Weak and strong fillability of higher dimensional contact manifolds

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Abstract For contact manifolds in dimension three, the notions of weak and strong symplectic fillability and tightness are all known to be inequivalent. We extend these facts to higher dimensions: in particular, we define a natural generalization of weak fillings and prove that it is indeed weaker (at least in dimension five), while also being obstructed by all known manifestations of "overtwistedness". We also find the first examples of contact manifolds in all dimensions that are not symplectically fillable but also cannot be called overtwisted in any reasonable sense. These depend on a higher dimensional analogue of Giroux torsion, which we define via the existence in all dimensions of exact symplectic manifolds with disconnected contact boundary.

1 Introduction

Contact structures in dimension $2n - 1$ arise naturally from symplectic structures in dimension $2n$ by considering symplectic manifolds with a convexity

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condition at the boundary. It has been understood since the work of Gromov [34] and Eliashberg [14] in the late 1980's that not every contact structure arises in this way, i.e. not all contact structures are *symplectically fillable*. Moreover, in dimension three, there are distinct notions of *strong* and *weak* fillability, and they are both closely related to the deep dichotomy between *tightness* and *overtwistedness*, which plays a crucial role in the problem of classifying contact structures. One has in particular the following inclusions among classes of contact structures on 3-manifolds:

$$\{\text{strongly fillable}\} \subset \{\text{weakly fillable}\} \subset \{\text{tight}\}.$$

Both of these are proper inclusions: in the first case this was shown by Eliashberg [16], and in the second by Etnyre and Honda [20], though today a simple alternative construction is also available using the notion of *Giroux torsion*. This invariant, introduced by Giroux in [31], measures the amount that a contact structure "twists" in neighborhoods of certain embedded 2-tori; it does not imply overtwistedness but does obstruct strong [22] and sometimes also weak [26] fillability. It also plays a key role in several classification results for tight contact structures, such as the "coarse" classification due to Colin, Giroux and Honda [10].

Most of the above discussion only makes sense so far in dimension three. This is partly because it is not known whether the tight/overtwisted dichotomy extends to higher dimensions, although recent work of the second author and others (e.g. [7, 47]) has revealed hints of "overtwistedness" in certain classes of examples. It also has not been clear up to now whether the notions of weak filling and Giroux torsion have any interesting higher dimensional counterparts. One of our main goals in this paper is to answer the latter question in the affirmative: we will show that several well known three-dimensional phenomena, such as the existence of tight but non-fillable or weakly but not strongly fillable contact manifolds, also occur in higher dimensions.

Let us begin the discussion with the phenomenon of contact structures that are tight but not (strongly) fillable. The emblematic example is the family of contact structures on $\mathbb{T}^3$ defined for $k \in \mathbb{N}$ by

$$\xi_k := \ker(\cos ks\, d\theta + \sin ks\, dt),$$

where we define $\mathbb{T}^3$ as $(\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})^2$ with coordinates (s, t, θ). These contact structures are all tight due to Bennequin's theorem [2], since they are covered by the standard contact structure on $\mathbb{R}^3$, but Eliashberg [16] showed that only ξ_1 has a strong symplectic filling. Despite this lack of fillability, they share other important properties that are incompatible with overtwistedness. For example, they are *hypertight*, i.e. they allow Reeb vector fields without contractible closed orbits, in contrast to Hofer's theorem [35] that such orbits

always exist in the overtwisted case. More importantly, they are not "flexible," meaning they are all homotopic as plane fields yet not isotopic [30], whereas overtwisted contact structures are maximally flexible due to Eliashberg's classification theorem [13].

In higher dimensions, it is an open question whether one can define a reasonable notion of tightness, but of course flexibility and contractible Reeb orbits are easy to define. Strong fillability can also be defined in the same way as in dimension three, by considering symplectic manifolds with convex boundary (see Definition 3 below). This allows us to compare the properties of the contact structures ξ_k on $\mathbb{T}^3$ discussed above with the following statement.

Theorem A *Identify the torus $\mathbb{T}^2$ with $(\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ with coordinates (s, t). In any odd dimension, there is a closed manifold M carrying two contact forms α_+ and α_- such that the formula*

$$\xi_k := \ker\left(\frac{1 + \cos ks}{2}\alpha_+ + \frac{1 - \cos ks}{2}\alpha_- + \sin ks \, dt\right)$$

for $k \in \mathbb{N}$ defines a family of contact structures on $\mathbb{T}^2 \times M$ with the following properties:

(1) *They all admit Reeb vector fields without contractible closed orbits.*
(2) *They are all homotopic as almost contact structures but not contactomorphic.*
(3) *$(\mathbb{T}^2 \times M, \xi_k)$ is strongly fillable only for $k = 1$.*

We recover the 3-dimensional case discussed above by taking $M = \mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ and $\alpha_\pm = \pm d\theta$ in the theorem.

The non-fillability of the above contact structures on $\mathbb{T}^3$ was later recognized to be a consequence of the positivity of their Giroux torsion, and we'd next like to generalize this fact. Let us briefly recall the definition of Giroux torsion, in language that is suitable for generalization to higher dimensions. Denote by (A, β) the cylinder $A := \mathbb{R} \times \mathbb{S}^1$ with coordinates (s, θ), together with the 1-form $\beta := s \, d\theta$, which makes it the completion of a Liouville domain (see Definition 3). The *contactization*[1] of (A, β) is the manifold $A \times \mathbb{S}^1 = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R} \times \mathbb{T}^2$ equipped with the contact structure $\ker(dt + \beta)$, where t denotes the coordinate on the new $\mathbb{S}^1$-factor. This contact structure is tangent to the $\mathbb{R}$-factor, and it makes a half twist as we move

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[1] Our use of the term "contactization" is slightly nonstandard, as the word is typically used in the literature to mean a product of a Liouville domain with $\mathbb{R}$ instead of with $\mathbb{S}^1$. In this paper, we shall go back and forth between both meanings of the term—it should always be clear from context which one is meant.

from $s = -\infty$ to $s = +\infty$. One can then compactify this domain by identifying it with the interior of $[0, \pi] \times \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates (s, t, θ) and contact structure

$$\ker(\cos s \, d\theta + \sin s \, dt).$$

This last contact manifold is called a *Giroux π-torsion* domain (or sometimes Giroux *half-torsion* domain). Such domains can be glued along boundary tori to achieve any number of half turns. The Giroux torsion of a contact 3-manifold (V, ξ) is defined to be the supremum of all integers n such that (V, ξ) contains $2n$ Giroux π-torsion domains glued together.

The idea described above can be conveniently rephrased in terms of *ideal Liouville domains*, a notion recently introduced by Giroux. We will review the precise definition in Sect. 5, but in a nutshell, an ideal Liouville domain is the compactification of a complete Liouville manifold that appears naturally e.g. as the closure of a page of a supporting open book decomposition, or more generally, the closure of any component of a ξ-convex hypersurface minus its dividing set. With this notion, a Giroux π-torsion domain can be viewed directly as the contactization of an ideal Liouville domain. In this paper, we shall refer to contactizations of ideal Liouville domains as *Giroux domains*. The fact that Giroux torsion is an obstruction to strong fillability [22] is then generalized to the following theorem.

Theorem B *If a contact manifold contains a connected codimension* 0 *submanifold with nonempty boundary obtained by gluing together two Giroux domains*, *then it is not strongly fillable*.

Observe that at least one of the Giroux domains in Theorem B must always have disconnected boundary. The existence of Liouville domains with disconnected boundary in dimensions four and higher is itself a nontrivial fact: the first examples were found by McDuff [43] in dimension four, and more were found by Geiges in dimensions four [24] and six [23], and Mitsumatsu [45] in dimension four. The following notion generalizes the construction of Geiges:

Definition 1 A *Liouville pair* on an oriented $(2n - 1)$-dimensional manifold M is a pair (α_+, α_-) of contact forms such that $\pm\alpha_\pm \wedge d\alpha_\pm^{n-1} > 0$, and the 1-form

$$\beta := e^{-s}\alpha_- + e^s\alpha_+$$

on $\mathbb{R} \times M$ satisfies $d\beta^n > 0$.

A Liouville pair allows us to construct Liouville domains with two boundary components (in fact, by attaching Stein 1-handles to these examples one

can obtain examples with any number of boundary components). These manifolds can then be used to build Giroux domains of the form $[0, \pi] \times \mathbb{S}^1 \times M$ with contact form

$$\lambda_{\mathrm{GT}} = \frac{1 + \cos s}{2} \alpha_+ + \frac{1 - \cos s}{2} \alpha_- + \sin s \, dt, \qquad (1.1)$$

which can be stacked together to produce the examples described in Theorem A.

In order to state an existence result[2] for Liouville pairs, recall that a *number field of degree n* is a field that is an *n*-dimensional vector space over $\mathbb{Q}$. Recall also that $\mathbb{R}$ contains number fields of arbitrary degree.

Theorem C *One can associate canonically to any number field* **k** *of degree n a* $(2n - 1)$*-dimensional closed contact manifold* $(M_{\mathbf{k}}, \xi_{\mathbf{k}})$. *If* **k** *can be embedded into* $\mathbb{R}$, *then* $M_{\mathbf{k}}$ *also admits a Liouville pair, hence* $\mathbb{R} \times M_{\mathbf{k}}$ *is Liouville.*

Corollary *There exist Liouville domains with disconnected boundary in all even dimensions.*

This corollary provides a source of examples[3] that can be plugged into Theorem B to construct nonfillable contact manifolds in all dimensions, and a special case of this leads to the examples of Theorem A as well as the higher dimensional version of Giroux torsion discussed in Sect. 9. The proof of Theorem B is in fact a generalization to higher dimensions of a construction that was used by the third author in [56] to show that every contact 3-manifold with Giroux torsion is weakly symplectically cobordant to one that is overtwisted. In higher dimensions, the overtwistedness will come from a generalization of the work of Atsuhide Mori in [46]. Note that already in dimension three, the cobordism argument requires the fact that overtwistedness obstructs *weak* (not only strong) fillability, a notion that has not previously been defined in any satisfactory way in higher dimensions. In dimension three of course, the subtle differences between weak and strong fillings are of interest in themselves, not only as a tool for understanding strong fillability.

As preparation for the definition of weak fillability that we will propose here, let us first have a look at the realm of (almost) complex manifolds.

Definition 2 One says that a contact manifold (V, ξ) is the *tamed pseudoconvex boundary* of an almost complex manifold (W, J) if $V = \partial W$ and

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[2]Our proof of Theorem C owes a considerable debt to Yves Benoist, who explained to us how to use number theory to find lattices in the groups considered by Geiges in [23].

[3]Actually this construction provides infinitely many examples with pairwise distinct fundamental groups. We thank Gaëtan Chenevier for arithmetic discussions clarifying this.

- ξ is the hyperplane field $TV \cap JTV$ of J-complex tangencies,
- W admits a symplectic form ω taming J, and
- V is J-convex.

The last point means that if we orient V as the boundary of W, then for any 1-form λ defining ξ (i.e. λ is a 1-form with $\xi = \ker \lambda$ as oriented hyperplanes), we have $d\lambda(v, Jv) > 0$ for every nonzero vector $v \in \xi$.

Note that there is no direct relation in the definition between the taming form ω and the contact structure ξ. It must also be pointed out that the existence of (W, J) is not very restrictive without the taming condition. For instance, the overtwisted contact structure on $\mathbb{S}^3$ that is homotopic to the standard contact structure can be realized as a pseudoconvex boundary of the ball for some almost complex structure, but the Eliashberg-Gromov theorem implies that this structure can never be tamed.

We now recall the standard definitions on the symplectic side.

Definition 3 Let V be a closed oriented manifold with a positive and co-oriented contact structure ξ. We say that a compact symplectic manifold (W, ω) is a *symplectic filling* of (V, ξ) if $\partial W = V$ as oriented manifolds and ω admits a primitive λ (a *Liouville form*) near ∂W which restricts to V as a contact form for ξ. We call (W, ω) an *exact filling* of (V, ξ), or a *Liouville domain*, if the Liouville form λ extends globally over W.

Note that a Liouville form λ gives rise (via the ω-dual) to a *Liouville vector field*, whose flow is a symplectic dilation, and the condition that $\lambda|_{TV}$ be a positive contact form means that the Liouville vector field points transversely outward at the boundary. For this reason we say in this case that (W, ω) has (symplectically) *convex* boundary. In dimension three, it is customary to distinguish this notion from the weaker version discussed below by calling (W, ω) a *strong filling* of (V, ξ), and we shall also apply this convention to higher dimensions in the present paper.

To obtain a weaker notion of symplectic filling, recall that every co-oriented contact structure ξ carries a natural conformal class CS_ξ of symplectic structures: indeed, if λ is any contact form for ξ, then $d\lambda|_\xi$ defines a symplectic bundle structure that is independent of the choice of λ up to scaling. If (W, ω) is a symplectic manifold and $V = \partial W$ carries a positive contact structure ξ, one says, following [17], that ω *dominates* ξ if the restriction $\omega_\xi := \omega|_\xi$ belongs to CS_ξ. This is always the case if (W, ω) is a strong filling of (V, ξ), and in dimension three it defines a strictly weaker notion of symplectic fillability, e.g. it is obstructed by overtwistedness but not by Giroux torsion. A symplectic 4-manifold (W, ω) dominating a contact structure ξ at its boundary $V = \partial W$ is therefore called a *weak filling* of (V, ξ). However,

McDuff proved [43, Lemma 2.1] that from dimension 5 upward, the dominating condition already implies that (W, ω) is a *strong* filling. In this paper, we propose the following weak filling condition for all dimensions.

Definition 4 Let ξ be a co-oriented contact structure on a manifold V. Denote by CS_ξ the canonical conformal class of symplectic structures on ξ. Let (W, ω) be a symplectic manifold with $\partial W = V$ as oriented manifolds and denote by ω_ξ the restriction of ω to ξ. We say that (W, ω) is a *weak filling* of (V, ξ) (and ω *weakly dominates* ξ) if ω_ξ is symplectic and $\omega_\xi + CS_\xi$ is a ray of symplectic structures on ξ.

The weak filling condition is thus equivalent to the requirement that

$$\alpha \wedge (d\alpha + \omega_\xi)^{n-1} \quad \text{and} \quad \alpha \wedge \omega_\xi^{n-1}$$

should be positive volume forms for every choice of contact form α for ξ. If one fixes a contact form α, then this is equivalent to requiring $\alpha \wedge (\omega_\xi + \tau \, d\alpha)^{n-1} > 0$ for all constants $\tau \geq 0$, and it holds for instance whenever

$$\alpha \wedge d\alpha^k \wedge \omega_\xi^{n-1-k} > 0$$

for all $k \in \{0, 1, \ldots, n-1\}$. In dimension three, weak domination is equivalent to domination, hence our definition of weak filling reduces to the standard one.

The first important result to state about this new definition is that it is the purely symplectic counterpart of tamed pseudoconvex boundaries.[4]

Theorem D *A symplectic manifold (W, ω) is a weak filling of a contact manifold (V, ξ) (Definition 4) if and only if it admits a smooth almost complex structure J that is tamed by ω and makes (V, ξ) the tamed pseudoconvex boundary of (W, J) (Definition 2).*

By contrast, weak fillings are not automatically strong fillings. Indeed, weak domination of a fixed ξ is an open condition on ω, so one can easily construct weak fillings that are non-exact at the boundary by taking small perturbations of strong fillings. The following less trivial examples of weak fillings non-exact at the boundary are inspired by Giroux's construction [29] of weak fillings for the tight contact structures ξ_k on $\mathbb{T}^3$.

Example 1.1 Starting from a closed contact manifold (V, ξ) and a supporting open book decomposition [32], Frédéric Bourgeois constructed in [4] a

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[4]We are deeply indebted to Bruno Sévennec and Jean-Claude Sikorav for discussions that led to the proof of Theorem D.

contact structure on $V \times \mathbb{T}^2$. It can be written as the kernel of the 1-form

$$\alpha_\varepsilon = \alpha_V + \varepsilon f \, dx_1 + \varepsilon g \, dx_2$$

for any $\varepsilon > 0$, where (x_1, x_2) are the coordinates on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, α_V is a contact form on V compatible with the given open book, and $f, g \colon V \to \mathbb{R}$ are functions associated to the open book. Now if (W, ω) is a weak filling of (V, ξ), one can check by examining the limit $\varepsilon \to 0$ that the Bourgeois contact structure on $V \times \mathbb{T}^2$ is weakly filled by $(W \times \mathbb{T}^2, \omega \oplus \omega_{\mathbb{T}^2})$, where $\omega_{\mathbb{T}^2}$ is an area form on $\mathbb{T}^2$.

The next result extends the fact that weak fillability is *strictly* weaker than strong fillability beyond dimension three. Though we prove this only for dimension five, it is presumably true in all dimensions; see Sect. 9 for further discussion.

Theorem E *There exist* 3-*manifolds M with Liouville pairs* (α_+, α_-) *such that the contact manifolds* $(\mathbb{T}^2 \times M, \xi_k)$ *of Theorem* A *are all weakly fillable. In particular, there exist contact* 5-*manifolds that are weakly but not strongly fillable.*

As in dimension three, one should expect that any notion of "overtwistedness" one might define in higher dimensions obstructs the existence of a weak filling. Here we have two possible notions in mind: recall first that the second author [47] has introduced a higher dimensional generalization of the overtwisted disk, called the *plastikstufe*. We shall introduce in Sect. 4 a natural generalization of this, called a *bordered Legendrian open book* (or "bLob" for short), and refer to contact manifolds that contain such objects as PS-*overtwisted*. An alternative (though not necessarily inequivalent) notion emerges from the observation that a contact 3-manifold is overtwisted if and only if it has a supporting open book that is the negative stabilization of another open book. The corresponding condition in higher dimensions is known to imply *algebraic* overtwistedness, i.e. vanishing contact homology [7]. We will show that each of these conditions gives an obstruction to semipositive[5] weak fillings:

Theorem F *If* (V, ξ) *is a closed contact manifold that either*

(i) *contains a contractible* PS-*overtwisted subdomain, or*

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[5]In Theorem F and several other results in this paper, we write the word "semipositive" in parentheses: this means that the condition is presently necessary for technical reasons, but should be removable in the future using the polyfold technology of Hofer-Wysocki-Zehnder, cf. [36].

(ii) *is obtained as the negative stabilization of an open book,*

then (V, ξ) *has no* (*semipositive*) *weak filling.*

   Hence any contact structure on a closed manifold V *with* $\dim V \geq 3$ *can be modified within its homotopy class of almost contact structures to one that admits no* (*semipositive*) *weak fillings.*

   We will also show in Sect. 2 that the weak filling condition is conveniently amenable to deformations near the boundary. An often used fact in dimension three, due originally to Eliashberg [15], is that any weak filling which is exact near the boundary can be deformed to a strong filling. This was extended in [50] to show that every weak filling can be deformed to make the boundary a *stable hypersurface*, so that weak fillings can be studied using the machinery of Symplectic Field Theory (SFT). Extending this idea to higher dimensions led to the notion of a *stable symplectic filling* defined in [39], and we will show:

Proposition 6 *Any weak filling can be deformed near its boundary to a stable filling. Moreover, if the symplectic form is exact near the boundary, then it can be deformed to a strong filling.*

   The fact that weak fillings can be "stabilized" means that they are obstructed by the invariants defined in [39], known as *algebraic torsion*. The following corollary, which we will not use in this paper, comes of course with the standard caveat about the analytical foundations of SFT:

Corollary 7 *If* (V, ξ) *has fully twisted algebraic torsion in the sense of* [39], *then it is not weakly fillable. In particular, this is the case if* (V, ξ) *has vanishing contact homology with fully twisted coefficients.*

   The contact structures defined in (1.1) can be used to define a higher dimensional version of the standard 3-dimensional Lutz twist along a pre-Lagrangian torus. Notably, whenever (V, ξ) contains a hypersurface H that is isomorphic to one of the boundary components of the domain $[0, 2\pi] \times \mathbb{S}^1 \times M$ with the contact structure given by λ_{GT}, we can cut V open along H and glue in an arbitrary number of such domains to modify the contact structure on V. The contact structure obtained from this operation will never be strongly fillable, and in some cases it is not even weakly fillable:

Theorem G *By inserting contact domains of the form* $([0, 2\pi k] \times \mathbb{S}^1 \times M, \ker \lambda_{GT})$ *for various* $k \in \mathbb{N}$, *one can construct closed manifolds in any dimension* $2n - 1 \geq 3$ *which admit infinite families of hypertight but not weakly fillable contact structures that are homotopic as almost contact structures but not contactomorphic.*

We will also discuss in Sect. 9 a "blown down" version of the above operation, which generalizes both the classical Lutz twist along transverse knots in dimension three and a 5-dimensional version recently introduced by A. Mori [46]. As we shall see, this operation always produces a contact structure that is in the same homotopy class of almost contact structures, but is PS-overtwisted and thus not weakly fillable. See also [21] for a completely different generalization of the Lutz twist to higher dimensions.

Organization Here is an outline of the remainder of the paper.

In Sect. 2 we establish some basic properties of the weak filling condition, including its relation to tamed pseudoconvexity and behavior under deformations in collar neighborhoods. This includes the proofs of Theorem D and Proposition 6.

Section 3 shows that weak fillings are obstructed by negatively stabilized open books. The technology here involves finite energy holomorphic planes in the noncompact completion of a weak filling; it is a minor adaptation of the contact homology computation due to Bourgeois and van Koert [7]. Instead of appealing to contact homology, however, we argue directly that the moduli space of holomorphic planes found in [7] cannot exist if there is a semipositive weak filling.

In Sect. 4, we introduce the bLob as a natural generalization of the plastik-stufe and adapt the standard "Bishop family of holomorphic disks" argument to prove the remainder of Theorem F.

The next three sections establish the proof of Theorem B, defining the first higher dimensional filling obstruction that is distinct from any notion of "overtwistedness". In Sect. 5, we discuss ideal Liouville domains and Giroux domains, and state a more precise version of Theorem B that can also be applied to weak fillings. The proof requires a surgery construction explained in Sect. 6, which is inspired by the construction in [56] of symplectic cobordisms from any contact 3-manifold with Giroux torsion to one that is overtwisted. In our case, we consider a contact manifold (V, ξ) which contains a region with nonempty boundary consisting of two Giroux domains $G_0 = \Sigma_0 \times \mathbb{S}^1$ and $G_1 = \Sigma_1 \times \mathbb{S}^1$ glued together. It turns out that one can attach along G_0 a symplectic "handle" of the form $\Sigma_0 \times \mathbb{D}^2$, the effect of which is to replace $G_0 \cup G_1$ with a region that is PS-overtwisted, thus a weak filling of (V, ξ) with suitable cohomological properties at the boundary gives rise to a larger weak filling of something PS-overtwisted and hence a contradiction. Note that since the new boundary is only weakly filled in general, the new notion of weak fillability plays a crucial role even just for proving that (V, ξ) is not *strongly* fillable. We shall also provide in Sect. 7 an alternative argument that avoids holomorphic disks and uses the somewhat simpler technology of closed holomorphic spheres; this allows us to overcome transversality problems using the recently developed polyfold machinery [37].

In Sect. 8 we switch gears and address the existence of Liouville pairs in all dimensions, proving Theorem C. For this we borrow an idea of Geiges from [23] to look for Liouville pairs among left-invariant 1-forms on noncompact Lie groups that admit co-compact lattices and hence compact quotients. Our examples of left-invariant Liouville pairs on Lie groups are quite easy to write down (see e.g. Eq. (8.1)), but in order to find co-compact lattices we'll need to apply some basic algebraic number theory.

Finally, Sect. 9 explains the most important special cases of the filling obstruction from Theorem B, leading to higher dimensional generalizations of Giroux torsion and the Lutz twist. From this follow the proofs of Theorems A, E and G.

The Appendix contains some technical results in symplectic linear algebra needed for the proof of Theorem D, relating weak symplectic fillings and tamed pseudoconvexity.

Notation   Unless otherwise indicated, throughout this paper we will assume (W, ω) is a compact symplectic manifold of dimension $2n \geq 4$, and (V, ξ) is a closed $(2n - 1)$-dimensional contact manifold, with ξ positive and co-oriented. In cases where V is identified with ∂W, we assume that this identification matches the orientation induced by ξ to the natural boundary orientation determined by ω. Also when $V = \partial W$, we will often use the abbreviations

$$\omega_V := \omega|_{TV} \quad \text{and} \quad \omega_\xi := \omega|_\xi.$$

2 The weak filling condition

2.1 Pseudoconvexity and weak filling

The aim of this section is to show that our definition of a weak filling (Definition 4 in the introduction) is in a certain sense the purely symplectic counterpart of a tamed almost complex manifold with pseudoconvex boundary.

Before proving the main theorem on this subject, we will need some important properties of complex structures on vector spaces which were explained to us by Bruno Sévennec and Jean-Claude Sikorav. We will give proofs of the following two propositions in Appendix A.1 and A.2 respectively.

Proposition 2.1 *The space of complex structures on a vector space E tamed by two given symplectic forms ω_0 and ω_1 is either empty or contractible.*

Proposition 2.2 *Let E be a real vector space equipped with two symplectic forms ω_0 and ω_1. The following properties are equivalent:*

1. *the linear segment between ω_0 and ω_1 consists of symplectic forms*

2. *the ray starting at ω_0 and directed by ω_1 consists of symplectic forms*
3. *there is a complex structure J on E tamed by both ω_0 and ω_1.*

Remark 2.3 When choosing an almost complex structure J on a symplectic manifold, for most applications it makes no difference whether one requires J to be *calibrated* (i.e. compatible with) or *tamed* by the symplectic structure, and typically very little attention is paid to this distinction in the literature. Note however that the *cotaming* condition is strictly weaker than *cocalibrating*, and in many cases it is not possible to require the latter. For instance, one can prove (by hand or using the previous proposition) that there exists a complex structure on $\mathbb{R}^4$ that is cotamed by the two forms $\omega_0 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ and $\omega_1 = dx_2 \wedge dx_1 + dx_3 \wedge dx_4$. On the other hand, one can use the fact that $\omega_0 \wedge \omega_1 = 0$ to show that there does not exist any complex structure that is both calibrated by ω_0 and tamed by ω_1.

The following is a restatement of Theorem D from the introduction.

Theorem 2.4 *A symplectic manifold (W, ω) is a weak filling of $(\partial W, \xi)$ if and only if there is an almost complex structure J on W which is tamed by ω and such that $(\partial W, \xi)$ is the strictly pseudoconvex boundary of (W, J).*

Proof We denote the boundary of W by V and use the notation of the introduction. Suppose we have a weak filling. From Proposition 2.2, using the fact that the cotaming property is open, it follows that every point in the manifold V has a small neighborhood on which there exists a complex structure J_ξ on ξ which is tamed by both ω_ξ and CS_ξ. Using the contractibility of the space of such J_ξ's (Proposition 2.1), we can then replace J_ξ with a global complex structure on ξ that has this property. Choose any vector field X on V that spans $\ker \omega_V$, and extend it to a collar neighborhood U of V. Let Y be a vector field on U that lies along V in the ω-orthogonal complement of ξ and that satisfies $\omega(X, Y) > 0$. We extend J_ξ to an almost complex structure J on U by setting $JX = Y$. Clearly, J is tamed by ω on a small neighborhood of V, and we can then extend J to the interior of W to obtain the desired tamed almost complex structure on the entire filling W. By construction, $\xi = TV \cap JTV$, and V is strictly J-pseudoconvex since J_ξ is tamed by CS_ξ.

Conversely, assume W has an almost complex structure J that is tamed by ω and makes the boundary strictly pseudoconvex, with ξ as the field of complex tangencies $TV \cap JTV$. We can then write ξ as the kernel of a nonvanishing 1-form α, and pseudoconvexity implies that we can choose the sign of α in such a way that $d\alpha|_\xi$ tames $J|_\xi$, and such that the natural orientation of ξ together with its co-orientation defined via α is compatible with the boundary orientation of W. Since ω tames J, ω_ξ also tames $J|_\xi$. We therefore have cotaming forms on ξ, so the easy implication (3) $\implies$ (2) of Proposition 2.2 guarantees that (W, ω) is a weak filling of (V, ξ). $\square$

Suppose now U is a domain inside a symplectic manifold (W, ω) and $V := \partial U$ is pseudoconvex for some tamed J. Using the easy direction of the preceding theorem, we see that $(U, \omega|_U)$ is a weak filling of $(V, TV \cap JTV)$. It is not true in general that it is a strong filling. This was observed first in [17, p. 158], where an example in $\mathbb{C}^n$ with its standard Kähler structure is discussed. In this example, Eliashberg proved that the relevant contact structure is actually Stein fillable, but weak fillability is much easier to check (recall that we used the easy direction). By Theorem F in the introduction, this already implies global information about the contact structure, such as the nonexistence of a contractible PS-overtwisted subdomain, or of a negatively stabilized supporting open book.

2.2 Magnetic collars and cones

Recall that for any co-oriented hyperplane field ξ on a manifold V, one can consider the annihilator of ξ in T^*V:

$$S\xi := \left\{ \lambda \in T^*V \mid \ker \lambda = \xi \text{ and } \lambda(v) > 0 \text{ if } v \text{ is positively transverse to } \xi \right\}.$$

The field ξ is a contact structure if and only if $S\xi$ is a symplectic submanifold of $(T^*V, \omega_{\mathrm{can}})$, and in this case $S\xi$ is called the *symplectization* of ξ. Any contact form α is a section of this $\mathbb{R}_+^*$-bundle, and thus determines a trivialization $S\xi \cong \mathbb{R}_+^* \times V$. In this trivialization, the restriction of the canonical symplectic form ω_{can} becomes $d(t\alpha)$, where t is the coordinate in $\mathbb{R}_+^*$.

In order to rephrase the definition of weak filling in these terms, we need to recall one further notion. Suppose ω_V is any closed 2-form on V, and denote the projection from T^*V to V by π. The 2-form $\omega_{\mathrm{can}} + \pi^* \omega_V$ is then a symplectic form on T^*V, which is called *magnetic*.

The definition of weak fillings can now be reformulated as follows.

Lemma 2.5 *Let (W, ω) be a symplectic manifold with $\partial W = V$. Denote by ω_V the restriction of ω to TV and by ω_ξ its restriction to a contact structure ξ on V. The manifold (W, ω) is a weak filling of (V, ξ) if and only if ω_ξ is symplectic and $S\xi$ is a symplectic submanifold of the magnetic cotangent bundle associated to ω_V.*

In the case where (W, ω) strongly fills (V, ξ), it admits a Liouville vector field X near V, which induces the contact form $\alpha = \iota_X \omega|_{TV}$ on V. Let φ_t denote the flow of X for time t. For sufficiently small $\varepsilon > 0$, the map $(t, m) \mapsto \varphi_{\ln t}(m)$ embeds $((1 - \varepsilon, 1] \times V, \omega_{\mathrm{can}})$ symplectically into W. This allows the completion of W by adding the positive half $(1, \infty) \times V$ of $S\xi$. To understand this from a magnetic point of view, observe that $\omega_V = d\alpha$, so the magnetic form on $S\xi$ is $\omega_V + d(t\alpha) = d((t + 1)\alpha)$. Thus $(t, m) \mapsto (t + 1, m)$ is a

symplectomorphism from the magnetic symplectization to the cylindrical end of the completed strong filling.

In the setting of weak fillings, we would similarly like to be able to complete (W, ω) by adding the magnetic symplectization. For this we need a suitable description of a collar neighborhood of the boundary: the following lemma has an obvious analogue for the situation where V is an oriented boundary component of a symplectic manifold (W, ω).

Lemma 2.6 *Suppose $V \subset W$ is an oriented hypersurface in the interior of a $2n$-dimensional symplectic manifold (W, ω), $\xi \subset TV$ is the co-oriented (and hence also oriented) hyperplane distribution induced by a nowhere vanishing 1-form λ on V, and the restriction of ω to ξ is symplectic and induces the positive orientation. Then a neighborhood of V in (W, ω) is symplectomorphic to*

$$\big((-\varepsilon, \varepsilon) \times V, \, d(t\lambda) + \omega_V\big),$$

for some $\varepsilon > 0$, where $\omega_V := \omega|_{TV}$, V is identified in the natural way with $\{0\} \times V$, and the direction of ∂_t is such that $\iota_{\partial_t}\omega^n = \lambda \wedge \omega^{n-1}$. Moreover, the vector field ∂_t in W can be chosen to extend any given vector field which has these properties on a neighborhood of some part of V.

Remark 2.7 The statement about the direction of ∂_t means that in the version of this lemma for the boundary of a weak filling, one obtains a neighborhood of the form $((-\varepsilon, 0] \times V, d(t\lambda) + \omega_V)$, so in particular ∂_t points outwards. There is a corresponding variation for negative boundary components of weak symplectic cobordisms, for which ∂_t points inwards.

Proof of Lemma 2.6 An identical proof has been given for the 3-dimensional case in [50]. We will first define a collar neighborhood of V by choosing a vector field that is transverse to V. Let $E \subset TW|_V$ be the ω-orthogonal complement of ξ along V. The intersection of E with TV is a 1-dimensional subbundle, and we can uniquely define a *Reeb-like* vector field X_ω by taking the section in $E \cap TV$ that satisfies $\lambda(X_\omega) \equiv 1$. By our definition, $\omega(X_\omega, \cdot)|_{TV} = 0$ holds. Choose now a second section Y in E that is transverse to V, and normalize it such that $\omega(Y, X_\omega) \equiv 1$. Note that if such a section is already given near some subset of V, then we can choose Y to be an extension of that section. We now have $\omega(Y, \cdot)|_{TV} = \lambda$, since both forms vanish on ξ and agree on X_ω.

Extend Y to a smooth vector field in a neighborhood of V, and use the flow φ^Y of this vector field to define a smooth diffeomorphism

$$\Phi \colon (-\varepsilon, \varepsilon) \times V \hookrightarrow W, \, (t, p) \mapsto \varphi_t^Y(p),$$

which agrees with the canonical identification on $\{0\} \times V$. Next, compare the 2-forms $\Phi^*\omega$ and $\omega_V + d(t\lambda)$ on $(-\varepsilon, \varepsilon) \times V$. Both forms coincide along $\{0\} \times V$, thus the linear interpolation of these forms is a path of symplectic structures (decreasing $\varepsilon > 0$ if necessary). We can then use the Moser trick to show that they are all symplectomorphic to each other (perhaps in a smaller neighborhood) by an isotopy that keeps the level set $\{0\} \times V$ fixed. $\qquad \square$

Corollary 2.8 *If (W, ω) is a weak filling of (V, ξ), then one can extend W to a magnetic completion $(\widehat{W}, \widehat{\omega})$ with $\widehat{W} = W \cup S\xi$, $\widehat{\omega}|_W = \omega$ and $\widehat{\omega}|_{S\xi} = \omega_{\mathrm{can}} + \omega_V$.*

Moreover, for every positive t, $(\{t\} \times V, \xi)$ is then weakly filled by $W \cup (0, t] \times V$ equipped with the restriction of $\widehat{\omega}$.

In the previous section we proved that whenever (W, ω) is a weak filling of (V, ξ), there is a J on ξ which is tamed by CS_ξ and also by the restriction of ω. However, it is sometimes desirable to fix a complex structure on ξ in advance. The following observation allows us to do this, at the price of first adding a sufficiently large part of the magnetic completion. The proof is a short computation using the fact that for $T \gg 0$, the restriction of $\omega_V + d(t\alpha)$ to $\{T\} \times V$ is dominated by the second term.

Lemma 2.9 *Suppose ω_V is a closed 2-form on V weakly dominating a contact structure ξ, α is a contact form for ξ and R_α is its Reeb vector field. Further, suppose J is an almost complex structure on $[0, \infty) \times V$ which preserves ξ such that $J|_\xi$ is tamed by $d\alpha|_\xi$ and $J\partial_t = R_\alpha$, with t denoting the coordinate on $[0, \infty)$. Then there exists a number $T \geq 0$ such that J is tamed by $\omega_V + d(t\alpha)$ on $[T, \infty) \times V$.*

2.3 Deformations of weak fillings

We now want to deform completions of weak fillings in order to obtain some flexibility for ω_V.

Lemma 2.10 *Let ω_V be a closed 2-form weakly dominating a contact structure $\xi = \ker \alpha$ on V, and suppose ω'_V is any closed 2-form on V that is cohomologous to ω_V. Then the symplectic structure $\omega_V + d(t\alpha)$ on $[0, \infty) \times V$ can be deformed away from $\{0\} \times V$ so that it coincides with $\omega'_V + d(t\alpha)$ on $(t_1, \infty) \times V$ for some large number $t_1 > 0$ and all levels $(\{t\} \times V, \xi)$ remain weakly filled.*

Proof Since ω_V and ω'_V are cohomologous, there exists a 1-form β on V such that $\omega'_V = \omega_V + d\beta$. Consider the closed 2-form $\omega' = d(t\alpha) + \omega_V + d(\rho\beta)$ on $[0, \infty) \times V$, where $\rho \colon [0, \infty) \to [0, 1]$ is a smooth monotone function

that is equal to 0 near $t = 0$ and to 1 for large values of t. We now show that, if the support of ρ is sufficiently far away from 0 and ρ increases sufficiently slowly, the new structure ω' will be symplectic. Since it is closed by construction, we only need to check nondegeneracy. We compute:

$$(\omega')^n = dt \wedge (\alpha + \rho'\beta) \wedge (t\, d\alpha + \omega_V + \rho\, d\beta)^{n-1}.$$

To prove that $t\, d\alpha + \omega_V + \rho\, d\beta$ is a symplectic form on $\ker(\alpha + \rho'\beta)$, choose an auxiliary norm on the space of differential forms on V, and set $c_1 := \|\beta\|$ and $c_2 := \|\omega_V\| + \|d\beta\|$.

The map $\Omega^1(V) \times \Omega^2(V) \to \Omega^{2n+1}(V)$, $(\gamma, \eta) \mapsto \gamma \wedge \eta^n$ is continuous, so that we find constants $\varepsilon_1, \varepsilon_2 > 0$ such that $\gamma \wedge \eta^n > 0$ for every pair $(\gamma, \eta) \in \Omega^1(V) \times \Omega^2(V)$ with $\|\gamma - \alpha\| < \varepsilon_1$, and $\|\eta - d\alpha\| < \varepsilon_2$. Then for $\eta = d\alpha + \omega_V/t + \rho\, d\beta/t$, we obtain $\|\eta - d\alpha\| = \|\omega_V/t + \rho\, d\beta/t\| \leq c_2/t$, and similarly, we find for $\gamma = \alpha + \rho'\beta$ that $\|\gamma - \alpha\| = \rho'c_1$.

The nondegeneracy of ω' is immediate whenever t lies outside the support of ρ. If we let ρ increase sufficiently slowly so that $\rho' < \varepsilon_1/c_1$ and also assume $\rho(t) = 0$ for $t < c_2/\varepsilon_2$, then the above calculation shows that ω' is nondegenerate everywhere. By the same reasoning, every hypersurface $(\{t\} \times V, \alpha)$ will be weakly filled in the new manifold. □

Remark 2.11 This lemma implies that a weak filling gives rise to a strong filling whenever $\omega|_{TV}$ is exact. This does not mean however that ω is a weak filling of a unique isotopy class of contact structures on the boundary—there are counter-examples in dimension 3. As explained for instance in [42, Sect. 4.2], any Seifert 3-manifold V is the boundary of a symplectic manifold (W, ω) such that $\ker\omega|_{TV}$ is tangent to the fibers. Thus any (positive) contact structure on V which is transverse to the fibers is weakly filled by (W, ω). If V is a Brieskorn sphere $-\Sigma(2, 3, 6n - 1)$, then the results of [27, 42] combine to prove that there are $n - 2$ isotopy classes of contact structures transverse to the fibers. Since those manifolds are homology spheres, $\omega|_{TV}$ is exact.

We now make the connection between weak fillings and stable hypersurfaces, establishing Proposition 6 and hence Corollary 7 from the introduction.

Corollary 2.12 *Any weak filling (W, ω) of a contact manifold (V, ξ) can be deformed to have the additional property that $\ker\omega_V = \ker d\alpha$ for some nondegenerate contact form α for ξ. In particular, (α, ω_V) is then a stable Hamiltonian structure on V, and (W, ω) is a stable filling of (V, ξ) in the sense of [39].*

Proof Since weak filling is an open condition, we can perturb ω so that without loss of generality it represents a rational cohomology class in $H^2_{\mathrm{dR}}(V)$.

Then by a result of Cieliebak and Volkov [9, Proposition 2.16], (V, ξ) admits a nondegenerate contact form α and a 2-form ω'_V cohomologous to ω_V such that the pair (α, ω'_V) define a stable Hamiltonian structure. The claim now follows by application of the preceding lemma. $\qquad\square$

3 Negative stabilizations

Corollary 7 in the introduction, together with the result of Bourgeois and van Koert [7] that negatively stabilized contact manifolds have vanishing contact homology (with full group ring coefficients), implies in principle that such manifolds are not weakly fillable and always admit contractible Reeb orbits. In this section we shall show how the computation from [7] can be modified to produce direct proofs of these facts without relying on SFT.

The simplest example of a negatively stabilized contact manifold is the sphere $(\mathbb{S}^{2n-1}, \xi_-)$ that is supported by the open book with page $T^*\mathbb{S}^{n-1}$ and monodromy isotopic to a single negative Dehn-Seidel twist. By an observation due to Giroux, we may for our purposes define an arbitrary closed $(2n-1)$-dimensional contact manifold to be *negatively stabilized* if and only if it is the contact connected sum of $(\mathbb{S}^{2n-1}, \xi_-)$ with some other closed contact manifold. Our goal is thus to prove the following:

Theorem 3.1 *For any closed $(2n-1)$-dimensional contact manifold (M, ξ), the contact connected sum $(M, \xi)\#(\mathbb{S}^{2n-1}, \xi_-)$ has no (semipositive) weak filling, and its Reeb vector fields always admit contractible closed orbits.*

To prepare the proof, recall that a 1-form λ and closed 2-form Ω on an oriented $(2n-1)$-dimensional manifold V form a *stable Hamiltonian structure* (λ, Ω) if $\lambda \wedge \Omega^{n-1} > 0$ and $\ker d\lambda \subset \ker \Omega$. Such a pair always determines a unique vector field R with the properties $\lambda(R) \equiv 1$ and $\Omega(R, \cdot) \equiv 0$. Note that if λ is also a contact form, then R is simply the Reeb vector field. We shall say that an almost complex structure J on $\mathbb{R} \times V$ is *adjusted to* (λ, Ω) if it is $\mathbb{R}$-invariant, maps the unit vector in the $\mathbb{R}$-direction to the vector field R, and restricts to an Ω-tame complex bundle structure on $\xi := \ker \lambda$.

Lemma 3.2 *Suppose (V, ξ) is a closed $(2n-1)$-dimensional contact manifold with nondegenerate contact form λ and closed 2-form Ω such that $\ker \lambda = \xi$ and (λ, Ω) forms a stable Hamiltonian structure on V. Suppose moreover that $\mathbb{R} \times V$ admits an almost complex structure J adjusted to (λ, Ω) with the following properties*:

- *There exists a finite energy J-holomorphic plane $u_0 : \mathbb{C} \to \mathbb{R} \times V$ which is Fredholm regular, has Fredholm index 1 and is asymptotic to a simply covered Reeb orbit γ.*

- *Other than $\mathbb{R}$-translations of u_0, $\mathbb{R} \times V$ admits no finite energy punctured J-holomorphic curves of genus zero with one positive end asymptotic to γ and no other positive ends.*

Then (V, ξ) does not admit any (semipositive) weak filling (W, ω) for which $\omega|_{TV}$ is cohomologous to Ω.

Proof Assume the contrary, that there exists a weak filling (W, ω) with $[\omega_V] = [\Omega] \in H^2_{\mathrm{dR}}(V)$. By Lemma 2.10, we can complete (W, ω) to an open symplectic manifold $(\widehat{W}, \omega)$ by attaching a cylindrical end $([0, \infty) \times V, \omega)$ such that for some $T > 0$, $\omega = \Omega + d(t\lambda)$ on $[T, \infty) \times V$. Assign to $(\widehat{W}, \omega)$ an ω-tame almost complex structure that matches the given $\mathbb{R}$-invariant structure J on $[T, \infty) \times M$ and is generic everywhere else; we shall denote this extension also by J. The point of assuming (λ, Ω) to be a stable Hamiltonian structure is that the compactness results of Symplectic Field Theory [8] are now valid for finite energy J-holomorphic curves in $(\widehat{W}, \omega)$.

The $\mathbb{R}$-translations of the J-holomorphic plane $u_0 \colon \mathbb{C} \to \mathbb{R} \times V$ asymptotic to the orbit γ now give rise to a smooth 1-dimensional family of J-holomorphic curves in $[T, \infty) \times V \subset \widehat{W}$. Let $\mathcal{M}$ denote the unique connected component of the moduli space of unparametrized finite energy J-holomorphic curves in $\widehat{W}$ that contains this family. All curves in $\mathcal{M}$ are planes asymptotic to the simply covered orbit γ and are thus somewhere injective. Let $\mathcal{M}_+ \subset \mathcal{M}$ denote the subset consisting of curves whose images are contained entirely in $[T, \infty) \times V$. By the uniqueness assumption for u_0, all of these are $\mathbb{R}$-translations of u_0, thus $\mathcal{M}_+ \cong [0, \infty)$. Then by genericity, all curves in $\mathcal{M} \setminus \mathcal{M}_+$ are also Fredholm regular, hence $\mathcal{M}$ is a smooth 1-dimensional manifold (without boundary). Observe that $\mathcal{M} \setminus \mathcal{M}_+$ is an open subset. Its closure $\overline{\mathcal{M} \setminus \mathcal{M}_+} \subset \mathcal{M}$ has exactly one boundary point, the unique curve in $\mathcal{M}_+$ that touches $\{T\} \times V$.

We claim that $\overline{\mathcal{M} \setminus \mathcal{M}_+}$ is compact. Indeed, by [8], any sequence $u_k \in \overline{\mathcal{M} \setminus \mathcal{M}_+}$ has a subsequence convergent to a J-holomorphic building u_∞ of arithmetic genus 0, with one positive end asymptotic to γ and no other ends. If u_∞ has any nontrivial upper level, then the uniqueness assumption implies that this level can only be an $\mathbb{R}$-translation of u_0, thus it has no negative ends and the main level of u_∞ must be empty. But this can happen only if u_k has its image in $[T, \infty) \times V$ for large k, hence $u_k \in \mathcal{M}_+$, giving a contradiction. Thus u_∞ has only a main level, and is at worst a nodal J-holomorphic curve in $\widehat{W}$, including exactly one component that is a plane asymptotic to γ, while all other components are spheres. The spheres are ruled out by semipositivity: since $\dim \mathcal{M} = 1$, any spheres that could appear in u_∞ would necessarily be covers of somewhere injective spheres with negative index, and thus cannot exist since J is generic. It follows that u_∞ is a smooth J-holomorphic plane, hence $\overline{\mathcal{M} \setminus \mathcal{M}_+}$ is compact as claimed.

The above shows that $\overline{\mathcal{M} \setminus \mathcal{M}_+}$ is diffeomorphic to a compact 1-dimensional manifold whose boundary is a single point. Since no such space exists, we have a contradiction and conclude that the filling (W, ω) cannot exist. $\qquad\square$

For the case $\Omega = d\lambda$, there is a variation on the above argument using a trick pioneered by Hofer in [35]. Instead of considering a completed filling $(\widehat{W}, \omega)$, one considers an exact cylindrical symplectic cobordism $(\mathbb{R} \times V, \omega)$ with $\omega = d(e^t \lambda)$ near $+\infty$ and $d(e^t \lambda')$ near $-\infty$, where λ' may be taken to be a constant multiple of any given contact form for ξ. Defining a moduli space of J-holomorphic planes in $\mathbb{R} \times V$ based on the $\mathbb{R}$-translations of u_0 as above, the same compactness argument goes through and produces a contradiction unless planes bubble off in the negative end, which means λ' must admit a contractible Reeb orbit. Note that in this case it's even easier to rule out sphere bubbling, as the exact cobordism $(\mathbb{R} \times M, \omega)$ does not admit *any* closed holomorphic curves. This proves:

Lemma 3.3 *If the assumptions of Lemma* 3.2 *are satisfied with* $\Omega = d\lambda$, *then every contact form on* (V, ξ) *admits a contractible closed Reeb orbit.*

Proof of Theorem 3.1 For the case of Ω exact, [7] establishes precisely the hypotheses of Lemma 3.2, thus proving that $(M, \xi)\#(\mathbb{S}^{2n-1}, \xi_-)$ is neither strongly fillable nor (by Lemma 3.3) hypertight. Specifically, Bourgeois and van Koert construct a contact form and suitable complex structure for $(\mathbb{S}^{2n-1}, \xi_-)$ such that there is a special Reeb orbit γ, which has smaller period than all other Reeb orbits in $\mathbb{S}^{2n-1}$, and is the asymptotic end of a unique J-holomorphic plane u_0. In the case of the connected sum $(M, \xi)\#(\mathbb{S}^{2n-1}, \xi_-)$, they also observe that $\gamma \subset \mathbb{S}^{2n-1}$ can be assumed to have smaller period than all other Reeb orbits except for a special set of orbits in the tube connecting $\mathbb{S}^{2n-1}$ to M, and there can be no holomorphic curves from γ to any combination of these orbits. To rule out weak fillings (W, ω) with arbitrary cohomology $\beta := [\omega|_{T(M\#\mathbb{S}^{2n-1})}] \in H^2_{\mathrm{dR}}(M\#\mathbb{S}^{2n-1})$, we now argue as follows. We can first perturb ω to assume without loss of generality that β is a rational cohomology class. Let $\beta' \in H^2_{\mathrm{dR}}(M)$ denote the image of β under the natural isomorphism $H^2_{\mathrm{dR}}(M\#\mathbb{S}^{2n-1}) \to H^2_{\mathrm{dR}}(M)$. Using the construction in [9, Proposition 2.16], we can find a stable Hamiltonian structure (λ', Ω') on M such that $\ker \lambda' = \xi$, $[\Omega'] = \beta'$ and $\Omega' = d\lambda'$ outside a tubular neighborhood $\mathcal{N}(\Sigma)$ of a contact submanifold $\Sigma \subset M$ such that $[\Sigma] \in H_{2n-3}(M)$ is Poincaré dual to a multiple of β'. The contact form λ' may also be chosen freely outside $\mathcal{N}(\Sigma)$, and we may assume that the ball deleted from M to form the connected sum is disjoint from $\mathcal{N}(\Sigma)$. The stable Hamiltonian structure (λ', Ω') can then be extended over $M\#\mathbb{S}^{2n-1}$ as a stable Hamiltonian structure (λ, Ω) such that $[\Omega] = \beta$, and outside of $\mathcal{N}(\Sigma)$, $\Omega = d\lambda$ with λ an

arbitrarily chosen contact form for $\xi \# \xi_-$. This construction can therefore be arranged to guarantee the same essential properties of the orbit γ and curve u_0 as in the exact case, thus establishing the hypotheses of Lemma 3.2.   □

4 Bordered Legendrian open books

In this section, we will first introduce a generalization of the plastikstufe that is more natural and less restrictive than the initial version introduced in [47]. Subsequently we will prove that these objects, under a certain homological condition (which is trivially satisfied for the overtwisted disk), represent obstructions to weak fillability.

Definition Let N be a compact manifold with nonempty boundary. A *relative open book* on N is a pair (B, θ) where:

- the *binding* B is a nonempty codimension 2 submanifold in the interior of N with trivial normal bundle;
- $\theta \colon N \setminus B \to \mathbb{S}^1$ is a fibration whose fibers are transverse to ∂N, and which coincides in a neighborhood $B \times \mathbb{D}^2$ of $B = B \times \{0\}$ with the normal angular coordinate.

Definition Let (V, ξ) be a $(2n + 1)$-dimensional contact manifold. A compact $(n + 1)$-dimensional submanifold $N \hookrightarrow V$ with boundary is called a *bordered Legendrian open book* (abbreviated bLob), if it has a relative open book (B, θ) such that:

 (i) all fibers of θ are Legendrian;
(ii) the boundary of N is Legendrian.

Remark 4.1 The binding B of a Legendrian open book is automatically isotropic because its tangent space is contained in the tangent space of the closure of all pages. Similarly, the fibers of θ and the boundary of N meet transversely in N, and saying that they are both Legendrian implies that the induced foliation on N is singular on B and ∂N.

A bLob is an example of a *maximally foliated* submanifold of (V, ξ), meaning that the singular distribution defined by intersecting its tangent spaces with ξ is integrable, thus forming an oriented singular foliation, and it has the largest dimension for which this is possible (see [47, Sect. 1] for further discussion). A bLob in a 3-dimensional contact manifold is the "flat version" of the overtwisted disk, the one where the characteristic foliation is singular along the boundary. This is a slight difference compared with the definition of plastikstufes in [47], where the boundary was a regular leaf of the induced

foliation, hence analogous to the "cambered version" of the overtwisted disk. This is a minor technical detail; each version can be deformed into the other one.

Definition 4.2 A contact manifold that admits a bLob is called *PS-overtwisted*.

Note that the definition of the bLob is topologically *much* less restrictive than the initial definition of the plastikstufe. For example, a 3-manifold admits a relative open book if and only if its boundary is a nonempty union of tori. On the other hand, a plastikstufe in dimension 5 is always diffeomorphic to a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$.

In this paper we will discuss one setting where we can find bLobs and are unable to find plastikstufes: in Proposition 5.9, we show that bLobs always exist in certain subdomains that are naturally associated to Liouville domains with disconnected boundary, a special case of which produces the Lutz-type twist due to Mori [46] (cf. Sect. 9.1).

Remark 4.3 Some bLobs also naturally arise in relation to the results of [48], where it is shown that sufficiently large neighborhoods of overtwisted submanifolds in higher dimensional contact manifolds give a filling obstruction. In [48] this required a rather technical argument involving holomorphic disks with an immersed boundary condition, but it can be simplified and strengthened by showing (using arguments similar to those of Proposition 5.9) that such neighborhoods always contain a bLob.

Of course, finding a bLob would be useless without the following theorem.

Theorem 4.4 *If a closed contact manifold is PS-overtwisted, then it does not have any (semipositive) weak symplectic filling (W, ω) for which ω restricted to the bLob is exact.*

Remark 4.5 The condition that the restriction of the symplectic form ω should be exact is trivially satisfied in dimension 5 for the plastikstufes defined in [47], which were all diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. In general however this condition could fail, and we believe that this could provide a hint as to varying degrees of filling obstructions or overtwistedness. Though it is unknown whether there is a *unique* natural notion of overtwistedness beyond dimension 3, or whether the different definitions known thus far are inequivalent, it would be interesting to speculate that a manifold can only be overtwisted in some "universal" sense if the bLob (or a similar object) can be *embedded into a ball* within the contact manifold. In this way the cohomological

condition is satisfied automatically, thus defining an obstruction to weak fillings due to the above theorem. We will refer to any bLob that lies inside a ball in the contact manifold as a *small* bLob.

If dim $V \geq 3$, then any contact structure ξ on V can be modified either by [49, 52] or by [21]—in the latter case without changing the homotopy class of almost contact structures—to produce one that is PS-overtwisted. In both cases, the change produces a small plastikstufe, hence Theorem 4.4 and the preceding section imply Theorem F stated in the introduction.

In the proof of Theorem 4.4 below, the general strategy is the same as in [47, 48], but there are differences coming from two sources: the need to handle weak rather than strong fillings, and bLobs rather than plastikstufes. Working with weak fillings complicates the question of energy bounds because the integral of ω on a holomorphic curve no longer has a direct relation to the integral of $d\alpha$. This is where the homological condition comes in. Further, it is no longer obvious that we can choose our almost complex structure to be both adapted to a contact form near the binding and boundary of the bLob and tamed by ω. As far as the differences between the plastikstufe and the bLob are concerned, the first is the singularity along the boundary, which makes energy control easier but makes it harder to ensure that holomorphic curves cannot escape through the boundary. This difference can be handled similarly to the analogous work in [48], which dealt with the case where the fibration of the bLob becomes trivial at the boundary. The general case additionally requires the somewhat technical Lemmas 4.6 and 4.7 below (though since we will not need this level of generality for our main results, the reader may skip these if desired). The second difference is of course that pages are more complicated and the interior monodromy can be anything, but this plays no role in the proof; what matters is the existence of a fibration over $\mathbb{S}^1$.

Proof of Theorem 4.4 Let N be a bLob in (V, ξ) with induced Legendrian open book (B, θ). Suppose that (W, ω) is a weak filling of V for which $\omega|_{TN}$ is exact. We choose a contact form α for ξ and attach to (W, ω) the corresponding conical end from Corollary 2.8. Since the restriction of ω to a neighborhood of the bLob is exact, we can choose a closed 2-form Ω on V that is cohomologous to $\omega|_{TV}$ and vanishes on a neighborhood of N. In a second step, we can deform the symplectic structure on the conical end to

$$\big([t_1, \infty) \times V, \Omega + d(t\alpha)\big)$$

for large t_1 as described in Lemma 2.10.

Identify the contact manifold (V, ξ) with a level set $\{T\} \times V$ in the conical end for sufficiently large $T > t_1$, and choose an almost complex structure J close to $\{T\} \times V$ that makes $(\{T\} \times V, \xi)$ pseudoconvex and is tamed by

$d(t\alpha)$. We require this J to be of the explicit form given in [47] in a neighborhood of the binding $\{T\} \times B$, which means the following. We can identify a neighborhood of $\{T\} \times B$ symplectically with an open set in $\mathbb{C}^2 \times T^*B$, with symplectic structure $\omega_0 \oplus d\lambda_{\mathrm{can}}$, such that the part of the bLob intersecting this neighborhood lies in $\mathbb{C}^2 \times B$. The desired almost complex structure is then the product of the standard structure i on the first factor with a tamed almost complex structure on the cotangent bundle. This choice simplifies the behavior of local holomorphic disks significantly: indeed, any disk lying entirely in this neighborhood and having boundary on the bLob projects to disks in $\mathbb{C}^2$ and T^*B, and the latter has boundary in the zero-section and must therefore be constant for energy reasons. In this way one can easily understand small disks close to the binding of the bLob, and in particular one obtains the existence of a Bishop family of holomorphic disks close to B, as well as the important fact that any holomorphic disk intersecting this model neighborhood must be part of the Bishop family. We refer to [47] for the full details.

Similarly, J should agree on a neighborhood of $\{T\} \times \partial N$ with an almost complex structure that we will describe in Lemma 4.7 below. As explained in Lemma 2.9, we can ensure by increasing T that the chosen J will not only be tamed by $d(t\alpha)$ but also by $\Omega + d(t\alpha)$ close to $\{T\} \times V$. Denote the symplectic manifold obtained by attaching $[0, T] \times V$ to W by $\widehat{W}$. We use contractibility of the space of tamed almost complex structure to extend J to the interior of the weak filling $\widehat{W}$.

As in [47], we now study the connected moduli space of J-holomorphic disks

$$u \colon \left(\mathbb{D}^2, \partial\mathbb{D}^2\right) \to \left(\widehat{W}, \{T\} \times (N \setminus B)\right)$$

emerging from a so-called *Bishop family* of disks in a neighborhood of some point on B. The boundaries of these disks necessarily intersect each page of the Legendrian open book exactly once.

We first establish the energy bound required for Gromov compactness. Any holomorphic disk u in the moduli space under consideration can be capped with a disk D lying in the bLob so that u together with D bounds a 3-ball B^3. Using Stokes' theorem,

$$0 = \int_{B^3} d\omega = \int_u \omega + \int_D \omega,$$

it then follows that the energy of the holomorphic disk is equal to minus the symplectic area of $D \subset N$. But since the restriction of ω to the bLob coincides with $T\,d\alpha$ in our construction, this quantity can be determined by integrating $T\alpha$ over the common boundary of the two disks u and D:

$$E_\omega(u) = \int_u \omega = -\int_D T\,d\alpha = T\int_{\partial u} \alpha.$$

Since the foliation on the bLob is given by $\xi \cap TN = \ker d\theta$, there is a continuous function $f: N \to \mathbb{R}$ that is everywhere nonnegative and vanishes only on $B \cup \partial N$ such that $\alpha|_{TN} = f \, d\theta$. The energy of u is thus bounded by

$$E_\omega(u) = T \int_{\partial u} \alpha \leq 2\pi T \max_{p \in N} f(p).$$

This leads to the same contradiction to Gromov compactness as in the proof for strong fillings [47], because by Lemma 4.7 below, the boundaries of the holomorphic disks are trapped between B and ∂N, and the topology of the Legendrian open book prevents bubbling of disks. $\qquad\square$

Lemma 4.6 *Suppose N is a manifold with boundary carrying a relative open book (B, θ) which embeds as a bLob into two contact manifolds (V_1, ξ_1) and (V_2, ξ_2). Then there are neighborhoods $U_1 \subset V_1$ and $U_2 \subset V_2$ of ∂N and a contactomorphism $\Phi: (U_1, \xi_1) \to (U_2, \xi_2)$ such that $\Phi(N \cap U_1) = N \cap U_2$.*

Proof Denote the two embeddings by $\iota_j: N \hookrightarrow V_j$ for $j = 1, 2$. The first step will be to prove the existence of contact forms α_1 and α_2 for ξ_1 and ξ_2 with $\iota_1^*\alpha_1 = \iota_2^*\alpha_2$ near ∂N. Start with any pair of contact forms α_1 and α_2. By the definition of a bLob, there are functions h_1 and h_2 which vanish exactly along ∂N such that $\iota_j^*\alpha_j = h_j \, d\theta$. We will prove shortly that h_1 and h_2 are both transverse to zero along ∂N. The implicit function theorem then guarantees the existence of a positive function f on V with $h_1 = f h_2$, allowing us to replace α_2 by $f\alpha_2$. The key point is that $\iota_j^* d\alpha_j = dh_j \wedge d\theta$, so dh_j cannot vanish anywhere along ∂N, otherwise TN would be an isotropic subspace of dimension $n + 1$ inside the symplectic vector space $(\xi_j, d\alpha_j)$ of dimension $2n$.

We now turn to the construction of the desired contactomorphism. We fix near ∂N a vector field X_r tangent to $\ker d\theta$ and a vector field X_θ tangent to ∂N such that $d\theta(X_\theta) = 1$. Then $d\alpha_j(X_r, X_\theta) = \iota_j^* d\alpha_j(X_r, X_\theta) = dh(X_r)$ is positive. We denote by $\mathcal{F}$ the foliation on ∂N induced by the pages, meaning $T\mathcal{F} = T\partial N \cap \ker d\theta$. Its tangent space is $d\alpha_j$-orthogonal to the symplectic subspace $\mathrm{span}(X_r, X_\theta)$, so we can construct for each $j = 1, 2$ a complex structure J_j on ξ_j which is compatible with $d\alpha_j$, such that $X_\theta = J_j X_r$ and the $d\alpha_j$-symplectic complement of $\mathrm{span}(X_r, X_\theta)$ in ξ_j is $T\mathcal{F} \oplus J_j T\mathcal{F}$. Denoting the Reeb vector field of α_j by R_j, we obtain the decomposition

$$TV_j|_{\partial N} = \mathrm{span}(X_r, X_\theta) \oplus T\mathcal{F} \oplus J_j T\mathcal{F} \oplus \mathrm{span}(R_j).$$

The first two summands span $TN|_{\partial N}$, and each $\nu_j := J_j T\mathcal{F} \oplus \mathrm{span}(R_j)$ can be identified with the normal bundle of N. Let τ_j be the restriction to ν_j of the exponential map for some auxilliary Riemannian metric. Each τ_j allows us to identify a tubular neighborhood of N with a neighborhood of the zero section

in ν_j. The bundles ν_1 and ν_2 are related by the bundle map $\Phi := \Psi \oplus \Phi_R$, where Φ_R sends $t \cdot R_1$ to $t \cdot R_2$ and $\Psi = \phi_2^{-1} \circ \phi_1$, with $\phi_j \colon J_j T\mathcal{F} \to T^*\mathcal{F}$ denoting the interior product with $d\alpha_j$. Thus $\tau_2 \circ \Phi \circ \tau_1^{-1}$ combines with the identity on N to give a diffeomorphism between tubular neighborhoods of N in V_1 and V_2 near ∂N. This map pulls α_2 back to α_1 and $d\alpha_2$ to $d\alpha_1$ for every $p \in \partial N$, so that the linear interpolation between both forms is a contact form, and we may apply the Moser trick.

Denoting by β_t with $t \in [0, 1]$ the interpolation between the pulled back contact forms, the Moser vector field Y_t is the unique solution to the two equations

$$\beta_t(Y_t) = 0 \quad \text{and} \quad (\iota_{Y_t} d\beta_t)|_{\ker \beta_t} = -\dot{\beta}_t|_{\ker \beta_t}.$$

From this we see that Y_t vanishes along ∂N, so that the isotopy φ_t is well defined on a small neighborhood U of ∂N and fixes ∂N pointwise. We now observe that $Y_t|_N$ lies in $\ker d\theta$, so that the isotopy preserves N. Indeed, if Y_t had any component in the complement of the Lagrangian subspace $\ker d\theta$, it would pair via $d\beta_t$ with a vector in $\ker d\theta$ and thus be different from $-\dot{\beta}_t$, which vanishes on TN. $\qquad\square$

We can now construct a suitable almost complex structure on a model which will be universal according to the preceding lemma.

Lemma 4.7 *Assume (W, ω) has a conical end, and identify (V, ξ) with a level set $\{T\} \times V$ of this conical end. Let α be any contact form for ξ. If N is a bLob in V, then we can choose an almost complex structure J_0 in a neighborhood $U_W \subset W$ of the boundary ∂N with the following properties*:

- *J_0 is compatible with the symplectization form $d(t\alpha)$ and it restricts to ξ.*
- *If J is any almost complex structure on W that makes (V, ξ) pseudoconvex and for which $J|_{U_W} = J_0$, then every compact J-holomorphic curve*

$$u \colon (\Sigma, \partial \Sigma) \to (W, N)$$

that intersects U_W and whose boundary lies in the bLob must be constant.

Proof The first step is to construct a model neighborhood for ∂N which is a bundle with exact symplectic fibers and holomorphic projection map. Let $\mathcal{F}_0$ be a fiber of the map $\theta|_{\partial N} \colon \partial N \to \mathbb{S}^1$. Then $\mathcal{F}_0$ is the intersection of ∂N with a page of the bLob, and ∂N is the mapping torus of some diffeomorphism $\psi \colon \mathcal{F}_0 \to \mathcal{F}_0$. We consider the $T^*\mathcal{F}_0$-fibration

$$\pi \colon \mathbb{C} \times \mathbb{R} \times \left(\mathbb{R} \times T^*\mathcal{F}_0\right)/\!\sim \quad \to \quad \mathbb{C} \times T^*\mathbb{S}^1,$$

$$(z, r; s; q, p) \quad \mapsto \quad (z; s, r),$$

where we use the equivalence relation $(z, r; s; q, p) \sim (z, r; s + 1; \psi(q),$ $(D\psi^{-1})^* p)$ on the total space.

Since $(\psi, (D\psi^{-1})^*)$ is symplectic, we get a symplectic structure $d\lambda_{\text{can}}$ on the vertical bundle $\ker D\pi$. Let $J_{\mathcal{F}}$ be a compatible complex structure on this bundle. Note that the directions ∂_r, and ∂_s are well defined, so that we can extend $J_{\mathcal{F}}$ to an almost complex structure $J = i \oplus i \oplus J_{\mathcal{F}}$ on the total space, where $i\partial_r = \partial_s$, and $i\partial_s = -\partial_r$. By construction, π is holomorphic with respect to J upstairs and $i \oplus i$ on $\mathbb{C} \times T^*\mathbb{S}^1$.

The next step consists in finding a J-plurisubharmonic function on a neighborhood of $\{1\} \times \{0\} \times (\mathbb{R} \times \mathcal{F}_0)/\sim$, where $\mathcal{F}_0$ denotes the 0-section in $T^*\mathcal{F}_0$. Define a function h on $\mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*\mathcal{F}_0)/\sim$ by using a metric on the *vector* bundle $\mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*\mathcal{F}_0)/\sim$ over $\mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times \mathcal{F}_0)/\sim$, and defining $h(v) = \|v\|^2/2$ for every vector v in this bundle. In a bundle chart, we obtain

$$h(z, r; s; q, p) = \frac{1}{2} \sum_{i,j} g_{i,j}(z, r; s; q) p_i p_j,$$

and it follows that $dd^c h = d(dh \circ J)$ simplifies on the 0-section of this bundle to

$$dd^c h = \sum_{i,j} g_{i,j} dp_i \wedge (dp_j \circ J).$$

We claim now that the function

$$F: \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*\mathcal{F}_0)/\sim \to [0, \infty),$$

$$(z, r; s; q, p) \mapsto |z|^2 + r^2 + h(z, r; s; q, p)$$

is J-plurisubharmonic in a neighborhood of $\{1\} \times \mathbb{R} \times (\mathbb{R} \times \mathcal{F}_0)/\sim$. Here one just needs to check that $-dd^c F$ simplifies near $\{1\} \times \mathbb{R} \times (\mathbb{R} \times \mathcal{F}_0)/\sim$ to

$$-dd^c F = 4\, dx \wedge dy + 2\, dr \wedge ds - dd^c h,$$

where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. This 2-form is positive on complex lines.

We find a neighborhood of $\{x = 1\}$ in the level set $F^{-1}(1)$, where the restriction of the 1-form $\alpha := -dF \circ J$ defines a contact structure. Furthermore, the submanifold $N' \subset F^{-1}(1)$ given by the embedding

$$[0, \varepsilon) \times (\mathbb{R} \times \mathcal{F}_0)/\sim \quad \hookrightarrow \quad \mathbb{C} \times \mathbb{R} \times (\mathbb{R} \times T^*\mathcal{F}_0)/\sim$$

$$(r; s; q) \quad \mapsto \quad (\sqrt{1 - r^2}, r; s; q, 0)$$

has $N' \cap \{r = 0\}$ as boundary and inherits a singular Legendrian foliation given by the form $r\, ds$. This foliation is diffeomorphic to the one
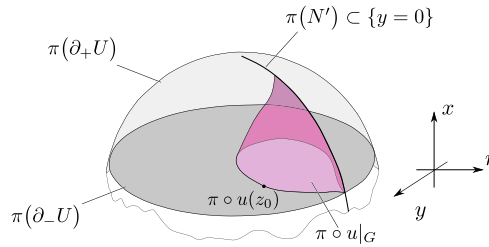
Fig. 1 The neighborhood of the boundary ∂N can be thought of as a $T^* \mathcal{F}_0$-bundle. We obtain a simple model by projecting this neighborhood and the holomorphic curve u to the base space. The holomorphic curve $\pi \circ u$ has to be cut off and will have two types of boundary: the original one that sits in the bLob and the boundary where the curve has been cut off. Along the cut-off boundary the x-value is minimal, and there will be a point where the y-value also becomes extremal, but this contradicts the boundary point lemma showing that the x- and y-values of u have to be constant

on the bLob N in the collar neighborhood of ∂N, so that by Lemma 4.6 above, there is a small relatively open set $U_W \subset F^{-1}((0, 1])$ containing $\{(1, 0)\} \times (\mathbb{R} \times \mathcal{F}_0)/\sim$ in the model such that $\partial_+ U := U_W \cap F^{-1}(1)$ with contact form α is contactomorphic to a neighborhood U_V of ∂N in V. Note that for $\delta > 0$ sufficiently small, the level set $\{x = 1 - \delta\}$ is a compact hypersurface with boundary in $\partial_+ U$, and we will set $\partial_- U := \{x = 1 - \delta\} \cap U_W$, writing from now on U_W for the compact set $U_W \cap \{x \geq 1 - \delta\}$.

Extending this contactomorphism, we can embed U_W into the symplectic manifold W such that $\partial_+ U$ lies in $\{T\} \times V$, and $N' \cap U_W$ is mapped to $N \cap U_V$. Choose the almost complex structure J on U_W constructed above, and extend it to one that makes the contact manifold (V, ξ) pseudoconvex.

Now let $u: (\Sigma, \partial \Sigma) \to (W, N)$ be any J-holomorphic curve that intersects the neighborhood U_W. Our aim is to show that u must be constant. Define $G := u^{-1}(U_W)$ and write $u|_G$ for the restriction of u. Perturbing δ slightly, we can assume that $u^{-1}(\partial_- U) \subset G$ is a properly embedded submanifold so that G has piecewise smooth boundary. Project the curve $u|_G$ via

$$\pi : \mathbb{C} \times \mathbb{R} \times \left(\mathbb{R} \times T^* \mathcal{F}_0\right)/\sim \quad \to \quad \mathbb{C} \times T^* \mathbb{S}^1,$$

and note that $\pi \circ u|_G$ is a holomorphic map with respect to the standard structure (see Fig. 1). The boundary $\pi \circ u(\partial G)$ lies in the union of

$$\pi\left(\partial_+ U \cap N'\right) = \left\{(z; s, r) \mid \mathrm{Re}\, z \geq 1 - \delta,\, \mathrm{Im}\, z = 0,\, r = -\sqrt{1 - |z|^2}\right\}$$

and

$$\pi(\partial_- U) = \left\{(z; s, r) \mid \mathrm{Re}\, z = 1 - \delta,\, |z|^2 + r^2 \leq 1\right\}.$$

Since both coordinate functions $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are harmonic, it follows that the maxima and minima are both attained on ∂G, so that if we assume y is not everywhere equal to 0, then u must intersect $\pi(\partial_- U)$, and in particular the minimum of x will be $1 - \delta$. Let $z_0 \in \partial G$ be a point for which $u(z_0)$ has both minimal x-coordinate and extremal y-coordinate. At this point, it follows that the derivative of $\pi \circ u|_G$ in the ∂G-direction has vanishing x and y-coordinates. Using the Cauchy-Riemann equation at the point z_0, we then see that the derivatives also vanish in the radial direction, thus contradicting the boundary point lemma, making both x and y constant on G.

It follows now that u is completely contained in $U_W \cap \{z = x_0\}$, and from this we immediately recover that the r-coordinate of $u|_{\partial \Sigma}$ is equal to $-\sqrt{1 - x_0^2}$. The r-coordinate is also harmonic, and it follows that $\pi \circ u$ must have constant r-coordinate everywhere, since both its maximum and its minimum are equal, and the Cauchy-Riemann equation then implies that the s-coordinate is also constant. This finishes the proof, because it follows that the projection $\pi \circ u$ is constant, so that u is completely contained in a fiber of π that is symplectomorphic to $T^*\mathcal{F}_0$ with exact symplectic form $d\lambda_{\mathrm{can}}$, but since $J_{\mathcal{F}}$ was compatible with $d\lambda_{\mathrm{can}}$, and since the boundary of u lies in the 0-section of $T^*\mathcal{F}_0$, it follows that u has no $d\lambda_{\mathrm{can}}$-energy, and hence must be constant. $\qquad\square$

5 Giroux domains

While the filling obstructions we've discussed so far (namely bLobs and negative stabilizations) were previously understood in less general forms, in this section we shall introduce a subtler class of filling obstructions that generalizes Giroux torsion in dimension three and is completely new in higher dimensions. The fundamental objects in this discussion are called Giroux domains and ideal Liouville domains. As was sketched in the introduction, an ideal Liouville domain is a natural compactification of a complete Liouville manifold, and its product with $\mathbb{S}^1$ naturally inherits a contact structure, producing what we call a Giroux domain. The definitions and elementary properties of these objects, including a blow-down operation along boundary components, are due to Giroux but cannot yet be found anywhere in the literature, so we will discuss them in some detail in Sect. 5.2 and the beginning of Sect. 5.3. Before that, in Sect. 5.1, we introduce for later convenience a slightly more general context for the blow-down operation. Some explicit examples of blown down Giroux domains have already appeared in the work of Mori [46], who showed that his examples always contain a plastikstufe. The notion of the bLob allows us to generalize this result using a purely topological description that we will explain in Sect. 5.4. The last subsection, cul-

minating with the statement of Theorem 5.13, defines a filling obstruction in terms of Giroux domains which refines Theorem B from the introduction and sets the stage for our higher dimensional generalization of Giroux torsion in Sect. 9.

5.1 Round hypersurfaces

We say that an oriented hypersurface H in a contact manifold (V, ξ) is a ξ-*round hypersurface modeled on* some closed contact manifold (M, ξ_M) if it is transverse to ξ and admits an orientation preserving identification with $\mathbb{S}^1 \times M$ such that $\xi \cap TH = T\mathbb{S}^1 \oplus \xi_M$. In this definition, the word "round" is used as in "round handle". In general, the orientation of a round hypersurface may be chosen at will, and we shall assume in particular that whenever H is a component of ∂V, its orientation is the *opposite* of the natural boundary orientation; see Remark 5.2 below. Observe that in dimension three, a ξ-round hypersurface is simply a pre-Lagrangian torus with closed characteristic leaves.

Lemma 5.1 *Any ξ-round hypersurface $H = \mathbb{S}^1 \times M$ in the interior (or boundary) of (V, ξ) has a neighborhood $(-\varepsilon, \varepsilon) \times H$ (or $[0, \varepsilon) \times H$ respectively) on which ξ can be defined by the contact form $\alpha_M + s\,dt$ where s is the coordinate on the interval, t the coordinate in $\mathbb{S}^1$ and α_M is a contact form for ξ_M.*

Proof Fix any tubular neighborhood (or collar neighborhood) of H with coordinate t. The 1-form described defines a contact form near H which induces the same hyperplane field as ξ on H, hence they are isotopic relative to H. Pulling back the neighborhood under this isotopy gives the desired neighborhood. $\square$

Suppose H is a ξ-round boundary component of (V, ξ), with orientation opposite the boundary orientation, and consider the collar neighborhood from the preceding lemma. We now explain how to modify (V, ξ) by *blowing down* H to M. Let D be the disk of radius $\sqrt{\varepsilon}$ in $\mathbb{R}^2$. The map $\Psi \colon (re^{i\theta}, m) \mapsto (r^2, \theta, m)$ is a diffeomorphism from $(D \setminus \{0\}) \times M$ to $(0, \varepsilon) \times \mathbb{S}^1 \times M$ which pulls back $\alpha_M + s\,dt$ to the contact form $\alpha_M + r^2\,d\theta$. Thus we can glue $D \times M$ to $V \setminus H$ to get a new contact manifold in which H has been replaced by M. This process is equivalent to performing a *contact cut* of V with respect to the (local) θ-action, as described in [40].

Remark 5.2 Topologically, the blow down operation glues $\mathbb{D}^2 \times M$ to V via the natural identification of $\partial(\mathbb{D}^2 \times M)$ with $\mathbb{S}^1 \times M = H \subset V$. This is why it is appropriate to assign to H the reverse of its natural boundary orientation with respect to V.

5.2 Ideal Liouville domains

The following notion is of central importance for the new filling obstructions that we will introduce.

Definition (Giroux) Let Σ be a compact manifold with boundary, ω a symplectic form on the interior $\mathring{\Sigma}$ of Σ and ξ a contact structure on $\partial\Sigma$. The triple (Σ, ω, ξ) is an *ideal Liouville domain* if there exists an auxiliary 1-form β on $\mathring{\Sigma}$ such that:

- $d\beta = \omega$ on $\mathring{\Sigma}$,
- For any smooth function $f : \Sigma \to [0, \infty)$ with regular level set $\partial\Sigma = f^{-1}(0)$, the 1-form $f\beta$ extends smoothly to $\partial\Sigma$ such that its restriction to $\partial\Sigma$ is a contact form for ξ.

In this situation, β is called a *Liouville form* for (Σ, ω, ξ).

Remark 5.3 In the above definition, the space of possible auxiliary Liouville forms β is contractible. Indeed, we first observe that if the second condition is satisfied for any given function f_1 as specified in the definition, then it is also satisfied for any other function f_2 with the required properties, as we then have $f_2 = g f_1$ for some smooth function $g : \Sigma \to (0, \infty)$. Thus we can fix a suitable function f and see that the set of admissible primitives β on $\mathring{\Sigma}$ is convex. An interesting variation on the above definition is obtained by also regarding ξ as auxiliary data: this still leaves a contractible space of auxiliary choices, but it is slightly less convenient for our purposes.

Remark 5.4 Note that for β and f as in the above definition, there is no requirement that $d(f\beta)$ should be symplectic, and in general it is not. It is true however that one can always find (using Lemma 5.5 below) suitable functions f for which $f\beta$ also defines a Liouville form on $\mathring{\Sigma}$, and Liouville forms of this type arise naturally in certain examples, cf. Example 5.7 and Remark 5.11.

One can check that a Liouville form β for an ideal Liouville domain Σ defines on the interior of Σ the structure of a *complete* Liouville manifold. This means that the flow of the vector field X which is ω-dual to β is complete, and in particular the interior of Σ has infinite volume with respect to ω. This follows from Lemma 5.5 below, which describes precisely what happens near $\partial\Sigma$. For our purposes, one may regard the statement of this lemma as part of the definition of an ideal Liouville domain, but keeping in mind that it is already implied by the definition above.

Lemma 5.5 (Giroux) *Suppose (Σ, ω, ξ) is an ideal Liouville domain with auxiliary Liouville form β, and let X denote its ω-dual vector field, i.e. the unique vector field on $\mathring{\Sigma}$ that satisfies $\iota_X \omega = \beta$.*

Choose any smooth function $f : \Sigma \to [0, \infty)$ with regular level set $\partial \Sigma = f^{-1}(0)$. Then the vector field $X_f := \frac{1}{f} X$ on $\mathring{\Sigma}$ extends smoothly over $\partial \Sigma$ so that it points transversely outward. Moreover, a collar neighborhood of $\partial \Sigma$ can be identified with $(0, 1] \times \partial \Sigma$ with coordinate $s \in (0, 1]$ such that $\beta = \frac{1}{1-s} \alpha$ on $(0, 1) \times \partial \Sigma$, where α is a contact form for ξ.

Proof By definition, the 1-form $\gamma := f\beta$ extends smoothly to Σ and restricts on the boundary $\partial \Sigma$ to a contact form for ξ. The smooth $2n$-form $\mu := f \, d\gamma^n - n \, df \wedge \gamma \wedge d\gamma^{n-1}$ on the domain Σ simplifies on the interior $\mathring{\Sigma}$ to

$$\mu = f^{n+1} \omega^n,$$

and is hence a volume form on $\mathring{\Sigma}$. It is also nondegenerate along $\partial \Sigma$, since $f \, d\gamma^n$ vanishes and $\gamma \wedge d\gamma^{n-1}$ is a volume form on $T(\partial \Sigma) = \ker df$. It follows that there is a unique vector field X_f on Σ satisfying the equation

$$\iota_{X_f} \mu = n\gamma \wedge d\gamma^{n-1}.$$

Using $\iota_X \omega^n = n\beta \wedge d\beta^{n-1}$ on the interior $\mathring{\Sigma}$, one can check that $X_f|_{\mathring{\Sigma}} = \frac{1}{f} X$, and since the first term of μ vanishes at $\partial \Sigma$ and f decreases in the outward direction, it follows that X_f points transversely outward through $\partial \Sigma$.

We now construct the collar neighborhood. The basic idea is to follow the flow of X_f starting from $\partial \Sigma$, but for a particular choice of the function $f : \Sigma \to [0, \infty)$ with regular level set $f^{-1}(0) = \partial \Sigma$. Starting from an arbitrary function f of this type, any other such function h can be written as $h = gf$ for some positive function g on Σ. We then seek h such that the vector field $X_h = \frac{1}{h} X$ satisfies

$$\mathcal{L}_{X_h}(h\beta) = 0.$$

This condition is equivalent to $dh(X_h) = -1$, which leads to the ordinary differential equation $dg(X_f) = -\frac{1+df(X_f)}{f} g$. The function f vanishes along $\partial \Sigma$, and by the construction above, we see that $df(X_f) = -1$ on $\partial \Sigma$, thus the differential equation is well behaved at $\partial \Sigma$ and can be solved with initial condition $g|_{\partial \Sigma} \equiv 1$.

We denote by α the contact form induced by $h\beta$ on $\partial \Sigma$. Since $\iota_{X_h} h\beta = 0$ and $\mathcal{L}_{X_h} h\beta = 0$, the flow $\varphi_t^{X_h}$ of X_h for negative t pulls $h\beta$ back to α. Further, from $dh(X_h) = -1$ we obtain $h \circ \varphi_t^{X_h} = -t$, so

$$\left(\varphi_t^{X_h}\right)^*(h\beta) = -t \cdot \left(\varphi_t^{X_h}\right)^* \beta = \alpha.$$

Reparameterizing the time variable, we finally obtain the map $\Phi(s, p) := \varphi_{s-1}^{X_h}(p)$ which gives the desired collar neighborhood with $(1-s)\Phi^*\beta = \alpha$. $\square$

5.3 Giroux domains

Given an ideal Liouville domain (Σ, ω, ξ) and a Liouville form β, one can endow $\Sigma \times \mathbb{R}$ with the contact structure $\ker(f\, dt + f\beta)$ for any smooth function $f : \Sigma \to [0, \infty)$ with regular level set $f^{-1}(0) = \partial \Sigma$. Over the interior of Σ, $\ker(f\, dt + f\beta) = \ker(dt + \beta)$, so one recovers the standard notion of the *contactization* of the Liouville manifold defined by β. On the boundary we have $f\, dt = 0$, so the contact hyperplanes are $\xi \oplus T\mathbb{R}$. Any two contact structures obtained in this way from different Liouville forms are isotopic relative to the boundary. Since the contact forms constructed on $\Sigma \times \mathbb{R}$ are $\mathbb{R}$-invariant, one can just as well replace $\mathbb{R}$ by $\mathbb{S}^1$. We will refer to $\Sigma \times \mathbb{S}^1$ with the contact structure defined in this way as the *Giroux domain* associated to (Σ, ω, ξ); see Example 5.7 below for our main motivation. Observe that the boundary is a ξ-round hypersurface modeled on $(\partial \Sigma, \xi)$.

Remark 5.6 The above is a special case of a more general construction, also due to Giroux, known as the *suspension* of a symplectomorphism φ with compact support in $\mathring{\Sigma}$. The result of this construction also has ξ-round boundary, and blowing it down gives the contact manifold associated to the abstract open book (Σ, φ). Observe that unlike Giroux's original construction of the contact structure associated to an open book (see e.g. in [25, Sect. 7.3]), this construction does not require any tweaking near the binding.

In a different direction, one can generalize the construction of Giroux domains to allow for nontrivial circle bundles over Σ using ideas from [11].

Example 5.7 We consider

$$\Sigma = \mathbb{S}^1 \times [0, \pi], \quad \omega = \frac{1}{\sin^2 s}\, d\theta \wedge ds,$$

where s is the coordinate in $[0, \pi]$ and θ the coordinate in $\mathbb{S}^1$, carrying the trivial contact structure $\ker \pm d\theta$. One can take as a Liouville form $\beta = \cot s\, d\theta$. Setting $f(\theta, s) = \sin s$, we get the contact form $f(\theta, s) \cdot (\beta + dt) = \cos s\, d\theta + \sin s\, dt$ on $\Sigma \times \mathbb{S}^1$. Thus the Giroux domain associated to this ideal Liouville domain is a Giroux π-torsion domain.

5.4 Blowing down

Let M be a union of connected components of the boundary of a Giroux domain $\Sigma \times \mathbb{S}^1$. These components are ξ-round hypersurfaces and can thus be blown down as described in Sect. 5.1. We shall denote the resulting manifold by $(\Sigma \times \mathbb{S}^1)//M$. It inherits a natural contact structure for which each of the blown down boundary components becomes a codimension two contact submanifold.

Example 5.8 Continuing the annulus example, a Giroux π-torsion domain with one boundary component blown down is a so-called *Lutz tube*, i.e. the solid torus that results from performing a Lutz twist along a transverse knot. With both boundary components blown down, it is the standard contact structure on $\mathbb{S}^2 \times \mathbb{S}^1$.

In the above example, when one boundary component is blown down but not the other, the resulting domain contains an overtwisted disk. We now generalize this to higher dimensions.

Proposition 5.9 *Suppose (V, ξ) is a contact manifold containing a subdomain G with nonempty boundary, obtained from a Giroux domain by blowing down at least one boundary component. Then (V, ξ) contains a small* bLob (*cf. Remark* 4.5).

The bLob in the above proposition will come from a Lagrangian submanifold in an ideal Liouville domain (Σ, ω, ξ). We first need a technical definition describing how these submanifolds will be allowed to approach the boundary. We say that a submanifold L properly embedded inside Σ and transverse to the boundary is a *Lagrangian with cylindrical end* if:

- $\mathring{L}$ is Lagrangian in $\mathring{\Sigma}$,
- ∂L is Legendrian in $\partial \Sigma$,
- There is a Liouville form β whose ω-dual vector field is tangent to L near $\partial \Sigma$. More precisely, there is a collar $(0, 1] \times \partial \Sigma$ as in Lemma 5.5 which intersects L along $(0, 1] \times \partial L$.

We will say that the Liouville form in this definition is *adapted* to L.

Lemma 5.10 *Let (Σ, ω, ξ) be an ideal Liouville domain. If L is a Lagrangian with cylindrical end in Σ, then $\widehat{L} := L \times \mathbb{S}^1$ inside the contactization $\Sigma \times \mathbb{S}^1$ is isotopic to a maximally foliated submanifold whose singular set is its boundary and whose foliation is otherwise defined via a fibration*

$$\vartheta \colon \widehat{L} \to \mathbb{S}^1, \quad (l, t) \mapsto F(l) + t,$$

for some smooth function $F \colon L \to \mathbb{S}^1$ that is constant on a neighborhood of ∂L.

Proof We first assume that there is a Liouville form β adapted to L which induces a rational cohomology class on L. This implies there is a real number $\hbar > 0$ such that $\hbar^{-1}$ times the cohomology class of the restriction of β to L is integral: $\hbar^{-1}[i^*\beta] \in H^1(L; \mathbb{Z})$. First note that $\hbar u \, dt + u\beta$ defines a contact structure isotopic to $\ker(u \, dt + u\beta)$ relative to the boundary of the Giroux domain $G := \Sigma \times \mathbb{S}^1$. Furthermore, the vector field constructed in the standard

proof of Gray's theorem vanishes along $\partial \Sigma \times \mathbb{S}^1$, so this isotopy is actually tangent to the identity along the boundary. We shall now prove the lemma using this contact form (and no further isotopy of $L \times \mathbb{S}^1$).

In the interior of G, the contact structure is defined by $\hbar dt + \beta$, which restricts to $\eta = \hbar dt + i^*\beta$ on $\widehat{L}$. Since η is closed, $\widehat{L}$ is foliated. Moreover, $\hbar dt$ never vanishes in $\widehat{L}$, so there is no singularity there. Along the boundary, the contact structure is defined by $u\beta$, whose restriction to $\widehat{L}$ vanishes, thus the singularities are as claimed.

We now define the fibration ϑ using Tischler's construction (cf. [54]). Let (l_0, t_0) be any base point in the interior of $\widehat{L}$. We define $\vartheta(l, t) = \frac{1}{\hbar} \int_\gamma \eta$, where γ is any path from (l_0, t_0) to (l, t). Since η is closed, Stokes' theorem guarantees that this is well defined modulo the integral of η along loops based at (l_0, t_0). If (γ_L, γ_t) is such a loop, then the integral over it is $\langle [\beta], [\gamma_L] \rangle + \hbar \langle [dt], [\gamma_t] \rangle$, which belongs to $\hbar \mathbb{Z} + \hbar \mathbb{Z} = \hbar \mathbb{Z}$, thus ϑ has a well-defined value in $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Observe that $\vartheta(l, t) = \vartheta(l, 0) + t$, and two points (l_1, t_1) and (l_2, t_2) lie in the same connected component of a fiber of ϑ if and only if they lie on the same leaf of the Legendrian foliation. On a suitable collar neighborhood of the boundary, the 1-form η simplifies to $\hbar dt$, so the behavior of ϑ is also as claimed.

We now explain how to enforce the rationality assumption by perturbation of the Liouville structure. Suppose β_0 is any Liouville form adapted to L, in which case $\beta_0|_{TL}$ is a closed 1-form that vanishes on a collar neighborhood of ∂L. For every $\varepsilon > 0$, we will find a closed 1-form λ_L on L with compact support in $\mathring{L}$ and $\|\lambda_L\| < \varepsilon$ (in the $\mathcal{C}^0$-norm with respect to a fixed auxiliary metric on L) such that $i^*\beta_0 + \lambda_L$ represents a rational cohomology class on L. Since the restriction of β_0 to L vanishes near ∂L, its cohomology class belongs to the kernel K of the map $H^1_{\mathrm{dR}}(L) \to H^1_{\mathrm{dR}}(\partial L)$ induced by inclusion. Let $\alpha_1, \ldots, \alpha_p$ be a set of closed 1-forms representing a basis of the image in K of $H^1(L; \mathbb{Z})$. By the definition of K, we can assume that all these 1-forms vanish near the boundary of L. The restriction of β_0 to L can be written as $\sum c_i \alpha_i + df$ for some real coefficients c_i and some function f. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, one can find arbitrarily small numbers ε_i such that $c_i + \varepsilon_i$ is rational for all i and then set $\lambda_L = \sum \varepsilon_i \alpha_i$.

We extend λ_L to a tubular neighborhood U of L in Σ by pulling it back to the normal bundle, and multiply it by a fixed cutoff function $\rho: U \to [0, 1]$ that has compact support on U and equals 1 on L. In this way we obtain a 1-form β_0' given by $\beta_0 + \rho \lambda_L$ on U that extends smoothly to β_0 on $\Sigma \setminus U$, and whose restriction to L yields the desired closed 1-form with compact support in $\mathring{L}$ that represents a rational cohomology class. We can choose ε above arbitrarily small, hence we can assume that all forms in the segment between $d\beta_0$ and $d\beta_0'$ are symplectic. The corresponding contact structures are then isotopic relative to $\Sigma \setminus U$ and $\partial \Sigma$. $\qquad\qquad \square$

Proof of Proposition 5.9 Let $(\Sigma, \omega, \xi_\partial)$ denote the ideal Liouville domain used to construct G. We will construct a Lagrangian $L \subset \Sigma$ with cylindrical end and blow down the foliated submanifold of Lemma 5.10 to find the desired bLob. If $\dim \Sigma = 2$, it suffices to take for L an embedded path between two distinct boundary components of Σ, where one corresponds to a blown down boundary component of G and the other does not. More generally, choose two disjoint boundary parallel Lagrangian disks L_{bd} and L_p with cylindrical ends in Σ such that ∂L_{bd} is a Legendrian sphere in one of the blown down boundary components of $\partial \Sigma$, and ∂L_p is a Legendrian sphere in another boundary component that is not blown down. By a symplectic isotopy supported in a tube connecting them, we can deform L_p away from ∂L_p so that it intersects L_{bd} transversely.

One can remove transverse self-intersection points between two Lagrangians L and L' using [51]. This construction works by removing for each intersection two small balls from L and L' containing this point, and gluing in a tube diffeomorphic to $[-\varepsilon, \varepsilon] \times \mathbb{S}^{n-1}$ joining the boundaries of the two balls. In fact, the construction is explicit: choose a Darboux chart around the intersection point such that L and L' are represented by the n-planes $\{(x_1, \ldots, x_n, 0, \ldots, 0)\}$ and $\{(0, \ldots, 0, y_1, \ldots, y_n)\}$ respectively. Remove a disk of radius ε around 0 in both planes and glue in the tube

$$(-\varepsilon, \varepsilon) \times \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^{2n},$$
$$(t; x_1, \ldots, x_n) \mapsto \big(\rho_1(t) \cdot (x_1, \ldots, x_n); \rho_2(t) \cdot (x_1, \ldots, x_n)\big)$$

for a smooth function $\rho_1 : (-\varepsilon, \varepsilon) \to [0, 1]$ that is 0 for values between $-\varepsilon$ and $-\varepsilon/2$, has positive derivative for $t > -\varepsilon/2$ and is the identity close to $+\varepsilon$. Define $\rho_2(t) := \rho_1(-t)$. This defines a Lagrangian manifold that glues well to $L \setminus \varepsilon \cdot \mathbb{D}^n$ for t close to ε and to $L' \setminus \varepsilon \cdot \mathbb{D}^n$ for t close to $-\varepsilon$.

The symplectic isotopy and the surgery process both took place far away from the boundary, so we obtain by this construction a Lagrangian that still has cylindrical ends. Lemma 5.10 then produces a foliated submanifold which becomes a bLob in the blown down Giroux domain. This bLob also embeds into a ball, because L is obtained from two Lagrangian disks parallel to the boundary and a thin tube that lies in the neighborhood of an embedded path, so that L lies in a ball of the form $[0, 1] \times \mathbb{D}^{2n-1} \subset \Sigma$. Moving to the contactization and blowing down the corresponding boundary components then gives a neighborhood diffeomorphic to a ball $\mathbb{D}^2 \times \mathbb{D}^{2n-1}$ that contains the bLob. $\qquad\square$

5.5 Convex hypersurfaces and gluing

Recall that a hypersurface Σ in a contact manifold (V, ξ) is said to be ξ-*convex* (cf. [28]) if there is a contact vector field X transverse to Σ. In

this situation, the *dividing set* associated to Σ and X is the hypersurface Γ in Σ where X is tangent to ξ. Observe that since the vector field X corresponds to a "contact Hamiltonian" function and can thus be cut off away from Σ, its flow identifies a neighborhood of Σ with $\Sigma \times \mathbb{R}$, with ξ defined by $\gamma + u\, dt$ where t denotes the coordinate on $\mathbb{R}$, γ is a 1-form on Σ and u a function on Σ. It follows from the computations in [28] that the 1-form γ induces a contact structure ξ_Γ on the dividing set Γ and, if Σ_0 denotes the closure of a connected component of $\Sigma \setminus \Gamma$, $(\Sigma_0, d(\gamma/u), \xi_\Gamma)$ is an ideal Liouville domain whose contactization is $\Sigma_0 \times \mathbb{R}$ equipped with the restriction of ξ.

If Σ is closed then Γ cannot be empty, otherwise Σ would be a closed exact symplectic manifold, contradicting Stokes' theorem. So in this case, $\Sigma \setminus \Gamma$ has at least one component on which u is positive and one where it is negative. One can then see $\Sigma \times \mathbb{R}$ as several contactizations of ideal Liouville domains glued together. Going in the opposite direction, we can take advantage of the fact that boundary components of Giroux domains are ξ-round hypersurfaces and use Lemma 5.1 to glue together any two Giroux domains along boundary components modeled on isomorphic contact manifolds.

Remark 5.11 One can check that the ideal Liouville domain $(\Sigma_0, d(\gamma/u), \xi_\Gamma)$ defined above depends only on the contact structure and contact vector field near Σ, not on the choice of contact form. For an arbitrary choice of contact form, one cannot expect $d\gamma$ itself to be symplectic everywhere on $\Sigma \setminus \Gamma$, but analogously to Remark 5.4, one can always choose a contact form for which this is true. The surface $\Sigma \times \{\text{const}\} \subset \Sigma \times \mathbb{S}^1$ in Example 5.7 provides a popular example.

5.6 Obstructions to fillability

We now want to state a non-fillability result. As preparation, note that any embedding of the interior of a Giroux domain $I_\Sigma := \mathring{\Sigma} \times \mathbb{S}^1$ into a contact manifold (V, ξ) determines a distinguished subspace $H_1(\Sigma; \mathbb{R}) \otimes H_1(\mathbb{S}^1; \mathbb{R}) \subset H_2(V; \mathbb{R})$. We call its annihilator in $H^2_{\mathrm{dR}}(V)$ the space of cohomology classes *obstructed* by I_Σ, and we denote it by $\mathcal{O}(I_\Sigma)$. Classes in $\mathcal{O}(I_\Sigma)$ are exactly those whose restriction to I_Σ can be represented by closed 2-forms pulled back from the interior of Σ. If $N \subset (V, \xi)$ is any subdomain resulting from gluing together a collection of Giroux domains $I_{\Sigma_1}, \ldots, I_{\Sigma_k}$ and blowing down some of their boundary components, then we define its obstructed subspace $\mathcal{O}(N) \subset H^2_{\mathrm{dR}}(V)$ to be $\mathcal{O}(I_{\Sigma_1}) \cap \cdots \cap \mathcal{O}(I_{\Sigma_k})$. We will say that such a domain is *fully obstructing* if $\mathcal{O}(N) = H^2_{\mathrm{dR}}(V)$.

Example 5.12 If Σ is homeomorphic to $[-1, 1] \times M$ for some closed manifold M, and N is the result of blowing down one boundary component of the Giroux domain I_Σ, then *any* embedding of N is fully obstructing. Indeed, any

class in $H_1(\Sigma; \mathbb{R}) \otimes H_1(\mathbb{S}^1; \mathbb{R})$ can be represented by a cycle in the $M \times \mathbb{D}^2$ part of the blown down Giroux domain and, of course, $H_1(\mathbb{S}^1; \mathbb{R})$ becomes trivial in $H_1(\mathbb{D}^2; \mathbb{R})$. For instance, a Lutz tube (see Example 5.8) in a contact 3-manifold is always fully obstructing, and the same is true for the higher dimensional generalization that we will discuss in Sect. 9.

Theorem 5.13 *Suppose (V, ξ) is a closed contact manifold containing a subdomain N with nonempty boundary, which is obtained by gluing and blowing down Giroux domains.*

(a) *If N has at least one blown down boundary component then it contains a small* bLob, *hence (V, ξ) does not have any (semipositive) weak filling.*

(b) *If N contains two Giroux domains $\Sigma^+ \times \mathbb{S}^1$ and $\Sigma^- \times \mathbb{S}^1$ glued together such that Σ^- has a boundary component not touching Σ^+, then (V, ξ) has no (semipositive) weak filling (W, ω) with $[\omega_V] \in \mathcal{O}(\Sigma^+ \times \mathbb{S}^1)$.*

In particular (V, ξ) has no (semipositive) strong filling in either case.

The first statement in this theorem follows immediately from Proposition 5.9 and Theorem 4.4. We will prove the second in Sect. 7, essentially by using the symplectic cobordism construction of the next section to reduce it to the first statement, though some care must be taken because the filling obtained by attaching our cobordism to a given semipositive filling need not always be semipositive. We will also give in Sect. 7 an alternative argument for both parts of Theorem 5.13 using J-holomorphic spheres: this requires slightly stricter homological assumptions than stated above, but has the advantage of not requiring semipositivity at all, due to the polyfold machinery recently developed in [37].

Without delving into the details, we should mention that we also expect the above filling obstruction to be detected algebraically in Symplectic Field Theory via the notion of *algebraic torsion* defined in [39]. Recall that a contact manifold is said to be *algebraically overtwisted* if it has algebraic 0-torsion (this is equivalent to having vanishing contact homology), but there are also infinitely many "higher order" filling obstructions known as algebraic *k-torsion* for integers $k \geq 1$. It turns out that one can always choose the data on a Giroux domain $\Sigma \times \mathbb{S}^1$ so that gradient flow lines of a Morse function on Σ give rise to holomorphic curves in the symplectization of $\Sigma \times \mathbb{S}^1$, and these can be counted in SFT. The expected result is as follows:

Conjecture 5.14 *Suppose (V, ξ) contains a subdomain N as in Theorem 5.13, choose any $c \in \mathcal{O}(N)$ and consider SFT with coefficients in $\mathbb{R}[H_2(V; \mathbb{R})/\ker c]$. Then (V, ξ) has algebraic 1-torsion, and it is also algebraically overtwisted if N contains any blown down boundary components.*

6 Surgery along Giroux domains

6.1 A handle attachment theorem

In this section, we explain a surgery procedure which removes the interior of a Giroux domain from a contact manifold and blows down the resulting boundary. This surgery corresponds to a symplectic cobordism that can be glued on top of any weak filling satisfying suitable cohomological conditions, leading to a proof of Theorem 5.13.

Suppose (V, ξ) is a $(2n - 1)$-dimensional contact manifold without boundary, containing a Giroux domain $G \subset V$, possibly with some boundary components blown down. Removing the interior of G, the boundary of $\overline{V \setminus G}$ is then a ξ-round hypersurface

$$\partial(\overline{V \setminus G}) = M \times \mathbb{S}^1,$$

where (M, ξ_M) is a (possibly disconnected) closed contact manifold. We can thus blow it down as described in Sect. 5.1, producing a new manifold

$$V' := (\overline{V \setminus G})//M$$

without boundary, which inherits a natural contact structure ξ'.

Topologically, the surgery taking (V, ξ) to (V', ξ') can be understood as a certain handle attachment. We now give a point-set description of this handle attachment which is sufficient to state the theorem below, and postpone the smooth description to the next subsection. Assume that G is obtained from the ideal Liouville domain $(\Sigma, \omega, \xi_\Sigma)$ with boundary $\partial \Sigma = M_p \sqcup M_{bd}$ by blowing down the Giroux domain $\Sigma \times \mathbb{S}^1$ at $M_{bd} \times \mathbb{S}^1$ but preserving $M_p \times \mathbb{S}^1$ as in Fig. 2(a) (here bd stands for "blown down", and p for "preserved"). Then topologically,

$$G = \left(M_{bd} \times \mathbb{D}^2\right) \cup_{M_{bd} \times \mathbb{S}^1} \left(\Sigma \times \mathbb{S}^1\right).$$

Note that M_{bd} can now be regarded as a codimension 2 contact submanifold of G, namely by identifying it with $M_{bd} \times \{0\}$.

Next, remove a small open collar neighborhood of M_{bd} from Σ and denote the resulting submanifold by Σ_h. We can regard $\Sigma_h \times \mathbb{S}^1$ as a subdomain of G, and consider the manifold with boundary and corners defined by

$$\left([0, 1] \times V\right) \cup_{\{1\} \times (\Sigma_h \times \mathbb{S}^1)} \left(\Sigma_h \times \mathbb{D}^2\right).$$

After smoothing the corners, this becomes a smooth oriented cobordism W with boundary (see Fig. 2(b)),

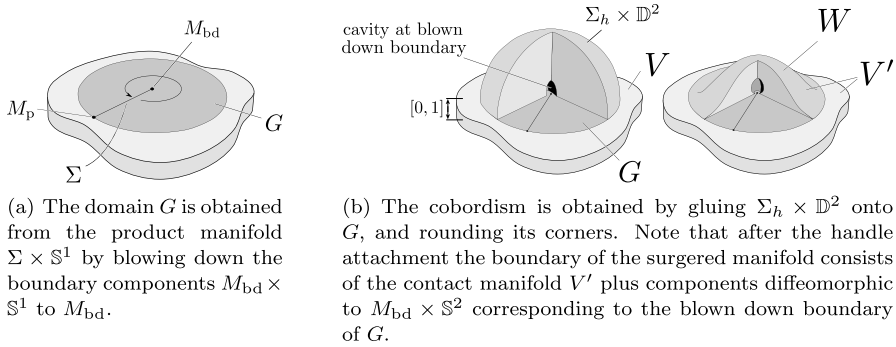$$\partial W = -V \sqcup V' \sqcup \left(M_{bd} \times \mathbb{S}^2\right).$$

(a) The domain G is obtained from the product manifold $\Sigma \times \mathbb{S}^1$ by blowing down the boundary components $M_{\mathrm{bd}} \times \mathbb{S}^1$ to M_{bd}.

(b) The cobordism is obtained by gluing $\Sigma_h \times \mathbb{D}^2$ onto G, and rounding its corners. Note that after the handle attachment the boundary of the surgered manifold consists of the contact manifold V' plus components diffeomorphic to $M_{\mathrm{bd}} \times \mathbb{S}^2$ corresponding to the blown down boundary of G.

Fig. 2 Handle attachment along a Giroux domain

We can now state the main theorem of this section.

Theorem 6.1 *Suppose W denotes the $2n$-dimensional smooth cobordism described above, and Ω is a closed 2-form on V such that*:

- *Ω weakly dominates ξ*
- *the cohomology class of Ω belongs to the obstructed subspace $\mathcal{O}(G)$, i.e. for every 1-cycle Z in Σ,*

$$\int_{Z \times \mathbb{S}^1} \Omega = 0.$$

Then W admits a symplectic structure ω with the following properties:

1. *$\omega|_{TV} = \Omega$.*
2. *The co-core $\Sigma_h \times \{0\} \subset \Sigma_h \times \mathbb{D}^2 \subset W$ is a symplectic submanifold weakly filling $(\partial \Sigma_h \times \{0\}, \xi_\Sigma)$.*
3. *(V', ξ') is a weakly filled boundary component of (W, ω) that is contactomorphic to the blown down manifold $(\overline{V \setminus G})//M_{\mathrm{p}}$.*
4. *A neighborhood of $M_{\mathrm{bd}} \times \mathbb{S}^2 \subset \partial W$ in (W, ω) can be identified symplectically with*

$$\big((-\delta, 0] \times M_{\mathrm{bd}} \times \mathbb{S}^2, \omega_0 \oplus \omega_{\mathbb{S}^2}\big)$$

for some $\delta > 0$, where $\omega_{\mathbb{S}^2}$ is an area form on $\mathbb{S}^2$ and ω_0 is a symplectic form on $(-\delta, 0] \times M_{\mathrm{bd}}$ for which the boundary $(M_{\mathrm{bd}}, \xi_\Sigma)$ is weakly filled. Moreover, the intersection of the co-core $\Sigma \times \{0\}$ with this neighborhood has the form $(-\delta, 0] \times M_{\mathrm{bd}} \times \{\mathrm{const}\}$.

Remark 6.2 Recall that due to Lemma 2.6, a pair of weak symplectic cobordisms can be smoothly glued together along a positive/negative pair of contactomorphic boundary components whenever the symplectic forms restricted to these boundary components match. Thus the symplectic cobordism of the

above theorem can be glued on top of any weak filling (W, ω) of (V, ξ) for which $[\omega|_{TV}] \in \mathcal{O}(G)$.

6.2 Construction of the symplectic cobordism

In this section we will give the proof of Theorem 6.1. The proof will consist of the following five steps:

1. Find a standardized model with a special contact form λ for tubular neighborhoods of ∂G and the blown down components M_{bd}.
2. Construct a symplectic form on our proto-cobordism $[0, 1] \times V$ that is well adjusted to both Ω and λ.
3. Carve out the interior of $\{1\} \times \Sigma \times \mathbb{S}^1$ from $[0, 1] \times V$. This creates a notch with corners along its edges, and we will then smoothly glue the handle $\Sigma \times \mathbb{D}^2$ into the cavity, creating a smooth manifold.
4. Study the symplectic form induced from the proto-cobordism on the glued part of the handle and extend it to the whole handle.
5. Check that the new boundary of the cobordism has the desired properties.

Step 1: Neighborhoods and contact form for G
For simplicity, we first pretend that G is a Giroux domain $\Sigma \times \mathbb{S}^1$ without blown down boundary components. Consider a collar neighborhood $(0, 1] \times \partial \Sigma$ associated to some Liouville form β by Lemma 5.5 and denote by α the corresponding contact form on $\partial \Sigma$. Let s be the coordinate in $(0, 1]$. We denote by u a smooth function $\Sigma \to [0, 1]$ which has the boundary $\partial \Sigma = u^{-1}(0)$ as a regular level set, equals $1 - s$ in the region $s \geq 3/4$ and 1 in the region $s \leq 1/4$ and outside the collar, and satisfies $u' \leq 0$ everywhere on the collar (see Fig. 3). We set $\gamma = u\beta$. The contact form on G associated to β and u is then $\lambda := u\beta + u \, d\theta = \gamma + u \, d\theta$, where θ denotes the coordinate on $\mathbb{S}^1$. In the collar one can set $f := u/(1 - s)$ so that $\lambda = f\alpha + u \, d\theta$. Note that the contact condition in $(0, 1] \times \partial \Sigma \times \mathbb{S}^1$ is equivalent to

$$f(f'u - u'f) > 0, \tag{6.1}$$

so appealing to Lemma 5.1, we can slightly extend our collar neighborhood embedded in (V, ξ) to one of the form $(0, 1 + \varepsilon] \times \partial \Sigma \times \mathbb{S}^1$, with λ written as above and u extended as $1 - s$ when $s > 1$.

In the more general case where G is a Giroux domain with some boundary components blown down, the function u becomes r^2 in $M_{bd} \times \mathbb{D}^2$, so λ is still a smooth contact form (but of course there is no extended collar).

Step 2: The symplectic form on $[0, 1] \times V$
The assumption that Ω weakly dominates ξ implies that the 2-form $\omega = d(t\lambda) + \Omega$ is symplectic on $(-\delta, 1] \times V$ for some small positive constant δ.

Fig. 3 The functions u and f
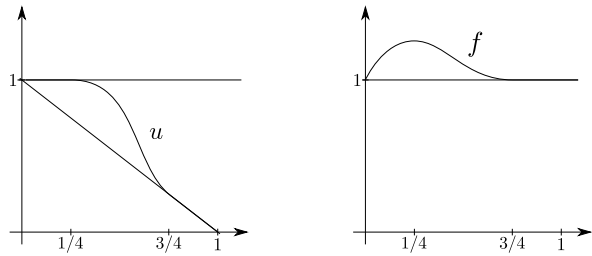
The cohomological assumption $[\Omega] \in \mathcal{O}(G)$ implies that Ω is cohomologous to some 2-form ω_0 such that $\omega_0|_G$ is the pull back of a 2-form on Σ. In addition, since the collar neighborhood $(0, 1] \times \partial \Sigma$ retracts to $\partial \Sigma$, we can assume that $\iota_{\partial_s} \omega_0 = 0$ when $s \geq 1/4$.

Lemma 6.3 *We can modify the form ω defined above to a new symplectic form on $(-\delta, 1] \times V$, keeping the assumption that ω restrict to Ω on $\{0\} \times V$ and ξ be weakly dominated by ω on each slice $\{t\} \times V$, but asking in addition that ω restrict to $C\, d(t\lambda) + \omega_0$ on $[1/2, 1] \times V$ for some large constant $C > 0$.*

Proof Using Lemma 2.10, we find a symplectic form ω' on $(-\delta, \infty) \times V$ such that each $\{t\} \times V$ is still weakly filled and ω' restricts to $d(t\lambda) + \omega_0$ for t greater than some large constant $C/2$. The scaling diffeomorphism $(t, v) \mapsto (t/C, v)$ pulls back ω' to the desired symplectic form. □

Step 3: Handle attachment
We now give a smooth description of the handle attachment which is compatible with the smooth description of the blow-down process for ξ-round hypersurfaces. For this, we will first create a small basin in the top of $[0, 1] \times V$ to which we can glue in the handle.

Recall that M_{bd} denotes the blown down boundary components of the Liouville domain Σ, and M_{p} denotes the other components. Let h be a smooth function from Σ to $(1/2, \infty)$ such that

- h restricts on the special collar of Step 1 to a function only depending on s with nonnegative derivative $h'(s)$,
- h is constant on $(0, 1/4) \times \partial \Sigma$ and outside the boundary collar,
- For $s \geq 1 - \varepsilon$, $h(s) = s$ near M_{p}, and $h(s) = s + \varepsilon$ near M_{bd}.

We denote by Σ_h the subset of Σ on which h is less than or equal to 1, and by $\mathcal{H} \subset [0, 1] \times V$ the product of $\mathbb{S}^1$ with the graph of h over Σ_h, see Fig. 4. We discard the region $\{t \geq h\}$ from $[0, 1] \times V$ to get an open manifold, to which we will glue the "handle" $\Sigma_h \times D$. Here D denotes the disk around the origin in $\mathbb{R}^2$ with radius $\sqrt{\varepsilon}$. In the following, we will find a symplectic vector field X in a neighborhood of the hypersurface $\mathcal{H}$ in $[0, 1] \times V$ that
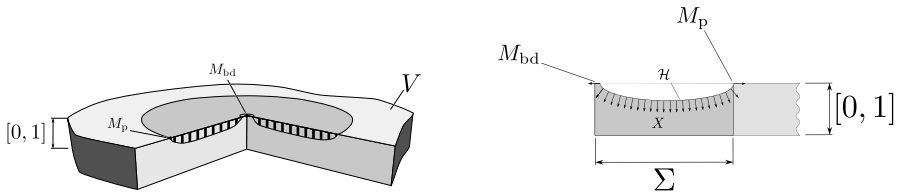
(a) The precise construction of the handle attachment sketched in Fig. 2(b) works by creating a trench on the top side of the cobordism $[0, 1] \times V$ to which we can glue in the handle. In the picture above we need to remove the hatched area, which corresponds to the Giroux domain $\Sigma \times \mathbb{S}^1$ minus a small neighborhood of the blown down boundary.

(b) The vector field X is tangent to the top face and transverse to the hypersurface $\mathcal{H}$, which is Σ_h pushed inside $[0, 1] \times \Sigma$ relative to its boundary. Everything above $\mathcal{H}$ has been discarded to make room for the handle.

Fig. 4 Smooth handle attachment

is transverse to $\mathcal{H}$, never points in the positive t-direction, and is tangent to $\{1\} \times V$ near the boundary of $\mathcal{H}$. Shrinking ε if needed, we may assume that the flow of X starting from $\mathcal{H}$ embeds $\mathcal{H} \times [0, \varepsilon]$ into $[1/2, 1] \times V$. We denote by φ_τ^X the flow of X at time τ. The manifold W' is obtained by attaching $\Sigma_h \times D$ to $([0, 1] \times V) \setminus \{t \geq h\}$ using the gluing map Ψ from $\Sigma_h \times D^*$ (where $D^* = D \setminus \{0\}$) to $[0, 1] \times V$ defined by

$$\Psi\big(\sigma, re^{i\theta}\big) = \varphi_{r^2}^X\big(h(\sigma), \sigma, \theta\big).$$

Note that as a point-set operation, the handle attachement reduces to the operation of adding the co-core $\Sigma_h \times \{0\}$ to the open manifold $([0, 1] \times V) \setminus \{t \geq h\}$.

The vector field X that we will use below coincides with ∂_s near $\{1\} \times \partial G$. This implies that the attachment using Ψ restricts precisely to the gluing map used to blow down the ξ-round hypersurface $M_p \times \mathbb{S}^1$.

As a gluing vector field X, we choose the ω-dual of $-C \, d\theta$, where C is the constant appearing in ω. Since this 1-form is closed, X is a symplectic vector field.

Lemma 6.4 *The vector field X is transverse to the hypersurface $\mathcal{H}$ and coincides with ∂_s near $\{1\} \times \partial G$ and $\{1\} \times M_{bd}$.*

Proof Away from the special collar neighborhoods considered in Step 1, $\lambda = d\theta + \beta$, and this gives $dt(X) = -1$. Elsewhere, on the collars $[0, 1] \times ([1/4, 1] \times \partial \Sigma \times \mathbb{S}^1)$, we use the ansatz $X = X^t \partial_t + X^s \partial_s$. Computing the interior product $\iota_X \omega$ using $\omega = C \, d(t(u \, d\theta + f\alpha)) + \omega_0$ and $\iota_{\partial_s} \omega_0 = 0$, we

find that X is indeed ω-dual to $-C\,d\theta$ provided

$$uX^t + tu'X^s = -1,$$

$$fX^t + tf'X^s = 0.$$

This system is everywhere nonsingular due to the contact condition (6.1). For $s \geq 3/4$ and $t = 1$, we have $X = \partial_s$ as promised. For $s < 3/4$, the conditions $f(s) > 0$ and $f'(s) > 0$ imply $X^t < 0$ and $X^s > 0$, hence X is transverse to $\mathcal{H}$. $\qquad\square$

Step 4: Symplectic form on the handle

Lemma 6.5 *The gluing map Ψ from $\Sigma_h \times D^*$ to $[0, 1] \times V$ pulls back ω to*

$$\Psi^*\omega = 2C\omega_D + C\,d(hu) \wedge d\theta + \Omega_0,$$

where $\omega_D := -r\,dr \wedge d\theta$ and Ω_0 is a symplectic form on Σ_h which weakly fills $(\partial\Sigma_h, \ker\gamma)$.

Proof Let $j_{\mathcal{H}}$ denote the embedding $\Sigma_h \times \mathbb{S}^1 \to \mathcal{H} \subset [0, 1] \times V, (\sigma, \theta) \mapsto (h(\sigma), \sigma, \theta)$. Then we can decompose Ψ as $\Psi = \Phi \circ P$, where P is the map from $\Sigma_h \times D^*$ to $\Sigma_h \times (0, \varepsilon] \times \mathbb{S}^1$ defined by $P(\sigma, re^{i\theta}) = (\sigma, r^2, \theta)$ and

$$\Phi(\sigma, \tau, \theta) := \varphi_\tau^X\big(h(\sigma), \sigma, \theta\big) = \varphi_\tau^X\big(j_{\mathcal{H}}(\sigma, \theta)\big).$$

Using the fact that the flow of $(\varphi_\tau^X)_*\partial_\tau = X$ preserves ω and $\iota_X\omega = -C\,d\theta$, we obtain for the pull back

$$\Phi^*\omega = -C\,d\tau \wedge d\theta + j_{\mathcal{H}}^*\omega,$$

and since the symplectic form ω is given in the range of $j_{\mathcal{H}}$ by $C\,d(t\lambda) + \omega_0$ with $\lambda = u\,d\theta + \gamma$, we can compute

$$j_{\mathcal{H}}^*\omega = C\,d(h\lambda) + \omega_0 = C\,d(hu) \wedge d\theta + \Omega_0,$$

where we have set $\Omega_0 = C\,d(h\gamma) + \omega_0$ (which is a 2-form on Σ_h).

Now since $P^*d\tau = 2r\,dr$, the only thing left to prove is that Ω_0 is a symplectic form which weakly fills $(\partial\Sigma_h, \ker\gamma)$. Since ω_D is the only term in $\Psi^*\omega$ that contains a dr-factor, and thus it follows that $(\Psi^*\omega)^n = 2nC\omega_D \wedge \Omega_0^{n-1} \neq 0$, we deduce that Ω_0 is symplectic.

The 2-form Ω_0 restricts on $\partial\Sigma_h$ to $C\,d\gamma + \omega_0$. Recall that the weakly dominating condition on $\{1\} \times V$ means that for any constant $\nu \geq 0$, $\lambda \wedge (\omega + \nu\,d\lambda)^{n-1} > 0$. Restricting to $\{1\} \times G$, where $\lambda = u\,d\theta + \gamma$ and $\omega = C\,d\lambda + \omega_0$, this becomes:

$$(u\,d\theta + \gamma) \wedge \big[(C + \nu)\,du \wedge d\theta + (C\,d\gamma + \omega_0 + \nu\,d\gamma)\big]^{n-1} > 0,$$

which we expand along $\{1\} \times \partial \Sigma_h \times \mathbb{S}^1$ as

$$(n-1)(C+v)\gamma \wedge du \wedge d\theta \wedge (C\,d\gamma + \omega_0 + v\,d\gamma)^{n-2} > 0.$$

In particular, this proves that $\gamma \wedge (\Omega_0 + v\,d\gamma)^{n-2}$ never vanishes. In order to check that it has the correct sign, it suffices to consider the case $v = 0$. $\qquad\square$

To finish the construction of the symplectic cobordism, we want to define a symplectic structure on $\Sigma_h \times D$ that agrees in a neighborhood of the boundary $\Sigma_h \times \partial D$ with $\Psi^*\omega$, and that has a split form near $\Sigma_h \times \{0\}$. Let ρ_1 and ρ_2 be functions from $[0, \sqrt{\varepsilon}]$ to $\mathbb{R}$ (constraints will be added later). We set:

$$\widetilde{\omega} := 2C\rho_1\omega_D + C\,d(\rho_2 hu) \wedge d\theta + \Omega_0$$

$$= g\omega_D + C\rho_2\,d(hu) \wedge d\theta + \Omega_0 \quad \text{with } g := \left(2\rho_1 - \frac{hu\rho_2'}{r}\right)C.$$

We choose $\rho_1(r) = \rho_2(r) = 1$ for r close to $\sqrt{\varepsilon}$, so that $\widetilde{\omega}$ extends $\Psi^*\omega$. Near 0, we choose ρ_1 to be a large positive constant and ρ_2 to vanish so that $\widetilde{\omega}$ makes sense near the center of D. One can compute $\widetilde{\omega}^n = ng\omega_D \wedge \Omega_0^{n-1}$. Since Ω_0 is symplectic on Σ_h, we see that $\widetilde{\omega}$ is symplectic as soon as g is positive. This condition is arranged by choosing ρ_1 sufficiently large away from $r = \sqrt{\varepsilon}$.

Step 5: Properties of the new boundary of W
We now consider in turns the two types of new boundary components resulting from the above construction: V' and $M_{\mathrm{bd}} \times \mathbb{S}^2$. Since hu is constant on $\partial \Sigma_h$, the restriction of $\widetilde{\omega}$ to $M_{\mathrm{p}} \times D$ is $g\omega_D + \Omega_0$. As we already noted, the gluing map Ψ extends the one used to define the blow-down, and the contact form on V' is $\lambda = \gamma - r^2 d\theta$. Thus in order to check the weak filling condition, we only need compute, for any constant $v \geq 0$,

$$\lambda \wedge (\widetilde{\omega} + v\,d\lambda)^{n-1} = (n-1)(g+2v)\omega_D \wedge \gamma \wedge (\Omega_0 + v\,d\gamma)^{n-2}.$$

This is indeed a positive volume form for any nonnegative v because (Σ_h, Ω_0) is a weak filling of $(M_{\mathrm{p}}, \ker\gamma)$ according to Lemma 6.5.

The situation along $M_{\mathrm{bd}} \times \mathbb{S}^2$ is very similar. There $\widetilde{\omega}$ induces $\omega_{\mathbb{S}^2} + \Omega_0$ for some area form $\omega_{\mathbb{S}^2}$. The distribution we consider is $\ker\gamma$. We compute:

$$\gamma \wedge (\widetilde{\omega} + v\,d\gamma)^{n-1} = (n-1)\omega_{\mathbb{S}^2} \wedge \gamma \wedge (\Omega_0 + v\,d\gamma)^{n-2}$$

so the restriction of $\widetilde{\omega}$ is symplectic on $\ker\gamma$ because of Lemma 6.5. Lemma 2.6 then gives us a neighborhood of $M_{\mathrm{bd}} \times \mathbb{S}^2$ in $(W', \widetilde{\omega})$ that can be identified symplectically with

$$\left((-\varepsilon, 0] \times M_{\mathrm{bd}} \times \mathbb{S}^2, \big(d(t\gamma) + \Omega_0\big) \oplus \omega_{\mathbb{S}^2}\right)$$

for $\varepsilon > 0$ sufficiently small. Observe also that $\widetilde{\omega}$ already takes this split form in a neighborhood of the co-core $\Sigma_h \times \{0\} \subset W'$, so we can apply the extension property of Lemma 2.6 to get a collar whose intersection with the co-core is precisely $(-\varepsilon, 0] \times M_{\mathrm{bd}} \times \{0\}$ with $0 \in \mathbb{D} \subset \mathbb{S}^2$.

7 Giroux domains and non-fillability

We now use the cobordism of the preceding section to prove Theorem 5.13 on filling obstructions. We will present two slightly different approaches in Sect. 7.1 and Sect. 7.2 respectively: the first uses holomorphic disks and the bLob, thus relying on a version of Theorem 4.4. The second approach uses holomorphic spheres and proves a slightly weaker result, as it requires stricter homological assumptions on the Giroux domains—though it should be mentioned that these assumptions are satisfied in all the interesting examples we know thus far, namely for the higher dimensional notions of Lutz twists and Giroux torsion defined in Sect. 9. The use of spheres instead of disks simplifies the proof in that it makes the Fredholm and compactness properties easier, while at the same time allowing the use of the recently completed polyfold technology [37] to handle transversality issues. For this reason the second proof does not require semipositivity.

7.1 Proof of Theorem 5.13 via the bLob

Part (a) of the theorem follows immediately from the fact that if (V, ξ) contains a Giroux domain N that has some boundary components that are blown down and others that are not, then by Proposition 5.9 it contains a small bLob, so Theorem 4.4 implies that (V, ξ) does not admit any semipositive weak filling.

To prove part (b), suppose N has the form

$$N = \left(\Sigma^+ \times \mathbb{S}^1 \right) \cup_{Y \times \mathbb{S}^1} \left(\Sigma^- \times \mathbb{S}^1 \right),$$

where $\Sigma^{\pm}$ are ideal Liouville domains with boundary $\partial \Sigma^{\pm} = \partial_{\mathrm{glue}} \Sigma^{\pm} \sqcup \partial_{\mathrm{free}} \Sigma^{\pm}$, $Y := \partial_{\mathrm{glue}} \Sigma^+ = \partial_{\mathrm{glue}} \Sigma^-$ carries the induced contact form α and $\partial_{\mathrm{free}} \Sigma^-$ is not empty. Arguing by contradiction, assume that (V, ξ) is weakly filled by a semipositive symplectic filling (W_0, ω) with $[\omega|_{TV}] \in \mathcal{O}(\Sigma^+)$. This establishes the cohomological condition needed by Theorem 6.1 on $\Sigma^+ \times \mathbb{S}^1$, so applying the theorem, we can enlarge (W_0, ω) by attaching $\Sigma^+ \times \mathbb{D}^2$, producing a compact symplectic manifold (W_1, ω) whose boundary (V', ξ') supports a contact structure that is weakly filled.

Since the boundary V' of the new symplectic manifold (W_1, ω) is contactomorphic to $\overline{(V \setminus (\Sigma^+ \times \mathbb{S}^1))} / / Y$, we find in (V', ξ') a domain isomorphic

to $(\Sigma^- \times \mathbb{S}^1)//Y$ that contains a small bLob. Unfortunately this does not directly obstruct the existence of the weak filling (W_1, ω), because even though W_0 was semipositive, W_1 might not be. We will follow the proof of Theorem 4.4, with the difference that we need to reconsider compactness to make sure that bubbling is still a "codimension 2 phenomenon".

Choose an almost complex structure J on (W_1, ω) with the following properties:

(i) J is tamed by ω and makes (V', ξ') strictly J-convex,
(ii) J is adapted to the bLob in the standard way, i.e. it is chosen close to the boundary of the bLob according to Lemma 4.7 and in a neighborhood of the binding according to [47] (cf. the proof of Theorem 4.4),
(iii) for some small radius $r > 0$, $J = J_{\Sigma^+} \oplus i$ on $\Sigma^+ \times \mathbb{D}_r^2 \subset W_1$, where J_{Σ^+} is a tamed almost complex structure on Σ^+ for which $\partial \Sigma^+$ is J_{Σ^+}-convex.

The third condition uses the fact from Theorem 6.1 that the co-core $\mathcal{K}' := \Sigma^+ \times \{0\}$ of the handle is a symplectic (and now also J-holomorphic) hypersurface weakly filling its boundary. The binding of the bLob lies in the boundary of the co-core $\mathcal{K}'_+$, and the normal form described in [47] is compatible with the splitting $\Sigma^+ \times \mathbb{D}_r^2$ so that (ii) and (iii) can be simultaneously achieved.

By choosing J_{Σ^+} generic, we can also assume that every somewhere injective J_{Σ^+}-holomorphic curve in Σ^+ is Fredholm regular and thus has nonnegative index. Note that any closed J-holomorphic curve in $\Sigma^+ \times \mathbb{D}_r^2$ is necessarily contained in $\Sigma^+ \times \{z\}$ for some $z \in \mathbb{D}_r^2$, and the index of this curve differs from its index as a J_{Σ^+}-holomorphic curve in Σ^+ by the Euler characteristic of its domain. This implies that every somewhere injective J-holomorphic sphere contained in $\Sigma^+ \times \mathbb{D}_r^2$ has index at least 2. Likewise, by a generic perturbation of J outside of this neighborhood we may assume all somewhere injective curves that are *not* contained entirely in $\Sigma^+ \times \mathbb{D}_r^2$ also have nonnegative index.

Now let $\mathcal{M}$ be the connected moduli space of holomorphic disks attached to the bLob that contains the standard Bishop family. We can cap off every holomorphic disk $u \in \mathcal{M}$ by attaching a smooth disk that lies in the bLob, producing a trivial homology class in $H_2(W_1)$. The cap and the co-core intersect exactly once, and it follows that u also must intersect the co-core $\mathcal{K}'_+$ exactly once, because u and $\mathcal{K}'_+$ are both J-complex.

To finish the proof, we have to study the compactness of $\mathcal{M}$ and argue that $\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of strata of codimension at least 2. A nodal disk u_∞ lying in $\overline{\mathcal{M}} \setminus \mathcal{M}$ has exactly one disk component u_0, which is injective at the boundary, and one component u_+ that intersects the co-core once; either $u_+ = u_0$ or u_+ is a holomorphic sphere. Every other nonconstant connected component v is a holomorphic sphere whose homology class has vanishing

intersection with the relative class $[\mathcal{K}'_+]$. So either v does not intersect the J-complex submanifold $\mathcal{K}'_+$ at all or v is completely contained in $\mathcal{K}'_+$. In either case, v is homotopic to a sphere lying in W_0: indeed, if v does not intersect the co-core, we can move it out of the handle by pushing it radially from $\Sigma^+ \times (\mathbb{D}^2 \setminus \{0\})$ into the boundary $\Sigma^+ \times \mathbb{S}^1 \subset W_0$, and if $v \subset \mathcal{K}'_+ = \Sigma_+ \times \{0\}$, then we can simply shift it to $\Sigma_+ \times \{1\} \subset W_0$. Using the fact that u_0 and u_+ are both somewhere injective, together with the semipositivity and genericity assumptions, we deduce that every connected component of u_∞ has nonnegative index, thus $\overline{\mathcal{M}} \setminus \mathcal{M}$ has codimension at least two in $\overline{\mathcal{M}}$. The rest of the proof is the same as for Theorem 4.4.

7.2 An alternative argument using holomorphic spheres

In this section we will prove the following variation on Theorem 5.13, which does not involve the word "semipositive" at all.

Theorem 7.1 *Suppose (V, ξ) is a closed contact manifold containing a subdomain N with nonempty boundary, which is obtained by gluing and blowing down Giroux domains. If either N has at least one blown down boundary component or it includes at least two Giroux domains glued together, then (V, ξ) does not admit any weak filling (W, ω) with $[\omega_V] \in \mathcal{O}(N)$. In particular (V, ξ) is not strongly fillable.*

Proof We consider first the case where N has no blown down boundary components but consists of at least two Giroux domains glued together: without loss of generality, we may then assume N has the form

$$N = \left(\Sigma^+ \times \mathbb{S}^1\right) \cup_{Y \times \mathbb{S}^1} \left(\Sigma^- \times \mathbb{S}^1\right),$$

where $\Sigma^\pm$ are ideal Liouville domains with boundary $\partial \Sigma^\pm = \partial_{\mathrm{glue}} \Sigma^\pm \sqcup \partial_{\mathrm{free}} \Sigma^\pm$, $Y := \partial_{\mathrm{glue}} \Sigma^+ = \partial_{\mathrm{glue}} \Sigma^-$ carries the induced contact form α and $\partial_{\mathrm{free}} \Sigma^+$ is not empty. Arguing by contradiction, assume also that (V, ξ) has a weak filling (W_0, ω) with $[\omega|_{TV}] \in \mathcal{O}(N)$. This establishes the cohomological condition needed by Theorem 6.1 on both $\Sigma^+ \times \mathbb{S}^1$ and $\Sigma^- \times \mathbb{S}^1$, so applying the theorem, we can enlarge (W_0, ω) by attaching $\Sigma^+ \times \mathbb{D}^2$ and $\Sigma^- \times \mathbb{D}^2$ in succession, producing a compact symplectic manifold (W_1, ω) whose boundary is a disjoint union of pieces

$$\partial W_1 = \partial_{\mathrm{sph}} W_1 \sqcup \partial_{\mathrm{ct}} W_1,$$

where $\partial_{\mathrm{ct}} W_1 \neq \emptyset$ supports a contact structure that is weakly dominated and $\partial_{\mathrm{sph}} W_1 \cong Y \times \mathbb{S}^2$ with symplectic fibers $\{*\} \times \mathbb{S}^2$ (here ct stands for "contact" and sph for "sphere"). Moreover, a neighborhood of $\partial_{\mathrm{sph}} W_1$ in (W_1, ω) can

be identified symplectically with the collar

$$\big((-\delta, 0] \times Y \times \mathbb{S}^2, \omega_Y \oplus \omega_{\mathbb{S}^2}\big),$$

where $\omega_{\mathbb{S}^2}$ is an area form on $\mathbb{S}^2$ and ω_Y is a symplectic form on $(-\delta, 0] \times Y$ with weakly filled boundary $(Y, \ker \alpha)$.

We can choose an almost complex structure J on (W_1, ω) with the following properties:

(i) J is tamed by ω and makes $\partial_{ct} W_1$ strictly J-convex,
(ii) $J = J_Y \oplus j$ on the collar $(-\delta, 0] \times Y \times \mathbb{S}^2$, where j is an $\omega_{\mathbb{S}^2}$-compatible almost complex structure on $\mathbb{S}^2$ and J_Y is an almost complex structure on $(-\delta, 0] \times Y$ which is tamed by ω_Y and makes $\{0\} \times Y$ strictly J_Y-convex.

By choosing J generic outside the collar neighborhood $(-\delta, 0] \times Y \times \mathbb{S}^2$, we may assume all somewhere injective J-holomorphic curves that aren't contained entirely in that region are Fredholm regular.

For each $(t, y) \in (-\delta, 0] \times Y$, there is now an embedded pseudoholomorphic sphere

$$u_{(t,y)} \colon \big(\mathbb{S}^2, j\big) \to \big((-\delta, 0] \times Y \times \mathbb{S}^2, J\big), \ z \mapsto (t, y, z).$$

These curves are all Fredholm regular: indeed, a neighborhood of $u_{(t,y)}$ in the moduli space of unparametrized J-holomorphic spheres in W_1 can be identified with a neighborhood of zero in the kernel of the linearized Cauchy-Riemann operator on its normal bundle, but the latter is a trivial bundle with the standard Cauchy-Riemann operator. Hence the operator splits into a direct sum of standard Cauchy-Riemann operators on trivial line bundles over $\mathbb{S}^2$, all of which have index 2 and are surjective by the Riemann-Roch theorem (cf. [44]). It follows also that the curves $u_{(t,y)}$ have index $2n - 2$.

Denote the co-cores of the two handles by

$$\mathcal{K}'_\pm := \Sigma^\pm \times \{0\} \subset \Sigma^\pm \times \mathbb{D}^2 \subset W_1.$$

The curves $u_{(t,y)}$ have exactly one transverse intersection with each of the two co-cores, so we have homological intersection numbers:

$$[u_{(t,y)}] \bullet [\mathcal{K}'_+] = [u_{(t,y)}] \bullet [\mathcal{K}'_-] = 1. \tag{7.1}$$

We claim that *every* somewhere injective J-holomorphic sphere in W_1 which intersects $\{0\} \times Y \times \mathbb{S}^2$ is equivalent to $u_{(0,y)}$ for some $y \in Y$. Indeed, if $u \colon \mathbb{S}^2 \to W_1$ is any such sphere, we define the open subset $\mathcal{U} = u^{-1}((-\delta, 0] \times Y \times \mathbb{S}^2)$, and observe that $u|_{\mathcal{U}}$ can be identified with a pair of maps $u_{\mathbb{S}^2} \colon \mathcal{U} \to \mathbb{S}^2$ and $u_Y \colon \mathcal{U} \to (-\delta, 0] \times Y$ which are j-holomorphic and J_Y-holomorphic respectively. But then u_Y touches the boundary of $(-\delta, 0] \times Y$ tangentially,

which is impossible due to pseudoconvexity unless u_Y is constant, so we conclude that $\mathcal{U} = \mathbb{S}^2$ and $u_{\mathbb{S}^2} \colon \mathbb{S}^2 \to \mathbb{S}^2$ is a degree 1 holomorphic map, hence the identity up to parameterization.

Let $\mathcal{M}$ denote the connected component of the moduli space of unparametrized J-holomorphic spheres containing the curves $u_{(t,y)}$, and define $\mathcal{M}_1$ to be the same space of curves but with one marked point, along with the natural compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_1$, consisting of nodal J-holomorphic spheres. Note that curves in $\overline{\mathcal{M}}$ can never touch $\partial_{\mathrm{ct}} W_1$ due to J-convexity. By (7.1), every curve in $\mathcal{M}$ intersects each of $\mathcal{K}'_+$ and $\mathcal{K}'_-$ algebraically once, thus all curves in $\mathcal{M}$ are somewhere injective, and the only nodal curves in $\overline{\mathcal{M}}$ intersecting $\{0\} \times Y \times \mathbb{S}^2$ are $u_{(0,y)}$ for $y \in Y$. Now our genericity assumptions for J, along with the Fredholm regularity of the special curves $u_{(t,y)}$, imply that $\mathcal{M}$ is a smooth $(2n-2)$-dimensional manifold with boundary, where the boundary is a single connected component consisting of the curves $u_{(0,y)}$. After perhaps shrinking $\delta > 0$, we claim in fact that every curve in $\mathcal{M}$ intersecting $(-\delta, 0] \times Y \times \mathbb{S}^2$ is one of the special curves $u_{(t,y)}$: were this not the case, we would find sequences of negative numbers $t_k \to 0$ and holomorphic spheres $u_k \in \mathcal{M}$ which are not equivalent to any $u_{(t,y)}$ but intersect $\{t_k\} \times Y \times \mathbb{S}^2$, and a subsequence then converges to a nodal curve in $\overline{\mathcal{M}}$ intersecting $\{0\} \times Y \times \mathbb{S}^2$, but the latter must be of the form $u_{(0,y)}$. We then obtain a contradiction from the implicit function theorem, as the $(2n-2)$-dimensional moduli space of curves close to $u_{(0,y)}$ consists only of curves of the form $u_{(t',y')}$ for some $(t', y') \in (-\delta, 0] \times Y$.

Although $\mathcal{M}$ and $\mathcal{M}_1$ are smooth as explained above, their compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_1$ need not be, due to the presence of nodal curves with multiply covered components for which transversality fails. This is exactly the kind of problem that the polyfold machinery of Hofer-Wysocki-Zehnder [37] is designed to solve: we perturb the nonlinear Cauchy-Riemann equation via a generic multisection of the appropriate polyfold bundle so that the compact space $\overline{\mathcal{M}}'$ of solutions to this perturbed equation is, in general, an oriented, weighted branched orbifold with boundary and corners. In the case at hand, the perturbation can be chosen to have support in a neighborhood of the nonsmooth part of $\overline{\mathcal{M}}$, thus we may assume in particular that elements of $\overline{\mathcal{M}}'$ approaching a neighborhood of the boundary are still actually J-holomorphic curves, so the uniqueness statements above continue to hold.

To conclude the proof, choose a smoothly embedded path $\ell \subset W_1$ with one boundary point in $\partial_{\mathrm{ct}} W_1$ and the other in $\partial_{\mathrm{sph}} W_1$, meeting both transversely, and define the compact space

$$\overline{\mathcal{M}}'_\ell = \mathrm{ev}^{-1}(\ell)$$

where $\mathrm{ev} \colon \overline{\mathcal{M}}'_1 \to W_1$ denotes the natural evaluation map. For generic choices, $\overline{\mathcal{M}}'_\ell$ is then a smooth, compact, oriented, weighted branched

1-dimensional manifold with boundary, the latter being $\text{ev}^{-1}(\partial\ell)$. By pseudoconvexity however, no curve in $\overline{\mathcal{M}}'$ intersects $\partial_{ct}W_1$, hence $\partial\overline{\mathcal{M}}'_\ell = \text{ev}^{-1}(\partial_{sph}W_1)$, but this consists of only one curve, namely the unique $u_{(0,y)}$ with $y \in \partial\ell$. Since there is no compact, oriented, weighted branched 1-manifold with connected boundary (see e.g. [53, Lemma 5.11]), this gives the desired contradiction.

The proof is essentially the same but slightly simpler if $N \subset (V, \xi)$ has any blown down boundary components. If suffices then to consider the case where N is a single blown down Giroux domain $(\Sigma \times \mathbb{S}^1)//M_{bd}$. Attaching $\Sigma \times \mathbb{D}^2$ via Theorem 6.1, we again obtain a symplectic manifold (W_1, ω) with $\partial W_1 = \partial_{ct}W_1 \sqcup \partial_{sph}W_1$, where $\partial_{ct}W_1 \neq \emptyset$ is weakly filled and $\partial_{sph}W_1 \cong M_{bd} \times \mathbb{S}^2$ is foliated by symplectic spheres that give rise to J-holomorphic spheres intersecting the co-core $\Sigma \times \{0\}$ exactly once. The rest of the argument is the same. $\square$

Remark 7.2 If the original filling is assumed semipositive, then the above proof can also be modified to take advantage of the symplectic co-core in the same way as Sect. 7.1 and thus avoid the need for polyfolds.

8 Construction of Liouville domains with disconnected boundary

8.1 Contact products and Liouville pairs

In this section we shall construct Liouville pairs on closed manifolds of every odd dimension; more precisely, we shall prove Theorem C from the introduction and thus lay the groundwork for our Giroux torsion construction in Sect. 9. Recall that the goal is to find positive/negative pairs of contact forms (α_+, α_-) on oriented odd-dimensional manifolds M with the property that, if $s \in \mathbb{R}$ denotes the coordinate on the first factor of $\mathbb{R} \times M$,

$$\beta := e^{-s}\alpha_- + e^s\alpha_+$$

defines a positively oriented Liouville form on $\mathbb{R} \times M$.

The first example of a Liouville pair is $\pm d\theta$ on $\mathbb{S}^1$. One can construct higher dimensional examples using contact products. The contact product of (M_1, ξ_1) and (M_2, ξ_2) is defined as the product of their symplectizations $S\xi_1 \times S\xi_2$ divided by the diagonal $\mathbb{R}$-action (cf. [33]). This describes a contact manifold but, since the Liouville pair condition is really about contact forms and not only contact structures, we want a more specific construction. Suppose we have contact forms α_1 and α_2. Those give identifications between $S\xi_i$ and $\mathbb{R} \times M_i$ with fiber coordinates t_i on $\mathbb{R}$. On the product, one has the Liouville form $\lambda = e^{t_1}\alpha_1 + e^{t_2}\alpha_2$ and its dual vector field $X = \partial_{t_1} + \partial_{t_2}$.

We shall say that a manifold V with a contact form λ is a *linear model* for the contact product of (M_1, α_1) and (M_2, α_2) if it is realized as a hypersurface in $S\xi_1 \times S\xi_2$ transverse to X and defined by a linear equation on t_1 and t_2. Concretely, this means $V = M_1 \times \mathbb{R} \times M_2$ is embedded into the product $(\mathbb{R} \times M_1) \times (\mathbb{R} \times M_2)$ by $\varphi(m_1, t, m_2) = (\mu t, m_1, \nu t, m_2)$ for some constants μ and ν. This gives a hypersurface positively transverse to X provided $\nu > \mu$. The contact form induced by λ on V is then $e^{\mu t}\alpha_1 + e^{\nu t}\alpha_2$.

Proposition 8.1 *If M_1 is $\mathbb{R}$ or $\mathbb{S}^1$ endowed with the Liouville pair $\alpha_\pm = \pm d\theta$ and (M_2, α_2) is any manifold with a contact form, then any linear model for the contact product inherits a Liouville pair $\pm e^{\mu t} d\theta + e^{\nu t} \alpha$.*

Proof We set $a = e^s + e^{-s}$, $b = e^s - e^{-s}$ and $e_\rho = e^{\rho t}$ for any real number ρ so that our candidate Liouville form on $\mathbb{R} \times M_1 \times \mathbb{R} \times M_2$ is $B = a e_\nu \alpha + b e_\mu d\theta$. One computes

$$dB = e_\nu(a\nu\, dt + b\, ds) \wedge \alpha + a e_\nu\, d\alpha + e_\mu(b\mu\, dt + a\, ds) \wedge d\theta$$

and then, denoting by $2q + 1$ the dimension of M_2,

$$dB^{q+2} = f\left(\nu a^2 - \mu b^2\right) ds \wedge d\theta \wedge dt \wedge \alpha \wedge d\alpha^q,$$

where $f = (q + 1)(q + 2)a^q e_{\mu+(q+1)\nu}$ and $\nu a^2 - \mu b^2$ is positive because $\nu > \mu$ and $a^2 - b^2 = 4$. $\qquad\square$

Remark 8.2 One can ask whether the above proposition extends to other Liouville pairs. It seems that not all linear models will be suitable for this. What we can prove, but will not use in this paper, is that if $\alpha_\pm$ is a Liouville pair on some manifold M_1 (of any dimension) then $\alpha_\pm + e^t \alpha$ is a Liouville pair on $M_1 \times \mathbb{R} \times M_2$.

Of course, the disadvantage of the contact product construction is that the resulting manifold is never compact, and there seems to be no general way of finding compact quotients of contact products. We shall therefore specialize further by seeking examples among Lie groups which can be seen as symplectizations of some subgroups that have co-compact lattices. (The idea to consider left-invariant contact forms on Lie groups is borrowed from Geiges [23].)

Before that, let us describe a corollary of the following algebraic constructions that has the advantage of seeming somewhat more concrete. We will not use this concrete description explicitly, so we leave it as an exercise to check that it can indeed be related to our abstract treatment below. Taking any integer $n \geq 0$, if we assign to $\mathbb{R}^n \times \mathbb{R}^{n+1}$ the coordinates $(t_1, \ldots, t_n, \theta_0, \ldots, \theta_n)$

then one can show using the above proposition or by an explicit calculation that for a suitable choice of orientation on $\mathbb{R}^n \times \mathbb{R}^{n+1}$,

$$\alpha_\pm := \pm e^{t_1 + \cdots + t_n} \, d\theta_0 + e^{-t_1} \, d\theta_1 + \cdots + e^{-t_n} \, d\theta_n. \qquad (8.1)$$

is a Liouville pair. We would like to prove the existence of compact quotients of $\mathbb{R}^n \times \mathbb{R}^{n+1}$ to which α_+ and α_- both descend. Observe that both are invariant under the group action of $\mathbb{R}^n$ on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ defined by

$$(\tau_1, \ldots, \tau_n) \cdot (t_1, \ldots, t_n, \theta_0, \theta_1, \ldots, \theta_n)$$
$$:= (t_1 + \tau_1, \cdots, t_n + \tau_n, e^{-(\tau_1 + \cdots + \tau_n)} \theta_0, e^{\tau_1} \theta_1, \ldots, e^{\tau_n} \theta_n). \quad (8.2)$$

What we will prove in the next few sections implies the following:

Lemma 8.3 *There exist lattices $\Lambda \subset \mathbb{R}^n$ and $\Lambda' \subset \mathbb{R}^{n+1}$ such that the group action of Λ on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ defined by* (8.2) *preserves $\mathbb{R}^n \times \Lambda'$.*

It follows that the action of Λ on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ descends to a smooth group action on $\mathbb{R}^n \times (\mathbb{R}^{n+1}/\Lambda')$, so dividing by this action we obtain a bundle with fiber $\mathbb{R}^{n+1}/\Lambda' \cong \mathbb{T}^{n+1}$ and base $\mathbb{R}^n/\Lambda \cong \mathbb{T}^n$, to which the Liouville pair (α_+, α_-) descends. In this way one obtains the following result, which suffices to prove the existence of Liouville domains with disconnected boundary in all dimensions:

Proposition 8.4 *For every integer $n \geq 0$, the Liouville pair defined by* (8.1) *on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ descends to a compact quotient which is a $\mathbb{T}^{n+1}$-bundle over $\mathbb{T}^n$.*

Lemma 8.3 is trivial when $n = 0$, and elementary when $n = 1$: for the latter case, one can choose $\Lambda \subset \mathbb{R}$ to be generated by any real number $\tau \neq 0$ such that e^τ is an eigenvalue of some matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. Then A may be viewed as the matrix of the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2 : (\theta_0, \theta_1) \mapsto (e^{-\tau}\theta_0, e^\tau\theta_1)$ in some other basis where it has integer coefficients. This transformation therefore preserves the lattice generated by that basis. This produces a Liouville pair on every $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with hyperbolic monodromy—these examples have appeared in the previous work of Geiges [24] and Mitsumatsu [45]. A hint of the general arithmetic strategy we will use below appears in this discussion, as the condition that e^τ should be an eigenvalue of some matrix in $\mathrm{SL}(2, \mathbb{Z})$ implies that e^τ belongs to a quadratic extension of the field $\mathbb{Q}$.

8.2 Some Lie groups as symplectizations

Denote by $\mathrm{Aff}^+(\mathbb{R})$ the group of orientation preserving affine transformations of the real line. Similarly, $\widetilde{\mathrm{Aff}}(\mathbb{C})$ will denote the universal cover of

the group Aff($\mathbb{C}$) of affine transformations of the complex plane, which can be identified with the semi-direct product $\mathbb{C} \ltimes \mathbb{C}$ by associating to any $(a, b) \in \mathbb{C} \times \mathbb{C}$ the transformation $z \mapsto e^a z + b$. Observe that the same trick identifies Aff$^+(\mathbb{R})$ with $\mathbb{R} \ltimes \mathbb{R}$.

Let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of the affine group Aff$^+(\mathbb{R})$. The identification Aff$^+(\mathbb{R}) = \mathbb{R} \ltimes \mathbb{R}$ defines global coordinates (t, θ) on Aff$^+(\mathbb{R})$ and hence a basis (T, Θ) of left-invariant vector fields that match $(\partial_t, \partial_\theta)$ at the identity; they satisfy $[T, \Theta] = \Theta$. Writing the dual Lie algebra as $\mathfrak{g}_{\mathbb{R}}^*$, its dual basis is the pair of left-invariant 1-forms

$$T^* = dt, \qquad \Theta^* = e^{-t}\, d\theta.$$

Thus we can associate to Aff$^+(\mathbb{R})$ the left-invariant Liouville forms Θ^* or $-\Theta^*$ and view it as the symplectization of $(\mathbb{R}, d\theta)$ or $(\mathbb{R}, -d\theta)$ respectively with fiber coordinate $-t = -\ln \circ \det$. Note that $(\mathbb{R}, \pm d\theta)$ has a canonical contact type embedding into (Aff$^+(\mathbb{R}), \pm\Theta^*$), namely as the unimodular subgroup $\{\det = 1\}$.

We denote by $\mathfrak{g}_{\mathbb{C}}$ the Lie algebra of $\widetilde{\text{Aff}}(\mathbb{C}) = \mathbb{C} \ltimes \mathbb{C}$. Using coordinates $(u + iv, x + iy)$ on $\mathbb{C} \ltimes \mathbb{C}$, the basis (U, V, X, Y) of $\mathfrak{g}_{\mathbb{C}}$ defined to match $(\partial_u, \partial_v, \partial_x, \partial_y)$ at the identity satisfies the relations

$$[U, X] = X, \qquad [V, X] = Y,$$
$$[U, Y] = Y, \qquad [V, Y] = -X,$$

with all other brackets vanishing. These relations give the following exterior derivatives for the dual basis of left-invariant 1-forms:

$$\begin{aligned} dU^* &= 0, & dX^* &= X^* \wedge U^* + V^* \wedge Y^*, \\ dV^* &= 0, & dY^* &= X^* \wedge V^* + Y^* \wedge U^*. \end{aligned} \tag{8.3}$$

Although this will not be used, we note for concreteness that in the coordinates defined above,

$$\begin{aligned} U^* &= du, & X^* &= e^{-u}(\cos v)\, dx + e^{-u}(\sin v)\, dy, \\ V^* &= dv, & Y^* &= -e^{-u}(\sin v)\, dx + e^{-u}(\cos v)\, dy. \end{aligned}$$

Now we can define a left-invariant Liouville form as, for instance,

$$\beta = X^* \quad \text{so that} \quad d\beta^2 = -2U^* \wedge V^* \wedge X^* \wedge Y^*.$$

The corresponding Liouville vector field is $\beta^\# = -U$, which is transverse to the unimodular subgroup $\{|\det|^2 = 1\} = i\mathbb{R} \ltimes \mathbb{C}$, whose Lie algebra is the kernel of U^*. So we have on $\widetilde{\text{Aff}}(\mathbb{C})$ a left-invariant symplectization structure

with fiber coordinate $-\ln \circ |\det|$, where det denotes the determinant of the projection in $\mathrm{Aff}(\mathbb{C})$.

We now combine any number of copies of the two preceding Lie groups as $\mathcal{G}^{r,s} := \mathrm{Aff}^+(\mathbb{R})^r \times \widetilde{\mathrm{Aff}}(\mathbb{C})^s$, and then consider the subgroup

$$\mathcal{G}_1^{r,s} = \left\{ (\varphi_1, \ldots, \varphi_r, \psi_1, \ldots, \psi_s) \in \mathcal{G}^{r,s} \ \middle| \ \prod_i \det \varphi_i \prod_j |\det \psi_j|^2 = 1 \right\},$$

where $\det \psi_j$ should be understood again as the determinant of the projection of ψ_j to $\mathrm{Aff}(\mathbb{C})$. The discussion above shows that this group can be seen as a linear model for a contact product. When r is positive, we can single out one of the $\mathrm{Aff}^+(\mathbb{R})$ factors and apply Proposition 8.1 to obtain:

Corollary 8.5 *For any positive r, the Lie group $\mathcal{G}_1^{r,s}$ admits a left-invariant Liouville pair.*

The goal of the next two sections is to prove the existence of co-compact lattices in this group in order to find closed manifolds with Liouville pairs. As preparation, it will be useful observe that both $\mathcal{G}^{r,s}$ and $\mathcal{G}_1^{r,s}$ can be viewed as semi-direct products: setting

$$\mathfrak{h}^{r,s} := \mathbb{R}^r \times \mathbb{C}^s,$$

$$\mathfrak{h}_1^{r,s} := \left\{ (t_1, \ldots, t_r, w_1, \ldots, w_s) \in \mathfrak{h}^{r,s} \ \middle| \ \sum_i t_i + 2 \sum_j \mathrm{Re}\, w_j = 0 \right\} \qquad (8.4)$$

and defining the action of each on $\mathbb{R}^r \times \mathbb{C}^s$ by

$$(t_1, \ldots, t_r, w_1, \ldots, w_s) \cdot (\theta_1, \ldots, \theta_r, z_1, \ldots, z_s)$$
$$:= \left(e^{t_1} \theta_1, \ldots, e^{t_r} \theta_r, e^{w_1} z_1, \ldots, e^{w_s} z_s \right),$$

we have natural isomorphisms $\mathcal{G}^{r,s} = \mathfrak{h}^{r,s} \ltimes (\mathbb{R}^r \times \mathbb{C}^s)$ and $\mathcal{G}_1^{r,s} = \mathfrak{h}_1^{r,s} \ltimes (\mathbb{R}^r \times \mathbb{C}^s)$.

Remark 8.6 For most of the following discussion, the reader is free to assume $s = 0$, since this suffices to prove Lemma 8.3 and thus the existence of closed manifolds admitting Liouville pairs in all dimensions. The case $s > 0$ is only really needed to relate our results to those of Geiges in Sect. 8.5.

8.3 Some number theory

In this section we will need some standard notions and results from algebraic number theory, e.g. Dirichlet's Unit Theorem; a good reference for this material is [41].

Throughout this section and the next, $\mathbf{k}$ will denote a number field, i.e. a finite degree extension of $\mathbb{Q}$, and n will denote its degree $[\mathbf{k} : \mathbb{Q}]$. Such a field is always isomorphic to $\mathbb{Q}[X]/(f)$ for some irreducible polynomial $f \in \mathbb{Q}[X]$ of degree n (with simple roots). We will denote by r the number of real roots and by s the number of complex conjugate pairs of nonreal roots, thus $n = r + 2s$. Each root α gives an embedding of $\mathbf{k}$ into $\mathbb{C}$, sending (the equivalence class of) X to α. These embeddings will be denoted by $\rho_1, \ldots, \rho_r$ and $\sigma_1, \ldots, \sigma_s, \bar{\sigma}_1, \ldots, \bar{\sigma}_s$. This method actually gives all embeddings of $\mathbf{k}$ into $\mathbb{C}$, and we can collect them to define an injective map

$$j : \mathbf{k} \to \mathbb{R}^r \times \mathbb{C}^s : x \mapsto \big(\rho_1(x), \ldots, \rho_r(x), \sigma_1(x), \ldots, \sigma_s(x)\big).$$

The norm of an element of $\mathbf{k}$ is defined as $N(x) = \prod_i \rho_i(x) \prod_j |\sigma_j(x)|^2$, and the fact that f is irreducible implies that $N(x)$ vanishes only when $x = 0$. The ring of integers $\mathcal{O}_{\mathbf{k}}$ of $\mathbf{k}$ is by definition the set of all elements in $\mathbf{k}$ which are roots of monic polynomials with coefficients in $\mathbb{Z}$. These all have integer-valued norms, and an important observation is that the map j defined above sends $\mathcal{O}_{\mathbf{k}}$ to a lattice in $\mathbb{R}^r \times \mathbb{C}^s$.

Invertible elements in the ring $\mathcal{O}_{\mathbf{k}}$ are called *units* of $\mathbf{k}$, and they form a (multiplicative) group denoted by $\mathcal{O}_{\mathbf{k}}^{\times}$. They all have norm ± 1 since $N(xy) = N(x)N(y)$. We denote by $\mathcal{O}_{\mathbf{k}}^{\times,+}$ the subgroup of *positive* units: $\mathcal{O}_{\mathbf{k}}^{\times,+} = \{x \in \mathcal{O}_{\mathbf{k}}^{\times} \mid \rho_i(x) > 0 \text{ for all } i\}$. Among units are the roots of unity, whose (finite) set is denoted by $\mathbb{U}_{\mathbf{k}}$. We also set $\mathbb{U}_{\mathbf{k}}^{+} = \mathbb{U}_{\mathbf{k}} \cap \mathcal{O}_{\mathbf{k}}^{\times,+}$. Dirichlet's Unit Theorem implies that $\mathcal{O}_{\mathbf{k}}^{\times}$ is a finitely generated abelian group with torsion $\mathbb{U}_{\mathbf{k}}$ and rank $r + s - 1$. Since $x^2 \in \mathcal{O}_{\mathbf{k}}^{\times,+}$ whenever $x \in \mathcal{O}_{\mathbf{k}}^{\times}$, it follows that $\mathcal{O}_{\mathbf{k}}^{\times,+}$ is similarly the product of the finite cyclic group $\mathbb{U}_{\mathbf{k}}^{+}$ with a free abelian group of rank $r + s - 1$. The map j restricts to an injective group homomorphism of $\mathcal{O}_{\mathbf{k}}^{\times,+}$ into the multiplicative group $H^{r,s} := (\mathbb{R}_+^*)^r \times (\mathbb{C}^*)^s$, and since $N(\mathcal{O}_{\mathbf{k}}^{\times,+}) = \{1\}$, its image lies in the subgroup

$$H_1^{r,s} = \left\{ (\rho_1, \ldots, \rho_r, \sigma_1, \ldots, \sigma_s) \in H^{r,s} \,\middle|\, \prod_i \rho_i \prod_j |\sigma_j|^2 = 1 \right\}.$$

The precise formulation of Dirichlet's theorem is that $j(\mathcal{O}_{\mathbf{k}}^{\times,+})$ is a lattice in $H_1^{r,s}$.

Examples We now discuss three examples of increasing complexity to see all the objects discussed above appearing. In the next subsection we will see the contact manifolds associated to these fields and the Liouville pair construction where applicable.

The very first example of a number field is $\mathbb{Q}$ itself. In this case $n = 1$, $f = X - 1$, $r = 1$, $s = 0$ and j is the inclusion of $\mathbb{Q}$ in $\mathbb{R}$. The ring of integers is $\mathcal{O}_{\mathbf{k}} = \mathbb{Z}$, with $\mathcal{O}_{\mathbf{k}}^{\times} = \{\pm 1\}$ and $\mathcal{O}_{\mathbf{k}}^{\times,+} = \{1\}$.

As a less trivial number field, we consider $\mathbf{k} = \mathbb{Q}[i]$. We have $n = 2$ and can set $f = X^2 + 1$, so the roots are $\pm i$, hence $r = 0$ and $s = 1$. Choosing i as a member of the complex conjugate pair $\pm i$, we have $j \colon \mathbf{k} \to \mathbb{C} \colon a + bX \mapsto a + ib$. The norm of $a + bX$ is $a^2 + b^2$. The integer ring $\mathcal{O}_{\mathbf{k}}$ is $\mathbb{Z} + \mathbb{Z}X$, and its image under j is the lattice $\mathbb{Z} + i\mathbb{Z}$ in $\mathbb{C}$. The group of units is $\mathcal{O}_{\mathbf{k}}^{\times} = \{\pm 1, \pm X\} = \mathbb{U}_{\mathbf{k}}$. All units are automatically positive since there is no real embedding. We have $H^{0,1} = \mathbb{C}^*$ and $H_1^{0,1} = \mathbb{S}^1$, in which $j(\mathcal{O}_{\mathbf{k}}^{\times,+})$ is indeed a lattice.

As a last example, we consider $\mathbf{k} = \mathbb{Q}[\sqrt{2}]$. Here $n = 2$ and $f = X^2 - 2$ with roots $\pm\sqrt{2}$, so $r = 2$ and $s = 0$. The j map is defined by $a + bX \mapsto (a + b\sqrt{2}, a - b\sqrt{2})$. The norm of $a + bX$ is $a^2 - 2b^2$. The integer ring $\mathcal{O}_{\mathbf{k}}$ is $\mathbb{Z} + \mathbb{Z}X$, and its image under j is the lattice

$$\left\{ (a + b\sqrt{2}, a - b\sqrt{2}) \mid a, b \in \mathbb{Z} \right\} = \mathbb{Z}(1, 1) + \mathbb{Z}(\sqrt{2}, -\sqrt{2}) \subset \mathbb{R}^2.$$

The group of units is $\mathcal{O}_{\mathbf{k}}^{\times} = \{\pm(1 + X)^k \mid k \in \mathbb{Z}\}$, and $\mathbb{U}_{\mathbf{k}} = \{\pm 1\}$. Restricting to positive elements, we have $\mathcal{O}_{\mathbf{k}}^{\times,+} = \{(3 + 2X)^k \mid k \in \mathbb{Z}\}$ and $\mathbb{U}_{\mathbf{k}}^+ = \{1\}$. The image of $\mathcal{O}_{\mathbf{k}}^{\times,+}$ in $H^{2,0} = (\mathbb{R}_+^*)^2$ is $j(\mathcal{O}_{\mathbf{k}}^{\times,+}) = \{((3 + 2\sqrt{2})^k, (3 - 2\sqrt{2})^k) \mid k \in \mathbb{Z}\}$, which is indeed a lattice in $H_1^{2,0} = \{(y, 1/y) \in (\mathbb{R}_+^*)^2 \mid y > 0\}$.

8.4 A manifold associated to a number field

The next result provides the desired co-compact lattices in the Lie groups $\mathcal{G}_1^{r,s}$.

Proposition 8.7 *Suppose $\mathbf{k} = \mathbb{Q}[X]/(f)$ is a number field of degree $n = r + 2s \geq 1$, where $f \in \mathbb{Q}[X]$ is an irreducible polynomial with r real and $2s$ complex roots (all simple). Then one can associate to $\mathbf{k}$ a lattice $G_{\mathbf{k}} \subset \mathcal{G}_1^{r,s}$ such that the quotient*

$$M_{\mathbf{k}} := \mathcal{G}_1^{r,s}/G_{\mathbf{k}}$$

is a $\mathbb{T}^n$-bundle over $\mathbb{T}^{n-1}$.

To prove this, we continue with the same notation as in the previous section and observe that the Lie algebras of $H^{r,s}$ and $H_1^{r,s}$ are precisely $\mathfrak{h}^{r,s}$ and $\mathfrak{h}_1^{r,s}$ respectively, defined in (8.4) above. Since $H^{r,s}$ is abelian, the exponential map

$$\exp \colon \mathfrak{h}^{r,s} \to H^{r,s} \colon (t_1, \ldots, t_r, w_1, \ldots, w_s) \mapsto \left(e^{t_1}, \ldots, e^{t_r}, e^{w_1}, \ldots, e^{w_s} \right)$$

is a surjective group homomorphism, as is its restriction to $\mathfrak{h}_1^{r,s} \to H_1^{r,s}$, and its kernel is the free abelian group $\{0\} \times 2\pi i \mathbb{Z}^s \subset \mathfrak{h}_1^{r,s}$. Thus the preimage

of $j(\mathcal{O}_{\mathbf{k}}^{\times,+})$ in $\mathfrak{h}_1^{r,s}$ is a rank $r + s - 1 + s = n - 1$ lattice, which we denote by $\Gamma_{\mathbf{k}}$.

The group $H^{r,s}$ acts on $\mathbb{R}^r \times \mathbb{C}^s$ via coordinate-wise multiplication, so pulling back this action via the exponential map defines an action of $\Gamma_{\mathbf{k}}$ on $\mathbb{R}^r \times \mathbb{C}^s$, which preserves the lattice $j(\mathcal{O}_{\mathbf{k}})$ since multiplication by elements of $\mathcal{O}_{\mathbf{k}}^{\times}$ preserves $\mathcal{O}_{\mathbf{k}}$. The inclusions $\Gamma_{\mathbf{k}} \hookrightarrow \mathfrak{h}_1^{r,s}$ and $j(\mathcal{O}_{\mathbf{k}}) \hookrightarrow \mathbb{R}^r \times \mathbb{C}^s$ can therefore be combined to an inclusion of the semi-direct product

$$G_{\mathbf{k}} := \Gamma_{\mathbf{k}} \ltimes j(\mathcal{O}_{\mathbf{k}}) \hookrightarrow \mathfrak{h}_1^{r,s} \ltimes \left(\mathbb{R}^r \times \mathbb{C}^s\right) = \mathcal{G}_1^{r,s},$$

forming a lattice. Proposition 8.7 now follows from the observation that the projection $\mathcal{G}_1^{r,s} = \mathfrak{h}_1^{r,s} \ltimes (\mathbb{R}^r \times \mathbb{C}^s) \to \mathfrak{h}_1^{r,s}$ descends to a well-defined projection

$$\mathcal{G}_1^{r,s}/G_{\mathbf{k}} \to \mathfrak{h}_1^{r,s}/\Gamma_{\mathbf{k}},$$

forming a bundle with fiber $(\mathbb{R}^r \times \mathbb{C}^s)/j(\mathcal{O}_{\mathbf{k}}) \cong \mathbb{T}^n$ and base $\mathfrak{h}_1^{r,s}/\Gamma_{\mathbf{k}} \cong \mathbb{T}^{n-1}$.

Note that the only choices we made in the construction of $M_{\mathbf{k}}$ were the ordering of the embeddings of $\mathbf{k}$ into $\mathbb{R}$ and $\mathbb{C}$, and which complex embedding we pick out of each complex conjugate pair. The manifold $M_{\mathbf{k}}$ does not depend on these choices up to diffeomorphism. Moreover, each choice of orientations for the factors of $\mathrm{Aff}^+(\mathbb{R})$ in $\mathcal{G}^{r,s}$ determines a contact structure on $M_{\mathbf{k}}$ uniquely up to isotopy. Indeed, aside from the orientation of $\mathrm{Aff}^+(\mathbb{R})$, the only other choices involved were the Liouville forms on the relevant Lie groups, but one can check that all possible left-invariant Liouville forms defining the same orientation are isotopic—they form a connected open subset of the dual Lie algebra. If we fix a single orientation of $\mathrm{Aff}^+(\mathbb{R})$ from the beginning, we then obtain the canonical contact structure promised in Theorem C from the introduction.

We remark that $G_{\mathbf{k}}$ is not the only possible lattice in $\mathcal{G}_1^{r,s}$. In the totally real case ($s = 0$) in particular, one can replace $\mathcal{O}_{\mathbf{k}}$ by any additive subgroup M of $\mathbf{k}$ which is a free abelian group of rank n, and $\mathcal{O}_{\mathbf{k}}^{\times,+}$ by any of its finite index subgroups preserving M. The contact manifolds obtained in this way are cusp cross sections of Hilbert modular varieties, see [55, Chap. 1]. In particular, they are Stein fillable and can be embedded as separating strictly pseudoconvex hypersurfaces in closed holomorphic manifolds.

For later use in Sect. 9, we note the following observation.

Lemma 8.8 *If $\mathbf{k}$ is a totally real number field which is not $\mathbb{Q}$, then $\pi_1(M_{\mathbf{k}})$ has trivial center.*

Proof Since $M_{\mathbf{k}} = \mathcal{G}_1^{r,s}/G_{\mathbf{k}}$ and $\mathcal{G}_1^{r,s}$ is simply connected, the lemma is equivalent to the claim that the group $G_{\mathbf{k}}$ has trivial center. In the totally real case (i.e. $s = 0$), we have $G_{\mathbf{k}} = \mathcal{O}_{\mathbf{k}}^{\times,+} \ltimes \mathcal{O}_{\mathbf{k}}$, so as a set $G_{\mathbf{k}} = \mathcal{O}_{\mathbf{k}}^{\times,+} \times \mathcal{O}_{\mathbf{k}}$, and the

composition law is $(u, x)(u', x') = (uu', x + ux')$. Suppose (u, x) is central in $G_{\mathbf{k}}$. This implies that for any (u', x'), $x + ux' = x' + u'x$.

We can apply this to $u' = 1$ to deduce that for any integer x', $ux' = x'$. Since $\mathcal{O}_{\mathbf{k}}$ is integral (recall it embeds in $\mathbb{R}$), we get $u = 1$ or $x' = 0$. Since $\mathcal{O}_{\mathbf{k}}$ is not a trivial group (it has rank $\deg(\mathbf{k})$), this implies $u = 1$.

Similarly, we can apply the above formula with $x' = 0$ to deduce that for any unit u', $u'x = x$. Hence $u' = 1$ or $x = 0$. Since $\mathcal{O}_{\mathbf{k}}^{\times,+}$ has rank $\deg(\mathbf{k}) - 1$ and we assume $\deg(\mathbf{k}) > 1$, we obtain $x = 0$, so $(1, 0)$ is the only central element. □

Examples Recall that our first example was $\mathbf{k} = \mathbb{Q}$. The corresponding $\mathcal{G}_1^{r,s}$ is the group of affine transformations of $\mathbb{R}$ with determinant one, i.e. it is $\mathbb{R}$. The unit group $\mathcal{O}_{\mathbf{k}}^{\times,+}$ is trivial, hence it acts trivially on $\mathbb{R}$, implying $M_{\mathbb{Q}} = \mathbb{R}/j(\mathcal{O}_{\mathbf{k}}) \cong \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. The resulting Liouville pair on $\mathbb{S}^1$ is $(d\theta, -d\theta)$.

Our second example was $\mathbf{k} = \mathbb{Q}[i]$, and the corresponding $\mathcal{G}_1^{r,s}$ is $\widetilde{\mathrm{Aff}}(\mathbb{C}) = i\mathbb{R} \ltimes \mathbb{C}$, whose elements (iv, w) correspond to affine transformations $z \mapsto e^{iv}z + w$. Note that since $r = 0$ in this case, all left-invariant contact forms on $\mathcal{G}_1^{r,s}$ induce the same orientation, so there can be no left-invariant Liouville pair, but we can still extract a co-compact lattice. We have $\mathfrak{h}_1^{r,s} = i\mathbb{R} \subset \mathbb{C} = \mathfrak{h}^{r,s}$, and $\Gamma_{\mathbf{k}} \subset i\mathbb{R}$ is spanned by $m := i\pi/2$, where m stands for "monodromy". The action of m on $\mathbb{C}$ is $z \mapsto e^{i\pi/2}z$, which does indeed preserve the lattice $j(\mathcal{O}_{\mathbf{k}}) = \mathbb{Z} + i\mathbb{Z}$. We conclude that $M_{\mathbf{k}}$ is a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ whose monodromy is a quarter turn. Observe that $M_{\mathbf{k}}$ is a finite quotient of $\mathbb{T}^3$, which cannot admit any Liouville pair due to [57, Example 2.14].

We proceed to the last example $\mathbf{k} = \mathbb{Q}[\sqrt{2}]$. The corresponding $\mathcal{G}_1^{r,s}$ is the unimodular subgroup within $\mathrm{Aff}^+(\mathbb{R})^2$, which is the solvable group of Thurston's geometries. In the hyperplane $\mathfrak{h}_1^{r,s} = \{(-t, t)\} \subset \mathbb{R}^2 = \mathfrak{h}^{r,s}$, $\Gamma_{\mathbf{k}}$ is spanned by $m := (\ln(3 + 2\sqrt{2}), \ln(3 - 2\sqrt{2}))$. The action of m on $\mathbb{R}^2$ is then $(y_1, y_2) \mapsto ((3 + 2\sqrt{2})y_1, (3 - 2\sqrt{2})y_2)$, and one can check by hand that it indeed preserves the lattice $j(\mathcal{O}_{\mathbf{k}})$. Recall that a basis of this lattice is $\{(1, 1), (\sqrt{2}, -\sqrt{2})\}$. In this basis, the matrix of m is $A = \left(\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right)$, so we see that $M_{\mathbf{k}}$ is a $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with monodromy A, which is hyperbolic. The Liouville pair we constructed yields two contact structures which rotate in opposite directions between the stable and unstable foliations of the Anosov flow defined by the monodromy (cf. [45]).

8.5 Geiges pairs and Geiges groups

The idea of Geiges in [23] was to consider a special class of Liouville pairs (without the general definition) that satisfy a much stronger algebraic condition, and to look for examples among left-invariant contact forms on Lie groups. The particular groups that Geiges considered turn out to be a subfamily of the ones that we've studied above.

Definition A *Geiges pair* on an oriented manifold M^{2n+1} is a pair of contact forms α_+ and α_- on M such that:

- $\alpha_+ \wedge d\alpha_+^n = -\alpha_- \wedge d\alpha_-^n > 0$, and
- for all $0 \leq k \leq n-1$, $\alpha_\pm \wedge d\alpha_\pm^k \wedge d\alpha_\mp^{n-k} = 0$.

A version of [23, Proposition 1] is then the simple observation that Geiges pairs are also Liouville pairs. Note that the Liouville pairs we constructed in the preceding section are Geiges pairs in dimensions 1 and 3, but not in higher dimensions in general.

Geiges constructed in each odd dimension $2n-1$ a Lie group G_{2n-1} admitting a left-invariant Geiges pair, and also found co-compact lattices in these groups in dimensions 3 and 5, thus giving examples of compact Liouville domains with two boundary components in dimensions 4 and 6. We shall now show that our number theoretic construction can also be used to find co-compact lattices for all the Geiges groups, implying the existence of Geiges pairs on some closed manifold in every odd dimension.

Proposition 8.9 *For any positive integer n, there is an isomorphism between G_{2n-1} and $\mathcal{G}_1^{r,s}$ where $r = 1$ if n is odd, $r = 2$ if n is even, and $s = (n-r)/2$.*

The remainder of this section is devoted to the proof of this isomorphism. The pairs constructed by Geiges have a nice form in the basis of the Lie algebra he considered, but our isomorphism will not preserve this basis in any nice way. Of course, the point of our description of these groups was that it makes the construction of co-compact lattices much easier.

First we recall the definition of the Geiges group G_{2n-1}. For each positive integer n, let A denote the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ -1 & 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

We define the $(2n-1)$-dimensional Geiges group G_{2n-1} as the semi-direct product $\mathbb{R}^{n-1} \ltimes_A \mathbb{R}^n$, where $(y_1, \ldots, y_{n-1})$ acts as $\exp(y_1 A + \cdots + y_{n-1} A^{n-1})$ on $\mathbb{R}^n$. Reversing the sign of the first vector in the canonical basis of $\mathbb{R}^n$, one sees that A is similar to the matrix of cyclic permutation of this basis. In particular, all powers of A appearing in the action have vanishing trace, because powers between 1 and $n-1$ of this permutation have no nontrivial fixed points, thus no diagonal term can appear.

The matrix A is orthogonal and has characteristic polynomial $X^n - 1$, so its eigenvalues are the nth roots of unity. We denote by R_α the 2×2 rotation

matrix with angle α and set $\theta = 2\pi/n$. Then A is similar to a block diagonal matrix $B = \mathrm{diag}(1, R_\theta, \ldots, R_{s\theta})$ if $r = 1$, or $B = \mathrm{diag}(1, -1, R_\theta, \ldots, R_{s\theta})$ if $r = 2$. Choose an invertible matrix P such that $A = P^{-1}BP$. The map $(y, x) \mapsto (y, Px)$ is now an isomorphism $\mathbb{R}^{n-1} \ltimes_A \mathbb{R}^n \to \mathbb{R}^{n-1} \ltimes_B \mathbb{R}^n$, where we define the latter group using B instead of A to construct the $\mathbb{R}^{n-1}$-action analogously.

To simplify the notation, we now assume that n is odd, so $r = 1$ and $s = (n-1)/2$; the other case is completely analogous. The matrix B can be seen as acting on $\mathbb{R} \times \mathbb{C}^s$, with R_θ acting as multiplication by $e^{i\theta}$. The matrix $\rho(y) = y_1 B + \cdots + y_{n-1} B^{n-1}$ for $y \in \mathbb{R}^{n-1}$ thus splits into block form as $\mathrm{diag}(\rho_0(y), \rho_1(y), \ldots, \rho_s(y))$ for some linear maps $\rho_0 \colon \mathbb{R}^{n-1} \to \mathbb{R}$ and $\rho_i \colon \mathbb{R}^{n-1} \to \mathbb{C}$, $i = 1, \ldots, s$. Using the identification $\mathcal{G}^{r,s} = (\mathbb{R} \times \mathbb{C}^s) \ltimes (\mathbb{R} \times \mathbb{C}^s)$, we can now write down an injective group homomorphism

$$\mathbb{R}^{n-1} \ltimes_B \left(\mathbb{R} \times \mathbb{C}^s \right) \to \mathcal{G}^{r,s}$$
$$\left(y, (x, z_1, \ldots, z_s) \right) \mapsto \left((\rho_0(y), \rho_1(y), \ldots, \rho_s(y)), (x, z_1, \ldots, z_s) \right).$$

Since B is similar to A and each power of A appearing in the definition of the Geiges group has vanishing trace, the same is true for B. After taking the exponential, this translates to the fact that the above map actually takes values in the subgroup $\mathcal{G}_1^{r,s}$: indeed, $|\exp(e^{i\theta})|^2 = \exp(2\,\mathrm{Re}\,e^{i\theta}) = \exp(\mathrm{tr}\,R_\theta)$. We conclude that it is an isomorphism to $\mathcal{G}_1^{r,s}$ since the dimensions match. The desired isomorphism from G_{2n-1} to $\mathcal{G}_1^{r,s}$ is now obtained by composing the two isomorphisms we've constructed.

9 Lutz twists and Giroux torsion in higher dimensions

In this section we examine the $(2n - 1)$-dimensional generalizations of Giroux torsion and Lutz twists that arise from any closed $(2n - 3)$-dimensional manifold with a Liouville pair. We will begin with general considerations and then turn to specific examples constructed using the Liouville manifolds of Sect. 8 to prove Theorems A and E (in Sect. 9.3) and G (in Sect. 9.4) from the introduction.

Throughout the following, we choose an integer $n \geq 2$ and assume M to be a closed oriented $(2n - 3)$-dimensional manifold with a fixed Liouville pair (α_+, α_-), writing the resulting positive/negative contact structures as $\xi_\pm = \ker \alpha_\pm$. We will often consider manifolds of the form $\mathbb{R} \times \mathbb{S}^1 \times M$ or $\mathbb{S}^1 \times \mathbb{S}^1 \times M$, with the natural coordinates on the first two factors denoted by s and t respectively.

9.1 Torsion domains and the Lutz-Mori twist

Given M with Liouville pair (α_+, α_-), we define a 1-form on $\mathbb{R} \times \mathbb{S}^1 \times M$ by

$$\lambda_{\mathrm{GT}} = \frac{1 + \cos s}{2} \alpha_+ + \frac{1 - \cos s}{2} \alpha_- + (\sin s)\, dt, \qquad (9.1)$$

and denote $\xi_{\mathrm{GT}} := \ker \lambda_{\mathrm{GT}}$.

Proposition 9.1 *The co-oriented distribution ξ_{GT} defined above is a positive contact structure on $\mathbb{R} \times \mathbb{S}^1 \times M$, which can be viewed as an infinite chain of Giroux domains $[k\pi, (k+1)\pi] \times \mathbb{S}^1 \times M = (M \times [k\pi, (k+1)\pi]) \times \mathbb{S}^1$ glued together.*

Proof Let $\varphi \colon (0, \pi) \to \mathbb{R}$ denote the orientation reversing diffeomorphism defined by $\varphi(s) = \ln \frac{1+\cos s}{\sin s}$. This induces an orientation preserving diffeomorphism from the interior of $\Sigma := M \times [0, \pi]$ to $\mathbb{R} \times M$, so pulling back $\beta := \frac{1}{2}(e^u \alpha_+ + e^{-u} \alpha_-)$ gives a Liouville form which defines on Σ the structure of an ideal Liouville domain. Regarding $\partial \Sigma$ as the zero-set of the function $\sin s$ and writing $u = \varphi(s)$, the Giroux domain $\Sigma \times \mathbb{S}^1$, then inherits the contact form

$$\lambda_{\mathrm{GT}} = (\sin s) \cdot \left[dt + \frac{1}{2} \left(e^u \alpha_+ + e^{-u} \alpha_- \right) \right],$$

proving that λ_{GT} is indeed a positive contact form on $M \times [0, \pi] \times \mathbb{S}^1 = [0, \pi] \times \mathbb{S}^1 \times M$. A similar argument proves the contact condition on $[\pi, 2\pi] \times \mathbb{S}^1 \times M$, and the rest follows by periodicity. $\square$

For any positive integer k, one can then define the *Giroux $2k\pi$-torsion domain* modeled on (M, α_+, α_-) as $([0, 2k\pi] \times \mathbb{S}^1 \times M, \lambda_{\mathrm{GT}})$.

The fact that Giroux torsion is a filling obstruction in dimension three now generalizes to the following immediate consequence of Theorem 5.13. Note that for the case $n = 2$, the additional topological condition giving an obstruction to weak fillability is equivalent to the condition that the embedding $[0, 2\pi] \times \mathbb{S}^1 \times M \hookrightarrow V$ should separate V.

Corollary 9.2 *If (V, ξ) is a closed $(2n - 1)$-dimensional contact manifold admitting a contact embedding $\iota \colon ([0, 2\pi] \times \mathbb{S}^1 \times M, \xi_{\mathrm{GT}}) \hookrightarrow (V, \xi)$, then (V, ξ) is not strongly fillable. Moreover, if $\iota_*([\mathbb{S}^1] \times C) = 0 \in H_2(V; \mathbb{R})$ for every $C \in H_1(M; \mathbb{R})$, then (V, ξ) is also not weakly fillable.*

The torsion domains $([0, 2k\pi] \times \mathbb{S}^1 \times M, \xi_{\mathrm{GT}})$ allow us to define a "twisting" operation on contact structures that generalizes the 3-dimensional

Lutz modification along a pre-Lagrangian torus with closed leaves, see [10, Sect. 1.4]. Note that for any k, both boundary components of $[0, 2k\pi] \times \mathbb{S}^1 \times M$ are ξ_{GT}-round hypersurfaces modeled on (M, ξ_+) (see Sect. 5.1). Now if (V, ξ) is any $(2n - 1)$-dimensional contact manifold containing a ξ-round hypersurface $H \subset V$ modeled on (M, ξ_+), then we can cut V open along H and insert $([0, 2k\pi] \times \mathbb{S}^1 \times M, \xi_{GT})$ such that the contact structures glue together smoothly. The resulting manifold is diffeomorphic to V, and it determines a new contact structure ξ_k on V uniquely up to isotopy. We shall say in this case that ξ_k is obtained from ξ by a k-fold *Lutz-Mori twist along H* (we use the name Mori to emphasize that A. Mori [46] introduced a similar modification along a codimension 2 contact submanifold in dimension 5, see below).

Recall that any positive co-oriented contact structure ξ on an oriented $(2n - 1)$-dimensional manifold V induces an *almost contact structure* on V, i.e. a reduction of the structure group of TV to $U(n - 1)$. For our purposes, we can regard an almost contact structure as equivalent to a choice of co-oriented hyperplane distribution $\xi \subset TV$ together with a symplectic structure on the bundle $\xi \to V$, and this choice is determined uniquely up to homotopy when ξ is contact. The homotopy class of almost contact structures amounts to a "classical" invariant that one can use to distinguish non-isotopic contact structures. As we will see, one of the important properties of the Lutz-Mori twist is that it does not change this invariant, though it can change the isomorphism class of the contact structure.

Theorem 9.3 *Suppose (V, ξ) is any contact manifold containing a closed ξ-round hypersurface H modeled on (M, ξ_+), where $\xi_+ = \ker \alpha_+$ and (α_+, α_-) is a Liouville pair on M. Then for any positive integer k, one can modify ξ near H by the k-fold Lutz-Mori twist as described above to define a contact structure ξ_k with the following properties*:

1. ξ *and* ξ_k *are homotopic through a family of almost contact structures.*
2. (V, ξ_k) *is not strongly fillable if V is closed.*
3. *If V is closed and the natural map $H_1(M; \mathbb{R}) \to H_2(V; \mathbb{R})$ induced by the inclusion $\mathbb{S}^1 \times M = H \hookrightarrow V$ is trivial, then (V, ξ_k) is not weakly fillable.*

Before proving this theorem, we note the following characterization of Liouville pairs, the proof of which is a simple computation. It will be useful for understanding homotopy classes of almost contact structures as well as Reeb vector fields in the next section.

Lemma 9.4 *A pair of 1-forms (α_+, α_-) on an oriented $(2n - 1)$-dimensional manifold M is a Liouville pair if and only if for every pair of constants $C_+, C_- \geq 0$ not both zero,*

$$(C_+\alpha_+ - C_-\alpha_-) \wedge (C_+ d\alpha_+ + C_- d\alpha_-)^{n-1} > 0.$$
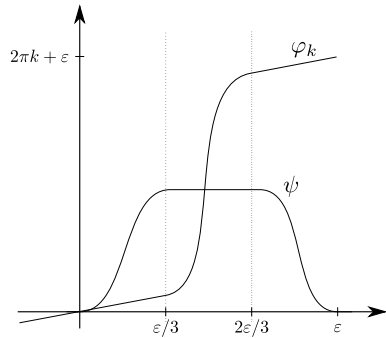
Proof of Theorem 9.3 The last two statements are simply Corollary 9.2. To prove the first, we can model the Lutz-Mori twist as follows. By Lemma 5.1, a neighborhood of H in (V, ξ) can be identified with a neighborhood of $\{0\} \times \mathbb{S}^1 \times M$ in $(\mathbb{R} \times \mathbb{S}^1 \times M, \xi_{\mathrm{GT}})$, i.e. with $((-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times M, \xi_{\mathrm{GT}})$ for $\varepsilon > 0$ sufficiently small. Then given k, choose a diffeomorphism $\varphi_k \colon (-\varepsilon, \varepsilon) \to (-\varepsilon, 2\pi k + \varepsilon)$ as shown in Fig. 5, with fixed slope 1 outside the interval $(\varepsilon/3, 2\varepsilon/3)$, and define a new contact form on $(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times M$ by

$$\lambda_k := \frac{1 + \cos \varphi_k(s)}{2} \alpha_+ + \frac{1 - \cos \varphi_k(s)}{2} \alpha_- + \big(\sin \varphi_k(s)\big) \, dt.$$

For convenience, let us also set $\varphi_0(s) = s$, so $\lambda_0 := \lambda_{\mathrm{GT}}$ and $\xi_0 := \xi_{\mathrm{GT}}$. Then it will suffice to show that for each integer $k \geq 0$, the almost contact structure induced by ξ_k on $(-\varepsilon, \varepsilon) \times \mathbb{S}^1 \times M$ admits a compactly supported homotopy through almost contact structures to a fixed almost contact structure independent of k.

To see this, choose a smooth function $\psi \colon [0, \varepsilon] \to [0, 1]$ which vanishes near the boundary and equals 1 precisely on $[\varepsilon/3, 2\varepsilon/3]$ (see Fig. 5), and for $\tau \in [0, 1]$, define a smooth 1-parameter family of nowhere zero 1-forms and co-oriented hyperplane fields by

$$\lambda_{k,\tau} = \big[1 - \tau \psi(s)\big] \lambda_k + \tau \psi(s) \, ds, \qquad \xi_{k,\tau} = \ker \lambda_{k,\tau}.$$

We have $\lambda_{k,\tau} = \lambda_k$ outside of some compact subset of $(0, \varepsilon) \times \mathbb{S}^1 \times M$ for all τ, while $\lambda_{k,0} \equiv \lambda_k$ and $\lambda_{k,1}$ is everywhere independent of k. This shows that the homotopy type of ξ_k as a co-oriented hyperplane field is independent of k. It remains only to show that the homotopy $\{\xi_{k,\tau}\}_{\tau \in [0,1]}$ can be accompanied by a homotopy $\{\Omega_{k,\tau}\}_{\tau \in [0,1]}$ of symplectic bundle structures such that $\Omega_{k,0} = d\lambda_k$ and $\Omega_{k,1}$ has no k-dependence.

We claim first that $\lambda_{k,\tau}$ is always contact, with the exception of $\lambda_{k,1} = ds$ on $[\varepsilon/3, 2\varepsilon/3] \times \mathbb{S}^1 \times M$. Indeed, $\lambda_{k,\tau} \wedge (d\lambda_{k,\tau})^{n-1} = (1 - \tau \psi)^n \lambda_k \wedge (d\lambda_k)^{n-1}$ since the term $\tau \psi(s) \, ds$ vanishes in this product; firstly it is closed,

and secondly the only term where dt appears in $d\lambda_k$ is a multiple of $ds \wedge dt$. Since λ_k is contact, it follows that $\lambda_{k,\tau}$ also is whenever $\tau\psi < 1$. Thus $d\lambda_{k,\tau}$ defines a suitable family of symplectic bundle structures for $\tau < 1$, which we next would like to modify so that it extends to $\tau = 1$. To facilitate this, observe that whenever τ and ψ are both close to 1, the projection along the s-direction restricts to a fiberwise isomorphism $\xi_{k,\tau} \to T(\mathbb{S}^1 \times M)$. Thus any symplectic bundle structure Ω on $\xi_{k,\tau}$ can be identified via this isomorphism with an s-dependent family of nondegenerate (but not necessarily closed) 2-forms $\hat{\Omega}(s)$ on $\mathbb{S}^1 \times M$. For $\Omega = d\lambda_{k,\tau}$ in particular, we find that $\widehat{d\lambda_{k,\tau}}(s)$ belongs to the contractible space Ξ of 2-forms on $\mathbb{S}^1 \times M$ having the form

$$\omega := C_+ \, d\alpha_+ + C_- \, d\alpha_- + \delta\alpha_+ \wedge \alpha_- + B \, dt \wedge (C_+\alpha_+ - C_-\alpha_-)$$

for some constants $C_+, C_- \geq 0$, $B > 0$ and $\delta \in \mathbb{R}$, where C_+ and C_- are assumed not both zero. In the case of $\widehat{d\lambda_{k,\tau}}(s)$, one can compute:

$$C_\pm = \frac{1 \pm \cos\varphi_k}{2}, \qquad \delta = -\frac{1 - \tau\psi}{2\tau\psi}\varphi_k' \sin\varphi_k \quad \text{and} \quad B = \frac{1 - \tau\psi}{\tau\psi}\varphi_k'.$$

It turns out that *any* 2-form in Ξ is nondegenerate since

$$\omega^n = (n-1)B \, dt \wedge (C_+\alpha_+ - C_-\alpha_-) \wedge (C_+ \, d\alpha_+ + C_- \, d\alpha_-)^{n-1}$$

is nonzero due to Lemma 9.4. We can therefore solve the extension problem to modify $\widehat{d\lambda_{k,\tau}}(s)$ for τ near 1 to a smooth homotopy of nondegenerate 2-forms that match $\widehat{d\lambda_{k,\tau}}(s)$ outside a neighborhood of $\{\varepsilon/3 \leq s \leq 2\varepsilon/3\}$ but also extend to $\tau = 1$ as nondegenerate forms with no dependence on k. Pulling back through the fiberwise isomorphism $\xi_{k,\tau} \to T(\mathbb{S}^1 \times M)$, this determines a homotopy of almost contact structures as desired. □

The original Lutz twist in dimension three modifies a contact structure in the neighborhood of a transverse knot to produce one that is always overtwisted, and Mori [46] generalized this to an operation on contact 5-manifolds along certain special contact submanifolds of codimension 2. In our context, Mori's construction generalizes as follows: suppose (V, ξ) is a $(2n - 1)$-dimensional contact manifold containing a contact submanifold $M \subset V$ of codimension 2 with trivial normal bundle such that $\xi \cap TM = \xi_+$. For any k, let (Y_k, ξ_{GT}) denote the result of blowing down $([0, 2k\pi] \times \mathbb{S}^1 \times M, \xi_{GT})$ along the ξ_{GT}-round hypersurface $\{0\} \times \mathbb{S}^1 \times M$ as described in Sect. 5.1. We can then remove a small neighborhood of M from (V, ξ) and glue in a

correspondingly small neighborhood of $(Y_k, \xi_{GT}) \subset (Y_{k+1}, \xi_{GT})$ such that the contact structures match up. The resulting manifold is again diffeomorphic to V and determines a new contact structure ξ_k up to isotopy, and we shall say that ξ_k is obtained from ξ by a k-fold *Lutz-Mori twist along M*.

Theorem 9.5 *Suppose (V, ξ) contains a closed codimension 2 contact submanifold $M \subset V$ with trivial normal bundle such that $\xi \cap TM = \xi_+$ where $\xi_+ = \ker \alpha_+$ for some Liouville pair (α_+, α_-) on M. Then for any positive integer k, one can modify ξ near M by a k-fold Lutz-Mori twist as described above to define a contact structure ξ_k with the following properties:*

1. *ξ_k and ξ are homotopic through a family of almost contact structures.*
2. *(V, ξ_k) is PS-overtwisted (cf. Definition 4.2) and not weakly fillable (if V is closed).*

Proof Since the homotopy of almost contact structures in our proof of Theorem 9.3 had compact support in $(0, \varepsilon) \times \mathbb{S}^1 \times M$, the argument can be carried over verbatim to the present context to prove the first statement. The presence of a bLob can be deduced from the general Proposition 5.9, but also much more directly for the concrete examples we discussed in Sect. 8. Indeed, one can check that the torus bundles on which we constructed Liouville pairs (α_+, α_-) always contain an n-torus T on which both α_+ and α_- vanish. In $[0, 2\pi] \times \mathbb{S}^1 \times M$, the contact form λ_{GT} induces on $[0, \pi] \times \mathbb{S}^1 \times T$ the integrable 1-form $(\sin s) \, dt$, whose kernel is singular exactly along the boundary. Blowing down $\{0\} \times \mathbb{S}^1 \times M$ turns this domain into a plastikstufe inside (Y_k, ξ_{GT}).

Non-fillability can also be deduced directly from Theorem 7.1, with the technical advantage that it does not require any semipositivity assumption thanks to the polyfold technology for holomorphic spheres [37]. (The corresponding technology for holomorphic disks remains under development.) □

Remark 9.6 In the 3-dimensional case one can also define the so-called "half-Lutz twist" along a positively transverse knot, which both changes the homotopy class of the contact structure and makes it overtwisted, producing a negatively transverse knot at the core of the inserted tube. The equivalent operation here would be defined by replacing a neighborhood of (M, ξ_+) in (V, ξ) with the domain $([\pi, 2\pi] \times \mathbb{S}^1 \times M, \xi_{GT})$ blown down along $\{\pi\} \times \mathbb{S}^1 \times M$. A variation on the above argument shows that the resulting contact manifold is also PS-overtwisted, and in this case the submanifold M at the center of the inserted "tube" inherits the *negative* contact structure ξ_- instead of ξ_+.

It is not remotely clear under what circumstances in general one can say that the modification from ξ to ξ_k or ξ_ℓ produces non-isomorphic contact structures for $k \neq \ell$, though we will show in the next few subsections that

this is at least sometimes the case for Lutz-Mori twists along round hypersurfaces. In light of the flexibility exhibited by overtwisted contact structures in dimension three, the following natural question arises:

Question If ξ_k and ξ_ℓ are obtained from the same contact structure by a k-fold and ℓ-fold Lutz-Mori twist respectively along a fixed contact submanifold of codimension 2, when are they isomorphic?

Remark 9.7 It should be emphasized that Lutz-Mori twists cannot be performed along arbitrary round hypersurfaces or codimension 2 contact submanifolds: we always need to assume that the contact structure restricted to the submanifold M admits a contact form belonging to a Liouville pair. This is a serious constraint, as there are many smooth manifolds that are known to admit contact structures but not Liouville pairs: for instance, by [19] and [1], this is the case for any 3-manifold whose contact structures are all known to be planar (e.g. $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$) or partially planar (e.g. $\mathbb{T}^3$), as these can never admit strong symplectic semifillings with disconnected boundary. The fact that 3-dimensional Lutz twists can be inserted along any contact submanifold (here transverse knots) can then be seen as a consequence of the fact that every contact form on every closed 1-dimensional manifold obviously belongs to a Liouville pair.

9.2 Liouville pairs and Reeb vector fields

In this section we describe the Reeb vector fields corresponding to contact forms coming from Liouville pairs.

Lemma 9.4 implies that for any Liouville pair (α_+, α_-) and constants $C_+, C_- \geq 0$ that do not both vanish, the 2-form $C_+ \, d\alpha_+ + C_- \, d\alpha_-$ has maximal rank. Its kernel therefore defines a nonsingular line field on M.

Definition A Liouville pair (α_+, α_-) is called *hypertight* if for every pair of constants $C_+, C_- \geq 0$ that are not both zero, M admits no contractible loops tangent to $\ker(C_+ \, d\alpha_+ + C_- \, d\alpha_-)$.

In particular, this condition implies that α_+ and α_- each admit no contractible closed Reeb orbits. As one can check, nonzero left-invariant vector fields on the Lie groups $\mathcal{G}_1^{r,s}$ of Sect. 8 never have closed orbits, thus we have the following useful observation:

Proposition 9.8 *All the Liouville pairs constructed in Sect.* 8 *are hypertight.*

The following computation will be useful for understanding Reeb vector fields on our examples in the next two subsections. As a simple application,

it immediately implies that whenever the pair (α_+, α_-) is hypertight, λ_{GT} admits no contractible closed Reeb orbits.

Lemma 9.9 *Suppose (α_+, α_-) is a hypertight Liouville pair on a manifold M of dimension $2n - 3 \geq 1$, and $f, g, h \colon \mathbb{R} \to \mathbb{R}$ are smooth functions such that f and g are both nonnegative and never vanish simultaneously, and the 1-form on $\mathbb{R} \times \mathbb{S}^1 \times M$ defined by*

$$\lambda := f(s)\alpha_+ + g(s)\alpha_- + h(s)\,dt$$

is contact. Then the Reeb vector field R_λ associated to λ has the form $R_\lambda(s, t, m) = X_s(m) + u(s)\partial_t$, where $u \colon \mathbb{R} \to \mathbb{R}$ is a smooth function and X_s is a smooth 1-parameter family of vector fields on M, each of which either vanishes identically or has no contractible closed orbits.

Proof Computing $\lambda \wedge (d\lambda)^{n-1}$, we find that the contact condition implies

$$\left[\left(hf' - h'f\right)\alpha_+ + \left(hg' - h'g\right)\alpha_-\right] \wedge (f\,d\alpha_+ + g\,d\alpha_-)^{n-2} \neq 0, \qquad (9.2)$$

thus there is for each $s \in \mathbb{R}$ a unique vector field X_s on M satisfying the conditions

$$\left(hf' - h'f\right)\alpha_+(X_s) + \left(hg' - h'g\right)\alpha_-(X_s) = -h',$$
$$f\,d\alpha_+(X_s, \cdot) + g\,d\alpha_-(X_s, \cdot) = 0.$$

This vector field vanishes precisely when $h'(s) = 0$, and otherwise it has no contractible orbits due to the hypertightness assumption. The relation (9.2) also implies that $h(s)$ and $h'(s)$ can never simultaneously vanish, thus one can define a function $u \colon \mathbb{R} \to \mathbb{R}$ by

$$u(s) = \begin{cases} \frac{1}{h}[1 - f\alpha_+(X_s) - g\alpha_-(X_s)] & \text{when } h(s) \neq 0, \\ -\frac{1}{h'}[f'\alpha_+(X_s) + g'\alpha_-(X_s)] & \text{when } h(s) = 0. \end{cases}$$

With these definitions, it is straightforward to check that $d\lambda(X_s + u(s)\partial_t, \cdot) = 0$ and $\lambda(X_s + u(s)\partial_t) = 1$. $\qquad\square$

9.3 A sequence of contact structures on $\mathbb{T}^2 \times M$

In order to prove Theorems A and E from the introduction, we now consider an example that generalizes the well-known sequence of weakly but not strongly fillable contact structures on $\mathbb{T}^3$ [16, 29]. Assume as usual that M has a Liouville pair (α_+, α_-), and define for each positive integer k a contact structure ξ_k on $\mathbb{T}^2 \times M$ by identifying the latter with $(\mathbb{R}/2k\pi\mathbb{Z}) \times \mathbb{S}^1 \times M$ and setting $\xi_k := \xi_{\mathrm{GT}}$ via (9.1). Theorems A and E are then consequences of the following result, together with Example 9.11 below.

Theorem 9.10 *For any closed manifold M with a Liouville pair (α_+, α_-), the sequence of contact structures $\{\xi_k\}_{k>0}$ on $\mathbb{T}^2 \times M$ defined above has the following properties*:

(1) $(\mathbb{T}^2 \times M, \xi_1)$ *is exactly fillable.*
(2) $(\mathbb{T}^2 \times M, \xi_k)$ *is not strongly fillable for any $k \geq 2$.*
(3) *For any k, ℓ, ξ_k and ξ_ℓ are homotopic through a family of almost contact structures.*
(4) *If (α_+, α_-) is hypertight (see Definition 9.2) then every ξ_k for $k \in \mathbb{N}$ is hypertight, and no two of these contact structures are isotopic. If additionally $\pi_1(M)$ has trivial center, then no two of these contact structures are contactomorphic.*
(5) *Suppose additionally that $\mathbb{S}^1 \times M$ admits a closed 2-form ω such that for some constants $c_+, c_- > 0$ and all sufficiently small $\varepsilon > 0$, $\varepsilon\omega + c_+ d\alpha_+ + c_- d\alpha_-$ is symplectic on $\mathbb{S}^1 \times M$. Then $(\mathbb{T}^2 \times M, \xi_k)$ is weakly fillable for every k.*

In particular, the first four statements are true for all the examples of Sect. 8 with $\dim M \geq 3$ and $s = 0$ (see Proposition 9.8 and Lemma 8.8), and the fifth statement is also true when $\dim M = 3$, so M may be any $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with hyperbolic monodromy (see the discussion following Proposition 8.4).

Example 9.11 We do not know any examples of Liouville pairs with $\dim M \geq 5$ for which we can verify the last condition, and this is why Theorem E in the introduction is stated only for dimension five. For $\dim M = 1$, the condition is the trivial observation that $\mathbb{T}^2$ admits an area form, and our argument will then reproduce Giroux's construction [29] of weak fillings for the tight contact structures on $\mathbb{T}^3$, which directly inspired our general case. Theorem E depends on finding closed 3-manifolds M with Liouville pairs such that $\mathbb{S}^1 \times M$ is symplectic, and this is also not hard. Every $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ with hyperbolic monodromy admits a hypertight Liouville pair that can be written as follows: on $\mathbb{R} \times \mathbb{R}^2$ with coordinates (t, x, y) let

$$\alpha_\pm = \pm e^t \, dx + e^{-t} \, dy.$$

Then if $A \in \mathrm{SL}(2, \mathbb{Z})$ has eigenvalues $e^{\pm\tau}$ for $\tau > 0$, one can find a lattice $\Lambda_A \subset \mathbb{R}^2$ which is preserved by the linear transformation $(x, y) \mapsto (e^{-\tau}x, e^\tau y)$, so that α_+ and α_- both descend to the mapping torus

$$M_A := \left(\mathbb{R} \times \left(\mathbb{R}^2/\Lambda_A\right)\right)/(t, x, y) \sim \left(t + \tau, e^{-\tau}x, e^\tau y\right).$$

Since M_A fibers over $\mathbb{S}^1$, $\mathbb{S}^1 \times M_A$ admits a symplectic form, and we can write it explicitly as

$$\omega = d\phi \wedge dt + dx \wedge dy,$$

where ϕ denotes the additional $\mathbb{S}^1$-coordinate. This form satisfies $\omega \wedge d\alpha_\pm = 0$, hence $\varepsilon\omega + c_+ \, d\alpha_+ + c_- \, d\alpha_-$ is symplectic for all constants $\varepsilon > 0$ and $c_\pm \in \mathbb{R}$.

Our argument for distinguishing the contact structures ξ_k for different values of k will use cylindrical contact homology as sketched by Eliashberg-Givental-Hofer [18], a theory which in its most general form has not yet been rigorously defined due to the difficulty of achieving transversality for multiply covered holomorphic curves. In our situation however, we are in the lucky position of being able to rule out multiply covered curves topologically. Suppose (V, ξ) is a closed contact manifold and $\bar{a}$ denotes a free homotopy class of loops $S^1 \to V$. We shall say that a contact form λ for (V, ξ) is $\bar{a}$-admissible if all its Reeb orbits in the homotopy class $\bar{a}$ are Morse-Bott and their periods are uniformly bounded, and there are no contractible Reeb orbits. The idea sketched in [18] is that if λ is nondegenerate, one should define a chain complex generated by a certain class of Reeb orbits homotopic to $\bar{a}$, with the differential counting rigid holomorphic cylinders in the symplectization for a generic choice of almost complex structure adapted to λ. The resulting homology is meant to depend only on (V, ξ) and $\bar{a}$ up to natural isomorphisms, so we denote it by $HC_*^{\bar{a}}(V, \xi)$. Bourgeois [3] has also explained how to extend this definition to Morse-Bott contact forms by counting so-called "holomorphic cascades."

Lemma 9.12 *Suppose $\bar{a}$ is a free homotopy class of loops in (V, ξ) which is primitive, i.e. it is not a positive multiple of any other homotopy class, and suppose (V, ξ) admits an $\bar{a}$-admissible contact form. Then the cylindrical contact homology $HC_*^{\bar{a}}(V, \xi)$ sketched in [18] is well defined and can be computed as described in [3] by counting holomorphic cascades for generic data associated to any $\bar{a}$-admissible contact form.*

Proof We only need to supplement the standard Floer-theoretic picture with the following observations. First, every Reeb orbit homotopic to $\bar{a}$ must be simply covered, thus every holomorphic curve having only one positive end, which is asymptotic to such an orbit, is guaranteed to be somewhere injective. Transversality for these curves can therefore be achieved via a generic perturbation of the almost complex structure, using the standard result of Dragnev [12] (see also the appendix of [5]). Secondly, if λ has no contractible Reeb orbits and λ' is a sufficiently small nondegenerate perturbation of it as in [3], then one may assume every contractible Reeb orbit for λ' to have arbitrarily large period. Then since the periods of Reeb orbits homotopic to $\bar{a}$ are bounded, one can choose a generic almost complex structure J adapted to λ' and define a subcomplex of the usual complex for the data (λ', J) by taking as generators all the Reeb orbits up to a certain period, chosen so that

all perturbations of the Morse-Bott orbits homotopic to $\bar{a}$ are included but holomorphic planes can never appear in the relevant compactifications because they have too much energy. For appropriate choices of the period cutoff, the standard construction of natural isomorphisms (i.e. by counting rigid holomorphic cylinders in symplectic cobordisms) suffices to prove that the homology is independent of auxiliary choices. □

Proof of Theorem 9.10 Statements (2) and (3) in the theorem follow immediately from Corollary 9.2 and Theorem 9.3 respectively. We shall now prove statements (1), (5) and (4), in that order.

Proof of (1). An exact filling of $(\mathbb{T}^2 \times M, \xi_1)$ can be constructed as the product of two Liouville domains of the form $([-1, 1] \times \mathbb{S}^1, \sigma \, d\theta)$ and $([-c, c] \times M, e^s \alpha_+ + e^{-s} \alpha_-)$ with rounded corners, where $c > 0$ may be assumed arbitrarily large and (σ, θ) denote the natural coordinates on $[-1, 1] \times \mathbb{S}^1$.

Proof of (5). If $\mathbb{S}^1 \times M$ also admits a 2-form ω as in the condition of statement (5), then we can modify the exact filling constructed above to define weak fillings of every $(\mathbb{T}^2 \times M, \xi_k)$, using the fact that the latter is naturally a k-fold cover of $(\mathbb{T}^2 \times M, \xi_1)$. Indeed, the assumption implies that we can find $s_0 \in (-1, 1)$ such that for any $\varepsilon > 0$ sufficiently small, the 2-form

$$\varepsilon \omega + e^{s_0} \, d\alpha_+ + e^{-s_0} \, d\alpha_- \tag{9.3}$$

is symplectic on $\mathbb{S}^1 \times M$. Now observe that since the weak filling condition is open with respect to the symplectic form, $([-1, 1] \times \mathbb{S}^1 \times [-c, c] \times M, \omega_\varepsilon)$ with rounded corners and

$$\omega_\varepsilon := d\big[e^s \, \alpha_+ + e^{-s} \, \alpha_- + \sigma \, d\theta \big] + \varepsilon \omega$$

is also a weak filling of $(\mathbb{T}^2 \times M, \xi_1)$ if $\varepsilon > 0$ is sufficiently small, and for any $\sigma_0 \in (-1, 1)$ its restriction to the interior submanifold

$$X_0 := \{\sigma_0\} \times \mathbb{S}^1 \times \{s_0\} \times M \subset \big([-1, 1] \times \mathbb{S}^1 \times [-c, c] \times M, \omega_\varepsilon\big)$$

is precisely (9.3). Thus we have a weak filling of $(\mathbb{T}^2 \times M, \xi_1)$ diffeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1 \times M$ and containing $\{0\} \times \mathbb{S}^1 \times M$ as a symplectic submanifold. For any k, the k-fold symplectic branched cover of this, branched at $\{0\} \times \mathbb{S}^1 \times M$, gives a weak filling of $(\mathbb{T}^2 \times M, \xi_k)$.

Proof of (4). Assume now that (α_+, α_-) is a hypertight Liouville pair. Lemma 9.9 then implies that λ_{GT} has no contractible Reeb orbits.

We next compute the cylindrical contact homology of $(\mathbb{T}^2 \times M, \xi_k)$, which is a straightforward adaptation of the calculation for the tight 3-tori explained in [6, § 4.2]. Let $\bar{a}$ denote the free homotopy class of the loop $\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1 \times M : \phi \mapsto (\text{const}, \phi, \text{const})$. Applying Lemma 9.9 again,

the Reeb orbits of λ_{GT} in homotopy class $\bar{a}$ on $\mathbb{R}/(2\pi k\mathbb{Z}) \times \mathbb{S}^1 \times M$ consist of precisely k Morse-Bott families foliating the submanifolds $\{\cos s = 0, \sin s = 1\} \cong \mathbb{S}^1 \times M$. Moreover, all of these orbits have the same period, thus our contact form is $\bar{a}$-admissible in the sense of Lemma 9.12. Now for any choice of admissible almost complex structure J on the symplectization of $(\mathbb{T}^2 \times M, \xi_k)$, there can never be any index 1 J-holomorphic cylinders connecting two orbits in homotopy class $\bar{a}$ since it would have zero energy. After making a nondegenerate perturbation as explained in [6], nondegenerate orbits in homotopy class $\bar{a}$ are in one-to-one correspondence with the critical points of a Morse function on the parameter space of the Morse-Bott families, i.e. on M. Similarly, the holomorphic cylinders for the perturbed data correspond to so-called "holomorphic cascades" for the unperturbed data, and in the absence of actual holomorphic cylinders, these are in one-to-one correspondence with gradient flow lines on M. We conclude that $HC_*^{\bar{a}}(\mathbb{T}^2 \times M, \xi_k)$ is isomorphic (up to a shift in the grading) to the direct sum of k copies of the Morse homology of M, which is simply the singular homology $H_*(M)$.

Observe also that if $\bar{b} \neq \bar{a}$ is any other free homotopy class of loops in $\mathbb{T}^2 \times M$ whose projections to M are contractible, then there are no Reeb orbits homotopic to $\bar{b}$ at all, hence $HC_*^{\bar{b}}(\mathbb{T}^2 \times M, \xi_k)$ is trivial.

The above computation shows that if $k \neq \ell$, then there can be no contactomorphism $(\mathbb{T}^2 \times M, \xi_k) \to (\mathbb{T}^2 \times M, \xi_\ell)$ whose action on $\pi_1(\mathbb{T}^2 \times M)$ preserves the subgroup

$$G := \pi_1(\mathbb{T}^2) \times \{1\} \subset \pi_1(\mathbb{T}^2) \times \pi_1(M) = \pi_1(\mathbb{T}^2 \times M).$$

Indeed, we have computed the cylindrical contact homology for all homotopy classes in this subgroup, and by Lemma 9.12, these computations would have to match if such a contactomorphism existed. This already implies that ξ_k and ξ_ℓ cannot be isotopic. To show that they are not even diffeomorphic, we add the assumption that $\pi_1(M)$ has trivial center: then the center of $\pi_1(\mathbb{T}^2 \times M)$ is G, which is therefore preserved by *every* automorphism of $\pi_1(\mathbb{T}^2 \times M)$. □

9.4 Hypertight but not weakly fillable

We now construct a family of examples in all dimensions that implies Theorem G from the introduction. Throughout this section, we denote by Σ_g the closed oriented surface of genus g, and by $\Sigma_{g,m}$ the compact oriented surface with genus g and m boundary components.

Theorem 9.13 *Suppose M is any closed $(2n - 3)$-dimensional manifold admitting a hypertight Liouville pair. Then for any integer $g > 0$, $\Sigma_{2g} \times M$ admits a sequence of contact structures $\{\xi_k\}_{k>0}$ with the following properties*:

(1) $(\Sigma_{2g} \times M, \xi_1)$ is exactly fillable.
(2) $(\Sigma_{2g} \times M, \xi_k)$ is not weakly fillable for any $k \geq 2$.
(3) $(\Sigma_{2g} \times M, \xi_k)$ is hypertight for all k.
(4) For any $k \neq \ell$, ξ_k and ξ_ℓ are homotopic through a family of almost contact structures but are not isotopic. If additionally $\pi_1(M)$ has trivial center and is solvable, then they are not even contactomorphic.

In particular, all of these statements are true for the Liouville pairs defined from totally real number fields in Sect. 8.

The contact structures ξ_k on $\Sigma_{2g} \times M$ will be constructed using a simple generalization of the blow-down operation along round hypersurfaces that was introduced in Sect. 5.1. To start with, we consider (Z_k, ξ_{GT}) where $Z_k := [0, (2k-1)\pi] \times \mathbb{S}^1 \times M$, so the two boundary components

$$\partial_+ Z_k := \{0\} \times \mathbb{S}^1 \times M, \qquad \partial_- Z_k := \{(2k-1)\pi\} \times \mathbb{S}^1 \times M$$

are ξ_{GT}-round hypersurfaces modeled on (M, ξ_+) and $(-M, \xi_-)$ respectively. At $\partial_+ Z_k$ in particular, we find by Lemma 5.1 a collar neighborhood identified with $([0, \varepsilon) \times \mathbb{S}^1 \times M, \ker(\alpha_+ + s\, dt))$ for some $\varepsilon > 0$. Now choose a Liouville form β on $\Sigma_{g,1}$ such that $\int_{\partial \Sigma_{g,1}} \beta = \varepsilon$. Then $\partial \Sigma_{g,1}$ has a neighborhood $\mathcal{N}(\partial \Sigma_{g,1}) \subset (\Sigma_{g,1}, \beta)$ that can be identified with $((0, \varepsilon] \times \mathbb{S}^1, s\, dt)$, defining a natural embedding $\Phi_+ : \mathcal{N}(\partial \Sigma_{g,1}) \times M \hookrightarrow (0, \varepsilon] \times \mathbb{S}^1 \times M \subset Z_k$ with $\Phi_+^* \xi_{\mathrm{GT}} = \ker(\beta + \alpha_+)$. Similarly, the other end of $\mathring{Z}_k$ admits an orientation preserving embedding $\Phi_- : \mathcal{N}(\partial \Sigma_{g,1}) \times (-M) \hookrightarrow \mathring{Z}_k$ such that $\Phi_-^* \xi_{\mathrm{GT}} = \ker(\beta + \alpha_-)$. We can therefore glue three pieces together to define

$$(\Sigma_{2g} \times M, \xi_k) := \big(\Sigma_{g,1} \times M, \ker(\beta + \alpha_+)\big) \cup_{\Phi_+} (\mathring{Z}_k, \xi_{\mathrm{GT}})$$

$$\cup_{\Phi_-} \big(\Sigma_{g,1} \times (-M), \ker(\beta + \alpha_-)\big).$$

Note that if $g = 0$, this construction is equivalent to blowing down (Z_k, ξ_{GT}) at both boundary components as defined in Sect. 5.1, and we shall think of the more general operation defined here as "blowing down with genus g."

We now proceed to construct a model of $(\Sigma_{2g} \times M, \xi_k)$ with a more tractable Reeb vector field. The disadvantage of using λ_{GT} for this purpose is that it cannot easily be related to the normal forms $\alpha_\pm + s\, dt$ coming from Lemma 5.1, as for instance near $\partial_+ Z_k$, the α_--term in λ_{GT} is small but not identically vanishing. The following lemma allows us to eliminate it entirely after a small adjustment which essentially replaces the Liouville form $e^s \alpha_+ + e^{-s} \alpha_-$ on $\mathbb{R} \times M$ by one which is explicitly the completion of a Liouville domain $[-c, c] \times M$.

Lemma 9.14 *Choose a smooth cutoff function $\psi : \mathbb{R} \to [0, 1]$ that equals 0 on $(-\infty, 0]$ and 1 on $[1, \infty)$. Then for any Liouville pair (α_+, α_-) on a $(2n -$*

1)-*dimensional manifold* M, *the* 1-*form*

$$\beta := \psi(c+s)\, e^s \alpha_+ + \psi(c-s)\, e^{-s} \alpha_-$$

is Liouville if $c > 0$ *is a sufficiently large constant.*

Proof The claim is immediate whenever $\psi' = 0$, so it will suffice to examine $d\beta$ on the segments $\{-c \le s \le -c+1\}$ and $\{c-1 \le s \le c\}$. On the former, we have $\beta = \psi_c(s) e^s \alpha_+ + e^{-s} \alpha_-$ where $\psi_c(s) := \psi(c+s)$. Thus

$$d\beta^n = n\, ds \wedge \left(\psi_c e^s \alpha_+ - e^{-s} \alpha_- + \psi_c' e^s \alpha_+\right) \wedge \left(\psi_c e^s\, d\alpha_+ + e^{-s}\, d\alpha_-\right)^{n-1}$$

$$= n e^{-ns}\, ds \wedge \left[\left(\psi_c e^{2s} \alpha_+ - \alpha_-\right) \wedge \left(\psi_c e^{2s}\, d\alpha_+ + d\alpha_-\right)^{n-1}\right.$$

$$\left. + e^{2s} \psi_c' \alpha_+ \wedge \left(\psi_c e^{2s}\, d\alpha_+ + d\alpha_-\right)^{n-1}\right].$$

In this last expression, the first term in the brackets can be made arbitrarily close to $-\alpha_- \wedge d\alpha_-^{n-1} > 0$ by assuming $c > 0$ large, while the second term can be made arbitrarily close to 0, hence the sum is positive. A similar argument also works for the segment $\{c-1 \le s \le c\}$. □

Combining this lemma with the reparametrization trick in the proof of Proposition 9.1, we can now introduce a convenient modification of the contact form λ_{GT}: on $Z_k = [0, (2k-1)\pi] \times \mathbb{S}^1 \times M$, there exists a contact form of type

$$\lambda_k = f(s)\, \alpha_+ + g(s)\, \alpha_- + h(s)\, dt$$

for some smooth functions $f, g, h \colon [0, (2k-1)\pi] \to \mathbb{R}$, such that for some small constant $\varepsilon > 0$:

- $\lambda_k = \lambda_{\mathrm{GT}}$ on $[2\varepsilon, (2k-1)\pi - 2\varepsilon] \times \mathbb{S}^1 \times M$,
- λ_k is everywhere C^1-close to λ_{GT},
- $\lambda_k = \alpha_+ + s\, dt$ on $[0, \varepsilon] \times \mathbb{S}^1 \times M$,
- $\lambda_k = \alpha_- + [(2k-1)\pi - s]\, dt$ on $[(2k-1)\pi - \varepsilon, (2k-1)\pi] \times \mathbb{S}^1 \times M$.

Then if β denotes the Liouville form on $\Sigma_{g,1}$ as described above with collar neighborhood $\mathcal{N}(\partial \Sigma_{g,1}) = (0, \varepsilon] \times \mathbb{S}^1$ in which $\beta = s\, dt$, we can smoothly glue $\Sigma_{g,1} \times M$ with contact form $\lambda_k := \alpha_+ + \beta$ to the interior of (Z_k, λ_k) along $(0, \varepsilon] \times \mathbb{S}^1 \times M$. Similarly, defining the auxiliary coordinate $s' := (2k-1)\pi - s \in [0, \varepsilon]$ on the opposite collar neighborhood, we can glue this neighborhood to $\Sigma_{g,1} \times M$ with contact form $\lambda_k := \alpha_- + \beta$ so that the coordinates (s', t) match the collar $\mathcal{N}(\partial \Sigma_{g,1}) = (0, \varepsilon] \times \mathbb{S}^1$. The kernel of λ_k is now isotopic to ξ_k.

Proof of Theorem 9.13 The claim regarding almost contact structures follows by the same argument as in Theorem 9.3. With this understood, we shall now

proceed to prove items (3) and (4) from the statement of the theorem, and after that prove items (1) and (2).

Proof of (3) and (4). The contact form λ_k constructed above determines a Reeb vector field R_{λ_k} that is given by Lemma 9.9 on $[\varepsilon, (2k-1)\pi - \varepsilon] \times \mathbb{S}^1 \times M$ and matches the Reeb vector fields of α_+ and α_- respectively on the two copies of $\Sigma_{g,1} \times M$. While this vector field does have nullhomologous closed orbits, none of them are contractible if $g > 0$ since $\partial \Sigma_{g,1}$ is not contractible in Σ_{2g}. Similarly, for $g > 0$ one can define the cylindrical contact homology $HC_*^{\bar{a}}(\Sigma_{2g} \times M, \xi_k)$ for any primitive homotopy class $\bar{a}$ due to Lemma 9.12. A repeat of the argument in the proof of Theorem 9.10 then shows that for $k \neq \ell$, there is no contactomorphism

$$(\Sigma_{2g} \times M, \xi_k) \to (\Sigma_{2g} \times M, \xi_\ell)$$

whose action on $\pi_1(\Sigma_{2g} \times M)$ preserves $\pi_1(\Sigma_{2g})$. So in particular, ξ_k and ξ_ℓ are not isotopic. Under the additional assumption on $\pi_1(M)$, they are not even contactomorphic due to Lemma 9.15 below.

Proof of (1). An exact filling of $(\Sigma_{2g} \times M, \xi_1)$ can be constructed as the product of the two Liouville domains $(\Sigma_{g,1}, \beta)$ and $([-c, c] \times M, e^s \alpha_+ + e^{-s} \alpha_-)$ for sufficiently large c.

Proof of (2). Corollary 9.2 implies that $(\Sigma_{2g} \times M, \xi_k)$ is not weakly fillable for $k \geq 3$; note that here we need the fact that for any 1-cycle C in M, the 2-cycle $\{\text{const}\} \times \mathbb{S}^1 \times C$ in $Z_k \subset \Sigma_{2g} \times M$ can be realized as the boundary of $\Sigma_{g,1} \times M$ and is thus nullhomologous.

At this point we've proved everything except the fact that $(\Sigma_{2g} \times M, \xi_2)$ is not weakly fillable. Since this already suffices to prove Theorem G, and the non-fillability of ξ_2 doesn't quite follow from our previous results as stated, we shall content ourselves with a sketch of the proof. The idea is analogous to the proof of Theorem 7.1, but using a straightforward generalization of the surgery in Sect. 6 to accommodate boundary components that are, in the terminology introduced above, blown down with genus. In particular, $(\Sigma_{2g} \times M, \xi_2)$ can be realized as a chain of three Giroux domains $G_0 \cup G_1 \cup G_2$ glued end to end, with the dangling ends of G_0 and G_2 blown down with genus g. Now if we perform surgery to remove the interiors of G_0 and G_1, we obtain a symplectic cobordism to a manifold with three connected components

$$(M \times \Sigma_g) \sqcup (M \times \mathbb{S}^2) \sqcup (V', \xi'),$$

where (V', ξ') is a weakly filled boundary component and the other two components are foliated by symplectic submanifolds $\{*\} \times \Sigma_g$ and $\{*\} \times \mathbb{S}^2$ respectively. Then if $(\Sigma_{2g} \times M, \xi_2)$ is assumed to be weakly fillable, one can derive a contradiction as in the proof of Theorem 7.1 by examining the moduli space of holomorphic spheres that emerge from the symplectic submanifolds

$\{*\} \times \mathbb{S}^2$. This only involves one feature not already present in the proof of Theorem 7.1: the holomorphic spheres cannot approach the boundary component $M \times \Sigma_g$. This is guaranteed if one uses a product complex structure near this boundary component, because then every somewhere injective holomorphic curve touching a neighborhood of it must be of the form $\{*\} \times \Sigma_g$, and no sequence of holomorphic spheres can converge to any cover of these curves since such a cover would necessarily have positive genus. $\square$

In the above proof we used the following algebraic lemma, whose proof was kindly explained to us by Yves de Cornulier.

Lemma 9.15 *Suppose Σ is a closed oriented surface of genus at least two. If G is any solvable group with trivial center, then any automorphism of $\pi_1(\Sigma) \times G$ preserves $\pi_1(\Sigma)$.*

Proof We set $H = \pi_1(\Sigma) \times G$. Our goal will be to show that G is the unique maximal normal solvable subgroup of H, thus G is preserved by any automorphism. Since G has trivial center, its centralizer in H is $\pi_1(\Sigma)$, which is therefore also preserved by any automorphism.

We now prove the claim about G. Suppose G_1 is a normal solvable subgroup of H. The projection $p(G_1)$ of G_1 into $\pi_1(\Sigma)$ is normal in $\pi_1(\Sigma)$ and solvable. We now view $\pi_1(\Sigma)$ as a Zariski dense subgroup of $\mathrm{PSL}(2, \mathbb{R})$. The Zariski closure of $p(G_1)$ is still solvable and is normal in the closure of $\pi_1(\Sigma)$, hence trivial because $\mathrm{PSL}(2, \mathbb{R})$ is simple and not solvable. Thus $p(G_1)$ is trivial and $G_1 \subset G$. $\square$

Appendix: Cotamed complex structures: existence and convexity

A.1 Contractibility of the space of cotamed almost complex structures

To go from the linear situation to global existence results on a manifold we will need the following result.

Proposition 2.1 (Sévennec) *The space of complex structures on a finite dimensional vector space tamed by two given symplectic forms is either empty or contractible.*

Using the fact that the space of complex structures tamed by a symplectic form is nonempty (which follows for instance by the linear Darboux theorem), and applying the proposition above twice to the same symplectic form, we recover as a special case the classical result of Gromov that states that the space of tamed complex structures is contractible. The proof of the proposition uses the following two lemmas, of which the first is more or less standard.

Lemma A.1 (Cayley, Sévennec) *Let V be a real finite dimensional vector space and $\mathcal{J}(V)$ the space of complex structures on V. We can define for any fixed $J_0 \in \mathcal{J}(V)$ a map*

$$\mu_{J_0}: J \mapsto (J + J_0)^{-1} \cdot (J - J_0)$$

which is a diffeomorphism from

$$\mathcal{J}_{J_0}^*(V) := \left\{ J \in \mathcal{J}(V) \mid J + J_0 \in \mathrm{GL}(V) \right\}$$

to

$$\mathcal{A}_{J_0}^*(V) := \left\{ A \in \mathrm{End}(V) \mid A J_0 = -J_0 A \text{ and } A - I \in \mathrm{GL}(V) \right\}.$$

The inverse of this map is given by $\mu_{J_0}^{-1}: A \mapsto (A - I)J_0(A - I)^{-1}$.

Proof One can view $\mathcal{A}_{J_0}^*(V)$ as the set of J_0-complex antilinear maps that do not have any eigenvalue equal to 1. Using the equations $(J - J_0)J_0 = -J(J - J_0)$ and $(J + J_0)J_0 = J(J + J_0)$, one sees that the image of μ_{J_0} consists of J_0-complex antilinear maps, and $\mu_{J_0}(J) - I = -2(J + J_0)^{-1}J_0$ is invertible. □

Lemma A.2 (Sévennec) *Let (V, ω) be a finite dimensional symplectic vector space and denote by $\mathcal{J}_t(\omega) \subset \mathcal{J}(V)$ the space of complex structures tamed by ω. Choosing any $J_0 \in \mathcal{J}_t(\omega)$, it follows that $\mathcal{J}_t(\omega)$ lies in $\mathcal{J}_{J_0}^*(V)$, and the image of $\mathcal{J}_t(\omega)$ under the associated map μ_{J_0} is a convex domain in $\mathcal{A}_{J_0}^*(V)$.*

We first explain how to prove Proposition 2.1 using the above lemma. Suppose there is a complex structure J_0 tamed by ω_0 and ω_1. The space of co-tamed complex structures $\mathcal{J}_t(\omega_0) \cap \mathcal{J}_t(\omega_1)$ is then diffeomorphic under the map μ_{J_0} to the intersection of the convex subsets given by the lemma. This intersection is again convex and hence contractible.

Proof of Lemma A.2 For any complex structure J tamed by ω, the endomorphism $J + J_0$ is invertible because for any nonzero w, we have $\omega(w, (J + J_0)w) > 0$, so in particular $(J + J_0)w$ is not zero. This proves the first part of the lemma.

Now fix a nonzero vector $v \in V$, and let C_v be the set of $A \in \text{End}(V)$ that anticommute with J_0, and that satisfy

$$\omega\big((A - I)v, (A - I)J_0 v\big) = -\omega\big((A - I)v, J_0(A + I)v\big) > 0.$$

We now prove that $C_v \subset \text{End}(V)$ is convex. Every segment $A_s = (1 - s)A_0 + sA_1$ with $s \in [0, 1]$ for arbitrary $A_0, A_1 \in C_v$ defines a polynomial of degree 2

$$P(s) = -\omega\big((A_s - I)v, J_0(A_s + I)v\big),$$

and the above inequality corresponds to checking that $P(s)$ is positive for all values $s \in [0, 1]$. The leading coefficient $-\omega((A_1 - A_0)v, J_0(A_1 - A_0)v)$ of $P(s)$ is never positive, because J_0 tames ω, so that $P(s)$ is either a line or a parabola facing downward. In both cases $P(s) \geq \min\{P(0), P(1)\} > 0$ for all $s \in (0, 1)$ so the inequality holds for the whole segment A_s.

Note that $C_v \neq \emptyset$ since $\mathbf{0} \in C_v$. Define the intersection

$$C^* := \bigcap_{v \neq 0} C_v,$$

which is a nonempty convex subset of $\text{End}(V)$. In fact, one has $C^* \subset \mathcal{A}_{J_0}^*(V)$, because if there were a matrix $A \in C^*$ with $\det(A - I) = 0$, then A would have an eigenvector $w \in V$ with eigenvalue 1, but then $-\omega((A - I)w, J_0(A + I)w) = 0$ so that $A \notin C_w$.

Since C^* lies in the domain of $\mu_{J_0}^{-1}$ and $\mathcal{J}_t(\omega)$ lies in the domain of μ_{J_0}, we have $C^* = \mu_{J_0}(\mathcal{J}_t(\omega))$, so that the image of the complex structures tamed by ω is convex as we wanted to show. $\qquad\square$

A.2 Existence of a cotamed complex structure

In this appendix, we prove Proposition 2.2, which we now recall:

Proposition 2.2 *Let V be a finite dimensional real vector space equipped with two symplectic forms ω_0 and ω_1. The following properties are equivalent:*

(1) *the segment between ω_0 and ω_1 consists of symplectic forms*
(2) *the ray starting at ω_0 and directed by ω_1 consists of symplectic forms*
(3) *there is a complex structure J on V tamed both by ω_0 and by ω_1.*

The equivalence between (1) and (3) was explained to us by Jean-Claude Sikorav. It relies on the simultaneous reduction of symplectic forms. Specifically, we need [38, Theorem 9.1] which we shall state (in a slightly weakened form) and reprove (in its full force) below as Proposition A.3, since the very general context of [38] makes it hard to read for people interested only in the symplectic case.

Recall that according to the linear Darboux theorem, any symplectic form on a $2n$-dimensional vector space is represented in some basis by the standard matrix

$$\Omega_{2n} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}.$$

We now want to understand what can be said for a pair of symplectic structures. Below we give an approximate normal form which is sufficient for our purposes and more pleasant to state than the precise result (cf. [38, Theorem 9.1]), though the precise result can also be extracted from the proof that we will give at the end of this section.

Proposition A.3 *Let ω_0 and ω_1 be symplectic forms on a finite dimensional vector space V. There exists a matrix A_1 that splits into blocks of the form*

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \quad and \quad \begin{pmatrix} 0 & 0 & \mu & \nu \\ 0 & 0 & -\nu & \mu \\ -\mu & \nu & 0 & 0 \\ -\nu & -\mu & 0 & 0 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R})$$

for $\lambda, \nu \neq 0$ with the following property: for any $\varepsilon > 0$, there is a basis of V such that ω_0 is represented by a block diagonal matrix with standard blocks Ω_{2k}, and ω_1 is represented by a matrix which is ε-close to A_1.

If the linear segment between ω_0 and ω_1 consists of symplectic forms, then the coefficients λ in the 2×2-blocks of A_1 described above cannot be negative.

The relation with cotamed complex structures will come from the following.

Proposition A.4

(a) *Let $V = \mathbb{R}^2$ with two antisymmetric bilinear forms ω_0 and ω_1 defined by $\omega_j(v, w) = v^t A_j w$, where*

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad A_1 = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}.$$

If $\lambda > 0$, then $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is tamed by both forms.

(b) *Let $V = \mathbb{R}^4$, and let ω_0 and ω_1 be antisymmetric bilinear forms defined by the matrixes*

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad and \quad A_1 = \begin{pmatrix} 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \\ -\lambda & \mu & 0 & 0 \\ -\mu & -\lambda & 0 & 0 \end{pmatrix},$$

with $\mu \neq 0$. Then there exists a complex structure J on $\mathbb{R}^4$ that is tamed by both forms.

Proof We only need to prove (b). For simplicity write V as $\mathbb{C}^2$, and the matrices A_0 and A_1 as

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and \quad A_1 = \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix}$$

with $z = \lambda + i\mu = re^{i\psi}$. The matrices

$$J_\phi = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix}$$

define complex structures on V, and it follows that $A_0 J_\phi = -\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$ is positive definite if $\cos\phi < 0$, and $A_1 J_\phi = -r\begin{pmatrix} e^{i(\psi-\phi)} & 0 \\ 0 & e^{i(\phi-\psi)} \end{pmatrix}$ is positive definite if $\cos(\psi - \phi) < 0$. As long as $\psi \neq \pi$ (which we have excluded by requiring that $\mu \neq 0$), it follows that we can choose ϕ such that $\phi \in (\pi/2, 3\pi/2)$ and $\phi - \psi \in (\pi/2, 3\pi/2) + 2\pi\mathbb{Z}$. $\qquad\square$

Proof of Proposition 2.2 We first explain the easy equivalence between (1) and (2). The (open) ray starting at ω_0 and directed by ω_1 and the open interval between ω_0 and ω_1 span the same cone in the space of antisymmetric bilinear forms. Since being symplectic is invariant under nonzero scalar multiplication, we have the equivalence.

The implication (3) $\Longrightarrow$ (1) is also direct because, for any $t \in [0, 1]$, we have

$$\big((1-t)\omega_0 + t\omega_1\big)(v, Jv) = (1-t)\omega_0(v, Jv) + t\omega_1(v, Jv),$$

which is positive whenever $v \in V$ is nonzero. So in particular, no such v can be in the kernel of an element of the segment between ω_0 and ω_1.

To prove (1) $\Longrightarrow$ (3), we use the fact that by Proposition A.3, there is a matrix A_1' that splits into certain standard blocks, such that we can find for any $\varepsilon > 0$ a basis of V for which ω_0 is in canonical form, and for which ω_1 is represented by a matrix that is ε-close to A_1'.

If condition (1) holds, then the blocks of A_1' correspond to the ones described in Proposition A.4, and we obtain the existence of a complex structure J on V that is tamed both by the standard symplectic form and by A_1'. By choosing $\varepsilon > 0$ sufficiently small, it follows that J is also tamed by ω_0 and ω_1, because tameness is an open condition. □

Proof of Proposition A.3 The proof will proceed in several steps.

Decomposition into generalized eigenspaces. In the first step we shall decompose V into suitable subspaces that are both ω_0- and ω_1-orthogonal.

Let $\varphi_r \colon V \to V^*$ for $r = 0, 1$ be the isomorphisms defined by $\varphi_r(v) := \omega_r(v, \cdot)$. We consider the endomorphism $B = \varphi_0^{-1} \circ \varphi_1$ of V so that $\omega_1(v, w) = \omega_0(Bv, w)$. The endomorphism B is invertible and it is ω_0-symmetric since:

$$\omega_0(Bv, w) = \omega_1(v, w) = -\omega_1(w, v) = -\omega_0(Bw, v) = \omega_0(v, Bw).$$

To define the generalized eigenspaces of B, complexify the vector space V to obtain $V^{\mathbb{C}}$, and extend the ω_r to sesquilinear forms $\omega_r^{\mathbb{C}}$. A computation analogous to the preceding one shows that B is $\omega_0^{\mathbb{C}}$-symmetric and we still have $\omega_0^{\mathbb{C}}(v, Bw) = \omega_1^{\mathbb{C}}(v, w)$.

The characteristic polynomial of B splits over $\mathbb{C}$ as $P(X) = \prod_\lambda (X - \lambda)^{m_\lambda}$, so we can decompose $V^{\mathbb{C}}$ into generalized eigenspaces

$$V^{\mathbb{C}} = \bigoplus_{\lambda \in Sp(B)} E_\lambda^{\mathbb{C}}; \quad E_\lambda^{\mathbb{C}} = \ker(B - \lambda)^{m_\lambda}.$$

Lemma A.5 *If λ and μ are eigenvalues of B such that $\lambda \neq \bar{\mu}$, then $E_\lambda^{\mathbb{C}}$ and $E_\mu^{\mathbb{C}}$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-orthogonal.*

Proof We prove by induction on k and l that $\ker(B - \lambda)^k$ and $\ker(B - \mu)^l$ are orthogonal.

To start the induction, note that if $v_\lambda \in \ker(B - \lambda)$, and $v_\mu \in \ker(B - \mu)$, then

$$(\bar{\lambda} - \mu)\omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = \omega_0^{\mathbb{C}}\big((B - \bar{\mu})v_\lambda, v_\mu\big) = \omega_0^{\mathbb{C}}\big(v_\lambda, (B - \mu)v_\mu\big) = 0,$$

thus since $\lambda \neq \bar{\mu}$, it follows that $\omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = 0$. Similarly, $\omega_1^{\mathbb{C}}(v_\lambda, v_\mu) = \omega_0^{\mathbb{C}}(v_\lambda, Bv_\mu) = \mu\omega_0^{\mathbb{C}}(v_\lambda, v_\mu) = 0$.

Assume now it has already been shown for the integers k and l that $\ker(B - \lambda)^k$ and $\ker(B - \mu)^l$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-orthogonal. Choose a vector $v_\lambda' \in \ker(B - \lambda)^{k+1}$ and use the fact that $Bv_\lambda' = \lambda v_\lambda' + w$ for some

$w \in \ker(B - \lambda)^k$. Then we obtain for any $v_\mu \in \ker(B - \mu)^l$,

$$
\begin{aligned}
(\bar{\lambda} - \mu)^l \omega_0^{\mathbb{C}}(v'_\lambda, v_\mu) &= (\bar{\lambda} - \mu)^{l-1} \omega_0^{\mathbb{C}}((B - \bar{\mu})v'_\lambda - w, v_\mu) \\
&= (\bar{\lambda} - \mu)^{l-1} \omega_0^{\mathbb{C}}((B - \bar{\mu})v'_\lambda, v_\mu) \\
&= \omega_0^{\mathbb{C}}(v'_\lambda, (B - \mu)^l v_\mu) = 0,
\end{aligned}
$$

and also $\omega_1^{\mathbb{C}}(v'_\lambda, v_\mu) = \omega_0^{\mathbb{C}}(Bv'_\lambda, v_\mu) = \bar{\lambda}\omega_0^{\mathbb{C}}(v'_\lambda, v_\mu) + \omega_0^{\mathbb{C}}(w, v_\mu) = 0$, which proves the induction step from (k, l) to $(k + 1, l)$. Since λ and μ have completely symmetric roles, this also explains how to go to $(k, l + 1)$. $\qquad\square$

We now relate this decomposition of $V^{\mathbb{C}}$ to the initial real vector space V. For a real eigenvalue λ, the intersection $V \cap E_\lambda^{\mathbb{C}}$ defines a real subspace E_λ with $\dim_{\mathbb{R}} E_\lambda = \dim_{\mathbb{C}} E_\lambda^{\mathbb{C}}$. Complex conjugation defines an isomorphism $E_\lambda^{\mathbb{C}} \to E_{\bar{\lambda}}^{\mathbb{C}}, v_\lambda \mapsto \bar{v}_\lambda$, and we can write $V \cap (E_\lambda^{\mathbb{C}} \oplus E_{\bar{\lambda}}^{\mathbb{C}})$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ as the direct sum of real subspaces $E_{\{\lambda, \bar{\lambda}\}} = \{v + \bar{v} \mid v \in E_\lambda^{\mathbb{C}}\} \oplus \{i(v - \bar{v}) \mid v \in E_\lambda^{\mathbb{C}}\}$.

This way we find a decomposition of V into pairwise ω_0- and ω_1-orthogonal subspaces

$$
E_{\mu_1} \oplus \cdots \oplus E_{\mu_k} \oplus E_{\{\lambda_1, \bar{\lambda}_1\}} \oplus \cdots \oplus E_{\{\lambda_l, \bar{\lambda}_l\}}
$$

with $\mu_1, \ldots, \mu_k \in \mathbb{R} \setminus \{0\}$, and $\lambda_1, \ldots, \lambda_l \in \mathbb{C} \setminus \mathbb{R}$.

Blocks with real eigenvalue. For the following considerations, we restrict to one of the subspaces E_{λ_j} with $\lambda_j \in \mathbb{R}$, and denote λ_j for simplicity just by λ. We will construct a basis of E_λ such that ω_0 and ω_1 have the particularly nice form described in the proposition. Note that ω_0 and ω_1 are both nondegenerate on E_λ.

Let $k + 1$ be the nilpotency index of $B - \lambda$, i.e. $(B - \lambda)^{k+1} = 0$ and $(B - \lambda)^k \neq 0$. Let v_0 be an element of E_λ not in $\ker(B - \lambda)^k$. We set $v_j := \varepsilon^{-j}(B - \lambda)^j v_0$ to define a collection of vectors $v_0, \ldots, v_k$. Choose now a vector $w_k \in E_\lambda$ with $\omega_0(v_k, w_k) = 1$ and $\omega_0(v_j, w_k) = 0$ for every $j \neq k$, and define inductively $w_{j-1} := \varepsilon^{-1}(B - \lambda)w_j$, or equivalently

$$
Bw_j = \lambda w_j + \varepsilon w_{j-1}
$$

for $j \geq 1$.

Lemma A.6 *The vectors* $v_0, \ldots, v_k, w_0, \ldots, w_k$ *are linearly independent and satisfy the relations* $\omega_r(v_j, v_{j'}) = \omega_r(w_j, w_{j'}) = 0$ *for all* $r = 0, 1$, *and* j, j', *and*

$$
\omega_0(v_j, w_{j'}) = \delta_{j,j'} \quad and \quad \omega_1(v_j, w_{j'}) = \lambda\delta_{j,j'} + \varepsilon\delta_{j,j'-1}.
$$

Proof We start by proving $\omega_r(v_j, v_{j'}) = 0$. For this we will use an induction on $|j - j'|$. If $j - j' = 0$ then the statement follows directly from the anti-symmetry of ω_r. Suppose that the claim is true for $j - j' \leq m$ and consider any j and j' with $j - j' = m + 1$ (in particular $j \geq 1$). We have

$$\varepsilon\omega_0(v_j, v_{j'}) = \omega_0\big((B - \lambda)v_{j-1}, v_{j'}\big) = \omega_1(v_{j-1}, v_{j'}) - \lambda\omega_0(v_{j-1}, v_{j'}) = 0$$

by the induction hypothesis. Using the fact that $Bv_{j'} = \varepsilon v_{j'+1} + \lambda v_{j'}$, we compute

$$\omega_1(v_j, v_{j'}) = \omega_0(v_j, Bv_{j'}) = \varepsilon\omega_0(v_j, v_{j'+1}) + \lambda\omega_0(v_j, v_{j'}).$$

The first term is zero by the induction hypothesis and the second one is zero because of the preceding computation. The proof of $\omega_r(w_j, w_{j'}) = 0$ follows the same lines, and will be omitted.

Note that

$$\omega_0(v_j, w_{j'}) = \varepsilon^{j'-k}\omega_0\big(v_j, (B - \lambda)^{k-j'}w_k\big)$$
$$= \varepsilon^{j'-k}\omega_0\big((B - \lambda)^{k-j'}v_j, w_k\big) = \omega_0(v_{k+j-j'}, w_k) = \delta_{j,j'},$$

and in particular this implies that $v_0, \ldots, v_k, w_0, \ldots, w_k$ are linearly independent vectors with respect to which ω_0 has standard form.

The remaining relation for ω_1 can be obtained by

$$\omega_1(v_j, w_{j'}) = \omega_0(v_j, Bw_{j'}) = \lambda\omega_0(v_j, w_{j'}) + \varepsilon\omega_0(v_j, w_{j'-1})$$
$$= \lambda\delta_{j,j'} + \varepsilon\delta_{j,j'-1}. \qquad \square$$

If we restrict ω_0 and ω_1 to the subspace $E = \mathrm{span}(v_0, \ldots, v_k, w_0, \ldots, w_k)$ and represent them in this basis, we now find that ω_0 is in standard form Ω_{2k} and ω_1 is represented by a matrix ε-close to $\lambda\Omega_{2k}$.

To continue the proof, restrict ω_0, ω_1, and B to the ω_0-symplectic complement E' of the space E. Note that E' is stable under B because for $u \in E'$,

$$\omega_0(v_j, Bu) = \omega_0(Bv_j, u) = \lambda\omega_0(v_j, u) + \varepsilon\omega_0(v_{j-1}, u) = 0,$$

and similarly for $\omega_0(w_j, Bu) = 0$. We can thus proceed as before to reduce all eigenspaces E_λ with $\lambda \in \mathbb{R}$ to ω_0-symplectic blocks in normal form.

Blocks with complex eigenvalue. We proceed now to the generalized complex eigenspace $E_\lambda^{\mathbb{C}}$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let k be the largest integer for which $E_\lambda^{\mathbb{C}} \neq \ker(B - \lambda)^k$, and construct as before a chain of vectors $v_0, \ldots, v_k \in E_\lambda^{\mathbb{C}}$ by starting with an element $v_0 \in E_\lambda^{\mathbb{C}} \setminus \ker(B - \lambda)^k$, and defining inductively

$$v_{j+1} := \varepsilon^{-1}(B - \lambda)v_j.$$

Using complex conjugation, we also find a chain $\bar{v}_0, \ldots, \bar{v}_k$ that lies in $E_{\bar{\lambda}}^{\mathbb{C}}$. Since B is the complexification of a real linear map, $\bar{v}_{j+1} := \varepsilon^{-1}(B - \bar{\lambda})\bar{v}_j$ holds.

Next, we define two chains $w_0, \ldots, w_k$ in $E_{\bar{\lambda}}^{\mathbb{C}}$ and $\bar{w}_0, \ldots, \bar{w}_k$ in $E_{\lambda}^{\mathbb{C}}$ by starting with a vector $w_k \in E_{\bar{\lambda}}^{\mathbb{C}}$ with $\omega_0^{\mathbb{C}}(v_k, w_k) = 1$ and $\omega_0^{\mathbb{C}}(v_j, w_k) = 0$ for every $j \neq k$, and defining $w_{j-1} := \varepsilon^{-1}(B - \bar{\lambda})w_j$, or equivalently

$$Bw_j = \bar{\lambda}w_j + \varepsilon w_{j-1}$$

for $j \geq 1$. Similarly, we obtain $\bar{w}_{j-1} = \varepsilon^{-1}(B - \lambda)\bar{w}_j$.

Lemma A.7

(a) *The space spanned by $v_0, \ldots, v_{k-1}, \bar{v}_0, \ldots, \bar{v}_{k-1}$ and the one spanned by $w_0, \ldots, w_{k-1}, \bar{w}_0, \ldots, \bar{w}_{k-1}$ are each isotropic with respect to both ω_0 and ω_1.*
(b) *The $\omega_0^{\mathbb{C}}$-pairings for these vectors are given by*

$$\omega_0^{\mathbb{C}}(v_j, \bar{w}_{j'}) = 0, \qquad \omega_0^{\mathbb{C}}(v_j, w_{j'}) = \delta_{j,j'},$$
$$\omega_0^{\mathbb{C}}(\bar{v}_j, w_{j'}) = 0, \qquad \omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j'}) = \delta_{j,j'}.$$

(c) *The $\omega_1^{\mathbb{C}}$-pairings for these vectors are given by*

$$\omega_1^{\mathbb{C}}(v_j, \bar{w}_{j'}) = 0, \qquad \omega_1^{\mathbb{C}}(v_j, w_{j'}) = \lambda\delta_{j,j'} + \varepsilon\delta_{j,j'-1},$$
$$\omega_1^{\mathbb{C}}(\bar{v}_j, w_{j'}) = 0, \qquad \omega_1^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j'}) = \bar{\lambda}\delta_{j,j'} + \varepsilon\delta_{j,j'-1}.$$

Proof To prove (a) note that since $\lambda \neq \bar{\lambda}$, the spaces $E_{\lambda}^{\mathbb{C}}$ and $E_{\bar{\lambda}}^{\mathbb{C}}$ are both $\omega_0^{\mathbb{C}}$- and $\omega_1^{\mathbb{C}}$-isotropic, so we only need to show that $\omega_r^{\mathbb{C}}(\bar{v}_j, v_{j'}) = \omega_r^{\mathbb{C}}(\bar{w}_j, w_{j'}) = 0$ for all j, j', and for $r = 0, 1$. If $j = j'$, we write v_j as $v_x + i v_y$, and we use sesquilinearity as follows:

$$\omega_0^{\mathbb{C}}(\bar{v}_j, v_j) = \omega_0^{\mathbb{C}}(v_x, v_x) + \omega_0^{\mathbb{C}}(v_x, iv_y) - \omega_0^{\mathbb{C}}(iv_y, v_x) - \omega_0^{\mathbb{C}}(iv_y, iv_y)$$
$$= \omega_0(v_x, v_x) + i\omega_0(v_x, v_y) + i\omega_0(v_y, v_x) - \omega_0(v_y, v_y)$$
$$= 0.$$

By the same computation, $\omega_1^{\mathbb{C}}(\bar{v}_j, v_j) = 0$.

If the statement is true for $j' - j = m \geq 0$, then

$$\varepsilon\omega_0^{\mathbb{C}}(\bar{v}_j, v_{j'+1}) = \omega_0^{\mathbb{C}}\big(\bar{v}_j, (B - \lambda)v_{j'}\big) = \omega_1^{\mathbb{C}}(\bar{v}_j, v_{j'}) - \lambda\omega_0^{\mathbb{C}}(\bar{v}_j, v_{j'})$$
$$= 0$$

and

$$\omega_1^{\mathbb{C}}(\bar{v}_j, v_{j'+1}) = \omega_0^{\mathbb{C}}(B\bar{v}_j, v_{j'+1}) = \omega_0^{\mathbb{C}}(\bar{\lambda}\bar{v}_j + \varepsilon\bar{v}_{j+1}, v_{j'+1})$$
$$= 0,$$

which finishes the induction. The argument for $\omega_r^{\mathbb{C}}(\bar{w}_j, w_{j'})$ is identical.

To prove (b), note first that the second two equations are the complex conjugate of the first two. Since $v_j, \bar{w}_{j'} \in E_\lambda^{\mathbb{C}}$, it also follows immediately that $\omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j'}) = 0$, so that we are only left with showing $\omega_0^{\mathbb{C}}(v_j, w_{j'}) = \delta_{j,j'}$, but the required computation is identical to the one used to show the analogous relation in the proof of Lemma A.6.

The equalities for (c) follow similarly.   □

We will now intersect the complex subspace spanned by the chains defined above with the initial real vector space V to finish the proof of the proposition. For this, define for all $j \leq k$ the real vectors

$$v_j^+ = \frac{1}{\sqrt{2}}(v_j + \bar{v}_j), \qquad v_j^- = \frac{i}{\sqrt{2}}(v_j - \bar{v}_j)$$

and

$$w_j^+ = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j), \qquad w_j^- = \frac{i}{\sqrt{2}}(w_j - \bar{w}_j)$$

which all lie in $E_{\lambda,\bar{\lambda}}$. Using the results deduced above, we obtain for all $r = 0, 1$, and j, j' the equations $\omega_r(v_j^+, v_{j'}^\pm) = \omega_r(v_j^-, v_{j'}^\pm) = 0$ and $\omega_r(w_j^+, w_{j'}^\pm) = \omega_r(w_j^-, w_{j'}^\pm) = 0$, and finally

$$2\omega_0(v_j^+, w_{j'}^+) = \omega_0^{\mathbb{C}}(v_j, w_{j'} + \bar{w}_{j'}) + \omega_0^{\mathbb{C}}(\bar{v}_j, w_{j'} + \bar{w}_{j'}) = 2\delta_{j,j'},$$
$$2\omega_0(v_j^+, w_{j'}^-) = i\omega_0^{\mathbb{C}}(v_j, w_{j'} - \bar{w}_{j'}) + i\omega_0^{\mathbb{C}}(\bar{v}_j, w_{j'} - \bar{w}_{j'}) = 0,$$
$$2\omega_1(v_j^+, w_{j'}^+) = \omega_1^{\mathbb{C}}(v_j, w_{j'} + \bar{w}_{j'}) + \omega_1^{\mathbb{C}}(\bar{v}_j, w_{j'} + \bar{w}_{j'})$$
$$= \omega_0^{\mathbb{C}}(v_j, Bw_{j'}) + \omega_0^{\mathbb{C}}(\bar{v}_j, B\bar{w}_{j'})$$
$$= \bar{\lambda}\omega_0^{\mathbb{C}}(v_j, w_{j'}) + \varepsilon\omega_0^{\mathbb{C}}(v_j, w_{j'-1})$$
$$+ \lambda\omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j'}) + \varepsilon\omega_0^{\mathbb{C}}(\bar{v}_j, \bar{w}_{j-1})$$
$$= (\lambda + \bar{\lambda})\delta_{j,j'} + 2\varepsilon\delta_{j,j'-1}$$

and similar computations for the other matrix elements, which prove the desired result with $\mu = \text{Re}\,\lambda$ and $\nu = \text{Im}\,\lambda$.

Sign of real eigenvalues. Assume that all 2-forms in the family

$$\omega_t := (1-t)\omega_0 + t\omega_1$$

for $t \in [0,1]$ are nondegenerate. The λ-coefficients in the 2×2-blocks of A_1' correspond to the real eigenvalues of the map B, so that if $\lambda < 0$ with eigenvector v, then we have $\omega_1(v, \cdot) = \omega_0(Bv, \cdot) = \lambda \omega_0(v, \cdot)$, and it follows that $\omega_t(v, \cdot) = (1 - t + t\lambda)\omega_0(v, \cdot)$ has to vanish for a certain value $t_0 \in (0, 1)$, so that ω_{t_0} is degenerate. $\qquad \square$

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