

Stability conditions and Stokes factors

Tom Bridgeland · Valerio Toledano Laredo

Dedicated to Graeme Segal, whose approach to mathematics has been an inspiration to both of us

Received: 15 October 2010 / Accepted: 10 March 2011 / Published online: 15 April 2011
© Springer-Verlag 2011

Abstract Let \mathcal{A} be the category of modules over a complex, finite-dimensional algebra. We show that the space of stability conditions on \mathcal{A} parametrises an isomonodromic family of irregular connections on \mathbb{P}^1 with values in the Hall algebra of \mathcal{A} . The residues of these connections are given by the holomorphic generating function for counting invariants in \mathcal{A} constructed by D. Joyce (in *Geom. Topol.* 11, 667–725, 2007).

Contents

1 Introduction	62
2 Irregular connections and Stokes phenomena	69
3 The Stokes map	75
4 Ringel–Hall algebras	81
5 Stability conditions and wall-crossing	88

T.B. supported by a Royal Society University Research Fellowship.
V.T.L. supported in part by NSF grants DMS-0707212 and DMS-0635607.

T. Bridgeland (✉)
All Souls College, Oxford, UK
e-mail: bridgeland@maths.ox.ac.uk

V. Toledano Laredo
Department of Mathematics, Northeastern University, 567 Lake Hall, 360 Huntington
Avenue, Boston, MA 02115, USA
e-mail: V.ToledanoLaredo@neu.edu

V. Toledano Laredo
School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

6 Stability conditions and Stokes data	93
Acknowledgements	97
References	97

1 Introduction

1.1

This paper stems from our attempt to understand the recent work of Joyce [13] on holomorphic generating functions for counting invariants in an abelian category \mathcal{A} . Somewhat unexpectedly, a more conceptual understanding of Joyce's formulae can be obtained by viewing them as defining an irregular connection on \mathbb{P}^1 with values in the Ringel–Hall Lie algebra of \mathcal{A} . This leads to a picture whereby stability conditions on \mathcal{A} can be naturally interpreted as defining Stokes data for such connections.

We begin with a leisurely introduction reviewing the salient points of Joyce's work and summarising our main results, a more precise formulation of which may be found in the body of the paper. From 1.6 on, our exposition assumes a passing knowledge of Stokes phenomena; the reader can find an introduction to this material in Sect. 2.

1.2

It is a familiar fact from Geometric Invariant Theory that if one wants to form moduli spaces parameterising algebro-geometric objects such as coherent sheaves or modules over an algebra, one first needs to restrict to some subclass of (semi)stable ones. The required notion of stability is usually not given *a priori*, but rather corresponds to some particular choice of weights. As these weights vary, the corresponding subclasses of semistable objects undergo discontinuous changes; in many cases the space of all possible weights has a wall-and-chamber decomposition such that the subclass of semistable objects is constant in each chamber but jumps as one moves across a wall.

More recently, these spaces of weights, or stability conditions, have been studied as interesting objects in their own right. Spaces of stability conditions on triangulated categories were introduced by the first author in [6] following earlier work by M. Douglas [9]. Considerations from Mirror Symmetry suggest that these spaces should have interesting geometric structures closely related to Frobenius structures [7]; in particular one expects that they should carry natural families of irregular connections. Recently, progress has been made towards defining such structures [13, 18] although the picture is still far from clear.

In this paper we shall be concerned with stability conditions on an abelian category \mathcal{A} , and in fact with the special case when \mathcal{A} is the category of finite-dimensional modules over a finite-dimensional algebra. We hope that, with further study these ideas can be applied to more general situations such as spaces of stability conditions on derived categories of coherent sheaves.

1.3

Suppose then that \mathcal{A} is the abelian category of finite-dimensional modules over a finite-dimensional associative \mathbb{C} -algebra R . Let $K(\mathcal{A})$ be the Grothendieck group of \mathcal{A} and $K_{>0}(\mathcal{A}) \subset K(\mathcal{A})$ the positive cone spanned by the classes of nonzero modules. Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half-plane. For our purposes, a *stability condition* on \mathcal{A} is just a homomorphism of abelian groups $Z : K(\mathcal{A}) \rightarrow \mathbb{C}$ such that

$$Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}.$$

In other words, a stability condition is a choice $Z(M) \in \mathbb{H}$ for each nonzero module M such that Z is additive on short exact sequences. Given such a stability condition Z each nonzero module M has a well-defined *phase*

$$\phi(M) = \frac{1}{\pi} \arg Z(M) \in (0, 1),$$

and a non-zero module M is said to be *semistable* with respect to Z if every non-zero submodule $A \subset M$ satisfies $\phi(A) \leq \phi(M)$.

Since the category \mathcal{A} has finite length, the Grothendieck group $K(\mathcal{A}) \cong \mathbb{Z}^N$ is freely generated by the classes of the simple modules, and the space of all stability conditions $\text{Stab}(\mathcal{A})$ can be identified with the complex manifold \mathbb{H}^N . It is easy to see that for each class $\alpha \in K_{>0}(\mathcal{A})$, there is a finite collection of codimension-one real submanifolds of $\text{Stab}(\mathcal{A})$ such that in each connected component of their complement the set of semistable modules of class α is constant. This is the wall-and-chamber structure referred to above. Moreover, it follows from results of King [17] that for any stability condition Z and class $\alpha \in K_{>0}(\mathcal{A})$, there is a projective scheme which is a coarse moduli space for semistable modules of type α . We can thus view this simple-minded example as a good model for studying wall-crossing phenomena.

1.4

From an algebraic perspective, these wall-crossing phenomena give rise to, and may be studied as change of bases within the *Ringel–Hall algebra* $\mathcal{H}(\mathcal{A})$ of \mathcal{A} , an idea which has its origins in the work of Reineke [21], and was greatly developed by Joyce [12, 14]. For a survey of Hall algebras over finite

fields see [26]. The variant we shall use here was sketched by Kapranov and Vasserot [16] and described in detail by Joyce [12].

Consider first the vector space $\mathcal{H}(\mathcal{A})$ of complex-valued constructible functions on the moduli stack of all R -modules. This vector space can be endowed with an associative product $*$ for which

$$(f_1 * \cdots * f_n)(M) = \int f_1(M_1/M_0) \cdots f_n(M_n/M_{n-1}) d\chi,$$

where the Euler characteristic integral is taken over the variety parameterising flags

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of M of length n . The characteristic function 1_0 of the zero module is the identity element. The resulting algebra has an obvious grading

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{\alpha \in K_{\geq 0}(\mathcal{A})} \mathcal{H}_\alpha(\mathcal{A}),$$

where $\mathcal{H}_\alpha(\mathcal{A})$ is the space of functions supported on modules of class α .

The algebra $\mathcal{H}(\mathcal{A})$ is usually much too big, and one considers instead the subalgebra $C(\mathcal{A})$ generated by the characteristic functions κ_α of the sets of modules of class α as α varies in $K_{>0}(\mathcal{A})$. The rule

$$\Delta(f)(M, N) = f(M \oplus N)$$

defines a coproduct $\Delta : C(\mathcal{A}) \rightarrow C(\mathcal{A}) \otimes C(\mathcal{A})$ on $C(\mathcal{A})$ which endows it with the structure of a cocommutative bialgebra. The corresponding Lie algebra of primitive elements $\mathfrak{n}(\mathcal{A})$ coincides with the space of functions supported on indecomposable objects, and the inclusion $\mathfrak{n}(\mathcal{A}) \subset C(\mathcal{A})$ identifies $C(\mathcal{A})$ with the universal enveloping algebra of $\mathfrak{n}(\mathcal{A})$.

The grading on $\mathcal{H}(\mathcal{A})$ induces gradings on $C(\mathcal{A})$ and $\mathfrak{n}(\mathcal{A})$. One can use the grading on $\mathfrak{n}(\mathcal{A})$ to define an extended Lie algebra $\mathfrak{b}(\mathcal{A}) = \mathfrak{h}(\mathcal{A}) \ltimes \mathfrak{n}(\mathcal{A})$ by endowing

$$\mathfrak{h}(\mathcal{A}) = \text{Hom}_{\mathbb{Z}}(K(\mathcal{A}), \mathbb{C})$$

with a trivial bracket, and setting $[Z, f] = Z(\alpha)f$ for $Z \in \mathfrak{h}(\mathcal{A})$ and $f \in \mathfrak{n}_\alpha(\mathcal{A})$.

It will also be necessary in what follows to consider the completions $\widehat{\mathfrak{n}}(\mathcal{A})$ and $\widehat{\mathfrak{b}}(\mathcal{A})$ of $\mathfrak{n}(\mathcal{A})$ and $\mathfrak{b}(\mathcal{A})$ with respect to their gradings. We shall collectively refer to these Lie algebras as Ringel–Hall Lie algebras of \mathcal{A} .

1.5

The original motivation of the present work was to understand Joyce’s remarkable paper [13], some features of which we now briefly recall.

Given a stability condition $Z \in \text{Stab}(\mathcal{A})$, the characteristic function of the set of semistable modules of a given class $\gamma \in K_{>0}(\mathcal{A})$ defines an element $\delta_\gamma \in C_\gamma(\mathcal{A})$ which plainly encodes the discontinuous behaviour of that set. Joyce defines closely related elements ϵ_α by the (finite) sum

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \alpha \\ Z(\gamma_i) \in \mathbb{R}_{>0} Z(\alpha)}} \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} * \dots * \delta_{\gamma_n},$$

and proves that they lie in the Lie algebra $\mathfrak{n}(\mathcal{A})$. Considered as a function $\text{Stab}(\mathcal{A}) \rightarrow \mathfrak{n}_\alpha(\mathcal{A})$, ϵ_α is constant on the chambers in $\text{Stab}(\mathcal{A})$ defined by α and exhibits discontinuous behaviour on their walls.

Joyce then considers, for any $\alpha \in K_{>0}(\mathcal{A})$, a generating function $f_\alpha: \text{Stab}(\mathcal{A}) \rightarrow \mathfrak{n}_\alpha(\mathcal{A})$ given by a Lie series of the form

$$f_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in K_{>0}(\mathcal{A}) \\ \alpha_1 + \dots + \alpha_n = \alpha}} F_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} * \dots * \epsilon_{\alpha_n}, \tag{1}$$

where $F_n: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ is a function on n complex variables, with $F_1 \equiv 1$. He then proves the following result (see [13, §3] for a more precise statement of (i)–(ii)).

Theorem (Joyce)

- (i) *The functions F_n can be chosen to be holomorphic with branchcuts which precisely balance the discontinuities of the ϵ_α , thus resulting in a continuous, holomorphic function f_α , independently of which algebra R (and hence which abelian category \mathcal{A}) one starts with.*
- (ii) *The functions F_n are uniquely characterised by the above requirements and a few additional mild assumptions.*
- (iii) *The functions F_n satisfy the differential equations*

$$dF_n(z_1, \dots, z_n) = \sum_{i=1}^{n-1} F_i(z_1, \dots, z_i) F_{n-i}(z_{i+1}, \dots, z_n) d \log \left(\frac{z_{i+1} + \dots + z_n}{z_1 + \dots + z_i} \right)$$

which implies that the functions f_α satisfy

$$df_\alpha = \sum_{\substack{\beta, \gamma \in K_{>0}(\mathcal{A}) \\ \beta + \gamma = \alpha}} [f_\beta, f_\gamma] d \log \gamma. \quad (2)$$

The remarkable point is that the specific jumping behaviour of the classes of semistable objects leads to universal holomorphic functions satisfying an interesting system of non-linear differential equations.

1.6

The main new idea of this paper is that a stability condition Z on \mathcal{A} can naturally be interpreted as defining *Stokes data* for an irregular connection on \mathbb{P}^1 with values in the Ringel–Hall Lie algebra $\widehat{\mathfrak{h}}(\mathcal{A})$. The discontinuous nature of the classes of semistables as Z varies corresponds to the discontinuous behaviour of the Stokes factors of an isomonodromic family of irregular connections as the Stokes rays collide and separate. Moreover, Joyce’s holomorphic functions f_α on $\text{Stab}(\mathcal{A})$ can be interpreted as defining the residues of this family of connections.

1.7

Our starting point was the observation that the differential equation (2) has the same form as the equation for isomonodromic deformations of irregular connections on \mathbb{P}^1 written down for the group $\text{GL}_n(\mathbb{C})$ by Jimbo–Miwa–Ueno [11] and extended to an arbitrary complex, reductive Lie group G by Boalch [5, Lemma 16].

In more detail, let \mathfrak{g} be the Lie algebra of G and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\Phi \subset \mathfrak{h}^*$ be the associated root system and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\text{od}}$, with $\mathfrak{g}_{\text{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ the corresponding root space decomposition. Define a meromorphic connection on the trivial G -bundle over \mathbb{P}^1 by

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt, \quad (3)$$

where $f = \sum_{\alpha \in \Phi} f_\alpha \in \mathfrak{g}_{\text{od}}$ and Z is a regular element of \mathfrak{h} . The connection ∇ has a pole of order 2 at the origin and a pole of order 1 at infinity.

The gauge equivalence class of such a connection is determined by its Stokes data. This data consists of a set of *Stokes rays*, namely the subsets of \mathbb{C} of the form $\mathbb{R}_{>0}Z(\alpha)$ for $\alpha \in \Phi$ and, for each such ray ℓ a corresponding *Stokes factor* $S_\ell \in G$. As Z varies the Stokes rays move, but if the element

$f \in \mathfrak{g}_{\text{od}}$ evolves according to the differential equation

$$df_\alpha = \sum_{\substack{\beta, \gamma \in \Phi \\ \beta + \gamma = \alpha}} [f_\beta, f_\gamma] d \log \gamma, \tag{4}$$

then the Stokes factors are locally constant, and when two rays collide or separate the product of the corresponding Stokes factors remains constant. Such deformations of ∇ are called *isomonodromic*.

1.8

The striking similarity of the differential equations (2) and (4) suggests that that the classes ϵ_α introduced by Joyce should be regarded as logarithms of Stokes factors for a connection of the form (3).

This interpretation is further corroborated by the following result of Reineke [21]. Since \mathcal{A} has finite length, all stability conditions Z on \mathcal{A} have the Harder–Narasimhan property: any non-zero module M has a unique filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M,$$

where the successive factors $F_i = M_i/M_{i-1}$ are Z -semistable and of strictly descending phases: $\phi(F_1) > \dots > \phi(F_n)$.

Using the product in $\mathcal{H}(\mathcal{A})$, this readily translates into the following identity [21, Proposition 4.8]

$$\kappa_\gamma = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1) > \dots > \phi(\gamma_n)}} \delta_{\gamma_1} * \dots * \delta_{\gamma_n}, \tag{5}$$

where κ_γ and δ_α are the elements defined in 1.4 and 1.5.

Reineke’s equation (5) can be rewritten in a more suggestive form as follows. Given a ray $\ell = \mathbb{R}_{>0} \exp(i\pi\phi) \subset \mathbb{C}^*$ with $\phi \in (0, 1)$, let SS_ℓ be the characteristic function of all semistable modules of phase ϕ (here we include the zero module). Similarly, let $1_{\mathcal{A}}$ be the function which is equal to 1 on all modules.¹ These functions define elements in the completion $\widehat{C}(\mathcal{A})$ of $C(\mathcal{A})$ with respect to its $K_{\geq 0}(\mathcal{A})$ -grading, and lie in fact in the pro-unipotent group $\widehat{N}(\mathcal{A})$ of invertible grouplike elements in $\widehat{C}(\mathcal{A})$ whose Lie algebra is $\widehat{\mathfrak{n}}(\mathcal{A})$. The relation (5) may then be rewritten as the following identity in $\widehat{N}(\mathcal{A})$

¹The function $1_{\mathcal{A}}$ should not be confused with the identity element of the Hall algebra $\mathcal{H}(\mathcal{A})$ which is the characteristic function 1_0 of the zero module.

$$1_{\mathcal{A}} = \prod_{\ell}^{\curvearrowright} \text{SS}_{\ell},$$

where the product over rays is taken in clockwise order.

The above equation is precisely what expresses the Stokes multiplier of a connection of the form (3) relative to the upper half-plane \mathbb{H} in terms of its Stokes factors S_{ℓ} . The analogy with Stokes phenomena proceeds further: there is a countable set of rays $\mathbb{R}_{>0}Z(\alpha)$ for $\alpha \in K_{>0}(\mathcal{A})$, which can collide or separate as Z varies, and to each such ray ℓ is associated an element SS_{ℓ} of a group with the property that the ordered product of these elements SS_{ℓ} remains constant.

1.9

To make these analogies precise, we study in [8] irregular connections of the form (3) with structure group an arbitrary complex, affine algebraic group G so as to encompass the pro-solvable group $\widehat{B}(\mathcal{A})$ corresponding to the Ringel–Hall Lie algebra $\widehat{\mathfrak{b}}(\mathcal{A})$. We also explicitly solve a Riemann–Hilbert problem for these connections by expressing their residue f at $t = 0$ in terms of their Stokes data.

We rely on these results to prove in this paper that the characteristic functions SS_{ℓ} of semistable objects of a given phase are the Stokes factors of a unique connection of the form (3). The residue f of this connection is given precisely by Joyce’s generating function (1). Moreover, as the stability condition varies, the connection varies isomonodromically, thus leading to a natural derivation of Joyce’s PDE as an isomonodromic deformation equation.

More precisely, let \widehat{P} be the holomorphically trivial, principal $\widehat{B}(\mathcal{A})$ -bundle on \mathbb{P}^1 . Let $Z \in \text{Stab}(\mathcal{A}) \subset \mathfrak{h}(\mathcal{A})$ be a stability condition and consider connections on \widehat{P} of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt, \quad (6)$$

where $f \in \widehat{\mathfrak{n}}(\mathcal{A})$. Our main result is the following

Theorem

- (i) *There exists a unique connection $\nabla_{\mathcal{A}, Z}$ of the form (6) with Stokes data given by either of the following equivalent conditions:*
 - (a) *the Stokes factor corresponding to a Stokes ray $\ell = \mathbb{R}_{>0} \exp(i\pi\phi)$ is the characteristic function SS_{ℓ} of Z -semistable modules of phase ϕ .*
 - (b) *The Stokes multipliers S_+ , S_- relative to the ray $r = \mathbb{R}_{>0}$ are the function $1_{\mathcal{A}}$ which takes the value 1 on every module, and the identity element 1_0 respectively.*

(ii) *The components of the residue $f = \sum_{\alpha \in K_{>0}(\mathcal{A})} f_\alpha$ of $\nabla_{\mathcal{A}, Z}$ at 0 are given by a Lie series in the elements $\{\epsilon_\beta\}$ of the form*

$$f_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in K_{>0}(\mathcal{A}) \\ \alpha_1 + \dots + \alpha_n = \alpha}} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} * \dots * \epsilon_{\alpha_n},$$

where $J_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ are holomorphic functions with branchcuts which coincide with Joyce’s functions F_n on the domain where they are holomorphic.

(iii) *As Z varies in $\text{Stab}(\mathcal{A})$, the family of connections $\nabla_{\mathcal{A}, Z}$ varies isomonodromically. In particular, $f_\alpha(Z)$ is a holomorphic function of Z and satisfies the PDE*

$$df_\alpha = \sum_{\beta + \gamma = \alpha} [f_\beta, f_\gamma] d \log \gamma.$$

1.10

We conclude with a detailed description of the contents of this paper. In Sect. 2, we review the definition of the Stokes data of an irregular connection. In Sect. 3, we state the results of [8] on the computation of the corresponding Stokes map and the Taylor series of its inverse in terms of multilogarithms. In Sect. 4, we review the construction of the Ringel–Hall algebra $\mathcal{H}(\mathcal{A})$ of an abelian category \mathcal{A} following [12]. In Sect. 5, we explain Joyce’s construction of $\mathcal{H}(\mathcal{A})$ -valued invariants which count semistable objects in \mathcal{A} . Section 6 contains our main result. We show that a stability condition A on \mathcal{A} defines Stokes data for an irregular connection on \mathbb{P}^1 with values in the Ringel–Hall Lie algebra of \mathcal{A} which varies isomonodromically with Z .

2 Irregular connections and Stokes phenomena

We review in this section the definition of Stokes data for irregular connections on \mathbb{P}^1 . Our exposition follows [8] which, in turn, is patterned on [4, 5]. Unlike [4, 5] and earlier treatments however (see, e.g. [2]), we do not restrict ourselves to connections whose structure group is reductive. We consider instead the case of an arbitrary algebraic group since this larger class encompasses the pro-solvable Ringel–Hall groups of abelian categories.

2.1 Algebraic groups

By an *algebraic group*, we shall always mean an affine algebraic group G over \mathbb{C} . By a *finite-dimensional representation* of G , we shall mean a rational representation, that is a morphism $G \rightarrow \text{GL}(V)$, where V is a finite-dimensional complex vector space. An algebraic group always possesses a

faithful finite-dimensional representation and may therefore be regarded as a linear algebraic group, that is a (Zariski) closed subgroup of some $GL(V)$, see e.g. [10].

Let \mathfrak{g} be the Lie algebra of G . If $\rho : G \rightarrow GL(V)$ is a finite-dimensional representation, we denote its differential $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ by the same symbol.

2.2

Let G be an algebraic group, $H \subset G$ a maximal torus in G and $\mathfrak{g}, \mathfrak{h}$ their Lie algebras. The following are the examples which will be most relevant to us

- (i) $G = GL_n(\mathbb{C})$ and H is the torus consisting of diagonal, invertible matrices.
- (ii) G is a complex, semisimple Lie group and $H \subset G$ is a maximal torus.
- (iii) $G = H \ltimes N$, where H is a torus acting on a unipotent group N .

As outlined in the Introduction, and further explained in Sect. 4, case (iii) arises naturally when studying the abelian category $\mathcal{A} = \text{Mod}(R)$ of finite-dimensional representations of a finite-dimensional algebra R over \mathbb{C} . In that case, $N = \widehat{N}(\mathcal{A})$ is the (pro-)unipotent group whose Lie algebra is the Ringel–Hall Lie algebra $\widehat{\mathfrak{n}}(\mathcal{A})$ of \mathcal{A} and H is the torus whose character lattice is the Grothendieck group $K(\mathcal{A})$.

2.3

Let $\mathbb{X}(H)$ be the group of characters of H and $\mathbb{X}(H) \cong \Lambda \subset \mathfrak{h}^*$ the lattice spanned by the differentials of elements in $\mathbb{X}(H)$. For any $\lambda \in \Lambda$ we denote the unique element of $\mathbb{X}(H)$ with differential λ by e^λ .

Decompose \mathfrak{g} as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\text{od}} \quad \text{with } \mathfrak{g}_{\text{od}} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \tag{7}$$

where $\Phi \subset \Lambda \setminus \{0\}$ is a finite subset and H acts on \mathfrak{g}_α via the character e^α . We refer to the elements of Φ as the roots of G . Since H is a maximal torus, Φ is independent of the choice of H .

2.4 The irregular connection ∇

Let P be the holomorphically trivial, principal G -bundle on \mathbb{P}^1 and consider the meromorphic connection on P given by

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt, \tag{8}$$

where $Z, f \in \mathfrak{g}$.

The connection ∇ has a pole of order 2 at $t = 0$ and a pole of order 1 at ∞ . We henceforth make the following assumptions:

- (Z) $Z \in \mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi} \text{Ker}(\alpha)$ is a regular element of \mathfrak{h} .
- (f) $f \in \mathfrak{g}_{\text{od}} \subset \mathfrak{g}$.

The reader unfamiliar with algebraic groups may wish to consider the case when $G = \text{GL}_n(\mathbb{C})$ and Z is a diagonal matrix with distinct eigenvalues. Condition (f) is then the statement that the diagonal entries of the matrix f are zero.

2.5 Stokes rays and sectors

Definition A ray is a subset of \mathbb{C}^* of the form $\mathbb{R}_{>0} \exp(i\pi\phi)$. The Stokes rays of the connection ∇ are the rays $\mathbb{R}_{>0}Z(\alpha)$, $\alpha \in \Phi$. The Stokes sectors are the open regions of \mathbb{C}^* bounded by them. A ray is called *admissible* if it is not a Stokes ray.

2.6 Canonical fundamental solutions

The Stokes data of the connection ∇ are defined using fundamental solutions with prescribed asymptotics. We first recall how these are characterised.

Given a ray r in \mathbb{C} , we denote by \mathbb{H}_r the corresponding half-plane

$$\mathbb{H}_r = \{z = uv : u \in r, \text{Re}(v) > 0\} \subset \mathbb{C}^*. \tag{9}$$

The following basic result is well-known for $G = \text{GL}_n(\mathbb{C})$ and Z regular (see, e.g. [28, pp. 58–61]) and was extended in [5] to the case of complex reductive groups. It is proved in [8] for an arbitrary algebraic group.

Theorem *Given an admissible ray r , there is a unique holomorphic function $Y_r : \mathbb{H}_r \rightarrow G$ such that*

$$\frac{dY_r}{dt} = \left(\frac{Z}{t^2} + \frac{f}{t} \right) Y_r, \tag{10}$$

$$Y_r \cdot e^{Z/t} \rightarrow 1 \quad \text{as } t \rightarrow 0 \text{ in } \mathbb{H}_r. \tag{11}$$

2.7

The uniqueness part of Theorem 2.6 relies upon the following statement which is proved in [8] (see [5, Lemma 22] for the case of G reductive).

Proposition *Let r, r' be two rays such that $r \neq -r'$, and $g \in G$ an element such that*

$$e^{-Z/t} \cdot g \cdot e^{Z/t} \rightarrow 1 \quad \text{as } t \rightarrow 0 \text{ in } \mathbb{H}_r \cap \mathbb{H}_{r'}.$$

Then, $g = \exp(X)$ where X lies in

$$\bigoplus_{\alpha: Z(\alpha) \in \overline{\Sigma}(r, r')} \mathfrak{g}_\alpha \subset \mathfrak{g},$$

with $\overline{\Sigma}(r, r') \subset \mathbb{C}^*$ the closed convex sector bounded by r and r' .

Proposition 2.7 implies in particular that if the rays r, r' are admissible and such that the sector $\overline{\Sigma}(r, r')$ does not contain any Stokes rays of ∇ , the element $g \in G$ determined by

$$Y_r(t) = Y_{r'}(t) \cdot g \quad \text{for } t \in \mathbb{H}_r \cap \mathbb{H}_{r'}$$

is equal to 1.

2.8 Stokes factors

Assume now that ℓ is a Stokes ray. Let r_\pm be small clockwise (resp. anti-clockwise) perturbations of ℓ such that the convex sector $\overline{\Sigma}(r_-, r_+)$ does not contain any Stokes rays of ∇ other than ℓ .

Definition The *Stokes factor* S_ℓ corresponding to ℓ is the element of G defined by

$$Y_{r_+}(t) = Y_{r_-}(t) \cdot S_\ell \quad \text{for } t \in \mathbb{H}_{r_+} \cap \mathbb{H}_{r_-}.$$

By Proposition 2.7, the definition of S_ℓ is independent of the choice of r_\pm .

2.9 Stokes multipliers

An alternative but closely related system of invariants are the Stokes multipliers of the connection ∇ . These depend on a choice of a ray r such that both r and $-r$ are admissible.

Definition The *Stokes multipliers* of ∇ corresponding to r are the elements $S_\pm \in G$ defined by

$$Y_{r,\pm}(t) = Y_{-r}(t) \cdot S_\pm, \quad t \in \mathbb{H}_{-r}$$

where $Y_{r,+}$ and $Y_{r,-}$ are the analytic continuations of Y_r to \mathbb{H}_{-r} in the anti-clockwise and clockwise directions respectively.

Note that the multipliers S_\pm remain constant under a perturbation of r so long as r and $-r$ do not cross any Stokes rays.

2.10

To relate the Stokes factors and multipliers, set $r = \mathbb{R}_{>0} \exp(i\pi\theta)$ and label the Stokes rays as $\ell_j = \mathbb{R}_{>0} \exp(i\pi\phi_j)$, with $j = 1, \dots, m_1 + m_2$ where

$$\theta < \phi_1 < \dots < \phi_{m_1} < \theta + 1 < \phi_{m_1+1} < \dots < \phi_{m_1+m_2} < \theta + 2.$$

The following result is immediate upon drawing a picture

Lemma *The following holds*

$$S_+ = S_{\ell_{m_1}} \cdots S_{\ell_1} \quad \text{and} \quad S_- = S_{\ell_{m_1+1}}^{-1} \cdots S_{\ell_{m_1+m_2}}^{-1}.$$

The Stokes factors therefore determine the Stokes multipliers for any ray r . In fact, conversely, the Stokes multipliers for a single ray r determine all the Stokes factors, although this is not so easy to see. It will follow from Proposition 3.9 below.

2.11 Isomonodromic families of connections

We discuss below isomonodromic deformations of ∇ . Their main interest from our point of view, indeed one of the starting points of the present work, is the isomonodromy equations (12) which bear a striking resemblance to the non-linear system of PDEs (2) appearing in Joyce’s work [13].

Let $\mathcal{U} \subset \mathfrak{h}_{\text{reg}}$ be an open set and consider a family of connections of the form

$$\nabla(Z) = d - \left(\frac{Z}{t^2} + \frac{f(Z)}{t} \right) dt,$$

where Z varies in \mathcal{U} and the dependence of $f(Z) \in \mathfrak{g}_{\text{od}}$ with respect to Z is arbitrary.

Definition The family of connections $\nabla(Z)$ is *isomonodromic* if for any $Z_0 \in \mathcal{U}$, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ and a ray r such that $\pm r$ are admissible for all $\nabla(Z)$, $Z \in \mathcal{U}_0$ and the Stokes multipliers $S_{\pm}(Z)$ of $\nabla(Z)$ relative to r are constant on \mathcal{U}_0 .

The isomonodromy of the family $\nabla(Z)$ may also be defined as the constancy of the Stokes factors. This requires a little more care since, as pointed out in [3, p. 190] for example, Stokes rays may split into distinct rays under arbitrarily small deformations of Z .

Call a sector $\Sigma \subset \mathbb{C}^*$ *admissible* if its boundary rays are admissible.

Proposition *The family of connections $\nabla(Z)$ is isomonodromic if, and only if, for any connected open subset $\mathcal{U}_0 \subset \mathcal{U}$ and any convex sector Σ which is admissible for all $\nabla(Z)$, $Z \in \mathcal{U}_0$, the clockwise product*

$$\prod_{\ell \subset \Sigma}^{\curvearrowright} S_{\ell}(Z)$$

of Stokes factors corresponding to the Stokes rays contained in Σ is constant on \mathcal{U}_0 .

Proof This follows from the fact that Stokes factors and multipliers determine each other by Lemma 2.10 and Proposition 3.9. □

2.12 Isomonodromy equations

The following characterisation of isomonodromic deformations was obtained by Jimbo–Miwa–Ueno in the case $G = GL_n$ [11] and adapted to the case of a complex, reductive group by Boalch [3, Appendix]. Its proof carries over verbatim to the case of a general algebraic group.

Theorem *Assume that f varies holomorphically in Z . Then, family of connections $\nabla(Z)$ is isomonodromic if, and only if f satisfies the PDE*

$$df_{\alpha} = \sum_{\substack{\beta, \gamma \in \Phi \\ \beta + \gamma = \alpha}} [f_{\beta}, f_{\gamma}] d \log \gamma. \tag{12}$$

Remark Equations (12) form a first order system of *integrable* non-linear PDEs and therefore possess a unique holomorphic solution $f(Z)$ defined in a neighbourhood of a fixed $Z_0 \in \mathfrak{h}_{\text{reg}}$ and subject to the initial condition $f(Z_0) = f_0 \in [\mathfrak{h}, \mathfrak{g}]$.

Remark Jimbo–Miwa–Ueno and Boalch also give an alternative characterisation of isomonodromy in terms of the existence of a flat connection on $\mathbb{P}^1 \times U$ which has a logarithmic singularity on the divisor $\{t = \infty\}$ and a pole of order 2 on $\{t = 0\}$, and restricts to $\nabla(Z)$ on each fibre $\{Z\} \times \mathbb{P}^1$. This connection is given by

$$\bar{\nabla} = d - \left[\left(\frac{Z}{t^2} + \frac{f}{t} \right) dt + \sum_{\alpha \in \Phi} f_{\alpha} \frac{d\alpha}{\alpha} + \frac{dZ}{t} \right].$$

One can check directly that the flatness of this connection is equivalent to (12).

3 The Stokes map

In this section we state the main results of [8]. These express the logarithms of the Stokes factors of the connection ∇ as explicit, universal Lie series in the variables f_α and, conversely, the f_α as Lie series in the logarithms of the Stokes factors, thus explicitly solving a Riemann–Hilbert problem.

3.1 Completion with respect to finite-dimensional representations

Our formulae are more conveniently expressed inside the completion $\widehat{U\mathfrak{g}}$ of $U\mathfrak{g}$ with respect to the finite-dimensional representations of G . We review below the definition of $\widehat{U\mathfrak{g}}$.

Let Vec be the category of finite-dimensional complex vector spaces and $\text{Rep}(G)$ that of finite-dimensional representations of G . Consider the forgetful functor

$$F : \text{Rep}(G) \rightarrow \text{Vec}.$$

By definition, $\widehat{U\mathfrak{g}}$ is the algebra of endomorphisms of F . Concretely, an element of $\widehat{U\mathfrak{g}}$ is a collection $\Theta = \{\Theta_V\}$, with $\Theta_V \in \text{End}_{\mathbb{C}}(V)$ for any $V \in \text{Rep}(G)$, such that for any $U, V \in \text{Rep}(G)$ and $T \in \text{Hom}_G(U, V)$, the following holds

$$\Theta_V \circ T = T \circ \Theta_U.$$

There are natural homomorphisms $U\mathfrak{g} \rightarrow \widehat{U\mathfrak{g}}$ and $G \rightarrow \widehat{U\mathfrak{g}}$ mapping $x \in U\mathfrak{g}$ and $g \in G$ to the elements $\Theta(x), \Theta(g)$ which act on a finite-dimensional representation $\rho : G \rightarrow \text{GL}(V)$ as $\rho(x)$ and $\rho(g)$ respectively. These homomorphisms are well-known to be injective (see e.g. [8, Lemma 4.1]) and we will use them to think of $U\mathfrak{g}$ as a subalgebra of $\widehat{U\mathfrak{g}}$ and G as a subgroup of the group of invertible elements of $\widehat{U\mathfrak{g}}$ respectively.

3.2 Representing Stokes factors

Fix a Stokes ray ℓ of the connection ∇ . We show below how to represent the corresponding Stokes factor S_ℓ in two different ways: by elements $\epsilon_\alpha \in \mathfrak{g}_{\text{od}}$ and by elements $\delta_\gamma \in U\mathfrak{g}$.

Consider the subalgebra

$$\mathfrak{n}_\ell = \bigoplus_{\alpha: Z(\alpha) \in \ell} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

The elements of \mathfrak{n}_ℓ are nilpotent, that is they act by nilpotent endomorphisms on any finite-dimensional representation of G . It follows that the exponential map $\exp : \mathfrak{n}_\ell \rightarrow G$ is an isomorphism onto the unipotent subgroup $N_\ell = \exp(\mathfrak{n}_\ell) \subset G$.

By Proposition 2.7, the Stokes factor S_ℓ lies in N_ℓ . For the first representation of S_ℓ , write

$$S_\ell = \exp\left(\sum_{\alpha: Z(\alpha) \in \ell} \epsilon_\alpha\right) \tag{13}$$

for uniquely defined elements $\epsilon_\alpha \in \mathfrak{g}_\alpha$.

For the second, we compute the exponential (13) in $\widehat{U}\mathfrak{g}$ and decompose the result along the weight spaces

$$\widehat{U}\mathfrak{g}_\gamma = \{x \in \widehat{U}\mathfrak{g} \mid \text{ad}(h)x = \gamma(h)x, \forall h \in \mathfrak{h}\}, \quad \gamma \in \mathfrak{h}^*$$

of the adjoint action of \mathfrak{h} . This yields elements $\delta_\gamma \in (U\mathfrak{n}_\ell)_\gamma$ such that

$$S_\ell = 1 + \sum_{\gamma \in \Lambda: Z(\gamma) \in \ell} \delta_\gamma, \tag{14}$$

where $\Lambda \subset \mathfrak{h}^*$ is the lattice generated by the set of roots Φ and the above identity is to be understood as holding in any finite-dimensional representation of G .

These two representations of S_ℓ are related as follows.

Lemma

(i) *Let $\gamma \in \Lambda$ be such that $Z(\gamma)$ lies on the Stokes ray ℓ . Then, δ_γ is given by the finite sum*

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi \\ Z(\alpha_i) \in \ell, \\ \alpha_1 + \dots + \alpha_n = \gamma}} \frac{1}{n!} \epsilon_{\alpha_1} \dots \epsilon_{\alpha_n}. \tag{15}$$

(ii) *Conversely, let $\alpha \in \Phi$ be such that $Z(\alpha) \in \ell$. Then, ϵ_α is given by the finite sum*

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\gamma_i \in \Lambda \\ Z(\gamma_i) \in \ell, \\ \gamma_1 + \dots + \gamma_n = \alpha}} \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} \dots \delta_{\gamma_n}. \tag{16}$$

Proof These are the standard expansions of $\exp: \mathfrak{n}_\ell \rightarrow N_\ell$ and $\log: N_\ell \rightarrow \mathfrak{n}_\ell$. □

3.3 Formula for the Stokes factors

We now give an explicit formula for the Stokes factors of the connection ∇ in terms of iterated integrals.

Definition Set $M_1(z_1) = 2\pi i$ and, for $n \geq 2$, define the function $M_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ by the iterated integral

$$M_n(z_1, \dots, z_n) = 2\pi i \int_C \frac{dt}{t - s_1} \circ \dots \circ \frac{dt}{t - s_{n-1}},$$

where $s_i = z_1 + \dots + z_i$, $1 \leq i \leq n$ and the path of integration C is the line segment $[0, s_n]$, perturbed if necessary to avoid any point $s_i \in [0, s_n]$ by small clockwise arcs.

Theorem ([8]) *The weight components δ_γ of the Stokes factor S_ℓ corresponding to the ray ℓ are given by*

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi \\ \alpha_1 + \dots + \alpha_n = \gamma}} M_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \dots f_{\alpha_n}, \quad (17)$$

where the equality is to be understood as holding in any finite-dimensional representation of G and the sum over n is absolutely convergent.

3.4 The Stokes map

Since the sets $\{\alpha \in \Phi : Z(\alpha) \in \ell\}$ partition Φ as ℓ ranges over the Stokes rays of ∇ , we may assemble the elements ϵ_α corresponding to different Stokes rays and form the sum

$$\epsilon = \sum_{\alpha \in \Phi} \epsilon_\alpha \in \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

For fixed $Z \in \mathfrak{h}$, we shall refer to the map

$$S : \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \longrightarrow \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

mapping f to ϵ as the *Stokes map*.

3.5 Formula for the Stokes map

We next state a formula for the Stokes map giving the element ϵ in terms of f . We first define the special functions appearing in this formula.

Definition The function $L_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ is given by $L_1(z_1) = 2\pi i$ and, for $n \geq 2$,

$$L_n(z_1, \dots, z_n) = \sum_{k=1}^n \sum_{\substack{0=i_0 < \dots < i_k=n \\ s_{i_j} - s_{i_{j-1}} \in \mathbb{R}_{>0} \cdot s_n}} \frac{(-1)^{k-1}}{k} \prod_{j=0}^{k-1} M_{i_{j+1}-i_j}(z_{i_j+1}, \dots, z_{i_{j+1}}),$$

where $s_j = z_1 + \dots + z_j$.

Remark Note that on the open subset

$$(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \quad \text{such that} \quad s_i \notin [0, s_n] \quad \text{for } 0 < i < n$$

the inner sum above is empty unless $k = 1$ and one therefore has

$$L_n(z_1, \dots, z_n) = M_n(z_1, \dots, z_n).$$

Thus L_n agrees with M_n on the open subset where it is holomorphic and differs by how it has been extended onto the cutlines.

3.6

The functions L_n are more complicated to define than the functions M_n . Unlike the latter however, they give rise to Lie series as the following result shows.

Theorem ([8])

- (i) Let x_1, \dots, x_n be elements in a Lie algebra \mathcal{L} . For any $(z_1, \dots, z_n) \in (\mathbb{C}^*)^n$, the finite sum

$$\sum_{\sigma \in \text{Sym}_n} L_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}) x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is a Lie polynomial in x_1, \dots, x_n and therefore lies in $\mathcal{L} \subset U\mathcal{L}$.

- (ii) The element $\epsilon = \mathcal{S}(f)$ is given by a Lie series in the variables $\{f_\alpha\}_{\alpha \in \Phi}$, given by

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi \\ \alpha_1 + \dots + \alpha_n = \alpha}} L_n(Z(\alpha_1), \dots, Z(\alpha_n)) f_{\alpha_1} f_{\alpha_2} \cdots f_{\alpha_n}. \quad (18)$$

As a series in n , (18) converges uniformly on compact subsets of \mathfrak{g}_{od} .

3.7 Inverse of the Stokes map

By Theorem 3.6, the Stokes map $\mathcal{S} : \mathfrak{g}_{\text{od}} \rightarrow \mathfrak{g}_{\text{od}}$ is holomorphic, satisfies $\mathcal{S}(0) = 0$ and its differential at $f = 0$ is invertible. By the inverse function Theorem, \mathcal{S} possesses an analytic inverse \mathcal{S}^{-1} defined on a neighborhood of $\epsilon = 0$.

Theorem ([8]) *The Taylor series of \mathcal{S}^{-1} at $\epsilon = 0$ is given by a Lie series in the variables $\{\epsilon_\alpha\}_{\alpha \in \Phi}$ of the form*

$$f_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_i \in \Phi \\ \alpha_1 + \dots + \alpha_n = \alpha}} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} \epsilon_{\alpha_2} \dots \epsilon_{\alpha_n} \tag{19}$$

for some special functions $J_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$.

Remark It follows from Theorem 3.7 that whenever the sum (19) is absolutely convergent over n , it inverts the Stokes map \mathcal{S} , in that the connection (8) determined by $f = \sum_\alpha f_\alpha$ has Stokes factors given by (13).

3.8 The functions J_n

The functions J_n appearing in Theorem 3.7 are explicitly described in [8] as sums of products of the functions L_n indexed by plane rooted trees. For example

$$\begin{aligned} &(2\pi i)^3 J_3(z_1, z_2, z_3) \\ &= L_2(z_1, z_2)L_2(z_1 + z_2, z_3) - L_3(z_1, z_2, z_3) + L_2(z_1, z_2 + z_3)L_2(z_2, z_3) \end{aligned}$$

corresponding to the three distinct plane rooted trees with 3 leaves.

Theorem ([8]) *The function $J_n : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$ is continuous and holomorphic on the complement of the hyperplanes*

$$H_{ij} = \{z_i + \dots + z_j = 0\}, \quad 1 \leq i < j \leq n$$

in the domain

$$\mathcal{D}_n = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \mid z_i/z_{i+1} \notin \mathbb{R}_{>0} \text{ for } 1 \leq i < n\}$$

Moreover, it satisfies the differential equation

$$\begin{aligned} &dJ_n(z_1, \dots, z_n) \\ &= \sum_{i=1}^{n-1} J_i(z_1, \dots, z_i) J_{n-i}(z_{i+1}, \dots, z_n) d \log \left(\frac{z_{i+1} + \dots + z_n}{z_1 + \dots + z_i} \right) \end{aligned}$$

together with the conditions $J_1(z) = 1/2\pi i$ and

$$J_n(z_1, \dots, z_n) = 0 \quad \text{if } z_1 + \dots + z_n = 0$$

for $n \geq 2$.

Remark It follows from Theorem 3.8 that the functions J_n are the same as the functions F_n appearing in Joyce’s paper [13] and alluded to in the Introduction, at least on the dense open subset where they are holomorphic.

3.9 Representing Stokes multipliers

We next show how to represent the Stokes multipliers S_{\pm} by an element $\kappa \in \widehat{U}\mathfrak{g}$.

Let $r = \mathbb{R}_{>0}e^{i\pi\theta}$ be the ray with respect to which S_{\pm} are defined and $\pm i\mathbb{H}_r$ the connected components of $\mathbb{C} \setminus \mathbb{R}e^{i\pi\theta}$. These determine a partition of $\Phi = \Phi_+ \sqcup \Phi_-$ given by

$$\Phi_{\pm} = \{\alpha \in \Phi : Z(\alpha) \in \pm i\mathbb{H}_r\}.$$

Let $\Lambda_{\pm} \subset \mathfrak{h}^* \setminus \{0\}$ be the cones spanned by the linear combinations of elements in Φ_{\pm} with coefficients in $\mathbb{N}_{>0}$. Similarly to Sect. 3.2, it follows from Proposition 2.7 that there is a unique element

$$\kappa = \sum_{\gamma \in \Lambda_+ \sqcup \Lambda_-} \kappa_{\gamma} \in \widehat{U}\mathfrak{g}$$

such that the Stokes multipliers S_{\pm} are equal to

$$S_+ = 1 + \sum_{\gamma \in \Lambda_+} \kappa_{\gamma}, \quad (S_-)^{-1} = 1 + \sum_{\gamma \in \Lambda_-} \kappa_{\gamma}.$$

Given $\gamma \in \Lambda_+$, set

$$\phi(\gamma) = \frac{1}{\pi} \arg Z(\gamma) \in (\theta, \theta + 1).$$

The following result gives the relation between the elements κ and δ .

Proposition

(i) For all $\gamma \in \Lambda_+$, there is a finite sum

$$\kappa_{\gamma} = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1) > \dots > \phi(\gamma_n)}} \delta_{\gamma_1} \dots \delta_{\gamma_n}, \tag{20}$$

where the sum is over elements $\gamma_i \in \Lambda_+$.

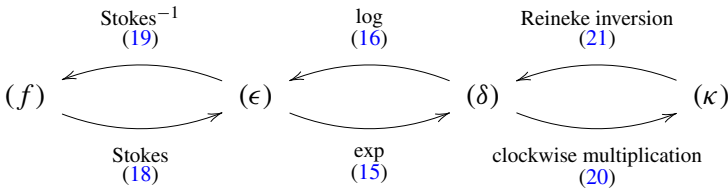
(ii) Conversely, for $\gamma \in \Lambda_+$

$$\delta_{\gamma} = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1 + \dots + \gamma_i) > \phi(\gamma)}} (-1)^{n-1} \kappa_{\gamma_1} \dots \kappa_{\gamma_n}. \tag{21}$$

Proof (i) follows from substituting (14) into the formula of Lemma 2.10. (ii) follows from Reineke’s inversion of formula (20) [21, Sect. 5]. \square

3.10

The following diagram summarizes the relationships between the elements δ, ϵ representing the Stokes factors, the element κ representing the Stokes multipliers, and the element $f \in \mathfrak{g}_{\text{od}}$ defining ∇ .



These related systems of invariants will appear again in Sect. 5 in the context of stability conditions on abelian categories.

4 Ringel–Hall algebras

In this section, we review the definition of the Hall algebra of an abelian category \mathcal{A} . We shall in fact restrict ourselves to the case where $\mathcal{A} = \text{Mod}(R)$ is the category of finite-dimensional, left modules over a fixed finite-dimensional, associative \mathbb{C} -algebra R . It should be possible to generalize our main results to include other abelian categories, for example categories of coherent sheaves, but rather than working in maximal generality we prefer to focus on a case where the underlying ideas are not clouded by technical issues.

4.1 The Grothendieck group $K(\mathcal{A})$

Since the category $\mathcal{A} = \text{Mod}(R)$ has finite length and finitely many simple modules S_1, \dots, S_N , up to isomorphism, the Grothendieck group $K(\mathcal{A})$ is a free abelian group of finite rank generated by the classes $[S_i]$

$$K(\mathcal{A}) = \mathbb{Z}[S_1] \oplus \dots \oplus \mathbb{Z}[S_N].$$

The positive and non-negative cones $K_{>0}(\mathcal{A}) \subset K_{\geq 0}(\mathcal{A}) \subset K(\mathcal{A})$ are defined by

$$K_{>0}(\mathcal{A}) = \{[M] : 0 \neq M \in \mathcal{A}\} \quad \text{and} \quad K_{\geq 0}(\mathcal{A}) = K_{>0}(\mathcal{A}) \sqcup \{0\}.$$

4.2 The Ringel–Hall algebra

There are many variants of the Hall algebra of \mathcal{A} , see for example [26] for a survey of Hall algebras over finite fields. We shall work over \mathbb{C} using constructible functions, an idea originally due to Schofield [27] and later taken up by Lusztig [19] and Riedtmann [23]. The precise construction we use was sketched by Kapranov and Vasserot [16] and described in detail by Joyce [12].

Recall that a complex-valued function $f: X \rightarrow \mathbb{C}$ on a variety X is *constructible* if it is of the form

$$f = \sum_{i=1}^k a_i 1_{Y_i}$$

for complex numbers a_1, \dots, a_k and locally-closed subvarieties $Y_i \subset X$. Such a function can be integrated by using the Euler characteristic as a measure [20]. By definition

$$\int_X f d\chi = \sum_{i=1}^k a_i \chi(Y_i),$$

where $\chi(Z)$ is the topological Euler characteristic of a complex variety Z endowed with the analytic topology.

Given an integer $d \geq 0$, there is an affine variety Rep_d parameterising R -module structures on the vector space \mathbb{C}^d . The moduli stack \mathcal{M}_d of R -modules of dimension d is the quotient

$$\mathcal{M}_d = \text{Rep}_d / \text{GL}_d(\mathbb{C}).$$

By definition, a constructible function on \mathcal{M}_d is a $\text{GL}_d(\mathbb{C})$ -equivariant constructible function on the affine variety Rep_d

We define $\mathcal{H}_d(\mathcal{A})$ to be the space of constructible functions on \mathcal{M}_d and set

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{H}_d(\mathcal{A}). \quad (22)$$

Note that elements of $\mathcal{H}(\mathcal{A})$ can be thought of as functions on modules that are constant on isomorphism classes. Given $f \in \mathcal{H}(\mathcal{A})$, we denote its value on a module M by $f(M)$. We say that an element of $\mathcal{H}(\mathcal{A})$ is *supported* on a certain class of modules to mean that its value on all other modules is zero.

Theorem (Kapranov-Vasserot [16], Joyce [12]) *There is an associative product*

$$*: \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{A})$$

for which

$$(f_1 * \cdots * f_n)(M) = \int f_1(M_1/M_0) \cdots f_n(M_n/M_{n-1}) d\chi,$$

where the integral is over the variety $\text{Flag}^n(M)$ parameterising flags

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of M of length n . The characteristic function of the zero module 1_0 is a unit for this multiplication.

Proof The case $n = 2$ is proved in [12, Theorem 4.3]. Joyce considers the morphism $\sigma(\{1, 2\})$ from the stack of short exact sequences in \mathcal{A} to the stack of objects of \mathcal{A} which on geometric points takes a short exact sequence of modules

$$0 \longrightarrow A \longrightarrow M \longrightarrow B \longrightarrow 0$$

to the module M . Note that this morphism induces injections on stabilizer groups since an isomorphism of short exact sequences is determined by its action on the middle term. Thus Joyce’s pushforward map on constructible functions is just given by integrating along the fibres of the morphism $\sigma(\{1, 2\})$ which are precisely the varieties $\text{Flag}^2(M)$ of the statement. The extension to the product of n elements follows by an induction argument. \square

Note that the algebra $\mathcal{H}(\mathcal{A})$ is graded by $K_{\geq 0}(\mathcal{A})$:

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{\gamma \in K_{\geq 0}(\mathcal{A})} \mathcal{H}_\gamma(\mathcal{A}),$$

where $\mathcal{H}_\gamma(\mathcal{A})$ is the subspace of functions supported on modules of class γ . This grading is a refinement of the $\mathbb{Z}_{\geq 0}$ -grading given by (22) via the homomorphism $K_{\geq 0}(\mathcal{A}) \rightarrow \mathbb{Z}_{\geq 0}$ mapping γ to the dimension of the modules of class γ .

4.3 The bialgebra $C(\mathcal{A})$

If the moduli stack \mathcal{M}_d has positive dimension, the space $\mathcal{H}_d(\mathcal{A})$ of constructible functions on it is very large since it contains the characteristic functions of points. It is therefore usual to consider a subalgebra generated by some natural set of elements.

For each $\gamma \in K_{\geq 0}(\mathcal{A})$, let $\kappa_\gamma \in \mathcal{H}_\gamma(\mathcal{A})$ be the characteristic function of the set of modules of class γ

$$\kappa_\gamma(M) = \begin{cases} 1 & \text{if } [M] = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

This is a constructible function because the class of a module in $K(\mathcal{A})$ is constant in families. Let $C(\mathcal{A}) \subset \mathcal{H}(\mathcal{A})$ be the subalgebra generated by the elements κ_γ :

$$C(\mathcal{A}) = \langle \kappa_\gamma : \gamma \in K_{\geq 0}(\mathcal{A}) \rangle \subset \mathcal{H}(\mathcal{A})$$

and note that $C(\mathcal{A})$ is a $\mathbb{Z}_{\geq 0}$ -graded (and, *a fortiori*, a $K_{\geq 0}(\mathcal{A})$ -graded) algebra with finite-dimensional homogeneous components.

The algebra $C(\mathcal{A})$ possesses the structure of a bialgebra. To see this, note first that the tensor product $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$ embeds into $\mathcal{H}(\mathcal{A} \times \mathcal{A})$ by setting

$$(f \otimes g)(M, N) = f(M)g(N).$$

Define a map $\Delta : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{A} \times \mathcal{A})$ by

$$\Delta(f)(M, N) = f(M \oplus N).$$

The image of Δ need not be contained in $\mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A}) \subset \mathcal{H}(\mathcal{A} \times \mathcal{A})$ in general. The following result is due to Joyce [12] and builds upon earlier work of Ringel [25].

Theorem *The map Δ restricts to a coassociative coproduct*

$$\Delta : C(\mathcal{A}) \rightarrow C(\mathcal{A}) \otimes C(\mathcal{A}),$$

preserving the $K_{\geq 0}(\mathcal{A})$ grading. The homomorphism $\eta : C(\mathcal{A}) \rightarrow \mathbb{C}$ given by evaluation on the zero module $\eta(f) = f(0)$ is a counit for Δ . The data $(, 1, \Delta, \eta)$ endows $C(\mathcal{A})$ with the structure of a cocommutative bialgebra.*

Proof This follows from the proof of [12, Theorem 4.20] using the fact that

$$\Delta(\kappa_\gamma) = \sum_{\gamma_1 + \gamma_2 = \gamma} \kappa_{\gamma_1} \otimes \kappa_{\gamma_2},$$

which is immediately verified by evaluating on a pair of modules (M, N) . \square

4.4 The Ringel–Hall Lie algebra

Recall that an element f of a bialgebra is *primitive* if $\Delta(f) = f \otimes 1 + 1 \otimes f$, and that the subspace of such elements is a Lie algebra under the commutator bracket. Recall also that a module $M \in \mathcal{A}$ is *indecomposable* if

$$M = N \oplus P \implies N = 0 \text{ or } P = 0.$$

In particular, the zero module is indecomposable.

Lemma *An element $f \in C(\mathcal{A})$ is primitive if, and only if it is supported on nonzero indecomposable modules.*

Proof According to the definition of the coproduct the primitive elements of $C(\mathcal{A})$ are those satisfying $f(M \oplus N) = f(M)1_0(N) + 1_0(M)f(N)$. In particular if M and N are nonzero then $f(M \oplus N) = 0$. Moreover $f(0) = f(0) + f(0)$ so $f(0) = 0$. Hence f is supported on nonzero indecomposable modules. The converse is easily checked. \square

We write $\mathfrak{n}(\mathcal{A})$ for the subspace of $C(\mathcal{A})$ consisting of primitive elements. Thus $\mathfrak{n}(\mathcal{A})$ is a Lie algebra which we call the *Ringel–Hall Lie algebra* of \mathcal{A} . Note that the grading on $C(\mathcal{A})$ induces a grading

$$\mathfrak{n}(\mathcal{A}) = \bigoplus_{\alpha \in K_{>0}(\mathcal{A})} \mathfrak{n}_\alpha(\mathcal{A}).$$

One can use the grading of $\mathfrak{n}(\mathcal{A})$ to form a larger Lie algebra $\mathfrak{b}(\mathcal{A}) = \mathfrak{h}(\mathcal{A}) \oplus \mathfrak{n}(\mathcal{A})$ by endowing

$$\mathfrak{h}(\mathcal{A}) = \text{Hom}_{\mathbb{Z}}(K(\mathcal{A}), \mathbb{C})$$

with the trivial bracket, and setting

$$[Z, f] = Z(\alpha)f \quad \text{for any } Z \in \mathfrak{h}(\mathcal{A}), f \in \mathfrak{n}_\alpha(\mathcal{A}).$$

We shall refer to $\mathfrak{b}(\mathcal{A})$ as the *extended Ringel–Hall Lie algebra* of \mathcal{A} .

Example Let Q be a finite quiver and R its path algebra. Assume that Q does not have oriented cycles, so that R is finite-dimensional. A simple argument due to Reineke [21, Lemma 4.4] shows that $C(\mathcal{A})$ coincides with the *composition algebra* of \mathcal{A} , that is the subalgebra of $\mathcal{H}(\mathcal{A})$ generated by the characteristic functions $\kappa_{[S_i]}$ of the simple modules. In this case, $\mathfrak{n}(\mathcal{A})$ is isomorphic to the positive part \mathfrak{n}_+ of the Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ corresponding to the undirected graph underlying Q and $\mathfrak{b}(\mathcal{A})$ to the corresponding Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}_+$. This result was first proved over a finite field by Ringel [24]. A characteristic zero result was later obtained by Schofield [27]. For the exact statement made above we refer to Joyce [12, Example 4.25] and in the finite-type case to Riedtmann [23].

4.5 Primitive generation of $C(\mathcal{A})$

Recall that a non-zero element c in a coalgebra C is *grouplike* if $\Delta(c) = c \otimes c$.

Lemma *The element $1 = 1_0$ is the only grouplike element in $C(\mathcal{A})$.*

Proof If $f \in C(\mathcal{A})$ is grouplike, then, for any module $M \in \mathcal{A}$ and $p \in \mathbb{N}^*$, $f(M^{\oplus p}) = f(M)^p$. Since f lies in $\mathcal{H}(\mathcal{A}) = \bigoplus_{d \in \mathbb{N}} \mathcal{H}_d(\mathcal{A})$, it is supported on modules of dimension $\leq D$ for large enough D . Choosing p such that $p \dim M > D$ shows that $f(M) = 0$ unless $M = 0$. In the latter case we have $f(0) = f(0 \oplus 0) = f(0)^2$, whence $f = 1$ since $f \neq 0$ and therefore $f = 1_0$. \square

Proposition *The inclusion $\mathfrak{n}(\mathcal{A}) \subset C(\mathcal{A})$ identifies $C(\mathcal{A})$ as a bialgebra with the universal enveloping algebra $U\mathfrak{n}(\mathcal{A})$ of $\mathfrak{n}(\mathcal{A})$.*

Proof We claim first that $C(\mathcal{A})$ is *connected*, that is that its coradical is one-dimensional. Indeed, since $C(\mathcal{A})$ is cocommutative and defined over an algebraically closed field, any simple subcoalgebra $C' \subset C(\mathcal{A})$ is one-dimensional [15, p. 8] and therefore spanned by an element c' which, up to a scalar, is necessarily grouplike. By Lemma 4.5, $C' = \mathbb{C}1_{\mathcal{M}_0}$. The proposition now follows from the Milnor–Moore Theorem (see, e.g. [15, Theorem 21] or [1, Theorem 2.5.3]). \square

4.6 Completion of $C(\mathcal{A})$

For each $d \geq 1$, the subspace $C_{>d}(\mathcal{A}) \subset C(\mathcal{A})$ of functions supported on modules of dimension $> d$ is an ideal. Consider the finite-dimensional algebra

$$C_{\leq d}(\mathcal{A}) = C(\mathcal{A})/C_{>d}(\mathcal{A})$$

and the corresponding inverse system $\cdots \rightarrow C_{\leq d}(\mathcal{A}) \rightarrow \cdots \rightarrow C_{\leq 0}(\mathcal{A}) = \mathbb{C}$. By definition, the completion $\widehat{C}(\mathcal{A})$ is the limit

$$\widehat{C}(\mathcal{A}) = \varprojlim_{d \geq 0} C_{\leq d}(\mathcal{A}) = \prod_{d \geq 0} C_d(\mathcal{A}).$$

For any $d \in \mathbb{N}$, set

$$C_{\leq d}(\mathcal{A})_+ = \{f \in C_{\leq d}(\mathcal{A}) \mid f(0) = 0\} \quad \text{and}$$

$$C_{\leq d}(\mathcal{A})^\times = \{f \in C_{\leq d}(\mathcal{A}) \mid f(0) = 1\}.$$

Then, $C_{\leq d}(\mathcal{A})_+$ is a Lie subalgebra of $C_{\leq d}(\mathcal{A})$ and $C_{\leq d}(\mathcal{A})^\times$ is a subgroup of the group of invertible elements in $C_{\leq d}(\mathcal{A})$ since $f \in C_{\leq d}(\mathcal{A})$ is invertible if, and only if, $f(0) \neq 0$. The following is standard.

Lemma *The standard exponential and logarithm functions*

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!} \quad \text{and} \quad \log(y) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (y - 1)^n$$

yield well-defined maps

$$\exp : C_{\leq d}(\mathcal{A})_+ \rightarrow C_{\leq d}(\mathcal{A})^\times \quad \text{and} \quad \log : C_{\leq d}(\mathcal{A})^\times \rightarrow C_{\leq d}(\mathcal{A})_+$$

which are each other's inverse.

Similarly, one can define a Lie subalgebra $\widehat{C}(\mathcal{A})_+$ of $\widehat{C}(\mathcal{A})$ and a subgroup $\widehat{C}(\mathcal{A})^\times$ of the group of invertible elements of $\widehat{C}(\mathcal{A})$. By Lemma 4.6, the exponential and logarithm functions give mutually inverse maps $\widehat{C}(\mathcal{A})_+ \rightleftarrows \widehat{C}(\mathcal{A})^\times$.

4.7 The pro-unipotent group $\widehat{N}(\mathcal{A})$

The completion $\widehat{C}(\mathcal{A})$ inherits a bialgebra structure from $C(\mathcal{A})$ since

$$\Delta(C_d(\mathcal{A})) \subset \bigoplus_{a+b=d} C_a(\mathcal{A}) \otimes C_b(\mathcal{A}).$$

Set $\mathfrak{n}_{>d}(\mathcal{A}) = \mathfrak{n}(\mathcal{A}) \cap C_{>d}(\mathcal{A})$ and

$$\mathfrak{n}_{\leq d}(\mathcal{A}) = \mathfrak{n}(\mathcal{A}) / \mathfrak{n}_{>d}(\mathcal{A}).$$

Then, the Lie algebra

$$\widehat{\mathfrak{n}}(\mathcal{A}) = \varprojlim_{d \geq 0} \mathfrak{n}_{\leq d}(\mathcal{A}) = \prod_{d \geq 0} \mathfrak{n}_d(\mathcal{A})$$

is the subspace of primitive elements in $\widehat{C}(\mathcal{A})$ and a Lie subalgebra of $\widehat{C}(\mathcal{A})_+$ by Lemma 4.4.

Define the *Ringel–Hall group* $\widehat{N}(\mathcal{A})$ of \mathcal{A} to be the set of grouplike elements of $\widehat{C}(\mathcal{A})$. Since any grouplike element $f \in \widehat{C}(\mathcal{A})$ satisfies $f(0) = \eta(f) = 1$, this is a subset of the set $\widehat{C}(\mathcal{A})^\times$ of invertible elements. It is easy to check using the bialgebra property that it is also a subgroup, i.e. is closed under multiplication.

Proposition

(i) *The exponential and logarithm maps restrict to isomorphisms*

$$\exp : \widehat{\mathfrak{n}}(\mathcal{A}) \rightarrow \widehat{N}(\mathcal{A}) \quad \text{and} \quad \log : \widehat{N}(\mathcal{A}) \rightarrow \widehat{\mathfrak{n}}(\mathcal{A}).$$

(ii) *The group $\widehat{N}(\mathcal{A})$ is a pro-unipotent group with Lie algebra $\widehat{\mathfrak{n}}(\mathcal{A})$.*

Proof (i) readily follows from Lemma 4.6 and the fact that $\Delta : \widehat{C}(\mathcal{A}) \rightarrow \widehat{C}(\mathcal{A}) \otimes \widehat{C}(\mathcal{A})$ is an algebra homomorphism (see, e.g. [22, Theorem 3.2]).

(ii) Let $\pi_{\leq d}$ be the projection $C(\mathcal{A}) \rightarrow C_{\leq d}(\mathcal{A})$. As an abstract subgroup of $\widehat{C}(\mathcal{A})$, $\widehat{N}(\mathcal{A})$ is the inverse limit of the groups $N_{\leq d}(\mathcal{A}) = \pi_{\leq d}(\widehat{N}(\mathcal{A}))$. By (i),

$$N_{\leq d}(\mathcal{A}) = \pi_{\leq d}(\widehat{N}(\mathcal{A})) = \pi_{\leq d}(\exp(\widehat{\mathfrak{n}}(\mathcal{A}))) = \exp(\mathfrak{n}_{\leq d}(\mathcal{A})).$$

Since any element $f \in C_{\leq d}(\mathcal{A})_+ \supset \mathfrak{n}_{\leq d}(\mathcal{A})$ is nilpotent, Lemma 4.6 implies that

$$N_{\leq d}(\mathcal{A}) = \{f \in C_{\leq d}(\mathcal{A})^\times \mid \log(f) \in \mathfrak{n}_{\leq d}(\mathcal{A})\}$$

is a Zariski closed, unipotent subgroup of $C_{\leq d}(\mathcal{A})^\times$ with Lie algebra $\mathfrak{n}_{\leq d}(\mathcal{A})$. The conclusion follows since the projection maps $N_{\leq d}(\mathcal{A}) \rightarrow N_{\leq d'}(\mathcal{A})$, $d \geq d'$ are clearly regular. □

4.8 The pro-solvable group $\widehat{B}(\mathcal{A})$

Let $H(\mathcal{A}) = \text{Hom}_{\mathbb{Z}}(K(\mathcal{A}), \mathbb{C}^*)$ be the torus of characters of $K(\mathcal{A})$. $H(\mathcal{A})$ acts by bialgebra automorphisms on $C(\mathcal{A})$ by

$$\sum_{\gamma \in K_{\geq 0}(\mathcal{A})} X_\gamma \mapsto \sum_{\gamma \in K_{\geq 0}(\mathcal{A})} \zeta(\gamma) X_\gamma,$$

where $\zeta \in H(\mathcal{A})$. This action extends to $\widehat{C}(\mathcal{A})$ and leaves $\widehat{N}(\mathcal{A})$ invariant. By definition, $\widehat{B}(\mathcal{A})$ is the semidirect product

$$\widehat{B}(\mathcal{A}) = H(\mathcal{A}) \ltimes \widehat{N}(\mathcal{A}) = \varprojlim H(\mathcal{A}) \ltimes N_{\leq d}(\mathcal{A}). \tag{23}$$

We refer to $\widehat{B}(\mathcal{A})$ as the *extended Ringel–Hall group* of \mathcal{A} . $\widehat{B}(\mathcal{A})$ is a pro-solvable, pro-algebraic group with maximal torus $H(\mathcal{A})$. Its Lie algebra is

$$\widehat{\mathfrak{b}}(\mathcal{A}) = \mathfrak{h}(\mathcal{A}) \ltimes \widehat{\mathfrak{n}}(\mathcal{A}),$$

where $\mathfrak{h}(\mathcal{A}) = \text{Hom}(K(\mathcal{A}), \mathbb{C})$ is the Lie algebra of $H(\mathcal{A})$.

5 Stability conditions and wall-crossing

5.1 Stability conditions

We shall define a *stability condition* on \mathcal{A} to be a group homomorphism

$$Z: K(\mathcal{A}) \rightarrow \mathbb{C}$$

such that $Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane.² Let $\text{Stab}(\mathcal{A})$ denote the set of all stability conditions on \mathcal{A} . Since the positive cone $K_{>0}(\mathcal{A})$ is generated by the classes of the simple modules S_1, \dots, S_N there is a bijection

$$\text{Stab}(\mathcal{A}) \cong \mathbb{H}^N$$

sending a stability condition Z to the N -tuple $(Z(S_1), \dots, Z(S_N))$. We may therefore regard $\text{Stab}(\mathcal{A})$ as a complex manifold.

Let Z be a stability condition on \mathcal{A} . Each nonzero module $M \in \mathcal{A}$ has a *phase*

$$\phi(M) = \frac{1}{\pi} \arg Z(M) \in (0, 1).$$

A module M is said to be Z -semistable if it is nonzero and if

$$0 \neq A \subset M \implies \phi(A) \leq \phi(M).$$

5.2 Wall-crossing

For any pair of classes $\beta, \gamma \in K_{>0}(\mathcal{A})$ which are not proportional over \mathbb{Q} , consider the real codimension one submanifold $W_{\beta,\gamma}$ of $\text{Stab}(\mathcal{A})$ given by

$$W_{\beta,\gamma} = \{Z \in \text{Stab}(\mathcal{A}) : Z(\beta)/Z(\gamma) \in \mathbb{R}_{>0}\}.$$

$W_{\beta,\gamma}$ is known as a *wall*.

Fix a class $\alpha \in K_{>0}(\mathcal{A})$ and consider the walls $W_{\beta,\gamma}$ as $\beta, \gamma \in K_{>0}$ vary over the finitely many pairs such that $\beta + \gamma = \alpha$. The connected components of the complement of these walls in $\text{Stab}(\mathcal{A})$ are called *chambers*. It is clear that in each chamber the set of semistable objects of type α is constant. However the set of semistable objects may change as one crosses a wall from one chamber to a neighbouring one. We refer to this behaviour as *wall-crossing*.

5.3 The functions δ_γ

The following result shows that semistability with respect to a given $Z \in \text{Stab}(\mathcal{A})$ is an open condition. Recall that a family of R -modules over a base variety S is a vector bundle M on S together with a ring homomorphism $R \rightarrow \text{End}_S(M)$.

Lemma *Given a family of modules over a variety S , the subset of points of S which correspond to Z -semistable modules is open.*

²This does not quite agree with the definition given in [6] where Z is allowed to take $K_{>0}(\mathcal{A})$ into the half-closed half-plane. The difference will be of no importance in this paper however.

Proof Fix a class $\gamma \in K_{>0}(\mathcal{A})$. It is immediate from the definitions that a module M of class γ is Z -semistable if, and only if it is θ -semistable in the sense of King [17], where $\theta: K(\mathcal{A}) \rightarrow \mathbb{R}$ is given by

$$\theta(\beta) = -\text{Im}(Z(\beta)/Z(\gamma)).$$

In turn, King shows that θ -semistability coincides with GIT semistability for the action of a reductive algebraic group on an affine variety [17, Proposition 3.1]. It follows from this that semistability is an open condition. \square

Given $Z \in \text{Stab}(\mathcal{A})$ and $\gamma \in K_{>0}(\mathcal{A})$ define $\delta_\gamma \in \mathcal{H}_\gamma(\mathcal{A})$ to be the characteristic function of Z -semistable modules of type $\gamma \in K(\mathcal{A})$.

$$\delta_\gamma(M) = \begin{cases} 1 & \text{if } M \text{ is } Z\text{-semistable and } [M] = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.3 implies that δ_γ is a constructible function. Clearly δ_γ depends on $Z \in \text{Stab}(\mathcal{A})$ in a discontinuous way because of wall-crossing behaviour.

5.4 Harder–Narasimhan filtrations and Reineke inversion

Since \mathcal{A} is a finite length category, the Harder–Narasimhan property holds (in this context see for example [6, Proposition 2.4]): every nonzero module M has a unique filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_n = M,$$

where the successive factors $F_i = M_i/M_{i-1}$ are Z -semistable and of strictly decreasing phases

$$\phi(F_1) > \dots > \phi(F_n).$$

The following result is due to Reineke [21].

Theorem *For every stability condition $Z \in \text{Stab}(\mathcal{A})$, and every $\gamma \in K_{>0}(\mathcal{A})$, the following holds in $\mathcal{H}(\mathcal{A})$.*

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1) > \dots > \phi(\gamma_n)}} \delta_{\gamma_1} * \dots * \delta_{\gamma_n}, \tag{24}$$

$$\delta_\gamma = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \gamma \\ \phi(\gamma_1 + \dots + \gamma_i) > \phi(\gamma)}} (-1)^{n-1} \kappa_{\gamma_1} * \dots * \kappa_{\gamma_n}, \tag{25}$$

where the (finite) sums are over elements $\gamma_i \in K_{>0}(\mathcal{A})$.

Proof The first identity follows immediately from the definition of the product in $\mathcal{H}(\mathcal{A})$ and the existence and uniqueness of Harder–Narasimhan filtrations. The second is proved in [21, Theorem 5.1]. \square

The following is a direct consequence of (25)

Corollary *The elements δ_γ lie in the subalgebra $C(\mathcal{A}) \subset \mathcal{H}(\mathcal{A})$ generated by the elements κ_γ .*

5.5 The elements SS_ℓ

Let $Z \in \text{Stab}(\mathcal{A})$ be a stability condition. Given a ray $\ell = \mathbb{R}_{>0}e^{i\pi\phi}$, the infinite sum

$$SS_\ell = 1 + \sum_{\substack{\gamma \in K_{>0}(\mathcal{A}) \\ Z(\gamma) \in \ell}} \delta_\gamma$$

defines an element of $\widehat{C}(\mathcal{A})$ which has the value 1 on a module M if it is zero or semistable of phase ϕ , and has value zero otherwise.

Lemma *The element SS_ℓ is grouplike and therefore lies in $\widehat{N}(\mathcal{A})$.*

Proof This amounts to the statement that a module $M \oplus N$ is semistable of phase ϕ precisely if M and N are. This is a standard fact but for the reader’s convenience we sketch the proof. The only non-obvious implication is that if M and N are semistable of phase ϕ then so is $M \oplus N$. Suppose $A \subset M \oplus N$ is a subobject with phase $\phi(A) > \phi$. We can assume that A is semistable: otherwise pass to a submodule of larger phase and repeat. The inclusion $A \subset M \oplus N$ gives a nonzero map $A \rightarrow M$ or $A \rightarrow N$. But the image of such a map has phase larger than ϕ (because it is a quotient of A which is semistable of phase $> \phi$) and smaller than ϕ (because it is a subobject of M or N which are semistable of phase ϕ). This gives a contradiction. \square

5.6 The element $1_{\mathcal{A}}$

Let now $1_{\mathcal{A}} \in \widehat{C}(\mathcal{A})$ be the element given by

$$1_{\mathcal{A}} = \sum_{\gamma \in K_{\geq 0}(\mathcal{A})} \kappa_\gamma.$$

The function $1_{\mathcal{A}}$ takes the value 1 on every module. It is therefore grouplike and lies in $\widehat{N}(\mathcal{A})$.

5.7 Clockwise product

The Harder–Narasimhan relation (24) can be written more compactly in terms of the elements SS_ℓ and $1_{\mathcal{A}}$ as follows

$$1_{\mathcal{A}} = \prod_{\ell}^{\curvearrowright} SS_\ell, \tag{26}$$

where the product is an infinite product in the group $\widehat{N}(\mathcal{A})$ over all rays of the form $\ell = \mathbb{R}_{>0}Z(\gamma)$ for some $\gamma \in K_{>0}(\mathcal{A})$, taken in clockwise order. This product makes sense in $\widehat{N}(\mathcal{A})$ because it is finite when evaluated on modules M of a fixed type $\gamma \in K_{\geq 0}(\mathcal{A})$.

5.8 Joyce’s elements ϵ_α

Given a class $\alpha \in K_{>0}(\mathcal{A})$, Joyce defines an element $\epsilon_\alpha \in C_\alpha(\mathcal{A})$ by the finite sum [13]

$$\epsilon_\alpha = \sum_{n \geq 1} \sum_{\substack{\gamma_1 + \dots + \gamma_n = \alpha \\ Z(\gamma_i) \in \mathbb{R}_{>0}Z(\alpha)}} \frac{(-1)^{n-1}}{n} \delta_{\gamma_1} * \dots * \delta_{\gamma_n}. \tag{27}$$

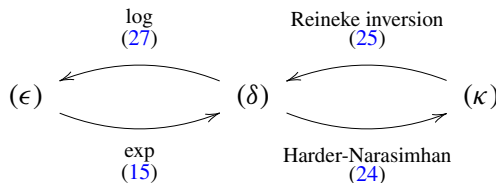
He then shows that ϵ_α is supported on indecomposable modules and hence defines an element $\epsilon_\alpha \in \mathfrak{n}_\alpha(\mathcal{A})$. From our Hopf algebraic perspective, this is clear: just as in formula (16), (27) expresses the grouplike element SS_ℓ as an exponential

$$SS_\ell = \exp \left(\sum_{\substack{\alpha \in K_{>0}(\mathcal{A}) \\ Z(\alpha) \in \ell}} \epsilon_\alpha \right). \tag{28}$$

Thus, the elements ϵ_α are primitive and, by Lemma 4.4 are supported on non-zero indecomposable objects.

5.9

The relations between the elements ϵ , δ and κ are summarised in the following diagram.



Note that it is identical to the right-hand part of the diagram of Sect. 3.10.

6 Stability conditions and Stokes data

We show in this section that a stability condition Z on \mathcal{A} defines Stokes data for an irregular connection on \mathbb{P}^1 with values in the Ringel–Hall Lie algebra of \mathcal{A} . We show moreover that this connection varies isomonodromically with Z .

6.1 Irregular $\widehat{B}\mathcal{A}$ -connections on \mathbb{P}^1

We begin by adapting the definition of Stokes data to irregular connections with values in the pro-solvable group $\widehat{B}(\mathcal{A})$ defined in Sect. 4.

Let

$$\Phi = \{\alpha \in K_{>0}(\mathcal{A}) \mid \mathfrak{n}_\alpha(\mathcal{A}) \neq 0\} \subset \mathfrak{h}(\mathcal{A})^*$$

be the set of roots of $\widehat{B}(\mathcal{A})$ relative to the torus $H(\mathcal{A})$ and set

$$\mathfrak{h}(\mathcal{A})_{\text{reg}} = \mathfrak{h}(\mathcal{A}) \setminus \bigcup_{\alpha \in \Phi} \text{Ker}(\alpha).$$

Let \widehat{P} be the holomorphically trivial, principal $\widehat{B}(\mathcal{A})$ -bundle on \mathbb{P}^1 . By this we mean the following: the group $\widehat{B}(\mathcal{A})$ is the inverse limit of the solvable algebraic groups $B_{\leq d}(\mathcal{A})$ and \widehat{P} is the limit of the corresponding principal bundles $P_{\leq d}$. In particular, a section of P is holomorphic if the induced section of each $P_{\leq d}$ is.

Fix $Z \in \mathfrak{h}(\mathcal{A})_{\text{reg}}$ and consider a connection on \widehat{P} of the form

$$\nabla = d - \left(\frac{Z}{t^2} + \frac{f}{t} \right) dt,$$

where $f \in \widehat{\mathfrak{n}}(\mathcal{A})$. ∇ is the inverse limit of the connections

$$\nabla_{\leq d} = d - \left(\frac{Z}{t^2} + \frac{\pi_{\leq d}(f)}{t} \right) dt$$

on $P_{\leq d}$, where $\pi_{\leq d} : \widehat{\mathfrak{n}}(\mathcal{A}) \rightarrow \mathfrak{n}_{\leq d}(\mathcal{A})$ is the projection, and each $\nabla_{\leq d}$ satisfies the assumptions of Sect. 2.4. Indeed, $H(\mathcal{A})$ is a maximal torus in $B_{\leq d}(\mathcal{A})$ with corresponding root space decomposition (7) given by

$$\mathfrak{b}_{\leq d}(\mathcal{A}) = \mathfrak{h}(\mathcal{A}) \oplus_{\substack{\alpha \in \Phi \\ \dim \alpha \leq d}} \mathfrak{n}_\alpha(\mathcal{A}),$$

$Z \in \mathfrak{h}(\mathcal{A})$ is a regular element and the projection of $\pi_{\leq d}(f)$ onto $\mathfrak{h}(\mathcal{A})$ is zero.

6.2 Canonical fundamental solutions

As in Sect. 2.8, we define a Stokes ray of ∇ to be a ray of the form $\ell = \mathbb{R}_{>0}Z(\alpha)$ for $\alpha \in \Phi$. Note that there may be an infinite number of such rays. A ray is admissible if it is not a Stokes ray. Since $\mathfrak{n}(\mathcal{A})$ is isomorphic to $\mathfrak{n}_{\leq d}(\mathcal{A}) \oplus \mathfrak{n}_{> d}(\mathcal{A})$ as $H(\mathcal{A})$ -module, an admissible ray for ∇ is admissible for each $\nabla_{\leq d}$. One can therefore use the existence and uniqueness statement of Theorem 2.6 to deduce the following

Theorem *Given an admissible ray r , there is a unique holomorphic fundamental solution*

$$Y_r : \mathbb{H}_r \rightarrow \widehat{B}(\mathcal{A})$$

of ∇ such that $Y_r(t) \cdot \exp(Z/t) \rightarrow 1$ as $t \rightarrow 0$ in \mathbb{H}_r .

6.3 Stokes factors

The definition of the Stokes factors of ∇ requires a little care since the set of Stokes rays of ∇ need not be discrete. If $r_1 \neq -r_2$ are two admissible rays however, ordered so that the closed sector $\overline{\Sigma}(r_1, r_2) \subset \mathbb{C}^*$ swept by clockwise rotation from r_1 to r_2 is convex, there is a unique element $S_{\overline{\Sigma}(r_1, r_2)} \in \widehat{B}(\mathcal{A})$ such that

$$Y_{r_2} = Y_{r_1} \cdot S_{\overline{\Sigma}(r_1, r_2)}$$

on $\mathbb{H}_{r_1} \cap \mathbb{H}_{r_2}$. By Proposition 2.7, $S_{\overline{\Sigma}(r_1, r_2)}$ is of the form $\exp(X)$ where

$$X \in \prod_{\substack{\alpha \in \Phi \\ Z(\alpha) \in \overline{\Sigma}(r_1, r_2)}} \mathfrak{n}_\alpha(\mathcal{A}).$$

Definition ∇ admits a Stokes factor $S_\ell \in \widehat{B}(\mathcal{A})$ along the Stokes ray ℓ if the elements $S_{\overline{\Sigma}(r_1, r_2)}$ tend to S_ℓ as the admissible rays r_1, r_2 tend to ℓ in such a way that $\ell \in \overline{\Sigma}(r_1, r_2)$.

Proposition

- (i) *The connection ∇ admits a Stokes factor S_ℓ along any Stokes ray ℓ .*
- (ii) *Given two admissible rays r_1, r_2 as above, one has*

$$S_{\overline{\Sigma}(r_1, r_2)} = \prod_{\ell \subset \overline{\Sigma}(r_1, r_2)}^{\curvearrowright} S_\ell.$$

Proof Both statements hold for each solvable quotient $B_{\leq d}(\mathcal{A})$ of $\widehat{B}(\mathcal{A})$. \square

6.4 Stokes multipliers

Unlike the definition of the Stokes factors, that of the Stokes multipliers S_{\pm} of ∇ relative to the choice of a ray r such that $\pm r$ are admissible is straightforward and given, as in Sect. 2.9 by

$$Y_{r,\pm}(t) = Y_{-r}(t) \cdot S_{\pm}, \quad t \in \mathbb{H}_{-r},$$

where $Y_{r,+}$ and $Y_{r,-}$ are the analytic continuations of Y_r to \mathbb{H}_{-r} in the anticlockwise and clockwise directions respectively. By Lemma 2.10, S_{\pm} are given by the clockwise products over Stokes factors

$$S_+ = \prod_{\ell \in \mathbb{H}_{ir}}^{\curvearrowright} S_{\ell} \quad \text{and} \quad (S_-)^{-1} = \prod_{\ell \in \mathbb{H}_{-ir}}^{\curvearrowright} S_{\ell}. \tag{29}$$

6.5 Stability conditions

Assume now that $Z \in \text{Stab}(\mathcal{A}) \subset \mathfrak{h}(\mathcal{A})$ is a stability condition. Note that $Z \in \mathfrak{h}(\mathcal{A})_{\text{reg}}$ since $Z(\alpha) \in \mathbb{H}$ for any $\alpha \in K_{>0}(\mathcal{A})$.

For any ray $\ell = \mathbb{R}_{>0}e^{i\pi\phi}$, let SS_{ℓ} be the characteristic function of semistable objects of phase ϕ defined in Sect. 5.5. By (28),

$$\text{SS}_{\ell} = \exp\left(\sum_{\alpha:Z(\alpha)\in\ell} \epsilon_{\alpha}\right),$$

where $\epsilon_{\alpha} \in \mathfrak{n}(\mathcal{A})_{\alpha}$. This shows in particular the following

Lemma *If ℓ is not a Stokes ray of ∇ , then $\text{SS}_{\ell} = 1$.*

Thus, only the elements SS_{ℓ} corresponding to Stokes rays of ∇ are non-trivial.

6.6

The following is the main result of this paper.

Theorem

- (i) *There exists a unique connection $\nabla_{\mathcal{A},Z}$ of the form*

$$\nabla_{\mathcal{A},Z} = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt \tag{30}$$

whose Stokes data is given by either of the following equivalent conditions:

- (a) *The Stokes factor corresponding to a Stokes ray $\ell = \mathbb{R}_{>0} \exp(i\pi\phi)$ is the characteristic function SS_ℓ of Z -semistable modules of phase ϕ .*
- (b) *The Stokes multipliers S_+, S_- relative to the ray $r = \mathbb{R}_{>0}$ are the function $1_{\mathcal{A}}$ which takes the value 1 on every module, and the identity element 1_0 respectively.*
- (ii) *The components of $f = \sum_{\alpha \in K_{>0}(\mathcal{A})} f_\alpha$ are given by the Lie series*

$$f_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_j \in K_{>0}(\mathcal{A}) \\ \alpha_1 + \dots + \alpha_n = \alpha}} J_n(Z(\alpha_1), \dots, Z(\alpha_n)) \epsilon_{\alpha_1} * \dots * \epsilon_{\alpha_n},$$

where the J_n are the functions appearing in Theorem 3.7.

- (iii) *As Z varies in $\text{Stab}(\mathcal{A})$, the family of connections $\nabla_{\mathcal{A}, Z}$ varies isomonodromically. In particular, $f_\alpha(Z)$ is a holomorphic function of Z and satisfies the PDE*

$$df_\alpha = \sum_{\beta + \gamma = \alpha} [f_\beta, f_\gamma] d \log \gamma.$$

Proof (i) We first show the equivalence of (a) and (b). Note that since $Z \in \text{Stab}(\mathcal{A})$, the Stokes rays of a connection ∇ of the form (30) are contained in the upper half plane \mathbb{H} . In particular the ray $r = \mathbb{R}_{>0}$ is such that $\pm r$ are admissible and the corresponding Stokes multiplier S_- is trivial by (29).

Assume that (a) holds and let \mathcal{R} be the set of Stokes rays of ∇ . By (29)

$$S_+ = \prod_{\ell \in \mathcal{R}: \ell \subset \mathbb{H}} S_\ell = \prod_{\ell \in \mathcal{R}: \ell \subset \mathbb{H}} SS_\ell = \prod_{\ell \subset \mathbb{H}} SS_\ell = 1_{\mathcal{A}},$$

where the third identity follows from Lemma 6.5 and the last one from the Harder–Narasimhan relation (26).

The implication (b) \Rightarrow (a) follows from (29), (26) and the fact that the relations (24) can be inverted.

The existence and uniqueness of f , and the fact that it is given by the Lie series (ii) follows from Theorem 3.7 (note that the series (19) is finite since any $\alpha \in K_{>0}(\mathcal{A})$ can only be decomposed as the sum of elements of $K_{>0}(\mathcal{A})$ in finitely many ways. Thus, the Taylor series of \mathcal{S}^{-1} yields a global inverse of the Stokes map in this case).

(iii) The first assertion follows from the fact that, by condition (b), the Stokes multipliers S_\pm are constant functions of Z . The second follows from Theorem 2.12. Indeed, since the Stokes map has a global inverse, $f = f(Z)$ varies holomorphically in Z by Remark 2.12 and satisfies the isomonodromy equations (12). □

Acknowledgements This project was begun while the first author visited Northeastern University in March of 2007. It was completed while the second author was a member of the Institute for Advanced Study in Princeton during the Academic year 2007–8. He is extremely grateful to the Institute for its financial support, hospitality and marvelous working conditions. We would very much like to thank Phil Boalch, Pierre Deligne, Dominic Joyce, Nikita Markarian and Simon Willerton for useful discussions during the writing of this paper. We are also grateful to M. Kontsevich and Y. Soibelman for sending us a preliminary version of [18] and to D. Huybrechts, B. Leclerc and J. Schroer for pointing out small inaccuracies in an earlier version of this paper.

References

1. Abe, E.: Hopf Algebras. Cambridge Tracts in Mathematics, vol. 74. Cambridge University Press, Cambridge (1980)
2. Balsler, W., Jurkat, W.B., Lutz, D.A.: Birkhoff invariants and Stokes multipliers for meromorphic linear differential equations. *J. Math. Anal. Appl.* **71**, 48–94 (1979)
3. Boalch, P.P.: Symplectic manifolds and isomonodromic deformations. *Adv. Math.* **163**, 137–205 (2001)
4. Boalch, P.P.: Stokes matrices, Poisson Lie groups and Frobenius manifolds. *Invent. Math.* **146**(3), 479–506 (2001)
5. Boalch, P.P.: G -bundles, isomonodromy, and quantum Weyl groups. *Int. Math. Res. Not.* **2002**, 1129–1166 (2002)
6. Bridgeland, T.: Stability conditions on triangulated categories. *Ann. Math.* **100**, 317–346 (2007)
7. Bridgeland, T.: Spaces of stability conditions. In: Algebraic Geometry—Seattle 2005. Part 1. Proc. Sympos. Pure Math., vol. 80, pp. 1–21. AMS, Providence (2009)
8. Bridgeland, T., Toledano Laredo, V.: Stokes factors and multilogarithms. [arXiv:1006.4623](https://arxiv.org/abs/1006.4623)
9. Douglas, M.R.: Dirichlet branes, homological mirror symmetry, and stability. In: Proceedings of the International Congress of Mathematicians, Beijing, 2002, vol. III, pp. 395–408. Higher Ed. Press, Beijing (2002)
10. Humphreys, J.E.: Linear Algebraic Groups. Graduate Texts in Mathematics, vol. 21. Springer, Berlin (1975)
11. Jimbo, M., Miwa, T., Ueno, K.: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. *Physica D* **2**, 306–352 (1981)
12. Joyce, D.: Configurations in abelian categories. II. Ringel–Hall algebras. *Adv. Math.* **210**, 635–706 (2007)
13. Joyce, D.: Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau 3-folds. *Geom. Topol.* **11**, 667–725 (2007)
14. Joyce, D.: Configurations in abelian categories. IV. Invariants and changing stability conditions. *Adv. Math.* **217**, 125–204 (2008)
15. Kaplansky, I.: Bialgebras. Lecture Notes in Mathematics. Department of Mathematics, University of Chicago (1975)
16. Kapranov, M., Vasserot, E.: Kleinian singularities, derived categories and Hall algebras. *Math. Ann.* **316**, 565–576 (2000)
17. King, A.D.: Moduli of representations of finite-dimensional algebras. *Q. J. Math. Oxf. Ser.* **45**, 515–530 (1994)
18. Kontsevich, M., Soibelman, Y.: Stability structures, motivic Donaldson–Thomas invariants and cluster transformations. [arXiv:0811.2435](https://arxiv.org/abs/0811.2435)
19. Lusztig, G.: Quivers, perverse sheaves, and quantized enveloping algebras. *J. Am. Math. Soc.* **4**, 365–421 (1991)

20. MacPherson, R.D.: Chern classes for singular algebraic varieties. *Ann. Math.* **100**, 423–432 (1974)
21. Reineke, M.: The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli. *Invent. Math.* **152**, 349–368 (2003)
22. Reutenauer, C.: *Free Lie Algebras*. London Mathematical Society Monographs. New Series. Oxford Science Publications, vol. 7. The Clarendon Press, Oxford University Press, Oxford (1993)
23. Riedtmann, C.: Lie algebras generated by indecomposables. *J. Algebra* **170**, 526–546 (1994)
24. Ringel, C.M.: Hall algebras and quantum groups. *Invent. Math.* **101**, 583–591 (1990)
25. Ringel, C.M.: Lie algebras arising in representation theory. In: *Representations of Algebras and Related Topics*, Kyoto, 1990. London Math. Soc. Lecture Note Ser., vol. 168, pp. 284–291. Cambridge University Press, Cambridge (1992)
26. Schiffmann, O.: Lectures on Hall algebras. [math/0611617](#)
27. Schofield, A.: Quivers and Kac–Moody Lie algebras. Unpublished manuscript
28. Wasow, W.: *Asymptotic Expansions for Ordinary Differential Equations*. Dover, New York (1987)