more direct construction which also yields the regularity of the solution at one stroke.

To this end, denote for each $m \geq 2$

$$\omega_0^{(m)}(x) = \sum_{j=1}^{m} \omega_j(x - x_j)$$

and let $\omega^{(m)}(t, x)$ be the corresponding solution to the Euler equation. Obviously for $0 \leq t \leq 1$,

$$\mathrm{supp}(\omega^{(m)}(t)) \subset \bigcup_{j=1}^{m} B(x_j, 3A_1C_1).$$

Now we define $\omega(t, x)$ as follows

$$\omega(t,x) = \begin{cases} \lim_{m\to\infty} \omega^{(m)}(t,x), & \text{if } x \in \bigcup_{j=1}^{\infty} B(x_j, 3A_1C_1), \\ 0, & \text{otherwise.} \end{cases}$$

We now justify that $\omega(t, x)$ is well-defined and is the desired solution.

Fix $j_0 \geq 1$ and consider the ball $B(x_{j_0}, 3A_1C_1)$. By (5.29) (setting $\omega = \omega^{(m)}$ and $\tilde{\omega} = \omega^{(m-1)}$), we have

$$\max_{0\leq t\leq 1} \|\omega^{(m)}(t) - \omega^{(m-1)}(t)\|_{H^2(B(x_{j_0}, 3A_1C_1))} \leq 2^{-m}, \quad \text{if } m \geq j_0 + 1.$$

By Lemma 5.1, we also have for any $k \geq 3$,

$$\max_{0\leq t\leq 1} \|\omega^{(m)}(t)\|_{H^k(B(x_{j_0}, 3A_1C_1))} \leq C_k = C_k(\|\omega_{j_0}\|_{H^k}, C_1), \quad \text{if } m \geq j_0 + 1.$$

Thus $(\omega^{(m)})$ forms a Cauchy sequence in $H^k(B(x_{j_0}, 3A_1C_1))$ for any $k \geq 2$ and hence converge to a unique limit $\omega(t, x) \in C^\infty(B(x_{j_0}, 3A_1C_1))$. Clearly (5.24) holds. Easy to check $\omega \in L^\infty$.

By using the Lebesgue Dominated Convergence Theorem, we have

$$\|\omega(t)\|_{L^1(B(x_{j_0}, 3A_1C_1))} \leq \lim_{m\to\infty} \|\omega^{(m)}(t)\|_{L^1(B(x_{j_0}, 3A_1C_1))} = \|\omega_{j_0}\|_{L^1}.$$

Summing in j_0 then gives us $\omega \in L^1$.

We now show that $\Delta^{-1}\nabla^{\perp}\omega^{(m)}$ converges locally uniformly to $\Delta^{-1}\nabla^{\perp}\omega$ on $\bigcup_{j=1}^{\infty} B(x_j, 3A_1C_1)$. By construction we can decompose

$$\omega^{(m)}(t,x) = \sum_{j=1}^{m} \omega_j^{(m)}(t,x),$$

where

$$\operatorname{supp}\left(\omega_j^{(m)}\right) \subset B(x_j, 3A_1C_1).$$

Also we have

$$\omega(t,x) = \sum_{j=1}^{\infty} \omega_j^{(\infty)}(t,x), \quad \operatorname{supp}\left(\omega_j^{(\infty)}\right) \subset B(x_j, 3A_1C_1). \tag{5.31}$$

The summation above is actually a finite sum since for each x there exists at most one j such that $\omega_j^{(\infty)}(t,x) \neq 0$.

Now fix $j_0 \geq 1$. Then for $x \in B(x_{j_0}, 2A_1C_1)$ and $m \geq j_0 + 1$, we have

$$\begin{aligned}
&\left|(\Delta^{-1}\nabla^{\perp}\omega^{(m)})(x) - (\Delta^{-1}\nabla^{\perp}\omega)(x)\right| \\
&\quad \leq \left|\left(\Delta^{-1}\nabla^{\perp}\left(\omega_{j_0}^{(m)} - \omega_{j_0}^{(\infty)}\right)\right)(x)\right| && (5.32)\\
&\quad + \sum_{\substack{j=1\\ j\neq j_0}}^{m} \left|\left(\Delta^{-1}\nabla^{\perp}(\omega_j^{(m)} - \omega_j^{(\infty)})\right)(x)\right| && (5.33)\\
&\quad + \sum_{j=m+1}^{\infty} \left|\left(\Delta^{-1}\nabla^{\perp}\omega_j^{(\infty)}\right)(x)\right|. && (5.34)
\end{aligned}$$

For (5.32), we use the inequality (5.1) to get

$$\begin{aligned}
&\left\|(\Delta^{-1}\nabla^{\perp}\left(\omega_{j_0}^{(m)} - \omega_{j_0}^{(\infty)}\right)(x)\right\|_{\infty} \\
&\quad \leq A_1 \cdot \left(\|\omega_{j_0}^{(m)} - \omega_{j_0}^{(\infty)}\|_1 + \|\omega_{j_0}^{(m)} - \omega_{j_0}^{(\infty)}\|_{\infty}\right) \\
&\quad \lesssim_{C_1} \|\omega_{j_0}^{(m)} - \omega_{j_0}^{(\infty)}\|_{\infty} \\
&\quad \lesssim_{C_1} \|\omega^{(m)} - \omega\|_{L^{\infty}(B(x_{j_0}, 3A_1C_1))} \\
&\qquad \to 0, \quad \text{as } m \to \infty,
\end{aligned}$$

since $\omega^{(m)}$ converges uniformly to ω on the ball $B(x_{j_0}, 3A_1C_1)$.

For (5.33), note that for $j \neq j_0$ [see (5.30)]

$$d\left(\mathrm{supp}(\omega_j^{(m)} - \omega_j^{(\infty)}),\ B(x_{j_0}, 3A_1C_1)\right) \geq 2^j.$$

Therefore by using an estimate similar to (5.10), we have

$$\begin{aligned}
(5.33) &\lesssim \sum_{\substack{j=1\\ j\neq j_0}}^{\infty} 2^{-j}\|\omega_j^{(m)} - \omega_j^{(\infty)}\|_{L^1\cap L^\infty} \\
&\lesssim_{C_1} \sum_{j=1}^{\infty} 2^{-j}\|\omega^{(m)} - \omega\|_{L^\infty(B(x_j, 3A_1C_1))} \\
&\to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

Similarly

$$(5.34) \lesssim_{C_1} \sum_{j=m+1}^{\infty} 2^{-j} \to 0, \quad \text{as } m \to \infty.$$

Hence we have shown that $\Delta^{-1}\nabla^\perp\omega^{(m)} \to \Delta^{-1}\nabla^\perp\omega$ locally uniformly on compact sets (and also uniformly in t) as m tends to infinity. By writing

$$\omega^{(m)}(t) = \omega^{(m)}(0) + \int_0^t (\Delta^{-1}\nabla^\perp\omega^{(m)} \cdot \nabla\omega^{(m)})(\tau)d\tau,$$

and sending m to infinity, we conclude that ω is the desired solution on the time interval [0, 1].

Finally (5.25) is a simple consequence of Lemma 5.2 and our choice of the centers x_j [see (5.27)]. □

We are now ready to complete the

Proof of Theorem 1.2 For each $j \geq 2$, we choose (by a slight abuse of notation) $h_j = h_{A_j}$ according to (3.4) with the parameter A_j to be taken sufficiently large. Consider the Euler equation

$$\begin{cases} \partial_t\omega + \Delta^{-1}\nabla^\perp\omega \cdot \nabla\omega = 0, & 0 < t \leq 1, \\ \omega\Big|_{t=0} = h_j. \end{cases}$$

By Proposition[6] 3.4, we obtain for some $t_j \in (0, \frac{1}{\log\log A_j})$,

$$\|(D\phi)(t_j,\cdot)\|_\infty > \log\log A_j,$$

where ϕ is defined in (3.11).

We then use Proposition 4.2 to find $\tilde{\omega}_j^{(0)} \in C_c^\infty(B(0,1))$, $\tilde{\omega}_j^{(0)}$ odd in both x_1 and x_2, such that

$$\begin{aligned}
&\|\tilde{\omega}_j^{(0)}\|_{L^1} \le 2\|h_j\|_{L^1}, \\
&\|\tilde{\omega}_j^{(0)}\|_{L^\infty} \le 2\|h_j\|_{L^\infty}, \\
&\|\tilde{\omega}_j^{(0)}\|_{\dot{H}^1} \le \|h_j\|_{\dot{H}^1} + 2^{-j}, \\
&\|\tilde{\omega}_j^{(0)}\|_{\dot{H}^{-1}} \le 2\|h_j\|_{\dot{H}^{-1}}, \\
&\|\tilde{\omega}_j(t_j,\cdot)\|_{\dot{H}^1} > j,
\end{aligned} \tag{5.35}$$

where $\tilde{\omega}_j(t)$ is the solution to the Euler equation

$$\begin{cases} \partial_t \tilde{\omega}_j + \Delta^{-1}\nabla^\perp \tilde{\omega}_j \cdot \nabla \tilde{\omega}_j = 0, \quad 0 < t \le 1, \\ \tilde{\omega}_j \Big|_{t=0} = \tilde{\omega}_j^{(0)}. \end{cases}$$

We then apply Proposition 5.3 to $\omega_1 = \omega_0^{(p)}$, $\omega_j = \tilde{\omega}_j^{(0)}$ for $j \ge 2$ and find the centers x_j. Obviously by (5.35) and (5.25), we have

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{H}^1} = +\infty, \quad \forall\, 0 < t_0 \le 1.$$

It is not difficult to check the $\dot{H}^{-1}$ regularity of the constructed solution by using conservation of L^2-norm of velocity. The theorem is proved. □

6 The 2D compactly supported case

Lemma 6.1 (Control of the support) *Suppose $\omega = \omega(t,x)$ is a smooth solution to the following equation:*

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega + (b_1 + b_2)\cdot\nabla\omega = 0, \\ \omega\Big|_{t=0} = f, \end{cases}$$

where $b_1 = b_1(t,x)$, $b_2 = b_2(t,x)$, $f = f(x)$ are smooth functions satisfying the following conditions:

[6] Note that the perturbation $\beta(x)$ therein can be chosen to be odd in x_1 and x_2.

- *$\|f\|_\infty \le C_f$ for some constant $C_f > 0$, and*

$$\mathrm{supp}(f) \subset B(0, R), \quad R > 0.$$

- *b_1, b_2 are incompressible, i.e. $\nabla \cdot b_1 = \nabla \cdot b_2 = 0$.*
- *For some $B_1 > 0$,*

$$|b_1(t, x)| \le B_1 |x|, \quad \forall x \in \mathbb{R}^2.$$

- *For some $B_2 > 0$,*

$$|b_2(t, x)| \le B_2 |x|^2, \quad \forall x \in \mathbb{R}^2.$$

Then there exists $R_0 = R_0(C_f, B_1, B_2) > 0$, $t_0 = t_0(C_f, B_1, B_2) > 0$, such that if $0 < R \le R_0$, then

$$\mathrm{supp}(\omega(t, \cdot)) \subset B(0, 2R), \quad \forall 0 \le t \le t_0.$$

Proof of Lemma 6.1 Define the forward characteristic lines $\phi = \phi(t, x)$ which solves the ODE

$$\begin{cases} \partial_t \phi(t, x) = (\Delta^{-1}\nabla^\perp \omega + b_1 + b_2)(t, \phi(t, x)), \\ \phi(t = 0, x) = x, \quad x \in \mathbb{R}^2. \end{cases}$$

By using the assumptions, we compute

$$\frac{d}{dt}\Big(|\phi(t, x)|^2\Big) \le \|(\Delta^{-1}\nabla^\perp \omega)(t, \cdot)\|_\infty |\phi(t, x)| + B_1 |\phi(t, x)|^2 + B_2 |\phi(t, x)|^3. \quad (6.1)$$

Since both b_1 and b_2 are incompressible, we have

$$\|\omega(t, \cdot)\|_{L^1} = \|\omega(t = 0, \cdot)\|_{L^1} = \|f\|_{L^1} \le C_f \cdot \pi R^2. \quad (6.2)$$

Then by interpolation and L^∞ conservation, we get

$$\begin{aligned} \|(\Delta^{-1}\nabla^\perp \omega)(t, \cdot)\|_\infty &\lesssim \|\omega(t, \cdot)\|_{L^1}^{\frac{1}{2}} \|\omega(t, \cdot)\|_{L^\infty}^{\frac{1}{2}} \\ &\lesssim \|f\|_{L^1}^{\frac{1}{2}} \|f\|_{L^\infty}^{\frac{1}{2}} \\ &\lesssim C_f R, \end{aligned} \quad (6.3)$$

where in the last inequality we have used (6.2) and all the implied constants are absolute constants.

Plugging (6.3) into (6.1), we obtain

$$\frac{d}{dt}\Big(|\phi(t,x)|\Big) \lesssim C_f R + B_1|\phi(t,x)| + B_2|\phi(t,x)|^2.$$

The desired result then follows from time integration and choosing R_0, t_0 sufficiently small. □

For the compactly supported case, we need to use a slight variant of the function h_A defined in (3.4). We now take any $A \gg 1$ and

$$g_A(x) = \frac{1}{\log\log\log\log A} \cdot \frac{1}{\sqrt{\log A}} \sum_{A \le k \le A+\log A} \eta_k(x), \tag{6.4}$$

where η_k was defined in (3.1).

It is easy to check that

$$\begin{aligned} \operatorname{supp}(g_A) &\subset B(0, R_A), \quad \text{with } R_A \sim 2^{-A}, \\ \|g_A\|_{H^1} &\lesssim \frac{1}{\log\log\log\log A}, \\ \|g_A\|_{L^\infty} &\lesssim \frac{1}{\sqrt{\log A}}, \\ \|D^2 g_A\|_{L^\infty} &\lesssim 2^{2(A+\log A)}. \end{aligned}$$

The main difference between g_A and h_A is that the former has weaker dependence on A in terms of the bounds on higher derivatives. This fact will be used in the perturbation theory later (see Lemma 6.3).

The following is a variant of Proposition 3.4. Note that the additional drift term has a special form which makes the class of odd flows invariant.

Lemma 6.2 *Let $\omega = \omega(t,x)$ be the smooth solution to the equation*

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega + b \cdot \nabla \omega = 0, \\ \omega\Big|_{t=0} = g_A, \end{cases}$$

where g_A is defined in (6.4), $b = b(t,x)$ takes the form

$$b(t,x) = b_0(t)\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}, \quad x \in \mathbb{R}^2; \tag{6.5}$$

and $b_0(t)$ is a smooth function satisfying

$$\|b_0\|_\infty \le B_0 < \infty. \tag{6.6}$$

Let $\phi = \phi(t,x)$ be the associated forward characteristic line which solves

$$\begin{cases} \partial_t \phi(t,x) = (\Delta^{-1}\nabla^{\perp}\omega + b)(t,\phi(t,x)), \\ \phi(t=0,x) = x, \quad x \in \mathbb{R}^2. \end{cases}$$

Then there exists $A_0 = A_0(B_0) > 0$ such that if $A > A_0$, then

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|(D\phi)(t,\cdot)\|_\infty > \log\log A. \tag{6.7}$$

Proof of Lemma 6.2 Thanks to the special assumption (6.5), it is easy to check that $\omega(t,x)$ is still an odd function in x_1 and x_2 for any t. We can then repeat the proof of Proposition 3.4 or use the simplified version as in the proof of Proposition 3.5. We omit the details. □

The next lemma shows that the patch dynamics can still be controlled under a suitable perturbation in the drift term. This will play an important role in our later constructions. Since we no longer have odd symmetry at our disposal, we need to carry out a perturbative analysis.

Lemma 6.3 *Let $W = W(t,x)$ be a smooth solution to the equation*

$$\begin{cases} \partial_t W + \Delta^{-1}\nabla^{\perp} W \cdot \nabla W + (b(t,x) + r(t,x)) \cdot \nabla W = 0, \\ W\Big|_{t=0} = W_0 = g_A, \end{cases}$$

where the functions g_A, b, r satisfies the following conditions:

- *g_A is the same as defined in (6.4);*
- *$b(t,x) = b_0(t)\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$, $\|b_0\|_\infty \le B_0 < \infty$;*
- *r is incompressible and*

$$\begin{aligned} &|r(t,x)| \le B_1 \cdot |x|^2, \\ &|(Dr)(t,x)| \le B_1 \cdot |x|, \\ &|(D^2 r)(t,x)| \le B_1, \quad \forall x \in \mathbb{R}^2,\ 0 \le t \le \frac{1}{\log\log A}. \end{aligned} \tag{6.8}$$

Here $B_1 > 0$ is a constant.

Let $\Phi = \Phi(t,x)$ be the characteristic line which solves the ODE

$$\begin{cases} \partial_t \Phi(t,x) = (\Delta^{-1}\nabla^{\perp} W + b + r)(t,\Phi(t,x)), \\ \Phi(t=0,x) = x, \quad x \in \mathbb{R}^2. \end{cases} \tag{6.9}$$

Then there exists $A_0 = A_0(B_0, B_1) > 0$ *such that if* $A > A_0$*, then*

$$\max_{0\le t\le \frac{1}{\log\log A}} \|(D\Phi)(t,\cdot)\|_\infty > \log\log\log A. \tag{6.10}$$

Proof of Lemma 6.3 We shall argue by contradiction. Assume (6.10) is not true, then

$$\max_{0\le t\le \frac{1}{\log\log A}} \|(D\Phi)(t,\cdot)\|_\infty \le \log\log\log A. \tag{6.11}$$

By the definition of the characteristic line Φ, we have $W(t,x) = W_0(\tilde{\Phi}(t,x))$ where $\tilde{\Phi}$ is the inverse map of Φ. By (6.11) and using a computation similar to (4.11), we get

$$\begin{aligned}\max_{0\le t\le \frac{1}{\log\log A}} \|DW(t,\cdot)\|_2 &\lesssim \|DW_0\|_2 \cdot \max_{0\le t\le \frac{1}{\log\log A}} \|D\Phi(t,\cdot)\|_\infty \\ &\lesssim \frac{1}{\log\log\log\log A}\cdot \log\log\log A \\ &\lesssim \log\log\log A. \end{aligned}\tag{6.12}$$

We shall need this estimate later.

The main idea is to compare W with the other solution ω which solves the "unperturbed" equation

$$\begin{cases} \partial_t\omega + \Delta^{-1}\nabla^\perp\omega\cdot\nabla\omega + b(t,x)\cdot\nabla\omega = 0, \\ \omega\Big|_{t=0} = g_A. \end{cases}$$

The perturbation theory requires a bit of work so we shall proceed in several steps.

`Step 1`. Set $\eta = W - \omega$. We first show that

$$\|\eta(t,\cdot)\|_{B^0_{\infty,1}} \lesssim 2^{-\frac{2}{3}A+}, \qquad \forall\, 0\le t\le \frac{1}{\log\log A}. \tag{6.13}$$

Here and below we use the notation $X+$ as in (2.1). Also to simplify notations we shall write $\lesssim_{B_1}$ as $\lesssim$ (i.e. we suppress the notational dependence on B_1) since A will be taken sufficiently large.

The equation for η takes the form

$$\begin{cases} \partial_t\eta + \Delta^{-1}\nabla^\perp\eta\cdot\nabla W + \Delta^{-1}\nabla^\perp\omega\cdot\nabla\eta + b\cdot\nabla\eta + r\cdot\nabla W = 0, \\ \eta(0) = 0. \end{cases}$$

By Lemma 6.1 and (6.8), we have

$$\begin{aligned}
&\|r(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 4^{-A},\\
&\|(Dr)(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 2^{-A},\\
&\|(D^2r)(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 1, \quad \forall\, 0 \le t \le \frac{1}{\log\log A}.
\end{aligned} \tag{6.14}$$

Let $1 < p < 2$. By Sobolev embedding and (6.14), we compute

$$\begin{aligned}
\frac{d}{dt}\left(\|\eta\|_p^p\right) &\lesssim \|\Delta^{-1}\nabla^\perp\eta\|_{(\frac{1}{p}-\frac{1}{2})^{-1}}\|\nabla W\|_2 \cdot \|\eta\|_p^{p-1} + 4^{-A}\|\nabla W\|_2 \cdot \|\eta\|_p^{p-1}\\
&\lesssim \|\nabla W\|_2 \cdot \|\eta\|_p^p + 4^{-A}\|\nabla W\|_2 \cdot \|\eta\|_p^{p-1}.
\end{aligned}$$

Therefore for $0 \le t \le \frac{1}{\log\log A}$, by using (6.12), we get

$$\begin{aligned}
\|\eta(t,\cdot)\|_p &\lesssim 4^{-A}\int_0^t e^{(t-s)\log\log\log A}ds \cdot \log\log\log A\\
&\lesssim 4^{-A}\cdot\frac{\log\log\log A}{\log\log A} \lesssim 4^{-A}.
\end{aligned} \tag{6.15}$$

This estimate is particularly good for $p = 2-$.

On the other hand, for any $2 \le q < \infty$, a standard energy estimate gives for any $0 \le t \le 1$,

$$\begin{aligned}
\|\eta(t,\cdot)\|_{W^{2,q}} &\lesssim \|W(t,\cdot)\|_{W^{2,q}} + \|\omega(t,\cdot)\|_{W^{2,q}}\\
&\lesssim \|g_A\|_{W^{2,q}}\\
&\lesssim \frac{1}{\sqrt{\log A}}2^{2(A+\log A)(1-\frac{1}{q})}\\
&\lesssim 4^A.
\end{aligned}$$

Interpolating the above with (6.15) then yields (6.13) (note that $\|\eta\|_{B^0_{\infty,1}(\mathbb{R}^2)} \lesssim \|\eta\|^{\frac{2}{3}}_{L^2(\mathbb{R}^2)}\|\Delta\eta\|^{\frac{1}{3}}_{L^\infty(\mathbb{R}^2)}$).

`Step 2.` Let ϕ be the characteristic line associated with the equation for ω, i.e.

$$\begin{cases}
\partial_t\phi(t,x) = (\Delta^{-1}\nabla^\perp\omega + b)(t,\phi(t,x)),\\
\phi(t=0,x) = x, \quad x \in \mathbb{R}^2.
\end{cases}$$

We show that

$$\max_{0\le t\le \frac{1}{\log\log A}} \|\phi(t,\cdot)-\Phi(t,\cdot)\|_\infty \lesssim 2^{-\frac{4}{3}A+}. \tag{6.16}$$

Set $Y(t,x)=\Phi(t,x)-\phi(t,x)$. By Lemma 6.1, we only need to consider $|x|\lesssim 2^{-A}$ sine $\phi(t,x)=\Phi(t,x)=x$ for $|x|\gg 2^{-A}$.

Then for $|x|\lesssim 2^{-A}$,

$$\begin{aligned}\frac{d}{dt}Y &= (\Delta^{-1}\nabla^\perp W)(\Phi)-(\Delta^{-1}\nabla^\perp \omega)(\phi)\\ &\quad + b_0(t)\begin{pmatrix}-Y_1\\ Y_2\end{pmatrix}+r(t,\Phi)\\ &= (\Delta^{-1}\nabla^\perp W)(\Phi)-(\Delta^{-1}\nabla^\perp W)(\phi)+(\Delta^{-1}\nabla^\perp(W-\omega))(\phi)\\ &\quad + b_0(t)\begin{pmatrix}-Y_1\\ Y_2\end{pmatrix}+r(t,\Phi).\end{aligned}$$

For $|x|\lesssim 2^{-A}$ and $0\le t\le \frac{1}{\log\log A}$, we have $|\Phi(t,x)|\lesssim 2^{-A}$. By (6.8), we get

$$\max_{|x|\lesssim 2^{-A}} |r(t,\Phi(t,x))|\lesssim 4^{-A},\qquad \forall\, 0\le t\le \frac{1}{\log\log A}.$$

Therefore

$$\frac{d}{dt}\Big(|Y(t,x)|\Big)\lesssim (B_0+\|\mathcal{R}W\|_\infty)\cdot|Y(t,x)|+\|\Delta^{-1}\nabla^\perp(W-\omega)\|_\infty+4^{-A}. \tag{6.17}$$

By using the usual log-interpolation inequality, we have

$$\begin{aligned}\|\mathcal{R}W\|_\infty &\lesssim \|W\|_2+\|W\|_\infty\log(10+\|W\|_{H^2})\\ &\lesssim A.\end{aligned} \tag{6.18}$$

On the other hand, by (6.13), we have

$$\begin{aligned}\|\Delta^{-1}\nabla^\perp(W-\omega)\|_\infty &\lesssim \|W-\omega\|_1^{\frac12}\|W-\omega\|_\infty^{\frac12}\\ &\lesssim 2^{-A}\cdot 2^{-\frac13 A+}\\ &\lesssim 2^{-\frac43 A+}.\end{aligned}$$

Plugging these estimates into (6.17) and integrating in time, we obtain

$$\begin{aligned}|Y(t,x)| &\lesssim \int_0^t e^{(t-s)C\cdot A}(4^{-A}+2^{-\frac{4}{3}A+})ds\\ &\lesssim \int_0^{\frac{1}{\log\log A}} e^{\frac{CA}{\log\log A}}(4^{-A}+2^{-\frac{4}{3}A+})ds\\ &\lesssim 2^{-\frac{4}{3}A+}.\end{aligned}$$

Step 3. Set $\tilde{J}(t)=(D\Phi)(t,x)$, $J(t)=(D\phi)(t,x)$, then obviously

$$\begin{aligned}\partial_t\tilde{J} &= (\mathcal{R}W)(\Phi)\tilde{J}+b_0(t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\tilde{J}+(Dr)(\Phi)\tilde{J},\\ \partial_t J &= (\mathcal{R}\omega)(\phi)J+b_0(t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}J.\end{aligned}$$

Let $q=\tilde{J}-J$. Then q satisfies the equation

$$\begin{aligned}\partial_t q = &\Big((\mathcal{R}W)(\Phi)-(\mathcal{R}W)(\phi)\Big)\tilde{J}+\Big((\mathcal{R}W)(\phi)-(\mathcal{R}\omega)(\phi)\Big)\tilde{J}\\ &+(\mathcal{R}\omega)(\phi)q+b_0(t)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}q+(Dr)(\Phi)\tilde{J}.\end{aligned}$$

By (6.13, 6.16, 6.18), we obtain

$$\begin{aligned}\partial_t|q| &\lesssim \|D\mathcal{R}W\|_\infty|\Phi-\phi|\cdot\|\tilde{J}\|_\infty+2^{-\frac{2}{3}A+}\|\tilde{J}\|_\infty\\ &\quad +A|q|+2^{-A}\|\tilde{J}\|_\infty\\ &\lesssim 2^{A+}2^{-\frac{4}{3}A+}\log\log\log A+2^{-\frac{2}{3}A+}\log\log\log A+A|q|\\ &\quad +2^{-A}\log\log\log A.\end{aligned}$$

Integrating in time and noting $t\le\frac{1}{\log\log A}$, we obtain (note $q(0)=0$)

$$\|q\|_\infty \lesssim \int_0^{\frac{1}{\log\log A}} e^{\frac{CA}{\log\log A}}2^{-\frac{1}{3}A+}\log\log\log A\,ds \lesssim 1.$$

But this obviously contradicts (6.7) and (6.11). □

The next lemma is the main building block for our construction in the compactly supported data case.

Lemma 6.4 *Suppose $f_{-1} \in C_c^\infty(\mathbb{R}^2)$ is a given real-valued function such that*

- *for some $R_0 > 0$,*

$$\mathrm{supp}(f_{-1}) \subset \{x = (x_1, x_2) : \quad x_1 \le -2R_0\};$$

- *f_{-1} is an odd function of x_2, i.e.*

$$f_{-1}(x_1, x_2) = -f_{-1}(x_1, -x_2), \quad \forall\, x \in \mathbb{R}^2.$$

Then for any $0 < \epsilon < \frac{R_0}{100}$, one can find $\delta_0 = \delta_0(f_{-1}, \epsilon, R_0) > 0$, $0 < t_0 = t_0(f_{-1}, \epsilon, R_0) < \epsilon$, and $f_0 \in C_c^\infty(B(0, \epsilon))$ (f_0 depends only on (f_{-1}, ϵ, R_0)) with the properties:

- *f_0 is an odd function of x_2;*
-

$$\|f_0\|_{L^1} + \|f_0\|_{L^\infty} + \|f_0\|_{H^1} + \|f_0\|_{\dot{H}^{-1}} \le \epsilon, \tag{6.19}$$

such that for any $f_1 \in C_c^\infty(\mathbb{R}^2)$ with

- $\mathrm{supp}(f_1) \subset \{x = (x_1, x_2) : \quad x_1 \ge R_0\}$*;*
- $\|f_1\|_{L^1} + \|f_1\|_{L^\infty} \le \delta_0$*,*

the following hold true:

Consider the Euler equation

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega = 0, \\ \omega\Big|_{t=0} = f_{-1} + f_0 + f_1, \end{cases}$$

then the smooth solution $\omega = \omega(t, x)$ satisfies the following properties:

(1) *for any $0 \le t \le t_0$, we have the decomposition*

$$\omega(t, x) = \omega_{-1}(t, x) + \omega_0(t, x) + \omega_1(t, x), \tag{6.20}$$

where

$$\mathrm{supp}(\omega_{-1}(t)) \subset B\left(\mathrm{supp}(f_{-1}),\ \frac{1}{8}R_0\right);$$
$$\mathrm{supp}(\omega_0(t)) \subset B\left(0, \epsilon + \frac{1}{8}R_0\right);$$
$$\mathrm{supp}(\omega_1(t)) \subset B\left(\mathrm{supp}\left(f_1, \frac{1}{8}R_0\right)\right).$$

(2) $\|\omega_0(t=0,\cdot)\|_{H^1} = \|f_0\|_{H^1} \le \epsilon$, *but*

$$\|\omega_0(t_0,\cdot)\|_{\dot{H}^1} > \frac{1}{\epsilon}. \tag{6.21}$$

Proof of Lemma 6.4 The decomposition (6.20) is a simple consequence of finite transportation speed. Therefore we only need to show how to choose f_0 to achieve (6.21) and the other conditions.

Consider first the equation

$$\begin{cases} \partial_t \omega^{(1)} + \Delta^{-1}\nabla^{\perp}\omega^{(1)} \cdot \nabla\omega^{(1)} = 0, \\ \omega^{(1)}\Big|_{t=0} = f_{-1} + g_A, \end{cases} \tag{6.22}$$

where g_A was defined in (6.4) and we shall choose A sufficiently large.

For $0 \le t \le \frac{1}{\log\log A}$,we decompose the solution $\omega^{(1)}(t)$ to (6.22) as

$$\omega^{(1)}(t,x) = \omega_{-1}^{(1)}(t,x) + \omega_0^{(1)}(t,x),$$

with

$$\begin{aligned} &\operatorname{supp}(\omega_{-1}^{(1)}(t,\cdot)) \subset B\left(\operatorname{supp}(f_{-1}), \frac{1}{10}R_0\right), \\ &\operatorname{supp}(\omega_0^{(1)})(t,\cdot)) \subset B\left(\operatorname{supp}(g_A), \frac{1}{10}R_0\right). \end{aligned} \tag{6.23}$$

Obviously $\omega_0^{(1)}(t)$ satisfies the equation

$$\begin{cases} \partial_t \omega_0^{(1)} + \Delta^{-1}\nabla^{\perp}\omega_0^{(1)} \cdot \nabla\omega_0^{(1)} + \Delta^{-1}\nabla^{\perp}\omega_{-1}^{(1)}(t) \cdot \nabla\omega_0^{(1)} = 0, \\ \omega_0^{(1)}\Big|_{t=0} = g_A. \end{cases} \tag{6.24}$$

Since by assumption f_{-1} is odd in x_2 and g_A is also odd in x_2, it is easy to check that both $\omega_{-1}^{(1)}(t)$ and $\omega_0^{(1)}(t)$ are odd functions of x_2. Therefore we have

$$\begin{aligned} &\left(\Delta^{-1}\partial_{22}\omega_{-1}^{(1)}\right)(t,x_1,0) = 0, \\ &\left(\Delta^{-1}\partial_{11}\omega_{-1}^{(1)}\right)(t,x_1,0) = 0, \\ &\left(\Delta^{-1}\partial_{1}\omega_{-1}^{(1)}\right)(t,x_1,0) = 0, \quad \forall\, 0 \le t \le \frac{1}{\log\log A},\ x_1 \in \mathbb{R}. \end{aligned} \tag{6.25}$$

Now let $\xi(t)$ solve the ODE

$$\begin{cases} \frac{d}{dt}\xi(t) = (\Delta^{-1}\partial_2\omega_{-1}^{(1)})(t, \xi(t), 0), \\ \xi(0) = 0. \end{cases} \tag{6.26}$$

Since for $0 \le t \le \frac{1}{\log\log A}$ and A sufficiently large, the function $\omega_{-1}^{(1)}$ is supported away from the origin [see (6.23)], it is easy to check that the function $(\Delta^{-1}\partial_2\omega_{-1}^{(1)})(t, \cdot)$ is smooth and has uniform (independent of A) Sobolev bounds in a small neighborhood of the origin. Thus $\xi(t)$ is well-defined and remains close to the origin for $t \le \frac{1}{\log\log A}$.

By (6.25), we have

$$\begin{aligned} \left(-\Delta^{-1}\partial_2\omega_{-1}^{(1)}\right)(t, \xi(t)+y_1, y_2) &= -\left(\Delta^{-1}\partial_2\omega_{-1}^{(1)}\right)(t, \xi(t), 0) \\ &\quad - \left(\Delta^{-1}\partial_{12}\omega_{-1}^{(1)}\right)(t, \xi(t), 0)y_1 + r_1^{(1)}(t, y), \\ \left(\Delta^{-1}\partial_1\omega_{-1}^{(1)}\right)(t, \xi(t)+y_1, y_2) &= \left(\Delta^{-1}\partial_{12}\omega_{-1}^{(1)}\right)(t, \xi(t), 0)y_2 + r_2^{(1)}(t, y), \end{aligned}$$

where for $0 \le t \le \frac{1}{\log\log A}$,

$$\begin{aligned} &|r_1^{(1)}(t, y)| + |r_2^{(1)}(t, y)| \lesssim_{f_{-1}, R_0} |y|^2, \\ &|(Dr_1^{(1)})(t, y)| + |\left(Dr_2^{(1)}\right)(t, y)| \lesssim_{f_{-1}, R_0} |y|, \\ &|(D^2 r_1^{(1)})(t, y)| + |\left(D^2 r_2^{(1)}\right)(t, y)| \lesssim_{f_{-1}, R_0} 1. \end{aligned}$$

By (6.26), we may write the above more compactly as

$$\begin{aligned} &(\Delta^{-1}\nabla^{\perp}\omega_{-1}^{(1)})(t, \xi(t) + y_1, y_2) \\ &\quad = \begin{pmatrix} -\frac{d}{dt}\xi(t) \\ 0 \end{pmatrix} + b_0(t)\begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} + r(t, y), \end{aligned} \tag{6.27}$$

where

$$\begin{aligned} &|b_0(t)| \lesssim_{f_{-1}, R_0} 1, \\ &|r(t, y)| \lesssim_{f_{-1}, R_0} |y|^2, \\ &|(Dr)(t, y)| \lesssim_{f_{-1}, R_0} |y|, \\ &|(D^2 r)(t, y)| \lesssim_{f_{-1}, R_0} 1. \end{aligned} \tag{6.28}$$

Now we make a change of variable and set

$$\begin{aligned} &x = (\xi(t) + y_1, y_2),\\ &\omega_0^{(1)}(t, x) = \omega_0^{(1)}(t, \xi(t) + y_1, y_2) =: W_0^{(1)}(t, y_1, y_2). \end{aligned} \tag{6.29}$$

By using (6.27) and the above expressions, we can write (6.24) as

$$\begin{aligned} &\partial_t W_0^{(1)}(t, y) + (\Delta^{-1}\nabla^{\perp} W_0^{(1)} \cdot \nabla W_0^{(1)})(t, y)\\ &\quad + b_0(t) \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} \cdot \nabla W_0^{(1)}(t, y) + r(t, y) \cdot \nabla W_0^{(1)}(t, y) = 0, \end{aligned}$$

where $b_0(t), r(t, y)$ satisfies (6.28).

By Lemma 6.1, we have for $0 \le t \le \frac{1}{\log\log A}$,

$$\operatorname{supp}(W_0^{(1)}(t, \cdot)) \subset B(0, \tilde{R}), \quad \text{with } \tilde{R} \sim 2^{-A}.$$

Therefore by (6.28) and Lemma 6.3, we have for A sufficiently large,

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|(D\Phi)(t, \cdot)\|_\infty > \log\log\log A, \tag{6.30}$$

were Φ is the forward characteristic line associated with $W_0^{(1)}$ [see (6.9)].

Let ϕ be the characteristic line solving the ODE

$$\begin{cases} \partial_t \phi(t, x) &= (\Delta^{-1}\nabla^{\perp}\omega_0^{(1)} + \Delta^{-1}\nabla^{\perp}\omega_{-1}^{(1)})(t, \phi(t, x)),\\ \phi(t = 0, x) = x. \end{cases}$$

Denote by $\tilde{\Phi}$, $\tilde{\phi}$ the inverse maps of Φ and ϕ respectively. By (6.29), it is easy to check that

$$\tilde{\Phi}(t, y) = \tilde{\phi}(t, \xi(t) + y_1, y_2), \qquad \text{for any } t \ge 0 \text{ and } y = (y_1, y_2) \in \mathbb{R}^2. \tag{6.31}$$

Therefore by (6.30),

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|(D\phi)(t, \cdot)\|_\infty > \log\log\log A. \tag{6.32}$$

Now we just need to modify slightly the proof of Proposition 4.2. Note that one can always choose the perturbation $\beta(x)$ [see (4.14)] to be odd in x_1 and x_2, for example,

$$\beta(x) = \frac{1}{k}\sin(kx_1)\sin(x_2)b(x)\frac{1}{\sqrt{M}}.$$

Denote by $\tilde{g}_A$ the perturbed initial data and let $\tilde{\omega}^{(1)}$ be the solution to

$$\begin{cases} \partial_t \tilde{\omega}^{(1)} + \Delta^{-1}\nabla^{\perp}\tilde{\omega}^{(1)} \cdot \nabla\tilde{\omega}^{(1)} = 0, \\ \tilde{\omega}^{(1)}\Big|_{t=0} = f_{-1} + \tilde{g}_A. \end{cases}$$

Similar to $\omega^{(1)}$ [see (6.22)], we also have the decomposition similar to that in (6.23):

$$\tilde{\omega}^{(1)} = \tilde{\omega}_{-1}^{(1)} + \tilde{\omega}_0^{(1)}.$$

By our choice of perturbation (and taking A sufficiently large), we have

$$\max_{0\le t\le \frac{1}{\log\log A}} \|\tilde{\omega}_0^{(1)}(t,\cdot)\|_{\dot{H}^1} > (\log\log\log A)^{\frac{1}{3}}.$$

Let $f_0 = \tilde{g}_A$. We then compare $\tilde{\omega}^{(1)}$ with ω which solves

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = 0, \\ \omega\Big|_{t=0} = f_{-1} + f_0 + f_1, \end{cases}$$

with $f_1 \in C_c^\infty(\mathbb{R}^2)$ satisfying

- $\mathrm{supp}(f_1) \subset \{x = (x_1, x_2) : \quad x_1 \ge \frac{1}{2}R_0\}$;
- $\|f_1\|_{L^1} + \|f_1\|_{L^\infty} \le \delta_0$

and δ_0 is to be taken sufficiently small.

By an argument similar to the proof of (5.18), we then have [see (6.20) for the definition of $\omega_0(t)$]

$$\max_{0\le t\le \frac{1}{\log\log A}} \|\omega_0(t) - \tilde{\omega}_0^{(1)}(t)\|_{H^2} \lesssim_{\epsilon, f_{-1}, R_0} \delta_0.$$

Therefore (6.21) follows by choosing δ_0 sufficiently small. □

To prove Theorem 1.5 we need the following C^0-perturbation lemma.

Lemma 6.5 *Let $R_0 > 0$ and $f \in C_c^\infty(B(0, R_0))$, $g \in C_c^\infty(B(0, R_0))$. Let ω^a and ω be smooth solutions to the following 2D Euler equations:*

$$\begin{cases} \partial_t \omega^a + (u^a \cdot \nabla)\omega^a = 0, \quad 0 < t \le 1,\ x \in \mathbb{R}^2, \\ u^a = \Delta^{-1}\nabla^\perp \omega^a, \\ \omega^a\Big|_{t=0} = f. \end{cases} \tag{6.33}$$

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0, \quad 0 < t \le 1,\ x \in \mathbb{R}^2, \\ u = \Delta^{-1}\nabla^\perp \omega, \\ \omega\Big|_{t=0} = f + g. \end{cases} \tag{6.34}$$

For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, R_0, f) > 0$ sufficiently small such that if

$$\|g\|_\infty < \delta,$$

then

$$\max_{0 \le t \le 1} \|\omega^a(t, \cdot) - \omega(t, \cdot)\|_\infty < \epsilon. \tag{6.35}$$

Proof of Lemma 6.5 By first taking $\|g\|_\infty \lesssim 1$, we have $\|f + g\|_\infty \lesssim_{f,R_0} 1$. Since $\mathrm{supp}(f) \subset B(0, R_0)$ and $\mathrm{supp}(g) \subset B(0, R_0)$, we get

$$\begin{aligned} &\mathrm{supp}(\omega(t, \cdot)) \subset B(0, R_1), \\ &\mathrm{supp}(\omega^a(t, \cdot)) \subset B(0, R_1), \quad \forall\, 0 \le t \le 1, \end{aligned}$$

where $R_1 > 0$ is some constant depending on R_0 and $\|f\|_\infty$ only.

Set $\eta = \omega^a - \omega$. Then η satisfies the equation

$$\begin{cases} \partial_t \eta + (\Delta^{-1}\nabla^\perp \eta \cdot \nabla)\omega^a + (u \cdot \nabla)\eta = 0, \\ \eta(0) = g. \end{cases} \tag{6.36}$$

By a simple energy estimate, we have

$$\max_{0 \le t \le 1} \|\nabla \omega^a(t, \cdot)\|_\infty \lesssim_f 1.$$

On the other hand, since $\mathrm{supp}(\eta(t,\cdot)) \subset B(0, R_1)$ for any $0 \le t \le 1$, we have

$$\begin{aligned}\|\Delta^{-1}\nabla^{\perp}\eta(t,\cdot)\|_\infty &\lesssim \|\eta(t,\cdot)\|_1 + \|\eta(t,\cdot)\|_\infty \\ &\lesssim_{R_1} \|\eta(t,\cdot)\|_\infty.\end{aligned}$$

By (6.36), we then get for any $0 \le t \le 1$,

$$\|\eta(t,\cdot)\|_\infty \lesssim_{R_1,f} \|g\|_\infty + \int_0^t \|\eta(s,\cdot)\|_\infty ds.$$

A Gronwall argument then yields

$$\max_{0\le t\le 1} \|\eta(t,\cdot)\|_\infty \lesssim_{R_1,f} \|g\|_\infty.$$

Therefore (6.35) follows by choosing $\|g\|_\infty$ sufficiently small. □

We now sketch the proof of Theorem 1.5.

Proof of Theorem 1.5 We begin by noting that the support condition in statement (1) of Theorem 1.5 ("compactly supported in a ball of radius ≤ 1") is rather easy to achieve: one only needs to change the parameters of the distances between the patch solutions in our construction below. Similar comment also applies to the condition "$\|\omega_0^{(p)}\|_{\dot{H}^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \epsilon$". Therefore we shall ignore all these conditions below. In particular to simplify notation we will construct $\omega_0^{(p)}$ of order 1. Also without loss of generality we may assume $\omega_0^{(g)}$ is supported (say) in a ball of radius $\le \frac{1}{1000}$.

Define $z_0 = (-2, 0)$, $z_1 = (0, 0)$. For each integer $j \ge 2$, define

$$z_j = (z_j^{(1)}, 0) = \left(\sum_{k=1}^{j-1} \frac{100}{2^k}, 0\right) \tag{6.37}$$

We shall choose z_j, $j \ge 0$ to be the center of the j^{th} patch.

Now define $h_0(x) = \omega_0^{(g)}(x - z_0)$ and $\delta_0 = 1$. By Lemma 6.4 with $f_{-1} = h_0$, $R_0 = \frac{1}{4}$, $\epsilon = 1/800$, we can find $\delta_1 > 0$, $0 < t_1 < \frac{1}{2}$, and $h_1 \in C_c^\infty(B(0, \frac{1}{800}))$ with the properties

- h_1 is an odd function of x_2;
- $\|h_1\|_{L^1} + \|h_1\|_{L^\infty} + \|h_1\|_{H^1} + \|h_1\|_{\dot{H}^{-1}} \le \frac{1}{8}$;

such that for any $\tilde{f} \in C_c^\infty(\mathbb{R}^2)$ with

- $\mathrm{supp}(\tilde{f}) \subset \{x = (x_1, x_2) : \quad x_1 \ge \frac{1}{4}\}$;

- $\|\tilde{f}\|_{L^1} + \|\tilde{f}\|_{L^\infty} \le \delta_1$,

the following hold true:

For the Euler equation

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega = 0, \\ \omega\Big|_{t=0} = h_0 + h_1 + \tilde{f}, \end{cases}$$

the smooth solution $\omega = \omega(t)$ satisfies:

(1) For any $0 \le t \le t_1$, $\omega(t)$ can be decomposed as

$$\omega(t,x) = \omega_{h_0}(t,x) + \omega_{h_1}(t,x) + \omega_{\tilde{f}}(t,x),$$

where

$$\mathrm{supp}(\omega_{h_0}(t,\cdot)) \subset B\left(0, -2 + \frac{1}{32}\right),$$
$$\mathrm{supp}(\omega_{h_1}(t,\cdot)) \subset B\left(0, \frac{1}{8} + \frac{1}{32}\right),$$
$$\mathrm{supp}(\omega_{\tilde{f}}(t,\cdot)) \subset \left\{x = (x_1,x_2): \quad x_1 \ge \frac{1}{4} - \frac{1}{32}\right\};$$

(2)

$$\|\omega_{h_1}(t_1,\cdot)\|_{\dot{H}^1} > 8.$$

We now inductively assume that for $1 \le i \le j$, we have chosen $h_i \in C_c^\infty(B(z_i, \frac{1}{2^{i+9}}))$ which is odd in x_2, $0 < t_i < \frac{1}{2^i}$, $\delta_i > 0$, with

$$\|h_i\|_{L^1} + \|h_i\|_{L^\infty} + \|h_i\|_{H^1} + \|h_i\|_{\dot{H}^{-1}} \le \frac{1}{2^i} \min_{0\le k<i} \delta_k, \tag{6.38}$$

such that for any $\tilde{f} \in C_c^\infty(\mathbb{R}^2)$ with

- $\mathrm{supp}(\tilde{f}) \subset \{x = (x_1,x_2): \quad x_1 \ge z_{i+1}^{(1)} - \frac{1}{2^i}\}$ [see (6.37)] for the definition of $z_j^{(1)}$);
- $\|\tilde{f}\|_{L^1} + \|\tilde{f}\|_{L^\infty} \le \delta_i$,

the solution $\omega(t)$ to the equation

$$\begin{cases} \partial_t \omega + \Delta^{-1}\nabla^\perp \omega \cdot \nabla \omega = 0, \\ \omega\Big|_{t=0} = \sum_{l=1}^{i-1} h_l + h_i + \tilde{f}, \end{cases}$$

satisfies the properties:

(1) for any $0 \le t \le t_i$, we have the decomposition

$$\omega(t,x) = \omega_{\le i-1}(t,x) + \omega_i(t,x) + \omega_{\tilde f}(t,x), \tag{6.39}$$

where

$$\operatorname{supp}(\omega_{\le i-1}(t,\cdot)) \subset \left\{ x=(x_1,x_2): \quad x_1 \le z_{i-1}^{(1)} + \frac{1}{2^i} \right\};$$
$$\operatorname{supp}(\omega_i(t,\cdot)) \subset \left\{ x=(x_1,x_2): \quad |x-z_i| \le \frac{1}{2^i} \right\};$$
$$\operatorname{supp}(\omega_{\tilde f}(t,\cdot)) \subset \left\{ x=(x_1,x_2): \quad x_1 \ge z_{i+1}^{(1)} - \frac{1}{2^i} \right\};$$

(2) $\|\omega_i(t_i,\cdot)\|_{\dot H^1} > 2^i$.

Then for $i = j+1$, by shifting the coordinate axis to z_{j+1} if necessary, we can apply Lemma 6.4 with $f_{-1} = \sum_{i=0}^{j} h_i$, $\epsilon \ll \frac{1}{2^{i+1}} \min_{0\le k\le i} \delta_k$, and choose $h_{j+1} \in C_c^\infty(B(z_{j+1}, \frac{1}{2^{j+9}}))$ to satisfy all the needed properties similar to the i^{th} step. This way we have completely specified the profiles of all h_j, $j = 0, 1, 2, \ldots$.

Now we define the initial data

$$\omega_0 = \sum_{j=0}^{\infty} h_j.$$

It is easy to check that ω_0 is compactly supported and $\omega_0 \in L^\infty \cap \dot H^{-1} \cap H^1$. Denote the approximating initial data

$$\omega_0^{(J)} = \sum_{j=0}^{J} h_j$$

and let $\omega^{(J)}$ be the solution to the Euler equation

$$\begin{cases} \partial_t \omega^{(J)} + \Delta^{-1}\nabla^\perp \omega^{(J)} \cdot \nabla \omega^{(J)} = 0, \\ \omega^{(J)}\Big|_{t=0} = \omega_0^{(J)}. \end{cases} \tag{6.40}$$

By using L^p, $1 \le p \le \infty$ conservation of vorticity, L^2 conservation of velocity, it is easy to check that

$$\sup_J \sup_{0\le t<\infty} \Big(\|\omega^{(J)}(t,\cdot)\|_{L^1} + \|\omega^{(J)}(t,\cdot)\|_{L^\infty} + \|\omega^{(J)}(t,\cdot)\|_{\dot{H}^{-1}} \Big) \lesssim 1. \quad (6.41)$$

Furthermore by Lemma 6.5, we can[7] guarantee that

$$\max_{0\le t\le 1} \|\omega^{(J)}(t,\cdot) - \omega^{(J-1)}(t,\cdot)\|_\infty \le 2^{-J}.$$

Note that by (6.41) we can always guarantee for some constant $R > 0$ that

$$\mathrm{supp}(\omega^{(J)}(t,\cdot)) \subset B(0,R), \quad \forall\, 0 \le t \le 1,\ J \ge 1.$$

We then view $(\omega^{(J)})_{J\ge 1}$ as a Cauchy sequence in the Banach space $C_t^0 C_x^0([0,1] \times \overline{B(0,R)})$ and extract the limit solution ω in the same space. By interpolation and Sobolev embedding, it is easy to check that $u^{(J)} = \Delta^{-1}\nabla^\perp \omega^{(J)}$ also forms a Cauchy sequence in $C_t^0 L_x^2 \cap C_t^0 C_x^\alpha([0,1] \times \mathbb{R}^2)$ for any $0 < \alpha < 1$. Therefore $u^{(J)}$ converges to the limit $u = \Delta^{-1}\nabla^\perp \omega$ and ω is the desired solution.

Set $x_* = \lim_{j\to\infty} z_j = (100, 0)$. We now prove statement (3) and (4) in Theorem 1.5. Fix any integer $n \ge 2$ and we choose $t_n < \frac{1}{2^n}$ in the same way as specified in (6.38). By our way of construction, the fact that $(\omega^{(J)})$ is Cauchy in C^0 and (a version of) Lemma 5.1, we have that the limit solution ω obeys a decomposition similar to that in (6.39). More precisely define $t_n^2 = t_n$, then for any $0 \le t \le t_n^2$, we have

$$\omega(t,x) = \omega_{<n}(t,x) + \omega_n(t,x) + \omega_{>n}(t,x), \quad (6.42)$$

where $\omega_{<n}(t,\cdot) \in C_c^\infty(\Omega_{<n})$, $\omega_n(t,\cdot) \in C_c^\infty(\Omega_n)$, and

$$\Omega_{<n} := \left\{ x = (x_1,x_2) : \quad |x| < 1000 \text{ and } x_1 < z_{n-1}^{(1)} + \frac{2}{2^n} \right\};$$

$$\Omega_n := \left\{ x = (x_1,x_2) : \quad |x - z_n| < \frac{2}{2^n} \right\};$$

$$\mathrm{supp}(\omega_{>n}(t,\cdot)) \subset \left\{ x = (x_1,x_2) : \quad x_1 \ge z_{n+1}^{(1)} - \frac{1}{2^n} \right\};$$

[7] One needs to inductively shrink the δ_i further (so that Lemma 6.5 can be applied) and re-choose the profiles h_j if necessary.

Furthermore we can choose $t_n^1 < t_n^2$ (t_n^1 is sufficiently close to t_n^2) such that

$$\|\omega_n(t,\cdot)\|_{\dot{H}^1} > n, \quad \forall t \in [t_n^1, t_n^2].$$

Therefore statement (4) in Theorem 1.5 is proved. Now for statement (3) we discuss two cases. If $x = (x_1, x_2) \neq x_* = (100, 0)$ and $x_1 \geq 100$, then by using finite transportation speed we can find a neighborhood N_x of x and $t_x > 0$ sufficiently small such that $\omega(t, y) = 0$ for any $0 \leq t \leq t_x$ and $y \in N_x$. Similarly we can treat the case $x = (x_1, x_2)$, $x_1 < 100$ and $|x| > 500$. On the other hand if $x = (x_1, x_2)$ and $x_1 < 100$ with $|x| \leq 500$, then we can find n sufficiently large such that $x \in \Omega_{<n}$. Obviously we just need to define $N_x = \Omega_{<n}$ and $t_x = t_n^2$ so that $\omega(t,\cdot) \in C^\infty(N_x)$ for all $0 \leq t \leq t_x$. □

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