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Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$

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Abstract We study stability and bifurcations in holomorphic families of polynomial automorphisms of $\mathbb{C}^2$. We say that such a family is weakly stable over some parameter domain if periodic orbits do not bifurcate there. We first show that this defines a meaningful notion of stability, which parallels in many ways the classical notion of J-stability in one-dimensional dynamics. Define the bifurcation locus to be the complement of the weak stability locus. In the second part of the paper, we prove that under an assumption of moderate dissipativity, the parameters displaying homoclinic tangencies are dense in the bifurcation locus. This confirms one of Palis' Conjectures in the complex setting. The proof relies on the formalism of semi-parabolic bifurcation and the construction of "critical points" in semi-parabolic basins (which makes use of the classical Denjoy–Carleman–Ahlfors and Wiman Theorems).

1 Introduction

One of the main goals in the modern theory of dynamical systems is to describe the dynamics of typical mappings in a representative family. Let us consider

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for instance the space of C^k diffeomorphisms ($k \geq 1$) of real compact surfaces. It was briefly believed in the 1960s that hyperbolicity was generically satisfied in $\mathrm{Diff}^k(M)$. This hope was discouraged fast, particularly with the discovery by Newhouse [46,47] of an open region $\mathcal{N}$ in $\mathrm{Diff}^k(M)$, $k \geq 2$ containing a dense subset of maps that display homoclinic tangencies. Moreover, a generic map in $\mathcal{N}$ has infinitely many sinks. (We will refer to $\mathcal{N}$ as the *Newhouse region*.)

A more refined picture of typical dynamics of diffeomorphisms then gradually emerged. It was articulated by Palis as a series of conjectures (see e.g. [48], [49, Chap. 7]). The first conjecture on this list is the following:

Conjecture (Palis). *Every $f \in \mathrm{Diff}^k(M)$, $k \geq 1$, can be C^k-approximated either by a hyperbolic diffeomorphism or by one exhibiting a homoclinic tangency.*

Here "homoclinic tangency" means a tangency between the stable and unstable manifolds of some saddle periodic point. Since hyperbolic diffeomorphisms are structurally stable, this singles out homoclinic tangencies as a basic phenomenon responsible for bifurcations. This conjecture was proven for $k = 1$ by Pujals and Sambarino [50], nevertheless it remains wide open for $k > 1$. More generally, there has been an important progress in the understanding of C^1-generic dynamics in the past few years (see [16] for a recent overview).

Another situation that has been extensively studied is one-dimensional dynamics, both real and complex. In fact, the early Density of hyperbolicity conjecture turned out to be true in the real one-dimensional case [27,34,42]. It is conjectured to be true in the complex case as well (this is known as the *Fatou Conjecture*), but this problem is still open.

Consider a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of rational mappings of degree d on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, parameterized by a complex manifold Λ (which may be the whole space of rational mappings of degree d). We say that the family is *J-stable* in a connected open subset $\Omega \subset \Lambda$ if in Ω the dynamics is structurally stable on the Julia set J. Work of Mañé et al. [43] and independently of the second author [40,41] implies that the J-stability locus is dense in Λ. In addition, parameters with preperiodic critical points (which is the one-dimensional counterpart of the homoclinic tangency) are dense in the bifurcation locus. We see that the Fatou Conjecture is reduced to the problem whether J-stability implies hyperbolicity (for a sufficiently generic family, like the whole space of polynomials or rational maps of a given degree).

In this paper we deal with families of polynomial automorphisms of $\mathbb{C}^2$, which shares features with both of the previous settings. Friedland and Milnor [22] showed that dynamically interesting automorphisms in $\mathbb{C}^2$ are conjugate to compositions of Hénon mappings $(z, w) \mapsto (aw + p(z), az)$, where a is a non-zero complex number and p is a polynomial of degree at least two. *In what follows, we assume without saying that all automorphisms under consideration are dynamically interesting*, and in particular, they have dynamical degree $d \geq 2$ (see Sect. 2 for a review of this notion).

Note that a polynomial automorphism f has constant complex Jacobian $\text{Jac } f = \det Df$. So $\text{Jac } f$ is a well-defined quantity attached to f. We work in the dissipative setting, and our main results actually require some stronger form of dissipation, namely we need

$$|\text{Jac } f| < \frac{1}{d^2}, \quad \text{where } d \text{ is the dynamical degree of } f. \tag{1}$$

We will call such maps *moderately dissipative*[1]

We denote by J^* the closure of saddle periodic points of f. It is unknown whether J^* is always equal to the "small Julia set" J, which can be defined in classical terms as the locus where both families $\{f^n\}_{n\geq 0}$ and $\{f^n\}_{n\leq 0}$ are not normal.

From one-dimensional holomorphic dynamics we borrow the idea of focusing on J-stability rather than hyperbolicity, and in accordance with the Palis program, we explain bifurcations by the presence of homoclinic tangencies. Our main result is the following, in the spirit of the Palis conjecture (the precise meaning of the terminology "weakly stable" will be explained shortly).

Theorem A. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of moderately dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then weakly stable maps, together with maps exhibiting non-persistent homoclinic tangencies form a dense subset of Λ.*

It is also true that *weakly stable maps, together with maps that have infinitely many sinks form a dense subset in Λ*. Somewhat surprisingly, this is just an observation obtained by analyzing the one-dimensional argument.

The set of locally weakly stable parameters will be simply referred to as the *stability locus*, and its complement is by definition the *bifurcation locus*. It is worth mentioning here that G. Buzzard [12] showed that *the Newhouse region is non-empty* in the space of polynomial automorphisms of sufficiently high degree. It follows that the stability locus is not dense in general.

Let us now discuss the notion of weak stability. To say it briefly, a family of polynomial automorphisms is weakly stable in some open set if periodic points do not bifurcate there. The first part of this paper is devoted to demonstrating that this defines a reasonable notion of stability in this context, parallel to the

[1] The word "moderately" was chosen to contrast with the very strong dissipativity assumptions that are often made in the study of real Hénon mappings.

usual J-stability in dimension 1. In particular we show that in a weakly stable family:

- there are no homoclinic bifurcations, and moreover all homoclinic and heteroclinic intersections can be followed holomorphically;
- the sets J^*, J^-, J^+, K move continuously in the Hausdorff topology;
- connectivity properties of the Julia sets are preserved.

Let us point out that these results are true for any dissipative family, without further assumption on the Jacobian. Naturally, these results are based on a generalization to two dimensions of the key idea of *holomorphic motion*. A fundamental problem here is that a holomorphic motion of a set X in higher dimension does not automatically admit an extension to a motion of $\overline{X}$. In practice, we work with a weaker notion of "branched holomorphic motion", in which collisions are allowed. Because of this, we have not been able to prove that weak stability implies structural J^*-stability.

An important special case is when f is uniformly hyperbolic on J^*. It then follows from the classical theory of hyperbolic dynamical systems that f is structurally stable on J^*. In addition it is known that J^* moves holomorphically (see Jonsson [31]), and that this holomorphic motion extends to a holomorphic motion of $J^+ \cup J^-$ (see Buzzard–Verma [13]).

The main point of this paper is to design a mechanism creating homoclinic tangencies from bifurcations of periodic points (for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$). Notice that conversely, the creation of sinks from (generic) homoclinic tangencies is classical and goes back to Newhouse [47] (see Gavosto [24] for a proof in our context). The theory of weak J^*-stability gives a fresh insight into this phenomenon as well.

Let us now formulate a more precise version of Theorem A:

Theorem A$'$. *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of moderately dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then parameters with non-persistent homoclinic tangencies are dense in the bifurcation locus.*

To understand the strategy of the proof of this theorem, let us first review the one-dimensional result that parameters with preperiodic critical points are dense in the bifurcation locus. The classical proof of this fact, based on Montel's theorem [38], does not seem to have an analogue in our context.

Let us outline an argument that admits a generalization to dimension two. If λ_0 belongs to the bifurcation locus, then some periodic point changes type near λ_0. In particular there exists λ_1 close to λ_0 such that at λ_1, there is a periodic point p whose multiplier crosses the unit circle at a rational parameter. The theory of parabolic implosion [37,52] describes how the dynamics in the basin $\mathcal{B}$ of the parabolic point can "implode" for some parameters close to λ_1. In particular, under generic assumptions, and replacing f by some iterate if needed, for well chosen sequences $\lambda_n \to \lambda_1$, $f^n_{\lambda_n}$ converges locally uniformly in $\mathcal{B}$ to some limiting holomorphic function $g : \mathcal{B} \to \mathbb{C}$, which we refer to as a *transit map*. Fix a repelling periodic point q, which necessarily persists as $q(\lambda)$ in the neighborhood of λ_1. By a classical theorem of Fatou, there exists a critical point c in $\mathcal{B}$. Then it is actually possible to adjust the sequence λ_n so that $g(c) = q$. From this we infer that for large n, there exists λ_n' close to λ_n, such that $f^n_{\lambda_n'}(c(\lambda_n')) = q(\lambda_n')$, which is precisely the result that we seek.

To prove Theorem A′, in the second part of the paper we design a two-dimensional generalization of this argument. In the dissipative regime, if some periodic point $p(\lambda)$ bifurcates at λ_0, then one multiplier of $p(\lambda)$ crosses the unit circle while the other stays smaller than 1. If furthermore $p(\lambda_0)$ has a root of unity as multiplier, it is said to be *semi-parabolic*, and we say that $p(\lambda)$ undergoes a *semi-parabolic bifurcation*. Then the proof is divided into two main steps:

- Step 1: prove the existence of "critical points" in the basins of semi-parabolic periodic points.
- Step 2: use "semi-parabolic implosion" to make these critical points leave the basin under small perturbations of λ_0, eventually creating tangencies.

The critical points in Step 1 are defined as follows. Let f be a polynomial automorphism with a semi-parabolic periodic point p, which we may assume is fixed. Then p admits a basin of attraction $\mathcal{B}$, which is endowed with a holomorphic *strong stable foliation*, whose leaves are characterized by the property that points in the same leaf approach one another exponentially fast under iteration. Then by definition a critical point is a point of tangency between the strong stable foliation in $\mathcal{B}$ and the unstable manifold of some saddle periodic point q.

We obtain the following result.

Theorem B. *Let f be a moderately dissipative polynomial automorphism of $\mathbb{C}^2$. Assume that f possesses a semi-parabolic periodic point with basin of attraction $\mathcal{B}$. Then for any saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point.*

Notice that this is precisely the place where the assumption on the Jacobian is required. Curiously, the proof relies on the classical theory of entire functions of finite order in one complex variable. The same idea was then used by Peters and the second author [39] to obtain a nearly complete classification of periodic Fatou components for moderately dissipative polynomial automorphisms of $\mathbb{C}^2$.

Remark The classical theory of entire functions was first applied to (one-dimensional) polynomial dynamics by Eremenko and Levin [20].

The second step relies on the construction of transit mappings in the context of semi-parabolic bifurcations. Semi-parabolic points are roughly classified according to the multiplicity of $f - \mathrm{id}$ at the periodic point under consideration. The theory of semi-parabolic implosion was recently developed by Bedford et al. [2] who obtained a satisfactory picture in the multiplicity two case. In particular, it generalizes a theorem of Lavaurs [37], thus obtaining a precise description of the transit behaviour in this setting. However, these results depend on certain explicit changes of variables that do not readily extend to the general case.

In our situation we have to deal with semi-parabolic points of arbitrary multiplicity, so we need to develop a more general method. It was inspired by a chapter of the celebrated Orsay Notes by Douady and Hubbard, cheerfully entitled "*un tour de valse*" [18] (written by Douady and Sentenac).

To be specific, if λ_0 is a parameter at which a semi-parabolic bifurcation occurs, replacing f by some iterate if needed, there exists a sequence of parameters $\lambda_n \to \lambda_0$ such that $f^n_{\lambda_n}$ converges in $\mathcal{B}$ to some holomorphic map $g : \mathcal{B} \to \mathbb{C}^2$. Notice that due to dissipation, g has 1-dimensional image. An important phenomenon here is that $g(\mathcal{B})$ need not be contained in $\mathcal{B}$: indeed f_{λ_n} shifts $\mathcal{B}$ slightly, which is then amplified by iteration. In this sense the limiting dynamics of f_{λ_n} is richer than that of f_0. Though these transit mappings g are not as explicit as in [2], they can still be well controlled (see Theorem 8.7). If now $q = q(\lambda_0)$ is any saddle point, and c is a critical point in $W^u(q)$, we can adjust the sequence λ_n so that $g(c) \in W^s(q)$. It is then easy to find parameters λ_n' close to λ_n for which $W^u(q(\lambda_n'))$ and $W^s(q(\lambda_n'))$ are tangent, thereby concluding the proof.

The plan of the paper is the following. The first section is devoted to some preliminaries on polynomial automorphisms of $\mathbb{C}^2$. The notion of branched holomorphic motion is explained in detail in Sect. 3. In Sect. 4, we define the notion of weak J^*-stability, which is the direct analogue of the one-dimensional notion of J-stability and study the properties of weakly J^*-stable families. In Sect. 5 we show that a weakly J^*-stable family is also weakly stable on J^+, J^-, and K. In particular this justifies the use of the more general "weakly stable" terminology. We also prove that if a dissipative family of polynomial automorphisms has persistently connected Julia set, then it is weakly stable (Theorem 5.7). This generalizes a well known result in dimension 1. The proof of Theorem A$'$ occupies Sect. 6 to 9. In Sect. 6, we recall some basics on semi-parabolic dynamics. The existence of critical points in semi-parabolic basins (Theorem B) is discussed in Sect. 7, which also includes some preparatory material on entire functions of finite order. A slight adaptation gives the existence of critical points in attracting basins. Details are given in Appendix A. Semi-parabolic implosion and transit mappings are studied in Sect. 8, and finally in Sect. 9 we assemble these results to prove Theorem A$'$.

Throughout the paper we use the following notation: if u and v are two real valued functions, we write $u \asymp v$ (resp $u \lesssim v$) if there exists a constant $C > 0$

such that $\frac{1}{C}u \leq v \leq Cu$ (resp. $u \leq Cv$). The disk in $\mathbb{C}$ of radius r centered at 0 is denoted by $\mathbb{D}_r$. Moreover, $\mathbb{D}$ stands for $\mathbb{D}_1$. Throughout the paper, Λ stands for a connected complex manifold, which serves as a parameter space.

Remark The results of this paper were first announced at the Balzan-Palis Symposium on Dynamical Systems (IMPA, June 2012) and at the Workshop on Holomorphic Dynamical Systems (Banff, July 2012).

2 Preliminaries

In this section we recall some basics on the dynamics of polynomial automorphisms of $\mathbb{C}^2$, and establish some preparatory results.

2.1 Basics

Let f be a polynomial automorphism of $\mathbb{C}^2$ with non-trivial dynamics. "Non-trivial dynamics" here means for instance that f has positive topological entropy, which then equals $\log d$, where

$$d = \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

is the *dynamical degree* of f. According to Friedland and Milnor [22] this happens if and only if f is conjugate to a composition of Hénon mappings

$$(z, w) \mapsto (p(z) - b\, w, z).$$

Let us recall the following basic dynamical objects and facts. The reader can consult [5,7,21,29] for details.

- $K^\pm$ are the forward and backward *filled Julia sets*, that is, the sets of points with bounded forward/backward orbits respectively.
- $U^\pm = \mathbb{C}^2\backslash K^\pm$ are the forward and backward *basins of infinity*.
- $J^\pm = \partial K^\pm$ are the forward and backward *Julia sets*. They can be also defined as the *sets of non-normality* for the families $\{f^{\pm n}\}_{n\geq 0}$ respectively. Note that in the dissipative case, K^- has empty interior so $J^- = K^-$.
- $K := K^+ \cap K^-$ is the *filled Julia set* consisting of points whose *two-sided orbits* do not escape.
- $J := J^+ \cap J^-$ is the *"little" Julia set* and $\widehat{J} := J^+ \cup J^-$ is the *"big"* one. In the complement of the former, at least one of the families, $\{f^n\}_{n\geq 0}$ or $\{f^n\}_{n\leq 0}$, is normal. In the complement of the latter, the whole two-sided family $\{f^n\}_{n\in\mathbb{Z}}$ is.

- $\mathfrak{S}$ is the set of saddle periodic points (*saddles*). As usual, $W^s(p)$ and $W^u(p)$ stand for the *stable* and *unstable* manifolds of a saddle[2] p. They are holomorphically immersed complex lines $\mathbb{C} \to \mathbb{C}^2$.
- Given a saddle p, $H(p)$ denotes the set of *homoclinic intersections* between $W^u(p)$ and $W^s(p)$, while $H^{\mathrm{tr}}(p)$ is the subset of *transverse* homoclinic intersection. For any $p \in \mathfrak{S}$ the closure of either of these sets coincides with J^* [7, Prop. 9.8], so according to the general dynamics terminology, J^* is the *homoclinic class of f*.

We do not devote a special notation for the set of *heteroclinic* intersections, but use the following abbreviated terminology: *s/u intersection* is a shorthand for "homoclinic or heteroclinic intersection of stable and unstable manifolds of saddle periodic orbits".

- J^* is the closure of $\mathfrak{S}$. It is contained in J, and it is an open problem (posed by Hubbard) whether $J = J^*$.
- $S^- = J^- \setminus K^+$ and $S^+ = J^+ \setminus K^-$.
- $G^\pm$ are the forward and backward *Green functions*. Their dynamical meaning is that of *escape rate* functions:

$$G^\pm(z) = \lim_{n\to+\infty} \frac{1}{d^n} \log^+ \| f^{\pm n}(z)\|.$$

  Moreover, they have the following properties:
  - $G^\pm$ are non-negative and vanish on $K^\pm$ respectively;
  - $G^\pm$ are pluri-subharmonic on the whole $\mathbb{C}^2$, and pluri-harmonic on $U^\pm$ respectively;
  - they satisfy the functional equations $G^\pm(f^{\pm 1}z) = d\, G^\pm(z)$.
- $\varphi^\pm$ are the forward and backward *Böttcher functions*. They are well defined and holomorphic in appropriate sectors in $U^\pm$ near infinity and satisfy $\log|\varphi^\pm| = G^\pm$. Moreover, they satisfy the *Böttcher functional equations*

$$\varphi^\pm(f^\pm z) = (\varphi^\pm(z))^d.$$

  Note that by means of this equation, $\varphi^\pm$ extend analytically to the whole basins $U^\pm$ as multi-valued functions with a single-valued absolute value > 1 (namely $\exp(G^\pm)$).
- Though the Böttcher functions do not coherently extend to the whole basins $U^\pm$, their level sets do (by means of the dynamics), defining holomorphic *Böttcher $\mathbb{C}$-foliations* [3] $\mathcal{F}^\pm$ in $U^\pm$.

[2] or for a more general periodic point whenever they exist.

[3] meaning that their leaves are conformally equivalent to $\mathbb{C}$.

- Stable and unstable *Green currents* $T^{\pm} := dd^c G^{\pm}$. They are supported on the forward and backward Julia sets $J^{\pm}$ respectively and satisfy the dynamical functional equations $f^* T^{\pm} = d^{\pm 1}$. Moreover, *the unstable manifold $W^u(p)$ of any saddle p is equidistributed with respect to T^-*, while the stable manifolds $W^s(p)$ are equidistributed with respect to T^+ [5,21]. It follows that any $W^u(p)$ is dense in J^-, while any $W^s(p)$ is dense in J^+.
- The *measure of maximal entropy* $\mu = T^+ \wedge T^-$. By [8], *saddles are equidistributed with respect to μ*. Moreover, supp $\mu = J^*$.

2.2 Families of compositions of Hénon maps

We will be interested in holomorphic families $(f_\lambda)_{\lambda \in \Lambda}$ of polynomial automorphisms, parameterized by some complex manifold Λ. We put a subscript λ to denote the parameter dependence of the corresponding objects, e.g., J_λ, μ_λ, etc.

The following proposition, which might be known to some experts (see e.g. [23], and also [56, Thm 1.6] for the birational case), asserts that as far as we are interested in properties of f_λ which are typical with respect to λ, it is not a restriction to assume that the f_λ are products of Hénon mappings.

Proposition 2.1 *Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of polynomial automorphisms in $\mathbb{C}^2$, parameterized by a connected complex manifold. There exists a Zariski open set $\Lambda' \subset \Lambda$ and an integer $d \geq 1$ such that for $\lambda \in \Lambda'$, f_λ has dynamical degree d.*

Furthermore, if $d \geq 2$, locally in Λ' we can write

$$f_\lambda = \varphi_\lambda^{-1} \circ h_\lambda^1 \circ \cdots \circ h_\lambda^m \circ \varphi_\lambda$$

where (φ_λ) is a polynomial automorphism and $(h_\lambda^i)_{i=1,\ldots,m}$ are Hénon mappings of degree d_i, with $\sum d_i = d$, all depending holomorphically on λ.

To prove the proposition we need to recall some ideas from [22]. Fix coordinates (z, w) on $\mathbb{C}^2$. We denote by E the group of automorphisms preserving the family of lines $\{w = C\}$. Such automorphisms are of the form $(z, w) \mapsto (\alpha z + p(w), \beta w + \gamma)$ and will be referred to as *elementary*. (More generally, an automorphisms is elementary if it can be put in this form in some system of coordinates (z, w).) The group of affine automorphisms will be denoted by A. It turns out that the group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$ is the free product of A and E, amalgamated along their intersection $S := A \cap E$, that is, every $f \in \mathrm{Aut}(\mathbb{C}^2) \backslash S$, can be written as a composition $f = g_k \circ \cdots \circ g_1$, where g_i belongs to $A \backslash S$ or $E \backslash S$. This decomposition is unique, up to simultaneously replacing g_i by $g_i \circ s$ and g_{i-1} by $s^{-1} \circ g_{i-1}$, for some $s \in S$. The degree of such a composition is to equal $\prod \deg(g_i)$ (of course only elementary automorphisms contribute to the degree). One has that $\deg(f^n) = (\deg f)^n$ if and only if f is cyclically reduced, that is the extreme factors g_1 and g_k belong to different subgroups A and E. In general, write

$$f = a_m \circ e_m \circ a_{m-1} \circ e_{m-1} \circ \cdots \circ e_1 \circ a_1, \text{ with } a_i \in A\backslash S \text{ and } e_i \in E\backslash S,$$

with possibly a_m or a_1 equal to the identity. We define the multidegree of f as $(d_m, \ldots, d_1)$ where $d_i = \deg(e_i)$.

Proof It is clear that there exists a Zariski open set $\Lambda_0 \subset \Lambda$ where the degree is constant, say, equal to d'. If $d' = 1$ there is nothing to prove so assume $d' \geq 2$. A theorem due to Furter asserts that in a connected holomorphic family of polynomial automorphisms, the degree is constant if and only if the multidegree is constant [23, Cor. 3]. Hence there exists an integer m such that for every $\lambda \in \Lambda_0$ we can write

$$f_\lambda = a_{m,\lambda} \circ e_{m,\lambda} \circ a_{m-1,\lambda} \circ e_{m-1,\lambda} \circ \cdots \circ e_{1,\lambda} \circ a_{1,\lambda}.$$

We claim that the factors $a_{i,\lambda}$ and $e_{i,\lambda}$ may be chosen to depend holomorphically on λ. This is not obvious since they are not unique. We can deal with the extreme factors a_m and a_1 as in [22, Lemma 2.4], by observing that the coset space A/S is isomorphic to $\mathbb{P}^1$ and that there is a well defined mapping $f_\lambda \mapsto (a_{1,\lambda}^{-1}S, a_{m,\lambda}S) \in \mathbb{P}^1 \times \mathbb{P}^1$. In a more explicit fashion, this mapping may be expressed as $f_\lambda \mapsto (I(f_\lambda), I(f_\lambda^{-1}))$, where $I(f)$ is the indeterminacy set of f viewed as a rational mapping on $\mathbb{P}^2$, and $\mathbb{P}^1$ is identified to the line at infinity. Since f_λ depends holomorphically on λ, so do $a_{1,\lambda}^{-1}S$ and $a_{m,\lambda}S$, hence absorbing some of the S factors in $e_{m,\lambda}$ and $e_{1,\lambda}$ if necessary, we infer that $a_{1,\lambda}$ and $a_{m,\lambda}$ depend holomorphically in λ. Thus we are left to proving that if f_λ is of the form $f_\lambda = e_{m,\lambda} \circ a_{m-1,\lambda} \circ \cdots \circ e_{1,\lambda}$, then the factors may be chosen to depend holomorphically on λ. By [22, Lemma 2.10], f_λ admits a *unique* decomposition of the form

$$f_\lambda = (\hat{s}_{m,\lambda} \circ \hat{e}_{m,\lambda}) \circ t \circ \hat{e}_{m-1,\lambda} \circ \cdots \circ t \circ \hat{e}_{1,\lambda},$$

where $\hat{s}_{m,\lambda}$ is affine with diagonal linear part, $\hat{e}_i$ is of the form $(z, w) \mapsto (z + p_i(w), w)$, with $p_i(0) = 0$ and $t(z, w) = (w, z)$. By uniqueness, the factors of this decomposition depend holomorphically on λ (see [23, p.909] for details) and our claim is proven.

From this point it is clear that the set of parameters such that f_λ is not cyclically reduced is Zariski closed in Λ_0. Indeed, conjugating f_λ by $a_{1,\lambda}$ we obtain an expression of the form

$$a_{1,\lambda} \circ a_{m,\lambda} \circ e_{m,\lambda} \circ \cdots \circ e_{1,\lambda},$$

which is not cyclically reduced if and only if $a_{1,\lambda} \circ a_{m,\lambda} \in S$, which is an analytic condition. If so, we absorb $a_{1,\lambda} \circ a_{m,\lambda}$ into $e_{m,\lambda}$ and infer that the resulting word is not cyclically reduced iff $e_{1,\lambda} \circ e_{m,\lambda} \in S$, and so on. Iterating this process we obtain a Zariski open set Λ' such that if $\lambda \in \Lambda'$, f_λ is cyclically reduced, and the first part of the proposition is proved.

To establish the second assertion, in Λ' we conjugate f_λ as above to make it cyclically reduced and of the form

$$(t \circ e_k) \circ \cdots \circ (t \circ e_1).$$

Then we argue as in [22, Theorem 2.6] that a mapping of the form $t \circ e_i$ is affinely conjugate to a Hénon mapping $(z, w) \mapsto (\delta_i z + p_i(w), z)$, which is unique up to finitely many choices if p_i is chosen to be monic and centered. □

Part 1. Holomorphic Motions and Stability

3 Branched Holomorphic Motions

Recall the notation Λ for the parameter domain, which is a connected complex manifold. It will often be pointed by a *base point* $\lambda_0 \in \Lambda$. In this case, if we have a family of objects parametrized by Λ, the base objects will often be simply labeled with 0, e.g., $f_0 \equiv f_{\lambda_0}$, $J_0 \equiv J_{\lambda_0}$, etc.

Recall that a *holomorphic motion* of a set A in $\mathbb{C}^d$ over Λ is a family of mappings $h_\lambda : A \to \mathbb{C}^d$ such that

- for fixed $a \in A$, $\lambda \mapsto h_\lambda(a)$ is holomorphic;
- for fixed $\lambda \in \Lambda$, $a \mapsto h_\lambda(a)$ is injective;

Holomorphic motions are often *pointed* by assuming that h_{λ_0} is the identity mapping.

The *total space* of a holomorphic motion of A over Λ is a family of disjoint holomorphic graphs over the first coordinate in $\Lambda \times \mathbb{C}^2$, which we endow with the topology of uniform convergence on compact subsets of Λ. Let us relax this notion as follows:

Definition 3.1 A *branched holomorphic motion* (abbreviated as "BHM " in the following) over Λ is a family of holomorphic graphs over the first coordinate in $\Lambda \times \mathbb{C}^2$.

Remark 3.2 This definition bears some similarity with the notion of "analytic multifunction", which was studied by Słodkowski, and others. In particular it appears in [53] under the name of "locally trivial analytic multifunction".

With $\mathcal{G}$ being such a family, we let $\mathcal{G}_\lambda = \{\gamma(\lambda),\ \gamma \in \mathcal{G}\}$ be the *sections* of the total space, and we say that the $\mathcal{G}_\lambda$ "move under the branched motion $\mathcal{G}$". We let $\Pi_\lambda : \mathcal{G} \to \mathcal{G}_\lambda$ be the natural projection, which is obviously continuous.

As in the one-dimensional setting, we will use extension properties of (branched) holomorphic motions. The classical λ-lemma asserts that a holomorphic motion of $A \subset \mathbb{C}$ extends to $\overline{A}$ and is automatically continuous. These virtues come from *Montel normality* of the family $\mathcal{G}$ of disjoint graphs $\Lambda \to \mathbb{C}$ and from the *Hurwitz Theorem* that ensures that disjointness is inherited by the closure $\overline{\mathcal{G}}$. Of course neither of these statements is true in higher dimension, which motivates our use of branched motions as well as the following definition:

Definition 3.3 A branched holomorphic motion $\mathcal{G}$ in $\mathbb{C}^2$ over Λ is called *normal* if $\mathcal{G}$ is a normal family of graphs $\gamma : \Lambda \to \mathbb{C}^2$.

Recall that *normality* means that from any sequence of graphs γ_n we can extract a subsequence γ_{n_k} which is either locally bounded (and hence locally equicontinuous) or else $\gamma_{n_k} \to \infty$ locally uniformly. In particular, this is the case if the whole family $\mathcal{G}$ is locally uniformly bounded, or more generally, if the sections $\mathcal{G}_\lambda$ belong to a Kobayashi hyperbolic domain $U \subset \mathbb{C}^2$.

With these definitions in hand, the following lemma is obvious.

Lemma 3.4 *If $\mathcal{G}$ is a normal branched holomorphic motion over Λ, then so is $\overline{\mathcal{G}}$.*

Recall that the *Hausdorff topology* on the space of subsets of $\mathbb{C}^2$ is defined by the following basis of neighborhoods: $\mathcal{U}_{r,\epsilon}(A)$ consists of subsets $X \subset \mathbb{C}^2$ such that the set $X \cap \mathbb{D}_r^2$ is contained in the ϵ-neighborhood of $A \cap \mathbb{D}_r^2$, and the other way around.

Lemma 3.5 *If $\mathcal{G}$ is a normal BHM then the sections $\mathcal{G}_\lambda$ depend continuously on λ in the Hausdorff topology.*

Proof This easily follows from the local equicontinuity of the truncated families

$$\mathcal{G}_\lambda(r, \delta) := \{\gamma \in \mathcal{G} :\ \gamma(\lambda) \in \mathbb{D}_r^2 \text{ for } \lambda \in \mathbb{D}_{1-\delta}\}. \tag{2}$$

□

Let us say that a BHM $\mathcal{G}$ is *unbranched* at some $\lambda \in \Lambda$ if the natural projection $\mathcal{G} \to \mathcal{G}_\lambda$ is injective. It is *unbranched along* $\gamma_0 \in \mathcal{G}$ if γ_0 does not cross any other graph $\gamma \in \mathcal{G}$.

Lemma 3.6 *Let $\mathcal{G}$ be a normal BHM in $\mathbb{C}^2$ over Λ . If $\overline{\mathcal{G}}$ is unbranched at some parameter $\lambda_0 \in \Lambda$ then the mappings $h_\lambda : \overline{\mathcal{G}}_{\lambda_0} \to \overline{\mathcal{G}}_\lambda$ defined by*

$$\gamma(\lambda_0) \mapsto \gamma(\lambda), \ for \ \gamma \in \overline{\mathcal{G}},$$

are continuous and depend holomorphically on $\lambda \in \Lambda$.

Proof The last statement is obvious from the definitions. To prove continuity of the h_λ, let us consider the functional space $\overline{\mathcal{G}}_0(r,\delta) \equiv \overline{\mathcal{G}}_{\lambda_0}(r,\delta)$ (defined in (2)), which is compact. By the unbranching assumption, the natural projection

$$\Pi_0 : \overline{\mathcal{G}}_0(r,\delta) \to \overline{\mathcal{G}}_0 \cap \mathbb{D}_r^2$$

is bijective and hence is a homeomorphism. It follows that the maps $h_\lambda = \Pi_\lambda \circ \Pi_0^{-1}$ are continuous. □

Corollary 3.7 *Under the circumstances of Lemma 3.6, if the motion of $\overline{\mathcal{G}}$ is unbranched, then the maps $h_\lambda : \overline{\mathcal{G}}_{\lambda_0} \to \overline{\mathcal{G}}_\lambda$ are homeomorphisms.*

More generally, let us say that a normal holomorphic motion $\mathcal{G}$ is *strongly unbranched* if for every $\gamma \in \mathcal{G}$, $\overline{\mathcal{G}}$ is unbranched along γ (notice that the whole $\overline{\mathcal{G}}$ is allowed to be branched).

We say that a holomorphic motion is *continuous* if all the maps $h_\lambda : \mathcal{G}_0 \to \mathcal{G}_\lambda$, $\lambda \in \Lambda$, are homeomorphisms.

Lemma 3.8 *A normal holomorphic motion $\mathcal{G}$ is continuous iff it is strongly unbranched.*

Proof Assume h_λ is discontinuous for some $\lambda \in \Lambda$. Then for some $\gamma \in \mathcal{G}$ there exists a sequence $\gamma_k \in \mathcal{G}$ such that $\gamma_k(\lambda_0) \to \gamma(\lambda_0)$ while $\|\gamma_k(\lambda) - \gamma(\lambda)\| \geq \delta > 0$. Since $\mathcal{G}$ is normal, we can pass to a limit $\gamma_\infty \in \overline{\mathcal{G}}$ such that $\gamma_\infty(\lambda_0) = \gamma(\lambda_0)$ while $\gamma_\infty(\lambda) \neq \gamma(\lambda)$. Thus, $\overline{\mathcal{G}}$ is branched at γ. The same argument shows that discontinuity of h_λ^{-1} implies branching of $\overline{\mathcal{G}}$ at some $\gamma \in \mathcal{G}$.

The reverse assertion is easily supplied. □

Next, let us formulate a simple consequence of the classical one-dimensional λ-lemma:

Lemma 3.9 *Let $\psi_\lambda : \mathbb{C} \to \mathbb{C}^2$, $\lambda \in \mathbb{D}$, be a holomorphic family of injectively immersed entire curves. Let $h_\lambda : A_0 \to \mathbb{C}^2$, $\lambda \in \mathbb{D}$, be a holomorphic motion in $\mathbb{C}^2$ such that $A_\lambda := h_\lambda(X_0) \subset \psi_\lambda(\mathbb{C})$. Then it extends to a holomorphic motion of $\psi_0(\mathbb{C})$ with values in $\psi_\lambda(\mathbb{C})$. Moreover, locally in λ (independently of the particular motion over Λ), there is a canonical extension which depends only on the images $\psi_\lambda(\mathbb{C})$ but not on the particular choice of the parametrizations ψ_λ.*

Proof Apply the Słodkowski λ-lemma [53] to the holomorphic motion in $\mathbb{C}$:

$$\psi_\lambda^{-1} \circ h_\lambda \circ \psi_0 : \ \psi_0^{-1}(A_0) \to \psi_\lambda^{-1}(A_\lambda), \quad \lambda \in \mathbb{D}.$$

Moreover, locally in λ, there is the *canonical* "harmonic" extension due to Bers and Royden [9] which is equivariant under complex affine changes of variable, so it is independent of the particular choice of the ψ_λ. □

We will refer to the above canonical extension as the *Bers–Royden motion*. It implies the following *foliated λ-lemma* (first considered in [13]).

Let us say that a family of holomorphic $\mathbb{C}$-foliations $\mathcal{F}_\lambda$ depends holomorphically on $\lambda \in \Lambda$ if the local defining functions ϕ_λ for the $\mathcal{F}_\lambda$ can be selected holomorphic in λ. Given a set A and a $\mathbb{C}$-foliation $\mathcal{F}$ we define the *leafwise closure* $\mathrm{cl}_\mathcal{F} A$ as $\bigcup_L (\mathrm{cl}_L(A \cap L)$, where the union is taken over all the leaves L of $\mathcal{F}$ and the closure cl_L is taken in the intrinsic topology of the leaf.

Corollary 3.10 *Let $h_\lambda : A_0 \to \mathbb{C}^2$ be a holomorphic motion in $\mathbb{C}^2$, and let $\mathcal{F}_\lambda$ be a holomorphic family of $\mathbb{C}$-foliations supported on open sets $U_\lambda \subset \mathbb{C}^2$ containing A_λ. Then h_λ extends to a holomorphic motion of the leafwise closure* $\mathrm{cl}_{\mathcal{F}_0} A_0$. *Moreover, locally in λ, it further extends to the motion of the whole leaves of $\mathcal{F}_\lambda$ that meet A_λ.*

Proof The extension to the leafwise closure is obvious by the simplest one-dimensional version of the λ-lemma. Further extension comes from Lemma 3.9 (it is important that this extension is canonical). □

4 Weak *J**-stability

4.1 Substantial families

From now on, $(f_\lambda)_{\lambda\in\Lambda}$ will stand for a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$ over a parameter domain Λ, which is a connected complex manifold.

We will often require an additional—presumably superfluous—assumption. We say that a holomorphic family of polynomial automorphisms is *substantial* if

- either all members of the family are dissipative
- or for any periodic point with eigenvalues α_1, α_2, no relation of the form $\alpha_1^a \alpha_2^b = c$, holds persistently in parameter space, where a, b, c are complex numbers and $|c| = 1$.

As an example, any open subset of the family of all polynomial automorphisms of dynamical degree d is substantial [11, Theorem 1.4]. On the other hand, a family of conservative polynomial automorphisms is not.

4.2 Stability and Newhouse phenomenon

A family $(f_\lambda)_{\lambda\in\Lambda}$ induces a *fibered map*

$$\widehat{f}: \Lambda \times \mathbb{C}^2 \to \Lambda \times \mathbb{C}^2, \quad \widehat{f}: (\lambda, z) \mapsto (\lambda, f_\lambda(z)), \tag{3}$$

which in turn, induces an action on the space of graphs $(\lambda, \gamma(\lambda))_{\lambda\in\Lambda}$ of holomorphic functions $\gamma : \Lambda \to \mathbb{C}^2$. A branched holomorphic motion $\mathcal{G}$ over Λ is called *equivariant* if $\widehat{f}(\mathcal{G}) = \mathcal{G}$.

Definition 4.1 A holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms of $\mathbb{C}^2$ is called *weakly J^*-stable* if the sets J^*_λ move under an equivariant[4] BHM . A map f_{λ_0} and the corresponding parameter $\lambda_0 \in \Lambda$ are called weakly J^*-stable if the family (f_λ) is weakly J^*-stable over a neighborhood $\Lambda_0 \subset \Lambda$ of λ_0, otherwise we say that a bifurcation occurs at λ_0.

If in this definition we require that the motion in question is *unbranched* then we obtain the usual notion of *J^*-stability*. Given any dynamical set X_f (e.g., K_f or $J^\pm_f$), we can define *(weak) X-stability* in the same way.

Remark Note that we do not assume that the BHM in question is normal. It turns out that for all dynamical sets considered in this paper (e.g., $X = \widehat{J}$), the BHM can be selected to be normal. (Of course, in case of $X = J^*$ it is automatically so.)

The following theorem is very much in the spirit of one dimensional dynamics [41,43]. It shows that weak J^*-stability is a reasonable notion of stability for polynomial automorphisms.

Theorem 4.2 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. The following are equivalent:*

- (i) *The family (f_λ) is weakly J^*-stable.*
- (ii) *Every periodic point stays of constant type (saddle, attracting, repelling, indifferent) throughout the family.*
- (iii) *J^*_λ moves continuously in the Hausdorff topology.*[5]
  If furthermore (f_λ) is dissipative, the following two conditions are equivalent, and imply the previous ones:
- (iv) *The number of attracting cycles is (finite and) locally constant.*
- (v) *The periods of attracting cycles are locally uniformly bounded.*

Most of the proof of this theorem is contained in Sects. 4.4 and 4.5 below (see in particular Proposition 4.14). The proof will be completed in Sect. 5.6.

[4] Later on we will see that equivariance is automatically satisfied.

[5] In general, J^*_λ moves lower semi-continuously, compare [17].

Corollary 4.3 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$. Then any bifurcation parameter $\lambda_0 \in \Lambda$ can be approximated by a parameter $\lambda \in \Lambda$ such that f_λ has an attracting cycle.*

Proof By item (ii) of the theorem, λ_0 can be a approximated by a parameter μ_0 such that f_{μ_0} has an non-persistently indifferent periodic point p_0 with multipliers $|\alpha_1| < |\alpha_2| = 1$. Such a point can be perturbed to an attracting one. □

The following consequence follows exactly as in dimension 1:

Corollary 4.4 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$. If the number of attracting cycles is locally uniformly bounded on Λ, then the locus of weak J^*-stability is open and dense.*

Proof By the above corollary, any bifurcation parameter $\lambda_0 \in \Lambda$ can be perturbed to a parameter λ_1 such that f_{λ_1} with an attracting cycle. If λ_1 is a also a bifurcation parameter then for the same reason, it can be further perturbed to a parameter λ_2 with two attracting cycles, and so on. Since the number of attracting cycles is locally uniformly bounded, this process must terminate, hence producing a stable parameter λ_n approximating λ_0. □

In particular we have the following nice corollary in the spirit of the Palis conjectures. We say that f_λ is a *Newhouse automorphism* if it possesses infinitely many sinks.

Corollary 4.5 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$. Then the set*

$$\{\textit{locally weakly}\, J^*\textit{-stable parameters}\} \cup \{\textit{Newhouse parameters}\}$$

is dense in Λ.

Proof Let $B \subset \Lambda$ be the open set where the number of attracting cycles is locally uniformly bounded. By Corollary 4.4, weak J^*-stability is dense in B. Now in B^c, the set

$$U_m = \{\lambda,\ f_\lambda \text{ possesses at least } m \text{ attracting cycles}\}$$

is relatively open and dense. We conclude by Baire's Theorem. □

According to the work of Buzzard [12], it is known that the Newhouse region, i.e. the closure of the set of Newhouse parameters, has non-empty interior in the space of polynomial automorphisms of sufficiently high degree.

On the other hand, it is an open question whether weak J^* stability implies that there are only finitely many sinks (i.e. whether conditions (i)–(v) in Theorem 4.2 are equivalent). (Of course, the Palis Conjecture would imply it).

It is also worthwhile to state the following result which will follow from the proof of Theorem 4.2 (see Proposition 4.14 below).

Corollary 4.6 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a weakly stable substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then:*

- *the BHM of the set J^* is unbranched over the set of periodic, homoclinic and heteroclinic points;*
- *homoclinic and heteroclinic tangencies are persistent.*

An important question that is left open after this analysis is whether branching can actually occur. Indeed, while the λ-lemma clearly fails for general two-dimensional holomorphic motions, we do not know any instance of branching in the dynamical context, or even any mechanism that may lead to it. Thus, it is tempting to believe that weak J^*-stability actually implies J^*-stability. (Again, the Palis Conjecture would imply that this must be true generically.)

4.3 Normality of motions in $S_\lambda^\pm$

According to Proposition 2.1, (f_λ) is conjugate to a holomorphic family of composition of Hénon mappings. From this it easily follows that the sets K_λ are locally uniformly bounded in $\mathbb{C}^2$. In particular, if $\mathcal{G}$ is a BHM such that $\mathcal{G}_\lambda \subset K_\lambda$ for all λ, then it is normal. The next lemma shows that it is also true if $\mathcal{G}_\lambda \subset S_\lambda^\pm$. This will be used in Sect. 5.7.

Lemma 4.7 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of polynomial automorphisms of dynamical degree $d \geq 2$. Then any family of holomorphic mappings $\gamma : \Lambda \to \mathbb{C}^2$ such that for every $\lambda \in \Lambda$, $\gamma(\lambda) \in S_\lambda^+ \cup S_\lambda^-$ is normal.*

We will make use of the following lemma, which is known as the *Zalcman Renormalization Principle* (see [57] and Berteloot [10] for the version that we state here).

Lemma 4.8 *Let M be a compact complex manifold and $(g_n)_{n\geq 1}$ be a sequence of holomorphic mappings from the unit disk to M. If (g_n) is not a normal family at $z_0 \in \mathbb{D}$ then there exists a sequence (z_n) converging to z_0 and a sequence of scaling factors $r_n > 0$ converging to 0 such that (after possible extraction) the sequence of holomorphic mappings $\zeta \mapsto g_n(z_n + r_n\zeta)$ converges uniformly on compact sets to a non-constant entire map $g : \mathbb{C} \to M$.*

Proof of Lemma 4.7 It is no loss of generality to assume that Λ is the unit disk in $\mathbb{C}$. Moreover, since the sets

$$\widehat{S}^{\pm} := \{(\lambda, z) \in \Lambda \times \mathbb{C}^2 : z \in S^{\pm}_{\lambda}\}$$

are relatively open in their union, each graph $\gamma \in \mathcal{G}$ is fully contained in one of them. Then we can assume without loss of generality that all the graphs $\gamma \in \mathcal{G}$ are contained in one of these sets. For definiteness, let it be $\widehat{S}^{+}$.

Assume by contradiction that the family of mappings given in the statement is not normal. Then by the Zalcman Lemma 4.8 there exist a sequence of holomorphic disks $\gamma_n : \Lambda \to \mathbb{C}^2$ such that $\gamma_n(\lambda) \in S^{+}_{\lambda}$, a sequence of parameters $\lambda_n \to \lambda_\infty \in \Lambda$, and a sequence $r_n \to 0$ such that rescaled holomorphic disks $\gamma_n(\lambda_n + r_n z)$ converge to a non-constant entire curve $\zeta : \mathbb{C} \to \overline{S^{+}_{\lambda_\infty}}$. Since the Zalcman Lemma requires the target manifold to be compact, the closure here is taken in $\mathbb{CP}^2$. Observe that that in $\mathbb{CP}^2$ we have $\overline{J^+} = J^+ \cup \{I^+\}$, where $I^+ = I^+_{\lambda}$ is a single point at infinity.

Consider now a sequence of positive harmonic functions H_n defined by

$$H_n(z) = G^{-}_{\lambda_n + r_n z}(\gamma_n(\lambda_n + r_n z)).$$

A first possibility is that the H_n diverge uniformly to $+\infty$. Then ζ would take its values in I^+, which is absurd. Hence the H_n are locally uniformly bounded and converge to $G^{-}_{\lambda_1}(\zeta(z))$. Being a limit of a sequence of harmonic functions, $G^{-}_{\lambda_1}(\zeta(z))$ is harmonic. But a non-negative harmonic function on $\mathbb{C}$ must be constant. On the other hand, $J^+ \cap \{G^- = c\}$ is compact, so again we arrive at a contradiction. □

Though normality is what we need, let us also make a related statement:

Lemma 4.9 *Let f be a product of Hénon mappings. For any $R > 0$, the domain*

$$\Omega = U^+ \cap (\{\max(|x|, |y|) < R\} \cup \{|y| < |x|\})$$

is Kobayashi hyperbolic (and similarly for U^-).

Proof Let us consider a domain

$$Q = \{z = (x, y) \in U^+ : \ |y| < R\,|\varphi^+(z)|\},$$

where φ^+ is the forward Böttcher function. It is well defined since $|\varphi^+|$ is such, and it contains Ω, by increasing R slightly if needed. Let $\widetilde{U}^+$ be the covering of U^+ that makes the Böttcher function φ^+ well defined. Then

$$\widetilde{Q} = \{|Y| < R\,|\Phi|^+\},$$

where capitals mean the lifts to $\widetilde{U}$. Then $Z \mapsto (\Phi^+(Z), Y/\Phi^+(Z))$ maps $\widetilde{Q}$ onto the bidisk $(\mathbb{C}\backslash\mathbb{D}) \times \mathbb{D}_R$, with discrete fibers. Since the latter is hyperbolic, so are $\widetilde{Q}$, Q, and Ω (see [33, Prop. 1.3.12 and 3.2.9]). □

4.4 Motion of saddle and heteroclinic points

Let us start with a result that shows that any branched holomorphic motion $\mathcal{G}_\lambda \subset K_\lambda$ is strongly unbranched (and hence continuous) on hyperbolic sets. In particular, it is strongly unbranched at saddles and heteroclinic points.

Lemma 4.10 *Let $\mathcal{G}$ be a BHM over Λ, such that for every $\lambda \in \Lambda$, $\mathcal{G}_\lambda \subset K_\lambda$ or $\mathcal{G}_\lambda \subset S^-_\lambda \cup S^+_\lambda$. Assume that (γ_k) is a sequence of graphs in** $\mathcal{G}$ such that for some $\lambda_0 \in \Lambda$, $\gamma_k(\lambda_0) \to p(\lambda_0)$ as $k \to \infty$, where $p(\lambda_0)$ belongs to some uniformly hyperbolic invariant compact set E_{λ_0}.*

Then there exists a unique holomorphic continuation $(p(\lambda))_{\lambda\in\Lambda}$ of $p(\lambda_0)$ such that $\gamma_k(\lambda) \to p(\lambda)$ as $k \to \infty$ uniformly on compact subsets of Λ. Furthermore $p(\lambda)$ coincides with the natural continuation of p near λ_0 as a point of the hyperbolic set E_λ that dynamically corresponds to E_{λ_0}. In particular, if $(\tilde{\gamma}_k)$ is any other sequence with $\tilde{\gamma}_k(\lambda_0) \to p(\lambda_0)$, then $\tilde{\gamma}_k(\lambda) \to p(\lambda)$ on the whole Λ.

This holds in particular when $p(\lambda_0)$ is a saddle periodic point or a transverse s/u intersection.

Proof Let N be a neighborhood of $\lambda_0 \in \Lambda$ where E_λ persists as a hyperbolic set. Then the point p admits a natural local continuation $(p(\lambda))_{\lambda\in N}$. We claim that in N, $\gamma_k(\lambda) \to p(\lambda)$ when $k \to \infty$. Then the other conclusions of the lemma follow.

Indeed consider any cluster value of the sequence of holomorphic maps $(\gamma_k(\lambda))_{\lambda\in\Lambda}$ (recall that from Lemma 4.7 and the remarks preceding it that this is a normal family). By our claim it has to coincide with $p(\lambda)$ in N. This in turn allows to define a holomorphic continuation $p(\lambda)$ of p throughout Λ.

It remains to prove our claim that $\gamma_k(\lambda) \to p(\lambda)$ in some neighborhood of λ_0. Let us first deal with the case where $\mathcal{G}_\lambda \subset K_\lambda$. The observation is that for $\lambda \in N$, the dynamics is locally expansive near $p(\lambda)$, that is: there exists $\delta > 0$, which can be chosen to be uniform in N (reducing N if needed), such that if $q(\lambda)$ is such that $d(f^n_\lambda(q(\lambda)), f^n_\lambda(p(\lambda))) \leq \delta$ for all $n \in \mathbb{Z}$, then $p(\lambda) = q(\lambda)$. Now let q be any cluster value of the sequence of graphs $\gamma_k(\lambda)$, and consider the family $(f^n_\lambda(q(\lambda)))_{n\in\mathbb{Z}}$. This is a bounded, hence normal, family of graphs (since they are contained in $\bigcup K_\lambda$), and by assumption, $f^n_{\lambda_0}(q)(\lambda_0) = f^n_{\lambda_0}(q(\lambda_0)) = f^n_{\lambda_0}(p(\lambda_0))$. Therefore by equicontinuity, for λ close to λ_0, $f^n_\lambda(q(\lambda))$ remains close to $f^n_\lambda(p(\lambda))$ and we are done.

Assume now that for all $\lambda \in \Lambda$, $\mathcal{G}_\lambda \subset J^+_\lambda \backslash K_\lambda$. By Lemma 4.7, we can extract a subsequence, still denoted by γ_k, such that γ_k converges to some

$\gamma : \Lambda \to \mathbb{C}^2$ with $\gamma(\lambda_0) = p(\lambda_0)$. We claim that for $\lambda \in \Lambda$, $\gamma(\lambda)$ is in K_λ. Indeed $\lambda \mapsto G^-_\lambda(\gamma_k(\lambda))$ is a sequence of positive harmonic functions, so its limit $\lambda \mapsto G^-_\lambda(\gamma(\lambda))$ is harmonic and non-negative, and we conclude by observing that $G^-_{\lambda_0}(\gamma(\lambda_0)) = 0$, whence $\lambda \mapsto G^-_\lambda(\gamma(\lambda))$ vanishes identically, so that $\gamma(\lambda) \in K_\lambda$. Then by applying the reasoning of the previous paragraph we deduce that $\gamma(\lambda) = p(\lambda)$, which was the desired result. □

We now show that in substantial families, saddles do not change their nature under holomorphic motions.

Lemma 4.11 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Let $\mathcal{P}_0$ be a set of periodic points for $f_0 \equiv f_{\lambda_0}$ which admits a continuation as a branched holomorphic motion $\mathcal{P}_\lambda \subset K_\lambda$ over Λ. If $q(\lambda_0) \in \mathcal{P}_0$ is a non-isolated saddle periodic point, then its unique continuation $q(\lambda) \in \mathcal{P}_\lambda$ remains a saddle for all $\lambda \in \Lambda$.*

Proof From Lemma 4.10 we know that $q(\lambda_0)$ admits a unique continuation $q(\lambda)$ to Λ which locally coincides with its continuation as a saddle point. By analytic continuation of the identity $f^N_\lambda(q(\lambda)) = q(\lambda)$, $q(\lambda)$ is periodic throughout the family, and we need to show that it remains of saddle type.

To illustrate the idea, assume first that f_λ is dissipative for any $\lambda \in \Lambda$. In this case, if a saddle bifurcates, it must become a sink for an open set of parameters. On the other hand, as $q(\lambda_0)$ is non-isolated in $\mathcal{P}_0$, there is a sequence of other saddles $p_n(\lambda_0) \in \mathcal{P}_0$ converging to $q(\lambda_0)$. By Lemma 4.10, $p_n(\lambda) \to q(\lambda)$ on the whole space Λ. Moreover, outside countably many exceptional parameters in Λ, the points $p_n(\lambda)$ remain different from $q(\lambda)$. It follows that there is a parameter $\lambda \in \Lambda$ for which $q(\lambda)$ is a sink that can be approximated by other periodic points, which is contradictory.

Let us now address the general case.[6] We start with a saddle periodic point q of period N, that we can follow holomorphically as $(q(\lambda))_{\lambda\in\Lambda}$. We want to show that it cannot change type, i.e. that neither of the eigenvalues of the differential Df^N at $q(\lambda)$ crosses the unit circle. Since these eigenvalues are not locally constant (this is forbidden by the "substantiality" assumption), at a bifurcating parameter they run through an open arc of the unit circle. Thus, we may always assume that we the eigenvalues are far from 1, so that we can follow them holomorphically as $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$. Without loss of generality we replace Λ by a one-dimensional submanifold with the property that no relation of the form $\alpha_1^a\alpha_2^b = c$ holds persistently in it.

If a bifurcation occurs in the locus where $|\mathrm{Jac}\ f_\lambda| \neq 1$ then a sink or source can be created, and we conclude as before. In the remaining case, elliptic points

[6] The argument is similar to that of [8, Theorem 3] but the possibility of persistent non-linearizability, e.g. persistent resonance between the eigenvalues, was overlooked there. This is the reason for the additional assumption that (f_λ) is substantial.

are created. Recall that the possibility of linearizing a periodic point depends on a Diophantine condition on the eigenvalues. To be specific, a sufficient condition for linearizability is that there exists $\nu > 0$ such that for $j_1, j_2 \geq 1$ and $k = 1, 2$, $|\alpha_1^{j_1}\alpha_2^{j_2} - \alpha_k| \geq \frac{C}{(j_1+j_2)^\nu}$.

Consider a piece C of the curve $\{|\mathrm{Jac}\ f_\lambda| = 1\}$ in parameter space, and a point $\lambda_1 \in C$ where $|\alpha_1| = |\alpha_2| = 1$. Recall that α_1 and α_2 are holomorphic and non-constant. There are two possibilities. Either $|\alpha_1| = |\alpha_2| = 1$ along C or not. In the latter case there is a branch of the curve $|\alpha_1| = 1$ having an isolated intersection with C, so we have bifurcations in the dissipative regime and we are done. In the first case, we claim that $q(\lambda)$ cannot be persistently non-linearizable along C. Then, at a parameter where $q(\lambda)$ is linearizable, it is the center of a Siegel ball, and we get a contradiction in the same way as in the dissipative case.

To prove our claim, note that for $\lambda \in C$ we can write $\alpha_k(\lambda) = e^{i\theta_k(\lambda)}$, $k = 1, 2$, where θ_k is real analytic. *Since the family is substantial*, $(\theta_1, \theta_2, 1)$ are linearly independent. It is then a theorem of Schmidt [51] (solving a conjecture of Sprindzhuk's, see also [32]) that for a.e. λ, $(\alpha_1(\lambda), \alpha_2(\lambda))$ is Diophantine. This concludes the proof of Lemma 4.11. □

Let us point out the following consequence of Lemmas 4.10 and 4.11.

Corollary 4.12 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Let $\mathcal{P}_0 \subset \mathfrak{S}_0$ be a set of saddles of $f_0 \equiv f_{\lambda_0}$ without isolated points, that admits a continuation as a branched holomorphic motion $\mathcal{P}$ with $\mathcal{P}_\lambda \subset K_\lambda$ for $\lambda \in \Lambda$.*

Then all points in $\mathcal{P}_0$ persist as saddles and $(\mathcal{P}_\lambda)_{\lambda\in\Lambda}$ is the corresponding holomorphic motion. It is strongly unbranched, and hence continuous. Moreover, it extends to a strongly unbranched holomorphic motion of all saddles that belong to $\overline{\mathcal{P}_0}$.

Proof If $p(\lambda_0) \in \mathcal{P}_0$, Lemma 4.10 implies that it admits a unique continuation $p(\lambda) \in \mathcal{P}_\lambda$, which is never isolated in $\mathcal{P}_\lambda$, and Lemma 4.11 says that $p(\lambda)$ is a saddle for all λ. So we can apply Lemma 4.10 at all parameters, and it follows that $\mathcal{P}$ is strongly unbranched. The same holds for every saddle point belonging to $\overline{\mathcal{P}_0}$. □

We will also need the following result.

Lemma 4.13 *Let $\mathcal{P}_\lambda$ be a holomorphic motion of a set of periodic points of f_λ, $\lambda \in \Lambda$, such that $\mathcal{P}_0 \equiv \mathfrak{S}_{\lambda_0}$ is the set of all saddles of $f_0 \equiv f_{\lambda_0}$. Then $\overline{\mathcal{P}_\lambda} \supset J_\lambda^*$ for all $\lambda \in \Lambda$.*

Proof Note first that the statement is not obvious since *a priori* f_0 may have infinitely many sinks that could transform into saddles during the deformation.

However, by [8], if we denote by $\mathcal{P}_{n,\lambda_0} = \mathfrak{S}_{n,\lambda_0}$ the set of saddle points with period dividing n, then

$$\frac{\#\mathfrak{S}_{n,\lambda_0}}{d^n} \to 1 \quad \text{and} \quad \frac{1}{d^n} \sum_{p \in \mathcal{S}_{n,\lambda_0}} \delta_p \to \mu_{\lambda_0}.$$

Hence the continuation $\mathcal{P}_{n,\lambda}$ of $\mathcal{P}_{n,\lambda_0}$ is a set of periodic points with $\#\mathcal{P}_{n,\lambda} = \#\mathcal{P}_{n,\lambda_0} \sim d^n$. Thus, by the Equidistribution Theorem of [8], applied to f_λ we obtain that

$$\frac{1}{d^n} \sum_{p \in \mathcal{P}_{n,\lambda}} \delta_p \to \mu_\lambda,$$

and the conclusion follows. □

4.5 From special motions to weak J^*-stability

The λ-lemma allows us to promote motion of saddles or s/u intersections to weak J^*-stability:

Proposition 4.14 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Assume there exists a BHM $\mathcal{G}$ such that:*

- *$\mathcal{G}_\lambda \subset K_\lambda$ for any $\lambda \in \Lambda$;*
- *$\mathcal{G}_0 \equiv \mathcal{G}_{\lambda_0}$ is dense in $J_0^* \equiv J_{\lambda_0}^*$ for some $\lambda_0 \in \Lambda$.*

Then:

(a) *$\overline{\mathcal{G}_\lambda} \supset J_\lambda^*$ for every $\lambda \in \Lambda$;*
(b) *no saddle point bifurcates in the family, and the motion of saddles of f_0 is an equivariant strongly unbranched (and hence continuous) holomorphic motion;*
(c) *the family $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J^*-stable;*
(d) *the motion of transverse s/u intersections is an equivariant strongly unbranched motion;*
(e) *homoclinic and heteroclinic tangencies are persistent.*

Conversely, each of the conditions (b), (c), *or* (d) *implies the others.*

Proof Let us consider the set $\mathfrak{S}_0$ of saddles of f_0. By Lemma 4.10, points of $\mathfrak{S}_0$ can be followed holomorphically along Λ, giving rise by Corollary 4.12 to a strongly unbranched holomorphic motion of saddles $\mathcal{P} = (\mathcal{P}_\lambda)$ over Λ. Moreover, $\mathcal{P} \subset \overline{\mathcal{G}}$, while by Lemma 4.13,

$$\overline{\mathcal{P}_\lambda} \supset J_\lambda^* \quad \text{for all} \quad \lambda \in \Lambda, \tag{4}$$

implying (a).

Since the motion $\mathcal{P}$ satisfies the assumptions of the proposition, while $\mathcal{G}$ is not part of any further assertion, from now on we can assume $\mathcal{G} = \mathcal{P}$. In particular, $\mathcal{G}$ is equivariant.

Since $\mathcal{G}$ is a motion of saddles, and since every saddle point belongs to J^*, we conclude from (4) that for every λ, $\overline{\mathcal{G}_\lambda} = J^*_\lambda$. Applying Corollary 4.12 once again (with different base points), we conclude that no saddle point can change type in the family, and that for any $\lambda \in \Lambda$, $\mathcal{G}_\lambda$ is the set of all saddles of f_λ, i.e., $\mathcal{G}_\lambda = \mathfrak{S}_\lambda$. This proves (b). Since $\mathcal{G}$ is an equivariant normal holomorphic motion, $\overline{\mathcal{G}}$ is an equivariant BHM of J^*, implying (c).

For (d), let q be a transverse point of intersection of $W^s(p_1)$ and $W^u(p_2)$ (for some parameter λ_0). By Lemma 4.10 q admits a unique continuation $q(\lambda)$ which locally coincides with its natural continuation as a s/u intersection. By (c), the saddle points p_1 and p_2 persist in the family. Since $f^n_\lambda(q(\lambda))$ is a normal family and $f^n_\lambda(q(\lambda)) \to p_1(\lambda)$ for λ close to λ_0, this convergence holds throughout Λ, and similarly for $f^{-n}_\lambda(q(\lambda))$. In particular $q(\lambda) \in W^s(p_1(\lambda)) \cap W^u(p_2(\lambda))$ for all parameters so it remains an s/u intersection.

Let us now show that this s/u intersection remains transverse or equivalently, that there are no collisions. This will establish (e) and at the same time that the motion of transverse s/u intersections is strongly unbranched by Lemma 4.10, thus completing the proof of (d). (Notice that tangencies are not a priori incompatible with the fact that intersections are moving holomorphically, due to the possibility of degenerate tangencies).

Consider a pair $q(\lambda_0)$, $q'(\lambda_0)$ of distinct transverse intersections of $W^s(p_1(\lambda_0))$ and $W^u(p_2(\lambda_0))$. We know that q, q' (as well as p_1, p_2) can be followed holomorphically. We have to show that q and q' stay distinct. For this we parameterize $W^u(p_2(\lambda))$ by some $\phi_\lambda : \mathbb{C} \to W^u(p_2(\lambda))$, depending holomorphically on λ (see the comments preceding Proposition 5.2 below), so we may identify $W^u(p_2(\lambda))$ with $\mathbb{C}$. Fix another saddle point $p_3(\lambda)$. Since $W^s(p_3(\lambda_0))$ intersects transversally $W^u(p_1(\lambda_0))$, by the Lambda (or inclination) lemma of hyperbolic dynamics (see [49, p. 155]) we get that $q(\lambda_0)$ is the limit, inside $\mathbb{C} \simeq W^u(p_2(\lambda_0))$ of a sequence of transverse intersection points $q_n(\lambda_0)$ of $W^s(p_3(\lambda_0)) \cap W^u(p_2(\lambda_0))$. These intersection points can be followed globally in Λ.

We claim that $q_n(\lambda)$ converges locally uniformly to $q(\lambda)$ in Λ (again here we work in $\mathbb{C}$). Indeed notice first that by Montel's theorem q_n is a normal family, since locally we can follow any finite set of transverse intersections of $W^s(p_1(\lambda)) \cap W^u(p_2(\lambda))$, and $q_n(\lambda)$ stays disjoint from them. Then we argue that $q_n(\lambda_0)$ converges to $q(\lambda_0)$ while $q_n(\lambda)$ is disjoint from $q(\lambda)$, so by Hurwitz' Theorem $q_n(\lambda)$ converges to $q(\lambda)$.

To conclude the argument, assume that there exists λ_1 such that $q(\lambda_1) = q'(\lambda_1)$, and let N be any neighborhood of λ_1. Now if for every $\lambda \in N$ and

$n \geq 0$, $q'(\lambda) \neq q_n(\lambda)$, we have a contradiction with Hurwitz' Theorem. Thus there exists $\lambda_2 \in N$ and an integer n such that $q'(\lambda_2) = q_n(\lambda_2)$ which is impossible because these points belong to different stable manifolds. Hence, item (e) is established.

Conversely, if one of the conditions (b), (c) or (d) holds, then the assumption of the proposition is satisfied, and we infer that the other conclusions hold. This completes the proof. □

Let us point out the following consequence of Proposition 4.14:

Corollary 4.15 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. If there exists a persistent set of saddle points, which is dense in J^* for some parameter, then the family is weakly J^*-stable.*

Proof Apply the implication (b) ⇒ (d) of Proposition 4.14 to the given set of saddles. □

Remark 4.16 Notice that in Proposition 4.14 we do not assume any equivariance for $\mathcal{G}$. This shows that the equivariance assumption is superfluous in Definition 4.1.

Remark 4.17 It follows from Proposition 4.14 and from the density of transverse homoclinic intersections in J^* at every parameter that if all homoclinic intersections can be followed in some $\Omega \subset \Lambda$, then (f_λ) is weakly J^*-stable there. This a priori does *not* imply that weak J^*-stability follows from the absence of homoclinic tangencies. Indeed, if there are no tangencies in Ω, every homoclinic intersection can be followed locally. However, intersections may disappear by "slipping off to infinity" inside stable and unstable manifolds. The methods that we develop in Part 2 of the paper should be seen as a way of circumventing this problem.

4.6 Further consequences

Corollary 4.18 *Under the assumptions of Proposition 4.14, if furthermore (f_λ) is J^*-stable in a neighborhood of λ_0 (for instance if f_{λ_0} is uniformly hyperbolic on $J^*_{\lambda_0}$), then for $\lambda \in \Lambda$ there is a semiconjugacy $J^*_{\lambda_0} \to J^*_\lambda$.*

Proof By Lemma 4.10 we can follow holomorphically all points of $J^*_{\lambda_0}$. The motion is continuous near λ_0, so by analytic continuation we easily deduce that it is continuous throughout Λ. Likewise, the compatibility between the motion and the dynamics holds near λ_0 so it holds everywhere, hence it defines a global semiconjugacy. □

Corollary 4.19 *If a substantial family has the property that its members are topologically conjugate on J^*, and the conjugating map depends continuously on λ, then it is J^*-stable.*

Proof Fix a parameter $\lambda_0 \in \Lambda$. For every λ there is a conjugacy $h_\lambda : J_0^* \to J_\lambda^*$ depending continuously on λ. If p is a saddle point for f_0, then $h_\lambda(p) = p(\lambda)$ is a continuously moving periodic point which is a limit of periodic points for all λ. By Lemma 4.11, $p(\lambda)$ is a saddle for all parameters. In particular the assumptions of Corollary 4.15 hold, and the family (f_λ) is weakly J^*-stable, thus all saddle points move holomorphically. To conclude, let $q \in J^*$ be any point, and let a sequence of saddle points $p_n \to q$. Then for all λ, $h_\lambda(p_n) \to h_\lambda(q)$, so $\lambda \mapsto h_\lambda(q)$ is holomorphic. Finally, it is obvious that the motion is injective since the h_λ are homeomorphisms. □

5 Holomorphic Motions in Unstable Manifolds and Persistent Connectivity of the Julia Set

In this section we capitalize on the idea that in a weakly J^*-stable family of polynomial automorphisms, we can apply the one-dimensional theory of holomorphic motions inside stable and unstable manifolds. In Sect. 5.2 we show that connectivity properties of Julia sets are preserved in a weakly J^*-stable family. Conversely, if in some dissipative family (f_λ), the Julia set is persistently connected, then the family is weakly J^*-stable (Theorem 5.7). In Sect. 5.6, we use these techniques to complete the proof of Theorem 4.2. Finally, in Sect. 5.7 we prove, using the Bers–Royden λ-lemma, that in a weakly J^*-stable family, the equivariant BHM of the little Julia set J^* actually extends to the big Julia set $\widehat{J} = J^+ \cup J^-$. Note that in the hyperbolic case, this method was first used by Buzzard and Verma [13] to show that $\widehat{J}$ moves under an actual holomorphic motion.

5.1 Motion of $\partial(W^u(p) \cap K^+)$

Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, and let p be a saddle periodic point. Then $W^u(p)$ is is dense in J^- (see Sect. 2.1). In particular, there are two distinct topologies on $W^u(p)$: the one induced by the isomorphism with $\mathbb{C}$, which we refer to as the *intrinsic topology*, and the topology induced from $\mathbb{C}^2$. Viewed as subsets of $\mathbb{C}$, the components of $W^u(p) \cap K^+$ are simply connected closed subsets that may be bounded or unbounded. Notice that, with our current state of knowledge, nothing prevents a component of $W^u(p) \cap K^+$ with non-empty interior from being fully contained in J or even in J^*.

Recall the notation $H(p)$ and $H^{\mathrm{tr}}(p)$ from Sect. 2.1 for the sets of homoclinic intersections associated with a saddle p. Throughout this section we use the notation $\mathrm{Int}_i X$, $\mathrm{cl}_i X$, and $\partial_i X$ for the intrinsic interior, closure, and boundary of a subset $X \subset W^u(p)$.

Lemma 5.1 *Relative to the intrinsic topology in $W^u(p) \simeq \mathbb{C}$ we have that*

$$\partial_i(W^u(p) \cap K^+) = \mathrm{cl}_i\, H(p) = \mathrm{cl}_i\, H^{\mathrm{tr}}(p).$$

Proof This is very similar to [7, Sect. 9] (see also the proof of [19, Cor. 1.9]). We freely use the formalism of laminar currents and Pesin boxes, the reader is referred to [7] for details.

Let $x \in \partial_i(W^u(p) \cap K^+)$ and let us show that $x \in \mathrm{cl}_i\, H^{\mathrm{tr}}(p)$. Since the dynamical Green function G^+ admits a non-trivial minimum at x, if $\Delta \subset W^u(p)$ is a disk containing x, then G^+ is not harmonic in Δ, i.e. $T^+ \wedge [\Delta] > 0$. Let ψ be a cut-off function in Δ, with $\psi = 1$ near x. By [6, Thm 1.6], $d^{-n}(f^n)_*(\psi\,[\Delta]) \to cT^-$ as $n \to \infty$, with $c = \int \psi[\Delta] \wedge T^+ > 0$. We now argue exactly as in [7, Lemma 9.1]. Let P be a Pesin box of positive μ-measure, and S^+ be the uniformly laminar current made of the local stable manifolds $W^s_{\mathrm{loc}}(z)$, $z \in P$, with transverse measure given by the unstable conditionals of μ. Then $0 < S^+ \leq T^+$ so S^+ has continuous potential [7, Lemma 8.2]. It follows that $d^{-n}(f^n)_*(\psi[\Delta]) \wedge S^+ \to cT^- \wedge S^+ > 0$ and we conclude that for large n, $f^n(\Delta)$ admits transverse intersection points with $W^s_{\mathrm{loc}}(z_0)$, for some $z_0 \in P$ (the transversality comes from [7, Lemma 6.4]).

We claim that iterating a bit further, the iterates of Δ intersect $W^s_{\mathrm{loc}}(z)$ transversely, for *every* $z \in P$. Indeed, for y sufficiently close to z_0, $f^n(\Delta)$ intersects $W^s_{\mathrm{loc}}(y)$ transversely. By Poincaré recurrence and the Pesin Stable Manifold Theorem, for typical y like this, there exists an infinite sequence $(n_j)_{j\geq 1}$, such that $f^{n_j}(y) \in P$ and $f^{n+n_j}(\Delta)$ contains a disk close to $W^u_{\mathrm{loc}}(f^{n_j}(y))$, and the claim follows.

We now apply exactly the same argument to a neighborhood of p in $W^s(p)$. In this way we obtain disks in $W^s(p)$, arbitrary close to $W^s_{\mathrm{loc}}(z)$ for some $z \in P$, from which we conclude that $f^{n+n_j}(\Delta)$, hence Δ, intersects $W^s(p)$ transversely, and we are done.

As was mentioned in Sect. 2.1, $\overline{H(p)} = \overline{H^{\mathrm{tr}}(p)}$ in $\mathbb{C}^2$ [7, Prop. 9.8]. In fact, the proof works in the intrinsic topology of $W^u(p)$ as well. For convenience, let us recall the argument: since $W^u(p)$ admits non-trivial transverse intersections with $W^s(p)$, it follows from the Hyperbolic λ-lemma that every disk $\Delta \subset W^u(p)$ is the limit in $\mathbb{C}^2$ of an infinite sequence of disjoint disks $\Delta_n \subset W^u(p)$. Hence the result follows from the instability of non-transverse intersections [7, Lemma 6.4].

To conclude the proof, let us show that $H^{\mathrm{tr}}(p) \subset \partial_i(W^u(p) \cap K^+)$. Observe first that $p \in \partial_i(W^u(p) \cap K^+)$ (which is well known). Otherwise p would lie in the interior of $W^u(p) \cap K^+$, hence the family of forward iterates f^n would be normal in some intrinsic neighborhood $V \subset W^u(p)$ of p, contradicting the growth of the derivatives $Df^n(p)$ in the direction of $W^u(p)$.

Let now $x \in H^{\mathrm{tr}}(p) \subset W^u(p) \cap K^+$ and let Δ be a small disk around x, which is thus transverse to the stable manifold of p. By the Hyperbolic λ-lemma, $f^n(\Delta)$ contains graphs arbitrary close in $\mathbb{C}^2$ to a neighborhood of p in $W^u(p)$, so $f^n(\Delta)$ must intersect U^+, and the conclusion follows. □

Let now $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of polynomial automorphisms with a holomorphically moving saddle point $(p(\lambda))$. Then there exists a holomorphic family of parametrizations $\psi_\lambda^u : \mathbb{C} \to W^u(p(\lambda))$ with $\psi_\lambda^u(0) = p(\lambda)$. Indeed, since the eigenvectors of $Df_\lambda(p(\lambda))$ depend holomorphically on λ, we can normalize the family so that $p(\lambda) = 0$ and the eigenbasis is equal to the standard basis at 0, with the vertical direction unstable. Then normalize ψ_λ^u so that $(\pi_2 \circ \psi_\lambda^u)'(0) = 1$. The Graph Transform construction of the local unstable manifold provides an explicit formula for the normalized parametrization:

$$\psi_\lambda(z) = \lim_{n \to +\infty} \frac{1}{\mu_n} f_\lambda^n(0, z), \quad \text{where } \mu_n = \left. \frac{d(\pi_2 \circ f_\lambda^n(0, z))}{dz} \right|_{z=0},$$

which is manifestly holomorphic in λ.

The following easy proposition is the starting point for most of the results in this section.

Proposition 5.2 *Let (f_λ) be a weakly J^*-stable family of polynomial automorphisms, (p_λ) be a holomorphically moving saddle point and (ψ_λ^u) be a holomorphic family of parameterizations of $W^u(p_\lambda)$, as above. Then $(\psi_\lambda^u)^{-1}(\partial_i(W^u(p_\lambda) \cap K_\lambda^+))$ moves holomorphically in $\mathbb{C}$.*

Proof By Proposition 4.14 (e) and (f), homoclinic intersections move holomorphically and without collisions. Therefore the result follows directly from Lemma 5.1 and the ordinary one-dimensional λ-lemma.

5.2 Preservation of connectivity under branched motions

Let us start with a few general comments on connectivity of Julia sets of polynomial automorphisms.

Of course, if J is totally disconnected, then so is J^*. Moreover, by [6], every component of K intersects J^*, so in particular if K is totally disconnected, then $J = J^* = K$. It follows from general topology that if J is totally disconnected then so is K: indeed the boundary of a non-trivial continuum cannot be totally disconnected. On the other hand it is unclear whether total disconnectedness of J^* implies that J is totally disconnected.

Following [3], we say that f is said to be *unstably connected* if $U^{\pm} \cap W^u(p)$ is simply connected for some (and then any) saddle point p, and *unstably disconnected* otherwise. By [3], J is disconnected iff f is stably and unstably disconnected. It is not difficult to see that K is disconnected in this case (an inclination lemma argument). Likewise, it follows from [3] that J is connected iff J^* is connected. We also remark that a dissipative map is always stably disconnected [3, Cor. 7.4].

We start by observing that the connectedness of J is preserved in weakly J^*-stable families.

Proposition 5.3 *Let $(f_\lambda)_{\lambda \in \Lambda}$ be a weakly J^*-stable substantial family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. Then stable and unstable connectivity are preserved in the family. In particular if for some parameter λ_0, J_{λ_0} is connected, then J_λ is connected for all λ.*

Proof Let us show that disconnectedness of J is preserved in a weakly J^*-stable family. So assume that for some λ, f_λ is stably and unstably disconnected, so for every saddle point p, $W^u(p) \cap K^+$ and $W^s(p) \cap K^-$ admit intrinsic compact components. By Proposition 5.2, if C_{λ_0} is any compact component of $W^u(p_{\lambda_0}) \cap K^+_{\lambda_0}$, $\partial_i C_{\lambda_0}$ moves holomorphically as the parameter evolves, without colliding with the other components, so its continuation $(\partial_i C)_\lambda$ bounds a compact component of $W^u(p_\lambda) \cap K^+_\lambda$. Hence f_λ is unstably disconnected at all parameters, and the same argument shows that it remains stably disconnected as well. We conclude that J_λ is disconnected for every $\lambda \in \Lambda$. □

In the same way we obtain that if $(f_\lambda)_{\lambda \in \Lambda}$ is weakly J^*-stable and if for some parameter λ_0, J_{λ_0} is totally disconnected, then for all λ, f_λ is stably and unstably totally disconnected. However, it is unclear whether stable and unstable total disconnectedness implies that J or J^* is totally disconnected.

5.3 Traces of attracting basins in unstable manifolds

Here we will establish the following useful property (perhaps, known to experts):

Lemma 5.4 *Let f be a polynomial automorphism possessing an attracting periodic point q. Then for every saddle point p, there is a non-empty component of $\mathrm{Int}_i(W^u(p) \cap K^+)$ contained in the basin $\mathcal{B}(q)$.*

If in addition f is unstably disconnected, then there exists such a component that is relatively compact in the intrinsic topology.

Proof Since the basin $\mathcal{B}(q)$ is biholomorphic to $\mathbb{C}^2$, it contains entire curves $\mathcal{E} : \mathbb{C} \to \mathbb{C}^2$. To any such curve $\mathcal{E}$, corresponds a positive closed *Ahlfors current* produced by averaging currents of integration over holomorphic disks $\mathcal{E} : \mathbb{D}_r \to \mathbb{C}^2$ (see e.g. [45, Sect. 7.4 pp. 349-350]). Since T^+ is a unique positive closed current of mass 1 supported on K^+ [21], the Ahlfors current must be equal to T^+.

We can now proceed as in the proof of Lemma 5.1: fix a Pesin box and construct a local laminar current $S^- \leq T^-$ made of local Pesin stable manifolds. Since S^- has continuous potential, the entire curve $\mathcal{E}$ must intersect it; more precisely we get a transverse intersection with some local unstable leaf $W^u_{\mathrm{loc}}(x)$. Since $W^u(p)$ contains disks arbitrary close to $W^u_{\mathrm{loc}}(x)$, $W^u(p)$ intersects $\mathcal{E}$ transversely as well. Thus, $W^u(p) \cap \mathcal{B}(q) \neq \emptyset$.

Finally, note that any component C of $\mathrm{int}_i\,(W^u(p) \cap K^+)$ is either entirely contained in the basin $\mathcal{B}(q)$ or disjoint from it. This follows easily from normality of the family of restrictions $f^n : C \to \mathbb{C}^2$, $n \geq 0$.

Now assume that f is unstably disconnected, or equivalently, that $K^+ \cap W^u(p)$ admits a compact component. Thus there exists an intrinsically bounded topological disk $\Delta \subset W^u(p)$ such that $\Delta \cap K^+ \neq \emptyset$ and $\partial_i \Delta \subset U^+$. We claim that there exists a component of $\mathcal{B}(q) \cap W^u(p)$ that is contained in Δ. By Lemma 5.1, $H^{\mathrm{tr}}(p) \cap \Delta \neq \emptyset$, so by the Hyperbolic λ-lemma, for large n, $f^n(\Delta)$ contains disks arbitrary C^1-close to any given disk in $W^u(p)$. Now the first part of the proof shows that there is a point of transverse intersection between $\mathcal{E}$ and $W^u(p)$. Therefore $f^n(\Delta)$ intersects $\mathcal{E}$, hence $\mathcal{B}(q)$ for large n. By invariance, Δ intersects $\mathcal{B}(q)$ as well, and since $\partial_i \Delta \cap K^+ = \emptyset$, we conclude that there is a component of $W^u(p) \cap \mathcal{B}(q)$ which is compactly contained in Δ. □

In fact, the above proof gives a more general statement:

Proposition 5.5 *Let $\mathcal{D}$ be a component of* Int K^+ *containing an entire curve $\mathcal{E} : \mathbb{C} \to \mathcal{D}$. Then for every saddle point p, there is a non-empty component of* $\mathrm{Int}_i(W^u(p) \cap K^+)$ *contained in $\mathcal{D}$. If in addition f is unstably disconnected, then there exists such a component that is relatively compact in the intrinsic topology.*

5.4 Persistent connectivity and moving Bedford–Smillie solenoids

We will now show that in the dissipative case, the preservation of connectivity properties of the Julia set implies stability. Our first statement is that persistent Cantor Julia sets are stable.

Proposition 5.6 *Let $(f_\lambda)_{\lambda \in \Lambda}$ be a family of dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$. If J^*_λ is totally disconnected for all λ, then (f_λ) is weakly J^*-stable.*

Proof If J^* is totally disconnected, then for any saddle p, $W^u(p) \cap J^*$ is totally disconnected as well. Together with Lemma 5.4, this implies that f does not have sinks. If this happens persistently over Λ, then no saddle point can bifurcate and the family $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J^*-stable by Proposition 4.14. □

Our next result asserts that persistent connectivity of J also implies stability. Similarly to the analogous statement for polynomials in $\mathbb{C}$, the argument is ultimately based on the absence of "escaping critical points".

Theorem 5.7 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of dissipative polynomial automorphisms of dynamical degree $d \geq 2$. If for every $\lambda \in \Lambda$, the Julia set J_λ is connected then the family (f_λ) is weakly J^*-stable.*

Remark 5.8 Note that this is the only moment in our argument that requires dissipativity.

As a preparation to the proof, let us recall the notion of *Riemann surface lamination*. It is a topological space S endowed with local charts $g_i : U_i \to D_i \times T_i$, where the U_i are open, D_i are domains in $\mathbb{C}$, and T_i are topological spaces (*transversals*) , such that the transit maps $g_i \circ g_j^{-1}$ (wherever they are defined) have form $(z, t) \mapsto (\gamma(z, t), h(t))$, where $\gamma(z, t)$ is conformal in z. Preimages of $D_i \times \{t\}$ in U_i are called *plaques* or *local leaves* ; they patch together to form global *leaves* endowed with a natural conformal structure. So, S is decomposed into Riemann surfaces, which is reflected in its name. We denote $L(z)$ the global leaf through a point $z \in S$.

If all the leaves are dense in S then S is called *minimal*. It is equivalent to saying that for any transversal T and any leaf L, the intersection $L \cap T$ is dense in T.

A minimal Riemann surface lamination S with Cantor transversals is called a *solenoid*. The leaves of a solenoid S can be topologically recognized as *path connected components* of S. It follows that any homeomorphism between solenoids $h : S \to S'$ maps homeomorphically leaves to leaves, $h : L(z) \to L(hz)$. If this leafwise map is conformal then h itself is called a *conformal* solenoidal homeomorphism. More generally, a *conformal solenoidal map* $h : S \to S'$ is a continuous map that induces, for any $z \in S$, a conformal isomorphism between the leaves $L(z)$ and $L(hz)$.

Let us now go back to Theorem 5.7. By Proposition 2.1, we can normalize the family (f_λ) so that f_λ is a product a Hénon mappings depending holomorphically on λ. This puts us in a position to apply the following Structure Theorem due to Bedford and Smillie [3].

Let f be a composition of Hénon maps that is unstably connected. Then the set $S^- = J^- \backslash K^+$ is a solenoid whose leaves are conformally equivalent to the upper half plane $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$. Furthermore, the Böttcher function

φ^+ admits a holomorphic extension to a neighborhood of S^- [3, Thm 6.3], and the map $\varphi^+ : S^- \to \mathbb{C}\backslash\overline{\mathbb{D}}$ is a locally trivial fibration with Cantor fibers $F(c) = F_f(c) := \{\varphi^+ = c\}$. Moreover, the restriction of φ^+ to each leaf of S^- is the universal covering over $\mathbb{C}\backslash\overline{\mathbb{D}}$. We will refer to $S^- = S_f^-$ as the *Bedford–Smillie solenoid of* f.

In particular, given a saddle p, any component L of $W^u(p)\backslash K^+$ provides us with a leaf of the solenoid S^-. It follows that L intersects any fiber F_c, $c \in \mathbb{C}\backslash\overline{\mathbb{D}}$, by a countable dense (in F_c) subset. Note also that by [3, Theorem 4.11], $W^u(p)\backslash K^+$ consists of only finitely many leaves.

The map f restricts to a conformal homeomorphism of S^- that maps fibers to the fibers, $f(F(c)) \subset F(c^d)$ (according to the Böttcher equation).

Theorem 5.7 will follow from the following result of independent interest. In the hyperbolic case, it was already established by Mummert [44].

Proposition 5.9 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a family of unstably connected polynomial automorphisms of dynamical degree $d \geq 2$. Then the Bedford–Smillie solenoid S_λ^- of f_λ moves under an equivariant holomorphic motion that preserves the fibers of the Böttcher function φ_λ^+.*

Proof Fix some $\lambda_0 \in \Lambda$; the objects corresponding to this parameter will be labeled by "0", e.g., $S_0^- \equiv S_{\lambda_0}^-$. It is enough to show that the solenoid S_λ^- moves holomorphically in some neighborhood of λ_0.

Pick a saddle point p_0 for $f_0 \equiv f_{\lambda_0}$, and let p_λ be its holomorphic continuation to some neighborhood of λ_0. The unstable manifold $W^u(p_\lambda)$ is parameterized by the (normalized) linearizing coordinate $\psi_\lambda^u : \mathbb{C} \to W^u(p_\lambda)$, which depends holomorphically on λ.

Let us consider a leaf $L_0 \subset W^u(p_0)$ of S_0^- and a point $z = \psi_0^u(t) \in L_0$. The map $g_0 := \phi_0^+ \circ \psi_0^u$ is univalent in some neighborhood of t, and so is its perturbation

$$g_\lambda := \varphi_\lambda^+ \circ \psi_\lambda^u, \quad |\lambda - \lambda_0| < \delta = \delta(z).$$

Hence the map g_λ^{-1} is well defined in some neighborhood of $c \equiv c(t) := g_0(t)$ and depends holomorphically on λ. Let

$$t_\lambda \equiv t_\lambda(c) := g_\lambda^{-1}(c), \quad z_\lambda \equiv z_\lambda(c) := \psi_\lambda^u(t_\lambda), \quad |\lambda - \lambda_0| < \delta(z).$$

Then for $\delta(z)$ small enough, we have

(i) $z_\lambda \in W^u(p_\lambda) \setminus K^+$, hence z_λ belongs to some leaf $L(z_\lambda) \subset W^u(p_\lambda)$ of S_λ^-;
(ii) $\varphi^+(z_\lambda) = c$, so z_λ belongs to the fiber F_c independently of λ;
(iii) z_λ depends holomorphically on λ.

Pick now a base point $z^* = \psi_0^u(t^*) \in L_0$, and let $c^* := \phi_0^+(z^*)$, z_λ^* be its motion as above, $L_\lambda^* \equiv L(z_\lambda^*)$. Since the maps $\varphi_\lambda^+ : (L_\lambda^*, z_\lambda^*) \to (\mathbb{C} \setminus \bar{\mathbb{D}}, c^*)$ are holomorphic universal coverings, for $|\lambda - \lambda_0| < \delta^* \equiv \delta(z^*)$ there exist conformal isomorphisms

$$h_\lambda : (L_0, z^*) \to (L_\lambda^*, z_\lambda^*) \quad \text{such that } \varphi_\lambda^+ \circ h_\lambda = \varphi_0^+. \tag{5}$$

Let us show that the maps $h_\lambda : L_0 \to \mathbb{C}^2$ form a holomorphic motion. Note that $\delta(z)$ can be selected so that it is lower semi-continuous (since the same $\delta = \delta(z)$ serves as $\delta(\zeta)$ for points ζ near z). Hence it is bounded away from 0 on compact subsets of L_0. Let us take a relative domain $D_0 \Subset L_0$ containing z^*, and let $\delta = \inf_{z \in D_0} \delta(z)$. Then for $|\lambda - \lambda_0| < \delta$, the maps

$$H_\lambda : D_0 \to \mathbb{C}^2, \quad z \mapsto z_\lambda, \ z \in D_0,$$

form a holomorphic motion of D_0. Since $H_\lambda(D_0)$ is contained in the solenoid S_λ, by Lemma 4.7 this motion of D_0 is normal. By Lemma 3.6, it is continuous in z. Since D_0 is connected, $H_\lambda(D_0)$ belongs to some leaf of S_λ, which must be $L(z_\lambda^*) \equiv L_\lambda^*$. Also, by definition,

$$\varphi_\lambda^+ \circ H_\lambda | D_0 = \varphi_0^+ | D_0 \quad \text{and} \quad H_\lambda(z^*) = z_\lambda^*.$$

Comparing this with (5), we conclude that $H_\lambda = h_\lambda | D_0$.

Consequently, there is $\delta_1 = \delta_1(z) > 0$ such that $h_\lambda(z)$ depends holomorphically on λ for $|\lambda - \lambda_0| < \delta_1(z)$. Finally, replacing λ_0 by any other parameter λ in the δ^*-neighborhood of λ_0, we conclude that $h_\lambda(z)$ depends holomorphically on λ for all λ in this neighborhood,

Applying Lemma 4.7 once again, we conclude that the motion $h_\lambda : L_0 \to \mathbb{C}^2$ is normal. By the λ-lemma (Lemma 3.4), h_λ extends to a BHM of $\overline{L}_0 \supset S_0^-$. To see that it gives an actual holomorphic motion of S_0^-, consider the Böttcher foliation $\mathcal{F}_\lambda^+$ of U_λ^+, notice that $\mathrm{cl}_{\mathcal{F}_\lambda}(L_\lambda) \supset S_\lambda^-$, and apply the foliated λ-lemma (Corollary 3.10).

Let us use the same notation $h_\lambda : S_0^- \to S_\lambda^-$ for the extended holomorphic motion. By continuity, it satisfies the covering property $\varphi_\lambda^+ \circ h_\lambda = \varphi_0^+$. Since φ_λ^+ is a leafwise covering over $\mathbb{C} \setminus \bar{\mathbb{D}}$, this property determines h_λ uniquely. Due to the Böttcher equation, this identity is inherited by the motion $\tilde{h}_\lambda := f_\lambda \circ h_\lambda \circ f_0^{-1}$,

$$\varphi_\lambda^+ \circ \tilde{h}_\lambda = (\varphi_\lambda^+ \circ h_\lambda \circ f_0^{-1})^d = (\varphi_0^+ \circ f_0^{-1})^d = \varphi_0^+,$$

implying $\tilde{h}_\lambda = h_\lambda$, which by definition means that the motion h_λ is equivariant. □

Proof of Theorem 5.7 Since for every $\lambda \in \Lambda$, f_λ is dissipative and J_λ is connected, it follows from [3] that f_λ is unstably connected. As was already mentioned in the proof of the above proposition, Lemma 3.4 implies that the equivariant holomorphic motion of the Bedford–Smillie solenoid S_λ extends to an equivariant BHM $\mathcal{G}$ of the closure $\bar{S}_\lambda$. The latter contains all saddle points, which are dense in J^*_λ. Moreover, by equivariance, these saddles remain being periodic under the motion, so they stay in K_λ. Now Proposition 4.14 implies the desired. □

5.5 Bers–Royden motion of unstable manifolds

Let us consider a weakly J^*-stable substantial family $(f_\lambda)_{\lambda\in\Lambda}$ of polynomial automorphisms. Fix a holomorphically moving saddle point p_λ, and consider a holomorphic family of parameterized unstable manifolds $\psi^u_\lambda : \mathbb{C} \to W^u(p_\lambda)$. By Proposition 5.2, the intrinsic boundary $\partial_i(W^u(p_\lambda) \cap K^+)$ moves holomorphically. Then locally in λ, this motion extends to the Bers–Royden holomorphic motion of the whole unstable manifold $W^u(p_\lambda) \approx \mathbb{C}$, see Lemma 3.9. Being canonical, it is automatically equivariant (where the dynamics is just multiplication by the unstable multiplier), so in this way we obtain an equivariant holomorphic motion of $W^u(p_\lambda)$ in $\mathbb{C}^2$.

Lemma 5.10 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a weakly J^*-stable substantial family of polynomial automorphisms, and let (p_λ) be a holomorphically moving saddle point as above. Then the Bers–Royden holomorphic motion of $W^u(p_\lambda)$ preserves the decomposition $\mathbb{C}^2 = K^+ \sqcup U^+$.*

Proof For $\lambda_0 \in \Lambda$, let $C_0 \equiv C_{\lambda_0}$ be an intrinsic connected component of $W^u(p_0) \cap K^+_0$, and let C_λ be its image under the Bers–Royden motion h_λ. We have to show that for every $\lambda \in \Lambda$, C_λ is an intrinsic connected component of $W^u(p_\lambda) \cap K^+_\lambda$.

By the one-dimensional λ-lemma, the maps h_λ are intrinsic homeomorphisms, so they preserve intrinsic topological properties of the subsets of $W^u(p_\lambda)$), e.g., $\partial_i C_\lambda = h_\lambda(\partial_i(C_0))$, C_λ is intrinsically bounded iff C_0 is, etc.

From Lemma 5.1 we know that $\partial_i C_0 \subset J^*_0$, hence for every λ, $\partial_i C_\lambda \subset J^*_\lambda \subset K^+_\lambda$. So, we need to show that for every $\lambda \in \Lambda$, $\mathrm{Int}_i(C_\lambda) \subset K^+_\lambda$.

Let Ω_0 be a connected component of $\mathrm{Int}_i\, C_0$, and let $\Omega_\lambda = h_\lambda(\Omega_0)$. If Ω_0 is intrinsically bounded in $W^u(p_0)$ then the Maximum Principle applied to the non-negative subharmonic function $G^+_\lambda | W^u(p_\lambda)$ implies that $\Omega_\lambda \subset K^+_\lambda$.

Assume Ω_0 is intrinsically unbounded. Then the Maximum Principle implies that Ω_0 is simply connected, so the same holds for Ω_λ.

Given a parameter λ_1, we will label the corresponding objects with "1", e.g., $f_1 \equiv f_{\lambda_1}$, $U_1 \equiv U_{\lambda_1}$. Assume by contradiction that $\Omega_1 \cap U^+_1 \neq \emptyset$ for some λ_1. We first claim that $\Omega_1 \subset U^+_1$. Otherwise Ω_1 would intersect

$\partial_i(W^u(p_1) \cap K_1^+) = \mathrm{cl}_i(H(p_1))$. Since the holomorphic motion preserves $H(p)$, Ω_0 would intersect $\mathrm{cl}_i(H(p_0))$ contradicting the fact that Ω_0 is contained in K^+.

It follows that Ω_1 is a simply connected intrinsic component of $W^u(p_1) \cap U_1^+$. The existence of such a component implies that f_1 is unstably connected [3, Theorem 0.1], so J_1 is connected. Since unstable connectivity is preserved in weakly J^*-stable families (see Proposition 5.3), f_0 is unstably connected, too.

On the other hand, for families of unstably connected polynomial automorphisms we have shown in Proposition 5.9 that every intrinsic component of $W^u(p_\lambda)\backslash K_\lambda^+$ can be followed by some holomorphic motion $\tilde{h}_\lambda$ coinciding with h_λ on $\partial_i(W^u(p_0) \cap K_0^+)$. But then the action of $\tilde{h}_\lambda$ on the space of connected components of $W^u(p_0) \setminus K_0^+$ must agree with that of h_λ, which is impossible for the component $\tilde{h}_1^{-1}(\Omega_1)$. This contradiction completes the proof. □

5.6 Proof of Theorem 4.2

Let us show that (i) $\Leftrightarrow$ (ii). First, it follows from Proposition 4.14 that (i) is equivalent to the statement

(ii′) *Saddle points stay of saddle type throughout the family.*

Obviously, (ii) implies (ii′). The reverse is also obvious in the dissipative case, since every bifurcation of a sink gives rise to a saddle. In general, we have to rule out the possibility that in a weakly J^*-stable substantial family, a periodic point q bifurcates from attracting to repelling through indifferent without ever turning into a saddle.

Assume by contradiction that such a scenario happens. Fix a parameter domain Λ' over which q can be followed holomorphically, and its eigenvalues cross the unit circle. So, inside Λ' there is a region Λ^- where $q(\lambda)$ is a sink and a region Λ^+ where $q(\lambda)$ is a source. Fix a (necessarily persistent) saddle point $p(\lambda)$. For $\lambda \in \Lambda^+$, since $|\mathrm{Jac}\, f_\lambda| > 1$, f_λ is unstably disconnected by [3, Cor. 7.4]. By the weak J^*-stability, the same is true for every $\lambda \in \Lambda$ (see Proposition 5.3). Then Lemma 5.4 implies that for $\lambda \in \Lambda^-$, there is a non-trivial bounded component Ω_λ of $W^u(p(\lambda)) \cap \mathcal{B}(q(\lambda))$.

By Lemma 5.10, we infer that under the Bers–Royden motion of $W^u(p(\lambda))$, Ω_λ persists throughout Λ as a bounded component of $W^u(p(\lambda)) \cap K_\lambda^+$. Let us consider the Bers–Royden orbit $z(\lambda)$ of some point of Ω_λ. Then the family of maps $\Lambda \to \mathbb{C}^2$, $\lambda \mapsto f_\lambda^n(z(\lambda))$, $n = 0, 1, \ldots$, is locally bounded and hence normal over Λ. Hence by analytic continuation the convergence $f_\lambda^n(z(\lambda)) \to q(\lambda)$ persists throughout Λ. But if $\lambda \in \Lambda^+$, $q(\lambda)$ is repelling, so we arrive at a contradiction, which finishes the proof of (ii) $\Leftrightarrow$ (ii′).

Condition (i) implies (iii) by Lemma 3.5. Conversely, (iii) implies (ii). Indeed if a periodic point changes type, then arguing as in Lemma 4.11, we see that for some λ, a multiplier of the cycle must cross the unit circle at a linearizable parameter. So at this parameter a Siegel ball or Siegel/attracting basin is created, and the corresponding periodic orbit jumps outside J^*, thus preventing continuity of J^*.

To conclude the proof we show the (rather obvious) chain of implications:

$$\text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(i)} + \text{ the number of non-saddle cycles is finite } \Rightarrow \text{(iv)}.$$

Indeed (iv) $\Rightarrow$ (v) is clear. Next, if (v) holds, then all periodic points of sufficiently high prime period are (necessarily persistent) saddles, so by Corollary 4.15, the family is weakly J^*-stable. Therefore, all periodic points are of constant type, hence (iv) holds.

The Theorem is proved. □

5.7 Motion of the big Julia set $\widehat{J} = J^+ \cup J^-$

Let us start with a simple observation:

Lemma 5.11 *Any equivariant normal BHM $\mathcal{G}$ preserves the sets*[7] $K^\pm$ *and hence preserves* K.

Proof For definiteness, let us treat the case of K^+. Let $\gamma = (\lambda, z(\lambda))$ be a graph of $\mathcal{G}$ such that $z(\lambda_0) \in K_0^+$ for some $\lambda_0 \in \Lambda$. Then the forward orbit $(f_\lambda^n(z(\lambda_0)))_{n\geq 0}$ is bounded. By the equivariance, all the graphs $\widehat{f}^n(\gamma) = (\lambda, (f_\lambda^n(z(\lambda))), n \geq 0$, belong to $\mathcal{G}$ as well. By normality of $\mathcal{G}$, the $\widehat{f}^n(\gamma)$ form a normal family. Consequently, this family is locally uniformly bounded on Λ, implying that $z(\lambda) \in K_\lambda^+$ for all $\lambda \in \Lambda$. □

Recall that a family (f_λ) is said to be weakly X-stable if the sets X_λ move under an equivariant BHM . We now prove the equivalence of several notions of weak stability.

Theorem 5.12 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a substantial family of polynomial automorphisms of dynamical degree $d \geq 2$. The following properties are equivalent:*

(i) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^*-stable.*
(ii) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^--stable.*
(iii) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly J^+-stable.*
(iv) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly K-stable.*
(v) $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly S^--stable (resp. S^+-stable).*

[7] Meaning that a point that begins in $K_0^\pm$ stays in $K_\lambda^\pm$ under the motion (it is not assumed that the motion is defined on the whole set $K^\pm$)

If $(f_\lambda)_{\lambda\in\Lambda}$ *is weakly* J*-stable, then the properties* (i)–(v) *hold.*

In items (i), (iv) *and* (v)*, the motions in question are automatically normal, while in items* (ii) *and* (iii) *they can be selected to be so. Moreover, in the latter items the motions preserve respectively the unstable and stable manifolds of all saddles.*

From now on a family satisfying the equivalent conditions (i)–(v) of this theorem will be simply referred to as *weakly stable.*

Remark 5.13 We do not know if weak stability implies weak J-stability since we cannot rule out a scenario where under a BHM of $K = K^+ \cap J^-$, a point in J moves out to $(\mathrm{Int}\, K^+) \cap J^-$.

Proof We start by proving that (i) $\Longrightarrow$ (ii) (of course (i) $\Longrightarrow$ (iii) for the same reason). So, assume $(f_\lambda)_{\lambda\in\Lambda}$ is weakly J^*-stable. Take a holomorphically moving saddle $p(\lambda)$, and consider the Bers–Royden equivariant holomorphic motion of its unstable manifold $W^u(p(\lambda))$. By Lemma 5.10, it respects the decomposition $K^+_\lambda \sqcup U^+_\lambda$ inside $W^u(p(\lambda))$. The motion of $K^+_\lambda \cap W^u(p(\lambda))$ is obviously normal, while the motion of $W^u(p(\lambda))\backslash K^+_\lambda$ is normal by Lemma 4.7. Hence the motion of the whole unstable manifold $W^u(p(\lambda))$ is normal as well. By the λ-lemma, it extends to an equivariant normal BHM of $\overline{W^u}(p(\lambda)) = J^-_\lambda$, as desired.

By Lemma 5.11, this motion preserves the decomposition $J^- = K \sqcup S^-$ (and similarly, for J^+), so (i) implies (iv) and (v) as well.

Let us show that (iii) $\Longrightarrow$ (i). Consider an arbitrary saddle $p(\lambda_0)$ and the graph $\gamma_0 = (\lambda, p(\lambda))$ of its motion. By Proposition 2.1, we may normalize our family so that each f_λ is a product of Hénon mappings. Then on a given compact subset of Λ, for sufficiently large R, the set $J^+ \cap \mathbb{D}^2_R$ is forward invariant. Hence the family of graphs $(f^n(\gamma_0))_{n\geq 0}$ is normal.[8] Let γ be a cluster graph for this family. Then $\gamma(\lambda) \in K_\lambda$ for any $\lambda \in \Lambda$, while $\gamma(\lambda_0) = p(\lambda_0)$ is an arbitrary saddle. Proposition 4.14 implies J^*-stability once again. Moreover, this argument shows that the motions in question preserve the stable manifolds of all saddles.

Of course, (ii) $\Longrightarrow$ (i) for the same reason. Furthermore, it follows directly from Proposition 4.14 that (iv) $\Longrightarrow$ (i). Likewise, weak J-stability implies (i) as well.

Let us show that (v) $\Longrightarrow$ (iv). Assume $(f_\lambda)_{\lambda\in\Lambda}$ is weakly S^--stable. By Lemma 4.7, the corresponding BHM is normal, so it extends to a normal BHM $\mathcal{G}$ of $\overline{S^-}$. By Lemma 5.1, $J^* \subset \overline{S^-}$. By Lemma 5.11, the graphs of $\mathcal{G}$ that begin in J^* for some $\lambda_0 \in \Lambda$ remain in K for all $\lambda \in \Lambda$. Applying Proposition 4.14 once again, we obtain the desired.

[8] Recall that we do not assume any normality in the definition of the weak J^+-stability.

The last assertion on normality and preservation of s/u manifolds has been established along the lines of the proof. □

Corollary 5.14 *If $(f_\lambda)_{\lambda\in\Lambda}$ is weakly stable, there exists an equivariant normal BHM of $\widehat{J}$ that preserves the stable and unstable manifolds of saddle periodic points. In particular $\widehat{J}$ moves continuously in the Hausdorff topology.*

Proof The former assertion follows directly from the theorem. The latter follows from Lemma 3.5. □

Remark 5.15 It is not difficult to show that (f_λ) is weakly stable if and only if $\lambda \mapsto J^+_\lambda$ is continuous for the Hausdorff topology (for J^* this was done in Theorem 4.2). On the other hand this is false for J^-. Indeed in the dissipative setting $K^- = J^-$, and it is classical that K^- moves upper semi-continuously while J^- moves lower semi-continuously (see e.g. [2, Prop. 4.7]). In particular $\lambda \mapsto J^-_\lambda$ is always continuous. Compare [17] in the one dimensional case.

According to [39], non-wandering components of Int (K^+) of a moderately dissipative polynomial automorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ can be classified as attracting, parabolic, or rotation basins. (For components of $\text{Int}_i\,(W^u(p) \cap K^+)$, there is one more theoretical option: they can be contained in the small Julia set J.) We cannot rule out that some of these components change type under a branched holomorphic motion of the Julia set (compare Remark 5.13). However, arguing as in Sect. 5.6, we obtain:

Proposition 5.16 *Under a branched holomorphic motion over a weakly stable domain, if for some parameter λ_0, z_0 is a point in $J^-_0 \cap K^+_0$ belonging to the basin of attraction of a sink q_0 then for every λ, $z(\lambda)$ stays in the basin of $q(\lambda)$.*

Part 2 Semi-parabolic Implosion and Homoclinic Tangencies

6 Semi-parabolic Dynamics

In this paragraph we collect some basic facts about semi-parabolic dynamics: basins, petals, etc., following the work of Ueda [54,55], Hakim [28] (see also [2]). Let f be a polynomial automorphism of $\mathbb{C}^2$. A periodic point p is *semi-parabolic* if its multipliers are 1 (or more generally a root of unity) and b with $|b| < 1$. Notice that this forces f to be dissipative. Replacing f by f^q for some $q \geq 1$ we can assume that p is fixed. Then, if we denote by $k + 1$ the multiplicity at 0 of $f - \text{id}$ (which is finite because f has no curve of fixed points), there exist local coordinates (x, y) in the neighborhood of p such that $p = (0, 0)$ and f is of the form

$$(x, y) \mapsto \left(x + x^{k+1} + Cx^{2k+1} + x^{2k+2}g(x, y), by + xh(x, y)\right), \quad (6)$$

where g and h are holomorphic near the origin and C is a complex number (see [28, Prop. 2.3]). Notice that in these coordinates, $\{x = 0\} = W^{ss}_{\mathrm{loc}}(0)$ is the local (strong) stable manifold of 0, and $f|_{\{x=0\}}$ is linear. For $r > 0$, the flower-shaped open set $\{x, \ |x^k + r^k| < r^k\}$ admits k connected components, which will be denoted by $P^\iota_{r,j}$, $0 \leq j \leq k-1$. Then for small $\eta > 0$, the domains $\mathcal{B}_{r,j,\eta} := P^\iota_{r,j} \times \mathbb{D}_\eta$ are attracted to the the origin under iteration. Finally, let $\mathcal{B}_j = \bigcup_{n\geq 0} f^{-n}(\mathcal{B}_{r,j,\eta})$. The open sets $\mathcal{B}_j$ are biholomorphic to $\mathbb{C}^2$ and are the components of the basin of attraction of p in $\mathbb{C}^2$.

To be more specific, in $\mathcal{B}_{r,j,\eta}$ we change coordinates by letting $(z, w) = ((kx^k)^{-1}, y)$, so that in the new coordinates, f assumes a form

$$(z, w) \longmapsto \left(z - 1 + \frac{c}{z} + O\left(\frac{1}{|z|^{1+1/k}}\right), bw + O\left(\frac{1}{|z|^{1/k}}\right)\right),$$

where c is a complex number depending on C. Notice that in the new coordinates, $\mathcal{B}_{r,j,\eta}$ corresponds to a region of the form $\{\mathrm{Re}(z) < -M\} \times \mathbb{D}_\eta$, with $M = (2kr^k)^{-1}$. Therefore if we set

$$w^\iota(x, y) = w^\iota(x) = \frac{1}{kx^k} + c \log \frac{1}{kx^k}$$

we infer that the limit

$$\varphi^\iota(x, y) = \lim_{n\to\infty} (w^\iota(f^n(x, y)) + n)$$

exists and satisfies the functional equation $\varphi^\iota \circ f = \varphi^\iota - 1$ (beware that this normalization differs from the references mentioned above). In addition, $\varphi^\iota - w^\iota$ is a bounded holomorphic function in $\mathcal{B}_{r,j,\eta}$. In the paper, the letter ι will stand for "incoming" and o for "outgoing", following a convenient notation from [2].

It easily follows that in the original coordinates, if $(x, y) \in \mathcal{B}$ then $f^n(x, y) = (x_n, y_n)$ with $x_n \sim (kn)^{-1/k}$ and $y_n = O(n^{-1/k})$, see [28, Prop. 3.1] (beware that y_n needn't be exponentially small).

Fix a component $\mathcal{B} = \mathcal{B}_j$ of the basin of attraction. By the iteration we can extend φ^ι to $\mathcal{B}$. It turns out that $\varphi^\iota : \mathcal{B} \to \mathbb{C}$ is a fibration [28, Thm 1.3], and that there exists a function $\phi_2 : \mathcal{B} \to \mathbb{C}$ such that $\Phi = (\varphi^\iota, \phi_2) : \mathcal{B} \to \mathbb{C}^2$ is a biholomorphism.

The following result is similar to [2, Thm 1.2], and its proof will be left to the reader.

Proposition 6.1 *If p_1 and p_2 are points in $\mathcal{B}$ such that $\varphi^\iota(p_1) = \varphi^\iota(p_2)$ then*

$$\lim_{n\to+\infty} \frac{1}{n} \log \mathrm{dist}(f^n(p_1), f^n(p_2)) = \log |b| < 0.$$

On the other hand, if $\varphi^\iota(p_1) \neq \varphi^\iota(p_2)$ *then* $\mathrm{dist}(f^n(p_1), f^n(p_2))$ *decreases like* $n^{-(1+1/k)}$.

From now on we will refer to the foliation $\{\varphi^\iota = C^{st}\}$ as the *strong stable foliation* in $\mathcal{B}$, and it will be denoted by $\mathcal{F}^{ss}$. Its structure near the origin is easy to describe. Indeed since $\varphi^\iota - w^\iota$ is bounded near the origin, it follows from Rouché's theorem that the leaf of $\mathcal{F}^{ss}$ through $(x_0, 0)$ in $\mathcal{B}_{r,j,\eta}$ is graph over the vertical direction, whose distance to the line $\{x = x_0\}$ tends to 0 as $x_0 \to 0$. In particular $\mathcal{F}^{ss}$ extends continuously to $\mathcal{B}_{r,j,\eta} \cup \{x = 0\} = \mathcal{B}_{r,j,\eta} \cup W^{ss}_{\mathrm{loc}}(0)$ by adding $W^{ss}_{\mathrm{loc}}(0)$ as a leaf.

On the "outgoing" side, there is also a notion of "repelling petal", which is defined as follows. With coordinates as in (6), denote by $P^o_{r,j}$, $0 \leq j \leq k-1$, the connected components of $\{x, \ \left|x^k - r^k\right| < r^k\}$. Then for small $r, \eta > 0$ the set $\Sigma_{j,\mathrm{loc}}$ defined by

$$\Sigma_{j,\mathrm{loc}} = \{q \in P^o_{r,j} \times \mathbb{D}_\eta : \forall n \geq 0,\ f^{-n}(q) \in P^o_{r,j} \times \mathbb{D}_\eta \text{ and } f^{-n}(q) \underset{n\to\infty}{\longrightarrow} 0\}.$$

Then $\Sigma_{j,\mathrm{loc}}$ is a graph $\{y = \psi(x)\}$ over the first coordinate, which extends continuously to the origin by putting $\psi(0) = 0$ (this extension cannot be made holomorphic, see [2, Prop. 1.3]). This is stated in [55, Thm 11.1] only for $k = 1$, however the proof relies on a more general result [55, Lemma 11.2] which allows to treat the general case as well. As usual we extend the petal globally to $\mathbb{C}^2$ by letting

$$\Sigma_j = \bigcup_{n\geq 0} f^n\left(\Sigma_{j,\mathrm{loc}}\right).$$

Then Σ_j is biholomorphic to $\mathbb{C}$ (this is stated only for $k = 1$ in [55, Thm 11.6] but the adaptation to the general case is straightforward). The union $\bigcup_j \Sigma_j$ is the set of points converging to the semi-parabolic point p under backward iteration, referred to as the *asymptotic curve* in [2,55]. We will also call it the *repelling petal* (or simply *unstable manifold*) of p.

7 Critical Points in Basins

In this section we prove the existence of critical points in semi-parabolic basins for moderately dissipative maps. Let f be a polynomial automorphism with a semi-parabolic basin $\mathcal{B}$. Recall that by a critical point, we mean a point of tangency between the strong stable foliation in $\mathcal{B}$ and the unstable manifold of some saddle periodic point. The argument is based on a refined version of some classical properties of entire functions of finite order: see Sect. 7.1. The proof of Theorem B comes in Sect. 7.2. These results will be generalized to attracting basins in Appendix A.

7.1 Entire functions of finite order

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. The *order* of f is defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, f)}{\log r}, \quad \text{where } M(r, f) = \max\{|f(z)|,\ |z| = r\}.$$

The class of entire functions of finite order is well-known to display a number of remarkable properties, some of which we recall now. We say that $a \in \mathbb{C}$ is an *asymptotic value* of f if there exists a continuous path $\gamma : [0, \infty) \to \mathbb{C}$ tending to infinity such that $f(\gamma(t)) \to a$ as $t \to \infty$. The famous Denjoy–Carleman–Ahlfors Theorem asserts that if f is an entire function of order $\rho < \infty$, then it admits at most 2ρ distinct asymptotic values (see e.g. [25, Chap. 5] or [35]). Another essentially equivalent formulation is that for every $R > 0$, the open set $\{z, |f(z)| > R\}$ admits at most $\max(2\rho, 1)$ connected components.

When $\rho < \frac{1}{2}$, we see that f has no asymptotic values. One can actually be more precise in this case. Indeed, Wiman's theorem [25, Chap. 5, Thm 1.3] asserts that there exists a sequence of circles $\{|z| = r_n\}$ with radii $r_n \to \infty$ such that $\min\{|f(z)|,\ |z| = r_n\} \to \infty$.

To prove the existence of critical points in semi-parabolic basins we will require a slight generalization of the Denjoy–Carleman–Ahlfors theorem on asymptotic values. We say that a is an *ε-approximate asymptotic value* of f if there exists a continuous path $\gamma : [0, \infty) \to \mathbb{C}$ tending to infinity such that $\limsup_{t\to\infty} |f(\gamma(t)) - a| < \varepsilon$. The statement is as follows:

Theorem 7.1 *Let f be an entire function of finite order. Assume that f admits n distinct ε-approximate asymptotic values $(a_i)_{i=1,\dots,n}$, with $\varepsilon < \min_{i\neq j} \frac{|a_i - a_j|}{5}$.*

Then the order of f is at least $n/2$.

In order to prove the theorem let us first recall the classical Phragmen–Lindelöf Principle.

Proposition 7.2 *Let D be an unbounded domain in $\mathbb{C}$. Let f be a bounded holomorphic function on D, such that $\limsup_{\partial D \ni z\to\infty} |f| \leq \delta$. Then $\limsup_{D\ni z\to\infty} |f| \leq \delta$.*

The following result is a version of a classical Lindelöf Theorem (the argument below is adapted from [35, Thm 12.2.2]).

Theorem 7.3 *Let D be a simply connected unbounded domain in $\mathbb{C}$, whose boundary consists of two simple curves γ_1, γ_2 both tending to infinity, and disjoint apart from their common starting point. Let f be holomorphic on D and continuous on ∂D, and assume that when z goes to infinity along γ_i, f has the property that* $\limsup_{t\to\infty} |f(\gamma_i(t)) - a_i| < \varepsilon$, *with $\varepsilon < \frac{|a_1-a_2|}{5}$. Then f is unbounded on D.*

Proof Assume by contradiction that f is bounded, and let $g(z) = (f(z) - a_1)(f(z) - a_2)$. Then g is bounded on D, and $\limsup |g(z)| \leq \delta$, as $z \to \infty$ along ∂D, for some $\delta < \frac{6}{5} |a_1 - a_2| \varepsilon$. It follows that $\limsup_{D\ni z\to\infty} |g| \leq \delta$.

Now for every $R > 0$ there exists a curve Γ in D joining γ_1 and γ_2 and staying at distance at least R from the origin. If R is large enough, we then have that $|g| < \frac{6}{5} |a_1 - a_2| \varepsilon$ along Γ. Furthermore at $\Gamma \cap \gamma_1$ (resp. $\Gamma \cap \gamma_2$), f is ε-close to a_1 (resp. a_2). So there exists $z_0 \in \Gamma$ such that $|f(z_0) - a_1| = |f(z_0) - a_2| \geq \frac{|a_1-a_2|}{2}$. We infer that

$$|g(z_0)| \geq \frac{|a_1 - a_2|^2}{4} \geq \frac{5}{4} |a_1 - a_2| \varepsilon,$$

a contradiction. □

Proof of Theorem 7.1 (compare [35, Cor. 14.2.3]) By assumption there are n curves γ_i going to infinity along which f ε-approximately converges to a_i. We may assume that all these curves are simple, start from 0, and intersect only at 0. We reassign the indices so that the curves are arranged in clockwise order. By the previous theorem f must be unbounded in the domain enclosed between γ_i and γ_{i+1} (here we put $\gamma_{n+1} = \gamma_1$). Therefore the order of f is at least $n/2$ by the ordinary Denjoy–Carleman–Ahlfors Theorem. □

Remark 7.4 Alex Eremenko has pointed out to us the following version of the Denjoy–Carleman–Ahlfors Theorem. Let f be an entire function bounded in the left-half plane and outside horizontal strips. Let

$$M(x) = \max_{\operatorname{Re} z=x} |f(z)| \text{ and } \rho = \rho(f) = \limsup_{x\to+\infty} \frac{\log M(x)}{x}.$$

Then f admits at most 2ρ asymptotic values. [This follows from the subharmonic version of the DCA Theorem applied to the function $u(z) = \log^+(f(\log z)/M)$, where M is the supremum of $|f|$ outside the half-strip $\Pi = \{\operatorname{Re} f > 1,\ |\mathrm{Imf}| < \pi\}$, and $\log z$ is the principal value of the logarithm in $\mathbb{C}\backslash\mathbb{R}_-$.]

This Theorem admits an ε-approximate version similar to Theorem 7.1.

7.2 Semi-parabolic basins

In this paragraph we prove Theorem B. We first recall that stable and unstable manifolds of saddle points, as well as strong stable manifolds of semi-parabolic points are entire curves, whose parameterizations are defined dynamically. More precisely, if p is a fixed point with an expanding eigenvalue κ^u, then the associated stable manifold is parameterized by an entire function $\psi^u : \mathbb{C} \to \mathbb{C}^2$ satisfying $\psi^u(0) = p$ and $f \circ \psi^u(t) = \psi^u(\kappa^u t)$ for every $t \in \mathbb{C}$, and similarly for a contracting eigenvalue. Let us now make an easy but important observation (see [20] in the one-dimensional setting, [3,30] in our setting, and [14] for automorphisms of compact projective surfaces).

Lemma 7.5 *Let q be a fixed point of a polynomial automorphism of dynamical degree $d \geq 2$ with an expanding eigenvalue κ^u, and $\psi^u : \mathbb{C} \to \mathbb{C}^2$ is a parameterization of the associated unstable manifold as above. Then the coordinates of ψ^u are entire functions of finite order*

$$\rho = \frac{\log d}{\log |\kappa^u|}.$$

Notice that any other parameterization has the same order, since two parameterizations differ from an affine map of $\mathbb{C}$. So we may speak of the order of the unstable manifold $W^u(q)$.

In the semi-parabolic case the result specializes as follows:

Corollary 7.6 *If p is a semi-parabolic periodic point for a polynomial automorphism of dynamical degree $d \geq 2$, then the order of $W^{ss}(p)$ is*

$$\frac{\log d}{\log |\mathrm{Jac}\ f|^{-1}}.$$

Proof Apply Lemma 7.5 to f^{-k}, where k is the period of p. □

Our use of this corollary will be the following.

Corollary 7.7 *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, with a semi-parabolic periodic point p. Assume that the Jacobian of f satisfies $|\mathrm{Jac}\ f| < \frac{1}{d^2}$. Then the connected component of p in $W^{ss}(p) \cap J^-$ is $\{p\}$.*

Proof The assumption on the Jacobian together with the previous corollary imply that the order of $W^{ss}(p)$ is smaller than $\frac{1}{2}$. Then by Wiman's theorem there exists a sequence of circles $\{|t| = r_n\}$ such that the second coordinate (say) of ψ^s satisfies $\min_{\{|t|=r_n\}} |\psi^s(t)| \to \infty$. Since K is bounded, these circles must be eventually disjoint from K. So we infer that the connected component of p in $(\psi^s)^{-1}(J^-)$ is bounded in $\mathbb{C}$. Since in addition the component is invariant under multiplication by κ^s, we are done. □

Theorem B now clearly follows from the previous corollary, together with the following result.

Proposition 7.8 *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing a semi-parabolic periodic point p with basin of attraction $\mathcal{B}$. Assume that the connected component of p in $W^{ss}(p) \cap J^-$ is reduced to $\{p\}$. Then for every saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point.*

Notice that in the above situation, Proposition 5.5 guarantees that $W^u(q) \cap \mathcal{B}$ is never empty.

Remark 7.9 Proposition 7.8 remains true in the case where q is a semi-parabolic point rather than a saddle (in particular, when $q = p$). The proof is the same except it makes use of the version of the Denjoy–Carleman–Ahlfors Theorem stated in Remark 7.4

Proof Let q be as in the statement of the proposition, and ψ^u as above be a parameterization of $W^u(q)$. Translating the coordinates and iterating if needed we may assume that the semi-parabolic point is fixed and equal to $0 \in \mathbb{C}^2$. Also we may assume that q is fixed. Let $\Omega \subset \mathbb{C}$ be a component of $(\psi^u)^{-1}(\mathcal{B})$—which must be non-empty by Proposition 5.5. By the maximum principle, Ω is biholomorphic to a disk. Recall that the incoming Fatou function φ^ι was defined in Sect. 6. Observe first that $\varphi^\iota \circ \psi^u : \Omega \to \mathbb{C}$ cannot be constant for otherwise $W^u(q)$ would coincide with a strong stable leaf, which cannot happen since it would then be contained in the compact set K. We argue by contradiction, so assume that Ω contains no critical point. Then $\varphi^\iota \circ \psi^u : \Omega \to \mathbb{C}$ is a locally univalent map. Since $\varphi^\iota \circ \psi^u$ cannot be a covering, it must possess asymptotic values, that is, there exists a path $\gamma : [0, \infty) \to \Omega$, tending to infinity in Ω (that is, converging to $\partial\Omega$ or to infinity in $\mathbb{C}$) such that $\varphi^\iota \circ \psi^u(\gamma(t))$ has a well defined limit in $\mathbb{C}$ as $t \to \infty$ (see [45, p. 284] or [26, Lemma 1.2] for a modern presentation). Notice that by definition, $\psi^u(\gamma)$, as well as all its iterates and cluster values, are contained in $K^+ \cap J^-$.

Recall that there exists a function $\phi_2 : \mathcal{B} \to \mathbb{C}$ such that $\Phi := (\varphi^\iota, \phi_2) : \mathcal{B} \to \mathbb{C}^2$ is a biholomorphism. At this point the proof splits into two cases.

Case 1: $\phi_2 \circ \psi^u(\gamma)$ is unbounded.

Observe that this case must occur when $\Phi \circ \psi^u : \Omega \to \mathbb{C}^2$ is proper, which happens for instance when Ω is relatively compact in $\mathbb{C}$. Consider a domain $\mathcal{B}_{r,j,\eta}$ as in Sect. 6, corresponding to the basin $\mathcal{B}$, and in which the strong stable foliation is made of vertical graphs, clustering at $W^{ss}_{\mathrm{loc}}(0) = \{x = 0\}$. For sufficiently large n, the connected component of $f^n(\psi^u(\gamma)) \cap \mathcal{B}_{r,j,\eta}$ containing $f^n(\psi^u(\gamma(0)))$ is a path which by our unboundedness assumption and the fact that φ^ι admits a finite limit along $\psi^u \circ \gamma$, goes up to the horizontal boundary $P^\iota_{r,j} \times \partial\mathbb{D}_\eta$ of $\mathcal{B}_{r,j,\eta}$. When n is large, this connected component is contained in a small neighborhood of $W^{ss}_{\mathrm{loc}}(0)$. In addition $f^n(\psi^u(\gamma(0)))$ converges to 0 when $n \to \infty$. Thus, taking a cluster value of this sequence of paths for the Hausdorff topology, we obtain a closed connected subset of $W^{ss}_{\mathrm{loc}}(0) \cap J^-$, containing 0, and touching the boundary, hence not reduced to a point. This contradicts Corollary 7.7, and finishes the proof in this case.

Case 2: $\phi_2 \circ \psi^u(\gamma)$ is bounded.

A first observation is that under this assumption, the path γ must go to infinity in $\mathbb{C}$. Indeed otherwise let (t_n) be a sequence such that $\gamma(t_n)$ converges to $\zeta \in \partial\Omega \subset \mathbb{C}$. Then $\psi^u(\gamma(t_n))$ converges to $\psi^u(\zeta) \notin \mathcal{B}$, contradicting the fact that $\Phi(\psi^u(\gamma(t_n)))$ stays bounded in $\mathbb{C}^2$. In particular γ is an asymptotic path for the entire mapping ψ^u.

For the sake of explanation, assume first that $\phi_2 \circ \psi^u(\gamma(t))$ admits a limit as $t \to \infty$. Thus $\Phi \circ \psi^u(\gamma(t))$ converges in $\mathbb{C}^2$, and $\psi^u(\gamma(t))$ converges to some limiting point $\omega \in \mathcal{B}$, which must be an asymptotic value of ψ^u (that is, both coordinates are asymptotic values of the coordinate functions of ψ^u). By the invariance of $W^u(p)$, all iterates $f^n(\omega)$, $n \in \mathbb{Z}$, are asymptotic values of ψ^u. Since ψ^u has finite order, this contradicts the Denjoy–Carleman–Ahlfors Theorem.

In the general case we use Theorem 7.1 instead. Let K be the cluster set of $\psi^u(\gamma)$, which is a compact subset of $\mathcal{B}$, contained in a leaf $\{\varphi^\iota = C^{st}\}$ of the strong stable foliation. Let us study the shape of $f^n(K)$. When n is large enough, $f^n(K)$ is contained in a small neighborhood of the origin, inside a local strong stable leaf. Perform a *linear* change of coordinates so that in the new coordinates, f expresses as $f(x, y) = (x, by) + h.o.t.$ These coordinates are tangent (at 0), but not equal, to the adapted local coordinates of Sect. 6. In a bidisk near 0, the strong stable foliation is made of vertical graphs and $W^{ss}_{\mathrm{loc}}(0)$ is a vertical graph tangent to the y-axis. Let $(x_0, y_0) \in K$ and $(x_n, y_n) = f^n(x_0, y_0)$. Then we infer that $x_n \sim (kn)^{-1/k}$ and $f^n(K)$ is a subset of the leaf $\mathcal{F}^{ss}(x_n, y_n)$ of size exponentially small with n. Denoting by $\mathrm{pr}_1 : (x, y) \mapsto y$ the first projection, we deduce that $\mathrm{pr}_1(f^n(K))$ is a set of exponentially small diameter about x_n. It follows that for every integer k there exists $\varepsilon > 0$ and integers $n_1, \ldots, n_k$ such that the sets $\mathrm{pr}_1(f^{n_j}(K))$, $1 \leq j \leq k$ are of diameter smaller than ε, and 5ε-apart from each other. Notice that the sets $\mathrm{pr}_1(f^{n_j}(K))$ are ε-approximate asymptotic values of $\mathrm{pr}_1 \circ \psi^u$ in the sense of Theorem 7.1. Now since pr_1 is linear, $\mathrm{pr}_1 \circ \psi^u$ is an entire function of finite order. Thus we obtain a contradiction with Theorem 7.1, and the proof is complete. □

8 Semi-parabolic Bifurcations and Transit Mappings

In this section we develop an analogue of the "tour de valse" of Douady and Sentenac [18] in the context of semi-parabolic implosion. When the family can be put in the form

$$f_\varepsilon(x, y) = (x + (x^2 + \varepsilon^2)\alpha_\varepsilon(x, y), b_\varepsilon(x)y + (x^2 + \varepsilon^2)\beta_\varepsilon(x, y)),$$

(this corresponds to the case $k = 1$ in (7) below), we may directly appeal to the results of Bedford, Smillie and Ueda [2]. In our setting, however, we have to deal with periodic points and bifurcations of a more general nature, and it is unclear how to extend the results of [2].

Consider a family $(f_\lambda)_{\lambda\in\Lambda}$ of dissipative polynomial automorphisms of $\mathbb{C}^2$, with a periodic point changing type, that is, one multiplier crosses the unit circle. It is no loss of generality to assume that Λ is the unit disk.

Let us start with a few standard reductions. Recall that for polynomial automorphisms all periodic points are isolated. Replacing f_λ by some iterate, we may assume the bifurcating periodic point is fixed. Passing to a branched cover of Λ if necessary, the fixed point moves holomorphically, so we assume it is equal to $0 \in \mathbb{C}^2$. Since f_λ is dissipative the multipliers depend holomorphically on λ, and we denote them by ρ_λ and b_λ, with $\rho_0 = e^{2\pi i \frac{p}{q}}$ and $|b_\lambda| < 1$ for all $\lambda \in \Lambda$. We further assume that ρ_λ crosses $\partial\mathbb{D}$ with non-zero speed, i.e. $\frac{\partial\rho}{\partial\lambda}\big|_{\lambda=0} \neq 0$.

8.1 Good local coordinates

The first step is to find adapted local coordinates. which is a parameterized version of the discussion in Sect. 6. This is analogous to Proposition 1 in [18].

Proposition 8.1 *If (f_λ) is as above, then for λ sufficiently close to 0, there exists a local change of coordinates $(x, y) = \varphi_\lambda(z, w)$ in which f_λ^q takes the form*

$$f_\lambda^q(x, y) = (\rho_\lambda^q x + x^{k+1} + x^{k+2} g_\lambda(x, y), b_\lambda^q y + x h_\lambda(x, y)) \tag{7}$$

with g_λ and h_λ holomorphic and $h_\lambda(0, 0) = 0$. Moreover $k = \nu q$ for some integer $\nu \geq 1$.

Proof Since we are working locally near $(0, 0) \in \Lambda \times \mathbb{C}^2$, we freely reduce the domains of definition in (λ, x, y) when necessary. We will also feel free to use to the same symbols for coordinates after successive changes of variable.

Recall from Sect. 6 that there exists an integer k and local coordinates in which f_0^q is of the form

$$(x, y) \mapsto (x + x^{k+1} + x^{k+2} g(x, y), b_0^q y + x h(x, y)). \tag{8}$$

Then f^q admits k (open) attracting petals and k repelling petals, which are permuted by f. These petals approach 0 at certain directions permuted by the differential $D_0 f$, so necessarily $k = \nu q$ for some nonzero integer ν.

From now on for notational ease we replace f^q by f, that is we assume $\rho_0 = 1$. Since the differential $D_0 f_\lambda$ is diagonalizable, there exists a (λ-dependent) linear change of coordinates so that $f_\lambda(x, y)$ takes the form $(\rho_\lambda x, b_\lambda y) + h.o.t$. There exists a local strong stable manifold tangent to the y-axis; we change coordinates so that it becomes $\{x = 0\}$. By the Schröder theorem we can linearize $f_\lambda|_{\{x=0\}}$, holomorphically in λ. Hence in the new coordinates, $f_\lambda(0, y) = (0, b_\lambda y)$, so that

$$f_\lambda(x, y) = (\rho_\lambda x(1 + O(x)), b_\lambda y + x h_\lambda(x, y)).$$

Of course $h_\lambda(0, 0) = 0$ since the linear part is $(\rho_\lambda x, b_\lambda y)$. From now on all changes of variables will be "horizontal", i.e. of the type $(x, y) \mapsto (x(1 + O(x, y)), y)$, so the form of the second coordinate persists and we focus on the first one.

Express $f_\lambda(x, y)$ as

$$\begin{aligned} f_\lambda(x, y) = (&\rho_\lambda a_1(\lambda, y) x + a_2(\lambda, y) x^2 + \cdots + a_j(\lambda, y) x^j \\ &+ \cdots, b_\lambda y + O_{\lambda, y}(x)), \end{aligned} \tag{9}$$

where the a_j are holomorphic and $a_1(\lambda, 0) = 1$. To start with, for $\lambda = 0$, we put f_0 in form (8), so that for $j \leq k$, $a_j(0, y) = 0$ and $a_{k+1}(0, y) = 1$.

The first task is to arrange so that $a_1 \equiv 1$. This is similar to [54]. For this we look for a change of coordinates of the form $(X, Y) = (x\varphi_\lambda(y), y)$, with $\varphi_\lambda(0) = 1$. Using the notation $f_\lambda(x, y) = (x_1, y_1)$ (and similarly in the (X, Y) variables), we infer that

$$X_1 = \rho_\lambda a_1(\lambda, Y) \frac{\varphi_\lambda(Y_1)}{\varphi_\lambda(Y)} X + O(X^2) = \rho_\lambda a_1(\lambda, Y) \frac{\varphi_\lambda(b_\lambda Y)}{\varphi_\lambda(Y)} X + O(X^2).$$

Therefore we see that to obtain the desired form, it is enough to choose

$$\varphi_\lambda(y) = \prod_{n=0}^{\infty} a_1(\lambda, b_\lambda^n y),$$

which is locally a convergent product since $a_1(\lambda, y) = 1+O_\lambda(y)$ and $|b_\lambda| < 1$. Notice also that for $\lambda = 0$, $\varphi_0(y) = 1$ so the change of variables is the identity. In particular f_0 remains of the form (8).

We then argue by induction. So assume that we have found coordinates such that for some $j \leq k$, $a_2(\lambda, y) = \cdots = a_{j-1}(\lambda, y) = 0$, and f_0 remains under the form (8) . Put

$$(X, Y) = \left(x + \frac{a_j(\lambda, y)}{\rho_\lambda - \rho_\lambda^j} x^j, y \right).$$

Notice that since $a_j(0) = 0$ and $\rho_\lambda - 1$ has a simple root at the origin, the change of coordinates is also well defined at $\lambda = 0$. Now, for $\lambda \neq 0$, since the term $a_j(\lambda, y)$ is non-resonant, a classical explicit computation (see e.g. [1, Thm 6.10.5.]) shows that it disappears in the new coordinates. Hence by continuity the same holds for $\lambda = 0$.

Moreover, for $\lambda = 0$ the change of coordinates is of the form $(X, Y) = (x + A_j(y)x^j, y)$, so $(x, y) = (X - A_j(Y)X^j + h.o.t., Y)$. In the new coordinates we obtain

$$\begin{aligned} X_1 &= x_1 + A_j(y_1)x_1^j = x + x^{k+1} + O(x^{k+2}) + A_j(y)(x + x^{k+1} + O(x^{k+2}))^j \\ &= x + A_j(y)x^j + x^{k+1} + O(x^{k+2}) \\ &= X + X^{k+1} + O(X^{k+2}), \end{aligned}$$

so f_0 remains of form (8) (observe that $j \leq k$ is used here).

Hence by induction we arrive at a situation where the first coordinate of $f_\lambda(x, y)$ is of the form $\rho_\lambda x + a_k(\lambda, y)x^k + O(x^{k+1})$, with $a_k(0, y) = 1$, and the desired form follows by putting $(X, Y) = (a_k(\lambda, y)^{\frac{1}{k-1}} x, y)$. □

Remark 8.2 Observe that the normal form (6) is more precise than the one that we obtain here for f_0. Indeed, as opposed to the case $\lambda = 0$, we cannot in general kill the terms $x^{k+2}, \ldots, x^{2k}$ in the first coordinate of (7).

In fact, the vanishing of these terms for $\lambda = 0$ is incompatible with keeping (f_λ) in form (7). Indeed, the change of variables required to kill these terms at $\lambda = 0$ is of the form $(x, y) \mapsto (x + \alpha_\lambda(y)x^j, y + h.o.t.)$, where $\alpha_0(0) \neq 0$ and $j \leq k$ (compare [1, Thm 6.5.7]). If $\rho_\lambda \neq 1$ for some λ, it brings back a non-zero term x^j in the first coordinate of f_λ.

On the other hand, if by chance the terms x^j, $j = k+2 \leq k \leq 2k$, vanish, we can reduce ourselves to [2] by letting $(x', y') = (k\rho_\lambda^{k-1} x^k, y)$ and then $(x'', y'') = (x' + (1 - \rho^k)/2, y')$. It is the presence of these extra non-vanishing terms that prevents us from using [2] directly.

8.2 Transit mappings in the one-dimensional case: un tour de valse

To fix the ideas, let us establish the statement we need in the one-dimensional case first. This is a refined version of [18]. Consider a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of mappings defined in some neighborhood of the origin in $\mathbb{C}$, of the form

$$f_\lambda(x) = \rho_\lambda x + x^{k+1} + x^{k+2} g_\lambda(x), \tag{10}$$

with g holomorphic. As before Λ is the unit disk. We assume that $\rho_0 = 1$ and $\frac{\partial\rho}{\partial\lambda}(0) \neq 0$ (this amounts to replacing f_λ by its q^{th} iterate in (7)).

Recall that for $\lambda = 0$ the repelling and attracting directions are respectively defined by the property that $(1 + x^k) \in \mathbb{R}^{+/-}$. We fix two consecutive such directions with respective angles 0 and $\frac{\pi}{k}$, and non-overlapping sectors about them by putting

$$S^\iota = \left\{\arg x \in \left(-\frac{5\pi}{4k}, -\frac{3\pi}{4k}\right)\right\} \text{ and } S^o = \left\{\arg x \in \left(-\frac{\pi}{4k}, \frac{\pi}{4k}\right)\right\}.$$

The result is as follows.

Theorem 8.3 *Let f_λ be as in (10) and $S^{\iota/o}$ be as above. There exists a neighborhood V of the origin in $\mathbb{C}$ with the following property: if Q^ι and Q^o are open topological disks with $Q^\iota \Subset S^\iota \cap V$ and $Q^o \Subset S^o \cap V$, then for every neighborhood W of* 0 *in Λ, there exists an integer N and a radius r such that if $n \geq N$ there exists a holomorphic map $\lambda_n : Q^\iota \times Q^o \to W$ such that for every $(z^\iota, z^o) \in Q^\iota \times Q^o$, $f^n_{\lambda_n(z^\iota,z^o)}$ is a well defined univalent function on $B(z^\iota, r)$, with $f^n_{\lambda_n}(z^\iota) = z^o$ and $\left|(f^n_{\lambda_n})' - 1\right| \leq \frac{1}{5}$.*

This statement being quite technical, a few words of explanation are in order. What this theorem says is that if a parabolic bifurcation of the form (10) occurs, then by carefully selecting the parameters λ_n, taking high iterates $f^n_{\lambda_n}$ we can map any point z^ι from an attracting sector to any point z^o in a consecutive repelling sector, with uniform control on the derivative $(f^n_{\lambda_n})'$ in the neighborhood of z^ι.

To prove the theorem we begin with some background and intermediate results. We work in the new coordinate $z = \frac{\rho_\lambda^{k+1}}{kx^k}$. Notice that for $\lambda = 0$, the change of variables maps the sector $\{\arg x \in (-\frac{3\pi}{2k}, \frac{\pi}{2k})\}$ onto $\mathbb{C}\backslash i\mathbb{R}^-$, hence for small enough λ, S^ι and S^o are contained in $(\frac{\rho_\lambda^{k+1}}{kx^k})^{-1}(\mathbb{C}\backslash i\mathbb{R}^-)$. In the new coordinates, S^ι and S^o are respectively perturbations of the sectors $\{\arg z \in (\frac{3\pi}{4}, \frac{5\pi}{4})\}$ and $\{\arg z \in (-\frac{\pi}{4}, \frac{\pi}{4})\}$.

Using the the classical notation $x_1 = f_\lambda(x)$ (and similarly for z), we infer that

$$\begin{aligned} z_1 &= \frac{\rho_\lambda^{k+1}}{kx_1^k} = \frac{\rho_\lambda^{k+1}}{k(f_\lambda(x))^k} = \frac{\rho_\lambda^{k+1}}{k\rho_\lambda^k x^k(1+\frac{x^k}{\rho_\lambda}+O(x^{k+1}))^k} \\ &= \frac{\rho_\lambda^{k+1}}{k\rho_\lambda^k x^k}\left(1-\frac{kx^k}{\rho_\lambda}+O(x^{k+1})\right) = \frac{\rho_\lambda}{kx^k} - 1 + O(x) \\ &= \rho_\lambda^{-k} z - 1 + \eta_\lambda(z), \text{with } \eta_\lambda(z) = O\left(\frac{1}{|z|^{1/k}}\right) \text{ as } z \to \infty, \text{ uniformly in } \lambda \in \Lambda \end{aligned}$$

The exponent $1/k$ will play a special role in the estimates to come, so for notational ease, from now on we put $\gamma = 1/k$. We also change coordinates in the parameter space by putting $u = \rho_\lambda^{-k} - 1$, so that u now ranges in some neighborhood W of the origin, and our mapping writes as

$$f_u(z) = (1+u)z - 1 + \eta_u(z), \text{ with } \eta_u(z) = O\left(\frac{1}{|z|^\gamma}\right).$$

In these coordinates, f_u is defined in an open set Ω_R of the form

$$\Omega_R = \left\{ z,\ |z| > R, \arg(z) \in \left(\frac{-3\pi}{8}, \frac{11\pi}{8}\right)\right\}, \tag{11}$$

for some $R = R_0$. Its complement is shaded on Fig. 1 and will be referred to as the “forbidden region”. We also pick two bounded open topological disks Q^ι and Q^o such that $Q^\iota \Subset S^\iota \cap \Omega_R$ and $Q^o \Subset S^o \cap \Omega_R$, where $R \geq R_0$ is to be fixed later (this corresponds to the choice of the neighborhood V in the statement of the theorem).

We fix a constant M such that for every parameter $u \in W$ and every $z \in \Omega_{R_0}$,

$$|\eta_u(z)| \leq \frac{M}{|z|^{1/k}} = \frac{M}{|z|^\gamma} \text{ and } \left|\eta_u'(z)\right| \leq \frac{M}{|z|^{1+\gamma}}. \tag{12}$$

We will let u vary in a small subset of W, of the form $W_n = B(-\frac{2\pi i}{n}, \frac{1}{n^{1+\gamma/2}})$. Notice that for $u \in W_n$ we have that

$$|1+u| = 1 + \frac{2\pi^2}{n^2} + o\left(\frac{1}{n^2}\right) \text{ and } \arg(1+u) = -\frac{2\pi}{n} + O\left(\frac{1}{n^{1+\gamma/2}}\right).$$

In the following we always consider n so large that $n^{\gamma/2} > 100$, $1 - \frac{30}{n^2} \leq |1+u| \leq 1 + \frac{30}{n^2}$ and $\left|\arg(1+u) + \frac{2\pi}{n}\right| \leq \frac{1}{100n}$.

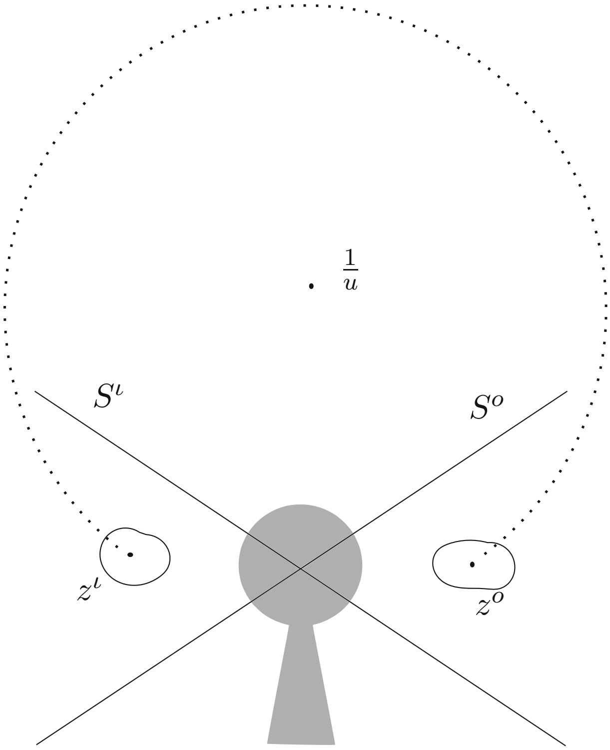

Fig. 1 Schematic view of the orbit connecting z^ι to z^o. It shadows the arc of a circle passing through the "gate" between the fixed points $1/u$ and ∞. The forbidden region is shaded

To understand the argument better, it is instructive to think of f_u as a perturbation of the affine map $\ell_u : z \mapsto (1+u)z - 1$. When $u \in W_n$ and n is large, ℓ_u is approximately a rotation by angle $-\frac{2\pi}{n}$ centered at $\frac{1}{u}$. Notice also that $\frac{1}{u}$ is close to $\frac{in}{2\pi}$ (see Fig. 1).

To fix the ideas, let us first analyze the linear case, dealing with ℓ_u instead of f_u.

Proposition 8.4 *With notation as above, there exists an integer N, and a radius r such that if $n \geq N$ then for every $(z^\iota, z^o) \in Q^\iota \times Q^o$, there exists a parameter $u = u(z^\iota, z^o) \in W_n$, depending holomorphically on (z^ι, z^o) and such that*

- $\ell_u^n(z^i) = z^o$;
- *for every $z \in B(z^\iota, r)$ the iterates $\ell_u^j(z)$, $j = 1, \ldots, n$ do not enter the forbidden region;*
- $\left|(\ell_u^n)' - 1\right| \leq \frac{1}{5}$ *on $B(z^\iota, r)$.*

Proof Let $l = \lfloor n/2 \rfloor$ and $m = n - l$. As said above, $\ell_u(z) = (1+u)(z - \frac{1}{u}) + \frac{1}{u}$ has its fixpoint at $\frac{1}{u}$. Write $u = \frac{-2\pi i}{n} + \frac{v}{n^{1+\gamma/2}}$, with $v \in \mathbb{D}$. For $j \leq n$ we have that

$$\begin{aligned}(1+u)^j &= \exp(j\log(1+u)) = \exp\left(j\log\left(1-\frac{2\pi i}{n}+\frac{v}{n^{1+\gamma/2}}\right)\right)\\ &= \exp\left(j\left(-\frac{2\pi i}{n}+\frac{v}{n^{1+\gamma/2}}+O\left(\frac{1}{n^2}\right)\right)\right)\\ &= \exp\left(-\frac{2j\pi i}{n}+\frac{jv}{n^{1+\gamma/2}}+O\left(\frac{j}{n^2}\right)\right),\end{aligned}$$

in particular for $j=n$

$$(1+u)^n = 1+\frac{v}{n^{\gamma/2}}+O\left(\frac{1}{n}\right), \tag{13}$$

where the $O(\cdot)$ is uniform with respect to $v\in\mathbb{D}$.

Simple geometric considerations (see [18]) then show that for $j\leq\lceil n/2\rceil$ $\ell_u^j(z^\iota)$ (resp. $\ell_u^{-j}(z^o)$) do not enter the forbidden area.

Let us prove that there exists $u(z^\iota,z^o)$, depending holomorphically on $(z^\iota,z^o)\in Q^\iota\times Q^o$ and such that $\ell_u^l(z^\iota)=\ell_u^{-m}(z^o)$. Then for such a parameter, by connecting the two pieces of orbits $1,\dots,l$ and $l+1,\dots,n$, we infer that the iterates $\ell_u^j(z^\iota)$ do not enter the forbidden area for $1\leq j\leq n$ and since ℓ_u is affine, the control of the derivative follows from (13).

To prove this, consider the expression

$$\frac{\ell_u^l(z^\iota)-\frac{1}{u}}{\ell_u^{-m}(z^o)-\frac{1}{u}} = (1+u)^n\frac{z^\iota-\frac{1}{u}}{z^o-\frac{1}{u}}.$$

A simple computation shows that

$$\frac{z^\iota-\frac{1}{u}}{z^o-\frac{1}{u}} = 1+(z^\iota-z^o)\frac{2\pi i}{n}+O\left(\frac{1}{n^{1+\gamma/2}}\right).$$

Therefore by (13), we infer that

$$\frac{\ell_u^l(z^\iota)-\frac{1}{u}}{\ell_u^{-m}(z^o)-\frac{1}{u}} = 1+\frac{v}{n^{\gamma/2}}+O\left(\frac{1}{n}\right), \tag{14}$$

where the $O(\cdot)$ is uniform with respect to $v\in\mathbb{D}$, $z^\iota\in Q^\iota$ and $z^o\in Q^o$. Thus when n is large enough the quantity in (14) winds once around 1 as v turns once around $\partial\mathbb{D}$, and the result follows from the Argument Principle. □

We now turn to f_u. Let us start with a technical lemma.

Lemma 8.5 *Fix $R \geq (10^5 kM)^k$. With notation as above, there exists an integer $N = N(R)$ depending only on R such that if $n \geq N$, $u \in W_n$ and if $z^\iota \in Q^\iota$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$:*

(i) *$f_u^j(z^\iota)$ stays outside the forbidden area;*
(ii) *$|f_u^j(z^\iota) - \frac{1}{u}| \geq \frac{n}{10}$;*
(iii) *writing $f_u^j(z^\iota) = z_j = x_j + iy_j$ we have that either $x_j \leq x_0 - \frac{j}{10}$ or $y_j \geq \frac{n}{10}$. In particular $\left|z_j\right| \geq \min(\frac{|z_0|}{2} + \frac{j}{10}, \frac{n}{10})$.*

The same results holds for $f_u^{-j}(z^o)$, when $z^o \in Q^o$ (in that case the last condition needs to be replaced by "either $x_j \geq x_0 + \frac{j}{10}$ or $y_j \geq \frac{n}{10}$")

Proof We first deal with the assertions (i) and (ii). We argue by induction so assume the result holds for $j \leq k-1$, for some $k \leq \lceil \frac{n}{2} \rceil$. Let us write

$$\frac{f_u^j(z^\iota) - \frac{1}{u}}{f_u^{j-1}(z^\iota) - \frac{1}{u}} = (1+u) + \frac{\eta_u(f_u^{j-1}(z^\iota))}{f_u^{j-1}(z^\iota) - \frac{1}{u}},$$

so that

$$\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}} = (1+u)^k \prod_{j=0}^{k} \left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right). \tag{15}$$

Considering the modulus of this expression, we see that

$$\begin{aligned}\left|f_u^k(z^\iota) - \frac{1}{u}\right| &\geq \left|z^\iota - \frac{1}{u}\right| \left(1 - \frac{30}{n^2}\right)^{\lceil \frac{n}{2} \rceil} \prod_{j=0}^{k-1} \left(1 - \frac{M}{(0.9)R^\gamma |f_u^j(z^\iota) - \frac{1}{u}|}\right) \\ &\geq \left|z^\iota - \frac{1}{u}\right| \left(1 - \frac{30}{n^2}\right)^{\lceil \frac{n}{2} \rceil} \left(1 - \frac{10M}{(0.9)R^\gamma n}\right)^{\lceil \frac{n}{2} \rceil},\end{aligned}$$

where the first estimate follows from bound (12) on η_u and the second estimate follows from the induction hypothesis. Since $(1 - \frac{30}{n^2})^{n/2} \to 1$ as $n \to \infty$, by our choice of R we see that when $n \geq N(R)$,

$$\left|f_u^k(z^\iota) - \frac{1}{u}\right| \geq \frac{9}{10} \left|z^\iota - \frac{1}{u}\right| \geq \frac{9}{10} d\left(\frac{1}{u}, S^\iota\right) \geq \frac{9}{10} \frac{n}{2\sqrt{2}\pi} \geq \frac{n}{10},$$

which proves (ii).

To prove that $f_u^k(z^\iota)$ does not enter the forbidden region, we look at the argument of $f_u^k(z^\iota) - \frac{1}{u}$. Recall that $\left|\arg(1+u) + \frac{2\pi}{n}\right| \leq \frac{1}{100n}$ so by (15)

$$\left|\arg\left(\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}}\right) - \left(\frac{-2k\pi}{n}\right)\right| \leq \frac{k}{100n} + \sum_{j=0}^{k-1}\left|\arg\left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right)\right|.$$

With our choice of R, $|\frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota)-\frac{1}{u})}| \leq \frac{1}{200n}$, so since $\log(1+z) = z + h.o.t.$, when n is large enough, we infer that

$$\left|\arg\left(1 + \frac{\eta_u(f_u^j(z^\iota))}{(1+u)(f_u^j(z^\iota) - \frac{1}{u})}\right)\right| \leq \frac{1}{100n}.$$

Thus we obtain that

$$\left|\arg\left(\frac{f_u^k(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}}\right) - \left(\frac{-2k\pi}{n}\right)\right| \leq \frac{k}{100n} + \frac{k}{100n} \leq \frac{k}{50n},$$

therefore arguing geometrically we see that $f_u^k(z^\iota)$ stays outside the forbidden region. The induction step is complete proving (i).

To establish (iii), let us first observe that due to the the above estimate on the argument, when $j \leq \lceil \frac{n}{2} \rceil$, $\arg(\frac{f_u^j(z^\iota)-\frac{1}{u}}{z^\iota - \frac{1}{u}})$ is equal to $\frac{-2j\pi}{n}$, up to an error of at most $\frac{1}{50}$. Expressing in coordinates, we see that

$$x_{j+1} = x_j - 1 + \frac{2\pi}{n} y_j + \varepsilon_j \text{ and } y_{j+1} = y_j - \frac{2\pi}{n} x_j + \varepsilon'_j,$$

with

$$|\varepsilon_j|, |\varepsilon'_j| \leq \max\left(\frac{1}{n^{1+\gamma/2}}, \frac{M}{|z_j|^\gamma}\right) \leq \frac{1}{1{,}000}$$

because $n^{\gamma/2} \geq 100$ and by the previous step, $z_j \in \Omega_R$. We see that, as soon as $y_j \leq \frac{n}{10}$, we have that $x_{j+1} \leq x_j - \frac{1}{10}$. Now when y_j reaches $\frac{n}{10}$, and until j is as large as $\frac{n}{4}$ (a time at which y_j is approximately equal to $\frac{n}{2\pi}$), since $\left|z_j - \frac{1}{u}\right| \geq \frac{9}{10}\left|z_0 - \frac{1}{u}\right|$, by expressing the distance in coordinates and using the estimate on the argument, we infer that $x_j \leq -\frac{n}{100}$ therefore $y_{j+1} \geq y_j$. The result follows.

The argument for $f_u^{-j}(z^o)$, $1 \leq j \leq \lceil \frac{n}{2} \rceil$ is similar, and is left to the reader. $\square$

Proof of Theorem 8.3 We argue as in Proposition 8.4. As before let $l = \lfloor n/2 \rfloor$ and $m = n - l$. Using (15) with $k = l$, we obtain

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{z^\iota - \frac{1}{u}} = (1+u)^l \prod_{j=1}^{l} \left(1 + \frac{\eta_u(f_u^{j-1}(z^\iota))}{(1+u)(f_u^{j-1}(z^\iota) - \frac{1}{u})} \right).$$

Hence, using Lemma 8.5 together with the inequality $\left|\prod(1+x_j) - 1\right| \leq \exp \sum \left|x_j\right| - 1$, we infer that

$$\begin{aligned}
\left| \frac{f_u^l(z^\iota) - \frac{1}{u}}{(1+u)^l(z^\iota - \frac{1}{u})} - 1 \right| &= \left| \frac{f_u^l(z^\iota) - \frac{1}{u}}{\ell_u^l(z^\iota) - \frac{1}{u}} - 1 \right| \\
&\leq \exp\left(\sum_{j=0}^{l} \frac{10M}{(0.9)n \min(\frac{|z^\iota|}{2} + \frac{j}{10}, \frac{n}{10})^\gamma} \right) - 1 \\
&\leq \exp\left(\sum_{j=0}^{\lceil \frac{n}{2} \rceil} \frac{100M}{9n \min(\frac{j}{10} + 1, \frac{n}{10})^\gamma} \right) - 1 \\
&\leq \exp\left(\frac{1{,}000M}{n^\gamma} \right) - 1 \\
&= O\left(\frac{1}{n^\gamma} \right),
\end{aligned} \tag{16}$$

where in the last inequality we use an elementary estimate

$$\sum_{j=0}^{\lceil \frac{n}{2} \rceil} \frac{1}{\min(\frac{j}{10} + 1, \frac{n}{10})^\gamma} \leq 50 n^{1-\gamma}.$$

Doing the same with $f_u^{-m}(z^o)$ we get that

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{f_u^{-m}(z^o) - \frac{1}{u}} \left(\frac{\ell_u^l(z^\iota) - \frac{1}{u}}{\ell_u^{-m}(z^o) - \frac{1}{u}} \right)^{-1} = 1 + O\left(\frac{1}{n^\gamma} \right).$$

Thus, from (14) we deduce that

$$\frac{f_u^l(z^\iota) - \frac{1}{u}}{f_u^{-m}(z^o) - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma} \right),$$

where the $O(\cdot)$ is uniform with respect to $(v, z^\iota, z^o) \in \mathbb{D}\times Q^\iota \times Q^o$. Therefore we conclude that if n is large enough, when u winds once around ∂W_n (i.e. v winds once around $\partial\mathbb{D}$), the curve $u \mapsto \dfrac{f_u^l(z^\iota) - 1/u}{f_u^{-m}(z^o) - 1/u}$ winds once around 1, so by the Argument Principle, there exists a unique $u = u(z^\iota, z^o) \in W_n$ (thus, depending holomorphically on (z^ι, z^o)), such that $f_u^l(z^\iota) = f_u^{-m}(z^o)$. Given such a u, we see that the iterates $f_u^j(z^\iota)$ $1 \leq j \leq n$ stay outside the forbidden region, and $f_u^n(z^\iota) = z^o$.

Let us now estimate the derivative $(f_u^l)'(z)$, for $z \in Q^\iota$ (this is place where we need to be precise on the value of R). Recall that for $1 \leq j \leq l$, $f_u^j(z)$ is well-defined, and write

$$(f_u^l)'(z) = \prod_{j=1}^{l} (f_u)'(f_u^{j-1}(z)), \quad \text{where } (f_u)'(z) = 1 + u + \eta_u'(z), \ \left|\eta_u'(z)\right| \leq \frac{M}{|z|^{1+\frac{1}{k}}}.$$

So we get that

$$(f_u^l)'(z) = (1+u)^l \prod_{j=1}^{l} \left(1 + \frac{\eta_u'(f_u^{j-1}(z))}{1+u}\right).$$

Our choice of u and l implies that $0.99 \leq \left|(1+u)^l\right| \leq 1.01$ for large n, while

$$\left|\prod_{j=1}^{l} \left(1 + \frac{\eta_u'(f_u^{j-1}(z))}{1+u}\right) - 1\right| \leq \exp\left(\sum_{j=1}^{\lceil \frac{n}{2}\rceil} \frac{10M}{9(\min(\frac{R}{2} + \frac{j}{10}, \frac{n}{10}))^{1+\gamma}}\right) - 1. \tag{17}$$

For large n (depending only on R) we have that

$$\sum_{j=1}^{\lceil \frac{n}{2}\rceil} \frac{1}{(\min(\frac{R}{2} + \frac{j}{10}, \frac{n}{10}))^{1+\gamma}} \leq \sum_{j=\lfloor \frac{R}{2}\rfloor}^{\infty} \frac{10^{1+\gamma}}{j^{1+\gamma}} + \sum_{j=\lfloor \frac{n}{10}\rfloor}^{\lceil \frac{n}{2}\rceil} \left(\frac{n}{10}\right)^{-(1+\gamma)} \leq 200kR^{-\gamma}. \tag{18}$$

Since $R \geq (10^5 kM)^k$, by (18) we infer that the right hand side of (17) is smaller than $\frac{1}{400}$, and finally we conclude that when $z \in Q^\iota$ and n is large enough, $\left|(f_u^l)'(z) - 1\right| \leq \frac{1}{50}$.

The following lemma is classical, for convenience we recall the proof below.

Lemma 8.6 *Let f be a holomorphic function on $\mathbb{D}_r$ such that $\left|f'-1\right| \leq a < 1$ on $\mathbb{D}_r$. Then f is injective on $\mathbb{D}_r$ and*

$$D(f(0),(1-a)r) \subset f(\mathbb{D}_r) \subset D(f(0),(1+a)r).$$

From this lemma we deduce that there exists $r > 0$ independent on n such that f_u^l is univalent on $B(z^\iota, r)$, and its image contains $B(f^l(z^\iota), r)$. Likewise, there exists $r > 0$ such that f_u^{-m} is univalent on $B(z^o, r)$, with derivative $\frac{1}{50}$-close to 1. Thus we conclude that f_u^l maps univalently $B(z^\iota, \frac{r}{2})$ into $B(z^o, r)$, and its derivative satisfies $\left|(f_u^n)'(z)-1\right| \leq \frac{1}{5}$ This completes the proof of the theorem. □

Proof of Lemma 8.6 Replacing f by $z \mapsto r^{-1}f(rz) - f(0)$ it is no loss of generality to assume that $f(0) = 0$ and $r = 1$. Since $z \mapsto z - f(z)$ is contracting, if $f(z) = f(z')$ we get that

$$\left|(z-f(z))-(z'-f(z'))\right| = \left|z-z'\right| \leq a\left|z-z'\right|,$$

whence $z = z'$. Thus f is injective. That $f(\mathbb{D}) \subset \mathbb{D}_{1+a}$ readily follows from the mean value inequality. Finally, to prove that for any $w \in \mathbb{D}_{1-a}$, the equation $f(z) = w$ admits a solution, it is enough to apply the Contraction Mapping Principle to $g : z \mapsto z - f(z) + w$ in $\mathbb{D}$. □

8.3 Transit mappings in dimension 2

We return to the two-dimensional setting. The treatment will be based on the observation that in a two-dimensional thickening of the domain Ω_R, the maps f_λ admit a *dominated splitting*, i.e., they have a horizontal cone field invariant under the forward dynamics, and moreover, they are contracting in the vertical direction.

Let us first fix some notation. As before the parameter space Λ is the unit disk. Changing coordinates and passing to an iterate if needed, by Proposition 8.1 we may assume that $(f_\lambda)_{\lambda\in\Lambda}$ is a holomorphic family of germs of diffeomorphisms in $(\mathbb{C}^2, 0)$ of the form

$$f_\lambda(x,y) = (\rho_\lambda x + x^{k+1} + x^{k+2}g_\lambda(x,y), b_\lambda y + xh_\lambda(x,y)), \qquad (19)$$

where $\rho_0 = 1$, $\frac{d\rho}{d\lambda}(0) \neq 0$, and $|b_\lambda| \leq b < 1$ for all λ.

As in the one-dimensional case, we consider two consecutive sectors $S^\iota = \{\arg x \in (-\frac{5\pi}{4k}, -\frac{3\pi}{4k})\}$ and $S^o = \{\arg x \in (-\frac{\pi}{4k}, \frac{\pi}{4k})\}$. For $\lambda = 0$, consider a bidisk $V = V_1 \times V_2$ around the origin such that:

- $(S^\iota \cap V_1) \times V_2$ is attracted to the origin under forward iteration;
- there exists a local repelling petal $\Sigma \subset V_1 \times V_2$, which is a graph over $S^o \cap V_1$, defined by the property that every orbit converging to 0 under backward iteration in $(S^o \cap V_1) \times V_2$ belongs to Σ.

The next theorem is the key technical mechanism which will allow us to create a homoclinic tangency from a critical point in a semi-parabolic basin. It is the counterpart of Theorem 8.3 in the dissipative 2-dimensional setting. Assuming that a semi-parabolic bifurcation of the form (19) occurs, it select parameters λ_n such that the iterates $f^n_{\lambda_n}$ map a given point p^ι in some semi-parabolic basin (almost) onto a given target p^o located in a repelling petal Σ, with a good control on the geometry of $f^n_{\lambda_n}$ near z^ι. This geometric control is expressed in terms of the pull-back action on a foliation transverse to Σ near p^o.

Theorem 8.7 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be as above. There exists a bidisk $V = V_1 \times V_2$ around $0 \in \mathbb{C}^2$ such that if $Q^\iota \Subset (S^\iota \cap V_1) \times V_2$, $Q^o \Subset \Sigma$, and $\mathcal{F}$ is a germ of holomorphic foliation transverse to Σ along Q^o, then for every neighborhood W of 0 in Λ, there exists an integer N and a radius r such that if $n \geq N$, there exists a holomorphic map $\lambda_n : Q^\iota \times Q^o \to W$ such that for every $(p^\iota, p^o) \in Q^\iota \times Q^o$, for $\lambda_n = \lambda_n(p^\iota, p^o)$, there exists a bidisk $D^2(p^\iota, r)$ around p^ι, and a neighborhood $D_\Sigma(p^o, r) = B(p^o, r) \cap \Sigma$ of p^o in Σ, such that the following properties hold:*

- *$f^n_{\lambda_n}(p^\iota)$ belongs to $\mathcal{F}(p^o)$, the leaf of $\mathcal{F}$ through p^o;*
- *the preimage of $\mathcal{F}$ under $f^n_{\lambda_n}$ defines a holomorphic foliation $\mathcal{F}^{-n}$ of $D^2(p^\iota, r)$ by vertical graphs along which $f^n_{\lambda_n}$ contracts by a factor b^n;*
- *the derivative of $f^n_{\lambda_n}$ along any horizontal line in $D^2(p^\iota, r)$ satisfies $\left|\frac{\partial f^n_{\lambda_n}}{\partial z} - 1\right| \leq \frac{1}{5}$.*

To prove the theorem, we consider the dynamics of f_λ in a domain of the form

$$\left\{\arg(x) \in \left(\frac{-3\pi}{2k}, \frac{\pi}{2k}\right)\right\} \times \mathbb{D}_{s_0}$$

and as in the previous section we change coordinates by putting $(z, w) = (\frac{\rho_\lambda^{k+1}}{kx^k}, y)$, and $u = \rho_\lambda^{-k} - 1$. In the new coordinates, u ranges in some small neighborhood W of the origin and f_u is defined in a domain of the form $\Omega_{R_0} \times \mathbb{D}_{s_0}$, where Ω_R is as in (11), $R_0 \geq 1$, $s_0 \leq 1$, and its expression becomes

$$f_u(z, w) = ((1 + u)z - 1 + \eta_u(z, w), b_u w + \theta_u(z, w)), \tag{20}$$

where $\eta_u(z,w)$ and $\theta_u(z,w)$ are of the form $\frac{1}{z^{1/k}}\varphi_u(\frac{1}{z^{1/k}}, w)$, with φ_u holomorphic in the neighborhood of the origin. In the new coordinates,

$$S^\iota = \left\{z,\ \arg(z) \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)\right\} \text{ and } S^o = \left\{z,\ \arg(z) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right\}.$$

As above we let $W = W_n = D(-\frac{2\pi i}{n}, \frac{1}{n^{1+\gamma/2}})$ (recall that $\gamma = 1/k$).

We will gradually adjust the parameters R and s. We fix M such that for $(z,w) \in \Omega_{R_0} \times \mathbb{D}_s$ and $u \in W$,

$$|\eta_u(z,w)|,\ \left|\frac{\partial \eta_u}{\partial w}(z,w)\right|,\ |\theta_u(z,w)|,\ \left|\frac{\partial \theta_u}{\partial w}(z,w)\right| \leq \frac{M}{|z|^\gamma},$$
$$\text{and } \left|\frac{\partial \eta_u}{\partial z}(z,w)\right| \leq \frac{M}{|z|^{1+\gamma}}.$$

Due to dissipation, there is now an asymmetry between positive and negative iterates. The idea of the construction of the transition mapping is now to pull back $n/2$ times a leaf of the foliation $\mathcal{F}$ from the "outgoing" region Q^o and to push forward $n/2$ times a point from the "incoming" region Q^ι, and use the Argument Principle to make the image of the point belong to the preimage of the leaf.

We will first prove Theorem 8.7 under a seemingly stronger assumption that the foliation $\mathcal{F}$ is composed of graphs over the second coordinate in $\Omega_R \times \mathbb{D}_s$, with slope bounded by $1/100$. We start by showing that the backward graph transform is well defined for such vertical graphs on an appropriate subregion of $\Omega_{R_0} \times D_{s_0}$ (as long as R_0 is large and s_0 is small). This is a standard technique for maps with dominated splitting, which is e.g. used to construct the strong stable foliation on forward invariant regions (this is not the case we are dealing with here).

Lemma 8.8 *Let* $\widetilde{\Omega}_R = \{\zeta \in \Omega_R :\ D(\frac{1+\zeta}{1+u}, 1) \subset \Omega_R\}$. *There exists* R_0 *and* s_0 *such that if* $R \geq R_0$ *and* $s \leq s_0$, *then if* Γ *is a vertical graph of slope* $\leq 1/100$ *in* $\widetilde{\Omega}_R \times \mathbb{D}_s$ *then* $f_u^{-1}(\Gamma) \cap (\Omega_R \times \mathbb{D}_s)$ *is a vertical graph in* $\Omega_R \times \mathbb{D}_s$ *of slope* $\leq 1/100$.

Proof Take $\Gamma = \{z = \psi(w)\}$ with $\psi(0) \in \widetilde{\Omega}_R$ and $|\psi'| \leq 1/100$. Then $f^{-1}(\Gamma)$ admits an equation of the form $\Psi(z,w) = 0$, where

$$\Psi(z,w) = (1+u)z - 1 + \eta_u(z,w) - \psi(b_u w + \theta_u(z,w)).$$

For $w = 0$, Rouché's theorem implies that for $R \geq R_0$, there exists z such that

$$\Psi(z, 0) = 0 \text{ and } \left| z - \frac{1+\psi(0)}{1+u} \right| \leq \frac{2M}{|z|^{\gamma}}. \tag{21}$$

In addition we have

$$\frac{\partial \Psi}{\partial z} = 1 + u + O(R^{-\gamma}) \text{ and } \left| \frac{\partial \Psi}{\partial w} \right| \leq \frac{b}{100} + O(R^{-\gamma}).$$

Thus, the result follows from the Implicit Function Theorem. □

From now on the parameter $s = s_0$ will be fixed, and for notational simplicity we denote the second factor $\mathbb{D}_{s_0}$ by D. Let $Q^\iota \Subset S^\iota \times D$. We will now state two different counterparts of Lemma 8.5: one for push-forwards, and the other one for pullbacks. For $p^\iota = (z^\iota, w^\iota) \in Q^\iota$, we denote by $p^\iota_j = (z^\iota_j, w^\iota_j)$ its j^{th} iterate under f_u.

Lemma 8.9 *With notation as above, fix $R \geq \max(M^k(1-b)^{-k} s_0^{-k}, (10^5 kM)^k)$. Then there exists an integer $N = N(R)$ such that if $n \geq N$, $p^\iota \in Q^\iota \times D$ and $u \in W_n$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$ we have that*

(i) *$f_u^j(p^\iota)$ belongs to $\Omega_R \times D$*
(ii) *$|z^\iota_j - \frac{1}{u}| \geq \frac{n}{10}$;*
(iii) *$|z^\iota_j| \geq \min(\frac{|z_0|}{2} + \frac{j}{10}, \frac{n}{10})$.*

Proof It follows from expression (20) for f_u that if $\frac{M}{R^\gamma} < (1-b)s_0$ and $(z, w) \in \Omega_R \times D$, then the second coordinate of $f_u(z, w)$ belongs to D. So we only need to focus on the first coordinate. By (20) we have that

$$\frac{z^\iota_{j+1} - \frac{1}{u}}{z^\iota_j - \frac{1}{u}} = 1 + u + \frac{\eta_u(f_u^j(p^\iota))}{z^\iota_j - \frac{1}{u}}, \tag{22}$$

with $|\eta_u(f_u^j(p^\iota))| \leq \frac{M}{R^\gamma}$ as soon as $f_u^j(p^\iota) \in \Omega_R \times D$. Then the proof is identical to that of Lemma 8.5. □

We now deal with pullbacks. Given $p^o \in Q^o$, we consider a holomorphic foliation $\mathcal{F}$ by vertical graphs of slope bounded by $1/100$ in a neighborhood of p^o, and by $\mathcal{F}(p)$ the leaf through p. Starting from $\mathcal{F}_0 = \mathcal{F}(p)$, by applying successive graph transforms we inductively define $\mathcal{F}_{-j-1} = f_u^{-1}(\mathcal{F}_{-j}) \cap (\Omega_R \times D)$. We also let $\zeta_{-j} = \mathcal{F}_{-j} \cap \{w = 0\}$.

Lemma 8.10 *Let R be as in Lemma 8.9. There exists an integer $N = N(R)$ such that if $n \geq N$, if $p \in Q^o$ and $u \in W_n$ then for every $1 \leq j \leq \lceil \frac{n}{2} \rceil$ we have*

(i) *$\mathcal{F}_{-j}(p)$ is a well-defined vertical graph in $\Omega_R \times D$, with slope bounded by $1/100$;*
(ii) *$|\zeta_{-j} - \frac{1}{u}| \geq \frac{n}{10}$;*
(iii) *$|\zeta_{-j}| \geq \min(\frac{|z_0|}{2} + \frac{j}{10}, \frac{n}{10})$.*

Proof From (21) we infer that

$$\left|\zeta_{-j-1} - \frac{1+\zeta_{-j}}{1+u}\right| \leq \frac{2M}{|\zeta_{-j}|^\gamma},$$

that is,

$$\frac{\zeta_{-j-1} - \frac{1}{u}}{\zeta_{-j} - \frac{1}{u}} = \frac{1}{1+u} + \frac{\varepsilon_j}{\zeta_{-j} - \frac{1}{u}}, \text{ with } |\varepsilon_j| \leq \frac{2M}{|\zeta_{-j}|^\gamma}, \tag{23}$$

so as before the result follows exactly as in the one-dimensional case (for (i) we also use Lemma 8.8). □

Pick now an R satisfying all the above requirements. Let as above $p^\iota = (z^\iota, w^\iota) \in Q^\iota$, $p^o \in Q^o$, and let $\mathcal{F}^o$ be the leaf of $\mathcal{F}$ through p^o. Let $l = \lfloor n/2 \rfloor$ and $m = n - l$. By Lemma 8.9, for $1 \leq j \leq l$ $f_u^j(p^\iota) \in \Omega_R$. Then, using (22) exactly as in (16) (i.e. by taking the product from 0 to $l-1$) we deduce that

$$\left|\frac{z_l^\iota - \frac{1}{u}}{(1+u)^l(z^\iota - \frac{1}{u})} - 1\right| = O\left(\frac{1}{n^\gamma}\right).$$

On the pullback side, recall that $\mathcal{F}_{-j}(p^o)$ denotes the j^{th} graph transform of $\mathcal{F}_0$, and let $\zeta_{-j} = \mathcal{F}_{-j}(p^o) \cap \{w = 0\}$. Using (23) and taking the product from $j = 0$ to $j = -m + 1$ we obtain:

$$\left|(1+u)^m \frac{\zeta_{-m} - \frac{1}{u}}{\zeta_0 - \frac{1}{u}} - 1\right| = O\left(\frac{1}{n^\gamma}\right).$$

Thus, writing $u = \frac{-2\pi i}{n} + \frac{v}{n^{1+\gamma/2}}$, from the two previous displayed equations together with (13), we obtain that

$$\frac{z_l^\iota - \frac{1}{u}}{\zeta_{-m} - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma}\right). \tag{24}$$

Now express the graph $\mathcal{F}_{-m}$ as $z = \psi(w)$, with $\psi(0) = \zeta_{-m}$. Since $\left|\psi(w_l^\iota) - \psi(0)\right| \leq 1/100$, we infer that

$$\frac{\psi(w_l^\iota) - \frac{1}{u}}{\zeta_{-m} - \frac{1}{u}} = 1 + O\left(\frac{1}{n}\right),$$

so from (24) we finally deduce that

$$\frac{z_l^\iota - \frac{1}{u}}{\psi(w_l^\iota) - \frac{1}{u}} = 1 + \frac{v}{n^{\gamma/2}} + O\left(\frac{1}{n^\gamma}\right).$$

By the Argument Principle we conclude that for every $(p^\iota, p^o) \in Q^\iota \times Q^o$, there exists a unique (hence, depending holomorphically on (p^ι, p^o)) $u = u(p^\iota, p^o) \in W_n$ such that $\psi(w_l^\iota) = z_l^\iota$, that is, $f_u^l(p^\iota) \in \mathcal{F}_{-m}(p^o)$.

For this parameter u we can pull back $\mathcal{F}_{-m}(p^o)$ under f_u^m, thus obtaining a vertical graph $\mathcal{F}_{-n}(p^o)$ through p^ι. It is clear that the derivative df^n contracts exponentially along this graph, more precisely $\left\|d(f^n|_{\mathcal{F}_{-n}(p^o)})\right\| \lesssim b^n$. Indeed, the tangent vectors to $\mathcal{F}_{-n}(p^o)$ remain in a cone field close to the vertical under iteration, and the second factor gets contracted at rate b.

From now on the parameter u is fixed. To simplify notation we drop the subscript u and write $f_u^j = (f_1^j, f_2^j)$ We will prove at the same time that f_u^n is defined in a fixed domain around p^ι and estimate its derivatives. Let $r > 0$ be such that all iterates f^j, $1 \leq j \leq n$ are well defined on $D(z^\iota, r) \times \{w^\iota\}$, and for all $j \leq n$, $\left\| f^j(p^\iota) - f^j(z, w^\iota)\right\|$ is bounded by, say, 1. For the moment r depends on n.

The estimate we need is contained in the following lemma. Denote $f^j(z, w^\iota)$ by (z_j, w_j^ι).

Lemma 8.11 *For r as above, let $K = 2(1-b)^{-1}M$. Then for every $z \in D(z^\iota, r)$, and every $1 \leq j \leq n$*

$$\frac{\partial f_1^j}{\partial z}(z, w^\iota) = (1+u)^j \prod_{i=1}^{j}(1+\delta_i), \text{ with } |\delta_i| \leq \frac{K}{|z_{i-1}|^{1+\gamma}},$$
$$\text{and } \left|\frac{\partial f_2^j}{\partial z}(z, w^\iota)\right| \leq \frac{K}{|z_{j-1}|^{1+\gamma}}$$

As a preliminary observation, notice that if $R \geq (10^6 k(1-b)^{-1}M)^k$, and δ_i is as in the statement of the lemma and n is large enough, then for every $1 \leq j \leq n$,

$$\left|(1+u)^j\prod_{i=1}^{j}(1+\delta_i)-1\right|\leq\frac{1}{5}. \tag{25}$$

Indeed, this follows from the proof of Theorem 8.3 (see (17) and (18) there; also if $j \geq l$ we need to split the product at l and to estimate separately the two terms).

Proof We argue by induction on j. The result holds true for $j=1$. So assume that it holds for some j. We compute

$$\begin{aligned}\frac{\partial f_1^{j+1}}{\partial z}(z,w^\iota) &= \left((1+u)+\frac{\partial\eta_u}{\partial z}(z_j,w_j^\iota)\right)\frac{\partial f_1^j}{\partial z}(z,w^\iota)\\ &\quad+\frac{\partial f_2^j}{\partial z}(z,w^\iota)\frac{\partial\eta_u}{\partial w}(z_j,w_j^\iota)\\ &=(1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)\left(1+\frac{1}{1+u}\frac{\partial\eta_u}{\partial z}(z_j,w_j^\iota)\right)\\ &\quad+\frac{\partial f_2^j}{\partial z}(z,w^\iota)\frac{\partial\eta_u}{\partial w}(z_j,w_j^\iota).\end{aligned}$$

By the induction hypothesis,

$$\left|\frac{\partial f_2^j}{\partial z}(z,w^\iota)\frac{\partial\eta_u}{\partial w}(z_j,w_j^\iota)\right|\leq\frac{K}{|z_{j-1}|^{1+\gamma}}\frac{M}{|z_j|^\gamma}.$$

Since $\frac{z_{j+1}}{z_j}$ is close to $1+u$ and $(1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)$ is close to 1, we can write

$$\frac{\partial f_1^{j+1}}{\partial z}(z,w^\iota)=(1+u)^{j+1}\prod_{i=1}^{j}(1+\delta_i)\left(1+\frac{1}{1+u}\frac{\partial\eta_u}{\partial z}(z_j,w_j^\iota)+\delta\right),$$
$$\text{with } |\delta|\leq\frac{2KM}{|z_j|^{1+2\gamma}}.$$

Thus if we put $\delta_{j+1}=\frac{1}{1+u}\frac{\partial\eta_u}{\partial z}(z_j,w_j^\iota)+\delta$ we get that

$$|\delta_{j+1}|\leq\left(\frac{M}{|1+u|}+\frac{2KM}{R^\gamma}\right)\frac{1}{|z_j|^{1+\gamma}},$$

which, from the choice of R and K is not greater than $\frac{K}{|z_j|^{1+\gamma}}$.

To get the bound on the derivative of f_2^{j+1}, we write

$$\frac{\partial f_2^{j+1}}{\partial z}(z, w^\iota) = b_u \frac{\partial f_2^j}{\partial z}(z, w^\iota) + \frac{\partial f_1^j}{\partial z}(z, w^\iota)\frac{\partial \theta_u}{\partial z}(z_j, w_j^\iota) + \frac{\partial f_2^j}{\partial z}(z, w^\iota)\frac{\partial \theta_u}{\partial w}(z_j, w_j^\iota),$$

and we get that

$$\left|\frac{\partial f_2^{j+1}}{\partial z}(z, w^\iota)\right| \leq b\frac{K}{|z_{j-1}|^{1+\gamma}} + \frac{6}{5}\frac{M}{|z_j|^{1+\gamma}} + \frac{K}{|z_{j-1}|^{1+\gamma}}\frac{M}{|z_j|^\gamma}$$
$$\leq \frac{K}{|z_j|^{1+\gamma}}\left(b\left|\frac{z_j}{z_{j-1}}\right|^{1+\gamma} + \frac{6M}{5K} + \frac{M}{R^\gamma}\left|\frac{z_j}{z_{j-1}}\right|^{1+\gamma}\right).$$

To conclude, we observe that when n is large enough, due to the choice of R and K, the expression within parentheses is smaller than 1. The proof of the lemma is complete. □

We are now in position to conclude the proof of Theorem 8.7. Let r_0 be the supremum of the radii $r > 0$ such that f^j is well-defined, and $f^j(z, w^\iota)$ stays at distance at most 1 from $f^j(z^\iota, w^\iota)$ for $1 \leq j \leq n$. By the above lemma and (25), $r_0 \geq \frac{2}{3}$. Then the image of $D(z^i, r_0) \times \{w^\iota\}$ under f^n is a graph over some neighborhood of z^o, which by Lemma 8.6 must contain $D(z^o, \frac{1}{2})$. Now since the repelling petal Σ is a graph (relative to the first coordinate) over $\{z,\ \mathrm{Re}(z) > R\}$, we infer that for $p \in B(p^o, \frac{1}{4}) \cap \Sigma$, $f^n(D(z^i, r_0) \times \{w^\iota\})$ intersects $\mathcal{F}(p)$ close to p. Therefore we can pull back $\mathcal{F}(p)$ under f^n to get a vertical graph intersecting $D(z^i, r_0) \times \{w^\iota\}$ along which (for the same reasons as before) the derivative of f^n along is smaller than b^n, and the proof is complete in the case where $\mathcal{F}$ is a foliation by vertical graphs in $\Omega_R \times D$.

What remains to be done is to remove the simplifying assumption on $\mathcal{F}$. For this we simply iterate backwards and use the previous analysis to show that for k large enough, $f^{-k}(\mathcal{F})|_{\Omega_R \times D}$ is made of vertical graphs of slope $\leq 1/100$. Indeed, let Δ be a germ of a holomorphic disk transverse to Σ at $p^o = (z_0, w_0) \in \Omega_R \times D$. Let $f_0^{-k}(p) = (z_{-k}, w_{-k})$ which (in our coordinates) converges to infinity by staying in $\Omega_R \times D$. Applying the reasoning of Lemma 8.11 for $u = 0$ (together with (25)) shows that for every $w \in D$, $f^k(D(z_{-k}, \frac{2}{3}) \times \{w\})$ is a horizontal graph over $D(z_0, \frac{1}{2})$, which is exponentially close to Σ due to vertical contraction. Thus when k is large enough, it intersects Δ exactly in one point, and the result follows. □

9 Proof of the Main Theorem on Homoclinic Tangencies

9.1 Creating tangencies between horizontal and vertical moving curves

Here we explain how to obtain a tangency between two holomorphically moving complex curves by using only "soft complex analysis", i.e. basically the Argument Principle. We work in the unit bidisk $\mathbb{B} = \mathbb{D}^2$. A subvariety V in $\mathbb{B}$ (or current, etc.) is *horizontal* if there exists some $\varepsilon > 0$ such that $V \subset \mathbb{D} \times \mathbb{D}_{1-\varepsilon}$. Vertical objects are defined similarly.

Following [29], we define the *horizontal (resp. vertical) Poincaré cone field* as the set of tangent vectors $v = (v_1, v_2) \in T_x\mathbb{B} \simeq \mathbb{C}^2$ such that $|v_1|_{\text{Poin}} > |v_2|_{\text{Poin}}$ (resp. $|v_2|_{\text{Poin}} > |v_2|_{\text{Poin}}$), where $|\cdot|_{\text{Poin}}$ denotes the Poincaré metric in $\mathbb{D}$. The contraction property of the Poincaré metric implies that if Γ is a horizontal graph in $\mathbb{B}$, then for every $x \in \Gamma$, $T_x\Gamma$ is contained in the horizontal Poincaré cone field.

A horizontal manifold (or subvariety) V in $\mathbb{B}$ has a *degree*, which is the degree of the branched cover $\pi_1 : V \to \mathbb{D}$ (here of course π_1 is the first coordinate projection). If V is irreducible and $d > 1$ then $\pi_1|_V$ must have critical points (indeed otherwise it would be a non-trivial covering of the unit disk). In particular, it admits tangent vectors in the vertical Poincaré cone field.

By definition a holomorphic family of submanifolds $(V_\lambda)_{\lambda\in\Lambda}$ of a complex manifold M is the data of a codimension 1 analytic set (which might be singular) $\widehat{V} \subset \Lambda \times M$ such that for every $\lambda \in \Lambda$, $V_\lambda = \widehat{V} \cap (\{\lambda\} \times \mathbb{B})$.

Here is the precise statement.

Proposition 9.1 *Let $(V_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of horizontal submanifolds of degree k in $\mathbb{B}$, parameterized by a connected Stein manifold Λ. We assume that:*

(i) *There exists a compact subset $\Lambda_0 \Subset \Lambda$ such that if $\lambda \notin \Lambda_0$, V_λ is the union of k graphs.*
(ii) *There exists $\lambda_0 \in \Lambda$ such that V_{λ_0} is not a union of graphs.*

Then, if $(W_\lambda)_{\lambda\in\Lambda}$ is any holomorphic family of vertical graphs in $\mathbb{B}$, there exists $\lambda_1 \in \Lambda$ such that V_{λ_1} and W_{λ_1} admit a point of tangency.

Using the above remarks, we see that condition (ii) could be replaced by "there exists $x \in \mathbb{B}$ and $\lambda_0 \in \Lambda$ such that $T_x V_{\lambda_0}$ is contained in the vertical Poincaré cone field". Notice also that (ii) implies that $k > 1$ in (i).

Proof To simplify the exposition, we assume that Λ is the unit disk (we will use the result in that case only). The proof in the general case is similar.

Notice first that if λ is close to $\partial\Lambda$, then there are no tangencies between V_λ and W_λ. Indeed the tangent vectors to V_λ and W_λ belong to disjoint cone fields. In particular, reducing Λ a little bit if needed, we may assume that the V_λ (resp. W_λ) are uniformly horizontal (resp. vertical), that is, that they are contained in $\mathbb{D} \times \mathbb{D}_{1-\varepsilon}$ (resp. $\mathbb{D}_{1-\varepsilon} \times \mathbb{D}$) for some fixed $\varepsilon > 0$.

If $V \subset \mathbb{B}$ is a smooth holomorphic curve, we let $\mathbb{P}TV$ be its lift (which is also a holomorphic curve) to the projectivized tangent bundle $\mathbb{P}T\mathbb{B} \simeq \mathbb{B} \times \mathbb{P}^1$. Notice that since V is smooth, $\mathbb{P}TV$ intersects every $\mathbb{P}^1$ fiber at a single point. If $(V_\lambda)_{\lambda\in\Lambda}$ is a holomorphic family of submanifolds, we obtain in this way a holomorphic family of submanifolds $(\mathbb{P}TV_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{B} \times \mathbb{P}^1$. In other words, there exists a subvariety of $\Lambda \times \mathbb{B} \times \mathbb{P}^1$, of dimension 2, which we denote $\widehat{\mathbb{P}TV}$ such that for every $\lambda \in \Lambda$,

$$\widehat{\mathbb{P}TV} \cap (\{\lambda\} \times \mathbb{B} \times \mathbb{P}^1) = \mathbb{P}TV_\lambda.$$

Let now $W = (W_\lambda)_{\lambda\in\Lambda}$ be any holomorphic family of vertical graphs in $\mathbb{B}$. An intersection point between $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$ corresponds to a parameter λ_0 at which V_{λ_0} and W_{λ_0} are tangent. We claim that then $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW}$ has dimension 0 (if non-empty). In particular, the varieties $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$ intersect properly in $\mathbb{B} \times \mathbb{P}^1 \times \Lambda$. Observe first that this intersection is compactly supported in $\Lambda \times \mathbb{B} \times \mathbb{P}^1$, indeed:

- as observed above, there are no tangencies between V_λ and W_λ when λ is close to $\partial\Lambda$;
- the intersection points between V_λ and W_λ are contained in $\mathbb{D}^2_{1-\varepsilon}$ for some $\varepsilon > 0$.

By the Maximum Principle, the projection of $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW}$ to $\Lambda \times \mathbb{B}$ is a finite set. Hence any component of $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW}$ of positive dimension is contained in a $\mathbb{P}^1$ fiber, which is impossible by definition of the lifts $\widehat{\mathbb{P}TV}$ and $\widehat{\mathbb{P}TW}$. This proves our claim.

By assumption, there exists λ_0 such that V_{λ_0} admits a vertical tangent vector, hence a tangency with some vertical line L. Let $\widehat{\mathbb{P}TL} \subset \Lambda \times \mathbb{B} \times \mathbb{P}^1$ be the surface corresponding to the trivial family where L is fixed. Then $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TL}$ is non-empty, therefore it is a finite set.

We can now deform L to W through some holomorphic family $(W_{\lambda,s})$ of vertical graphs with $W_{\lambda,0} = L$ and $W_{\lambda,1} = W_\lambda$, and s ranges in some neighborhood $\mathbb{D}_{1+\varepsilon}$ of the closed unit disk. For this we can simply take a linear interpolation. In this way we obtain a holomorphic deformation $(\widehat{\mathbb{P}TW_s})_{s\in\overline{\mathbb{D}}_\varepsilon}$ from $\widehat{\mathbb{P}TL}$ to $\widehat{\mathbb{P}TW}$, parameterized by a neighborhood of the unit disk.

To conclude that $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW} \neq \emptyset$, we argue that the set of parameters $s \in \mathbb{D}_{1+\varepsilon}$ such that $\widehat{\mathbb{P}TV} \cap \widehat{\mathbb{P}TW_s} \neq \emptyset$ is open in $\mathbb{D}_{1+\varepsilon}$ by the continuity of the intersection index of properly intersecting analytic sets of complementary dimensions (see [15, Prop. 2 p.141]) and closed because intersection points stay compactly contained in $\mathbb{B}$. This completes the proof. □

9.2 From critical points to tangencies

In this section we settle the second main step of Theorem A′. If p_λ is a holomorphically varying periodic point for a holomorphic family $(f_\lambda)_{\lambda\in\Lambda}$ of dissipative polynomial automorphisms of $\mathbb{C}^2$, we say that p_λ *undergoes a non-degenerate semi-parabolic bifurcation at* λ_0 if one of the multipliers ρ_λ of p_λ satisfies $\rho_{\lambda_0} = 1$ and $\frac{\partial \rho}{\partial \lambda}|_{\lambda=\lambda_0}$ is a submersion $\Lambda \to \mathbb{C}$.

Proposition 9.2 *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of dissipative polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree d with positive entropy, parameterized by the unit disk. Assume that:*

- *there exists a holomorphically varying periodic point p_λ which admits a non-degenerate semi-parabolic bifurcation at λ_0;*
- *for $\lambda = \lambda_0$, there is a critical point in one of the basins of attraction of p_{λ_0}.*

Then λ_0 can be approximated by parameters possessing non-persistent homoclinic tangencies.

Proof Without loss of generality we may assume that Λ is the unit disk, $\lambda_0 = 0$, and p_λ is fixed. Normalize the situation so that f_λ is locally of the form (19). Conjugating by a rotation, we may assume that the critical point lies in the basin $\mathcal{B}$ corresponding to the attracting direction $\{(x, 0),\ \arg(x) = -\frac{\pi}{k}\}$. Let q_0 be a saddle point such that $W^u(q_0)$ admits a point of tangency with the strong stable foliation in $\mathcal{B}$. (Then, by the hyperbolic λ-lemma (see e.g. [49, Thm 2, p. 155]) and the fact that any pair of saddle points have transverse heteroclinic intersections, any saddle point would do.) Let t be such a point of tangency.

The global repelling petal Σ in the direction of $\{(x, 0), \arg(x) = 0\}$ is an immersed curve biholomorphic to $\mathbb{C}$ ([54], see Sect. 6). Hence, using the theory of laminar currents, exactly as in [7], it admits transversal intersections with $W^s(q_0)$. We fix a transverse intersection point $m \in \Sigma \cap W^s(q_0)$.

Fix a neighborhood W of 0 in Λ. Iterating t forward and m backward a few times, we may assume that both points are close to 0. Theorem 8.7 thus provides us with an integer N and a radius r, so that for $n \geq N$, a transition mapping $f^n_{\lambda_n}$ is defined from $D^2(t, r)$ to $D^2(m, r)$ such that $f^n_{\lambda_n}(t) = m$. Let Γ^u_λ be the component of $W^u_\lambda(q_\lambda) \cap D^2(t, r)$ containing t. Reducing W if necessary, Γ^u_λ can be followed holomorphically as $\lambda \in W$.

To make the situation visually clearer, we consider adapted coordinates close to t and m. These changes of coordinates have bounded derivatives. Abusing slightly, we declare that in the new coordinates, the neighborhoods remain of size r. Near t we choose (z^ι, w^ι) so that $t = (0,0)$, the strong stable foliation $\mathcal{F}^{ss}$ becomes the vertical foliation $\{z^\iota = C^{st}\}$ and for $\lambda \in W$, Γ^u_λ is a horizontal manifold in $D^2(t,r)$ of some degree $d \geq 2$, which is transverse to $\mathcal{F}^{ss}$ outside t. Near m we choose local coordinates (z^o, w^o) such that $m = (0,0)$ $\Sigma^0 = \{w^o = 0\}$, and the component of $W^s(q_\lambda)$ containing m is $\{z^o = 0\}$. Denote by $\mathcal{F}$ the vertical foliation in the target bidisk $D^2(m,r)$.

For $|z^o| \leq r$, let us consider the parameter $\lambda_n = \lambda_n(t, z^o)$ given by Theorem 8.7 such that the first coordinate of $f^n_{\lambda_n}(t)$ is z^o. For every such parameter, by Theorem 8.7 $f^n_{\lambda_n}$ realizes a crossed mapping of degree 1 [29] from $D^2(t,r)$ to $D(z^o, \frac{r}{2}) \times \mathbb{D}_r$. So when $|z^o| \leq \frac{r}{4}$ we get by restriction a crossed mapping from $D^2(t,r)$ to $\mathbb{D}_{\frac{r}{4}} \times \mathbb{D}_r$. In particular, we infer that for $|z^o| \leq \frac{r}{4}$, $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ is a horizontal submanifold of degree d in $\mathbb{D}_{\frac{r}{4}} \times \mathbb{D}_r$.

To conclude the argument, let us show that when z^o ranges in $\mathbb{D}_{\frac{r}{4}}$ and n is large, the family of curves $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ satisfies the assumptions of Proposition 9.1 in $\mathbb{D}_{\frac{r}{8}} \times \mathbb{D}_r$.

The first observation is that the preimage $\mathcal{F}^{-n}_{\lambda_n}$ of $\mathcal{F}$ under $f^n_{\lambda_n}$ converges to the strong stable foliation associated to f_0 in $D^2(t,r)$, uniformly in z^o. Indeed, we know that the leaves of $\mathcal{F}_{-n}$ are graphs with bounded geometry over some fixed direction, and the $f^n_{\lambda_n}$ contract exponentially along these graphs, with uniform bounds. So any cluster limit of $\mathcal{F}^{-n}$ must be $\mathcal{F}^{ss}(f_0)$, which proves our claim.

Now, when $|z^o| = \frac{r}{4}$, for every z such that $|z| < \frac{3r}{16}$, when n is large, for $\lambda_n = \lambda_n(t, z^o)$ $f^{-n}_{\lambda_n}(\{z\} \times \mathbb{D}_r)$ is close to a leaf of $\mathcal{F}^{ss}$ which intersects $\mathbb{D}_r \times \{0\}$ transversely at a distance $\geq \frac{r}{8}$ from t. Therefore, $f^{-n}_{\lambda_n}(\mathbb{D}_{\frac{3r}{16}} \times \mathbb{D}_r) \cap \Gamma^u_{\lambda_n}$ is the union of d graphs, over a disk of radius greater than $\frac{3r}{16} \cdot \frac{4}{5} = \frac{3r}{20}$ by Lemma 8.6. Pushing by $f^n_{\lambda_n}$ and applying the lemma again, we conclude that $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ is a union of d graphs over $\mathbb{D}_{\frac{3r}{25}}$.

On the contrary, when $z^o = 0$, for $\lambda_n = \lambda_n(t, 0)$, $\mathrm{pr}_1(f^n_{\lambda_n}) = 0$. Since $\mathcal{F}^{-n}$ converges to $\mathcal{F}^{\mathrm{ss}}$, when n is large $\Gamma^u_{\lambda_n}$ has a tangency with $\mathcal{F}^{-n}$ close to t, hence $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ has a vertical tangency close to m.

We see that the assumptions of Proposition 9.1 are satisfied so there exists a parameter $\lambda_n = \lambda_n(t, z^o)$ such that $f^n_{\lambda_n}(\Gamma^u_{\lambda_n})$ has a tangency with $W^s(q_{\lambda_n})$ close to m, and we are done. □

Proof of Theorem A′. Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of polynomial automorphisms of $\mathbb{C}^2$ of dynamical degree $d \geq 2$ with a bifurcation at λ_0. By Proposition 2.1 we may assume that the f_λ are products of Hénon mappings. Then close to λ_0 there is a periodic point with a multiplier ρ crossing the unit circle. Without loss of generality we may assume that $\dim(\Lambda) = 1$. Hence there exists λ_1 close to λ_0 such that at λ_1, the multiplier is a root of unity, different from 1, so that this periodic point p_λ can be followed holomorphically close to

λ_1. In addition we may assume that $\frac{\partial\rho}{\partial\lambda}|_{\lambda=\lambda_1} \neq 0$. Replacing f_λ by f_λ^k for some k, we may assume that p_λ is fixed and $\rho_{\lambda_1} = 1$ (we keep the same notation for the new multiplier, which is the k^{th} power of the previous one) . Notice that for the new multiplier we still have that $\frac{\partial\rho}{\partial\lambda}|_{\lambda=\lambda_1} \neq 0$.

Since the condition that $|\text{Jac } f_\lambda| < \deg(f_\lambda)^{-2}$ is preserved under iteration, Theorem B asserts that there is a critical point in every component of the attracting basin of p_{λ_1}. Thus the result follows from Proposition 9.2. □

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Appendix A. Attracting basins

The methods of Sect. 7.2 also give the existence of "critical points" in attracting basins, under certain minor hypotheses (that are needed to even define the critical points). Though these results are not used in the paper, they are interesting on their own right.

Let f be a polynomial automorphism of dynamical degree $d \geq 2$ with an attracting point p. As usual, we may assume that p is fixed, and we denote by $\mathcal{B}$ its basin of attraction. It is classical that there is a local holomorphic change of coordinates which puts f in a simple normal form (this result goes apparently back to Lattès [36]). Let κ_1 and κ_2 be the eigenvalues of Df_p, ordered so that $0 < |\kappa_2| \leq |\kappa_1| < 1$. We say that (κ_1, κ_2) is resonant if there exists an integer $i \geq 1$ such that $\kappa_2 = \kappa_1^i$ (notice that $i = 1$ is allowed). Then there exists a local change of coordinates near p such that in the new coordinates (z_1, z_2), f expresses as

$$f(z_1, z_2) = \begin{cases} (\kappa_1 z_1, \kappa_2 z_2) \text{ if } (\kappa_1, \kappa_2) \text{ is not resonant,} \\ (\kappa_1 z_1, \kappa_2 z_2 + \alpha z_1^i) \text{ otherwise, where } i \text{ is as above,} \\ \quad \text{and } \alpha \in \{0, 1\}. \end{cases}$$

In any case, we see that the vertical foliation $\{z_1 = C\}$ is invariant under f. If $|\kappa_2| < |\kappa_1|$ this is the "strong stable foliation", characterized by the property that points in the same leaf approach each other at the fastest possible rate κ_2^n. As before, it will be denoted by $\mathcal{F}^{ss}$. Using the dynamics, the coordinates (z_1, z_2) extend to the basin and define a biholomorphism $\mathcal{B} \simeq \mathbb{C}^2$. In the non-resonant (i.e. linearizable) case, the foliation $\{z_2 = C\}$ is invariant as well. We then simply refer to $\{z_1 = C\}$ and $\{z_2 = C\}$ as the invariant coordinate foliations in $\mathcal{B}$.

We give two statements on the existence of critical points. The first one parallels Theorem B

Theorem A.1 *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing an attracting point p, whose eigenvalues satisfy $0 < |\kappa_2| < |\kappa_1| < 1$, with basin of attraction $\mathcal{B}$. Assume that $|\text{Jac } f| < d^{-4}$, or more generally that the connected component of p in $W^{ss}(p) \cap J^-$ is $\{p\}$. Then for every saddle periodic point q, every component of $W^u(q) \cap \mathcal{B}$ contains a critical point, that is, a point of tangency with the strong stable foliation in $\mathcal{B}$.*

The second statement concerns the hyperbolic case.

Theorem A.2 *Let f be a polynomial automorphism of $\mathbb{C}^2$ of dynamical degree $d \geq 2$, possessing an attracting point p with basin $\mathcal{B}$. Assume that f is uniformly hyperbolic on J, and fix any saddle periodic point q.*

If the eigenvalues of p satisfy $|\kappa_2| < |\kappa_1|$ (resp. are non-resonant), then every component of $W^u(q) \cap \mathcal{B}$ admits a tangency with the strong stable foliation of $\mathcal{B}$ (resp. with both invariant coordinate foliations).

Here is an interesting geometric consequence. Recall that if f is dissipative and hyperbolic, J^+ is (uniquely) laminated by stable manifolds. Let us denote by $W^s(J)$ this lamination. It is natural to wonder whether the strong stable foliation in $\mathcal{B}$ matches continuously with the lamination of J^+ (recall that $\partial\mathcal{B} = J^+$). The existence of critical points implies that this is never the case (compare [4, Cor. A.2]).

Corollary A.3 *Let f be as in the previous theorem, in particular f is hyperbolic on J. Then if p is an attracting point with eigenvalues $|\kappa_2| < |\kappa_1|$ and basin $\mathcal{B}$, then for every $x \in J$, $W^s(J) \cup \mathcal{F}^{ss}(\mathcal{B})$ does not define a lamination near x. If p is linearizable, the same holds for both invariant coordinate foliations.*

Proof Let us deal with the case where $|\kappa_1| < |\kappa_2|$. It is enough to assume that x is a saddle periodic point. Hyperbolicity implies that $W^u(J)$ and $W^s(J)$ are transverse near x, so if $W^s(J) \cup \mathcal{F}^{ss}(\mathcal{B})$ is a lamination near x, $\mathcal{F}^{ss}(\mathcal{B})$ must be transverse to $W^u(J)$ near x. On the other hand, there exist critical points on $W^u(x)$ arbitrary close to x (obtained from the previous ones by iterating backwards). This contradiction finishes the proof. □

Proof of Theorems A.1 and A.2 This is very similar to Proposition 7.8 so the proof is merely sketched. Let us first deal with the case where $|\kappa_2| < |\kappa_1|$, with f hyperbolic or not. Let $\pi_1 : \mathcal{B} \to \mathbb{C}$ be the projection along the strong stable foliation. In the coordinates (z_1, z_2), it simply expresses as $(z_1, z_2) \mapsto z_1$. Assume by contradiction that there is no critical point in Ω. Then $\pi_1 \circ \psi^u$: $\Omega \backslash (\psi^u)^{-1}(W^{ss}(p)) \to \mathbb{C}^*$ is a locally univalent map. Since it cannot be a covering it must possess an asymptotic value, hence there is a diverging path γ in Ω such that the limit $\lim_{t\to\infty} \pi_1 \circ \psi^u(\gamma(t)) = \omega$ exists in $\mathbb{C}^*$. Let $\pi_2 : \mathcal{B} \to \mathbb{C}$ be the second coordinate projection. As before, we split the argument according to the bounded or unbounded character of $\pi_2 \circ \psi^u(\gamma)$.

If $\pi_2 \circ \psi^u(\gamma)$ is unbounded, we iterate forward and take cluster values to create an unbounded component C of $J^- \cap W^{ss}(p)$ containing p. Now if $|\mathrm{Jac}\ f| < d^{-4}$, then $|\kappa_2| < d^{-2}$, so by Corollary 7.7, the component of p in $W^{ss}(p) \cap J^-$ is a point, and we get a contradiction.

If f is hyperbolic we argue as follows: in $\mathcal{B}\backslash\{p\}$, J^- is laminated by unstable manifolds. In particular by [7, Lemma 6.4] the set of tangencies between $W^{ss}(p)$ and the unstable lamination is discrete. Pick $c \in C\backslash\{p\}$ such that $W^{ss}(p)$ and the unstable lamination are transverse near c. There exist coordinates (x, y) close to c in which $W^{ss}(p)$ is $\{x = 0\}$, $c = (0, 0)$, and the leaves of the unstable lamination close to c are horizontal in the unit bidisk $\mathbb{B}$. By construction, there is a sequence of integers n_j such that $f^{n_j}(\psi^u(\gamma))$ has a component C_j, vertically contained in $\mathbb{B}$, touching the boundary, and passing close to c. On the other hand C_j must be contained in a leaf of the unstable foliation so we get a contradiction.

If $\pi_2 \circ \psi^u(\gamma)$ is bounded, then as before the path γ must be unbounded in $W^u(q)$. Let E be the cluster set of $\psi^u(\gamma)$, which is a compact subset of the strong stable leaf $\{z_1 = \omega\}$. If $|\kappa_2| < |\kappa_1|$, then as in Proposition 7.8 we make a linear change of coordinates close to p such that in the new coordinates, f expresses as $f(x, y) = (\kappa_1 x, \kappa_2 y) + h.o.t.$ We see that in these coordinates, $\mathrm{pr}_1(f^n(E))$ is a set of diameter $\lesssim \kappa_2^n$ about $x_n \sim c\kappa_1^n$, which leads to a contradiction with Theorem 7.1, exactly as in Proposition 7.8.

It remains to treat the case where f is hyperbolic, p is linearizable, and we look for tangencies with any of the invariant coordinate foliations. We argue exactly as before, with (π_1, π_2) being the linearizing coordinate projections, in any order, and keep the same notation. The case where $\pi_2 \circ \psi^u(\gamma)$ is unbounded is dealt with exactly as above. If now $\pi_2 \circ \psi^u(\gamma)$ is bounded and E denotes its cluster set in the leaf $\{z_1 = \omega\}$, we observe that as in the unbounded case, the laminar structure of J^- outside p forces E to be reduced to a point. Therefore ψ^u admits an asymptotic value in $\mathcal{B}\backslash\{p\}$ and the contradiction arises by iterating and applying the ordinary Denjoy–Carleman–Ahlfors theorem. □

References

1. Beardon, A.F.: Iteration of Rational Functions. Complex Analytic Dynamical Systems. Graduate Texts in Mathematics, vol. 132. Springer, New York (1991)
2. Bedford, E., Smillie, J., Ueda, T.: Parabolic bifurcations in complex dimension 2 (2011, preprint)

3. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$. VI. J. Ann. Math. (2) **148**, 695–735 (1998)
4. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$. VII. Hyperbolicity and external rays. Ann. Sci. École Norm. Sup. (4) **32**, 455–497 (1999)
5. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity. Invent. Math. **103**, 69–99 (1991)
6. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$. III. Ergodicity, exponents and entropy of the equilibrium measure. Math. Ann. **294**, 395–420 (1992)
7. Bedford, E., Lyubich, M., Smillie, J.: Polynomial diffeomorphisms of $\mathbb{C}^2$. IV. The measure of maximal entropy and laminar currents. Invent. Math. **112**, 77–125 (1993)
8. Bedford, E., Lyubich, M., Smillie, J.: Distribution of periodic points of polynomial diffeomorphisms of $\mathbb{C}^2$. Invent. Math. **114**, 277–288 (1993)
9. Bers, L., Royden, H.L.: Holomorphic families of injections. Acta Math. **157**, 259–286 (1986)
10. Berteloot, F.: Méthodes de changement d'échelles en analyse complexe. Ann. Fac. Sci. Toulouse Math. (6) **15**, 427–483 (2006)
11. Buzzard, G.T., Hruska, S.L., Ilyashenko, Y.: Kupka-Smale theorem for polynomial automorphisms of $\mathbb{C}^2$ and persistence of heteroclinic intersections. Invent. Math. **161**, 45–89 (2005)
12. Buzzard, G.T.: Infinitely many periodic attractors for holomorphic maps of 2 variables. Ann. Math. (2) **145**, 389–417 (1997)
13. Buzzard, G.T., Verma, K.: Hyperbolic automorphisms and holomorphic motions in $\mathbb{C}^2$. Mich. Math. J. **49**, 541–565 (2001)
14. Cantat, S.: Croissance des variétés instables. Ergod. Theory Dyn. Syst. **23**(4), 1025–1042 (2003)
15. Chirka, E.M.: Complex Analytic Sets. Mathematics and its Applications (Soviet Series), vol. 46. Kluwer Academic Publishers Group, Dordrecht (1989)
16. Crovisier, S.: Perturbation de la dynamique de difféomorphismes en topologie C^1. Astérisque **354** (2013)
17. Douady A.: Does a Julia set depend continuously on a polynomial? In: Complex Dynamical Systems (Cincinnati, OH), pp. 91–138. Proceedings of Symposia Applied Mathematics, vol. 49. American Mathematical Society, Providence, RI (1994)
18. Douady, A., Sentenac, P.: Un tour de valse. In: Douady, A., Hubbard, J.H. (eds.) Étude dynamique des polynômes complexes, Partie II. Publications Mathématiques d'Orsay, Paris (1985)
19. Dujardin, R.: Some remarks on the connectivity of Julia sets for 2-dimensional diffeomorphisms. In: Devaney, R.L., Keen, L. (eds.) Complex Dynamics, pp. 63–84. Contemporary Mathematics, vol. 396. American Mathematical Society, Providence (2006)
20. Eremenko, A., Levin, G.: On periodic points of a polynomial. English translation: Ukr. Math. J. **41**, 1258–1262 (1989)
21. Fornæss, J.E., Sibony, N.: Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains. Duke Math. J. **65**, 345–380 (1992)
22. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. Ergod. Theory Dyn. Syst. **9**, 67–99 (1989)
23. Furter, J.-P.: Plane polynomial automorphisms of fixed multidegree. Math. Ann. **343**(4), 901–920 (2009)
24. Gavosto, E.A.: Attracting basins in $\mathbf{P}^2$. J. Geom. Anal. **8**, 433–440 (1998)
25. Goldberg, A.A., Ostrovskii, I.V.: Value Distribution of Meromorphic Functions. Translations of Mathematical Monographs, vol. 236. American Mathematical Society, Providence (2008)
26. Goldberg, L.R., Keen, L.: A finiteness theorem for a dynamical class of entire functions. Ergod. Theory Dyn. Syst. **6**(2), 183–192 (1986)

27. Graczyk, J., Swiatek, G.: Generic hyperbolicity in the logistic family. Ann. Math. **146**, 1–52 (1997)
28. Hakim, M.: Attracting domains for semi-attractive transformations of $\mathbf{C}^p$. Publ. Mat. **38**(2), 479–499 (1994)
29. Hubbard, J.H., Oberste-Vorth, R.W.: Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. In: Branner, B., Hjorth, P. (eds.) Real and Complex Dynamical Systems, pp. 89–132. NATO Advanced Science Institutes Series C Mathematical and Physical Sciences, vol. 464. Kluwer Academic Publications, Dordrecht (1995)
30. Jin, T.: Unstable manifolds and the Yoccoz inequality for complex Hénon mappings. Indiana Univ. Math. J. **52**, 727–751 (2003)
31. Jonsson, M.: Holomorphic motions of hyperbolic sets. Mich. Math. J. **45**, 409–415 (1998)
32. Kleinbock, D., Margulis, G.A.: Flows on homogeneous spaces and Diophantine approximation on manifolds. Ann. Math. (2) **148**, 339–360 (1998)
33. Kobayashi, S.: Hyperbolic Complex Spaces. Grundlehren der Mathematischen Wissenschaften, vol. 318. Springer, Berlin (1998)
34. Kozlovski, O., Shen, W., van Strien, S.: Density of hyperbolicity in dimension one. Ann. Math. **166**, 145–182 (2007)
35. Langley, J.K.: Postgraduate notes on complex analysis. http://www.maths.nottingham.ac.uk/personal/jkl/pg1
36. Lattès, S.: Sur les formes réduites des transformations ponctuelles dans le domaine d'un point double. Bull. Soc. Math. France **39**, 309–345 (1911)
37. Lavaurs, P.: Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques. Thèse, Université Paris-Sud (1989)
38. Levin, G.M.: Irregular values of the parameter of a family of polynomial mappings. Russ. Math. Surv. **36**(6), 189–190 (1981)
39. Lyubich, M., Peters, H.: Classification of invariant Fatou components for dissipative Hénon maps. Geom. Func. Anal. **24**, 887–915 (2014)
40. Lyubich, M.: An analysis of stability of the dynamics of rational functions. Teoriya Funk., Funk. Anal. & Prilozh., vol. 42, pp. 72–91 (1984, Russian) (English translation: Sel. Math. Sov. **9**, 69–90 (1990))
41. Lyubich, M.: Some typical properties of the dynamics of rational mappings. Russ. Math. Surv. **38**(5), 154–155 (1983)
42. Lyubich, M.: Dynamics of quadratic polynomials, I-II. Acta Math. **178**, 185–297 (1997)
43. Mañé, R., Sad, P., Sullivan, D.: On the dynamics of rational maps. Ann. Sci. École Norm. Sup. **16**, 193–217 (1983)
44. Mummert, P.P.: The solenoid and holomorphic motions for Hénon maps. Indiana Univ. Math. J. **56**, 2739–2761 (2007)
45. Nevalinna, R.: Analytic Functions. Grundlehren der Mathematischen Wissenschaften, vol. 162. Springer, New York (1970)
46. Newhouse, S.: Non-density of axiom $A(a)$ on S^2. In: Global Analysis. In: Proceedings of the Symposium on Pure Mathematics, vol. XIV, pp. 191–203. American Mathematical Society, Providence (1970)
47. Newhouse, S.: Diffeomorphisms with infinitely many sinks. Topology **13**, 9–18 (1974)
48. Palis, J.: A global view of dynamics and a Conjecture of the denseness of finitude of attractors. In: Asterisque volume in honor of Douady's 60th birthday. Géométrie complexe et systémes dynamiques, vol. 261, pp. 335–348 (2000)
49. Palis, J., Takens, F.: Hyperbolicity and Sensitive-Chaotic Dynamics at Homoclinic Bifurcations. Cambridge University Press, London (1993)
50. Pujals, E.R., Sambarino, M.: Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. Ann. Math. (2) **151**(3), 961–1023 (2000)

51. Schmidt, W.M.: Metrische Sätze über simultane Approximation abhängiger Grössen. Monatsh. Math. **68**, 154–166 (1964)
52. Shishikura, M.: Bifurcation of parabolic fixed points. In: Tan, L. (ed.) The Mandelbrot Set, Theme and Variations, pp. 325–363. London Mathematical Society Lecture Note Series, vol. 274. Cambridge University Press, Cambridge (2000)
53. Słodkowski, Z.: Approximation of analytic multifunctions. Proc. Am. Math. Soc. **105**(2), 387–396 (1989)
54. Ueda, T.: Local structure of analytic transformations of two complex variables. I. J. Math. Kyoto Univ. **26**(2), 233–261 (1986)
55. Ueda, T.: Local structure of analytic transformations of two complex variables. II. J. Math. Kyoto Univ. **31**(3), 695–711 (1991)
56. Xie, J.: Periodic points of birational maps on projective surfaces (2011, preprint). ath.AG:1106.1825
57. Zalcman, L.: Normal families: new perspectives. Bull. Am. Math. Soc. (N.S.) **35**, 215–230 (1998)