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The Hesselink stratification of nullcones and base change

Matthew C. Clarke · Alexander Premet

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Abstract Let G be a connected reductive algebraic group over an algebraically closed field of characteristic $p \geq 0$. We give a case-free proof of Lusztig's conjectures (Lusztig in Transform. Groups 10:449–487, 2005) on so-called unipotent pieces. This presents a uniform picture of the unipotent elements of G which can be viewed as an extension of the Dynkin–Kostant theory, but is valid without restriction on p. We also obtain analogous results for the adjoint action of G on its Lie algebra $\mathfrak{g}$ and the coadjoint action of G on $\mathfrak{g}^*$.

1 Introduction and statement of results

Notation. In what follows $\Bbbk$ will denote an algebraically closed field of arbitrary characteristic $p \geq 0$, unless stated otherwise. Let $\mathfrak{g}$ denote the Lie algebra of G, and let G_{uni} and $\mathfrak{g}_{\mathrm{nil}}$ denote the unipotent variety of G and nilpotent variety of $\mathfrak{g}$ respectively. By an $\mathfrak{sl}_2$-triple of $\mathfrak{g}$ we mean elements $e, f, h \in \mathfrak{g}$ such that $\langle e, f, h\rangle \cong \mathfrak{sl}_2(\Bbbk)$. We say that p is *good* for G if $p = 0$ or p is greater than the coefficient of the highest root in each component of the root system of G, expressed as an integer combination of simple roots. We denote by G' a connected reductive group over the complex numbers with the

M.C. Clarke
Trinity College, Cambridge CB2 1TQ, UK
e-mail: m.clarke@dpmms.cam.ac.uk

A. Premet (✉)
School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK
e-mail: sasha.premet@manchester.ac.uk

same root datum as G, and $\mathfrak{g}'$ its Lie algebra. We use $\mathrm{Hom}(A,B)$ to denote the set of algebraic group homomorphisms between algebraic groups A and B, and set $X(G)=\mathrm{Hom}(G,\Bbbk^\times)$, and $Y(G)=\mathrm{Hom}(\Bbbk^\times,G)$. We use $\langle\ ,\ \rangle$ to denote the natural pairing $X(G)\times Y(G)\to\mathbb{Z}$. We let G (resp. $\mathbb{Z}$) act on $Y(G)$ by $g\cdot\lambda:\xi\mapsto g\lambda(\xi)g^{-1}$ (resp. $n\lambda:\xi\mapsto\lambda(\xi)^n$) for all $\xi\in\Bbbk^\times$. The identity element of G will be denoted by 1_G. When G acts on a set X, we let X/G denote the set of G-orbits in X. We use the convention that $\mathbb{N}=\mathbb{Z}_{\geq 0}$. If $f:\Bbbk^\times\to V$ is a morphism of varieties and $v\in V$, then we use the notation $\lim_{\xi\to 0}f(\xi)=v$ to mean that f may be extended to a morphism $\tilde{f}:\Bbbk\to V$ such that $\tilde{f}(0)=v$.

1.1 We begin by briefly reviewing some classical results about unipotent elements of G'. First we assume that G' is a simple algebraic group of adjoint type over $\mathbb{C}$. Springer has shown that there exists a G'-equivariant bijective morphism $\sigma:G'_{\mathrm{uni}}\to\mathfrak{g}'_{\mathrm{nil}}$, a *Springer morphism*. (Cf. [47, Theorem III.3.12]. Usually the group is required to be simply connected but in characteristic zero the unipotent and nilpotent varieties of isogenous groups are naturally isomorphic so we may drop that requirement in this case.) Hence, the study of unipotent classes is equivalent to the study of nilpotent orbits. Let $e\in\mathfrak{g}_{\mathrm{nil}}$. Then, by the Jacobson–Morozov theorem, e lies in an $\mathfrak{sl}_2$-triple of $\mathfrak{g}'$. Kostant [22] has shown that this induces a bijection between G'-orbits of nilpotent elements and G'-orbits of subalgebras of $\mathfrak{g}'$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In [8] Dynkin determined the latter in terms of characteristic diagrams (now called weighted Dynkin diagrams), and showed that by considering $\mathfrak{g}'$ as an $\mathfrak{sl}_2(\mathbb{C})$-module, one can naturally define an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{g}'$. Thus, one obtains a homomorphism of algebraic groups $\mathrm{SL}_2(\mathbb{C})\to(\mathrm{Aut}\,\mathfrak{g}')^\circ=G'$. Let

$$\tilde{D}_{G'}=\left\{\omega\in Y\big(G'\big)\;\middle|\;\begin{array}{l}\exists\tilde{\omega}\in\mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}),G')\\ \text{with }\omega(\xi)=\tilde{\omega}\begin{bmatrix}\xi & \\ & \xi^{-1}\end{bmatrix}\end{array}\right\}. \tag{1}$$

Then we have the following bijection of finite sets:

$$\big\{\text{unipotent classes of }G'\big\}\stackrel{1-1}{\longleftrightarrow}\tilde{D}_{G'}/G'. \tag{2}$$

In fact (2) holds even when we relax the assumption that G' is simple and adjoint, by well-known reduction arguments; see, e.g., [5, Chap. 5].

1.2 Now assume that $p\geq 0$. It has been shown by Springer and Steinberg in [47] that if $p>3(h-1)$, where h is the Coxeter number of G, then everything described in the previous subsection remains true, by essentially the same proofs. Importantly, the analogue of $\tilde{D}_{G'}/G'$ for $p>3(h-1)$ is naturally in bijection with $\tilde{D}_{G'}/G'$, which can be seen by identifying both with certain subsets of Weyl group orbits on one parameter subgroups of a maximal torus. (We will consider a more precise correspondence of one parameter subgroups attached to fixed tori in Sect. 4 by taking a scheme-theoretic approach.) When $p \leq 3(h-1)$ the $\mathfrak{sl}_2$-theory may no longer be available and so an entirely different approach is necessary. However, Pommerening's theorem (which extends the Bala–Carter theorem) implies that, in fact, this parametrisation of unipotent classes extends to any good p. This means that one may take $\tilde{D}_{G'}/G'$ to be a parameter set for the unipotent classes of any connected reductive group with the same Dynkin diagram as G', independent of good characteristic. More recently, a case-free proof of Pommerening's theorem was found in [38] and simplified further in [48]. A Springer morphism also exists in good characteristic and so $\tilde{D}_{G'}/G'$ also parametrises the nilpotent orbits. Spaltenstein has shown further that this parametrisation preserves the poset structure and dimensions of classes, as well as certain compatibility relations between parabolic subgroups, across different ground fields of good characteristic ([43, Théorème III.5.2]).

When p is a bad prime for G, the number of unipotent classes is often bigger (but never less) than $|\tilde{D}_{G'}/G'|$, and, since Springer morphisms do not exist when p is bad, they need not be in bijection with the nilpotent orbits. Both have been determined in all cases, however. (See [5, pp. 180–183] for a bibliographic account.) By a classical result of Lusztig [25], based on the theory of complex representations of finite Chevalley groups, the orbit set G_{uni}/G is finite in all characteristics. The orbit set $\mathfrak{g}_{\mathrm{nil}}/G$ is always finite as well. Unfortunately, the only available proof of this fact for groups of types E_7 and E_8 relies very heavily on computer-aided computations; see [14]. It turns out that in all cases the cardinality of the set G_{uni}/G is less than or equal to that of $\mathfrak{g}_{\mathrm{nil}}/G$.

1.3 Following [26] we now define unipotent pieces. First note that $Y(G)/G$ is naturally isomorphic to $Y(G')/G'$. (Indeed, in each case we may restrict to one parameter subgroups of a fixed maximal torus, say T and T', since all maximal tori are conjugate. Then the orbits are precisely the Weyl group orbits on the $\mathbb{Z}$-modules $Y(T)$, $Y(T')$, which can be identified unambiguously.) We let $\tilde{D}_G$ denote the unique G-stable subset of $Y(G)$ whose image in $Y(G')/G'$ corresponds to $\tilde{D}_{G'}/G'$ under this bijection. Corresponding to $\tilde{D}_G$ we define D_G to be the set of sequences

$$\triangle = \left(G_0^{\triangle} \supset G_1^{\triangle} \supset G_2^{\triangle} \supset \cdots\right)$$

of closed connected subgroups of G such that for some $\omega \in \tilde{D}_G$ we have

$$\operatorname{Lie} G_i^{\triangle} = \left\{x \in \mathfrak{g} \;\middle|\; \lim_{\xi \to 0} \xi^{1-i}\big(\operatorname{Ad}\omega(\xi)\big)x = 0\right\}.$$

The obvious map $\tilde{D}_G \to D_G$ induces a bijection $\tilde{D}_G/G \xrightarrow{\sim} D_G/G$ on the set of G-orbits. Assume that $\omega \in \tilde{D}_G$ corresponds to some $G_0^{\triangle}$, and T is a maximal torus of $G_0^\triangle$ containing $\operatorname{Im}\omega$, and let Σ denote the root system of G relative to T. Then one can show that

$$G_0^\triangle = \big\langle T, U_\alpha | \alpha \in \Sigma, \langle \alpha, \omega \rangle \ge 0 \big\rangle, \quad \text{and}$$
$$G_i^\triangle = \big\langle U_\alpha | \alpha \in \Sigma, \langle \alpha, \omega \rangle \ge i \big\rangle \quad \text{for } i \ge 1,$$

where the U_α are the root subgroups of G relative to T. From this characterisation we see that $G_0^\triangle$ is a parabolic subgroup of G, with unipotent radical $G_1^\triangle$, and that $G_i^\triangle$ is normalised by $G_0^\triangle$ for any $i \ge 0$.

For any G-orbit $\blacktriangle \in D_G/G$, let $\tilde{H}^\blacktriangle = \bigcup_{\triangle \in \blacktriangle} G_2^\triangle$. It is straightforward to see that each set $\tilde{H}^\blacktriangle$ is a closed irreducible variety stable under the conjugation action of G; see Lemma 5.2. We now define

$$H^\blacktriangle := \tilde{H}^\blacktriangle \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'},$$

where the union is taken over all $\blacktriangle' \in D_G/G$ such that $\tilde{H}^{\blacktriangle'} \subsetneq \tilde{H}^\blacktriangle$. The subsets $H^\blacktriangle$ are called the *unipotent pieces* of G. We also define

$$X^\triangle := G_2^\triangle \cap H^\blacktriangle,$$

for each $\triangle \in D_G$, where $\blacktriangle$ is the G-orbit of $\triangle$. Since $H^\blacktriangle$ is the complement of finitely many non-trivial closed subvarieties of $\tilde{H}^\blacktriangle$, it is open and dense in $\tilde{H}^\blacktriangle$, hence it is locally closed in G_{uni}. The subset $H^\blacktriangle$ is G-stable since its complement in $\tilde{H}^\blacktriangle$ is. Consequently, $X^\triangle$ is open and dense in $G_2^\triangle$, and stable under conjugation by $G_0^\triangle$. It is worth mentioning that $G_0^\triangle$ coincides with the stabiliser of $\triangle$ in G.

1.4 In [26], Lusztig has stated the following five properties and conjectured that they should hold for all connected reductive groups G over algebraically closed fields.

$\mathfrak{P}_1$. The sets $X^\triangle$ ($\triangle \in D_G$) form a partition of G_{uni}, that is $G_{\text{uni}} = \bigsqcup_{\triangle \in D_G} X^\triangle$.

$\mathfrak{P}_2$. For every $\blacktriangle \in D_G/G$ the sets $X^\triangle$ ($\triangle \in \blacktriangle$) form a partition of $H^\blacktriangle$, i.e. $H^\blacktriangle = \bigsqcup_{\triangle \in \blacktriangle} X^\triangle$.

$\mathfrak{P}_3$. The locally closed subsets $H^\blacktriangle$ ($\blacktriangle \in D_G/G$) form a (finite) partition of G_{uni}, that is $G_{\text{uni}} = \bigsqcup_{\blacktriangle \in D_G/G} H^\blacktriangle$.

$\mathfrak{P}_4$. For any $\triangle \in D_G$ we have that $G_3^\triangle X^\triangle = X^\triangle G_3^\triangle = X^\triangle$.

$\mathfrak{P}_5$. Suppose $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$ and let $F\colon G \to G$ be the Frobenius endomorphism corresponding to a split $\mathbb{F}_q$-rational structure. Let $\triangle \in D_G$ be such that $F(G_i^\triangle) = G_i^\triangle$ for all $i \ge 0$ and let $\blacktriangle$ be the G-orbit of $\triangle \in D_G$. Then there exist polynomials $\varphi^\blacktriangle(t)$ and $\psi^\triangle(t)$ in $\mathbb{Z}[t]$ with coefficients independent of p such that $\varphi^\blacktriangle(q) = |H^\blacktriangle(\mathbb{F}_q)|$ and $\psi^\triangle(q) = |X^\triangle(\mathbb{F}_q)|$.

When p is good, properties $\mathfrak{P}_1$–$\mathfrak{P}_5$ follow from Pommerening's classification; see [34], [35], [18], [38]. Lusztig has proved in [26], [27] and [29] that $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for classical groups (any p) by a case-by-case analysis. For groups of type E (any p) properties $\mathfrak{P}_1$–$\mathfrak{P}_5$ can be deduced from [31], although this is unsatisfactory since the extensive computations which the results of that paper are based on are largely omitted, and these results are known to contain many misprints. As mentioned in [26, p. 451] it is desirable to have an independent verification of properties $\mathfrak{P}_1$–$\mathfrak{P}_5$ for groups of type E.

More recently Lusztig has introduced natural analogues of the unipotent pieces $X^{\triangle}$ ($\triangle \in D_G$) and $H^{\blacktriangle}$ ($\blacktriangle \in D_G/G$) for the adjoint G-module $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ and called them *nilpotent pieces* of $\mathfrak{g}$ and $\mathfrak{g}^*$. Replacing G_{uni} by the nilpotent varieties $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ (see Sect. 2.1) he conjectured that properties $\mathfrak{P}_1$–$\mathfrak{P}_5$ should hold for them as well. We stress that the G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$ are *very* different when $p = 2$ and G is of type B, C or F_4 and when $p = 3$ and G is of type G_2. In all other cases there is a G-equivariant bijection $\mathcal{N}_{\mathfrak{g}} \xrightarrow{\sim} \mathcal{N}_{\mathfrak{g}^*}$ which restricts to a bijection between the corresponding nilpotent pieces and induces a 1–1 correspondence between the orbit sets $\mathcal{N}_{\mathfrak{g}}/G$ and $\mathcal{N}_{\mathfrak{g}^*}/G$; see [39, Sect. 5.6] for more details. It is worth mentioning that the coadjoint action of G on $\mathfrak{g}^*$ plays a very important role in studying irreducible representations of the Lie algebra $\mathfrak{g}$.

In [27], [29] and [28], Lusztig proved that $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for $\mathcal{N}_{\mathfrak{g}}$ in the case where G is a classical group and for $\mathcal{N}_{\mathfrak{g}^*}$ in the case where G is a group of type C. Very recently the coadjoint case for groups of type B was settled by Ting Xue, a former PhD student of Lusztig; see [49]. In proving $\mathfrak{P}_1$–$\mathfrak{P}_5$ for classical groups Lusztig and Xue relied on intricate counting arguments involving linear algebra in characteristic 2 and combinatorics.

The main goal of this paper is to give a uniform proof of the following using Hesselink's theory of the stratification of nullcones.

Theorem *Let G be a reductive group over an algebraically closed field of characteristic $p \geq 0$ and $\mathfrak{g} = \operatorname{Lie} G$. Let $\mathcal{G}$ be one of G, $\mathfrak{g}$ or $\mathfrak{g}^*$ and write $X^{\triangle}(\mathcal{G})$ for the piece $X^{\triangle}$ of $\mathcal{G}$ labelled by $\triangle \in D_G$. Then $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for $\mathcal{G}$ and the stabiliser in G of any element in $X^{\triangle}(\mathcal{G})$ is contained in $G_0^{\triangle}$.*

We mention for completeness that the definition of nilpotent pieces used by Lusztig and Xue for G classical differs formally from Lusztig's original definition in [26] which we follow. However, Theorem 1.4 implies that both definitions give rise to the same partitions of $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$; see Remark 14 for more details. It is far from clear whether the definition of Lusztig and Xue can be used for exceptional groups in arbitrary characteristic.

Remark Regarding $\mathfrak{P}_2$, Lusztig has also conjectured that each piece $H^{\blacktriangle}(\mathcal{G})$ is a smooth variety and there exists a G-equivariant fibration $f\colon H^{\blacktriangle}(\mathcal{G}) \twoheadrightarrow$

$\blacktriangle \cong G/G_0^{\triangle}$ such that $f^{-1}(\triangle) \cong X^{\triangle}$ for all $\triangle \in \blacktriangle$. As far as we know, the smoothness of $H^{\blacktriangle}(\mathcal{G})$ is still an open problem in bad characteristic. Using the techniques of [1, Sect. 4] one can show that there always exists a projective homogeneous G-variety $Y \cong G/P$, where P is a parabolic group scheme with $P_{\mathrm{red}} = G_0^{\triangle}$, and a G-equivariant fibration φ: $H^{\blacktriangle}(\mathcal{G}) \twoheadrightarrow Y$ whose fibres are isomorphic to $X^{\triangle}$ where $\triangle \in \blacktriangle$. However, we do not know whether φ can be chosen to be separable, hence the smoothness of $H^{\blacktriangle}(\mathcal{G})$ is not guaranteed. On the other hand, in the Lie algebra case there exist nilpotent pieces $H^{\blacktriangle} = H^{\blacktriangle}(\mathfrak{g})$ which are not G-equivariantly isomorphic to the geometric quotients $G \times^{G_0^{\triangle}} X^{\triangle}$ with $\triangle \in \blacktriangle$. The simplest example occurs when $\mathrm{char}\, \Bbbk = 2$, $G = \mathrm{PSL}_2(\Bbbk)$ and $X^{\triangle} = \Bbbk^{\times} e$ where e is a nonzero nilpotent element of $\mathfrak{g} = \mathfrak{pgl}_2(\Bbbk)$. To see this it suffices observe that $\tilde{H}^{\blacktriangle} = \mathcal{N}_{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ is an abelian ideal of $\mathfrak{g}$ and hence the derived action of $[\mathfrak{g}, \mathfrak{g}] \subset \mathrm{Lie}(G)$ on the function algebra $\Bbbk[H^{\blacktriangle}] \subset \Bbbk(\mathcal{N}_{\mathfrak{g}})$ is trivial, whereas the action of $[\mathfrak{g}, \mathfrak{g}]$ on $\Bbbk[G \times^{G_0^{\triangle}} X^{\triangle}] = H^0(G/G_0^{\triangle}, \Bbbk[X^{\triangle}])$ is not trivial.

It is well-known that the sets G_{uni}, $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ coincide with the subvarieties of G-unstable elements of the G-varieties G, $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (we assume that G acts on itself by conjugation). Therefore each set admits a natural stratification coming from the Kempf–Rousseau theory, which we review in Sect. 2. In fact, such a stratification was defined by Hesselink [13] for any affine G-variety V with a distinguished point $*$ fixed by the action of G. It is often referred to as the *Hesselink stratification* of the variety of Hilbert nullforms of V. In Sect. 5 we show that every piece $H^{\blacktriangle}(\mathcal{G})$ coincides with a Hesselink stratum of $\mathcal{G}$ and conversely every Hesselink stratum of $\mathcal{G}$ has the form $H^{\blacktriangle}(\mathcal{G})$ for a unique $\blacktriangle \in D_G/G$. We also identify the subsets $X^{\triangle}(\mathcal{G})$ ($\triangle \in D_G$) with the *blades* of the variety of nullforms of $\mathcal{G}$. (As in the theorem we assume here that $\mathcal{G}$ is one of G, $\mathfrak{g}$ or $\mathfrak{g}^*$.)

In order to relate the pieces $H^{\blacktriangle}(\mathcal{G})$ ($\blacktriangle \in D_G/G$) with Hesselink strata we first upgrade certain reductive subgroups of G involved in the Kempf–Ness criterion for optimality of one parameter subgroups to reductive $\mathbb{Z}$-group schemes split over $\mathbb{Z}$, and then make use of a well-known result of Seshadri [41] on invariants of reductive group schemes. This is done in Sect. 4. After relating unipotent and nilpotent pieces with Hesselink strata we deduce rather quickly that $\mathfrak{P}_1$–$\mathfrak{P}_4$ hold for G, $\mathfrak{g}$ and $\mathfrak{g}^*$.

1.5 Proving that $\mathfrak{P}_5$ holds for G, $\mathfrak{g}$ and $\mathfrak{g}^*$ requires more effort. Since our arguments involve induction on the rank of the group we have to look at a much larger class of finite-dimensional rational G-modules.

Let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme split over $\mathbb{Z}$ and suppose that $\Bbbk$ contains an algebraic closure of $\mathbb{F}_p$. Set $G' := \mathfrak{G}(\mathbb{C})$ and $G := \mathfrak{G}(\Bbbk)$. We say that a rational G-module V is *admissible* if there is a finite-dimensional G'-module V' and an admissible lattice $V'_{\mathbb{Z}}$ in V' such that $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Recall that a $\mathbb{Z}$-lattice in V' is called *admissible* if it is stable under the action of the distribution algebra $\mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G})$; see [16] for more details. For any pth power q we may regard the finite vector space $V(\mathbb{F}_q) := V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_q$ as an $\mathbb{F}_q$-form of the $\mathbb{k}$-vector space V.

Since G is a reductive group, the invariant algebra $\mathbb{k}[V]^G$ is generated by finitely many homogeneous polynomial functions $f_1, \ldots, f_m$ on V. The G-*nullcone* of V, denoted $\mathcal{N}_{G,V}$ or simply $\mathcal{N}_V$, is defined as the zero locus of $f_1, \ldots, f_m$ in V. We set $\mathcal{N}_V(\mathbb{F}_q) := \mathcal{N}_V \cap V(\mathbb{F}_q)$.

Theorem *For every admissible G-module V there exists a polynomial $n_V(t) \in \mathbb{Z}[t]$ such that $|\mathcal{N}_V(\mathbb{F}_q)| = n_V(q)$ for all $q = p^l$. The polynomial $n_V(t)$ depends only on the G'-module V', but not on the choice of an admissible lattice $V'_{\mathbb{Z}}$, and is the same for all primes $p \in \mathbb{N}$.*

In fact, a more general version of Theorem 1.5 is established in Sect. 6.2 which takes care of non-split Frobenius actions on G. Property $\mathfrak{P}_5$ for $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ now follows almost at once since both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible G-modules; see Sect. 7. Proving $\mathfrak{P}_5$ for G_{uni} requires some extra work; see Corollary 7. Theorem 1.5 enables us to show that the classical results of Steinberg and Springer on the cardinality of $G_{\mathrm{uni}}(\mathbb{F}_q)$ and $\mathcal{N}_{\mathfrak{g}}(\mathbb{F}_q)$, respectively, are equivalent. It also enables us to compute the cardinality of $\mathcal{N}_{\mathfrak{g}^*}(\mathbb{F}_q)$ thereby generalising a recent result of Lusztig proved for G classical; see [28] and [49].

Corollary *Let $N = \dim G - \mathrm{rk} G$. Then $|\mathcal{N}_{\mathfrak{g}}(\mathbb{F}_q)| = |\mathcal{N}_{\mathfrak{g}^*}(\mathbb{F}_q)| = q^N$ for any pth power q and any prime $p \in \mathbb{N}$.*

Once we observe that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible G-modules coming from the adjoint G'-module $\mathfrak{g}'$, Corollary 1.5 follows immediately from Steinberg's formula $|G_{\mathrm{uni}}(\mathbb{F}_q)| = q^N$ and the existence for $p \gg 0$ of a G-equivariant isomorphism between $\mathcal{N}_{\mathfrak{g}}$ and G_{uni} defined over $\mathbb{F}_q$. Indeed, Theorem 1.5 then ensures that the polynomial $n_{\mathfrak{g}}(t) = n_{\mathfrak{g}^*}(t)$ has coefficients independent of p.

2 The Kempf–Rousseau theory

Although much of this theory goes back to Mumford [32], Kempf [20] and Rousseau [40], our set-up here is inspired by Hesselink [13], Slodowy [42] and Tsujii [48].

2.1 Let V be a pointed G-variety, i.e. a G-variety with a distinguished point $* \in V$ fixed by the action of G. We will assume further that V is affine and non-singular at $*$, although many results still hold even when $*$ is singular. Let H be a Zariski closed reductive subgroup of G. Then a point $v \in V$ is called *H-unstable* if there exists some $\lambda \in Y(H)$ such that $\lim_{\xi\to 0}\lambda(\xi)\cdot v = *$. Otherwise we say that v is *H-semistable*.

Theorem (The Hilbert-Mumford criterion (cf. [30])) *The following are equivalent.*

(i) *v is H-unstable.*
(ii) *$f(v)=0$ for each regular function $f \in \Bbbk[V]^H$ which vanishes at $*$.*
(iii) $* \in \overline{H\cdot v}$.

The set of all G-unstable elements is called the *nullcone*, denoted $\mathcal{N}_V$. It is well-known that $\Bbbk[V]^H$ is generated (as a $\Bbbk$-algebra with 1) by finitely many elements, and so $\mathcal{N}_V$ is Zariski-closed in V. (In positive characteristic this requires the Mumford conjecture proved by Haboush in [11].) If we take $V=\mathfrak{g}$, with adjoint G-action and $*=0$, then in all characteristics $\mathcal{N}_{\mathfrak{g}}=\mathfrak{g}_{\mathrm{nil}}$. Similarly, if $V=G$, with the conjugation action and $*=1_G$, then in all characteristics $\mathcal{N}_G = G_{\mathrm{uni}}$.

2.2 Let $\psi : X\to Y$ be a morphism of affine varieties, and let $\psi^* : \Bbbk[Y]\to \Bbbk[X]$ be its comorphism. Let $y\in Y$ and let I_y be the maximal ideal of y in $\Bbbk[Y]$. We define the *coordinate ring* of the schematic fibre $\psi^{-1}(y)$ to be $\Bbbk[X]/\psi^*(I_y)\Bbbk[X]$ (cf. [9, Sect. 14.3]). Now let $v\in V$ and $\lambda\in Y(G)$. If $\lim_{\xi\to 0}\lambda(\xi)\cdot v = *$ and $v\neq *$, then the fibre of the extended morphism at $*$ has coordinate ring $\Bbbk[T]/(T^m)$ for some m, where T is an indeterminate.

We now define a function which can be used to measure instability. Given $\lambda\in Y(G)$ we define a function $m(-,\lambda)\colon V \longrightarrow \mathbb{Z}_{\geq 0}\sqcup\{\pm\infty\}$ as follows:

$$m(v,\lambda) := \begin{cases} -\infty & \text{if } \lim_{\xi\to 0}\lambda(\xi)\cdot v \text{ does not exist;}\\ 0 & \text{if } \lim_{\xi\to 0}\lambda(\xi)\cdot v = v' \neq *;\\ m \text{ (as above)} & \text{if } \lim_{\xi\to 0}\lambda(\xi)\cdot v = *(v\neq *);\\ +\infty & \text{if } v = *. \end{cases}$$

Note that $v\in V$ is H-unstable if and only if $m(v,\lambda)\geq 1$ for some $\lambda\in Y(H)$. For a set $X\subset V$ we also define $m(X,\lambda)=\inf_{v\in X} m(v,\lambda)$, and say that X is *uniformly unstable* if $m(X,\lambda)\geq 1$ for some $\lambda\in Y(G)$.

2.3 Let $\lambda\in Y(G)$. We define some subgroups of G associated to λ as follows:

$$P(\lambda) := \Big\{ g\in G \;\Big|\; \lim_{\xi\to 0}\lambda(\xi)g\lambda(\xi)^{-1} \text{ exists} \Big\},$$
$$L(\lambda) := C_G(\operatorname{Im}\lambda),$$
$$U(\lambda) := \Big\{ g\in G \;\Big|\; \lim_{\xi\to 0}\lambda(\xi)g\lambda(\xi)^{-1} = 1_G \Big\}.$$

Let T be a maximal torus of $L(\lambda)$ (and therefore a maximal torus of G). If Σ is the root system of G relative to T, then

$$P(\lambda) = \big\langle T, U_\alpha | \alpha \in \Sigma, \langle \alpha, \lambda \rangle \geq 0 \big\rangle,$$
$$L(\lambda) = \big\langle T, U_\alpha | \alpha \in \Sigma, \langle \alpha, \lambda \rangle = 0 \big\rangle,$$
$$U(\lambda) = \big\langle U_\alpha | \alpha \in \Sigma, \langle \alpha, \lambda \rangle \geq 1 \big\rangle.$$

Hence $P(\lambda)$ is a parabolic subgroup of G with unipotent radical $U(\lambda)$. The following is now a straightforward exercise.

Lemma *Let $v \in V$ and $\lambda \in Y(G)$. Then $m(g \cdot v, \lambda) = m(v, g \cdot \lambda) = m(v, \lambda)$ for all $g \in P(\lambda)$. In particular, for $i \geq 0$, the set of $v \in V$ such that $m(v, \lambda) \geq i$ is $P(\lambda)$-invariant.*

2.4 We define the set of *virtual one parameter subgroups* of G as follows. Let

$$Y_{\mathbb{Q}}(G) = \big(\mathbb{N} \times Y(G)\big)/\sim,$$

where $\sim$ is the equivalence relation on $\mathbb{N} \times Y(G)$ such that $(n, \lambda) \sim (m, \mu)$ if and only if $n\mu = m\lambda$. Note that $Y(G)$ is naturally a subset of $Y_{\mathbb{Q}}(G)$ and the action of G on $Y(G)$ naturally induces an action on $Y_{\mathbb{Q}}(G)$. If T is a torus, then $Y(T)$ is a free $\mathbb{Z}$-module, and so $Y_{\mathbb{Q}}(T) \cong Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ may be regarded as a $\mathbb{Q}$-vector space. We extend our measure of instability to $Y_{\mathbb{Q}}(G)$ as follows. For $\lambda \in Y_{\mathbb{Q}}(G)$, we have that $n\lambda \in Y(G)$ for some $n \in \mathbb{N}$ and so we may define $m(v, \lambda) = n^{-1} m(v, n\lambda)$.

A *squared norm mapping* on $Y_{\mathbb{Q}}(G)$ is a G-invariant function $q : Y_{\mathbb{Q}}(G) \to \mathbb{Q}_{\geq 0}$ whose restriction to $Y_{\mathbb{Q}}(T)$ for any maximal torus T is a positive definite quadratic form. By an averaging trick (cf. [46, Sect. 7.1.7]) one can always define a W-invariant positive definite quadratic form q on $Y_{\mathbb{Q}}(T)$. For an arbitrary $\lambda \in Y_{\mathbb{Q}}(G)$, let $g \in G$ be such that $g \cdot \lambda \in Y_{\mathbb{Q}}(T)$. Then define $q(\lambda) = q(g \cdot \lambda)$. One checks that this defines a squared norm mapping on $Y_{\mathbb{Q}}(G)$ by observing that the G-orbits on $Y_{\mathbb{Q}}(G)$ restrict to the W-orbits on $Y_{\mathbb{Q}}(T)$. We define a map $\|\cdot\|_q : Y_{\mathbb{Q}}(G) \to \mathbb{R}_{\geq 0}$ by $\|\lambda\|_q := \sqrt{q(\lambda)}$ for all $\lambda \in Y_{\mathbb{Q}}(G)$, which we call a *norm* on $Y_{\mathbb{Q}}(G)$. From now on we will fix such a norm, and drop the subscript q. Let $X \subset V$ and $\lambda \in Y(G) \setminus \{0\}$. We say that λ is *optimal* for X if

$$\frac{m(X, \lambda)}{\|\lambda\|} \geq \frac{m(X, \mu)}{\|\mu\|} \quad \text{for all } \mu \in Y(G) \setminus \{0\}.$$

If $v \in V$ then, for ease of notation, we will often identify it with the set $\{v\}$ and thus talk about one parameter subgroups which are optimal for v. Usually the notion of optimality depends on the norm, but in the special case that $V = \mathfrak{g}_{\mathrm{nil}}$ or G_{uni}, with adjoint or conjugation action respectively, or when V is a G-module, it is independent of the norm by [12, Theorem 7.2]. Note that if λ is optimal for some set, then so is any non-zero scalar multiple of λ. It will be convenient therefore to have a canonical way of choosing an element in $(\mathbb{Q}^{\times}\lambda) \cap Y(G)$ and for this we use the following notion from [42]. We say that λ is *primitive* if we cannot write $\lambda = n\mu$ for any integer $n \geq 2$ and $\mu \in Y(G)$. If $X \subset V$ is uniformly unstable, we let Δ_X denote the set of all primitive elements in $Y(G)$ which are optimal for X.

Remark Hesselink has defined a similar set in [13], denoted $\Delta(X)$. This corresponds to a canonical choice for optimal *virtual* one parameter subgroups. Let $\lambda \in \Delta_X$. Then $\Delta(X) = \frac{1}{m(X,\lambda)}\Delta_X$. We will need to use both sets later. To avoid confusion we will use $\tilde{\Delta}_X$ to denote $\Delta(X)$, except in Sect. 6.1, where it would be cumbersome to do so.

Theorem (Kempf [20], Rousseau [40]) *Let $X \subset V$ be uniformly unstable.*

(i) *We have $\Delta_X \neq \emptyset$ and there exists a parabolic subgroup $P(X)$ in G such that $P(X) = P(\lambda)$ for all $\lambda \in \Delta_X$.*
(ii) *We have $\Delta_X = \{g \cdot \lambda | g \in P(X)\}$ for any $\lambda \in \Delta_X$.*
(iii) *If T is a maximal torus of $P(X)$, then $Y(T) \cap \Delta_X$ contains exactly one element, which we denote by $\lambda_T(X)$.*
(iv) *For any $g \in G$ we have that $\Delta_{g\cdot X} = g\Delta_X g^{-1}$ and $P(g \cdot X) = gP(X)g^{-1}$. The stabiliser $G_X = \{g \in G | g \cdot X = X\}$ is contained in $P(X)$.*

2.5 We now restrict to the special case where V is a finite-dimensional rational G-module with $* = 0$. Let T be a maximal torus of G with Weyl group W. A very useful set of tools for analysing the T-instability and optimality of subsets of V are certain polytopes in $Y_{\mathbb{Q}}(T)$ defined in terms of weights of the T-action on V. Let $X_{\mathbb{Q}}(T) = X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $(\ ,\ )$ be a W-invariant inner product on $Y_{\mathbb{Q}}(T)$ induced by the norm $\|\cdot\|$. Then there is a $\mathbb{Q}$-linear isomorphism $\phi_T \colon X_{\mathbb{Q}}(T) \to Y_{\mathbb{Q}}(T)$ defined uniquely by the relation $\langle \chi, \lambda\rangle = (\phi_T(\chi), \lambda)$ for all $\chi \in X_{\mathbb{Q}}(T)$ and $\lambda \in Y_{\mathbb{Q}}(T)$.

Consider the weight space decomposition $V = \bigoplus_{\chi \in X(T)} V_\chi$ of V with respect to T, where

$$V_\chi = \left\{v \in V | t \cdot v = \chi(t)v \text{ for all } t \in T\right\}.$$

Then for any $v \in V$ we may uniquely write $v = \sum_{\chi \in X(T)} v_\chi$ with $v_\chi \in V_\chi$. If $X \subset V$, we define $S_T(X) := \{\chi \in X(T) | v_\chi \neq 0 \text{ for some } v \in X\}$, and let $K_T(X)$ denote the convex hull (or Newton polytope) of $\phi_T(S_T(X))$ in $Y_{\mathbb{Q}}(T)$. Then we have the following.

Lemma (Cf. [42]) *Let $X \subset V$ and T be a maximal torus of G.*

(i) *If $\lambda \in Y(T)$, then $m(X, \lambda) = \min_{\mu \in \phi_T(S_T(X))}(\mu, \lambda) = \min_{\mu \in K_T(X)}(\mu, \lambda)$.*

(ii) *There exists a unique element* $\mu_T(X) \in K_T(X)$ *of minimal norm.*
(iii) *The set* X *is uniformly* T*-unstable if and only if* $\mu_T(X) \neq 0$, *in which case we have that* $\|\mu_T(X)\|^2 = m(X, \mu_T(X))$.
(iv) *If* X *is* T*-unstable and* $\lambda_T(X)$ *is the unique primitive scalar multiple of* $\mu_T(X)$, *then* $\Delta_{X,T} = \{\lambda_T(X)\}$.

We mention that when X is uniformly G-unstable and $T \subset P(X)$, then $\Delta_{X,T} = Y(T) \cap \Delta_X$ and $\lambda_T(X)$ is nothing but the element from Theorem 2.4(iii).

2.6 Resume the more general assumption that V is a G-variety. For $i \geq 0$ and $\lambda \in Y_{\mathbb{Q}}(G)$, we denote by $V(\lambda)_i$ be the set of elements $v \in V$ with $m(v, \lambda) \geq i$, a closed subvariety of V. Let $X \subset V$ be uniformly unstable and suppose that $\lambda \in \Delta_X$ and $k = m(X, \lambda)$. Then we define the *saturation* of X to be $S(X) = V(\Delta_X)_k$. This is well-defined by Theorem 2.4(ii) and Lemma 2. We call a set *saturated* if it is uniformly unstable and equal to its own saturation.

Assume, temporarily, again that V is a G-module with $* = 0$. We may grade V, with respect to λ, as a direct sum of subspaces

$$V(\lambda, i) = \left\{v \in V \,|\, \lambda(\xi) \cdot v = \xi^i v \text{ for all } \xi \in \Bbbk^\times\right\},$$

for $i \in \mathbb{Z}$. Then a saturated set $X \subset V$ may be written as

$$X = V(\Delta_X)_k = \bigoplus_{i \geq k} V(\lambda, i),$$

where $\lambda \in \Delta_X$ and $k = m(X, \lambda)$. Letting T be a maximal torus of $C_G(\operatorname{Im} \lambda)$, it is not hard to see that the $V(\lambda, i)$ are sums of weight subspaces of V. More precisely,

$$X = \bigoplus_{\langle \chi, \lambda \rangle \geq k} V_\chi.$$

Since all maximal tori of G are conjugate and V has finitely many T-weights, the number of conjugacy classes of saturated subsets of V is finite.

The following result of Hesselink shows that the description of saturated sets in the general situation, in which V is a G-variety, may be reduced to the above consideration. (Note that since $*$ is G-invariant, the tangent space $\mathrm{T}_* V$ naturally becomes a G-module.)

Proposition (Hesselink [13, Proposition 3.8]) *If* X *is a saturated subset of* V, *then* $\mathrm{T}_* X$ *is a saturated subset of* $\mathrm{T}_* V$ *which is isomorphic to* X *and satisfies* $\Delta_{\mathrm{T}_* X} = \Delta_X$. *The application of* T_* *is a bijection from the class of saturated subsets of* V *to the class of saturated subsets of* $\mathrm{T}_* V$.

In particular, the saturated sets in the adjoint action of G on itself are connected unipotent subgroups.

By virtue of Proposition 2.6 we may implicitly identify a saturated set with its tangent space, so that Lemma 2.5 now makes sense for arbitrary saturated sets. We now gather some basic facts about saturated sets that will be useful later. First we need the following definitions. Given a uniformly G-unstable subset X of V we define

$$\|X\| := \min\bigl\{\bigl\|\mu_T(g\cdot X)\bigr\| \,|\, g\in G, 0\notin K_T(g\cdot X)\bigr\}.$$

Note that $\|X\|$ is the minimal distance from the origin to a point in a finite union of polytopes of the form $K_T(g\cdot X)$ for *some* $g\in G$, and it is independent of the choice of T. It follows from Lemma 2.5 that $\|X\| = \inf\{\|\lambda\| \,|\, \lambda\in Y(G), m(X,\lambda)\geq 1\}$ (cf. [13], p. 143).

Lemma *Let X and Y be uniformly unstable subsets of V.*

(i) *$S(X)$ is uniformly unstable, $\Delta_{S(X)} = \Delta_X$ and $\tilde{\Delta}_{S(X)} = \tilde{\Delta}_X$.*
(ii) *$\tilde{\Delta}_X = \tilde{\Delta}_Y$ if and only if $Y\subset S(X)$ and $\|X\| = \|Y\|$.*
(iii) *$X\subset S(X) = S(S(X))$.*
(iv) *If $X\subset Y$, then $\|X\|\geq\|Y\|$.*
(v) *If $g\in G$, then $g\cdot S(X) = S(g\cdot X)$.*

Proof This is a straightforward exercise. Cf. [13, Lemma 2.8]. □

2.7 Following [13, Sect. 4] now define some equivalence relations on $\mathcal{N}_V$. For $x, y\in\mathcal{N}_V$ we set

$$\begin{aligned} x\approx y \quad &\Leftrightarrow \quad \tilde{\Delta}_x = \tilde{\Delta}_y;\\ x\sim y \quad &\Leftrightarrow \quad \tilde{\Delta}_{g\cdot x} = \tilde{\Delta}_y \quad \text{for some } g\in G. \end{aligned}$$

We call an equivalence class $[v] = \{x \,|\, x\approx v\}$ a *blade* and an equivalence class $G[v] = \{x \,|\, x\sim v\}$ a *stratum*. Hesselink gives the following description of blades and strata.

Lemma *Let $v\in\mathcal{N}_V$. Then*

(i) $[v] = \{x\in S(v)\colon \|x\| = \|v\|\}$.
(ii) *$[v]$ is open and dense in $S(v)$.*
(iii) *$GS(v)$ is an irreducible closed subset of $\mathcal{N}_V$.*
(iv) $G[v] = \{x\in GS(v) \,|\, \|x\| = \|v\|\}$.
(v) *$G[v]$ is open and dense in $GS(v)$.*

We will eventually show that when $V = G_{\mathrm{uni}}$ the strata are precisely Lusztig's unipotent pieces. To that end the following result will be crucial.

Proposition *Let $v \in V$. Then*

$$G[v] = GS(v) \setminus \bigcup GS(v'),$$

where the union is taken over all saturated sets $S(v')$ such that $GS(v') \subsetneq GS(v)$.

Proof Let $v, v' \in \mathcal{N}_V$ be such that $GS(v') \subseteq GS(v)$. In order to prove the proposition, it is sufficient to show that $GS(v') = GS(v)$ if and only if $\|v\| = \|v'\|$.

Suppose that $GS(v') = GS(v)$. Then there exists $g \in G$ such that $g \cdot v' \in S(v)$. Hence $\|v'\| = \|g \cdot v'\| \geq \|S(v)\| = \|v\|$ by Lemma 2.6. Similarly we can find $h \in G$ such that $h \cdot v \in S(v')$ and deduce that $\|v'\| \leq \|v\|$, and thus $\|v'\| = \|v\|$.

Conversely, suppose that $\|v'\| = \|v\|$. Since $GS(v') \subseteq GS(v)$, there exists $g \in G$ such that $g \cdot v' \in S(v)$. Then Lemma 2.6(ii) yields $\tilde{\Delta}_{g \cdot v'} = \tilde{\Delta}_v$, and so $S(g \cdot v') = S(v)$. Hence $g \cdot S(v') = S(v)$ by Lemma 2.6(v). It follows that $GS(v') = GS(v)$. □

3 A modification of the Kirwan–Ness theorem

3.1 Let $\lambda \in Y(G) \setminus \{0\}$ and let T be a maximal torus of G containing $\operatorname{Im}\lambda$. (This is equivalent to T being a maximal torus of $L(\lambda)$.) Then we define

$$T^\lambda := \langle \operatorname{Im}\mu \mid \mu \in Y(T), (\mu, \lambda) = 0 \rangle,$$
$$L^\perp(\lambda) := \langle T^\lambda, \mathcal{D}L(\lambda) \rangle.$$

Note that $L^\perp(\lambda)$ is independent of the choice of T since $(gTg^{-1})^\lambda = gT^\lambda g^{-1}$ for all $g \in G$. Also, T^λ is a subtorus of T and $L^\perp(\lambda) = T^\lambda \cdot \mathcal{D}L(\lambda)$ is a connected reductive group by [46, Corollary 2.2.7], [3, Sect. IV.14.2].

3.2 We now restrict to the special case where V is a G-module with $* = 0$. In [42], [36], [48] the following generalisation of the Kirwan–Ness theorem is proved.

Theorem (Cf. Kirwan [21], Ness [33]) *Let $v \in V \setminus \{0\}$ and $\lambda \in Y(G) \setminus \{0\}$. Assume that $k = m(v, \lambda) \geq 1$ and write $v = \sum_{i \geq k} v_i$ with $v_i \in V(\lambda, i)$ (and $v_k \neq 0$). Then λ is optimal for v if and only if v_k is $L^\perp(\lambda)$-semistable.*

Our goal is to obtain an analogous result for the conjugation action of G on the unipotent variety. Our proof is modelled on the proof in [48] of the above result. We will need the following lemmas from [42] and [48] for this task.

3.3 We continue to assume that V is a G-module with $* = 0$. It follows from [3, Proposition 8.2(c)] that an element of $X_{\mathbb{Q}}(T^{\lambda})$ may be lifted to an element of $X_{\mathbb{Q}}(T)$. In fact, $X_{\mathbb{Q}}(T^{\lambda})$ may be naturally identified with the orthogonal projection of $X_{\mathbb{Q}}(T)$ onto the hyperplane $\{\chi \in X_{\mathbb{Q}}(T) | (\chi, \lambda) = 0\}$. The following lemma shows that this projection behaves well with respect to optimality.

Lemma (Cf. [42]) *Let $\lambda \in Y(G) \setminus \{0\}$ and $v \in V(\lambda, k)$ for some $k \in \mathbb{N}$. If T is a maximal torus of G containing* $\operatorname{Im}\lambda$ *then $\mu_{T^{\lambda}}(v) = \mu_T(v) - \frac{k}{(\lambda,\lambda)}\lambda$.*

3.4 We continue to assume that V is a G-module with $* = 0$. The following is the key lemma used in the proof of Theorem 3.2.

Lemma ([48, Lemma 2.6]) *Let T be a maximal torus of G and assume that $v \in V \setminus \{0\}$ is T-unstable. Let $k = m(v, \lambda_T(v))$ and $v' \in v + \bigoplus_{i>k} V(\lambda_T(v), i)$. Then $\lambda_T(v) = \lambda_T(v')$.*

3.5 We now assume that $V = G_{\mathrm{uni}}$ with $* = 1_G$. Let $\lambda \in Y(G)$ and let T be a maximal torus of $L(\lambda)$ with corresponding G-root system Σ. Recall that for each root $\alpha \in \Sigma$ we denote the corresponding root subgroups by U_α, and we have that

$$R_u\big(P(\lambda)\big) = U(\lambda) := \big\langle U_\alpha | \alpha \in \Sigma, \langle \alpha, \lambda\rangle \geq 1\big\rangle,$$

where $R_u(P(\lambda))$ denotes the unipotent radical of $P(\lambda)$. In fact, $U(\lambda)$ is directly spanned by the root subgroups U_α with $\langle \alpha, \lambda\rangle \geq 1$; see [3, Sect. IV.14]. Let $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ be all such root subgroups and assume further that $\langle \alpha_i, \lambda\rangle \leq \langle \alpha_j, \lambda\rangle$ whenever $i \leq j$. Since each $U_\alpha = \langle x_\alpha(t) | t \in \mathbb{k}\rangle$ is isomorphic to the additive group $\mathbb{k}^+$, this gives an isomorphism of affine varieties $f: U(\lambda) \xrightarrow{\sim} \mathbb{A}^n(\mathbb{k})$. Consider $\mathbb{A}^n(\mathbb{k})$ as a vector space with basis indexed by the set $\{1, 2, \ldots, n\}$. It becomes a T-module by letting $t \in T$ act on the ith basis vector by scalar multiplication by $\alpha_i(t)$. With respect to this f is T-equivariant. For $k \geq 1$, we set $U_k(\lambda) := \langle U_\alpha | \alpha \in \Sigma, \langle \alpha, \lambda\rangle \geq k\rangle$, a connected normal subgroup of $U(\lambda)$. For $\lambda \neq 0$ and $u \neq 1_G$ define $m'(u, \lambda) := \min\{i | u \in U_i(\lambda)\}$ and $m'(u, \lambda) := +\infty$ for $u = 1_G$. Then we have the following.

Lemma *Let $\lambda \in Y(G) \setminus \{0\}$ and $u \in U(\lambda)$. Then $m'(u, \lambda) = m(u, \lambda)$.*

Proof If $u = 1_G$, the statement is obvious, so suppose $u \neq 1_G$. For each root α_i let $m_i = \langle \alpha_i, \lambda\rangle$. Then we have a morphism of varieties $\ell : \mathbb{A}^1(\mathbb{k}) \to U(\lambda)$ given by $t \longmapsto \lambda(t)u\lambda(t)^{-1}$ for $t \in \mathbb{k}^\times$ and $\ell(0) = 1_G$. Writing $u = u_{\alpha_1} \cdot u_{\alpha_2} \cdots u_{\alpha_n}$ with $u_{\alpha_i} = x_{\alpha_i}(\xi_i) \in U_{\alpha_i}$, we have

$$\begin{aligned}\ell(t) &= \lambda(t)u_{\alpha_1}\lambda(t)^{-1}\cdot\lambda(t)u_{\alpha_2}\lambda(t)^{-1}\cdot\dots\cdot\lambda(t)u_{\alpha_n}\lambda(t)^{-1}\\ &= x_{\alpha_1}\big(\xi_1 t^{\langle\alpha_1,\lambda\rangle}\big)\cdot x_{\alpha_2}\big(\xi_2 t^{\langle\alpha_2,\lambda\rangle}\big)\cdot\dots\cdot x_{\alpha_n}\big(\xi_n t^{\langle\alpha_n,\lambda\rangle}\big)\\ &= x_{\alpha_1}\big(\xi_1 t^{m_1}\big)\cdot x_{\alpha_2}\big(\xi_2 t^{m_2}\big)\cdot\dots\cdot x_{\alpha_n}\big(\xi_n t^{m_n}\big).\end{aligned}$$

Note that $m_1 \le m_2 \le \dots \le m_n$ and $m'(u,\lambda) = m_k$ for some $k \le n$, so that $\xi_i = 0$ for $i < k$. Then, identifying $\Bbbk[U(\lambda)]$ and $\Bbbk[\mathbb{A}^1(\Bbbk)]$ with the polynomial rings $\Bbbk[T_1,\dots,T_n]$ and $\Bbbk[T]$ respectively, the comorphism ℓ^* sends $g = g(T_1,\dots,T_n) \in \Bbbk[U(\lambda)]$ to $g(0,\dots,0,\xi_k T^{m_k},\dots,\xi_n T^{m_n})$. Hence, if $I = \langle T_1,\dots,T_n\rangle$ is the maximal ideal of $1_G \in U(\lambda)$, then the ideal $\ell^*(I)$ of the schematic fibre $\ell^{-1}(u)$ is generated by $\xi_k T^{m_k},\dots,\xi_n T^{m_n}$. As $\xi_k \ne 0$, it follows that the coordinate ring of the schematic fibre $\ell^{-1}(u)$ equals $\Bbbk[T]/(T^{m_k})$.

Now consider the composition $\mathbb{A}^1(\Bbbk) \xrightarrow{\ell} U(\lambda) \xrightarrow{\iota} G_{\mathrm{uni}}$. If $\iota(1_G) = 1_G$ has maximal ideal I' of $\Bbbk[G_{\mathrm{uni}}]$, then $\iota^*(I') = I$, so that $(\iota\circ\ell)^*(I') = \ell^*\circ\iota^*(I') = \ell^*(I)$, which completes the proof. □

3.6 For each $i \ge 1$, the group $L(\lambda)$ acts rationally on the affine variety $V_i(\lambda) := U_i(\lambda)/U_{i+1}(\lambda)$. The variety $V_i(\lambda)$ is a connected abelian unipotent group. It may be regarded as a vector space over $\Bbbk$ with basis $v_1,\dots,v_{l(i)}$ consisting of the images of $x_{\beta_1}(1),\dots,x_{\beta_{l(i)}}(1)$ in $U_i(\lambda)/U_{i+1}(\lambda)$. Our convention here is that $\xi_1 v_1 + \dots + \xi_{l(i)}v_{l(i)}$ is the image of $\prod_{j=1}^{l(i)} x_{\beta_j}(\xi_j)$ in $U_i(\lambda)/U_{i+1}(\lambda)$ for all $\xi_i \in \Bbbk$. The preceding remarks then imply that the torus $T \subset L(\lambda)$ acts linearly on $V_i(\lambda)$ with the v_j being weight vectors of $V_i(\lambda)$ with respect to T. In view of Chevalley's commutator relations it is straightforward to see that each root subgroup U_α with $\langle\alpha,\lambda\rangle = 0$ acts linearly on $V_i(\lambda)$ as well. It follows that the group $L(\lambda)$ acts linearly and rationally on $V_i(\lambda)$. In other words, each vector space $V_i(\lambda)$ is a rational $L(\lambda)$-module.

We are now ready to state and prove the following version of the Kirwan–Ness theorem.

Theorem *Let $u \ne 1_G$ be a unipotent element of G and $\lambda \in Y(G)\setminus\{0\}$. Assume that $u \in U(\lambda)$ and let $k = m(u,\lambda)$. Then λ is optimal for u if and only if the image of u in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$ is $L^\perp(\lambda)$-semistable.*

Proof In proving the theorem we may assume without loss of generality that λ is primitive. We follow Tsujii's arguments from [48, Theorem 2.8] very closely.

First suppose λ is optimal for u and let $k = m(u,\lambda)$. Then $u \in U_k(\lambda)\setminus U_{k+1}(\lambda)$ by Lemma 3.5. Let $\bar{u}$ denote the image of u in the $L^\perp(\lambda)$-module $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$. We must show that $\bar{u}$ is semistable with respect to all maximal tori of $L^\perp(\lambda)$. Of course, each of these has the form T^λ for some maximal torus T of $L(\lambda)$. In particular, $\lambda \in Y(T)$ and hence $\lambda = \lambda_T(u)$

by our assumption on λ. Note that Lemma 2.5 can be used in our present (non-linear) situation in view of Proposition 2.6 applied with $G=T$. Then $k=(\mu_T(u),\lambda_T(u))$, so that

$$\mu_T(u)\in\big\{\mu\in K_T(u)\,\big|\,\big(\mu,\lambda_T(u)\big)=k\big\}=K_T(\bar u).$$

Therefore $\mu_T(u)=\mu_T(\bar u)$ and $\lambda_T(u)=\lambda_T(\bar u)$. Let $\mu\in Y(T)\setminus\{0\}$. Then Lemma 2.5 implies that

$$\frac{m(u,\lambda_T(u))}{\|\lambda_T(u)\|}=\frac{k}{\|\lambda_T(u)\|}=\frac{m(\bar u,\lambda_T(u))}{\|\lambda_T(u)\|}=\frac{m(\bar u,\lambda_T(\bar u))}{\|\lambda_T(\bar u)\|}\geq\frac{m(\bar u,\mu)}{\|\mu\|}.$$

Since $S_T(\bar u)\subseteq S_T(u)$ we have that $m(\bar u,\mu)\geq m(u,\mu)$. Then $\lambda_T(\bar u)\in\Delta_{T,u}=\{\lambda_T(u)\}$, implying that $\mu_{T^\lambda}(\bar u)$ and λ are proportional; see Lemma 3.3. Since λ is orthogonal to $\mu_{T^\lambda}(\bar u)\in Y(T^\lambda)$ it must be that $\|\mu_{T^\lambda}(\bar u)\|=0$. Hence $\bar u$ is T^λ-semistable by Lemma 2.5(iii).

Conversely, suppose that $\bar u$ is $L^\perp(\lambda)$-semistable. The parabolic subgroups $P(\lambda)$ and $P(u)$ have a maximal torus in common, T' say; see [15, Corollary 28.3]. We may choose $w\in U(\lambda)$ with $T:=wT'w^{-1}\subset L(\lambda)$ so that $\lambda\in Y(T)$. Then $\bar u$ is T^λ-semistable by the assumption and hence $\mu_{T^\lambda}(\bar u)=0$ by Lemma 2.5. Applying Lemma 3.3 we now get $\mu_T(\bar u)=\frac{k}{(\lambda,\lambda)}\lambda$. It follows that $\lambda=\lambda_T(\bar u)$. We claim that also $\lambda=\lambda_T(wuw^{-1})$.

In order to prove the claim we first recall that $U(\lambda)\cong\mathbb{A}^n(\Bbbk)$ can be regarded as a T-module in such a way that each $U_i(\lambda)$ is a T-submodule of $U(\lambda)$; see Sect. 3.5. Moreover, $U_i(\lambda)/U_{i+1}(\lambda)\cong V_i(\lambda)$ as T-modules for all $i\geq 1$. Let $U^k(\lambda)$ be the T-stable complement of the T-module $U_k(\lambda)$ spanned by the U_{α_i}'s with $\langle\alpha_i,\lambda\rangle=k$ and write $u=u_k\cdot u'$ with $u_k\in U^k(\lambda)$ and $u'\in U_{k+1}(\lambda)$. Since $U^k(\lambda)\cong V_k(\lambda)$ as T-modules, we have that $\lambda_T(\bar u)=\lambda_T(u_k)$. In view of Lemma 3.4, we now need to show that the first projection of wuw^{-1} associated with the decomposition $U_k(\lambda)\cong U^k(\lambda)\times U_{k+1}(\lambda)$ is u_k. Write $u=\prod_{\langle\alpha,\lambda\rangle\geq k}u_\alpha$ and assume that $w=\prod_{i=1}^n x_{\alpha_i}(\zeta_i)$ for some $\zeta_i\in\Bbbk$. Then Chevalley's commutator relations yield

$$\begin{aligned}wuw^{-1}=\prod_{\substack{\alpha\in\Sigma\\ \langle\lambda,\alpha\rangle\geq k}}wu_\alpha w^{-1}&\in\prod_{\substack{\alpha\in\Sigma\\ \langle\lambda,\alpha\rangle\geq k}}\Bigg(u_\alpha\prod_{\substack{i,j>0\\ i\alpha+j\beta\in\Sigma}}U_{i\alpha+j\beta}\Bigg)\\ &\subseteq\Bigg(\prod_{\substack{\alpha\in\Sigma\\ \langle\lambda,\alpha\rangle\geq k}}u_\alpha\Bigg)\cdot U_{k+1}(\lambda)\subseteq u_kU_{k+1}(\lambda).\end{aligned}$$

Hence $\lambda=\lambda_T(wuw^{-1})$ as claimed. To complete the proof of the theorem note that $T\subset wP(\lambda)w^{-1}=P(wuw^{-1})$, and so $\lambda\in\Delta_{wuw^{-1}}=\Delta_u$ by Theorem 2.4. □

Remark For each $\beta \in \Sigma$ with $\langle \beta, \lambda \rangle = k$ we let v_β denote the image of $x_\alpha(1)$ in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$ and write X_β for the tangent vector of the root subgroup $U_\beta = \langle x_\beta(t) | t \in \Bbbk \rangle$ in $\mathfrak{g} = \operatorname{Lie} G$, so that

$$\big(\operatorname{Ad} x_\beta(t)\big) y \equiv y + t[X_\beta, y] \quad \big(\operatorname{mod} \mathfrak{g} \otimes t^2 \Bbbk[t]\big) \quad \big(\forall y \in \mathfrak{g} \otimes \Bbbk[t]\big).$$

The map $v_\beta \mapsto X_\beta$ extends uniquely up to a linear isomorphism between $V_k(\lambda)$ and the subspace $\mathfrak{g}(\lambda, k) = \operatorname{span}\{X_\beta | \langle \beta, \lambda \rangle = k\}$; we call it η_k. Using Chevalley's commutator relations and our definition of the vector space structure on $V_k(\lambda)$ at the beginning of this subsection it is straightforward to see that η_k is an isomorphism of $L(\lambda)$-modules. If G and T are defined over $\mathbb{Z}$, then so is η_k.

4 Reductive group schemes and Seshadri's theorem

We now briefly review reductive group schemes before stating a result of Seshadri which we will need later. For a general reference see [16], for example.

4.1 For an affine variety X over $\Bbbk$, we say that X is *defined over* $\mathbb{Z}$ if there is an embedding of X into some affine space $\mathbb{A}^n(\Bbbk)$ such that the radical ideal $I(X)$ of X is generated by elements of $\mathbb{Z}[X_1, \ldots, X_n]$. (This is the same as requiring that $\Bbbk[X] \cong \mathbb{Z}[X] \otimes_{\mathbb{Z}} \Bbbk$, where $\mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_n]/(I(X) \cap \mathbb{Z}[X_1, \ldots, X_n])$.) A morphism $\phi : X \to Y$ of $\Bbbk$-varieties defined over $\mathbb{Z}$ is said to be defined over $\mathbb{Z}$ if it can be written in terms of elements of $\mathbb{Z}[X_1, \ldots, X_n]$. (This is the same as requiring that its comorphism restricts to a homomorphism $\phi^* : \mathbb{Z}[Y] \to \mathbb{Z}[X]$ of $\mathbb{Z}$-algebras.)

When X is defined over $\mathbb{Z}$ we may associate to it a reduced affine algebraic $\mathbb{Z}$-scheme, i.e. a functor $\mathfrak{X} : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}$ such that if A, A' are $\mathbb{Z}$-algebras and $\psi : A \to A'$ is a $\mathbb{Z}$-algebra homomorphism then $\mathfrak{X}(A) = \operatorname{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[X], A)$ and $\mathfrak{X}(\psi) : \alpha \mapsto \psi \circ \alpha$ for each $\alpha \in \operatorname{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[X], A)$. We identify $\mathfrak{X}(A)$ with the set $\{a \in A^n | f(a) = 0 \text{ for all } f \in I(X) \cap A[X_1, \ldots, X_n]\}$.

If G is an affine algebraic group over $\Bbbk$, then we say that G is defined over $\mathbb{Z}$ if it is so as a variety and the product and inverse morphisms are defined over $\mathbb{Z}$. (This is the same as requiring that the Hopf algebra structure on $\Bbbk[G]$ restricts to one on $\mathbb{Z}[G]$.) In this case we may associate to it (using Jantzen's terminology) a reduced algebraic $\mathbb{Z}$-group, i.e. a functor $\mathfrak{G} : \mathrm{Alg}_{\Bbbk} \to \mathrm{Grp}$ defined as above, with the group structure on $\mathfrak{G}(A)$ defined via the Hopf algebra structure on $A[G] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ for each $\mathbb{Z}$-algebra A. From now on we call such a functor a $\mathbb{Z}$-*group scheme*. G is said to be $\mathbb{Z}$-*split* if there exists a maximal torus T of G such that there is an isomorphism $T \xrightarrow{\sim} \Bbbk^\times \times \cdots \times \Bbbk^\times$ which is defined over $\mathbb{Z}$ and the root morphisms of T are defined over $\mathbb{Z}$.

It has been shown by Chevalley ([6]) that every connected reductive algebraic group over an algebraically closed field $\Bbbk$ may be obtained by extension of scalars from a reduced algebraic $\mathbb{Z}$-group, and that many familiar subgroups and actions are also defined over $\mathbb{Z}$. This allows one to pass information between the characteristic zero and prime characteristic settings; see [16]. We will use this to relate optimal one parameter subgroups of reductive groups G in arbitrary characteristic to those of reductive groups G' with the same root system defined over $\mathbb{C}$. This will eventually allow us to use the parameter set $\tilde{D}_{G'}/G'$ from Sect. 1 in arbitrary characteristic.

4.2 Let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme and let $\mathfrak{X}$ be a reduced affine algebraic $\mathbb{Z}$-scheme. We will say that $\mathfrak{G}$ *acts* on $\mathfrak{X}$ if, for any $\mathbb{Z}$-algebra A, there is a map $\phi_A : \mathfrak{G}(A) \times \mathfrak{X}(A) \to \mathfrak{X}(A)$, functorial in A, given by polynomials over A. If $\mathfrak{G}$ acts on an affine space $\mathbb{A}^n_{\mathbb{Z}}$ (regarded as a $\mathbb{Z}$-scheme) then we say that this action is linear if, for any $\mathbb{Z}$-algebra A, $g \in \mathfrak{G}(A)$, the map $\phi_A(g) : \mathbb{A}^n_{\mathbb{Z}}(A) \to \mathbb{A}^n_{\mathbb{Z}}(A)$ is A-linear.

We now state a result of Seshadri ([41]) which allows one to pass information about semistability between characteristics.

Theorem (Cf. [41, Proposition 6]) *Let $\Bbbk$ be an algebraically closed field and let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme acting linearly on $\mathbb{A}^n_{\mathbb{Z}}$. Suppose that $\mathfrak{X}$ is a $\mathfrak{G}$-stable open subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ and $x \in \mathfrak{X}(\Bbbk)$ is a semistable point. Then there exists a $\mathfrak{G}$-invariant $F \in \mathbb{Z}[\mathbb{A}^n_{\mathbb{Z}}] = \mathbb{Z}[X_1, \dots, X_n]$ such that $F(x) \neq 0$. Furthermore, there is an open subscheme $\mathfrak{X}^{ss}$ of $\mathfrak{X}$ such that for any algebraically closed field $\Bbbk'$, the set $\mathfrak{X}^{ss}(\Bbbk')$ consists of the semistable points of $\mathfrak{X}(\Bbbk')$.*

4.3 In the next section we will prove our main result by applying Theorem 4.2 to a reductive $\mathbb{Z}$-group scheme associated with $L^{\perp}(\lambda)$. To that end we will now construct such a scheme. From now on assume that we have a fixed reductive $\mathbb{Z}$-group scheme $\mathfrak{G}$, which determines the reductive groups G, G' that we are interested in. In addition, let us fix a maximal torus $\mathfrak{T}$ of $\mathfrak{G}$. Then there is a natural identification of the one parameter subgroups of $\mathfrak{T}(\Bbbk)$ as $\Bbbk$ varies. It follows that there is a reductive $\mathbb{Z}$-group scheme $\mathfrak{L}$, the scheme-theoretic centraliser of a one parameter subgroup λ of $\mathfrak{T}$, which gives rise to the groups $L(\lambda)$. The groups $L^{\perp}(\lambda)$ may also be obtained from a reductive $\mathbb{Z}$-group scheme, but since this is not a standard result we will now give an explicit construction.

Recall that a root datum of a connected reductive group, or reductive $\mathbb{Z}$-group scheme, is a quadruple $(X(T), \Sigma, Y(T), \Sigma^{\vee})$, with respect to a fixed maximal torus, together with the perfect pairing $X(T) \times Y(T) \to \mathbb{Z}$ and the associated bijection $\Sigma \to \Sigma^{\vee}$ between the roots and coroots of G with respect to T. If we forget about the fixed torus T and merely regard $X(T)$ and $Y(T)$ as abstract free abelian groups with finite subsets Σ and $\Sigma^{\vee}$ respectively, then the datum is unique and moreover any such abstract root datum gives rise to a connected reductive group, or reductive group

$\mathbb{Z}$-scheme. If G' is another such group, or $\mathbb{Z}$-group scheme, with datum $(X(T'), \Sigma', Y(T'), \Sigma'^{\vee})$, then a *homomorphism of root data* is a group homomorphism $f: X(T') \to X(T)$ that maps Σ' bijectively to Σ and such that the dual homomorphism $f^{\vee}: Y(T) \to Y(T')$ maps $f(\beta)^{\vee}$ to $\beta^{\vee}$ for each $\beta \in \Sigma'$. A morphism of algebraic groups $\psi: T \to T'$ is said to be *compatible with the root data* if the induced homomorphism $\psi^*: X(T') \to X(T)$ is a homomorphism of root data.

Proposition *The connected reductive group $L^{\perp}(\lambda)$ is a $\mathbb{Z}$-scheme theoretic subgroup of $L(\lambda)$. In other words, if $\mathfrak{L}$ is a $\mathbb{Z}$-group scheme such that $\mathfrak{L}(\Bbbk) = L(\lambda)$, then there exists a $\mathbb{Z}$-subgroup scheme $\mathfrak{L}^{\perp}$ of $\mathfrak{L}$ such that $\mathfrak{L}^{\perp}(\Bbbk) = L^{\perp}(\lambda)$.*

Proof Suppose that $(X(T), \Sigma, Y(T), \Sigma^{\vee})$ is the root datum of $L(\lambda)$. It follows then that the root datum of $L^{\perp}(\lambda)$, with respect to the maximal torus T^{λ}, is $(X(T^{\lambda}), \{\alpha|_{T^{\lambda}} | \alpha \in \Sigma\}, Y(T^{\lambda}), \Sigma^{\vee})$. We may also construct reductive $\mathbb{Z}$-group schemes from these data, say $\mathfrak{L}$ (as above) for the former and $\tilde{\mathfrak{L}}^{\perp}$ for the latter. We now need to construct a subgroup scheme $\mathfrak{L}^{\perp}$ of $\mathfrak{L}$, isomorphic to $\tilde{\mathfrak{L}}^{\perp}$ which gives rise to $L^{\perp}(\lambda)$. We start by showing that T^{λ} is defined over $\mathbb{Z}$ as a subgroup of T, so that we may construct a $\mathbb{Z}$-group scheme $\mathfrak{T}$ with subgroup scheme $\mathfrak{T}^{\lambda}$ which give rise to T and T^{λ} respectively.

We know that T^{λ} is a subtorus of codimension 1 in T (for it is a connected subgroup of T and $Y(T^{\lambda})$ has rank equal to $l-1$ where $l = \dim T$). Therefore T/T^{λ} is a 1-dimensional torus. By [3, Corollary 8.3] the natural short exact sequence $1 \to T^{\lambda} \to T \to T/T^{\lambda} \to 1$ gives rise to a short exact sequence of character groups $0 \to X(T/T^{\lambda}) \to X(T) \to X(T^{\lambda}) \to 0$. Since T/T^{λ} is a one dimensional torus, its character group $X(T/T^{\lambda})$ is generated by one element, say η. By the above η can be regarded as a rational character of T and

$$X(T) \cong \mathbb{Z}\eta \oplus X\big(T^{\lambda}\big). \tag{3}$$

(One should keep in mind here that $X(T^{\lambda})$ is a free $\mathbb{Z}$-module of rank $l-1$.) By construction, η vanishes on T^{λ}.

On the other hand, [3, Proposition 8.2(c)] shows that T^{λ} coincides with the intersection of the kernels of rational characters of T, say $T^{\lambda} = \bigcap_{\chi \in A} \ker\chi$ where A is a non-empty subset of $X(T)$. If A contains a character of the form $a\eta + \mu$ for some non-zero $\mu \in X(T^{\lambda})$ then $T^{\lambda} \subseteq \ker\eta \cap \ker\mu$. But then $\dim T^{\lambda} \le l-2$ because η and μ are linearly independent in $X_{\mathbb{Q}}(T)$. Since this is false, it must be that $A \subseteq \mathbb{Z}\eta$. As a result, $T^{\lambda} = \ker\eta$.

The above argument is characteristic-free since η can be described as the unique, up to a sign, primitive element of $X(T)$ proportional to λ in $X_{\mathbb{Q}}(T)$, which we identify with $Y_{\mathbb{Q}}(T)$ by means of our W-invariant inner product. In view of (3) we may regard η as one of the standard generators of the Laurent polynomial ring $\mathbb{C}[T]$. This implies that $\eta - 1 \in \mathbb{Z}[T]$ generates a prime ideal of $\mathbb{C}[T]$, thus showing that $T^\lambda = \ker\eta$ is defined over $\mathbb{Z}$. This enables us to construct the desired subgroup scheme $\mathfrak{T}^\lambda$ of $\mathfrak{T}$.

The inclusion $\mathfrak{T}^\lambda \subset \mathfrak{T}$ induces a homomorphism of root data, and by [16, Proposition II.1.15] (and the proof) there exists an injective homomorphism of $\mathbb{Z}$-group schemes $\iota : \tilde{\mathfrak{L}}^\perp \hookrightarrow \mathfrak{L}$ which agrees on the root subgroups. We may therefore take $\mathfrak{L}^\perp$ to be the functor defined by $A \mapsto \iota(\tilde{\mathfrak{L}}^\perp)(A)$ for any $\mathbb{Z}$-algebra A. We know that this gives rise precisely to $L^\perp(\lambda)$ since the restriction of the functor ι to the root subgroups determines it uniquely by [16, II.1.3(10)]. □

5 Unipotent pieces in arbitrary characteristic

5.1 We will need the following result, due to H. Kraft, during the proof of our next theorem. This was not published by Kraft but the details can be found in [12]; see Theorem 11.3 and the remarks in Sect. 12. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}'$ and assume that we have the usual grading on $\mathfrak{g}'$ given by $\mathfrak{g}'(i) = \{x \in \mathfrak{g}' | [h, x] = ix\}$ for all $i \in \mathbb{Z}$. Let $\rho : \mathbb{C}^\times \to (\mathrm{Aut}\mathfrak{g}')^\circ$ be defined by $\rho(\xi)x = \xi^i x$ if $x \in \mathfrak{g}'(i)$. It follows that there is a one parameter subgroup $\lambda' \in Y(G')$ such that $\rho = \mathrm{Ad} \circ \lambda'$. We then say that λ' is *adapted to* e. (For full details see [47, Sect. E, p. 238].) If $\nu \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$, then we define $\nu_* \in Y(G')$ by composing ν with the map $\xi \mapsto \left[\begin{smallmatrix} \xi & \\ & \xi^{-1} \end{smallmatrix}\right]$.

Theorem (H. Kraft, unpublished) *The following are true.*

(i) *Let $e \in \mathfrak{g}'_{\mathrm{nil}}$ and assume that $\lambda' \in Y(G')$ is a one parameter subgroup adapted to e. Then $\frac{1}{2}\lambda' \in \tilde{\Delta}_e$.*
(ii) *Let $u \in G'_{\mathrm{uni}}$ and assume that we have $\nu \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$ such that $\nu\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right] = u$. Then $\frac{1}{2}\nu_* \in \tilde{\Delta}_u$.*

5.2 We now turn our attention to the conjugation action of G on itself, that is we assume that $V = G_{\mathrm{uni}}$ and $* = 1_G$. Recall the subsets $X^\triangle$ ($\triangle \in D_G$) and $H^\blacktriangle$ ($\blacktriangle \in D_G/G$) introduced in Sect. 1.3.

Lemma *Each set $\tilde{H}^\blacktriangle$ is a closed irreducible variety stable under the conjugation action of G.*

Proof It is clear that the set $\tilde{H}^\blacktriangle$ is G-stable. To see that it is closed, consider the set

$$\mathcal{S} = \left\{\left(gG_0^\triangle, x\right) \mid g^{-1}xg \in G_2^\triangle\right\} \subset G/G_0^\triangle \times \tilde{H}^\blacktriangle.$$

If we show that $\mathcal{S}$ is closed, then $\tilde{H}^\blacktriangle$ is closed since it is the image under the second projection of a closed set, and $G/G_0^\triangle$ is a complete variety.

In fact it is sufficient to show that $\mathcal{S}' := \{(g,x) | g^{-1}xg \in G_2^{\triangle}\}$ is closed in $G \times G$. Indeed, $\mathcal{S}$ is isomorphic to the image of $\mathcal{S}'$ under the quotient map $\eta\colon G \times G \to G/G_0^{\triangle} \times G$ and it is explained in [45, p. 67], for instance, that η maps closed subsets of $G \times G$ consisting of complete cosets of $G_0^{\triangle} \times \{1_G\}$ to closed subsets of $G/G_0^{\triangle} \times G$. The set $\mathcal{S}'$ is closed as it is the inverse image of $G_2^{\triangle}$ under the conjugation morphism $G \times G \to G$. Finally, the set $\tilde{H}^{\blacktriangle}$ is irreducible since the product map $G \times G_2^{\triangle} \to \tilde{H}^{\blacktriangle}$ is a surjective morphism from an irreducible variety. $\square$

Next we show that the sets from Sect. 1.3 defined by Lusztig are precisely the sets from Sect. 2.7 defined by Hesselink.

Theorem *The following are true.*

(i) *The sets $G_2^{\triangle}$ ($\triangle \in D_G$) are the saturated sets of G_{uni}.*
(ii) *The sets $H^{\blacktriangle}$ ($\blacktriangle \in D_G/G$) are the strata of G_{uni}.*
(iii) *The sets $X^{\triangle}$ ($\triangle \in D_G$) are the blades of G_{uni}.*

Furthermore, if $\tilde{\Delta}_G$ denotes the subset of $Y(G)$ consisting of elements which are in some $\tilde{\Delta}_X$, for a uniformly unstable set X, then $\tilde{\Delta}_G = \frac{1}{2}\tilde{D}_G$.

Proof Let $\triangle \in D_G$, and assume that $\mu \in Y(G)$ is associated to $\triangle$ under the natural map described in Sect. 1.3. Assume that $\omega \in Y(G')$ comes from the same $\mathbb{Z}$-scheme theoretic one parameter subgroup of $\mathfrak{T}$ as μ. (Then $G\mu$ is identified with $G'\omega$ under the canonical bijection $Y(G)/G \leftrightarrow Y(G')/G'$.) So there exists $\tilde{\omega} \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$ such that $\tilde{\omega}_* = \omega$, as in (1). Let $u' = \tilde{\omega}\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right] \in G'$. Then $\frac{1}{2}\omega \in \tilde{\Delta}_{u'}$ by Theorem 5.1(ii).

Recall that $U(\omega)$ is the unipotent radical of $(G')_0^{\triangle} = P(\omega)$ and let $U_k(\omega)$ have the same meaning as in Sect. 3.6. Let $\bar{u}'$ denote the image of u' in $V_2(\omega) := U_2(\omega)/U_3(\omega)$. Recall that $V_2(\omega) \cong \mathfrak{g}'(\omega, 2)$ as $L^{\perp}(\omega)$-modules; see Remark 2. By Theorem 3.6 the vector $\bar{u}'$ is $L^{\perp}(\omega)$-semistable. Since $V_2(\omega) \cong \mathfrak{g}'(\omega, 2)$ and the action on it by $L^{\perp}(\omega)$ are defined over $\mathbb{Z}$ there exists an affine scheme $\mathcal{V}_2(\omega)_{ss}$, acted on by $\mathfrak{L}^{\perp}$, such that $\mathcal{V}_2(\omega)_{ss}(\mathbb{C}) = V_2(\omega)_{ss}$. (One should keep in mind here that $L^{\perp}(\omega) = \mathfrak{L}^{\perp}(\Bbbk)$ thanks to Proposition 4.3.) Since $\bar{u}' \in V_2(\omega)$, applying Theorem 4.2 shows that $\mathcal{V}_2(\omega)_{ss}$ has content over any algebraically closed field. So over $\Bbbk$, there exists $\bar{u} \in V_2(\mu) \cong \mathfrak{g}(\mu, 2)$ which is $L^{\perp}(\mu)$-semistable. Let u be a preimage of $\bar{u}$ in $U_2(\mu)$. By applying Theorem 3.6 again we see that μ is optimal for u. Also, since $\frac{1}{2}\omega \in \tilde{\Delta}_{u'}$, we see that $\frac{1}{2}\mu \in \tilde{\Delta}_u$. Hence $G_2^{\triangle} = U_2(\mu)$ is a saturated set.

Conversely, suppose that S is a non-trivial saturated set in G_{uni}. We may assume that $S = S(u)$ for some unipotent element $u \neq 1_G$; see Lemma 2.6(ii), for example. Let $\lambda \in \Delta_u$ and $k = m(\lambda, u)$. Then $S = U_k(\lambda)$. Replacing u by a G-conjugate we may assume further that $\lambda \in Y(T)$. As before, we identify $Y(T)$ and $Y(T')$. Let $\bar{u}$ denote the image of u in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$.

Theorem 3.6 then implies that $\bar{u} \in V_k(\lambda)$ is $L^{\perp}(\lambda)$-semistable. Since $V_k(\lambda) \cong \mathfrak{g}(\lambda, k)$ as $L^{\perp}(\lambda)$-modules by Remark 2, we may again obtain an affine scheme $\mathcal{V}_k(\lambda)_{ss}$, defined over $\mathbb{Z}$ and acted on by $\mathfrak{L}^{\perp}$, such that $\mathcal{V}_k(\lambda)_{ss}(\mathbb{k}) = V_k(\lambda)_{ss}$. Applying Theorem 4.2 we again see that $\mathcal{V}_k(\lambda)_{ss}$ has content over any algebraically closed field, and may therefore find $e' \in \mathfrak{g}'(\lambda, k)_{ss} \cong \mathcal{V}_k(\lambda)_{ss}(\mathbb{C})$; see Remark 2.

By applying Theorem 3.6 we see that λ is optimal and primitive for e'. Since we are now in characteristic zero, the Jacobson–Morozov theorem yields that there exist $f', h' \in \mathfrak{g}'$ such that (e', h', f') is an $\mathfrak{sl}_2$-triple. Now let $\lambda' \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$ be such that $\lambda'_* \in Y(G')$ is adapted to e', so that $e' \in \mathfrak{g}'(\lambda'_*, 2)$. Applying Theorem 5.1 we see that $\frac{1}{2}\lambda'_* \in \tilde{\Delta}_{e'}$. Hence $P(\frac{1}{2}\lambda'_*) = P(\lambda) = P(e')$. Since all maximal tori in $P(e') = L(\lambda) \cdot R_u(P(e'))$ are conjugate we can find $g \in R_u(P(e'))$ such that $\mathrm{Im}(\lambda'_*)$ and $g(\mathrm{Im}\,\lambda)g^{-1}$ lie in the same maximal torus, T say. Note that $g \cdot \lambda$ is optimal for $(\mathrm{Ad}\, g)e' \in e' + \sum_{i>k} \mathfrak{g}'(\lambda, i)$. Applying Lemma 3.4 we see that $g \cdot \lambda$ is optimal for e' as well. Then $g \cdot \lambda \in \mathbb{Q}^{\times}\lambda'_*$ by Theorem 2.4(iii). It is well-known that $\lambda'_* \in \tilde{D}_{G'}$ (see, e.g., [5, Proposition 5.5.6]), hence $g^{-1} \cdot \lambda'_* \in \tilde{D}_{G'}$. But $g^{-1} \cdot \lambda'_* = \lambda$ if λ'_* is primitive and $g^{-1} \cdot \lambda'_* = 2\lambda$ otherwise. So we conclude that $\frac{2}{k}\lambda \in \tilde{D}_{G'}$ in all cases. Then, associating a suitable $\triangle \in D_G$ to $\frac{2}{k}\lambda$, we have that $S = U_2(\frac{2}{k}\lambda) = G_2^{\triangle}$. This completes the proof of (i). The claim that $\tilde{\Delta}_G = \frac{1}{2}\tilde{D}_G$ also easily follows from these arguments. Part (ii) now follows from (i) and Proposition 2. Part (iii) then follows from (i) and (ii). □

5.3 We are now in a position to prove one of our main results.

Theorem *Properties $\mathfrak{P}_1$–$\mathfrak{P}_4$ hold for any connected reductive group over any algebraically closed field. Moreover, $C_G(u) \subset G_0^{\triangle}$ for any $u \in X^{\triangle}$.*

Proof Properties $\mathfrak{P}_1$ and $\mathfrak{P}_3$ are immediate by Theorem 5.2 since the blades and strata are equivalence classes on G_{uni}. That the sets $X^{\triangle}$ ($\triangle \in \blacktriangle$) form a partition of $H^{\blacktriangle}$ for any $\blacktriangle \in D_G/G$ is also clear since $H^{\blacktriangle} = \bigsqcup_{\triangle \in \blacktriangle} X^{\triangle}$. Let $g \in G_3^{\triangle}$ and $u \in X^{\triangle}$. Clearly $gu \in G_2^{\triangle}$. Let $\lambda \in \Delta_u$ and let u_k be the minimal component of u with respect to λ. By the commutator relations u_k is also the minimal component of gu with respect to λ. By Theorem 3.6 we see that $\Delta_u = \Delta_{gu}$. Now $\|u\|$, $\|gu\|$ are determined by the minimal component with respect to (any) optimal one parameter subgroup. Hence, $\|u\| = \|gu\|$ by Lemma 2.6(ii), and so $gu \in H^{\blacktriangle}$ by Proposition 2(iv) and Theorem 5.2. Hence $G_3^{\triangle}X^{\triangle} = X^{\triangle}$. Similarly $X^{\triangle}G_3^{\triangle} = X^{\triangle}$, and so $\mathfrak{P}_4$ holds for G. Since the parabolic subgroup $G_0^{\triangle} = P(\lambda)$ is optimal for u, Theorem 2.4(iv) implies that $C_G(u) \subset G_0^{\triangle}$. □

6 Admissible modules and the Hesselink stratification

6.1 Previously we did not restrict char $\Bbbk$ but for this section it will be convenient to assume that char $\Bbbk = p > 0$. As in Sect. 1.5 we denote by $\mathfrak{G}$ a reductive $\mathbb{Z}$-group scheme split over $\mathbb{Z}$ and write $G' = \mathfrak{G}(\mathbb{C})$ and $G = \mathfrak{G}(\Bbbk)$. Then G' and G are connected reductive groups over $\mathbb{C}$ and $\Bbbk$ respectively. Let V' be a finite-dimensional rational G'-module. Given an admissible lattice $V'_{\mathbb{Z}}$ in V' we set $V := V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. We call V an *admissible* G*-module*. Since the lattice $V'_{\mathbb{Z}}$ is stable under the action of the distribution $\mathbb{Z}$-algebra $\mathrm{Dist}(\mathfrak{G})$, the $\Bbbk$-vector space V is a module over $\mathrm{Dist}(G) = \mathrm{Dist}(\mathfrak{G}) \otimes_{\mathbb{Z}} \Bbbk$. This gives V a rational G-module structure; see [16, Sect. II.1] for more details.

Let $\mathfrak{T}$ be a toral group subscheme of $\mathfrak{G}$ such that $T' := \mathfrak{T}(\mathbb{C})$ is a maximal torus of G' and $T := \mathfrak{T}(\Bbbk)$ is a maximal torus of G. We may and will identify the groups of rational characters $X(T')$ and $X(T)$ and their duals $Y(T')$ and $Y(T)$. The lattice $V'_{\mathbb{Z}}$ decomposes over $\mathbb{Z}$ into a direct sum $V'_{\mathbb{Z}} = \bigoplus_{\mu\in X(T)} V'_{\mathbb{Z},\mu}$ of common eigenspaces for the action of the distribution algebra $\mathrm{Dist}(\mathfrak{T}) \subset \mathrm{Dist}(\mathfrak{G})$ and base-changing this direct sum decomposition we obtain the weight space decompositions $V' = \bigoplus_{\mu\in X(T)} V'_{\mu}$ and $V = \bigoplus_{\mu\in X(T)} V_{\mu}$ of V' and V with respect to T' and T respectively; see [16, II1.1(2)]. We mention for completeness that $\dim_{\mathbb{C}} V'_{\mu} = \dim_{\Bbbk} V_{\mu}$ for all $\mu \in X(T)$.

Theorem *The following are true.*

(i) *Let $\mathcal{S}'$ and $\mathcal{S}$ denote the collections of saturated sets of V' and V associated with the one parameter subgroups in $Y(T')$ and $Y(T)$ respectively. There exists a collection $\mathfrak{S}$ of* $\mathrm{Dist}(\mathfrak{T})$*-stable direct summands of $V'_{\mathbb{Z}}$ such that*

$$\mathcal{S}' = \{S \otimes_{\mathbb{Z}} \mathbb{C} \mid S \in \mathfrak{S}\} \quad \textit{and} \quad \mathcal{S} = \{S \otimes_{\mathbb{Z}} \Bbbk \mid S \in \mathfrak{S}\}.$$

(ii) *For every $S \in \mathfrak{S}$ we have that $\Delta(S \otimes_{\mathbb{Z}} \mathbb{C}) \cap Y_{\mathbb{Q}}(T') = \Delta(S \otimes_{\mathbb{Z}} \Bbbk) \cap Y_{\mathbb{Q}}(T)$.*

(iii) *The strata of V are parametrised by those of V'.*

(iv) *The parametrisation from* (*iii*) *respects the dimensions of the strata. In particular, the dimensions of the nullcones of V' and V agree.*

Proof (i) Let $v' \in V'$ and $v \in V$ be unstable relative to T' and T respectively. Let λ' and λ be the sole elements of $\tilde{\Delta}_{v',T'}$ and $\tilde{\Delta}_{v,T}$ respectively. Then $S(v') = \bigoplus_{\langle\mu,\lambda'\rangle\geq 1} V'_{\mu}$ and $S(v) = \bigoplus_{\langle\mu,\lambda\rangle\geq 1} V_{\mu}$. As we mentioned earlier, for every $\mu \in X(T)$ we have that $V'_{\mu} = V_{\mu,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $V_{\mu} = V_{\mu,\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Since the sets of weights of V' and V in $X(T') = X(T)$ coincide, part (i) follows.

(ii) Let $S \in \mathfrak{S}$. Our proof of part (i) and Remark 2.4 then show that $S = V'(\lambda)_k \cap V'_{\mathbb{Z}}$ for some $\lambda \in Y(T') = Y(T)$ and some positive integer k. Put

$\mathfrak{L}^{\perp} = \mathfrak{L}^{\perp}(\lambda)$ and consider the actions of $\mathfrak{L}^{\perp}(\mathbb{C})$ and $\mathfrak{L}^{\perp}(\Bbbk)$ on $V'(k,\lambda)$ and $V(k,\lambda)$ respectively. By Theorem 4.2, there is an open subscheme $\mathcal{V}(\lambda,k)_{ss}$ of $V_{\mathbb{Z}}(\lambda,k) := V'(k,\lambda) \cap V'_{\mathbb{Z}}$ with the property that $\mathcal{V}(\lambda,k)_{ss}(\mathbb{C})$ is the set of $\mathfrak{L}^{\perp}(\mathbb{C})$-semistable vectors of $V'(k,\lambda)$ and $\mathcal{V}(\lambda,k)_{ss}(\Bbbk)$ is the set of $\mathfrak{L}^{\perp}(\Bbbk)$-semistable vectors of $V(k,\lambda)$. On the other hand, Theorem 3.2 tells us that λ is optimal for an element in $V'(\lambda)_k$ (resp. in $V(\lambda)_k$) if and only if $\mathcal{V}(\lambda,k)_{ss}(\mathbb{C}) \neq \emptyset$ (resp. $\mathcal{V}(\lambda,k)_{ss}(\Bbbk) \neq \emptyset$). This shows that either both sets $\Delta(S \otimes_{\mathbb{Z}} \mathbb{C}) \cap Y_{\mathbb{Q}}(T)$ and $\Delta(S \otimes_{\mathbb{Z}} \Bbbk) \cap Y_{\mathbb{Q}}(T)$ are empty or there exists a natural number $m = m(S)$ such that

$$\Delta(S \otimes_{\mathbb{Z}} \mathbb{C}) \cap Y_{\mathbb{Q}}(T) = \Delta(S \otimes_{\mathbb{Z}} \Bbbk) \cap Y_{\mathbb{Q}}(T) = \frac{1}{m}\lambda.$$

This proves part (ii).

(iii) Consider a stratum $G'[v] \subset V'$. Without loss of generality we may assume that the blade $[v]$ is T'-unstable, since all maximal tori are conjugate in G'. Then part (ii) gives us a blade $[w] \subset V$ corresponding to $[v]$. Since all maximal tori in G are conjugate as well, part (ii), in conjunction with our discussion in Sect. 2.7, shows that any stratum $G[w] \subset V$ is obtained by the above construction in a unique way. Then the map $G'[v] \mapsto G[w]$ defines the required parametrisation.

(iv) With $[v] \subset V'$ and $[w] \subset V$ as above we have that

$$\dim_{\mathbb{C}} G'[v] = \dim_{\mathbb{C}} G' - \dim_{\mathbb{C}} P(v) + \dim_{\mathbb{C}} S(v)$$

and

$$\dim_{\Bbbk} G[w] = \dim_{\Bbbk} G - \dim_{\Bbbk} P(w) + \dim_{\Bbbk} S(w)$$

by [13, Proposition 4.5(c)]. By part (i) we have that $\dim_{\mathbb{C}} S(v) = \dim_{\Bbbk} S(w)$, whilst the equality $\dim_{\mathbb{C}} P(v) = \dim_{\Bbbk} P(w)$ follows from the definition of $P(\lambda)$ in Sect. 2.3. Hence $\dim_{\mathbb{C}} G'[v] = \dim_{\Bbbk} G[w]$, as required.

Since the set of T'-weights of V' is finite, so is the set $\{K_T(v') | v' \in V'\}$. Then Lemma 2.5 implies that the number of $S \in \mathfrak{S}$ with $\Delta(S \otimes_{\mathbb{Z}} \mathbb{C}) \cap Y_{\mathbb{Q}}(T) \neq \emptyset$ is finite, too. In view of our earlier remarks in this part we now get $\dim_{\mathbb{C}} \mathcal{N}_{V'} = \dim_{\Bbbk} \mathcal{N}_V$. □

Remark

1. In general, different lattices $V'_{\mathbb{Z}}$ may give rise to non-isomorphic G-modules. On the other hand, the theorem implies that the stratification does *not* depend on the choice of lattice and remains essentially the same over any algebraically closed field.
2. Let $E(\lambda)$ denote the finite dimensional irreducible rational G-module with highest weight $\lambda \in X(T)$. Then it is well-known that λ is a dominant weight and there exists an admissible lattice, $V''_{\mathbb{Z}}(\lambda)$, in the irreducible finite dimensional $\mathfrak{g}'$-module $V'(\lambda)$ with highest weight λ such that $E(\lambda)$

is isomorphic to a submodule of the G-module $V''_{\Bbbk}(\lambda) := V''_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \Bbbk$; see [44, Sect. 12, Exercise after Theorem 39]. If $\nu \in Y(G)$ is optimal for a non-zero G-unstable vector $v \in E(\lambda)$, then the definition in Sect. 2.4 shows that it remains so for v regarded as a vector of $V''_{\Bbbk}(\lambda)$. Therefore the Hesselink strata of $E(\lambda)$ are precisely the intersections of those of $V''_{\Bbbk}(\lambda)$ with $E(\lambda)$. Now Theorem 6.1(iii) implies the Hesselink strata of $E(\lambda)$ are parametrised by a subset of the Hesselink strata of the $\mathfrak{g}'$-module $V'(\lambda)$.

6.2 In this subsection we assume that $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$. Keeping the notation of Sect. 4.3 we assume that $(X(T), \Sigma, Y(T), \Sigma^\vee)$ is the root datum of the reductive group scheme $\mathfrak{G}$. Let $G = \mathfrak{G}(\Bbbk)$ and write $x_\alpha(t)$ for Steinberg's generators of the unipotent root subgroups U_α of G; see [44]. Choose a basis of simple roots Π in Σ and denote by $Y^+(T)$ the Weyl chamber in $Y(T)$ associated with Π. (It consists of all $\mu \in Y(T)$ such that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Pi$.) Let τ be an automorphism of the lattice $X(T)$ and denote by τ^* the natural action of τ on $Y(T) = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$. Assume further that τ preserves both Σ and Π and τ^* preserves $\Sigma^\vee$. Finally, assume that the quadratic form q from Sect. 2.4 is invariant under τ^*.

Now fix a pth power $q = p^l$. Then it is well-known that τ gives rise to a Frobenius endomorphism $F = F(\tau, l)\colon x \mapsto F(x)$, of the algebraic $\Bbbk$-group $G = \mathfrak{G}(\Bbbk)$. The endomorphism F is uniquely determined by the following properties:

1. $(\tau\eta)(F(x)) = \eta(x)^q$ for all $\eta \in X(T)$ and $x \in T$;
2. $F(\lambda(t)) = (\tau^*\lambda)(t^q)$ for all $\lambda \in Y(T)$ and $t \in \Bbbk^\times$;
3. $F(x_\alpha(t)) = x_{\tau\alpha}(t^q)$ for all $\alpha \in \Sigma$ and $t \in \Bbbk$;

see [7, Theorem 3.17] for instance. Let V be an admissible G-module endowed with an action of F such that

$$F\big(g(v)\big) = \big(F(g)\big)\big(F(v)\big) \quad \text{for all } g \in G \text{ and } v \in V. \tag{4}$$

As usual we require that the action of F is *q-linear*, that is $F(\lambda v) = \lambda^q F(v)$ for all $\lambda \in \Bbbk$ and $v \in V$, and that each vector in V is fixed by a sufficiently large power of F. In this situation one knows that the fixed point space V^F is an $\mathbb{F}_q$-form of V. In particular, $\dim_{\mathbb{F}_q} V^F = \dim_{\Bbbk} V$; see [7, Corollary 3.5]. We mention, for use later, that there is a natural q-linear action of F on the dual space V^*, compatible with that of G (recall that G acts on V^* via $(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$ for all $g \in G$, $\xi \in V^*$, $v \in V$). Since V^F is an $\mathbb{F}_q$-form of V, the dual space $(V^F)^*$ contains a $\Bbbk$-basis of V^*, say $\xi_1, \dots, \xi_m$. Then every $\xi \in V^*$ can be uniquely expressed as a linear combination $\xi = \sum_{i=1}^m \lambda_i \xi_i$ with $\lambda_i \in \Bbbk$ and we can define $F\colon V^* \to V^*$ by setting $F(\xi) := \sum_{i=1}^m \lambda_i^q \xi_i$. Verifying (4) for this action of F reduces to showing that $g^{-1}F(g)(\xi) = \xi$ for all $\xi \in (V^F)^*$ and $g \in G$, which is clear because $F(g^{-1})g(v) = v$ for all $v \in V^F$.

There are many reasons to be interested in the cardinality of the finite set ${\mathcal{N}_V}^F = \mathcal{N}_V \cap V^F$, and here we can offer the following general result.

Theorem *Under the above assumptions on F and V there exists a polynomial $n_V(t) \in \mathbb{Z}[t]$ such that $|{\mathcal{N}_V}^F| = n_V(q)$ for all $q = p^l$. The polynomial $n_V(t)$ depends only on V' and τ, but not on the choice of an admissible lattice $V'_{\mathbb{Z}}$, and is the same for all primes $p \in \mathbb{N}$.*

Proof Let $\Lambda(V)$ denote the set of pairs (λ, k) where $\lambda \in Y^+(T)$ is primitive and k is a positive integer such $\mathcal{V}(\lambda, k)_{ss}(\Bbbk) \neq \emptyset$ (the notation of Sect. 6.1). Set $\Lambda(V, \tau) = \{(\lambda, k) \in \Lambda(V) | \tau^*\lambda = \lambda\}$ and define

$$\mathcal{H}(\lambda, k) := G \cdot \left(\mathcal{V}(\lambda, k)_{ss}(\Bbbk) \oplus \bigoplus_{i>k} V(\lambda, i) \right),$$

the Hesselink stratum associated with $(\lambda, k) \in \Lambda(V)$. Recall that $\mathcal{V}(\lambda, k)_{ss}(\Bbbk) = V(\lambda, k) \setminus \mathcal{N}_{V(\lambda,k)}$ where $\mathcal{N}_{V(\lambda,k)}$ is the set of all $L^{\perp}(\lambda)$-unstable vectors of $V(\lambda, k)$. To ease notation we set

$$V(\lambda, \geq k)_{ss} := \mathcal{V}(\lambda, k)_{ss}(\Bbbk) \oplus \bigoplus_{i>k} V(\lambda, i).$$

If $\mu \in Y(G)$ is optimal for a non-zero vector $v \in {\mathcal{N}_V}^F$, then so is $F(\mu)$, forcing $P(v) = P(\mu) = P(F(\mu)) = F(P(v))$. So the optimal parabolic subgroup of v is F-stable. But then $P(v)$ contains an F-stable Borel subgroup which, in turn, contains an F-stable maximal torus of G; we shall call it T_1. Since both T and T_1 are F-stable maximal tori contained in an F-stable Borel subgroup of G, there is an element $g_1 \in G^F$ such that $T_1 = g_1^{-1} T g_1$; see [7, 3.15]. Then $Y(T)$ contains an optimal one parameter subgroup for $g_1(v) \in V^F$, say μ_1. Lemma 2.5(iv) yields $\tau^*\mu_1 = \mu_1$. Since the unipotent radical $U(\mu_1)$ of $P(\mu_1)$ is contained in the Borel subgroup of G associated with our basis of simple roots Π, we see that $\mu_1 \in Y^+(T)$.

Now suppose $v \in \mathcal{H}(\lambda, k)^F$, so that $v = gw$ for some $w \in V(\lambda, \geq k)_{ss}$ and $g \in G$. Let $g_1 \in G^F$ and $\mu_1 \in Y^+(T)$ be as above (so that μ_1 is optimal for $v_1 = g_1(gw) \in V^F$). Note that $T \subset L(\mu_1) \subset P(v_1)$. We may assume without loss of generality that μ_1 is primitive in $Y(G)$. Since w and v_1 are in the same Hesselink stratum of V it must be that $G \cdot \Delta_{v_1} = G \cdot \Delta_w$. This yields the equality $(G \cdot \mu_1) \cap Y(T) = (G \cdot \lambda) \cap Y(T)$ which, in turn, implies that μ_1 and λ are conjugate under the action of the Weyl group W on $Y(T)$. Since both λ and μ_1 are in $Y^+(T)$, we get $\mu_1 = \lambda$.

As a result, we deduce that $\tau^*\lambda = \lambda$. Hence both $P(\lambda)$ and $V(\lambda, \geq k)_{ss}$ are F-stable. Applying [13, Proposition 4.5(b)] now yields that $F(g) \in gP(w)$. We choose in G^F a set of representatives $\mathcal{X}(\lambda, \tau, q)$ for $G^F/P(\lambda)^F$, so that

$$\left|\mathcal{X}(\lambda, \tau, q)\right| = \left|G^F/P(\lambda)^F\right|.$$

As $P(\lambda)$ is an F-stable connected group, the Lang–Steinberg theorem shows that $g^{-1}F(g) = x^{-1}F(x)$ for some $x \in P(\lambda)$; see [7, Theorem 3.10] for instance. Then $gx^{-1} \in P(\lambda)^F$ and hence no generality will be lost by assuming that $g \in \mathcal{X}(\lambda, \tau, q)$.

According to [13, Proposition 4.5(b)] there is an F-equivariant bijection between the fibre product $G \times^{P(\lambda)} V(\lambda, \geq k)_{ss} \cong (G/P(\lambda)) \times V(\lambda, \geq k)_{ss}$ and the stratum $\mathcal{H}(\lambda, k)$. Since $v \in V^F$ and $g \in G^F$ we have that $g(F(w)) = gw$, which shows that $w \in V(\lambda, \geq k)_{ss}{}^F$. As a consequence,

$$\begin{aligned}\left|\mathcal{H}(\lambda, k)^F\right| &= \left|\mathcal{X}(\lambda, \tau, q)\right| \cdot \left|V(\lambda, \geq k)_{ss}{}^F\right| \\ &= f_{\tau,\lambda}(q) \cdot q^{N(\lambda,k)}\left(q^{n(\lambda,k)} - \left|\mathcal{N}_{V(\lambda,k)}{}^F\right|\right) \end{aligned} \tag{5}$$

where $f_{\tau,\lambda}(q) = |\mathcal{X}(\lambda, \tau, q)| = |G^F/P(\lambda)^F|$, $N(\lambda, k) = \sum_{i>k} \dim V(\lambda, i)$, and $n(\lambda, k) = \dim V(\lambda, k)$.

After these preliminary remarks we are going to prove our theorem by induction on the rank of G. If $\mathrm{rk} G = 0$, then $G = \{1_G\}$ and hence $\Bbbk[V]^G = \Bbbk[V]$. Therefore $\mathcal{N}_V{}^F = \{0\}$ and we can take 1, a constant polynomial, as $n_V(t)$. Now suppose that $\mathrm{rk} G > 0$ and our theorem holds for all connected reductive groups of rank $< \mathrm{rk} G$. Since for every $(\lambda, k) \in \Lambda(V, \tau)$ we have that $\mathrm{rk} L^{\perp}(\lambda) < \mathrm{rk} G$ and each $L^{\perp}(\lambda)$-module $V(\lambda, i)$ is admissible by our discussion in Sect. 6.1, there exist polynomials $n_{V(\lambda,i)}(t) \in \mathbb{Z}[t]$ with coefficients independent of p and our choice of an admissible lattice $V'_{\mathbb{Z}}(\lambda, i)$ in $V'(\lambda, i)$ such that $|\mathcal{N}_{V(\lambda,i)}{}^F| = n_{V(\lambda,i)}(q)$.

Next we note that for every $\lambda \in Y(T)$ with $\tau^*\lambda = \lambda$ there is a polynomial $f_{\tau,\lambda} \in \mathbb{Z}[t]$ with coefficients independent of p such that $f_{\tau,\lambda}(q) = |G^F/P(\lambda)^F|$ for all pth powers q and all p. Indeed, it is immediate from [7, Proposition 3.19(ii)] that $f_{\tau,\lambda}$ can be chosen as a quotient $a_{\tau,\lambda}/b_{\tau,\lambda}$ of two coprime polynomials $a_{\tau,\lambda}, b_{\tau,\lambda} \in \mathbb{Z}[t]$ with coefficients independent of p. Since $f_{\tau,\lambda}(q) \in \mathbb{Z}$ for infinitely many $q \in \mathbb{Z}$, it must be that $\deg b_{\tau,\lambda} = 0$. Therefore $f_{\tau,\lambda} \in \mathbb{Q}[t]$. On the other hand, G^F/P^F is the set of $\mathbb{F}_q$-rational points of a smooth projective variety defined over $\mathbb{F}_p$. Applying [10, Lemma 2.12] one obtains that $f_{\tau,\lambda} \in \mathbb{Z}[t]$, as stated.

Putting everything together we now get

$$\begin{aligned}\left|\mathcal{N}_V{}^F\right| &= 1 + \sum_{(\lambda,k)\in\Lambda(V,\tau)} \left|\mathcal{H}(\lambda, k)^F\right| \\ &= 1 + \sum_{(\lambda,k)\in\Lambda(V,\tau)} f_{\lambda,\tau}(q) \cdot q^{N(\lambda,k)}\left(q^{n(\lambda,k)} - n_{V(\lambda,k)}(q)\right). \end{aligned}$$

Since the data $\{(n(\lambda, k), N(\lambda, k)) | (\lambda, k) \in \Lambda(V, \tau)\}$ arrives unchanged from the G'-module V' and is independent of p by Theorem 6.1, the RHS is a polynomial in q with integer coefficients independent of p and the choice of admissible lattice in V'. □

Remark In the notation of Sect. 6.1, the distribution algebra $\mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G})$ acts naturally on the $\mathbb{Z}$-algebra $\mathbb{Z}[V'_{\mathbb{Z}}]$ and we may consider the invariant algebra of this action, which coincides with $\mathbb{Z}[V'_{\mathbb{Z}}]^{\mathfrak{G}}$. According to [41, Sect. II], the algebra $\mathbb{Z}[V'_{\mathbb{Z}}]^{\mathfrak{G}}$ is generated over $\mathbb{Z}$ by finitely many homogeneous elements. The ideal of $\mathbb{Z}[V'_{\mathbb{Z}}]$ generated by these elements defines a closed subscheme of the affine scheme $\mathrm{Spec}\,\mathbb{Z}[V'_{\mathbb{Z}}]$ which we denote by $\mathcal{N}(V'_{\mathbb{Z}})$. It follows from [41, Proposition 6(2)] that for any prime $p \in \mathbb{N}$ the null-cone $\mathcal{N}_V$ coincides with the variety of closed points of the affine $\Bbbk$-scheme $\mathcal{N}(V'_{\mathbb{Z}}) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\Bbbk$. At this point Theorem 6.2 shows that the affine $\mathbb{Z}$-scheme $\mathcal{N}(V'_{\mathbb{Z}})$ is *strongly polynomial-count* in the terminology of N. Katz. Applying [19, Theorem 1(3)] we now deduce that the polynomial $n_V(t)$ from Theorem 6.2 is closely related with the *E-polynomial* $E(\mathcal{N}_{V'}; x, y) = \sum_{i,j} e_{i,j} x^i y^j \in \mathbb{Z}[x, y]$ of the complex algebraic variety $\mathcal{N}_{V'}$. More precisely, we have that $E(\mathcal{N}_{V'}; x, y) = n_V(xy)$ as polynomials in x, y; see [19, p. 618] for more details. This shows that the coefficients of $n_V(t)$ are determined by Deligne's mixed Hodge structure on the compact cohomology groups $H^k_c(\mathcal{N}_{V'}, \mathbb{Q})$.

Define $n'_V(t) := (n_V(t) - 1)/(t - 1)$. As $n'_V(q) = \mathrm{Card}\{\mathbb{F}_q^\times v \mid v \in \mathcal{N}_V{}^F, v \neq 0\}$ for all pth powers q, it is straightforward to see that $n'_V(t)$ is a polynomial in t. The long division algorithm then shows that $n'_V(t) \in \mathbb{Z}[t]$. We conjecture that the polynomial $n'_V(t)$ has *non-negative* coefficients. This conjecture holds true for $\mathfrak{G} = \mathbf{SL}_2$ where one can compute $n'_V(t)$ explicitly for any admissible G-module V. The details are left as an exercise for the interested reader.

7 Nilpotent pieces in $\mathfrak{g}$ and $\mathfrak{g}^*$

7.1 We now define nilpotent pieces in the Lie algebra $\mathfrak{g}$ completely analogously to the definition of unipotent pieces, that is, we partition $\mathfrak{g}_{\mathrm{nil}} = \mathcal{N}_{\mathfrak{g}}$ into locally closed G-stable pieces, indexed by the unipotent classes in $G' = \mathfrak{G}(\mathbb{C})$. For convenience, we now allow $\mathrm{char}\,\Bbbk = p \geq 0$. For $\triangle \in D_G$ and $i \geq 0$ we define $\mathfrak{g}_i^\triangle = \mathrm{Lie}\, G_i^\triangle$. For any G-orbit $\blacktriangle \in D_G$, let $\tilde{H}^\blacktriangle(\mathfrak{g}) = \bigcup_{\triangle \in \blacktriangle} \mathfrak{g}_2^\triangle$. This is a closed irreducible G-stable variety by the proof of Lemma 1.3. We define the *nilpotent pieces* of $\mathfrak{g}$ to be the sets

$$H^\blacktriangle(\mathfrak{g}) := \tilde{H}^\blacktriangle(\mathfrak{g}) \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'}(\mathfrak{g}),$$

where the union is taken over all $\blacktriangle' \in D_G/G$ such that $\tilde{H}^{\blacktriangle'}(\mathfrak{g}) \subsetneq \tilde{H}^\blacktriangle(\mathfrak{g})$. We also define

$$X^\triangle(\mathfrak{g}) := \mathfrak{g}_2^\triangle \cap H^\blacktriangle(\mathfrak{g}),$$

for each $\triangle \in D_G$, where $\blacktriangle$ is the G-orbit of $\triangle$. Since $H^{\blacktriangle}(\mathfrak{g})$ is the complement of finitely many non-trivial closed subvarieties of $\tilde{H}^{\blacktriangle}(\mathfrak{g})$, it is open and dense in $\tilde{H}^{\blacktriangle}(\mathfrak{g})$, hence it is locally closed in $\mathfrak{g}_{\mathrm{nil}}$. The subset $H^{\blacktriangle}(\mathfrak{g})$ is G-stable since its complement in $\tilde{H}^{\blacktriangle}(\mathfrak{g})$ is. Consequently, $X^{\triangle}(\mathfrak{g})$ is open and dense in $\mathfrak{g}_2^{\triangle}$, and stable under the adjoint action of $G_0^{\triangle}$.

Recall from Sects. 5.1 and 5.2 that for any $\triangle \in D_G$ there is an element $g \in G$ and a one parameter subgroup $\omega \in Y(T) = Y(T')$, coming from a rational homomorphism $\mathrm{SL}_2(\mathbb{C}) \to G'$, such that $\frac{1}{2}\omega \in \tilde{\Delta}_x$ for some $x \in \mathfrak{g}'(\omega, 2)$ and $\mathfrak{g}_k^{\triangle} = \bigoplus_{i \geq k} \mathfrak{g}(g \cdot \omega, i)$ for all $k \in \mathbb{Z}$. Note that different $g \in G$ with this property have the same image in $G/G_0^{\triangle}$. Given $\mu \in Y(G)$ and $i \in \mathbb{Z}$ we denote by $\mathfrak{g}^*(\mu, i)$ the subspace in $\mathfrak{g}^*$ consisting of all linear functions that vanish on each $\mathfrak{g}(\mu, j)$ with $j \neq -i$. Now define $(\mathfrak{g}^*)_k^{\triangle} := \bigoplus_{i \geq k} \mathfrak{g}^*(g \cdot \omega, i)$, for $k \in \mathbb{Z}$. The preceding remark shows that this is independent of the choice of $g \in G$ and therefore the subspaces $(\mathfrak{g}^*)_k^{\triangle}$ are well-defined.

In a completely analogous way we now define the *nilpotent pieces* of the dual space $\mathfrak{g}^*$. For any G-orbit $\blacktriangle \in D_G$, we let $\tilde{H}^{\blacktriangle}(\mathfrak{g}^*) = \bigcup_{\triangle \in \blacktriangle} (\mathfrak{g}^*)_2^{\triangle}$, a closed irreducible G-stable subset of $\mathfrak{g}^*$, and put

$$H^{\blacktriangle}(\mathfrak{g}^*) := \tilde{H}^{\blacktriangle}(\mathfrak{g}^*) \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'}(\mathfrak{g}^*),$$

where the union is taken over all $\blacktriangle' \in D_G/G$ with $\tilde{H}^{\blacktriangle'}(\mathfrak{g}^*) \subsetneq \tilde{H}^{\blacktriangle}(\mathfrak{g}^*)$. We define

$$X^{\triangle}(\mathfrak{g}^*) := (\mathfrak{g}^*)_2^{\triangle} \cap H^{\blacktriangle}(\mathfrak{g}^*),$$

for each $\triangle \in D_G$. Arguing as before we observe that each $H^{\blacktriangle}(\mathfrak{g}^*)$ is a G-stable, locally closed subset of $\mathcal{N}_{\mathfrak{g}^*}$. Hence $X^{\triangle}(\mathfrak{g}^*)$ is open and dense in $\mathfrak{g}_2^{\triangle}$, and stable under the coadjoint action of $G_0^{\triangle}$.

7.2 In the next two subsections we study the nullcone $\mathcal{N}_{\mathfrak{g}^*}$ associated with the coadjoint action of G on the dual space $\mathfrak{g}^* = \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, \Bbbk)$. Recall that $(g \cdot \xi)(x) = \xi((\mathrm{Ad}\, g^{-1})x)$ for all $g \in G$, $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. It is immediate from the Hilbert–Mumford criterion (our Theorem 2.1) that $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ if and only if ξ vanishes on the Lie algebra of a Borel subgroup of G. The nilpotent linear functions $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ play an important role in the study of the centre of the enveloping algebra $U(\mathfrak{g})$ and were first investigated in our setting by Kac and Weisfeiler in [24]. In characteristic zero the Killing form induces a G'-equivariant isomorphism $\mathfrak{g}' \cong (\mathfrak{g}')^*$. However, in positive characteristic it may happen that $\mathfrak{g} \ncong \mathfrak{g}^*$ as G-modules.

We first assume that the group G is simple and simply connected. Rather than study $\mathfrak{g}^*$ directly, we will present a slightly different construction which will allow us to combine Theorems 6.1 and 6.2 with classical results of Dynkin [8] and Kostant [22] on $\mathcal{N}_{\mathfrak{g}'}$. As before, we fix a set of simple roots Π in Σ and denote the corresponding set of positive roots by Σ^+. Let

$\mathcal{C}' = \{X_\alpha, H_\beta \mid \alpha \in \Sigma, \beta \in \Pi\}$ be a Chevalley basis of $\mathfrak{g}'$ and denote by $\mathfrak{g}'_{\mathbb{Z}}$ the $\mathbb{Z}$-span of $\mathcal{C}'$ in $\mathfrak{g}$. Then the following equations hold in $\mathfrak{g}'_{\mathbb{Z}}$:

(i) $[H_\alpha, X_\beta] = \langle \beta, \alpha \rangle X_\beta$ for all $\alpha \in \Pi$, $\beta \in \Sigma$;
(ii) $[X_\beta, X_{-\beta}] = H_\beta$ for all $\beta \in \Pi$, where $H_\beta = \mathrm{d}_e\beta^\vee$ is an integral linear combination of $H_\alpha = \mathrm{d}_e\alpha^\vee$ with $\alpha \in \Pi$;
(iii) $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha + \beta \in \Sigma$, where $N_{\alpha,\beta} = \pm(q+1)$ and q is the maximal integer for which $\beta - q\alpha \in \Sigma$;
(iv) $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \notin \Sigma$;

see [44, Sect. 1], for example. As usual, $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\alpha, \alpha)$, where $(\ ,\ )$ is a scalar product on the $\mathbb{R}$-span of Π, invariant under the action of the Weyl group W of Σ. We may assume, by rescaling if necessary, that $(\alpha, \alpha) = 2$ for every short root α of Σ. Let $\tilde{\alpha}$ denote the maximal root, and α_0 the maximal short root in Σ^+ respectively, and set $d := (\tilde{\alpha}, \tilde{\alpha})/(\alpha_0, \alpha_0)$. Recall that a prime $p \in \mathbb{N}$ is called *special* for Σ if $d \equiv 0 \pmod p$. The special primes are 2 and 3. To be precise, 2 is special for Σ of type B_ℓ, C_ℓ, $\ell \geq 2$, and F_4, whilst 3 is special for Σ of type G_2.

Since G is assumed to be simply connected, we have that $\mathfrak{g} = \operatorname{Lie} G = \mathfrak{g}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ (cf. [2, Sect. 2.5] or [17, Sect. 1.3]). Also, the distribution algebra $\mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G})$ identifies canonically with the unital $\mathbb{Z}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g}')$ generated by all $X_\beta^n/n!$ with $\beta \in \Sigma$ and $n \in \mathbb{N}$. The algebra $U_{\mathbb{Z}}$ is known as *Kostant's* $\mathbb{Z}$*-form* of $U(\mathfrak{g})$ and was first introduced in [23]. Thus, a $\mathbb{Z}$-lattice $V'_{\mathbb{Z}}$ in a finite-dimensional $\mathfrak{g}'$-module V' is admissible if and only if it is invariant under all operators $X_\alpha^n/n!$ $(n \in \mathbb{N})$ under the obvious action of $U(\mathfrak{g}')$ on V'. For instance, $\mathfrak{g}'_{\mathbb{Z}}$ itself is admissible, since $\mathfrak{g}'_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot X_{\tilde{\alpha}}$.

We now recall very briefly how admissible lattices give rise to rational G-modules. Let $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Since $\mathrm{Dist}_{\Bbbk}(G) = \mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G}) \otimes_{\mathbb{Z}} \Bbbk = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$, the action of $U_{\mathbb{Z}}$ on $V'_{\mathbb{Z}}$ gives rise to a representation of $\mathrm{Dist}_{\Bbbk}(G)$ on $\mathrm{End}_{\Bbbk} V$, and hence to a rational linear action of G on V; see [16, Sects. II.1.12 and II.1.20] for more details. Given $X \in U_{\mathbb{Z}}$ we denote the induced linear transformations on $V'_{\mathbb{Z}}$ and V by $\rho_{\mathbb{Z}}(X)$. We then define invertible linear transformations $x_\beta(t) = \sum_{n \geq 0} t^n \rho_{\mathbb{Z}}(X_\beta^n/n!)$ on V, for each $\beta \in \Sigma$, where $t \in \Bbbk$. (Note that the sum is finite since the X_β act nilpotently on V'.) The set $\{x_\beta(t) \mid \beta \in \Sigma, t \in \Bbbk\}$ generates a Zariski-closed, connected subgroup $G(V)$ of $\mathrm{GL}(V)$. Since G is simply connected and hence a universal Chevalley group in the sense of [44], the linear group $G(V)$ is a homomorphic image of G. For any admissible lattice $V'_{\mathbb{Z}}$ in a finite-dimensional $\mathfrak{g}'$-module V', we thus obtain a G-module structure on $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$.

Define a symmetric bilinear form $\langle\ ,\ \rangle\colon \mathfrak{g}'_{\mathbb{Z}} \times \mathfrak{g}'_{\mathbb{Z}} \to \mathbb{Z}$ by setting

$$\langle X_\alpha, X_\beta\rangle = 0 \qquad \text{if } \alpha+\beta \neq 0,$$

$$\langle H_\alpha, H_\beta\rangle = \frac{4d(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)} \qquad \text{for all } \alpha,\beta\in\Sigma,$$

$$\langle X_\alpha, X_{-\alpha}\rangle = \frac{2d}{(\alpha,\alpha)} \qquad \text{for all } \alpha\in\Sigma,$$

and extending to $\mathfrak{g}'_{\mathbb{Z}}$ by $\mathbb{Z}$-bilinearity. Note that this is well-defined, since the condition $(\alpha_0,\alpha_0)=2$ ensures that the image is indeed in $\mathbb{Z}$; see Bourbaki's tables in [4]. Obviously we may extend $\langle\ ,\ \rangle$ to symmetric bilinear forms $\langle\ ,\ \rangle_{\mathbb{C}}$ on $\mathfrak{g}'=\mathfrak{g}'_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}$, and $\langle\ ,\ \rangle_{\Bbbk}$ on $\mathfrak{g}=\mathfrak{g}'_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$.

It is proved in [37, p. 240] that the bilinear form $\langle\ ,\ \rangle_{\mathbb{C}}$ is a scalar multiple of the Killing form κ of $\mathfrak{g}'=\operatorname{Lie}G'$. In particular, $\langle\ ,\ \rangle_{\mathbb{C}}$ is G'-invariant. This, in turn, implies that

$$\big\langle X(u), v\big\rangle = \big\langle u, X^{\top}(v)\big\rangle \quad \text{for all } u,v\in V'_{\mathbb{Z}} \text{ and } X\in U_{\mathbb{Z}}, \tag{6}$$

where $\top$ stands for the canonical anti-automorphism of $U(\mathfrak{g})$. Since $x^{\top}=-x$ for all $x\in\mathfrak{g}'$, it is straightforward to see that $\top$ preserves the $\mathbb{Z}$-form $U_{\mathbb{Z}}$ of $U(\mathfrak{g}')$. (In fact, the map $\top\colon U_{\mathbb{Z}}\to U_{\mathbb{Z}}$ is nothing but the antipode of the Hopf algebra $U_{\mathbb{Z}}=\operatorname{Dist}_{\mathbb{Z}}(\mathfrak{G})$.) As a consequence, the bilinear form $\langle\ ,\ \rangle_{\Bbbk}$ on $\mathfrak{g}=\operatorname{Lie}G$ is G-invariant.

Lemma *If p is non-special for Σ, then the radical of $\langle\ ,\ \rangle_{\Bbbk}$ coincides with the centre $\mathfrak{z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. If p is special for $\mathfrak{g}$, then* $\operatorname{Rad}\langle\ ,\ \rangle_{\Bbbk}\not\subseteq\mathfrak{z}(\mathfrak{g})$.

Proof The first statement of the lemma is [37, Lemma 2.2(ii)]. For the second statement, we note that the image of X_{α_0} in $\mathfrak{g}=(\mathfrak{g}'_{\mathbb{Z}}/p\mathfrak{g}'_{\mathbb{Z}})\otimes_{\mathbb{F}_p}\Bbbk$ lies in the radical of $\langle\ ,\ \rangle_{\Bbbk}$, but not in the centre of $\mathfrak{g}$. (Recall that G is assumed to be simply connected.) □

The lemma hints at the fact that $\mathfrak{g}$ and $\mathfrak{g}^*$ are similar as G-modules if p is non-special, but very different if p is special. Nevertheless, as we will see, we may construct an alternative admissible lattice $\mathfrak{g}''_{\mathbb{Z}}\subset\mathfrak{g}'$ which gives rise to another G-module $\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ such that $\langle\ ,\ \rangle$ induces a non-degenerate pairing between $\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ and $\mathfrak{g}$ in all cases. This will enable us to identify the G-modules $\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ and $\mathfrak{g}^*$.

7.3 We define $\mathfrak{g}''_{\mathbb{Z}}:=\{x\in\mathfrak{g}'\mid\langle x,y\rangle\in\mathbb{Z},\forall y\in\mathfrak{g}'_{\mathbb{Z}}\}$, a $\mathbb{Z}$-lattice in $\mathfrak{g}'$. It is immediate from (6) that $\mathfrak{g}''_{\mathbb{Z}}$ is an admissible lattice. Consequently, we obtain a G-module structure on the vector space $\mathfrak{g}'_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$. We also obtain a G-invariant pairing

$$\langle\ ,\ \rangle^*_{\Bbbk}\colon \mathfrak{g}\times\big(\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk\big)\longrightarrow\Bbbk. \tag{7}$$

We will now exhibit a basis of $\mathfrak{g}''_{\mathbb{Z}}$ dual to our Chevalley basis $\mathcal{C}'$, with respect to $\langle\ ,\ \rangle$. Thus, we will show that the pairing $\langle\ ,\ \rangle^*_{\Bbbk}$ is non-degenerate. Let $\mathfrak{t}'$ be the Cartan subalgebra of $\mathfrak{g}'$ spanned by $\{H_\alpha|\alpha\in\Pi\}$. Let $\{H'_\alpha|\alpha\in\Pi\}$ be the dual basis of $\mathfrak{t}'$ with respect to the restriction of $\langle\ ,\ \rangle_{\mathbb{C}}$ to $\mathfrak{t}'$. (These may be thought of as the fundamental weights of the dual root system $\Sigma^\vee$.) This extends to a basis

$$\mathcal{C}=\left\{H'_\alpha|\alpha\in\Pi\right\}\sqcup\{X_\beta|\beta\in\Sigma\text{ long}\}\sqcup\left\{(1/d)X_\beta|\beta\in\Sigma\text{ short}\right\}$$

of $\mathfrak{g}$ which is dual to our Chevalley basis $\mathcal{C}'$ with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$. Specifically, the corresponding pairing of basis elements is as follows:

$$\begin{aligned} &H_\alpha\leftrightarrow H'_\alpha && \text{if } \alpha\in\Pi,\\ &X_\beta\leftrightarrow X_{-\beta} && \text{if } \beta\in\Sigma \text{ is long},\\ &X_\beta\leftrightarrow (1/d)X_{-\beta} && \text{if } \beta\in\Sigma \text{ is short}. \end{aligned}$$

Moreover, it is easy to check that $\mathcal{C}$ is a $\mathbb{Z}$-basis of $\mathfrak{g}''_{\mathbb{Z}}$, as required. Since the lattice $\mathfrak{g}''_{\mathbb{Z}}$ is admissible, we see that the bases $\mathcal{C}'\otimes 1$ of $\mathfrak{g}=\mathfrak{g}'_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ and $\mathcal{C}\otimes 1$ of $\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ are dual to each other with respect to $\langle\ ,\ \rangle^*_{\Bbbk}$. This shows that $\mathfrak{g}$ and $\mathfrak{g}^*\cong\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ are admissible G-modules associated with different admissible lattices in $\mathfrak{g}'$.

Now suppose that G is semisimple and simply connected. Then G is a direct product of simple, simply connected groups and the above arguments carry over to G in a straightforward fashion. In particular, (7) is still available for a suitable choice of an admissible lattice $\mathfrak{g}''_{\mathbb{Z}}\subset\mathfrak{g}'$ and $\mathfrak{g}^*\cong\mathfrak{g}''_{\mathbb{Z}}\otimes_{\mathbb{Z}}\Bbbk$ as G-modules.

Theorem *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic $p\geq 0$ and let $\mathcal{G}$ be $\mathfrak{g}$ or $\mathfrak{g}^*$. If $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$, assume further that we have a Frobenius endomorphism $F\colon G\to G$ corresponding to an $\mathbb{F}_q$-rational structure of G. Then $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for $\mathcal{G}$ and the stabiliser G_x of any element $x\in X^\triangle(\mathcal{G})$ is contained in the parabolic subgroup $G_0^\triangle$ of G.*

Proof Let U be an F-stable maximal connected unipotent subgroup of G. It follows from the Hilbert–Mumford criterion (our Theorem 2.1) that $\mathcal{N}_{\mathfrak{g}}=(\operatorname{Ad}G)\cdot\mathfrak{u}$ where $\mathfrak{u}=\operatorname{Lie}U$. Since $U\subset\mathcal{D}G$, we have that $\mathcal{N}_{\mathfrak{g}}\subseteq\mathcal{N}_{\bar{\mathfrak{g}}}$ where $\bar{\mathfrak{g}}=\operatorname{Lie}\mathcal{D}G$. As any $\xi\in\mathcal{N}_{\mathfrak{g}*}$ vanishes on a Borel subalgebra of $\mathfrak{g}$, the restriction map $\mathfrak{g}^*\to\bar{\mathfrak{g}}^*$, $\xi\mapsto\xi|_{\bar{\mathfrak{g}}}$, induces a G-equivariant injection $\eta\colon\mathcal{N}_{\mathfrak{g}^*}\to\mathcal{N}_{\bar{\mathfrak{g}}^*}$. But η is, in fact, a bijection since every linear function on $\mathfrak{u}$ can be extended to a nilpotent linear function on $\mathfrak{g}$.

Let $\tilde{G}$ be a semisimple, simply connected group isogeneous to $\mathcal{D}G$. Let $\iota\colon\tilde{G}\to\mathcal{D}G$ be an isogeny and let $\tilde{U}$ be the connected unipotent subgroup of $\tilde{G}$ with $\iota(\tilde{U})=U$. Let $\tilde{\mathfrak{g}}=\operatorname{Lie}\tilde{G}$ and $\tilde{\mathfrak{u}}=\operatorname{Lie}\tilde{U}$. Then $\mathrm{d}_e\iota\colon\tilde{\mathfrak{g}}\to\bar{\mathfrak{g}}$ maps

$\tilde{\mathfrak{u}}$ isomorphically onto $\mathfrak{u}$ and induces a $\tilde{G}$-equivariant bijection between $\mathcal{N}_{\tilde{\mathfrak{g}}}$ and $\mathcal{N}_{\bar{\mathfrak{g}}} = \bar{\mathfrak{g}}_{\mathrm{nil}}$. Let $\tilde{T}$ be a maximal torus of $\tilde{G}$ normalising $\tilde{\mathfrak{u}}$ and $T = \iota(\tilde{T})$, a maximal torus of G normalising $\mathfrak{u}$. We regard $\mathfrak{u}^*$ and $\tilde{\mathfrak{u}}^*$ as subspaces of $\bar{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}^*$ respectively, by imposing that every $\xi \in \mathfrak{u}^*$ vanishes on the T-invariant complement of $\mathfrak{u}$ in $\mathfrak{g}$ and every $\tilde{\xi} \in \tilde{\mathfrak{u}}^*$ vanishes on the $\tilde{T}$-invariant complement of $\tilde{\mathfrak{u}}$ in $\tilde{\mathfrak{g}}$. Then the linear map $(\mathrm{d}_e\iota)^*\colon \bar{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$ induced by $\mathrm{d}_e\iota$ restricts to a linear isomorphism between $\mathfrak{u}^*$ and $\tilde{\mathfrak{u}}^*$. Since the map $(\mathrm{d}_e\iota)^*$ is $\tilde{G}$-equivariant, it induces a natural bijection between $\mathcal{N}_{\tilde{\mathfrak{g}}^*} = (\mathrm{Ad}^*\,\tilde{G}) \cdot \tilde{\mathfrak{u}}^*$ and $\mathcal{N}_{\mathfrak{g}^*} = (\mathrm{Ad}^*\,G) \cdot \mathfrak{u}^*$. It is clear from our description of F in Sect. 6.2 that there is a Frobenius endomorphism $\tilde{F}\colon \tilde{G} \to \tilde{G}$ such that $\iota \circ \tilde{F} = F|_{\mathcal{D}G}$. Furthermore, $\tilde{T}$ and $\tilde{U}$ can be chosen to be $\tilde{F}$-stable.

The above discussion shows that in proving the theorem we may assume that the group G is semisimple and simply connected. Then both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible G-modules. More precisely, $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{g}^* = \mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ for some admissible lattices $\mathfrak{g}'_{\mathbb{Z}}$ and $\mathfrak{g}''_{\mathbb{Z}}$ in $\mathfrak{g}'$. Then Theorem 6.1 shows that the subsets $H^{\blacktriangle}(\mathcal{G})$ ($\blacktriangle \in D_G/G$) are the Hesselink strata of $\mathcal{N}_{\mathcal{G}}$ and for each $\blacktriangle \in D_G/G$ the subsets $X^{\triangle}(\mathcal{G})$ with $\triangle \in \blacktriangle$ are the blades of $\mathcal{N}_{\mathcal{G}}$ contained in $H^{\blacktriangle}(\mathcal{G})$. In particular, $\mathcal{N}_{\mathcal{G}} = \bigsqcup_{\blacktriangle \in D_G/G} H^{\blacktriangle}(\mathcal{G})$, showing that $\mathfrak{P}_3$ holds for $\mathcal{G}$. It follows from [12, Proposition 4.5] that for every $\blacktriangle \in D_G/G$ there is a surjective G-equivariant map $H^{\blacktriangle} \twoheadrightarrow G/G_0^{\triangle'}$, $\triangle' \in \blacktriangle$, whose fibres are exactly the blades $X^{\triangle}$ with $\triangle \in \blacktriangle$ (this map is not a morphism, in general). So $\mathfrak{P}_1$ and $\mathfrak{P}_2$ hold for $\mathcal{G}$ as well. In order to show that $\mathfrak{P}_4$ holds for $\mathcal{G}$ it suffices to establish that for every $x \in X^{\triangle}(\mathcal{G})$ the optimal parabolic subgroup $P(x)$ coincides with $G_0^{\triangle}$. This is completely analogous to our arguments at the end of the proof of Theorem 5.2. Of course it is much easier since we may use Tsujii's result (Theorem 3.2) in its original form, and there is no need for Sect. 3. The inclusion $G_x \subset G_0^{\triangle}$ follows from Theorem 2.4(iv).

It remains to show that $\mathfrak{P}_5$ holds for $\mathcal{G}$, so suppose from now on that $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$ and $F = F(\tau, l)$ where $q = p^l$; see Sect. 6.2. As explained there, we have a natural q-linear action of F on $\mathfrak{g}^*$ compatible with the coadjoint action of G. We adopt the notation introduced in the course of proving Theorem 6.2. It follows from Theorem 5.1 that the set $\Lambda(\mathfrak{g}, \tau) = \Lambda(\mathfrak{g}^*, \tau)$ consists of all pairs $(\lambda'_{\blacktriangle}, k)$ such that $\lambda'_{\blacktriangle} \in Y^+(T)$ is primitive, $k \in \{1, 2\}$ and $\frac{2}{k}\lambda'_{\blacktriangle}$ is adapted by a suitable nilpotent element in the adjoint G'-orbit labelled by $\blacktriangle$. Then (5) yields

$$\begin{aligned}\varphi_{\mathcal{G}}^{\blacktriangle}(q) &:= \left|H^{\blacktriangle}(\mathcal{G})^F\right| = f_{\tau,\lambda'_{\blacktriangle}}(q) \cdot q^{N(\lambda'_{\blacktriangle},k)}\left(q^{n(\lambda'_{\blacktriangle},k)} - \left|\mathcal{N}_{\mathcal{G}(\lambda'_{\blacktriangle},k)}{}^F\right|\right) \\ &= f_{\tau,\lambda'_{\blacktriangle}}(q) \cdot q^{N(\lambda'_{\blacktriangle},k)}\left(q^{n(\lambda'_{\blacktriangle},k)} - n_{\mathcal{G}(\lambda'_{\blacktriangle},k)}(q)\right).\end{aligned}$$

If $\triangle \in \blacktriangle$ is such that $F(G_i^{\triangle}) = G_i^{\triangle}$ for all $i \geq 0$, then the proof of Theorem 6.2 also yields that $\tau^*(\lambda'_{\blacktriangle}) = \lambda'_{\blacktriangle}$ and

$$\psi_{\mathcal{G}}^{\triangle}(q) := \left|X^{\triangle}(\mathcal{G})^F\right| = q^{N(\lambda'_{\blacktriangle},k)}\left(q^{n(\lambda'_{\blacktriangle},k)} - n_{\mathcal{G}(\lambda'_{\blacktriangle},k)}(q)\right).$$

As the $L^\perp(\lambda'_\blacktriangle)$-modules $\mathfrak{g}(\lambda'_\blacktriangle, k)$ and $\mathfrak{g}^*(\lambda'_\blacktriangle, k)$ come from different admissible lattices of the $(\mathfrak{L}^\perp(\lambda'_\blacktriangle))(\mathbb{C})$-module $\mathfrak{g}'(\lambda'_\blacktriangle, k)$, applying Theorem 6.2 shows that $\psi^\triangle_{\mathfrak{g}}(q) = \psi^\triangle_{\mathfrak{g}^*}(q)$ are polynomials in q with integer coefficients independent of p. This, in turn, implies that so are $\varphi^\blacktriangle_{\mathfrak{g}}(q) = \varphi^\blacktriangle_{\mathfrak{g}^*}(q)$, completing the proof. □

Corollary *Let G be a connected reductive group defined over an algebraic closure of $\mathbb{F}_p$ and assume that we have a Frobenius endomorphism $F\colon G \to G$ corresponding to an $\mathbb{F}_q$-rational structure on G. Then $\mathfrak{P}_5$ holds for G.*

Proof Let $\triangle \in D_G$ be such $F(G_i^\triangle) = G_i^\triangle$ for all $i \geq 0$ and let $\blacktriangle$ be the orbit of $\triangle$ in D_G/G. Then $gG_0^\triangle g^{-1} = P(\lambda'_\blacktriangle)$ and $gG_i^\triangle g^{-1} = U_i(\lambda'_\blacktriangle)$ for some $g \in G$, where $i \geq 1$. If s is the order of τ^*, then there exists $r \in \mathbb{N}$ with $r \equiv 1 (\mathrm{mod}\, s)$ such that $X^\triangle(G)^{F^r} \neq \emptyset$. Then $H^\blacktriangle(G)^{F^r} \neq \emptyset$ and the argument used in the proof of Theorem 6.2 shows that $\tau^{*r}(\lambda'_\blacktriangle) = \lambda'_\blacktriangle$. Since $\tau^{*r}(\lambda'_\blacktriangle) = \tau^*(\lambda'_\blacktriangle)$ by our choice of r, we see that $P(\lambda'_\blacktriangle)$ is F-stable. Hence $F(g)G_0^\triangle(F(g))^{-1} = gG_0g^{-1}$ forcing $g^{-1}F(g) \in N_G(G_2^\triangle) = G_0^\triangle$. As $G_0^\triangle$ is connected and F-stable, the Lang–Steinberg theorem shows that $g^{-1}F(g) = x^{-1}F(x)$ for some $x \in G_0^\triangle$; see [7, Theorem 3.10]. Replacing g by gx^{-1} we thus may assume that $g \in G^F$. In conjunction with Theorems 3.6 and 5.2 this shows that

$$\left|X^\triangle(G)^F\right| = \left|\pi^{-1}\left(V_2\left(\lambda'_\blacktriangle\right)_{ss}{}^F\right)\right| \tag{8}$$

where $V_2(\lambda'_\blacktriangle)_{ss}$ stands for the set of all $L^\perp(\lambda'_\blacktriangle)$-semistable vectors of the $L(\lambda'_\blacktriangle)$-module $V_2(\lambda'_\blacktriangle) = U_2(\lambda'_\blacktriangle)/U_3(\lambda'_\blacktriangle)$ and $\pi\colon U_2(\lambda'_\blacktriangle)^F \to V_2(\lambda'_\blacktriangle)^F$ is the map induced by the canonical homomorphism $U_2(\lambda'_\blacktriangle) \twoheadrightarrow V_2(\lambda'_\blacktriangle)$. Now the argument used in the proof of Theorem 6.2 yields

$$\left|H^\blacktriangle(G)^F\right| = \left|G^F/P\left(\lambda'_\blacktriangle\right)^F\right| \cdot \left|\pi^{-1}\left(V_2\left(\lambda'_\blacktriangle\right)_{ss}{}^F\right)\right|. \tag{9}$$

In view of Remark 2 we have that

$$\left|V_2\left(\lambda'_\blacktriangle\right)_{ss}{}^F\right| = \left|\mathfrak{g}\left(\lambda'_\blacktriangle, 2\right)_{ss}{}^F\right|. \tag{10}$$

Since the group $U_3(\lambda'_\blacktriangle)$ is connected and F-stable, the Lang–Steinberg theorem shows that for every $v \in V_2(\lambda'_\blacktriangle)_{ss}{}^F$ there is an element $\tilde{v} \in V_2(\lambda'_\blacktriangle)_{ss}{}^F$ such that $\pi(\tilde{v}) = v$. From this it is immediate that

$$\pi^{-1}(v) = \tilde{v} \cdot U_3\left(\lambda'_\blacktriangle\right)^F \quad \left(\forall v \in V_2\left(\lambda'_\blacktriangle\right)_{ss}{}^F\right). \tag{11}$$

Combining (8), (10) and (11) we obtain that

$$\left|X^\triangle(G)^F\right| = \left|\pi^{-1}\left(V_2\left(\lambda'_\blacktriangle\right)_{ss}{}^F\right)\right| = \left|\mathfrak{g}\left(\lambda'_\blacktriangle, 2\right)_{ss}{}^F\right| \cdot \left|U_3\left(\lambda'_\blacktriangle\right)^F\right|. \tag{12}$$

As we know by Remark 2, for each $i \geq 3$ the connected abelian group $V_i(\lambda'_\blacktriangle) = U_i(\lambda'_\blacktriangle)/U_{i+1}(\lambda'_\blacktriangle)$ is a vector space over $\Bbbk$ isomorphic to $\mathfrak{g}(\lambda_\blacktriangle, i)$. Since $\tau^*\lambda'_\blacktriangle = \lambda'_\blacktriangle$, it is equipped with a q-linear action of F. Therefore

$$\left|V_i\left(\lambda'_\blacktriangle\right)^F\right| = q^{\dim \mathfrak{g}(\lambda'_\blacktriangle, i)}, \quad i \geq 3; \tag{13}$$

see [7, Corollary 3.5], for example. Since every group $U_i(\lambda'_\blacktriangle)$ with $i \geq 3$ is connected and F-stable, the Lang–Steinberg theorem yields that for every $u \in V_i(\lambda'_\blacktriangle)^F$ there exists $\tilde{u} \in U_i(\lambda'_\blacktriangle)^F$ whose image in $V_i(\lambda'_\blacktriangle)^F$ equals u. This, in turn, implies that every quotient $V_i(\lambda'_\blacktriangle)^F$ with $i \geq 3$ has a section in $U_i(\lambda'_\blacktriangle)^F$; we call it $\widetilde{V}_i(\lambda'_\blacktriangle)$. Then

$$\left|U_3\left(\lambda'_\blacktriangle\right)^F\right| = \prod_{i \geq 3} \left|\widetilde{V}_i\left(\lambda'_\blacktriangle\right)^F\right|. \tag{14}$$

Together (12), (13) and (14) show that

$$\begin{aligned}\left|X^\triangle(G)^F\right| &= \left|\pi^{-1}\left(V_2(\lambda'_\blacktriangle)_{ss}{}^F\right)\right| \\ &= \left(q^{\dim \mathfrak{g}(\lambda'_\blacktriangle, 2)} - \left|\mathcal{N}_{\mathfrak{g}(\lambda'_\blacktriangle, 2)}{}^F\right|\right) \cdot q^{\dim \mathfrak{g}(\lambda'_\blacktriangle, \geq 3)}.\end{aligned}$$

As a result, $|X^\triangle(G)^F| = |X^\triangle(\mathfrak{g})^F| = \psi_\mathfrak{g}^\triangle(q)$ for every $\triangle$ as above. Now (9) yields $|H^\blacktriangle(G)^F| = |H^\blacktriangle(\mathfrak{g})^F| = \varphi_\mathfrak{g}^\blacktriangle(q)$. In view of Theorem 7.3 this implies that $\mathfrak{P}_5$ holds for G. □

Remark

1. In the appendix to [29] and more recently in [28], Lusztig and Xue proposed for G classical a definition of nilpotent pieces which avoids the partial ordering of nilpotent orbits. Given $\triangle \in D_G$ choose $g \in G$ as in Sect. 7.1 and define $\mathfrak{g}_2^{\triangle!}$ to be the set of all $x = \sum_{i \geq 2} x_i \in \mathfrak{g}_2^\triangle$ with $x_i \in \mathfrak{g}(g \cdot \omega, i)$ and $C_G(x_2) \subset G_2^\triangle$. Similarly, let $(\mathfrak{g}^*)_2^{\triangle!}$ be the set of all $\xi = \sum_{i \geq 2} \xi_i \in (\mathfrak{g}^*)_2^\triangle$ with $\xi_i \in \mathfrak{g}^*(g \cdot \omega, i)$ such that the stabiliser of ξ_2 in G is contained in $G_0^\triangle$. According to the definition of Lusztig and Xue, the nilpotent pieces of $\mathfrak{g}$ and $\mathfrak{g}^*$ are

$$\left\{\mathfrak{g}_2^{\triangle!} \mid \triangle \in D_G\right\} \quad \text{and} \quad \left\{(\operatorname{Ad} G) \cdot \mathfrak{g}_2^{\triangle!} \mid \blacktriangle \in D_G/G\right\}$$

and

$$\left\{\left(\mathfrak{g}^*\right)_2^{\triangle!} \mid \triangle \in D_G\right\} \quad \text{and} \quad \left\{\left(\operatorname{Ad}^* G\right) \cdot \left(\mathfrak{g}^*\right)_2^{\triangle!} \mid \blacktriangle \in D_G/G\right\},$$

respectively, where $\triangle$ is implicitly taken to be a representative of $\blacktriangle$ in each case. Lusztig and Xue proved that for G classical these subsets stratify $\mathcal{N}_\mathfrak{g}$ and $\mathcal{N}_{\mathfrak{g}^*}$. On the other hand, Theorem 7.3 implies that $X^\triangle(\mathfrak{g}) \subseteq \mathfrak{g}_2^{\triangle!}$ and $X^\triangle(\mathfrak{g}^*) \subseteq (\mathfrak{g}_2^*)^{\triangle!}$ for every $\triangle \in D_G$. But equality must hold in each case because the blades, too, stratify the nullcones. This shows that for G classical both definitions lead to the same stratifications of $\mathcal{N}_\mathfrak{g}$ and $\mathcal{N}_{\mathfrak{g}^*}$.

2. The proof of Corollary 7 shows that for any $p > 0$ there exists a bijection between ${G_{\mathrm{uni}}}^F$ and ${\mathfrak{g}_{\mathrm{nil}}}^F$ which maps every non-empty subset $X^{\triangle}(G)^F$ onto $X^{\triangle}(\mathfrak{g})^F$ and every non-empty subset $H^{\blacktriangle}(G)^F$ onto $H^{\blacktriangle}(\mathfrak{g})^F$. Furthermore, it is immediate from Remark 2 and the definition of $X^{\triangle}(\mathcal{G})$, that there is a bijection between G_{uni} and $\mathfrak{g}_{\mathrm{nil}}$ which maps $X^{\triangle}(G)$ onto $X^{\triangle}(\mathfrak{g})$ for every $\triangle \in D_G$.
3. It follows from [41, Proposition 6(2)] that for every $\blacktriangle \in D_G/G$ there is a homogeneous regular function $f_{\blacktriangle} \in \mathbb{Z}[\mathfrak{g}'_{\mathbb{Z}}(\lambda'_{\blacktriangle}, 2)]$ invariant under the natural action of the group scheme $\mathfrak{L}^{\perp}(\lambda'_{\blacktriangle})$ and such that for any algebraically closed field $\Bbbk$ the variety $\mathcal{N}_{\mathfrak{g}(\lambda'_{\blacktriangle},2)}$ coincides with the zero locus of the image of $f_{\blacktriangle}$ in $\Bbbk[\mathfrak{g}(\lambda'_{\blacktriangle}, 2)] = \mathbb{Z}[\mathfrak{g}'_{\mathbb{Z}}(\lambda'_{\blacktriangle}, 2)] \otimes_{\mathbb{Z}} \Bbbk$; see [38, Sect. 2.4] for a related discussion.

7.4 In this subsection, we give an application of our results to the centralisers of elements in connected reductive groups. This was suggested by the first referee.

Proposition *Let g be an element of a connected reductive group G defined over an algebraically closed field of arbitrary characteristic. Then the centraliser $C_G(g)$ is reductive if and only if the element g is semisimple.*

Proof If the element g is semisimple, then it is contained in a maximal torus of G and the group $C_G(g)$ is reductive. This is standard fact of the theory of algebraic groups which follows, for instance, from the argument used in [47, I.4.1].

Now suppose that g is not semisimple. We need to show that the group $C_G(g)$ is not reductive. Write $g = s \cdot u$ for the Jordan decomposition of g in G. Then $1_G \neq u \in G_{\mathrm{uni}}$ and s is a semisimple element of $C_G(u)$. In view of our results in Sect. 5 we may assume further that $u \in X^{\triangle}$ for some $\triangle \in D_G$. Let $\mu \in Y(G)$ be as in the proof of Theorem 5.2, so that $\frac{1}{2}\mu \in \widetilde{\Delta}_u$ and $U_2(\mu) = G_2^{\triangle}$. Since μ is optimal for u, Theorem 2.4(iv) yields $C_G(u) \subset P(u) = P(\mu)$. In particular, $s \in P(u)$. As s is a semisimple element, it is contained in a maximal torus T' of $P(u)$; see [3, Corollary 11.12]. Since all optimal, primitive cocharacters of u are conjugate under $P(u)$ and $Y(T') \cap \Delta_u$ is a singleton by Theorem 2.4, we may assume without loss of generality that $\mu(\Bbbk^{\times}) \subset T'$. In other words, we may assume that $T' = T$ and $s \in L(\mu)$. Since T normalises $U_2(\mu)$ we may express u uniquely as a product of elements $x_\alpha(\lambda_\alpha) \in U_\alpha$ with $\lambda_\alpha \in \Bbbk$ and $\langle \alpha, \mu \rangle \geq 2$. The uniqueness of such a presentation implies that $\alpha(s) = 1$ whenever $\lambda_\alpha \neq 0$. This, in turn, yields $u \in C_G(s)^{\circ}$.

Since $C_G(s)^{\circ}$ is a connected reductive group, we may assume from now that $s \in Z(G)$, the centre of G. Let $\ell = \max\{\langle \alpha, \mu \rangle | \alpha \in \Sigma\}$. Since $u \in G_2^{\triangle}$ and $s \in Z(G)$, we have that $G_\ell^{\triangle} \subseteq C_G(g)$. Since $C_G(g) \subset P(u) = G_0^{\triangle}$, we now deduce that $G_\ell^\triangle \neq \{1_G\}$ is a connected normal unipotent subgroup of $C_G(g)$. Hence the group $C_G(g)$ is not reductive, as wanted. □

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