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A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds

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Abstract We consider Kerr spacetimes with parameters a and M such that $|a| \ll M$, Kerr-Newman spacetimes with parameters $|Q| \ll M$, $|a| \ll M$, and more generally, stationary axisymmetric black hole exterior spacetimes $(\mathcal{M}, g)$ which are sufficiently close to a Schwarzschild metric with parameter $M > 0$ and whose Killing fields span the null generator of the event horizon. We show uniform boundedness on the exterior for solutions to the wave equation $\Box_g \psi = 0$. The most fundamental statement is at the level of energy: We show that given a suitable foliation Σ_τ, then there exists a constant C depending only on the parameter M and the choice of the foliation such that for all solutions ψ, a suitable energy flux through Σ_τ is bounded by C times the initial energy flux through Σ_0. This energy flux is positive definite and does not degenerate at the horizon, i.e. it agrees with the energy as measured by a local observer. It is shown that a similar boundedness statement holds for all higher order energies, again without degeneration at the horizon. This leads in particular to the pointwise uniform boundedness of ψ, in terms of a higher order initial energy on Σ_0. Note that in view of the very general assumptions, the separability properties of the wave equation or geodesic flow on the Kerr background are not used. In fact, the physical mechanism for boundedness uncovered in this paper is independent of the dispersive properties of waves in the high-frequency geometric optics regime.

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Contents

1 Introduction

The Kerr family, discovered in 1963 [36], comprises perhaps the most important family of exact solutions to the Einstein vacuum equations

$$R_{\mu\nu} = 0, \tag{1}$$

the governing equations of general relativity. For parameter values $0 \le |a| < M$ (here M denotes the mass and a the angular momentum per unit mass), the Kerr solutions represent black hole spacetimes: i.e. asymptotically flat spacetimes possessing a region which cannot communicate with future null infinity. The celebrated Schwarzschild family sits as the one-parameter subfamily of Kerr corresponding to $a = 0$. Much of current theoretical astrophysics is based on the hypothesis that isolated systems described by Kerr metrics are ubiquitous in the observable universe.

Despite the centrality of the Kerr family to the general relativistic world picture, the most basic questions about the behaviour of linear waves on Kerr backgrounds have remained to this day unanswered. This behaviour is in turn intimately connected to the stability properties of the Kerr metrics themselves as solutions of (1), and thus, with the very physical tenability of the notion of black hole. In particular, even the question of the uniform boundedness (pointwise, or in energy) of solutions ψ to the linear wave equation

$$\Box_g \psi = 0 \tag{2}$$

in the domain of outer communications has not been previously resolved, except for the Schwarzschild subfamily.

The main theorems of this paper give the resolution of the boundedness problem for (2), for the case $|a| \ll M$. Solutions to (2) arising from regular initial data remain uniformly bounded in the domain of outer communications. The most fundamental statement is Theorem 1.1. This says that there is a constant C depending on M and a suitable foliation Σ_τ such that for all solutions ψ of (2), a positive definite energy flux through Σ_τ and through null infinity $\mathcal{I}^+$ is bounded by C times the initial energy. Theorem 1.2 then states that there exists again such a constant C such that ψ^2 is bounded pointwise by C times an initial higher order energy.

In fact, the results of this paper apply to a much more general setting than the specific Kerr metric: Theorems 1.1 and 1.2 follow as special cases of analogous statements concerning solutions of (2) on stationary axisymmetric spacetimes whose metrics are C^1-close to a Schwarzschild exterior spacetime with mass $M > 0$, and whose Killing fields span the null generator of an event horizon. This gives the proof a certain robustness; in particular, and very importantly, *the proof does not depend on the hidden symmetries of Kerr*—i.e. the existence of an additional non-trivial Killing tensor and the resulting separability of the wave equation and geodesic flow—and in fact, *the proof is completely independent of the dispersive properties of waves in the geometric optics limit*, properties which are complicated by the presence of so-called trapped null geodesics, and would appear to be governed by higher regularity of the metric. This robustness may be of relevance in the ultimate goal of this analysis: understanding the dynamics of the Einstein equations (1) in a neighborhood of a Kerr metric. Cf. [16].

We first give a statement of the main results for the special case of Kerr and the related Kerr-Newman family (this is a family of solutions to the coupled Einstein-Maxwell system).

1.1 Statement of the theorem for Kerr and Kerr-Newman

We refer the reader to [13, 32] for an introduction to the Kerr-Newman geometry. Let $(\mathcal{M}, g)$ here denote the Kerr manifold with metric parameters in the subextremal range

$$0 \le |a| < M$$

or more generally the Kerr-Newman manifold with metric parameters (a, Q, M)[1] satisfying

$$0 \le \sqrt{a^2 + Q^2} < M.$$

For definiteness, for us the "Kerr manifold" or "Kerr-Newman manifold" is by definition the maximal Cauchy development of a complete asymptotically flat spacelike hypersurface with two asymptotically flat ends. This is a globally hyperbolic subdomain of the maximal analytic Kerr-Newman manifold

[1] The additional parameter Q is known as the charge.

of [32], proper (as a subset) except in the case $a = 0$, where it corresponds precisely to the Kruskal extension of the Schwarzschild solution.

Let $\mathcal{D} \subset \mathcal{M}$ denote the closure (in $\mathcal{M}$) of a domain of outer communications. Recall that this domain is most naturally characterized in terms of the asymptotic structure at infinity.[2] Concretely, the region $\mathcal{D}$ can alternatively be represented as the closure (in $\mathcal{M}$) of a standard Boyer-Lindquist coordinate chart $(\hat{r}, \hat{t}, \hat{\theta}, \hat{\phi})$

$$\{\hat{r} > M + \sqrt{M^2 - a^2 - Q^2}\} \times \{-\infty < \hat{t} < \infty\} \times \{0 < \hat{\theta} < \pi\} \times \{0 < \hat{\phi} < 2\pi\}.$$

For convenience, we may here take this latter representation as our definition of $\mathcal{D}$. We note for reference the explicit form of the Kerr-Newman metric in this coordinate chart:

$$-\frac{\Delta}{\rho^2}\left(d\hat{t} - a\sin^2\hat{\theta}d\hat{\phi}\right)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\hat{\theta}^2 + \frac{\sin^2\hat{\theta}}{\rho^2}\left(ad\hat{t} - (\hat{r}^2 + a^2)d\hat{\phi}\right)^2, \tag{3}$$

where $\Delta = \hat{r}^2 - 2M\hat{r} + a^2 + Q^2$, $\rho^2 = \hat{r}^2 + a^2\cos^2\hat{\theta}$.[3]

The boundary of $\mathcal{D}$ in $\mathcal{M}$ is then a bifurcate null hypersurface $\mathcal{H}^+ \cup \mathcal{H}^-$, where $\mathcal{H}^+ \cap \mathcal{H}^- = \partial\mathcal{H}^\pm$ is a topological 2-sphere (the bifurcation sphere) and $\mathcal{H}^\pm$ are null hypersurfaces with boundary, each characterised as the future (resp. past) boundary of $\mathcal{D}$ in $\mathcal{M}$. We call $\mathcal{H}^+$ the *future event horizon* and $\mathcal{H}^-$ the *past event horizon*. Let Σ be a Cauchy hypersurface in $(\mathcal{M}, g)$ such that $\Sigma \cap \mathcal{H}^- = \emptyset$ and such that $\Sigma \cap \mathcal{D}$ coincides with a constant-$\hat{t}$ hypersurface outside a compact set. Note that under our assumptions, $\Sigma \cap \mathcal{D}$ is a past Cauchy hypersurface for $J^+(\Sigma) \cap \mathcal{D}$.[4]

The reader familiar with Penrose diagrammatic notation may wish to refer to the Fig. 1 representing $\mathcal{M}$.

Recall that the Kerr-Newman metrics possess a 2-dimensional Killing algebra spanned by a stationary Killing field $\mathbb{T}$ and an axisymmetric Killing field Φ. In the domain of the Boyer-Lindquist chart referred to above, these Killing fields correspond precisely to the coordinate vector fields $\mathbb{T} = \partial_{\hat{t}}$, $\Phi = \partial_{\hat{\phi}}$.[5] We have that $J^+(\Sigma) \cap \mathcal{D}$ is foliated by Σ_τ for $\tau \geq 0$, where

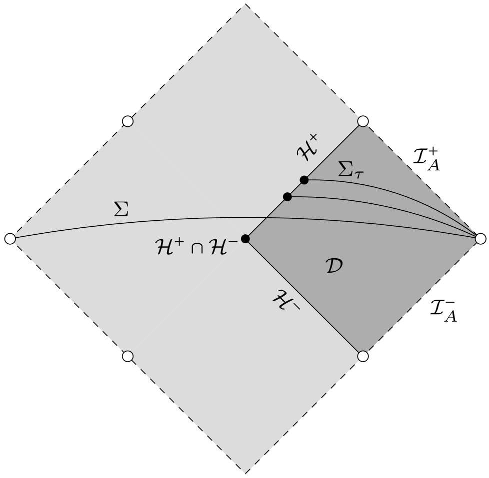
[2]With respect to suitable asymptotic notions defining null infinity $\mathcal{I}^\pm$, $\mathcal{D}$ can be realised as the closure (in $\mathcal{M}$) of $J^-(\mathcal{I}^+_A) \cap J^+(\mathcal{I}^-_A)$. where $\mathcal{I}^\pm_A$ denote connected components of $\mathcal{I}^\pm$ associated to one of the ends.

[3]We shall require this explicit form only insofar as to show that the Kerr-Newman family indeed satisfies the assumptions of Sect. 3.2.

[4]Here and in what follows, J^+ denotes causal future [32], not to be confused with currents $\mathbf{J}_\mu$ to be defined later.

[5]In the case where $a = 0$, Φ is not unique, and the choice of a particular Boyer-Lindquist coordinate system can be thought to determine then Φ. See Sect. 3.1.

Fig. 1 A domain of outer communications $\mathcal{D}$



$\Sigma_\tau = \rho_\tau(\Sigma \cap \mathcal{D})$ is the future translation of $\Sigma \cap \mathcal{D}$ by the flow ρ_τ generated by $\mathbb{T}$. Let $\mathbb{n}$ denote the unit future normal of Σ_τ. Let $\mathbb{n}_{\mathcal{H}}$ denote a ρ_τ-invariant null generator for $\mathcal{H}^+$, and give $\mathcal{H}^+ \cap \mathcal{D}$ the induced volume form from g and $\mathbb{n}_{\mathcal{H}}$. Let $\mathbf{T}_{\mu\nu}[\psi]$ denote the standard energy momentum tensor associated to a real-valued solution ψ of the wave equation (2)

$$\mathbf{T}_{\mu\nu}[\psi] = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \psi \partial_\alpha \psi,$$

and define the currents

$$\mathbf{J}^{\mathbb{n}}_\mu[\psi] \doteq \mathbf{T}_{\mu\nu}[\psi]\mathbb{n}^\nu, \qquad \mathbf{J}^{\mathbb{T}}_\mu[\psi] \doteq \mathbf{T}_{\mu\nu}[\psi]\mathbb{T}^\nu. \tag{4}$$

Note that since $\mathbb{n}$ is future-timelike, the former current is positive definite when contracted with a future-timelike vector field, but is not conserved, whereas the latter current is conserved as $\mathbb{T}$ is Killing, but in general not positive definite when so contracted, since there is a non-empty subset of $\mathcal{D}$ where $\mathbb{T}$ is spacelike, unless $a = 0$ (see the discussion of superradiance in Sect. 1.4.1 below).

Theorem 1.1 *Let $(\mathcal{M}, g)$ be the subextremal Kerr-Newman manifold with parameters $M > 0$, Q, a, and let $\mathcal{D}$, Σ_τ, $\mathbb{n}$, etc. be as above. There exists a universal positive constant $\epsilon > 0$, and a constant C depending on M and the choice of Σ_0 such that if*

$$0 \le |a| \le \epsilon M, \qquad 0 \le Q \le \epsilon M, \tag{5}$$

then the following statement holds:

Let ψ be a solution of (2) *on* $(\mathcal{M}, g)$ *such that* $\int_{\Sigma_0} \mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu < \infty$. *Then*

$$\int_{\Sigma_\tau} \mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu \le C \int_{\Sigma_0} \mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu, \tag{6}$$

$$\left| \int_{\mathcal{H}^+ \cap J^+(\Sigma_0)} \mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu_{\mathcal{H}} \right| \le C \int_{\Sigma_0} \mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu, \tag{7}$$

$$\int_{\mathcal{I}^+} \mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu_{\mathcal{I}} \le C \int_{\Sigma_0} \mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu. \tag{8}$$

In the above theorem, the integrals are to be taken with respect to the induced volume forms. The integral on the left hand side of (8) is here to be understood as a suggestive notation for the following limiting integral on the cone in $\mathcal{D}$ defined by $K_{R_1,R_2} = \partial J^+(\Sigma \cap \{\hat{r} \ge R_2\}) \cap \{\hat{r} \ge R_1\}$, for $R_2 > R_1$:

$$\lim_{R_1 \to \infty} \lim_{R_2 \to \infty} \int_{K_{R_1,R_2}} \mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu_K.$$

We note that the above expression represents the energy 'radiated to infinity'.

The hypothesis of Theorem 1.1 can be re-expressed as the statement that local energy as measured by a local observer be finite, i.e. that $\nabla^{\Sigma_0}\psi|_{\Sigma_0}$, $\mathrm{n}\psi|_{\Sigma_0}$ be in $L^2_{\mathrm{loc}}(\Sigma_0)$, together with the global assumption that

$$\int_{\Sigma_0} \mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu < \infty.$$

In view of the fact that Σ_0 coincides with a constant-$\hat{t}$ hypersurface for large $\hat{r}$, one easily sees that this condition is equivalent to $\psi|_{\Sigma_0} \in \dot{H}^1(\Sigma_0)$, $(\mathrm{n}\psi)|_{\Sigma_0} \in L^2(\Sigma_0)$. Here $L^2(\Sigma_0)$, etc., denotes the natural L^2 norm defined by the induced Riemannian metric on Σ_0.

A version of Theorem 1.1 holds for all higher energies, that is to say, one can control all higher order derivatives of ψ in $L^2(\Sigma_\tau)$, including as above transversal derivatives without degeneration at $\mathcal{H}^+$, from an initial higher order energy. We omit here in this introduction the precise statement, but give only the following important corollary:

Theorem 1.2 *Let* $(\mathcal{M}, g)$, M, Q, a, Σ_0 *etc. be as before. Then there exists a universal constant* $\epsilon > 0$ *and a constant* C *depending on* M *and the choice of* Σ_0 *such that if* (5) *is satisfied then the following statement holds*:

Let ψ be a solution of the wave equation (2) *on* $(\mathcal{M}, g)$ *such that*

$$\mathbf{Q}_1 \doteq \sup_{\Sigma_0} |\psi|^2 + \int_{\Sigma_0} \left(\mathbf{J}^{\mathrm{n}}_\mu[\psi] + \mathbf{J}^{\mathrm{n}}_\mu[\mathrm{n}\psi]\right)\mathrm{n}^\mu < \infty.$$

Then

$$\psi^2 \leq C\mathbf{Q}_1$$

in $\mathcal{D} \cap J^+(\Sigma_0)$.

The hypothesis of Theorem 1.2 is satisfied if $\psi|_{\Sigma_0} \in \dot{H}^1(\Sigma_0)$, $(\mathbb{n}\psi)|_{\Sigma_0} \in H^1(\Sigma_0)$, $(\mathbb{n}^2\psi)|_{\Sigma_0} \in L^2(\Sigma_0)$.

Finally, note that given an arbitrary Cauchy surface $\tilde{\Sigma}$ for Kerr, sufficiently well behaved at i_0, it follows that the right hand side of (6) is bounded by

$$C(\Sigma_0, \tilde{\Sigma}) \int_{\tilde{\Sigma}\cap(J^-(\Sigma_0)\cup J^+(\Sigma_0))} \mathbf{J}^{\tilde{\mathbb{n}}}_\mu[\psi]\tilde{\mathbb{n}}^\mu,$$

thus the above regularity assumptions could be imposed on an arbitrary Cauchy surface. In particular, there are no unphysical restrictions on the support of the solution in a neighborhood of the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$.

1.2 Statement for general stationary axisymmetric perturbations of Schwarzschild

Theorems 1.1 and 1.2 follow as special cases from analogous statements in the much more general setting of the wave equation (2) on arbitrary stationary axisymmetric black hole exterior metrics C^1-close to Schwarzschild, and with suitable assumptions on the geometry of the Killing fields. In particular, in addition to closeness, it is required that—as in the Kerr solution—the horizon is null and its null generator is contained in the span of the Killing fields.

As the results are new for the Kerr and Kerr-Newman cases and resolve a longstanding open question, we have preferred to specialise the theorems of the introduction directly to these cases and defer a formal discussion of the more general class to the body of the paper. We emphasize however that the generality of the assumptions is a fundamental element of this paper and key for future applications of the method. This shall be clear from the overview to follow in Sect. 1.4, where we shall describe precisely what structure from the Kerr solution is necessary for each part of the argument. The precise assumptions are given in Sect. 3. The main results of the paper for this general class are then the statements of Theorems 4.1, 9.1 and Corollary 9.1 (corresponding to the statement of Theorem 1.1), Theorem 10.1 (giving a generalisation of the previous to non-degenerate energies of all orders), and Theorem 10.2 (giving pointwise bounds to all orders, including thus the analogue of the statement of Theorem 1.2).

1.3 Previous results

Before proceeding to give an overview of the proof, we will review in detail previous work on this and related problems. Results analogous to Theorems 1.1 and 1.2 for static perturbations of Minkowski space pose little difficulty. (Indeed, the analogue of Theorem 1.1 is immediate, and Theorem 1.2 can be proven with the help of Sobolev inequalities after commuting the equation with the static Killing field.) Thus, we shall pass directly to the black hole case.

1.3.1 Schwarzschild: boundedness

The characteristic feature which makes the Schwarzschild boundedness problem much simpler than for Kerr is that in Schwarzschild, the stationary Killing field $\mathbb{T}$ is timelike everywhere in the domain of outer communications (i.e. the interior of $\mathcal{D}$), becoming null however on the horizon $\mathcal{H}^+ \cup \mathcal{H}^-$. This is clear from the explicit form of the metric in the interior of $\mathcal{D}$, which becomes:

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2). \tag{9}$$

Recall that the interior of $\mathcal{D}$ corresponds to $r > 2M$.

[Note: In anticipation of what follows, it is useful to fix a region of a Schwarzschild manifold and view the Kerr metrics (and eventually, the more general class of Sect. 3.2) as all living on the same underlying region defined by a Schwarzschild coordinate system. Thus, when referring to Schwarzschild, we have here dropped the hats from the coordinates. These should not be confused with Boyer-Lindquist coordinates of a nearby Kerr (Sect. 3.3), for which we retain hats. For the purpose of this introduction, the reader may prefer not to think about the precise relation of various coordinates until the discussion in Sect. 3.1.]

In view of the consequent nonnegative definiteness of the flux of the conserved current $\mathbf{J}^{\mathbb{T}}$ (recall the definition (4)) on spacelike hypersurfaces in $\mathcal{D}$ and through the horizon $\mathcal{H}^+$, one immediately obtains

$$\int_{\Sigma_\tau}\mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu \le \int_{\Sigma_0}\mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu. \tag{10}$$

Recall, however, that at the horizon, $\mathbf{J}^{\mathbb{T}}_\mu$ degenerates with respect to $\mathbf{J}^{\mathrm{n}}_\mu$, and thus, although we have

$$\int_{\Sigma_\tau}\mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu \le C\int_{\Sigma_\tau}\mathbf{J}^{\mathrm{n}}_\mu[\psi]\mathrm{n}^\mu,$$

the reverse inequality does not hold. Thus, (10) does *not* imply (6). (We note that from the nonnegative definiteness of $\mathbf{J}^{\mathbb{T}}$ we do immediately obtain the estimates (7) and (8).)

The original approach in the Schwarzschild case for pointwise boundedness of ψ (as in the statement of Theorem 1.2) was to try and obtain such bounds directly from the degenerate estimate (10). Clearly, commuting (2) with $\mathbb{T}$ and applying (10), elliptic estimates and Sobolev inequalities, one easily obtains pointwise bounds away from the horizon. Thus, the only essential difficulty is obtaining uniform bounds for ψ up to the horizon, exactly where $\mathbf{J}^{\mathbb{T}}$ degenerates compared with $\mathbf{J}^{\mathrm{n}}$.

This difficulty was resolved in the celebrated paper of Kay and Wald [35], building on previous work of Wald [52] where uniform pointwise boundedness had been proven for the restricted class of solutions ψ whose support was assumed not to contain the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$. The arguments of Kay and Wald to prove the analogue of Theorem 1.2 relied on the staticity to realize a solution ψ as $\partial_t \tilde{\psi}$ where $\tilde{\psi}$ is again a solution of (2) constructed by inverting an elliptic operator acting on initial data. In addition, it was necessary to commute (2) with the full Lie algebra associated to spherical symmetry. Finally, a pretty geometric construction exploiting the discrete symmetries of maximal Schwarzschild was used to remove the unphysical restriction on the support near $\mathcal{H}^+ \cap \mathcal{H}^-$ necessary for constructing $\tilde{\psi}$ in the original [52].

The non-degenerate energy estimate (6) of Theorem 1.1 for Schwarzschild was first proven as part of the decay results of [19] to be discussed below, exploiting in particular the red-shift effect. See Sect. 1.3.4. This led to a new proof of Theorem 1.2 for Schwarzschild which avoided the construction of $\tilde{\psi}$ and appeal to discrete symmetries, but still required commutation with the Lie algebra associated to spherical symmetry.

As we shall see below, one side benefit of the proofs of Theorems 1.1 and 1.2 of the present paper, is that, when specialised to Schwarzschild, they yield a novel method, which is both more elementary and more robust than the previous proofs [19, 35] (and in fact reduces to be quite short) and should perhaps be thought of as the definitive boundedness argument. See the discussion in Sects. 1.4.5 and 1.4.6.

1.3.2 Schwarzschild: integrated local energy decay

The problem of understanding boundedness beyond Schwarzschild is intimately connected to understanding, at least in part, the dispersive properties of solutions of (2). We thus must also review results of this type.

The programme of proving quantitative dispersive estimates on Schwarzschild was initiated in [7, 38].[6] These papers introduced to this problem what

[6] For earlier nonquantitative statements of decay see for instance Twainy [50].

in the language of the present paper would correspond to energy currents $\mathbf{J}^{\mathbb{X}}_\mu[\psi]=\mathbf{T}_{\mu\nu}[\psi]\mathbb{X}^\nu$, where $\mathbb{X}$ is a vector field in the direction of ∂_r (with respect to Schwarzschild coordinates), chosen such that the associated 0-currents

$$\mathbf{K}^{\mathbb{X}}[\psi]=\mathbf{T}_{\mu\nu}[\psi]\nabla^\nu\mathbb{X}^\mu$$

enjoy positivity properties. (In fact, more general currents of the form (46) of Sect. 4.4 must be used, but we shall suppress this for the purpose of the present discussion.) The construction of such 'virial' currents in the context of the wave equation on Minkowski space goes back to seminal work of Morawetz [43].

In brief, the point of this construction is as follows: For solutions ψ of the wave equation (2), the energy identity

$$\nabla^\mu\mathbf{J}^{\mathbb{X}}_\mu[\psi]=\mathbf{K}^{\mathbb{X}}[\psi] \tag{11}$$

holds. Integrating (11) in a spacetime domain, it follows that, if indeed the right hand side enjoys positivity properties, one obtains that a positive definite *spacetime* integral of a weighted energy density associated to ψ is controlled by boundary terms.[7] Under appropriate assumptions on $\mathbf{J}^{\mathbb{X}}$, these boundary terms are in turn controlled by the initial $\mathbf{J}^{\mathbb{T}}$-energy with the help of the conservation of the non-negative definite $\mathbf{J}^{\mathbb{T}}$.

One has obtained thus a spacetime-integral estimate, which can be thought to embody a weak statement of dispersion. In particular, this statement immediately excludes "fixed-frequency" obstructions to decay, for instance stationary or periodic solutions. There is a much more subtle obstruction to decay, however, which occurs in the "high frequency" limit, arising from the presence of trapped null geodesics associated with the photon sphere $r=3M$. (These are geodesics which neither reach the event horizon nor escape to null infinity.) One can indeed construct high-frequency finite energy solutions of (2) which remain localised near such geodesics for arbitrary long time. Thus, the spacetime estimates associated to the energy identity of $\mathbf{J}^{\mathbb{X}}$ must degenerate at $r=3M$. See [45]. This degeneration must in turn be reflected in the construction of the currents by the vanishing of the vector field $\mathbb{X}$ precisely at $r=3M$. As we shall see, this degeneration makes the construction of $\mathbf{J}^{\mathbb{X}}$ delicate and the nonnegativity properties of $\mathbf{K}^{\mathbb{X}}$ fragile.

Recall that Schwarzschild coordinates degenerate on $\mathcal{H}^+$, and ∂_r becomes colinear with $\partial_t=\mathbb{T}$ in the limit as the horizon is approached. This means that $\mathbf{K}^{\mathbb{X}}$ will also degenerate at $r=2M$. The significance of this only became apparent later in the context of the red-shift effect. See Sect. 1.3.4.

[7] Note, in comparison, that for a Killing field, for instance $\mathbb{T}$, we have $\mathbf{K}^{\mathbb{T}}=0$, and the associated energy identity expresses conservation of the $\mathbf{J}^{\mathbb{T}}$ current.

1.3.3 Schwarzschild: energy-flux and pointwise decay

Starting from a new construction of a current $\mathbf{J}^{\mathbb{X}}$ of the type described above and the resulting spacetime integral estimate, [19] showed quantitative decay for the energy flux of ψ through a suitable foliation connecting the future event horizon to null infinity, as well as pointwise decay, in particular, the uniform decay result $\psi^2 \leq C\mathbf{Q}v_+^{-2}$ in the domain of outer communications. Here v is an Eddington-Finkelstein advanced time coordinate and $\mathbf{Q}$ is an appropriate quantity computable on initial data, and v_+ denotes say $\max\{v, 1\}$. Similar decay results were proven independently by Blue and Sterbenz [10] for initial data vanishing on $\mathcal{H}^+ \cap \mathcal{H}^-$, but with control which degenerates on the horizon. See Sect. 1.3.4 immediately below. Stronger decay results for spherically symmetric solutions had been proven previously in [18] by different methods, as a byproduct of a result concerning the coupled spherically-symmetric Einstein-(Maxwell)-scalar field system.

1.3.4 Schwarzschild: the red-shift and the vector fields $\mathbb{Y}$, $\mathbb{N}(= \mathbb{Y} + \mathbb{T})$

In the course of the above study, the work [19] introduced the use of a vector field multiplier current $\mathbf{J}^{\mathbb{Y}}_\mu$ associated to a vector field $\mathbb{Y}$ which becomes null on (and transversal to) the horizon $\mathcal{H}^+$, such that the flux of $\mathbf{J}^{\mathbb{N}}$ where $\mathbb{N} = \mathbb{T} + \mathbb{Y}$ gives the non-degenerate energy at the horizon as measured by a local observer. The associated 0-current $\mathbf{K}^{\mathbb{Y}}(= \mathbf{K}^{\mathbb{N}})$ enjoys positivity properties near the horizon which can be thought to capture the red-shift effect. It is use of this current which allowed one to deduce the boundedness statement (6) for the non-degenerate energy, as well as suitable decay statements for a non-degenerate energy flux. The positivity property of the current $\mathbf{K}^{\mathbb{N}}$ near $\mathcal{H}^+$ is manifestly stable to perturbation of the metric. In particular, one can use a small amount of this current to 'stabilise' the nonnegativity properties of $\mathbf{K}^{\mathbb{X}}$.[8] This will be of critical importance in the present paper.

The introduction of the current $\mathbf{J}^{\mathbb{Y}}_\mu$ to capture the red-shift was inspired by an analogous weighted estimate for null derivates of spherically symmetric *self-gravitating* scalar fields [17, 18] (with the possible presence of charge) near their event horizons. In fact, the first use of these estimates was to control the solution in a region of the black hole *interior* [17]; these estimates were used to show for instance the formation of an apparent horizon, and the persistence of decay properties deep inside the black hole. The results of [17], when specialised to the simpler case of a scalar field on a fixed Schwarzschild or Reissner-Nordström background (with charge Q) lead to the statement that boundedness and decay properties propagate from $\mathcal{H}^+$ to the region

[8] For an explicit use of $\mathbf{K}^{\mathbb{Y}}$ as a stabiliser, one can compare with [21] (in the context of Schwarzschild-de Sitter, see Sect. 1.5.4) where failure of positivity of $\mathbf{K}^{\mathbb{X}}$ very near $\mathcal{H}^+$ was compensated for by the positivity of $\mathbf{K}^{\mathbb{N}}$.

$r > M - \sqrt{M^2 - Q^2} + \epsilon$, for any $\epsilon > 0$, but with degeneration as $\epsilon \to 0$. In the presense of charge, however, the red-shift effect gives way to a blue-shift effect as the Cauchy horizon $r = M - \sqrt{M^2 - Q^2}$ is approached, leading to instabilities at the level of derivatives [17]. The interesting problem for black hole interiors is to understand the precise behaviour of ψ (and, in the coupled case, the behaviour of the geometry) as this horizon is approached. See [17].

1.3.5 Schwarzschild: technical refinements

There have been more recent technical refinements of the above results: The works [8, 10, 19] had controlled trapping effects with the help of vector field multipliers which must be carefully chosen for each spherical harmonic separately. An alternative proof of such estimates not relying on spherical harmonic decompositions is provided by our more recent [20], and independently, by the subsequent Marzuola et al. [41], where further important refinements are given, including applications to Strichartz estimates and nonlinear wave equations. Both [20] and [41] rely on the red-shift construction introduced in [19]. For other refinements, as well as extensions to the Reissner-Nordström metric, see [9].

1.3.6 Kerr: separation and statements for fixed modes

In contrast to Schwarzschild, all previous work on (2) for Kerr with $|a| \neq 0$ is restricted to fixed modes of various type.

This approach begins with work of Carter [12] who showed that the wave equation on a Kerr metric with parameters M, a can be formally separated in Boyer-Lindquist coordinates, in the sense that the expression:

$$\mathfrak{R}(a\omega, \lambda, \hat{r}) \mathfrak{S}_{k\ell}(a\omega, \hat{\theta}) e^{ik\hat{\phi}} e^{i\omega\hat{t}} \tag{12}$$

provides a complex-valued solution to the wave equation (2) for arbitrary integer k and complex number ω, where $\mathfrak{S}_{k\ell}(a\omega, \hat{\theta})$ are the so-called oblate spheroidal harmonics with oblateness parameter $a\omega$, and $\mathfrak{R}(a\omega, \lambda, \hat{r})$ is a complex function satisfying a certain second order ordinary differential equation with respect to $\hat{r}$ (with the coefficients of the equation depending on $a\omega$ and λ, where $\lambda = \lambda(a\omega, k, \ell)$ is the corresponding eigenvalue to $\mathfrak{S}_{k\ell}$).

Carter's separation was quite unexpected in view of the fact that for $a \neq 0$, the dimension of the Lie algebra of isometries is only 2. The geometric origin of the separation (12) lies in the existence of an additional 'hidden' symmetry [53], associated to a non-trivial Killing tensor, now known as the Carter tensor.

In Whiting's seminal [54], it is shown that, for no subextremal value of the Kerr parameters $|a| < M$ does there exist a solution of the form (12) with a

$\mathrm{Im}(\omega) < 0$ and finite initial energy on $\{\hat{t} = 0\}$. The argument relies on the algebraic symmetries enjoyed by the class of ode's which $\mathfrak{R}(a\omega, \lambda, r)$ satisfies. Whiting's statement is often known as "mode stability". This statement is of course retrieved in particular by Theorem 1.1 for $|a| \ll M$.

Finster et al. [27, 28] consider azimuthal modes on Kerr spacetimes for the general subextremal parameter range $|a| < M$, i.e. the case of solutions of (2) of the form

$$\mathfrak{P}_k(\hat{r}, \hat{\theta}, \hat{t})e^{ik\hat{\phi}} \tag{13}$$

for fixed k, which are further restricted by the requirement that they be smooth and vanish identically in a neighborhood of the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$. Using [54] and spectral theoretic techniques, the authors express that class of azimuthal modes (13) as a superposition of modes (12) over real ω, and study properties of $\mathfrak{R}(a\omega, \lambda, \hat{r})$. The paper [27] contains the following non-quantitative statement: For fixed $\hat{r} > M + \sqrt{M^2 - a^2}$ and $\hat{\theta}$,

$$\lim_{\hat{t} \to \infty} \mathfrak{P}_k(\hat{t}, \hat{r}, \hat{\theta}) \to 0. \tag{14}$$

Of course, in the absence of a quantitative boundedness statement, no finiteness or decay statement could then be inferred from (14) for general solutions ψ of (2) unconstrained by (13), because the lim of (14) does not *a priori* commute with summation over k.

Even results of the above type seem to be previously unknown for Kerr-Newman, for any $Q \neq 0$.

1.4 Overview of the proof

In this section, we give a brief overview of the proof of Theorems 1.1 and 1.2. In the process, we shall motivate the more general framework of Sect. 3 to which our results apply, introducing the necessary assumptions as they are used. For the convenience of the reader, we will highlight the key novel ideas separated from the main body of the text.

1.4.1 The ergoregion and superradiance

The elusiveness of any sort of boundedness-type result whatsoever stems from the well-known phenomenon of *superradiance*. This is related to the fact that, unlike in the Schwarzschild case (see Sect. 1.3.1 above), for $a \neq 0$, the stationary Killing field $\mathbb{T}$ fails to be everywhere timelike in the domain of outer communications. In particular, there is a non-empty subset $\mathcal{E} \subset \mathcal{D}$ where $\mathbb{T}$ is spacelike, the so-called *ergoregion*.

The significance of the ergoregion was first understood from the point of view of point particles: the presence of $\mathcal{E}$ allows a particle coming in from infinity to split (consistent with conservation of energy-momentum) into one of negative energy entering the black hole and one of greater positive energy returning to infinity; this is known as the *Penrose process*. The pioneering study by Christodoulou [14] of the "black hole transformations" obtainable via a Penrose process led to a subject now known as "black hole thermodynamics".

For solutions ψ of the wave equation (2), the presence of the ergoregion $\mathcal{E}$ implies that the energy current $\mathbf{J}^{\mathbb{T}}_\mu[\psi]$ fails to be nonnegative definite when integrated over spacelike hypersurfaces. Thus, the conservation of $\mathbf{J}^{\mathbb{T}}_\mu$ does not imply *a priori* bounds on an L^2-based quantity, analogous to the estimate (10) in Schwarzschild. In particular, the local energy of the solution (or alternatively, the energy radiated to infinity, i.e. the left hand side of (8)) can be greater than the initial total energy, even if the energy is initially supported in the region where $\mathbf{J}^{\mathbb{T}}_\mu$ is positive definite; hence the term 'superradiance'.

The fundamental problem is then to find a mechanism which gives a quantitative bound on how large this energy can become. Indeed, a priori, the supremum of the left hand side of (10), or, alternatively, the left hand side of (8), can in fact be infinite. As a result, there is no analogue of the a priori pointwise bounds on ψ away from the horizon, which are immediately available in Schwarzschild. For all fixed r, the supremum of ψ in t is a priori infinite.

1.4.2 The stabilising role of the red-shift effect

The first key to the desired mechanism is provided by the stability of considerations near the horizon, related to the redshift.

First a remark: The positive-definite property of the 0-current $\mathbf{K}^{\mathbb{X}}$ of Sect. 1.3.2 is stable to perturbation of the metric everwhere except where it degenerates, namely near the Schwarzschild horizon $r = 2M$ and near the Schwarzschild photon sphere $r = 3M$. Let us suppose for the time being that one could perturb the construction of $\mathbf{J}^{\mathbb{X}}$ so that the nonnegative-definiteness property of $\mathbf{K}^{\mathbb{X}}$ continued to hold on Kerr spacetimes with $|a| \ll M$ in a neighborhood of $r = 3M$, and turn to the issue of the horizon.

For Kerr metrics satisfying $|a| \ll M$, then, arguing by continuity from Schwarzschild, the positivity properties of the 0-current $\mathbf{K}^{\mathbb{N}}$ associated to the vector field $\mathbb{N}$ (see Sect. 1.3.4) are stable in a neighborhood of the horizon, containing in particular the ergoregion $\mathcal{E}$. Considering then the current

$$\mathbf{J}^{\mathbb{X}} + e\mathbf{J}^{\mathbb{N}},$$

for a small parameter e, then for even smaller Kerr rotation parameter $a \ll e$, it would follow that the associated $\mathbf{K}^{\mathbb{X}+e\mathbb{N}}$ is globally positive, degenerating neither at the horizon nor (in view of the assumption of the previous paragraph) near $r = 3M$. Note that the 1-current $\mathbf{J}^{e\mathbb{N}}$ yields positive definite boundary terms on and near the horizon. Coupling this with the use of the identity for $\mathbf{J}^{\mathbb{T}}$ (whose failure to be positive definite at the horizon can now be absorbed for small a by the positivity properties of $\mathbf{J}^{e\mathbb{N}}$), it follows that one may estimate *both* boundary and bulk terms. This would at the same time give us not only the desired boundedness statement, but an integrated decay statement as well. To summarise:

> *With the help of the redshift effect, the fundamental difficulty of the absence of a conserved energy can be circumvented for Kerr with $|a| \ll M$, provided one can perturb the construction of the Schwarzschild virial current so as to still yield a nonnegative bulk term near $r = 3M$.*

It cannot be stressed too heavily that even given a virial identity with positive definite bulk, one still needs an argument ensuring the positivity of the boundary terms, and smallness of a is used again above in a crucial way.

If we define the ergoregion more generally for stationary metrics g to be the region where the stationary Killing field $\mathbb{T}$ is spacelike, we note that the above considerations have nothing to do with the Kerr metric per se and hold for spacetimes $(\mathcal{M}, g)$ $\epsilon_{\mathrm{close}}$-close to Schwarzschild in a suitable C^1 sense.[9] The presence of a stationary Killing field and the closeness to Schwarzschild are the first restrictions on the general class of spacetimes described in Sect. 3.2 to which our arguments will apply.

1.4.3 Trapped null geodesics?

The above section reduces the problem to perturbing the Schwarzschild construction of $\mathbf{J}^{\mathbb{X}}$ near $r = 3M$. As remarked previously, to obtain a nonnegative definite $\mathbf{K}^{\mathbb{X}}$ near $r = 3M$, one must 'see' the obstruction to dispersion provided by high frequency solutions localised near trapped null geodesics. In contrast to Schwarzschild, in the Kerr case for $|a| \neq 0$, the 'limit' r-values of such geodesics fill out an open subset in r. This is related to the fact that in Kerr the codimensionality of the space of trapped geodesics can only be properly understood in phase space. In fact, Alinhac [1] has shown that no current of the form $\mathbf{J}^{\mathbb{X}}$ (more generally, of the form considered in Sect. 4.4) can yield the necessary positivity for Kerr. This indicates that controlling trapping for Kerr requires a far more delicate analysis.[10]

Turning to more general, $\epsilon_{\mathrm{close}}$-close C^1 perturbations of the Schwarzschild metric, the situation appears even worse. It is not clear at all what dispersive properties one should expect, as it is now difficult to make any

[9] The significance of C^1 is that this makes positivity at the level of energy identities stable.

[10] See Sect. 1.7.

statements about the properties of geodesics. Since the previous section 'couples' understanding of boundedness to understanding of dispersion, it seems one must give up on proving boundedness in such a class.

1.4.4 Superradiant frequencies are not trapped!

Perhaps the main insight of the present paper is that, despite the above appearances, the problem of boundedness can in fact be decoupled from the problem of dispersion in the geometric optics limit, and thus, from the problem of trapping.

To see this, let us first return to the Schwarzschild case. Taking the Fourier transform in time t and expanding in modes associated to the azimuthal coordinate ϕ, one may decompose the solution ψ into two pieces, each again solving (2), as follows

$$\psi = \psi_\flat + \psi_\sharp \tag{15}$$

where $\psi_\flat$ is to be supported in frequency space (real frequencies ω and integer k are Fourier variables dual to t and ϕ) only in the range

$$\omega^2 \lesssim \omega_0^2 k^2, \tag{16}$$

whereas $\psi_\sharp$ is to be supported in frequency space only in the range

$$\omega^2 \gtrsim \omega_0^2 k^2, \tag{17}$$

for ω_0 a parameter to be determined.

The crucial point is then the following: For ω_0 sufficiently small, one can in fact construct (see Sect. 5.3) a current[11] $\mathbf{J}^{\mathbb{X}}$ such that $\mathbf{K}^{\mathbb{X}}[\psi_\flat]$ is essentially nonnegative after suitable integration in t and ϕ, and moreover, this current degenerates only at the horizon, *and not at* $r = 3M$. This is related to the fact that the frequency range (16) is in fact 'elliptic' near $r = 3M$, and, in particular, does not 'see' trapping.

Now let us turn to Kerr. Recall that one retains as Killing fields ∂_t and ∂_ϕ; suppressing some important technical issues, let us thus pretend for the purpose of the present discussion that a decomposition of the form (15) is still possible (see Sect. 1.4.7!). Since $\mathbf{K}^{\mathbb{X}}[\psi_\flat]$ is positive definite without degeneration near $r = 3M$, this positivity property is stable to Kerr, for parameters $|a| \ll M$. The considerations of Sect. 1.4.2 should then apply for the $\psi_\flat$, and thus, it would appear that the entire problem can be reduced to understanding $\psi_\sharp$.

[11] Again, the construction in fact concerns a more general type of current of the form described in Sect. 4.4, but we shall continue to suppress this in the discussion here.

What would be the significance of such a reduction? For $|a| \ll \omega_0$, then the so-called 'superradiant frequency range' $0 \le k\omega \le \frac{ak^2}{2Mr_+}$ is completely contained in the range (16), or to say it equivalently, all frequencies in the range (17) are non-superradiant.[12] The embodiment of the property of 'non-superradiance' which will be useful for us is simply:

$$\int_{S\subset\mathcal{H}^+} \mathbf{J}^{\mathbb{T}}[\psi_\sharp]\mathrm{n}^{\mu}_{\mathcal{H}^+} \gtrsim 0 \tag{18}$$

for suitable sufficiently large subsets S of the event horizon. (See Sect. 1.4.7 for a brief discussion of our use here of $\gtrsim$.) The inequality (18) suggests that study of boundedness for $\psi_\sharp$ is more akin to the Schwarzschild case, and thus, easier. See Sect. 1.4.5 immediately below.

To summarise:

Superradiant frequencies are not trapped. Restricted to these frequencies, the dispersive mechanism from Schwarzschild, as captured by virial identities, is heuristically stable in the sense required in Sect. 1.4.2.

The above arguments would hold more generally for spacetimes ϵ_{close}-close to Schwarzschild in C^1, with $\epsilon_{\text{close}} \ll e$, $\epsilon_{\text{close}} \ll \omega_0$, as long as ∂_t and ∂_ϕ are retained as Killing fields—necessary to define (15)—and the span of ∂_t and ∂_ϕ is a null plane tangent to the horizon. It is precisely the latter condition on the span which (together with the closeness) ensures that the 'non-superradiance' property (18) holds for $\psi_\sharp$. With this added assumption we have now essentially completely described the class of metrics considered in Sect. 3.2.

1.4.5 A new boundedness argument for Schwarzschild and the non-superradiant regime

This leaves then the non-superradiant part $\psi_\sharp$.

As noted already in Sect. 1.4.4, the idea is that this case should be similar to the Schwarzschild problem, as the usual $\mathbf{J}^{\mathbb{T}}[\psi_\sharp]$ energy is essentially nonnegative definite, at least when considering the flux through the horizon. At best, however, the expected resulting boundedness statement would be analogous to the statement (10) on Schwarzschild. (Recall that our previous [19] inferred the boundedness of the non-degenerate energy (6) only after construction a $\mathbf{J}^{\mathbb{X}}$ current with nonnegative $\mathbf{K}^{\mathbb{X}}$, in accordance with the insight of Sect. 1.4.2. It is precisely this nonnegativity property which is not available here.) Nonetheless, as we shall see, we can indeed obtain the full (6).

[12] The notion of superradiant frequency is typically discussed in terms of the separation (12). See for instance [54]. For the purpose of the present paper, one should consider (18) as the defining property of non-superradiance.

Before discussing $\psi_\sharp$, let us first return to the argument for Schwarzschild. There is in fact another way of using the red-shift identity of Sect. 1.4.2, which makes use only of (10), and *not* a $\mathbf{J}^{\mathbb{X}}$ current. Indeed, the red-shift identity immediately yields that the spacetime integral of $\mathbf{K}^{\mathbb{N}}$ restricted to a neighborhood of the horizon grows at most linearly in time. Revisiting the identity, using the comparability of $\mathbf{J}^{\mathbb{N}}_{\mu} n^{\mu}_{\Sigma}$ and $\mathbf{K}^{\mathbb{N}}$ near $\mathcal{H}^+$, one can in fact show that the energy flux associated to $\mathbf{J}^{\mathbb{N}}$ through Σ_τ is bounded, without showing uniform boundedness of the spacetime integral of $\mathbf{K}^{\mathbb{N}}$ (which would have required the construction of $\mathbf{J}^{\mathbb{X}}$). This yields a new, simpler and more robust proof of Theorem 1.1 for Schwarzschild.

The above argument in fact would apply more generally to solutions ψ of (2) on small stationary perturbations of Schwarzschild such that the flux of $\mathbf{J}^{\mathbb{T}}[\psi]$ though suitable subsets of $\mathcal{H}^+$ is nonnegative.

To summarise:

For stationary perturbations of Schwarzschild, the boundedness of the non-degenerate energy can be directly inferred from the red-shift energy identity alone, provided that the sign of the flux of the conserved energy through suitable subsets of the horizon is nonnegative.

It should be clear that, in view then of the defining property (18) of 'non-superradiance', the above considerations should apply in particular to the non-superradiant $\psi_\sharp$ under the assumptions of the previous section. See Sects. 7.2 and 8.4. Putting this together thus with Sect. 1.4.4, one would obtain uniform energy boundedness for $\psi = \psi_\flat + \psi_\sharp$, in accordance with Theorem 1.1.

1.4.6 Red-shift commutation

To obtain also Theorem 1.2, there is one final new ingredient. The red-shift technique, introduced in [19], is further extended here to commutators. See Sect. 10.1. In brief, in addition to applying $\mathbb{Y}$ as a multiplier, we may also commute the wave equation with $\mathbb{Y}$, that is, we may consider the energy identity associated to the second order current $\mathbf{J}^{\mathbb{Y}}[\mathbb{Y}\psi]$. The most dangerous term on the right hand side again comes with a favourable sign. In conjunction with application of $\mathbb{T}$ as a commutator, this allow us to estimate all resulting terms in a manner consistent with obtaining boundedness for the energy flux on Σ_τ. In particular, since $\mathbb{T}+\mathbb{Y}$ is timelike on $\mathcal{H}^+$, this means that we always can commute with a strictly timelike direction. Using elliptic estimates and a standard Sobolev estimate, one has then a direct approach to pointwise estimates as in Theorem 1.2.

The red-shift effect ensures that one can commute ψ (arbitrarily many times) with a vector field timelike up to and including $\mathcal{H}^+$ and again derive estimates, giving non-degenerate energy boundedness to all orders. In particular, pointwise estimates for ψ (and arbitrary derivatives, including transversal to $\mathcal{H}^+$) follow naturally.

Note that, even when specialised to Schwarzschild, this allows for a novel proof of Theorem 1.2, which, in addition to avoiding appeal to $\mathbf{J}^{\mathbb{X}}$ (see Sect. 1.4.5 above), now entirely avoids commutation with angular momentum operators.

1.4.7 Technical issues

There are various technical issues that arise in implementing the strategy outlined above.

To even define $\psi_\flat$ and $\psi_\sharp$, we must take the Fourier transform in t. On the other hand, a priori we do not have finiteness properties (that is what we are trying to prove!) let alone the requisite integrability. To define thus the Fourier transform, we first cutoff ψ in a region of interest of "time-length" τ to form what we shall denote as $\psi_{\text{✂}}^{\tau}$ (see Sect. 6.1). Moreover, we will apply smooth cutoffs in frequency space to define $\psi_\flat$ and $\psi_\sharp$ so as to be able to appropriately localise considerations in physical space. In particular, various error terms coupling $\psi_\flat$ and $\psi_\sharp$ are well localised near the cutoff region. Because these error terms do not decay in r in a manner compatible with our estimates, we must artificially induce additional decay in r on the error terms by widening the cutoff regions. Thus, the cutoffs are not defined with respect to Σ_τ but with respect to the foliations $\Sigma_\tau^\pm$, which diverge from Σ_τ by a factor $\sim r^{1/2}$.

Let us mention also that the defining nonnegativity property (18) characterizing 'non-superradiance', as well as the positivity property of $\mathbf{K}^{\mathbb{X}}[\psi_\flat]$ are again true modulo errors that arise from localising the Plancherel formula in physical space. See Sect. 6.3. In view of the smooth cutoffs in frequency space, the resulting error terms are such that they are integrable in time, and can thus be controlled from the boundedness of an energy quantity.

Finally, to absorb the cutoff errors, it is essential to have a smallness parameter. For this, we exploit the following fundamental fact: given any fixed τ_{step}, for ϵ_{small} depending on this choice, one has a certain priori control on the solution for time-interval τ_{step}, following essentially from continuity from the analogous control in the Schwarzschild case (see Sect. 8.1).

1.5 Related problems

1.5.1 Klein-Gordon

A related problem to the wave equation is that of the Klein-Gordon equation

$$\Box_g \psi = m^2 \psi \tag{19}$$

with $m > 0$. There is a well-developed scattering theory on Schwarzschild for the class of solutions of (19) with finite energy associated to the Killing $\mathbb{T}$. In particular, an asymptotic completeness statement has been proven in [4].

This analysis in of itself, however, when specialised to H^1_{loc} solutions in the geometric sense, only gives very weak information about the solution. In particular, it does not give L^2 control of ψ or its angular derivatives on $\mathcal{H}^+$.

In the case of Kerr, there are again certain partial results for (19) in the direction of scattering for a "non-superradiant" subspace of initial data [30]. These interesting results do not, however, address the characteristic difficulties of superradiance. See also [5].

1.5.2 Maxwell

Decay estimates for the Maxwell equation on Schwarzschild have been obtained by Blue. See [6].

1.5.3 Dirac on Kerr

Finally, we mention that there has been a series of interesting papers concerning the Dirac equation on Kerr and Kerr-Newman. See [26, 31]. For Dirac, considerations turn out to be much easier as this equation does not exhibit the phenomenon of superradiance. We shall not comment more about this here but refer the reader to [31].

1.5.4 $\Lambda \neq 0$

It is also interesting to consider (2) and related equations on background solutions of the Einstein equations with a cosmological constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$, for $\Lambda \neq 0$. In the $\Lambda = 0$ case, the natural analogue of the Schwarzschild family is the so-called Schwarzschild-de Sitter class. Boundedness in the region between the black hole and cosmological horizons, analogous to the statement of Theorem 1.1, was proven in [21]. Various dispersive estimates and decay results are proven independently in [11, 21].

1.6 Heuristic and numerical work

We cannot do justice here to the vast work on this subject in the physics literature. See [37] for a nice survey.

1.7 Addendum: subsequent developments

Since the original appearance of this paper on the arxiv in May 2008, there have been rapid further developments in this subject which we briefly mention.

1.7.1 Dispersion on exactly Kerr spacetimes

Recall that as described in Sects. 1.4.2 and 1.4.3, one would be able to prove integrated decay estimates (and together energy boundedness) for solutions of (2) on perturbations of Schwarzschild, as long as one could perturb the construction of $\mathbf{J}^{\mathbb{X}}$ near $r = 3M$ so as for $\mathbf{K}^{\mathbb{X}}$ to retain its nonnegativity properties. This approach for understanding dispersion has now been carried out independently by [22, 48] for Kerr spacetimes with $|a| \ll M$. (Our [22] uses Carter's separation (12) to frequency-localise the construction of $\mathbf{J}^{\mathbb{X}}$, whereas [48] uses a more standard pseudo-differential construction, exploiting however the complete integrability of geodesic flow. See also [49].) Both proofs indeed use the red-shift energy identity discussed in Sect. 1.4.2 to stabilise the construction at the horizon, as well as the red-shift commutation property introduced in the present paper (see Sect. 1.4.6) to obtain pointwise bounds up to the horizon. These proofs were followed later by [2], where frequency localisation on Kerr with $|a| \ll M$ is achieved by an alternative, quite attractive method based on higher order energy currents constructed via commutation with the Carter tensor. The result obtained, however, is weaker, requiring many derivatives on ψ, weights at infinity, and with estimates degenerating at the horizon. Finally, the general subextremal case $|a| < M$ is considered in [24]. It turns out that the insight that superradiant frequencies are not trapped (see Sect. 1.4.4) persists throughout the subextremal range of parameters, and this plays a fundamental role in the argument.[13]

1.7.2 More general spacetimes?

All three proofs [2, 22, 48] for $|a| \ll M$ and the proof [24] for the general case $|a| < M$ use in one form or another the hidden symmetries of Kerr,[14] and thus, as such, do not carry over to the more general class considered here. The very recent [55] strongly suggests that one might be able to enlarge the class of metrics for which one could show dispersion by appealing to structural stability results of geodesic flow. Not surprisingly, these results require however a very high amount of regularity of the metric g. This confirms then the point of view advanced in this paper, that the boundedness property is fundamentally more robust than dispersive properties.

[13] This insight can be thought of as a generalisation of the 'restricted pseudo-convexity' property which plays a role in unique continuation for stationary solutions of wave equations on black hole backgrounds [34]. Indeed, this latter property concerns the special case $\omega = 0$.

[14] In fact, for Ricci flat stationary axisymmetric spacetimes, separability of the wave equation (used in [22]), separability of geodesic flow (used in [48]), and the existence of a non-trivial additional Killing tensor (used in [2]) are equivalent. See [29].

1.7.3 Energy-flux and pointwise decay

The problem of passing from integrated decay to decay proper has been further studied in [2, 22, 40] extending ideas from the Schwarzschild case [10, 19]. The current state of the art is represented by two independent and very different techniques, developed in our [23] and Tataru [47], both giving a general framework for obtaining definitive decay-type estimates from the following three ingredients: (1) a quantitative boundedness statement of the type proven in the present paper, (2) a Morawetz-type integrated decay estimate, (3) good asymptotics at infinity.[15] The results of [47], based on Fourier-methods and resolvent estimates, are more refined but more fragile, rely heavily on exact stationarity, and yield sharp results in the context of smooth data of compact support. They can be compared with the strong Huygens principle. The results of [23] are completely physical-space based, rougher but more robust, in particular they do not depend at all on stationarity, and are sharp in the context of the norms typically used in nonlinear stability proofs. Indeed, the methods of [23] can be used to extend the domain of stability results for quasilinear wave equations on background metrics which do not tend to stationarity [56].

1.7.4 The redshift and surface gravity

The positivity properties associated to the use of $\mathbb{Y}$ as a multiplier as introduced in [19] and as a commutator as introduced in the present paper (see Sect. 1.4.6) have been extended in [22] to the neighborhood of general Killing horizons with positive surface gravity. The resultant estimates are thus applicable in particular to all stationary non-extremal classical black hole solutions of the Einstein equations. This immediately allows one to infer a variety of novel boundedness statements. See [22].

1.7.5 Other spacetimes

We mention extensions to n-dimensional Schwarzschild [39, 46], as well as the Klein-Gordon equation on Kerr-AdS (i.e. $\Lambda \neq 0$) for masses above the so-called Breitenlohner-Freedman bound [33]. Other more recent results for non-zero cosmological constant are contained in [25, 42, 51]. Another case of intense interest in the high-energy physics community is that of extremal black hole spacetimes, characterized precisely by the vanishing of surface

[15] We note that, to apply either method in the presence of horizons, it is fundamental that one control (in the boundedness and integrated decay estimates) transversal derivatives. In particular, both the original use of the red-shift vector field as a multiplier [19] and the use as a commutator introduced here (see Sect. 1.4.6) are essential.

gravity: Boundedness, but also blow up results, for the most basic example, namely extremal Reissner-Nordström, are contained in [3]. The characteristic difficulty of the extremal case is precisely the degeneration of the red-shift estimates of Sect. 1.7.4. A similar difficulty occurs for Kerr with $|a| = M$.

2 Constants, parameters and notational conventions

We summarise in this section some conventions regarding notations.

2.1 General constants

In the next section we shall fix once and for all a Schwarzschild metric with parameter $M > 0$. We shall use the notation B and b for general positive constants which only depend on the choice of M. An inequality true with a constant B will be true if B is replaced by a larger constant, and similarly, for b if b is replaced by a smaller positive constant.[16] We shall use the notation $f_1 \sim f_2$ to denote

$$bf_1 \le f_2 \le Bf_1.$$

Since B and b denote general constants, we shall apply without comment the obvious algebraic rules $B^2 = B$, $B^{-1} = b$, $b^2 = b$, etc.

2.2 Fixed parameters and functions

We will also require various particular fixed parameters which can be chosen depending only on M. From these parameters are defined various fixed functions.

Most parameters are not explicitly computed but are determined implicitly by various constraints. Before choosing a parameter, say parameter α, we shall use notation like $B(\alpha)$, $b(\alpha)$ to denote constants depending only on M and the as of yet unchosen α. It is to be understood that again here, the notation B indicates that the constant can always be replaced by a bigger one, and b by a smaller one. We shall also use the notation $R_1(\alpha)$ to indicate that the parameter R_1 depends on the still unchosen α. Once α is determined, we may replace the expressions $B(\alpha)$, $R(\alpha)$ etc., with B, R, etc.

Let us draw the reader's attention to what are perhaps the most important parameters:

$$\omega_0, e, \epsilon_{\text{close}} \text{ (small)}, \qquad \tau_{\text{step}} \text{ (large)}$$

The role of each of these parameters have already been discussed in Sect. 1.4.

[16] In the case of chains of inequalities, e.g. $f_1 \le Bf_2 \le Bf_3$ this convention is obviously violated and has to be reinterpreted appropriately.

2.3 Other notational conventions

As the reader may already have noticed, we have systematically used blackboard bold to denote vector fields, e.g. $\mathbb{T}$, $\mathbb{N}$, and bold to denote tensorial quantities quadratic in a function Ψ and its derivatives, for instant $\mathbf{T}_{\mu\nu}[\Psi]$, $\mathbf{K}[\Psi]$, $\mathbf{J}_\mu[\Psi]$. The lowercase $\mathbf{q}$, $\mathbf{q}_e$, are reserved for quadratic scalar quantities which are positive definite. These can be interpreted as energy densities. The uppercase $\mathbf{Q}$ are positive definite energies. We shall use ψ for the solution to (2) on an admissible spacetime as described in Sect. 3, whereas we shall use Ψ for an arbitrary spacetime function which could appear as an argument of $\mathbf{J}$, etc., to be thought of as quadratic functionals.

3 The class of spacetimes

In this section we shall describe the general class of metrics for which our results will apply. To set the stage, we must first fix some structures associated to a Schwarzschild metric.

3.1 Schwarzschild

We refer the reader to our previous [19] for a review of the geometry of Schwarzschild. We must first fix a certain subregion of Schwarzschild with parameter $M > 0$, relevant coordinates, and a choice of axisymmetric Killing field. This will provide the underlying manifold with stratified boundary[17] for the class of metrics to be considered later. Let us use the notation g_M to denote the Schwarzschild metric.

Refer to the Fig. 2 below.

In what follows, we specialise the constructions of Sect. 1.1 to the case $Q = a = 0$. We thus will denote by $\mathcal{D}$ the closure of a domain of outer communications in maximally extended Schwarzschild $(\mathcal{M}, g_M)$ with mass M.

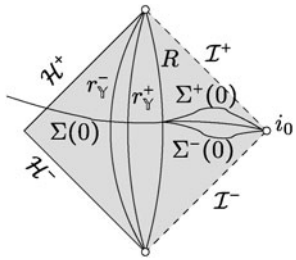


Fig. 2 The underlying differentiable structure and Schwarzschild metric on $\mathcal{D}$

[17]The boundary will be the union of two manifolds with boundary intersecting along their common boundary.

Recall that $\mathcal{D}$ can be alternatively characterized as the closure of a coordinate chart

$$\{r > 2M\} \times \{-\infty < t < \infty\} \times \{0 < \theta < \pi\} \times \{0 < \phi < 2\pi\} \tag{20}$$

in which the metric takes the form (9). For definiteness, let us fix a particular such chart, referring to it as 'standard Schwarzschild coordinates' and notating the coordinates without hats. Recall the notations $\mathcal{H}^{\pm}$ from Sect. 1.1.

The stationary Killing field $\mathbb{T}$ discussed in Sect. 1.1 is hypersurface orthogonal with respect to g_M. We say thus that Schwarzschild is 'static'. Moreover, $\mathbb{T}$ is strictly timelike on $\mathrm{int}(\mathcal{D})$ and null on $\mathcal{H}^+ \cup \mathcal{H}^-$, vanishing on $\mathcal{H}^+ \cap \mathcal{H}^-$. We recall finally the notation for the associated one-parameter family of diffeomorphisms $\rho_s : \mathcal{D} \to \mathcal{D}$ generated by $\mathbb{T}$. We note that in the Schwarzschild coordinate chart, ρ_s corresponds to translation of the t-coordinate by s. The choice of coordinate t is in fact unique up to overall translation. Recall that the coordinate t is not well defined on $\mathcal{H}^+ \cup \mathcal{H}^-$: Indeed, if $\gamma : [-1, 1] \to \mathcal{D}$, is a curve with $\gamma(\pm 1) \in \mathcal{H}^{\pm} \setminus \mathcal{H}^{\mp}$, $\gamma(-1, 1) \subset \mathrm{int}(\mathcal{D})$ then $\lim_{s \to \pm 1} t(\gamma(s)) = \pm\infty$. This is of course related to the degeneration of the expression (9).

The metric element (9) is manifestly spherically symmetric. We have in fact that $SO(3)$ acts on maximally extended Schwarzschild $(\mathcal{M}, g_M)$ by isometry, preserving $\mathcal{D}$. The choice of our Schwarzschild coordinate system (20) distinguishes a Killing field Φ as the unique Killing extension of the coordinate vector field ∂_ϕ to $\mathcal{M}$, and thus in particular, to $\mathcal{D}$.

The coordinate function r extends to a smooth function on $\mathcal{D}$. This function has the invariant geometric characterization $r(p) = \sqrt{4\pi \,\mathrm{Area}(S(p))}$, where $S(p)$ here denotes the unique $SO(3)$ symmetry group orbit containing p. On the boundary $\mathcal{H}^+ \cup \mathcal{H}^-$ we have $r = 2M$. We will use the notation μ for the function defined by $\mu = 2M/r$.

Associated to Schwarzschild will be the constants $2M < r_{\mathbb{Y}}^- < r_{\mathbb{Y}}^+$ determined in Sect. 5.2. We may assume say that

$$r_{\mathbb{Y}}^- \le \frac{9M}{4}. \tag{21}$$

We shall fix a hypersurface $\Sigma(0)$ as follows: Let $z(r)$ be a smooth function in $r > 2M$ such that

$$z(r) = 2M \log(r - 2M) - 2M \log((r_{\mathbb{Y}}^- - 2M)/2).$$

for $r \le 2M + (r_{\mathbb{Y}}^- - 2M)/4$, and

$$z(r) = 0$$

for $r \geq r^-_{\mathbb{Y}}$, and such that the hypersurface

$$t = -z(r) \tag{22}$$

is spacelike. Define now the hypersurface $\Sigma(0)$ to be the closure in $\mathcal{D}$ of the subset of $\mathrm{int}(\mathcal{D})$ defined by (22). One sees easily that $\Sigma(0)$ becomes a spacelike manifold with boundary, where $\partial\Sigma(0)$ is a sphere on $\mathcal{H}^+ \setminus \mathcal{H}^-$. Note of course that in view of the support of z, it follows that in the region $r \geq r^-_{\mathbb{Y}}$, $\Sigma(0)$ coincides with the constant $t = 0$ hypersurface. Thus, $\Sigma(0)$ can be thought of as a specific choice of what in Sect. 1.1 was more generally denoted Σ_0.

We may define a new coordinate

$$t^* \doteq t + z(r),$$

or more precisely, the smooth extension of this expression to $\mathcal{D} \setminus \mathcal{H}^-$. The coordinate t^* is thus regular on $\mathcal{H}^+ \setminus \mathcal{H}^-$. We have that

$$\Sigma(0) = \{t^* = 0\}.$$

Let us define

$$\Sigma(\tau) = \{t^* = \tau\}.$$

Clearly $\Sigma(\tau) = \rho_\tau(\Sigma(0))$.

We have that

$$B \geq -g_M(\nabla t^*, \nabla t^*) \geq b > 0 \tag{23}$$

for some constants B, b. Recall here the conventions of Sect. 2.

For technical reasons, we shall require two auxiliary sets of spacelike hypersurfaces. Let $\chi(x)$ be a cutoff function such that $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Let us define

$$t^+ = t^* - \chi(-r + R)(1 + r - R)^{1/2}$$

and

$$t^- = t^* + \chi(-r + R)(1 + r - R)^{1/2}$$

for an R to be determined later with $R \geq r^-_{\mathbb{Y}} + 1$. Let us define

$$\Sigma^+(\tau) \doteq \{t^+ = \tau\}, \qquad \Sigma^-(\tau) \doteq \{t^- = \tau\}.$$

For R sufficiently large, we have that $\Sigma^\pm$ are spacelike, in fact

$$B \geq -g_M(\nabla t^+, \nabla t^+) \geq b > 0, \qquad B \geq -g_M(\nabla t^-, \nabla t^-) \geq b > 0. \tag{24}$$

In what follows we shall restrict to

$$\mathcal{R} \doteq \mathcal{D} \cap J^+_{g_M}(\Sigma^-(0)).$$

The set $\mathcal{R}$ is again a manifold with stratified boundary (as was $\mathcal{D}$), where the boundary is given by $\Sigma^-(0) \cup (\mathcal{H}^+ \cap J^+_{g_M}(\Sigma^-(0)))$.

We note that $\mathcal{R}$ is mapped by coordinates

$$(r, t^*, \theta, \phi) \tag{25}$$

into

$$\{r \geq 2M\} \times \{t^* \geq -\chi(-r+R)(1+r-R)^{1/2}\} \times \{0 \leq \theta \leq \pi\} \times \{0 \leq \phi \leq 2\pi\}$$

with the usual caveat that (25) is only a valid coordinate system where $\pi \neq 0, \pi$, and $\phi \neq 0, 2\pi$. We note that $\partial_{t^*} = \mathbb{T}$, $\partial_\phi = \Phi$ in this coordinate chart. In view of the above degeneration, the coordinates (25) will not be useful for formulating closeness assumptions, and we must introduce yet another coordinate system.

Choosing then a coordinate atlas consisting of two charts (ξ^A, ξ^B), $(\tilde{\xi}^A, \tilde{\xi}^B)$ on the standard sphere $\mathbb{S}^2$, then setting $x^A = r^{-1}\xi^A$, $\tilde{x}^A = r^{-1}\tilde{\xi}^A$, it follows that

$$(r, t^*, x^A, x^B), \qquad (r, t^*, \tilde{x}^A, \tilde{x}^B) \tag{26}$$

form a coordinate atlas for $\mathcal{R}$. We can ensure moreover that the regions of the sphere covered by the charts are restricted so that the metric functions satisfy

$$(g_M)_{ij} \leq B, \qquad g_M^{ij} \leq B \tag{27}$$

in these coordinates. Note that with respect to both these charts, we have $\partial_{t^*} = \mathbb{T}$. We will use the above coordinate atlas (26) in formulating our closeness assumptions.

Finally, we shall also at times refer to so-called Regge-Wheeler coordinates

$$(r^*, t, x^A, x^B).$$

Here t is the standard Schwarzschild time defined previously and the coordinate r^* is defined by

$$r^* \doteq r + 2M\log(r-2M) - 3M - 2M\log M.$$

Note that this coordinate is regular in $\mathcal{R} \setminus \mathcal{H}^+$, but sends the boundary component $\mathcal{H}^+$ to $r^* = -\infty$. With respect to Regge-Wheeler coordinates, we note that the coordinate vector field ∂_{r^*} extends to a smooth vector field on $\mathcal{H}^+ \cap \mathcal{R}$, and in fact, in the limit, $\partial_{r^*} = \mathbb{T}$ on $\mathcal{H}^+ \cap \mathcal{R}$.

This last coordinate system is not useful for formulating closeness assumptions in view of the fact that it breaks down on the horizon. We shall only use Regge-Wheeler coordinates for making calculations with respect to the Schwarzschild metric.

Finally, a word of caution. Since we have several coordinate systems which will be considered, coordinate vectors like ∂_{t^*} will always be referred to in conjunction with a specific coordinate system.

3.2 The general class

We now describe the class of metrics to be allowed.

We consider the manifold with stratified boundary $\mathcal{R}$ defined above, with boundary $(\mathcal{H}^+ \cap \mathcal{R}) \cup \Sigma^-(0)$. Let us denote in what follows $\mathcal{H}^+ \cap \mathcal{R}$ more simply as $\mathcal{H}^+$. We consider fixed the Schwarzschild metric g_M, the Schwarzschild coordinates (20), the vector fields $\mathbb{T}$ and Φ, and all derived coordinate systems as in Sect. 3.1. (Recall that the Schwarzschild Killing fields correspond to the vector fields $\mathbb{T} = \partial_{t^*}$, $\Phi = \partial_\phi$ with respect to (r, t^*, θ, ϕ) coordinates, with the usual caveat of the degeneration of the coordinates on the sphere.) We consider now the class of all C^1 Lorentzian metrics g on $\mathcal{R}$ such that:

(1) For $\epsilon_{\mathrm{close}} > 0$ sufficiently small,

$$|g_{ij} - (g_M)_{ij}| \le \epsilon_{\mathrm{close}} r^{-2}, \qquad |g^{ij} - (g_M)^{ij}| \le \epsilon_{\mathrm{close}} r^{-2}, \tag{28}$$

$$|\partial_m g_{ij} - \partial_m (g_M)_{ij}| \le \epsilon_{\mathrm{close}} r^{-2} \tag{29}$$

with respect to the atlas (26).[18]

(2) The Schwarzschild Killing fields $\mathbb{T}$ and Φ are again Killing with respect to g.

(3) There is a C^1 function γ defined on $\mathcal{H}^+$ such that $\mathbb{T} + \gamma\Phi$ is null on the horizon, and

$$|\gamma| < \epsilon_{\mathrm{close}}. \tag{30}$$

In particular, Assumption 3.2 above implies that $\mathcal{H}^+$ is null with respect to g and its null generator lies in the span of $\mathbb{T}$ and Φ. We may define the *ergoregion* $\mathcal{E} \subset \mathcal{R}$ corresponding to g to be the closure of the region where $\mathbb{T}$ is spacelike with respect to g.

For sufficiently small $\epsilon_{\mathrm{close}}$, assumptions (28) and (23) imply that $\Sigma(0)$ is spacelike with respect to g, in fact, with our conventions on constants,

$$B \ge -g(\nabla t^*, \nabla t^*) \ge b. \tag{31}$$

[18] When specialised to the case of Kerr-Newman, this clearly will *not* be the Boyer-Lindquist $\hat{r}$ referred to previously. Hence our distinction in notation. For the relation to Kerr-Newman, see Sect. 3.3.

Similarly, we have from (24) that for ϵ_{close} sufficiently small, $\Sigma^+(\tau)$ and $\Sigma^-(\tau)$ are spacelike, in fact

$$B \geq -g(\nabla t^{\pm}, \nabla t^{\pm}) \geq b, \tag{32}$$

independently of the choice of (sufficiently large) R.

Note that $\Sigma(\tau)$ is again isometric to $\Sigma(0)$ with respect to g, and similarly $\Sigma^{\pm}(\tau)$ is isometric to $\Sigma^{\pm}(0)$. We will denote by $\mathbb{n}_{\Sigma}$ the future normal of $\Sigma(\tau)$:

$$\mathbb{n}^{\mu}_{\Sigma} \doteq (-g(\nabla t^*, \nabla t^*))^{-1/2} \nabla^{\mu} t^*.$$

This defines a translation invariant smooth timelike unit vector field on $\mathcal{R}$. Similarly, we define

$$\mathbb{n}^{\mu}_{\Sigma^{\pm}} \doteq (-g(\nabla t^{\pm}, \nabla t^{\pm}))^{-1/2} \nabla^{\mu} t^{\pm}.$$

We will use the notations

$$\mathcal{R}(\tau', \tau'') \doteq \bigcup_{\tau' \leq \bar{\tau} \leq \tau''} \Sigma(\bar{\tau}),$$

$$\mathcal{R}^+(\tau', \tau'') \doteq \bigcup_{\tau' \leq \bar{\tau} \leq \tau''} \Sigma^+(\bar{\tau}),$$

$$\mathcal{R}^-(\tau', \tau'') \doteq \bigcup_{\tau' \leq \bar{\tau} \leq \tau''} \Sigma^-(\bar{\tau}),$$

$$\mathcal{H}(\tau', \tau'') \doteq \mathcal{H}^+ \cap \mathcal{R}(\tau', \tau'').$$

All integrals without an explicit measure of integration are to be taken with respect to the volume form in the case of a region of spacetime or a spacelike hypersurface, and an induced volume form connected to the choice of a ρ_t-invariant tangential vector field $\mathbb{n}^{\mu}_{\mathcal{H}}$, in the case of $\mathcal{H}(\tau', \tau'')$.

Note the following property of the volume integral with respect to the (almost) global (t, r, ϕ, θ) coordinate system: There exist smooth $\nu(\theta, r) \geq 0$, $\tilde{\nu}(\theta) \geq 0$ such that for all continuous f:

$$\int_{\mathcal{R}(\tau', \tau'')} f = \int_{2M}^{\infty} \int_0^{\pi} \nu(\theta, r) \left(\int_{\tau'}^{\tau''} \left(\int_0^{2\pi} f \, d\phi \right) dt^* \right) d\theta \, dr,$$

$$\int_{\mathcal{H}(\tau', \tau'')} f = \int_0^{\pi} \tilde{\nu}(\theta) \left(\int_{\tau'}^{\tau''} \left(\int_0^{2\pi} f \, d\phi \right) dt^* \right) d\theta.$$

Also let us note that

$$\int_{\mathcal{R}(\tau',\tau'')} f = \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \left(-g(\nabla t^*, \nabla t^*)\right)^{-1/2} f \right) d\bar\tau.$$

By (31), it follows that if $f_1 \sim f_2$ in the sense $0 < bf_1 \leq f_2 \leq Bf_1$, it follows that

$$\int_{\mathcal{R}(\tau',\tau'')} f_1 \sim \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} f_2 \right) d\bar\tau.$$

A similar relation holds with $\mathcal{R}^\pm$ and $\Sigma^\pm$.

3.3 The Kerr and Kerr-Newman metrics

Proposition 3.1 *Given arbitrary $\epsilon_{\mathrm{close}} > 0$, $M > 0$, then for $Q \ll M$, $|a| \ll M$ depending on $\epsilon_{\mathrm{close}}$, the Kerr-Newman metric $g = g_{M,Q,a}$ with parameters Q, a can be defined on $\mathcal{R}$ so as to satisfy the assumptions of Sect.* 3.2.

Let us sketch how one can implicitly define a Kerr-Newman metric on $\mathcal{R}$ in our (r, t^*, θ, ϕ) coordinate system.

For convenience, let us do this by defining a new set of coordinates on $\mathrm{int}(\mathcal{R})$ which are to represent Boyer-Lindquist coordinates $(\hat r, \hat t, \hat\theta, \hat\phi)$. For this define $\hat r$ by

$$r^2 - 2Mr = \hat r^2 - 2M\hat r + Q^2 + a^2.$$

Let $z(r, a, Q)$ be a function smooth in its arguments for $r > 2M$, $|a| \ll M$, $Q \ll M$, such that $z(r, 0, 0) = z(r)$ defined previously, and such that

$$\frac{dz}{d\hat r}(r(\hat r), a, Q) = \frac{2M\hat r - Q^2}{\hat r^2 - 2M\hat r + Q^2 + a^2}$$

for $r \leq 2M + (r_{\mathbb{Y}}^- - 2M)/4$, and $\frac{dz}{d\hat r} = 0$ for $r \geq r_{\mathbb{Y}}^-$, and $|\frac{dz}{d\hat r}| \leq B(a + Q)$ for $2M + (r_{\mathbb{Y}}^- - 2M)/4 \leq r(\hat r) \leq r_{\mathbb{Y}}^-$. Define now $\hat t$ by

$$\hat t = t^* - z(r, a, Q).$$

Define $\hat\phi$ by

$$\hat\phi = \phi - P(\hat r)$$

where $\frac{dP}{d\hat r} = \frac{a}{\hat r^2 - 2M\hat r + Q^2 + a^2}$, and $\hat\theta$ by

$$\hat\theta = \theta.$$

Note that $\partial_t = \partial_{\hat{t}}$, $\partial_\phi = \partial_{\hat{\phi}}$ in the intersection of the domains of the coordinate systems. Now consider the metric on $\mathrm{int}(\mathcal{R})$ defined in these new coordinates by the expression (3). Rewriting the metric in (r, t^*, θ, ϕ) coordinates, and then relating this form in turn to the coordinates of (26) one sees immediately that

$$r^2(g_{ij} - (g_M)_{ij}) \to 0, \qquad r^2(g^{ij} - g_M^{ij}) \to 0$$

uniformly as $a \to 0$, and

$$r^3(\partial_k g_{ij} - \partial_k (g_M)_{ij}) \to 0$$

uniformly as $a \to 0$, where i, j, k denote coordinates of (26). It follows that given $\epsilon_{\mathrm{close}}$, the assumptions (28) and (29) hold for $Q \ll M$, $|a| \ll M$ sufficiently small. The remaining assumptions are well-known properties of Kerr which are manifest from the Boyer-Lindquist form (3).

We remark that the above prescription is in no way unique. Among other properties, the above prescription satisfies that when applied to $g_{M,0,0}$ one obtains $g_{M,0,0} = g_M$ (as opposed to $g_{M,0,0} = \varphi^* g_M$ for some diffeomorphism φ), and also, that $\hat{t} = c$ hypersurfaces coincide with $t^* = c'$ (and thus also with $t = c'$) for large r. By introducing a cutoff in the definition of P, one could also ensure that $\phi = \hat{\phi}$ for large r, if desired. Similarly, one could ensure that c' be independent of parameters a, Q. None of these properties is in fact of much significance.

4 Preliminaries

4.1 Well posedness and the class of admissible solutions ψ

Let $(\mathcal{R}, g)$, $\Sigma(\tau)$ be as in Sect. 3.2, and let Ψ be an H^1_{loc} function on $\Sigma(0)$, and let Ψ' be an L^2_{loc} function on $\Sigma(0)$. Here the L^2 norm is defined naturally with respect to the induced Riemannian metric on $\Sigma(0)$. By standard theory, there exists a unique solution ψ of the initial value problem

$$\Box_g \psi = 0, \qquad \psi|_{\Sigma(0)} = \Psi, \qquad n_\Sigma \psi|_{\Sigma(0)} = \Psi', \tag{33}$$

with the property that

$$\psi \in C^1(H^1_{\mathrm{loc}}(\Sigma(\tau)), \qquad n_\Sigma \psi \in C^0(L^2_{\mathrm{loc}}(\Sigma(\tau)).$$

For our most basic boundedness result, we shall require precisely the C^1 regularity of the metric as in Sect. 3.2 and the following regularity and boundedness for initial data:

$$\nabla^\Sigma \Psi \in L^2(\Sigma(0)), \qquad \Psi' \in L^2(\Sigma(0)). \tag{34}$$

For higher energy and pointwise boundedness we shall require progressively greater regularity for both the underlying metric and for initial data.

By density arguments, one can in fact prove all results under the assumption that the metric g and data ψ, ψ' are in fact smooth, and thus, that ψ is smooth. Moreover, we can safely assume that $\nabla^{\Sigma}\psi$ and ψ' are supported away from infinity. Let us assume this in what follows so as not to have to comment on regularity issues or the a priori finiteness of certain quantities. It follows in particular from this assumption that

$$\nabla\psi \in L^2(\Sigma(\tau)), \qquad \nabla\psi \in L^2(\Sigma^{\pm}(\tau)), \tag{35}$$

moreover, that $\nabla\psi$ is supported away from spatial infinity.

4.2 The uniform energy boundedness theorem

For a sufficiently regular function Ψ, let us define

$$\mathbf{T}_{\mu\nu}[\Psi] \doteq \partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi \tag{36}$$

and for $\mathbb{V}^\mu$ a vector field,

$$\mathbf{J}^{\mathbb{V}}_\mu[\Psi] \doteq \mathbf{T}_{\mu\nu}[\Psi]\mathbb{V}^\nu. \tag{37}$$

In addition, let us define the quantity

$$\mathbf{q}[\Psi] \doteq \mathbf{J}^{\mathbb{n}_\Sigma}_\mu[\Psi]\mathbb{n}_\Sigma{}^\mu.$$

Note that this is non-negative. Moreover, in the coordinate charts of the atlas (26), we have

$$\mathbf{q}[\Psi] \sim \sum_i(\partial_i\Psi)^2. \tag{38}$$

We have

$$\int_{\Sigma(0)}\mathbf{q}[\psi] \sim B\left(\|\psi'\|^2_{L^2} + \|\nabla^\Sigma\psi\|^2_{L^2}\right)$$

and thus, by (34) and (35), for all $\tau \geq 0$,

$$\int_{\Sigma(\tau)}\mathbf{q}[\psi] < \infty, \qquad \int_{\Sigma^{\pm}(\tau)}\mathbf{q}[\psi] < \infty.$$

The first main result of our paper concerns the uniform boundedness of this quantity.

Theorem 4.1 *There exist positive constants ϵ_{close}, C depending only on $M>0$ such that the following holds. Let g, $\Sigma(\tau)$ be as in Sect. 3.2 and let ψ, ψ', ψ be as in Sect. 4.1 where ψ satisfies (2). Then, for $\tau \geq 0$,*

$$\int_{\Sigma(\tau)} \mathbf{q}[\psi] \leq C \int_{\Sigma(0)} \mathbf{q}[\psi]. \tag{39}$$

When specialised to the Kerr and Kerr-Newman case, Theorem 4.1 yields precisely inequality (6) of Theorem 1.1. (The reduction to our particular $\Sigma(0)$ of Theorem 4.1 from the arbitrary Σ_0 of Theorem 1.1 follows from the remark at the end of Sect. 1.1 while the universality of the constant ϵ in the statement of that theorem follows a posteriori from a simple scaling argument.)

4.3 The auxiliary positive definite quadratic quantities $\mathbf{q}_e$ and $\mathbf{q}_e^\star$

We note that given $e>0$, for small enough $\epsilon_{\text{close}} \ll e$, the vector field $\mathbb{T}+e\mathbb{n}_\Sigma$ is timelike. For sufficiently regular Ψ, let us define

$$\mathbf{q}_e[\Psi] = \mathbf{J}_\mu^{\mathbb{T}+e\mathbb{n}_\Sigma}[\Psi]\mathbb{n}_\Sigma^\mu.$$

Note that

$$eb\mathbf{q}[\Psi] \leq \mathbf{q}_e[\Psi] \leq B\mathbf{q}[\Psi]. \tag{40}$$

Thus, to prove Theorem 4.1, it is sufficient to prove (39) with $\mathbf{q}_e$ replacing $\mathbf{q}$. The significance of the parameter e has been discussed in Sect. 1.4.2 and will become clear in the context of the proof.

We shall need also a weaker positive definite quantity defined as follows. Let $\chi_{\mathbb{Y}} = \chi_{\mathbb{Y}}(r)$ be a cutoff function such that $\chi_{\mathbb{Y}} = 1$ for $r \leq 2M + (r_{\mathbb{Y}}^- - 2M)/2$ and $\chi_{\mathbb{Y}} = 0$ for $r \geq r_{\mathbb{Y}}^-$. For a sufficiently regular function Ψ, define

$$\mathbf{q}_e^\star[\Psi] = r^{-2}\mathbf{J}_\mu^{(1-\chi_{\mathbb{Y}})\mathbb{T}+e\mathbb{n}_\Sigma}[\Psi]\mathbb{n}_\Sigma^\mu.$$

This quantity is related to lower bounds for the 0-current $\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e}$ to be constructed in Sect. 5.3, by inequalities (72), (73) and (74).

Note that we have

$$ebr^{-2}\mathbf{q}[\Psi] \leq \mathbf{q}_e^\star[\Psi] \leq Br^{-2}\mathbf{q}[\Psi].$$

Note also that for $r \geq r_{\mathbb{Y}}^-$, we have

$$\mathbf{q}_e[\Psi] \sim \mathbf{q}[\Psi]$$

and

$$\mathbf{q}_e^\star[\Psi] \sim r^{-2}\mathbf{q}[\Psi] \sim r^{-2}\mathbf{q}_e[\Psi]. \tag{41}$$

For all $r \geq 2M$, we have

$$\mathbf{q}_e[\Psi] \leq Be^{-1} r^2 \mathbf{q}_e^{\bigstar}[\Psi], \tag{42}$$

$$\mathbf{q}[\Psi] \leq Be^{-1} r^2 \mathbf{q}_e^{\bigstar}[\Psi]. \tag{43}$$

4.4 The basic identity for currents

For an arbitrary suitably regular function Ψ such that $\nabla\Psi$ is supported away from spatial infinity, recall from (36) and (37) the definitions of $\mathbf{T}_{\mu\nu}$ and $\mathbf{J}_\mu$. Define also

$$\mathbf{K}^{\mathbb{V}}[\Psi] \doteq \mathbf{T}_{\mu\nu}[\Psi]\nabla^\mu \mathbb{V}^\nu.$$

We have

$$\nabla^\mu \mathbf{J}^{\mathbb{V}}_\mu[\Psi] = \mathbf{K}^{\mathbb{V}}[\Psi] + F\mathbb{V}^\nu \Psi_\nu$$

where

$$F \doteq \Box_g \Psi.$$

Thus, setting

$$\mathbf{Er}^{\mathbb{V}}[\Psi] = -F\mathbb{V}^\nu\Psi_\nu, \tag{44}$$

we have the identity

$$\begin{aligned} &\int_{\mathcal{H}(\tau',\tau'')} \mathbf{J}^{\mathbb{V}}_\mu[\Psi]\mathrm{n}^\mu_{\mathcal{H}} + \int_{\Sigma(\tau'')} \mathbf{J}^{\mathbb{V}}_\mu[\Psi]\mathrm{n}^\mu_\Sigma + \int_{\mathcal{R}(\tau'.\tau'')} \mathbf{K}^{\mathbb{V}}[\Psi] \\ &\quad = \int_{\Sigma(\tau')} \mathbf{J}^{\mathbb{V}}_\mu[\Psi]\mathrm{n}^\mu_\Sigma + \int_{\mathcal{R}(\tau',\tau'')} \mathbf{Er}^{\mathbb{V}}[\Psi]. \end{aligned} \tag{45}$$

We will also consider currents modified as follows. Given a function w, define $\mathbf{J}^{\mathbb{V},w}_\mu$ by

$$\mathbf{J}^{\mathbb{V},w}_\mu[\Psi] = \mathbf{J}^{\mathbb{V}}_\mu[\Psi] + \frac{1}{8} w\partial_\mu(\Psi^2) - \frac{1}{8}(\partial_\mu w)\Psi^2, \tag{46}$$

$$\mathbf{K}^{\mathbb{V},w}[\Psi] = \mathbf{K}^{\mathbb{V}}[\Psi] - \frac{1}{8}(\Box_g w)\Psi^2 + \frac{1}{4} w\nabla^\alpha\Psi\nabla_\alpha\Psi,$$

$$\mathbf{Er}^{\mathbb{V},w}[\Psi] = \mathbf{Er}^{\mathbb{V}}[\Psi] - \frac{1}{4} w\Psi F. \tag{47}$$

Identity (45) also holds for $\mathbf{J}^{\mathbb{V},w}$ as long as appropriate assumptions are made in a neighborhood of spatial infinity. We will always apply $\mathbf{J}^{\mathbb{V},w}$ to Ψ with $\Psi_0 = 0$, and thus, by our assumptions in Sect. 4.1 on $\nabla\Psi$, such Ψ will in fact be supported away from spatial infinity.

It will be useful to have a separate notation for currents as defined with respect to the Schwarzschild metric. For these we use the notation $(\mathbf{J}^{\mathbb{V}}_{g_M})_\mu$, $\mathbf{K}^{\mathbb{V}}_{g_M}$, $(\mathbf{J}^{\mathbb{V},w}_{g_M})_\mu$, etc.

Suppose that $\mathbb{V}$ is a vector field such that its components $\mathbb{V}^i$ are bounded in the atlas (26). It follows from (28) that

$$\left|(\mathbf{J}^{\mathbb{V},w}_{g_M})_\mu[\Psi]\mathrm{n}^\mu - \mathbf{J}^{\mathbb{V},w}_\mu[\Psi]\mathrm{n}^\mu\right| \le B\epsilon_{\mathrm{close}} r^{-2}\max_i |\mathbb{V}_i| \sum (\partial_i\Psi)^2. \tag{48}$$

The above applies in particular if $w=0$, i.e. for the case $\mathbf{J}^{\mathbb{V}}_{g_M}$. (In fact, the w term disappears from the difference above.) Note that if the components of $\mathrm{n}_\mu - \tilde{\mathrm{n}}_\mu$ are less than $B\epsilon_{\mathrm{close}} r^{-2}$ we have by the triangle inequality

$$\left|(\mathbf{J}^{\mathbb{V},w}_{g_M})_\mu[\Psi]\mathrm{n}^\mu - \mathbf{J}^{\mathbb{V},w}_\mu[\Psi]\tilde{\mathrm{n}}^\mu\right| \le B\epsilon_{\mathrm{close}} r^{-2}(|w| + \max_i(|\mathbb{V}_i| + |\partial_i w|)) \cdot \sum (\partial_i\Psi)^2. \tag{49}$$

Note also that if $\mathbb{V}^j$, $\partial_i\mathbb{V}^j$, w, $\partial_i w$ and $\partial_i\partial_j w$ are bounded with respect to (26), where then from (28), (29), we obtain

$$\left|\mathbf{K}^{\mathbb{V},w}_{g_M}[\Psi] - \mathbf{K}^{\mathbb{V},w}[\Psi]\right| \le B\epsilon_{\mathrm{close}} r^{-2}\left(\max_{ij}\max_{p_k=0,1} |\partial_j^{p_j}\mathbb{V}^i| + |\partial_i^{p_i}\partial_j^{p_j} w|\right) \cdot \sum (\partial_i\Psi)^2. \tag{50}$$

If F above vanishes, then $\mathbf{J}^{\mathbb{V},w}_\mu$ are examples of *compatible currents* in the sense of [15]. This is a unifying principle for understanding the structure behind much of the analysis for Lagrangian equations like (2).

5 The vector fields and their currents

5.1 The vector field $\mathbb{T}$

Since $\mathbb{T}$ is Killing we have

$$\mathbf{K}^{\mathbb{T}}[\Psi] = 0.$$

In $r \ge r^-_{\mathbb{Y}}$, $\mathbb{T}$ is timelike and moreover we have

$$\mathbf{J}^{\mathbb{T}}_\mu[\Psi]\mathrm{n}^\mu_\Sigma \sim \mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi]\mathrm{n}^\mu_\Sigma$$

in that region. In all regions we have

$$\left|\mathbf{J}^{\mathbb{T}}_\mu[\Psi]\mathrm{n}^\mu_\Sigma\right| \le B\mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi]\mathrm{n}^\mu_\Sigma, \qquad \left|\mathbf{J}^{\mathbb{T}}_\mu[\Psi]\mathrm{n}^\mu_{\mathcal{H}}\right| \le B\mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi]\mathrm{n}^\mu_{\mathcal{H}}.$$

For $\epsilon_{\text{close}} \ll e$ we have

$$\left|\mathbf{J}^{\mathbb{T}}_{\mu}[\Psi]\mathrm{n}^{\mu}_{\Sigma}\right| \le B\mathbf{q}_e[\Psi]. \tag{51}$$

5.2 The vector fields $\mathbb{Y}$ and $\mathbb{N}_e = \mathbb{T} + e\mathbb{Y}$

Let (u, v) denote Eddington-Finkelstein null coordinates[19] on $\text{int}(\mathcal{D})$ and let r^* denote the Regge-Wheeler coordinate. In the paragraph that follows, coordinate derivatives are with respect to say (u, v, x^A, x^B) coordinates, whereas y_1', y_2' denote $\frac{dy_1}{dr^*}$, etc.

Recall from [19] that for a vector field $\mathbb{Y}$ of the form:

$$\mathbb{Y} = y_1(r^*)\frac{1}{1-\mu}\frac{\partial}{\partial u} + y_2(r^*)\frac{\partial}{\partial v},$$

we have

$$\begin{aligned}\mathbf{K}_{g_M}[\Psi] = {} & \frac{(\partial_u\Psi)^2}{2(1-\mu)^2}\left(\frac{y_1\mu}{r} - y_1'\right) + (\partial_v\Psi)^2\frac{y_2'}{2(1-\mu)} \\ & + \frac{1}{2}|\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{y_1'}{1-\mu} - \frac{(y_2(1-\mu))'}{1-\mu}\right) \\ & - \frac{1}{r}\left(\frac{y_1}{1-\mu} - y_2\right)\partial_u\Psi\partial_v\Psi.\end{aligned}$$

Let us define $y_1 = \tilde{\chi}_{\mathbb{Y}}(r)(1 + (1-\mu))$, $y_2 = \tilde{\chi}_{\mathbb{Y}}(r)\delta^{-1}(1-\mu)$ where $\tilde{\chi}_{\mathbb{Y}}$ is a cutoff function such that $\tilde{\chi}_{\mathbb{Y}} = 1$ for $r \le r^-_{\mathbb{Y}}$, and $\tilde{\chi}_{\mathbb{Y}} = 0$ for $r \ge r^+_{\mathbb{Y}}$, for two parameters $2M < r^-_{\mathbb{Y}} < r^+_{\mathbb{Y}}$, and a small constant δ. (We then consider y_i as a function of r^* in the usual manner $y_i(r^*) = y_i(r(r^*))$.) One sees easily that there exist such parameters such that for $r \le r^-_{\mathbb{Y}}$,

$$\left(\frac{y_1\mu}{r} - y_1'\right) \ge b, \qquad \frac{y_2'}{2(1-\mu)} \ge b, \qquad \frac{y_1'}{1-\mu} - \frac{(y_2(1-\mu))'}{1-\mu} \ge b,$$

$$\begin{aligned}&\left|-\frac{1}{r}\left(\frac{y_1}{1-\mu} - y_2\right)\partial_u\Psi\partial_v\Psi\right| \\ &\quad \le \frac{1}{2}\left(\frac{(\partial_u\Psi)^2}{2(1-\mu)^2}\left(\frac{y_1\mu}{r} - y_1'\right) + (\partial_v\Psi)^2\frac{y_2'}{2(1-\mu)}\right).\end{aligned}$$

[19] See [20, 21]. Our use of this terminology is somewhat non-standard. Here $v \doteq (t + r^*)/2$, $u \doteq (t - r^*)/2$.

Let us return now to the coordinate charts of our (26). We see from the above that the vector field $\mathbb{Y}$ has the property that in $r \le r_{\mathbb{Y}}^-$,

$$B \sum_i (\partial_i \Psi)^2 \ge \mathbf{K}^{\mathbb{Y}}_{g_M}[\Psi] \ge b \sum_i (\partial_i \Psi)^2 \tag{52}$$

where i, j refer to the coordinate charts of (26) whereas we easily see also that in $r_{\mathbb{Y}}^- \le r \le r_{\mathbb{Y}}^+$

$$|\mathbf{K}^{\mathbb{Y}}_{g_M}| \le B \sum_i (\partial_i \Psi)^2. \tag{53}$$

Finally, for $r \ge r_{\mathbb{Y}}^+$, $\mathbb{Y} = 0$.

Moreover, we note that $\mathbb{Y}$ is a regular vector field, in particular, when expressed with respect to the coordinates of (26), we have $\max |\mathbb{Y}^i| \le B$, $\max |\partial_i \mathbb{Y}^j| \le B$.

Because all derivatives appear on the right hand sides of (52) and (53), these inequalities are stable, i.e. it follows from (50) that for ϵ_{close} sufficiently small,

$$\mathbf{K}^{\mathbb{Y}}[\Psi] \sim \sum_i (\partial_i \Psi)^2 \sim \mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi] \mathrm{n}^\mu_\Sigma \tag{54}$$

in $r \le r_{\mathbb{Y}}^-$, and

$$\left| \mathbf{K}^{\mathbb{Y}} \right| \le B \sum_i (\partial_i \Psi)^2 \le B \mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi] \mathrm{n}^\mu_\Sigma \tag{55}$$

in $r_{\mathbb{Y}}^- \le r \le r_{\mathbb{Y}}^+$, while certainly $\mathbf{K}^{\mathbb{Y}} = 0$ for $r \ge r_{\mathbb{Y}}^+$.

Define

$$\mathbb{N}_e = \mathbb{T} + e\mathbb{Y}.$$

Note that

$$\mathbf{K}^{\mathbb{N}_e} = \mathbf{K}^{\mathbb{T}} + e\mathbf{K}^{\mathbb{Y}} = e\mathbf{K}^{\mathbb{Y}}.$$

In the region $r_{\mathbb{Y}}^- < r < r_{\mathbb{Y}}^+$, we have by (55)

$$\left| \mathbf{K}^{\mathbb{N}_e}[\Psi] \right| \le Be \mathbf{J}^{\mathrm{n}_\Sigma}_\mu[\Psi] \mathrm{n}^\mu_\Sigma \le eB \mathbf{q}^\bigstar_e[\Psi].$$

Note the factor of e. In the region $r \le r_{\mathbb{Y}}^-$, we certainly have by (54)

$$\mathbf{K}^{\mathbb{N}_e}[\Psi] \ge b \mathbf{q}^\bigstar_e[\Psi]. \tag{56}$$

For $r \ge r_{\mathbb{Y}}^+$, we have of course

$$\mathbf{K}^{\mathbb{N}_e} = e\mathbf{K}^{\mathbb{Y}} = 0.$$

In particular, the bound

$$-\mathbf{K}^{\mathbb{N}_e}[\Psi] \leq eB\mathbf{q}_e^{\bigstar}[\Psi] \tag{57}$$

holds in all regions.

With the help of (54) and (57), we obtain easily that

$$\mathbf{q}_e[\Psi] \leq B\left(\mathbf{K}^{\mathbb{N}_e}[\Psi] + \mathbf{J}_\mu^{\mathbb{T}}[\Psi]\mathrm{n}_\Sigma^\mu\right) \tag{58}$$

holds everywhere, if e is sufficiently small.

By similar considerations to the above, we see that given e, by requiring $\epsilon_{\text{close}} \ll e$ sufficiently small, we have that $\mathbb{N}_e$ is timelike everwhere up to the boundary, and in fact

$$\mathbf{J}_\mu^{\mathbb{N}_e}[\Psi]\mathrm{n}_\Sigma^\mu \sim \mathbf{q}_e[\Psi]. \tag{59}$$

On the other hand, since by Assumption 3.2, $\mathcal{H}^+$ is null, $\mathbf{J}_\mu^{\mathbb{N}_e}[\Psi]\mathrm{n}_{\mathcal{H}}^\mu$ controls all tangential derivatives. More precisely, we have in particular

$$(\partial_{t^*}\psi)^2 \leq (B + B\epsilon_{\text{close}}e^{-1})\mathbf{J}_\mu^{\mathbb{N}_e}[\Psi]\mathrm{n}_{\mathcal{H}}^\mu \leq B\mathbf{J}_\mu^{\mathbb{N}_e}[\Psi]\mathrm{n}_{\mathcal{H}}^\mu, \tag{60}$$

$$(\partial_\phi\psi)^2 \leq Be^{-1}\mathbf{J}_\mu^{\mathbb{N}_e}(\Psi)\mathrm{n}_{\mathcal{H}}^\mu, \tag{61}$$

on $\mathcal{H}^+$. For the above we have used the full content of Assumption 3.2, as well as the ρ_s-translation invariance of $\mathrm{n}_{\mathcal{H}}$, n_Σ, ∂_ϕ, ∂_{t^*} and $\mathbb{N}_e$, which allows us to choose uniform constants B.

5.3 The vector fields $\mathbb{X}_a$ and $\mathbb{X}_b$

In this section we shall often use Regge-Wheeler coordinates as many of the computations refer to the Schwarzschild metric g_M.

In particular, we will consider vector fields of the form $\mathbb{V} = f(r^*)\partial_{r^*}$. In what follows f' will denote $\frac{df}{dr^*}$.

In (t, r^*, x^A, x^B) coordinates[20] we have

$$\mathbf{K}_{g_M}^{\mathbb{V}} = \frac{f'}{1-\mu}(\partial_{r^*}\Psi)^2 + \frac{1}{2}|\nabla\!\!\!/\Psi|_{g_M}^2\left(\frac{2-3\mu}{r}\right)f$$
$$-\frac{1}{4}\left(2f' + 4\frac{1-\mu}{r}f\right)g_M^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \tag{62}$$

[20]Careful, t not the t^* of our chart! Of course, t^* coincides with t for $r \geq r_{\mathbb{Y}}^-$.

where $|\nabla\!\!\!/\Psi|^2_{g_M}$ denotes the induced metric from g_M on the spheres. We may rewrite the above as

$$\mathbf{K}^{\mathbb{V}}_{g_M} = \left(\frac{f'}{2(1-\mu)} - \frac{f}{r}\right)(\partial_{r^*}\Psi)^2 + |\nabla\!\!\!/\Psi|^2_{g_M}\left(-\frac{\mu}{2r}f - \frac{1}{2}f'\right) + \left(\frac{f'}{2(1-\mu)} + \frac{f}{r}\right)(\partial_t\Psi)^2. \tag{63}$$

Let α, $R_1(\alpha) \gg M$ be parameters to be chosen in what follows. Let $R(\alpha) = \exp(4)R_1(\alpha)$. Given these, we define a function f_a such that

$$f_a = -r^{-4}(r_{\mathbb{Y}}^-)^4, \quad \text{for } r \le r_{\mathbb{Y}}^-,$$
$$f_a = -1, \quad \text{for } r_{\mathbb{Y}}^- \le r \le R_1(\alpha),$$
$$f_a = -1 + \int_{R_1(\alpha)}^{r} \frac{d\tilde{r}}{4\tilde{r}}, \quad \text{for } R_1(\alpha) \le r \le R(\alpha),$$
$$f_a = 0, \quad \text{for } r \ge R(\alpha).$$

(One can smooth this function, although this is irrelevant.) As before, we may consider f_a as a function of r^* defined by $f(r^*) = f(r(r^*))$. We call the resulting vector field $\mathbb{X}_a = f_a\partial_{r^*}$.

We obtain that in $R_1(\alpha) \ge r > r_{\mathbb{Y}}^-$

$$\mathbf{K}^{\mathbb{X}_a}_{g_M}[\Psi] = |\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{\mu}{2r}\right) + r^{-1}|\partial_{r^*}\Psi|^2 - r^{-1}|\partial_t\Psi|^2. \tag{64}$$

Since $t = t^*$ for $r \ge r_{\mathbb{Y}}^-$, we can rewrite this as

$$\mathbf{K}^{\mathbb{X}_a}_{g_M}(\Psi) = |\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 - r^{-1}|\partial_{t^*}\Psi|^2, \tag{65}$$

where the coordinate derivatives in the last line can now be understood with respect to the atlas (26). For $\epsilon_{\mathrm{close}}$ sufficiently small we obtain from (50)

$$\mathbf{K}^{\mathbb{X}_a}[\Psi] \ge |\nabla\!\!\!/\Psi|^2\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 - r^{-1}|\partial_{t^*}\Psi|^2 - \epsilon_{\mathrm{close}}B\mathbf{q}_e^{\star}[\Psi] \tag{66}$$

in this region, where we have used (41).

By (57), it follows that in $r_{\mathbb{Y}}^- \le r \le R_1(\alpha)$

$$\mathbf{K}^{\mathbb{X}_a}[\Psi] + \mathbf{K}^{\mathbb{N}_e}[\Psi] \ge |\nabla\!\!\!/\Psi|^2\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 - r^{-1}|\partial_{t^*}\Psi|^2 - eB\mathbf{q}_e^{\star}[\Psi] \tag{67}$$

for small enough $\epsilon_{\mathrm{close}} \ll e$.

Consider now the region $2M \leq r \leq r_{\mathbb{Y}}^-$. We have

$$f' = 4r^{-5}(r_{\mathbb{Y}}^-)^4(1-\mu),$$

and thus

$$\left(\frac{f'}{2(1-\mu)} + \frac{f}{r}\right) = (r_{\mathbb{Y}}^-)^4 r^{-5},$$
$$\left(\frac{f'}{2(1-\mu)} - \frac{f}{r}\right) = 3(r_{\mathbb{Y}}^-)^4 r^{-5},$$
$$\left(-\frac{\mu}{2r}f - \frac{1}{2}f'\right) = (r_{\mathbb{Y}}^-)^4\left(\frac{5\mu-4}{2}r^{-5}\right).$$

We have thus

$$\mathbf{K}_{g_M}^{\mathbb{X}_a}[\Psi] \geq 0$$

in this region.

Thus, by (50), (38) and (43) we have

$$\mathbf{K}_g^{\mathbb{X}_a}[\Psi] \geq -\epsilon_{\text{close}} e^{-1} B \mathbf{q}_e^{\bigstar}[\Psi]$$

in this region. It follows now from (56) that

$$\mathbf{K}^{\mathbb{X}_a}[\Psi] + \mathbf{K}^{\mathbb{N}_e}[\Psi] \geq b\mathbf{q}_e^{\bigstar}[\Psi] \tag{68}$$

in this region, for small enough $\epsilon_{\text{close}} \ll e$.

In view of (64), (67) and (68), $\mathbf{K}^{\mathbb{X}_a+\mathbb{N}_e}[\Psi]$ will "have[21] a sign" when applied to $\Psi = \psi_\flat^\tau$ (see Sect. 7.1) except for very large values of r, namely $r \geq R_1(\alpha)$. To control the behaviour there we will need an additional current. First, let us notice that for the $\mathbb{X}_a$ we have selected, the coefficient of $(\partial_r\Psi)^2$ in (63) is always nonnegative. Finally we notice that for $r \geq R_1(\alpha)$, the coefficient of $|\nabla\!\!\!/\Psi|^2$ in (63) satisfies

$$-\frac{\mu}{2r}f_a - \frac{1}{2}f_a' \geq -\frac{1}{8r}. \tag{69}$$

To choose an additional vector field, let us define

$$f_b \doteq \chi(r^*)\pi^{-1}\int_0^{r^*} \frac{\alpha\, dx}{x^2+\alpha^2},$$

[21] After integration over appropriate domains and modulo error terms.

where χ is a smooth cutoff with $\chi = 0$ for $r^* \le 0$ and $\chi = 1$ for $r^* \ge 1$, and let $\mathbb{X}_b$ be the vector

$$\mathbb{X}_b = f_b \partial_{r^*}.$$

Finally, define the function

$$w_b \doteq f_b' + 2\frac{1-\mu}{r} f_b - \frac{2M(1-\mu)f_b}{r^2}$$

and consider the modified current $\mathbf{J}_\mu^{\mathbb{X}_b, w_b}$ defined by (46), as well as the associated $\mathbf{K}^{\mathbb{X}_b, w_b}$ and $\mathbf{Er}^{\mathbb{X}_b, w_b}$.

Note that for general f, we can rewrite

$$\begin{aligned}\mathbf{K}^{\mathbb{V}}_{g_M} &= \left(\frac{f'}{1-\mu}\right)(\partial_{r^*}\Psi)^2 + \frac{1}{2}\left(\frac{2-3\mu}{r}\right) f |\nabla\!\!\!/ \Psi|^2_{g_M} \\ &\quad - \frac{M(1-\mu)f}{r^2} g_M^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \\ &\quad - \frac{1}{8}\left(2f' + 4\frac{1-\mu}{r} f - \frac{4M(1-\mu)f}{r^2}\right)(\Box_{g_M}\Psi^2 - 2\Psi F) \end{aligned} \tag{70}$$

from which we see

$$\begin{aligned}\mathbf{K}^{\mathbb{X}_b, w_b}_{g_M}[\Psi] &= \left(\frac{f_b'}{1-\mu} - \frac{M f_b}{r^2}\right)(\partial_{r^*}\Psi)^2 + \frac{M f_b}{r^2}(\partial_t\Psi)^2 \\ &\quad + \left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b |\nabla\!\!\!/ \Psi|^2_{g_M} \\ &\quad - \frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)}{r^2} f_b\right)\Psi^2. \end{aligned}$$

Note also the modified error term

$$\mathbf{Er}^{\mathbb{X}_b, w_b}[\Psi] = \mathbf{Er}^{\mathbb{X}_b}[\Psi] - \frac{1}{4}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)f_b}{r^2}\right)\Psi F.$$

Finally, let us define the currents

$$\mathbf{J}^{\mathbb{X}}_\mu = \mathbf{J}^{\mathbb{X}_a}_\mu + \mathbf{J}^{\mathbb{X}_b, w_b}_\mu, \qquad \mathbf{K}^{\mathbb{X}} = \mathbf{K}^{\mathbb{X}_a} + \mathbf{K}^{\mathbb{X}_b, w_b}, \qquad \mathbf{Er}^{\mathbb{X}} = \mathbf{Er}^{\mathbb{X}_a} + \mathbf{Er}^{\mathbb{X}_b, w_b}.$$

By our previous remarks, (45) holds for $\mathbf{J}^{\mathbb{X}}$. Also, in view of the definition of w, identities (48), (49) and (50) hold for $\mathbf{J}^{\mathbb{X}}$, $\mathbf{K}^{\mathbb{X}}$.

Let us expand

$$\mathbf{K}^{\mathbb{X}}_{g_M}[\Psi] = H_1(\partial_{r^*}\Psi)^2 + H_2(\partial_t\Psi)^2 + H_3|\nabla\!\!\!/ \Psi|^2_{g_M} + H_4\Psi^2$$

where

$$H_1 = \frac{f_a'}{2(1-\mu)} - \frac{f_a}{r} + \frac{f_b'}{1-\mu} - \frac{Mf_b}{r^2},$$
$$H_2 = \frac{f_a'}{2(1-\mu)} + \frac{f_a}{r} + \frac{Mf_b}{r^2},$$
$$H_3 = -\frac{\mu}{2r} f_a - \frac{1}{2} f_a' + \left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b,$$
$$H_4 = -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)}{r^2} f_b\right).$$

Note that for $r^* \geq 1$, we have

$$f_b' = \frac{1}{\pi}\frac{\alpha}{(r^*)^2 + \alpha^2}.$$

In particular, for $r \geq R_1(\alpha)$ for sufficiently large $R_1(\alpha)$ we have that

$$H_1 = \frac{f_a'}{2(1-\mu)} - \frac{f_a}{r} + \frac{f_b'}{1-\mu} - \frac{Mf_b}{r^2} \geq \frac{\alpha}{2\pi r^2}$$

while in $r_{\mathbb{Y}}^- \leq r \leq R_1(\alpha)$, we have

$$H_1 = \frac{1}{r} + \frac{f_b'}{1-\mu} - \frac{Mf_b}{r^2} \geq \frac{1}{2r}.$$

For H_2, let us simply remark that for $r \geq R(\alpha)$, we have

$$H_2 = \frac{Mf_b}{r^2} \geq b(\alpha) r^{-2}.$$

For H_3, we note first that we have the following asymptotic formula

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \sim \frac{1}{r},$$

i.e. for $r \geq R_1(\alpha)$ for sufficiently big $R_1(\alpha)$, we have

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{7}{8r}$$

and thus by (69)

$$H_3 = -\frac{\mu}{2r} f_a - \frac{1}{2} f_a' + \left(\frac{2-3\mu}{r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{3}{4r}.$$

To consider the behaviour for $r \leq R_1(\alpha)$, let us first note that there exists an R_0 depending only on M—i.e. independent of α if we require α to be sufficiently large—such that for $r > R_0$ we have

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq 0$$

and thus, in $R_0 \leq r \leq R_1(\alpha)$ we have

$$H_3 = \frac{\mu}{2r} + \left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{M}{r^2}.$$

For $r_{\mathbb{Y}}^- \leq r \leq R_0$ we have

$$\left|\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b\right| \leq B\alpha^{-1}$$

and thus, say

$$H_3 \geq \frac{M}{2r^2}$$

for α sufficiently large.

Turning to H_4, we note first

$$\begin{aligned}
&-\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)}{r^2} f_b\right) \\
&= -\frac{1}{4}\frac{1}{1-\mu} f_b''' - \frac{1}{r} f_b'' + \frac{\mu'}{r(1-\mu)} f_b' \\
&\quad - \frac{1}{2(1-\mu)r}\left(\frac{\mu'(1-\mu)}{r} - \mu''\right) f_b + \frac{1}{2}\Box_{g_M}\left(\frac{M(1-\mu)}{r^2} f_b\right) \\
&\sim \frac{7\alpha}{2\pi r^4}
\end{aligned}$$

for large r, i.e. we have

$$H_4 = -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)}{r^2} f_b\right) \geq \frac{7\alpha}{4\pi r^4}$$

for $r \geq R_1(\alpha)$ for $R_1(\alpha)$ suitably chosen. On the other hand, one sees easily that R_0 before could have been chosen such that for all α we have

$$H_4 = -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r} f_b - \frac{4M(1-\mu)}{r^2} f_b\right) \geq 0$$

for $r \geq R_0$. For $r_{\mathbb{Y}}^- \leq r \leq R_0$, we have

$$\left| -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)}{r^2}f_b\right)\right| \leq B\alpha^{-1}.$$

We may thus choose α large enough so that in this region

$$|H_4| \leq \frac{M}{8r^4} \leq \frac{1}{4r^2}H_3.$$

Let α be now chosen. It follows that $R_1 = R_1(\alpha)$ and $R = R(\alpha)$ can be chosen. These choices thus can be made to depend only on M.

Let us assume in what follows in this section that $\Psi_0 \doteq \int_0^{2\pi} \Psi\, d\phi = 0$. (In later sections, we shall always apply the considerations below to $\Psi_\flat$ which will indeed have this property; see Sect. 6.2.) By orthogonality we have

$$\int_0^{2\pi} \Psi^2\, d\phi \leq \int_0^{2\pi} (\partial_\phi \Psi)^2 d\phi. \tag{71}$$

It follows that

$$\int_0^{2\pi} \Psi^2\, d\phi \leq \int_0^{2\pi} (\partial_\phi \Psi)^2\, d\phi \leq r^2 \int_0^{2\pi} |\nabla\!\!\!/ \Psi|^2_{g_M}\, d\phi.$$

Thus, in the region $r_{\mathbb{Y}}^- \leq r \leq R$, we have

$$\int_0^{2\pi} (H_3|\nabla\!\!\!/ \Psi|^2_{g_M} + H_4\Psi^2)\, d\phi \geq \frac{1}{2}\int_0^{2\pi} H_3|\nabla\!\!\!/ \Psi|^2_{g_M}\, d\phi.$$

Note that, in the support of f_b, we have

$$(\partial_r^*\Psi)^2 \sim (\partial_r\Psi)^2, \qquad |\nabla\!\!\!/ \Psi|^2 \sim |\nabla\!\!\!/ \Psi|^2_{g_M}.$$

We have then by the above bounds and (50), (38) and (41) that for $r \geq R$,

$$\begin{aligned}
&\int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\Psi]\, d\phi \\
&\quad \geq \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}}_{g_M} + \mathbf{K}^{\mathbb{N}_e}_{g_M})[\Psi]\, d\phi - \int_0^{2\pi} \epsilon_{\mathrm{close}} B\mathbf{q}_e^{\bigstar}[\Psi]\, d\phi \\
&\quad = \int_0^{2\pi} \left(H_1(\partial_{r^*}\Psi)^2 + H_2(\partial_t\Psi)^2 + H_3|\nabla\!\!\!/ \Psi|^2_{g_M} + H_4\Psi^2\right) d\phi \\
&\qquad - \int_0^{2\pi} \epsilon_{\mathrm{close}} B\mathbf{q}_e^{\bigstar}[\Psi]\, d\phi
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^{2\pi} \left(b\mathbf{q}_e^\bigstar[\Psi] - \epsilon_{\text{close}} B \mathbf{q}_e^\bigstar[\Psi]\right) d\phi \\
&\geq \int_0^{2\pi} b\mathbf{q}_e^\bigstar[\Psi]\, d\phi
\end{aligned} \tag{72}
$$

for ϵ_{close} suitably small, whereas for $r_{\mathbb{Y}}^- \leq r \leq R$ we may write

$$
\begin{aligned}
\int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\Psi]\, d\phi &\geq b \int_0^{2\pi} \mathbf{q}_e^\bigstar[\Psi]\, d\phi + \int_0^{2\pi} (b|\nabla\!\!\!/\Psi|^2 - B(\partial_t\Psi)^2)\, d\phi \\
&\quad - be \int_0^{2\pi} \mathbf{q}_e^\bigstar[\Psi] \\
&\geq b \int_0^{2\pi} \mathbf{q}_e^\bigstar[\Psi]\, d\phi \\
&\quad + \int_0^{2\pi} (b|\nabla\!\!\!/\Psi|^2 - B(\partial_t\Psi)^2)\, d\phi
\end{aligned} \tag{73}
$$

where for the second inequality we require that e be sufficiently small. From (68) and the fact that f_b vanishes identically in $r \leq r_{\mathbb{Y}}^-$, we have

$$
\begin{aligned}
\int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\Psi]\, d\phi &= \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}_a} + \mathbf{K}^{\mathbb{N}_e})[\Psi]\, d\phi \\
&\geq b \int_0^{2\pi} \mathbf{q}_e^\bigstar[\Psi]\, d\phi,
\end{aligned} \tag{74}
$$

in the region $r \leq r_{\mathbb{Y}}^-$.

To give bounds for the boundary terms, note first that $\mathbb{X}_a = -(\frac{r_{\mathbb{Y}}^-}{2M})^4\mathbb{T}$ on $\mathcal{H}^+$. It follows that on the horizon, we have

$$
\mathbf{J}_\mu^{\mathbb{X}_a} n_{\mathcal{H}}^\mu = -\left(\frac{r_{\mathbb{Y}}^-}{2M}\right)^4 \mathbf{J}_\mu^{\mathbb{T}} \mathbb{n}_{\mathcal{H}}^\mu.
$$

One sees easily that for $\mathcal{H}^+$ or $\Sigma(\tau)$ where $\mathbb{n}^\mu = \mathbb{n}_{\mathcal{H}}^\mu$ or $\mathbb{n}^\mu = \mathbb{n}_\Sigma^\mu$, we have

$$
\left|\mathbf{J}_\mu^{\mathbb{T}} \mathbb{n}^\mu\right| \leq \mathbf{J}_\mu^{\mathbb{T}} \mathbb{n}^\mu + B\epsilon_{\text{close}} e^{-1} \mathbf{J}_\mu^{\mathbb{N}_e} \mathbb{n}^\mu \leq B\mathbf{J}_\mu^{\mathbb{N}_e} n^\mu,
$$

and thus

$$
\left|\mathbf{J}_\mu^{\mathbb{X}_a} \mathbb{n}^\mu\right| \leq B\left|\mathbf{J}_\mu^{\mathbb{T}} \mathbb{n}^\mu\right| + B\epsilon_{\text{close}} e^{-1} \mathbf{J}_\mu^{\mathbb{N}_e} \mathbb{n}^\mu \leq B\mathbf{J}_\mu^{\mathbb{N}_e} \mathbb{n}^\mu
$$

for $\epsilon_{\text{close}} \ll e$ sufficiently small.

On the other hand, in view of the assumption $\Psi_0 = 0$, we have similarly

$$\left|\int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{X}_b} \mathrm{n}^\mu \, d\phi\right| \leq B \left|\int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{T}} \mathrm{n}^\mu d\phi\right| + B\epsilon_{\text{close}} e^{-1} \int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{N}_e} \mathrm{n}^\mu \, d\phi$$
$$\leq B \int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{N}_e} \mathrm{n}^\mu \, d\phi.$$

It follows from the above inequalities that

$$\left|\int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{X}} \mathrm{n}^\mu \, d\phi\right| \leq B \int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{N}_e} \mathrm{n}^\mu d\phi \tag{75}$$

on both $\Sigma(\tau)$ and $\mathcal{H}^+$.

6 The superradiant/non-superradiant frequency decomposition

As explained in the introduction, the arguments of this paper hinge on separating the "superradiant" part of the solution from the non-"superradiant" part, and then exploiting dispersion for the former and positive definiteness for the $\mathbf{J}^{\mathbb{T}}$ flux through the event horizon for the latter. These two parts will be characterized by their support in frequency space. As we certainly do not know, however, a priori that ψ is in $L^2(t^*)$, we will first need to cut off ψ in t^*. This construction, together with propositions which control the errors that arise, are given in this section.[22]

6.1 ψ cut off: the definition of $\psi_{✂}^{\tau}$

Let $\chi(x)$ be a cutoff function such that $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$ (for instance, the same cutoff function defined in Sect. 3.1). Given $\tau \geq 2$, define

$$\psi_{✂}^{\tau} = \chi(t^+ + 1 - \tau)\chi(-t^- + 1)\psi.$$

We may express this as

$$\psi_{✂}^{\tau} = \chi_{✂}^{\tau}\psi = ({}^{+}\chi_{✂}^{\tau} + {}^{-}\chi_{✂}^{\tau})\psi,$$

[22] In our decomposition, $\psi_\flat$ also contains non-superradiant frequencies, thus it is technically not correct to refer to this as the 'superradiant' part. Nonetheless, we shall for convenience use this terminology.

where ${}^+\chi^\tau_{\text{✂}}$ and ${}^-\chi^\tau_{\text{✂}}$ are smooth functions on $\mathcal{R}$ with

$$\begin{aligned}&\mathrm{supp}({}^+\chi^\tau_{\text{✂}}(1-{}^+\chi^\tau_{\text{✂}}))\subset\mathcal{R}^+(\tau-1,\tau),\\&\mathrm{supp}({}^-\chi^\tau_{\text{✂}}(1-{}^-\chi^\tau_{\text{✂}}))\subset\mathcal{R}^-(0,1),\\&0\le{}^-\chi^\tau_{\text{✂}}\le 1,\qquad 0\le{}^+\chi^\tau_{\text{✂}}\le 1\end{aligned}$$

and

$$|\partial_{(i)}{}^+\chi^\tau_{\text{✂}}|\le B,\qquad|\partial_{(i)}{}^-\chi^\tau_{\text{✂}}|\le B,$$

with respect to the charts of (26), for any multi-index $|(i)|\le 2$. Moreover,

$$\partial_\theta{}^+\chi^\tau_{\text{✂}}=0,\qquad\partial_\theta{}^-\chi^\tau_{\text{✂}}=0.\tag{76}$$

The reader may wonder why the cutoff region is related to $\Sigma^\pm$, indeed, why $\Sigma^\pm$ have been introduced in the first place. Essentially, this is necessary to achieve the propositions of Sects. 6.4–6.6. We would like to express all errors in terms of the positive definite quantity $\mathbf{q}_e(\psi)$. This quantity does not contain ψ itself but only derivatives. Of course, in view of the fact that, as we shall see, the spherical average ψ_0 does not give rise to errors, this does not generate problems for the region $r\le R$ for $(\psi-\psi_0)^2$ can be controlled by $\mathbf{q}_e(\psi)$ via a Poincaré inequality. As $r\to\infty$, one needs extra negative powers of r. Our cutoff region diverges from $\mathcal{R}(0,\tau)$ as $r\to\infty$ and this allows us to "gain" powers of r necessary to control 0'th order terms via a Poincaré inequality in $\mathcal{R}(0,\tau)$. One can then retrieve estimates all the way to the boundary of the cutoff region using the positive definiteness of $\mathbf{J}^{\mathbb{T}}$ for large r.

6.2 Definition of $\Psi_\flat$ and $\Psi_\sharp$

Let ζ be a smooth cutoff supported in $[-2,2]$ with the property that $\zeta=1$ in $[-1,1]$ and $\zeta(\omega)=\zeta(-\omega)$, and let $\omega_0>0$ be a parameter to be determined later.

For a smooth real function $\Psi(t^*,\phi,\cdot)$ of compact support in t^*, let Ψ_k denote its k'th azimuthal mode:

$$\Psi_k(t^*,\cdot)=\frac{1}{2\pi}\int_0^{2\pi}\Psi(t^*,\phi,\cdot)e^{-ik\phi}d\phi.$$

Note that Ψ_k is complex-valued. Let $\hat{\Psi}$ denote the Fourier transform of Ψ in t^*. Note that $\widehat{\Psi_k}=\hat{\Psi}_k$.

Define

$$\Psi_\flat(t^*,\cdot)\doteq\sum_{k\neq 0}e^{-ik\phi}\int_{-\infty}^{\infty}\zeta((\omega_0k)^{-1}\omega)\hat{\Psi}_k(\omega,\cdot)e^{i\omega t^*}d\omega,$$

$$\Psi_\sharp(t^*,\cdot) \doteq \Psi_0 + \sum_{k\neq 0} e^{-ik\phi}\int_{-\infty}^{\infty}\left(1-\zeta((\omega_0 k)^{-1}\omega)\right)\hat{\Psi}_k(\omega,\cdot)e^{i\omega t^*}\,d\omega.$$

Note of course

$$\Psi_\flat + \Psi_\sharp = \Psi, \tag{77}$$

and $\Psi_\flat$ and $\Psi_\sharp$ are real-valued. Note in addition that

$$(\Psi_\flat)_0 = 0 \tag{78}$$

whereas

$$(\Psi_\sharp)_0 = \Psi_0. \tag{79}$$

In the application to $\Psi = \psi^\tau_{\text{✂}}$, we shall write simply $\psi^\tau_\sharp$ and $\psi^\tau_\flat$. Note finally, that in view of (76), $(\psi_k)^\tau_{\text{✂}} = (\psi^\tau_{\text{✂}})_k$.

Note that for $k\neq 0$,

$$(\Psi_\flat)_k(t^*) = \int_{-\infty}^{\infty}\zeta((\omega_0 k)^{-1}\omega)\hat{\Psi}_k(\omega)e^{i\omega t^*}\,d\omega = \int_{-\infty}^{\infty}P_k^<(t^*-s^*)\Psi_k(s^*)\,ds^*,$$

where

$$P_k^<(t^*) = \omega_0 k\int_{-\infty}^{\infty}\zeta(\omega)e^{-i\omega(\omega_0 k t^*)}\,d\omega.$$

The kernel $P_k^<(t^*)$ is a rescaled copy of a Schwarz function of t^*. As a consequence, for any $m, q\geq 0$,

$$|\partial^m_{t^*}P_k^<(t^*)| \leq B_{mq}(\omega_0|k|)^{m+1}\left(1+|\omega_0 k t^*|\right)^{-q}. \tag{80}$$

On the other hand, let $\tilde{\zeta}$ be a smooth cut-off function supported in $(-3,3)$ such that $\tilde{\zeta}=1$ on $[-2,2]$. Then, since $\tilde{\zeta}\zeta=\zeta$, we have the reproducing formula

$$\begin{aligned}(\Psi_\flat)_k(t^*) &= \int_{-\infty}^{\infty}\tilde{\zeta}((\omega_0 k)^{-1}\omega)(\hat{\Psi}_\flat)_k(\omega)e^{i\omega t^*}\,d\omega\\ &= \int_{-\infty}^{\infty}\tilde{P}_k^<(t^*-s^*)(\Psi_\flat)_k(s^*)\,ds^*,\end{aligned}$$

where the kernel $\tilde{P}_k^<$ also satisfies (80).

Finally, let $\xi(\omega)$ be a function smooth away from $\omega=0$ and with the property that $\xi(\omega)=\omega^{-1}$ for $|\omega|\leq 1/2$ and $\xi(\omega)=1$ for $|\omega|\geq 1$. In particular, the function $\tilde{\xi}(\omega) = \omega\xi(\omega)$ is smooth and $\tilde{\xi}(\omega) = 1$ for $|\omega| \leq 1/2$ and $\tilde{\xi}(\omega) = \omega$ for $|\omega| \geq 1$. Since $\xi(1-\zeta) = 1-\zeta$, we can write for $k \neq 0$,

$$(\Psi_\sharp)_k(t^*) = \int_{-\infty}^{\infty} Q_k^>(t^* - s^*)(\Psi_\sharp)_k(s^*)\,ds^*,$$

where

$$Q_k^>(t^*) = \omega_0 k \int_{-\infty}^{\infty} \xi(\omega) e^{i\omega(\omega_0 k t^*)}\,d\omega.$$

Furthermore,

$$\partial_{t^*}(\Psi_\sharp)_k(t^*) = \omega_0 k \int_{-\infty}^{\infty} \tilde{Q}_k^>(t^* - s^*)(\Psi_\sharp)_k(s^*)\,ds^*,$$

where

$$\tilde{Q}_k^>(t^*) = \omega_0 k \int_{-\infty}^{\infty} \tilde{\xi}(\omega) e^{i\omega(\omega_0 k t^*)}\,d\omega$$

and

$$(\Psi_\sharp)_k(t^*) = (\omega_0 k)^{-1} \int_{-\infty}^{\infty} R_k^>(t^* - s^*)\partial_{s^*}(\Psi_\sharp)_k(s^*)\,ds^*,$$

where

$$R_k^>(t^*) = \omega_0 k \int_{-\infty}^{\infty} \left(\tilde{\xi}(\omega)\right)^{-1} e^{i\omega(\omega_0 k t^*)}\,d\omega.$$

The function $a(\omega) = (\tilde{\xi}(\omega))^{-1}$ is equal to one on $(-1/2, 1/2)$ and ω^{-1} for $|\omega| \geq 1$. The kernel $R_k^>(t^*)$ satisfies

$$|R_k^>(t^*)| \leq B_q(\omega_0|k|)^{1-q}|t^*|^{-q}$$

for any $q > 0$. In addition, we have a uniform bound (coming from $1/\omega$ decay)

$$|R_k^>(t^*)| \leq B\omega_0|k|(1 + |\log(\omega_0|k|t^*)|).$$

Combining we obtain

$$|R_k^>(t^*)| \leq B_q\omega_0|k|(1 + |\log(\omega_0|k|t^*)|)(1 + |\omega_0 k t^*|)^{-q}.$$

6.3 Comparing $\partial_{t^*}\Psi$ and $\partial_\phi\Psi$

The decomposition of Ψ into $\Psi_\flat$ and $\Psi_\sharp$ is motivated by the desire to compare various L^2-type norms of the ∂_ϕ and ∂_{t^*} derivatives. Since this is required at a localised level, however, error terms arise. The precise relations one can make are recorded in this section. The estimates of this section employ standard techniques of elementary Fourier analysis. We must be careful, however, to express all "error terms" in a form which can be related to our bootstrap assumptions which will be introduced later on.

6.3.1 Comparisons for $\Psi_\flat$

First a lemma.

Lemma 6.1 *Let $\tau'' \geq \tau'$, $\omega_0 \leq 1$, and let Ψ be smooth and of compact support in t^*. Then*

$$\begin{aligned}
&\int_{\mathcal{R}(\tau',\tau'')\cap\{r_{\mathbb{Y}}^- \leq r \leq R\}} (\partial_{t^*}\Psi_\flat)^2 \\
&\quad \leq B{\omega_0}^2 \int_{\mathcal{R}(\tau',\tau'')\cap\{r_{\mathbb{Y}}^- \leq r \leq R\}} (\partial_\phi \Psi_\flat)^2 \\
&\qquad + B\omega_0 \sup_{-\infty \leq \bar{\tau} \leq \infty} \int_{\bar{\tau}}^{\bar{\tau}+1} \left(\int_{\Sigma(\tilde{\tau})\cap\{r_{\mathbb{Y}}^- \leq r \leq R\}} (\partial_\phi \Psi_\flat)^2 \right) d\tilde{\tau}.
\end{aligned}$$

Proof Recall (78). Note first that by the relations of Sect. 6.2, it follows that for any $q > 0$, we have

$$\left|\partial_{t^*}(\Psi_\flat)_k(t^*,\cdot)\right| \leq B_q(\omega_0 k)^2 \int_{-\infty}^{\infty} \left(1 + \left|\omega_0 k(t^* - s^*)\right|\right)^{-q} \left|(\Psi_\flat)_k\right|(s^*,\cdot)ds^*.$$

We have thus

$$\begin{aligned}
\left|\partial_{t^*}(\Psi_\flat)_k(t^*,\cdot)\right| &\leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (1+|\ell|)^{-q} \left|(\Psi_\flat)_k\right|(s^*,\cdot)ds^* \\
&\leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} \left|(\Psi_\flat)_k\right|(s^*,\cdot)ds^* \\
&\leq B_q(\omega_0|k|)^{3/2} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \\
&\qquad \times \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\Psi_\flat)_k|^2(s^*,\cdot)ds^* \right)^{1/2}.
\end{aligned}$$

It follows that, for $q > 1$,

$$
\begin{aligned}
&\int_{\tau'}^{\tau''} (\partial_{t^*}(\Psi_\flat)_k)^2(t^*,\cdot)\,dt^* \\
&\quad\le B_q(\omega_0|k|)^3\int_{\tau'}^{\tau''}\left(\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\right. \\
&\qquad\times\left.\left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\right)^{1/2}\right)^2 dt^* \\
&\quad\le B_q(\omega_0|k|)^3\int_{\tau'}^{\tau''}\left(\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\right) \\
&\qquad\times\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,dt^* \\
&\quad\le B_q(\omega_0|k|)^3\int_{\tau'}^{\tau''}\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,dt^* \\
&\quad= B_q(\omega_0|k|)^3\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'}^{\tau''}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,dt^* \\
&\quad= B_q(\omega_0|k|)^3\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\frac{\ell}{\omega_0|k|}}^{\frac{\ell+1}{\omega_0|k|}}\int_{\tau'}^{\tau''}|(\Psi_\flat)_k|^2(s^*+t^*,\cdot)\,dt^*\,ds^* \\
&\quad\le B_q(\omega_0 k)^2\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}}|(\Psi_\flat)_k|^2(t^*,\cdot)\,dt^*.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\tau'}^{\tau''}|\partial_{t^*}(\Psi_\flat)_k|^2(t^*,\cdot)\,dt^* \\
&\quad\le B_q(\omega_0 k)^2\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}}\chi_{\tau',\tau''}(s^*)|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^* \\
&\qquad+ B_q(\omega_0 k)^2\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}
\end{aligned}
$$

$$\times \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}} (1-\chi_{\tau',\tau''}(s^*))|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*$$

$$\doteq \mathbf{S}_{1,k} + \mathbf{S}_{2,k} \tag{81}$$

where $\chi_{\tau',\tau''}(s^*) = 1$ if $s^* \in [\tau', \tau'']$ and 0 otherwise.

To prove the lemma, in view of the comments in Sect. 3.2 on the volume form and Plancherel, it would suffice to show that

$$\sum_{|k|\geq 1} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \mathbf{S}_{1,k}\,d\phi\,d\theta\,dr$$

$$\leq B{\omega_0}^2 \int_{\tau'}^{\tau''} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} (\partial_\phi \Psi_\flat)^2\,d\phi\,d\theta\,dr\,dt^*, \tag{82}$$

$$\sum_{|k|\geq 1} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \mathbf{S}_{2,k}\,d\phi\,d\theta\,dr$$

$$\leq B\omega_0 \sup_{-\infty\leq\bar{\tau}\leq\infty} \int_{\bar{\tau}}^{\bar{\tau}+1} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} (\partial_\phi \Psi_\flat)^2\,d\phi\,d\theta\,dr\,d\tilde{\tau}. \tag{83}$$

The first term on the right hand side of (81) is bounded by

$$\mathbf{S}_{1,k} \leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tau'}^{\tau''} |(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*$$

$$\leq B_q(\omega_0 k)^2 \int_{\tau'}^{\tau''} |(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*.$$

Thus, it follows that

$$\sum_{|k|\geq 1} \int_0^{2\pi} \mathbf{S}_{1,k} d\phi \leq \sum_{|k|\geq 1} B_q \omega_0^2 k^2 \int_0^{2\pi} \int_{\tau'}^{\tau''} |(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^* d\phi$$

$$\leq B_q \omega_0^2 \int_0^{2\pi} \int_{\tau'}^{\tau''} (\partial_\phi \Psi_\flat)^2(s^*,\cdot)\,ds^* d\phi.$$

We have established (82).

The second term on the right hand side of (81) is bounded by

$$\mathbf{S}_{2,k} \leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau'} |(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*$$

$$+ B_q(\omega_0 k)^2 \sum_{\ell=0}^{\infty} (1+|\ell|)^{-q} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0|k|}} |(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*$$

$$\doteq \mathbf{S}_{21,k} + \mathbf{S}_{22,k}. \tag{84}$$

We have

$$\sum_k \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \mathbf{S}_{21,k}\,d\phi\,d\theta\,dr$$

$$\leq B_q\omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r)$$

$$\times \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau'} k^2|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,d\phi\,d\theta\,dr$$

$$\leq B_q\omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r)$$

$$\times \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} k^2|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,d\phi\,d\theta\,dr$$

$$\leq B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \sum_{|k|\geq 1} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r)$$

$$\times \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} k^2|(\Psi_\flat)_k|^2(s^*,\cdot)\,ds^*\,d\phi\,d\theta\,dr$$

$$\leq B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r)$$

$$\times \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} (\partial_\phi\Psi_\flat)^2(s^*,\cdot)\,ds^*\,d\phi\,d\theta\,dr$$

$$\leq B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q}\,|\ell|\omega_0^{-1} \sup_{-\infty\leq\bar{\tau}\leq\infty} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta,r)$$

$$\times \int_0^{2\pi} \int_{\bar{\tau}}^{\bar{\tau}+1} (\partial_\phi \Psi_\flat)^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$\leq B\omega_0 \sup_{-\infty \leq \bar{\tau} \leq \infty} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r) \int_0^{2\pi} \int_{\bar{\tau}}^{\bar{\tau}+1} (\partial_\phi \Psi_\flat)_k^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr, \tag{85}$$

for q chosen sufficiently large, where we have used that $\omega_0^{-1}|\ell| \geq 1$.

As for $\mathbf{S}_{22,k}$, we have

$$\sum_k \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r) \int_0^{2\pi} \mathbf{S}_{22,k} d\phi\, d\theta\, dr$$

$$\leq B_q \omega_0^2 \sum_{|k| \geq 1} \sum_{\ell=0}^\infty (1+\ell)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r)$$

$$\times \int_0^{2\pi} \int_{\tau''}^{\tau'' + \frac{\ell+1}{\omega_0 |k|}} k^2 |(\Psi_\flat)_k|^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$\leq B_q \omega_0^2 \sum_{|k| \geq 1} \sum_{\ell=0}^\infty (1+\ell)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r)$$

$$\times \int_0^{2\pi} \int_{\tau''}^{\tau'' + \frac{\ell+1}{\omega_0}} k^2 |(\Psi_\flat)_k|^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$\leq B_q \omega_0^2 \sum_{\ell=0}^\infty (1+\ell)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r)$$

$$\times \sum_{|k| \geq 1} \int_0^{2\pi} \int_{\tau''}^{\tau'' + \frac{\ell+1}{\omega_0}} k^2 |(\Psi_\flat)_k|^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$= B_q \omega_0^2 \sum_{\ell=0}^\infty (1+\ell)^{-q} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r)$$

$$\times \int_0^{2\pi} \int_{\tau''}^{\tau'' + \frac{\ell+1}{\omega_0}} (\partial_\phi \Psi_\flat)^2 (s^*, \cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$\leq B_q \omega_0^2 \sum_{\ell=0}^\infty (1+\ell)^{-q} \frac{1+\ell}{\omega_0} \sup_{-\infty \leq \bar{\tau} \leq \infty} \int_{r_{\mathbb{Y}}^-}^R \int_0^\pi \nu(\theta, r)$$

$$\times \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi \Psi_\flat)^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr$$

$$\leq B_q\omega_0 \sup_{-\infty\leq\bar\tau\leq\infty} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi \Psi_\flat)^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr.$$

The above and (85) give (83). □

Lemma 6.2 *Under the assumptions of the previous lemma then*

$$\int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau-1}^{\bar\tau} (\partial_\phi \Psi_\flat)^2 dt^*\, d\phi\, d\theta\, dr$$

$$\leq B\omega_0^{-1} \sup_{-\infty\leq\tilde\tau\leq\infty} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\tilde\tau-1}^{\tilde\tau} (\partial_\phi \Psi)^2 dt^*\, d\phi\, d\theta\, dr.$$

Proof For any $q>0$, we have

$$|(\Psi_\flat)_k(t^*,\cdot)| \leq B_q(\omega_0|k|) \int_{-\infty}^{\infty} (1+|\omega_0 k(t^*-s^*)|)^{-q} |\Psi_k|(s^*,\cdot)\, ds^*$$

$$\leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|(s^*,\cdot)\, ds^*$$

$$\leq B_q(\omega_0|k|)^{\frac12} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\, ds^* \right)^{\frac12}.$$

It follows, with the help of Cauchy-Schwarz, that for $q>1$,

$$|(\Psi_\flat)_k(t^*,\cdot)|^2 \leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\, ds^*,$$

and thus,

$$\int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_{\bar\tau-1}^{\bar\tau} |(\Psi_\flat)_k(t^*,\cdot)|^2 dt^*\, d\theta\, dr$$

$$\leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{r_{\mathbb{Y}}^-}^{R} \int_0^{\pi} \nu(\theta,r)$$

$$\times \int_{\bar\tau-1}^{\bar\tau} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\, ds^*\, dt^*\, d\theta\, dr$$

$$
\begin{aligned}
&\le B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\\
&\quad\times\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}\int_{s^*-\frac{\ell+1}{\omega_0|k|}}^{s^*-\frac{\ell}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,dt^*\,ds^*\,d\theta\,dr\\
&\le B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*\,d\theta\,dr.
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
&\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1}^{\bar\tau}(\partial_\phi\Psi_\flat)^2dt^*\,d\phi\,d\theta\,dr\\
&\quad=\sum_{|k|\ge1}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1}^{\bar\tau}k^2|(\Psi_\flat)_k(t^*,\cdot)|^2dt^*\,d\theta\,dr\\
&\quad\le B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\sum_{|k|\ge1}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\\
&\qquad\times\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}k^2|\Psi_k|^2(s^*,\cdot)\,ds^*\,d\theta\,dr\\
&\quad\le B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\sum_{|k|\ge1}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1-\frac{\ell}{\omega_0}}^{\bar\tau+\frac{\ell+1}{\omega_0}}k^2|\Psi_k|^2(s^*,\cdot)\,ds^*\,d\theta\,dr\\
&\quad=B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1-\frac{\ell}{\omega_0}}^{\bar\tau+\frac{\ell+1}{\omega_0}}(\partial_\phi\Psi)^2\,ds^*\,d\phi\,d\theta\,dr\\
&\quad\le B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}(2\ell+2){\omega_0}^{-1}\sup_{-\infty\le\tilde\tau\le\infty}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\\
&\qquad\times\int_0^{2\pi}\int_{\tilde\tau-1}^{\tilde\tau}(\partial_\phi\Psi)^2\,ds^*\,d\phi\,d\theta\,dr\\
&\quad\le B_q\omega_0^{-1}\sup_{-\infty\le\tilde\tau\le\infty}\int_{r_{\mathbb{Y}}^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\tilde\tau-1}^{\tilde\tau}(\partial_\phi\Psi)^2\,dt^*\,d\phi\,d\theta\,dr,
\end{aligned}
$$

where we have assumed q sufficiently large, and have used ${\omega_0}^{-1}\ge1$. The lemma follows after fixing q. $\square$

6.3.2 Application to $\psi_\flat^\tau$

From the above lemmas, we easily obtain the following statement, which is the form we shall use later in this paper:

Proposition 6.1 *Let $\tau'' \geq \tau'$ and $\omega_0 \leq 1$. Then*

$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r_{\mathbb{Y}}^- \leq r \leq R\}} (\partial_{t^*}\psi_\flat^\tau)^2 \leq B{\omega_0}^2 \int_{\mathcal{R}(\tau',\tau'')\cap\{r_{\mathbb{Y}}^- \leq r \leq R\}} (\partial_\phi \psi_\flat^\tau)^2 + B \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof In view of (76), it follows that

$$(\partial_\phi \psi_{\text{✂}}^\tau)^2 = (\chi_{\text{✂}}^\tau)^2 (\partial_\phi \psi)^2 \leq B\,\mathbf{q}_e(\psi)$$

in the region $r_{\mathbb{Y}}^- \leq r \leq R$. In view also of the support of $\psi_{\text{✂}}^\tau$, we may thus bound the right hand side of the statement of Lemma 6.2 applied to $\psi_{\text{✂}}^\tau$ by

$$B\omega_0^{-1} \sup_{1\leq\bar\tau\leq\tau} \int_{\bar\tau-1}^{\bar\tau} \left(\int_{\Sigma(\tilde\tau)} \mathbf{q}_e(\psi) \right) d\tilde\tau.$$

The proposition now follows from Lemmas 6.1 and 6.2. □

6.3.3 Comparisons for $\Psi_\sharp$

First a lemma:

Lemma 6.3 *Let $\tau' \leq \tau''$ let Ψ be smooth and of compact support in t^*. Then*

$$\int_{\mathcal{H}(\tau',\tau'')} (\partial_{t^*}\Psi_\sharp)^2 \geq B{\omega_0}^2 \int_{\mathcal{H}(\tau',\tau'')} (\partial_\phi \Psi_\sharp)^2 - B{\omega_0}^{-1} \sup_{-\infty\leq\bar\tau\leq\infty} \int_{\mathcal{H}(\bar\tau,\bar\tau+1)} (\partial_{t^*}\Psi_\sharp)^2.$$

Proof Note first that the lemma holds trivially for $(\Psi_\sharp)_0$. We may thus assume that $(\Psi_\sharp)_0 = 0$. For $|k| \geq 1$, we note first that from Sect. 6.2 we obtain

$$|(\Psi_\sharp)_k(t^*,\cdot)| \leq B_q \omega_0^{-1} |k|^{-1} \int_{-\infty}^{\infty} \omega_0 |k| \frac{1 + |\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^*.$$

Thus,

$$
\begin{aligned}
&\int_{\tau'}^{\tau''} k^2 |(\Psi_\sharp)_k|^2(t^*,\cdot)\, dt^* \\
&\quad \le B_q \omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{-\infty}^{\infty} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \right. \\
&\qquad \left. \times \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^* \right)^2 dt^* \\
&\quad \le B_q \omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{\tau'}^{\tau''} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \right. \\
&\qquad \left. \times \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^* \right)^2 dt^* \\
&\qquad + B_q \omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{-\infty}^{\tau'} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \right. \\
&\qquad \left. \times \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^* \right)^2 dt^* \\
&\qquad + B_q \omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{\tau''}^{\infty} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \right. \\
&\qquad \left. \times \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^* \right)^2 dt^* \\
&\quad \doteq \mathbf{S}_{1,k} + \mathbf{S}_{2,k} + \mathbf{S}_{3,k}.
\end{aligned}
$$

We obtain immediately that for sufficiently large q, since

$$
\int_{\tau'}^{\tau''} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q}\, ds^* \le B_q
$$

then

$$
\begin{aligned}
&\int_{\tau'}^{\tau''} \left(\int_{\tau'}^{\tau''} \omega_0 |k| \frac{1+|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| ds^* \right)^2 dt^* \\
&\quad \le B_q \int_{\tau'}^{\tau''} |(\partial_t \Psi_\sharp)_k|^2(t^*,\cdot)\, dt^*
\end{aligned}
$$

and thus

$$\mathbf{S}_{1,k} \le B_q \omega_0^{-2} \int_{\tau'}^{\tau''} |\partial_{t^*}(\Psi_\sharp)_k)|^2(t^*, \cdot)\, dt^*.$$

On the other hand,

$$\begin{aligned}
&\sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta)\mathbf{S}_{2,k}\, d\theta \\
&\quad = B_q \omega_0^{-2} \sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau'}^{\tau''} \\
&\qquad \times \left(\int_{-\infty}^{\tau'} \omega_0 k \frac{1 + |\log|\omega_0 k(t^* - s^*)||}{(1 + |\omega_0 k(t^* - s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*, \cdot)\right| ds^* \right)^2 dt^*\, d\theta \\
&\quad = B_q \omega_0^{-2} \sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau'}^{\tau''} \left(\sum_{\ell=0}^{\infty} \int_{\tau' - \frac{\ell+1}{\omega_0|k|}}^{\tau' - \frac{\ell}{\omega_0|k|}} \omega_0 k \frac{1 + |\log|\omega_0 k(t^* - s^*)||}{(1 + |\omega_0 k(t^* - s^*)|)^q} \right. \\
&\qquad \left. \times \left|\partial_{s^*}(\Psi_\sharp)_k(s^*, \cdot)\right| ds^* \right)^2 dt^*\, d\theta \\
&\quad \le B_q \omega_0^{-2} \sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau'}^{\tau''} \left(\sum_{\ell=0}^{\infty} \left(\int_{\tau' - \frac{\ell+1}{\omega_0|k|}}^{\tau' - \frac{\ell}{\omega_0|k|}} \omega_0^2 k^2 \right.\right. \\
&\qquad \left. \times \frac{1 + |\log^2|\omega_0 k(t^* - s^*)||}{(1 + |\omega_0 k(t^* - s^*)|)^{2q}} ds^* \right)^{1/2} \\
&\qquad \left. \times \left(\int_{\tau' - \frac{\ell+1}{\omega_0|k|}}^{\tau' - \frac{\ell}{\omega_0|k|}} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*, \cdot)\right|^2 ds^* \right)^{1/2} \right)^2 dt^*\, d\theta \\
&\quad \le B_q \omega_0^{-2} \sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau'}^{\tau''} \omega_0 |k| \left(\sum_{\ell=0}^{\infty} \frac{1 + |\log|\omega_0 k(t^* - \tau') + \ell)||}{(1 + |\omega_0 k(t^* - \tau') + \ell|)^q} \right. \\
&\qquad \left. \times \left(\int_{\tau' - \frac{\ell+1}{\omega_0|k|}}^{\tau' - \frac{\ell}{\omega_0|k|}} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*, \cdot)\right|^2 ds^* \right)^{1/2} \right)^2 dt^*\, d\theta \\
&\quad \le B_q \omega_0^{-2} \sum_{|k|\ge 1} \int_0^\pi \tilde{\nu}(\theta) \int_0^\infty \left(\sum_{\ell=0}^{\infty} \frac{1 + |\log|t^* + \ell)||}{(1 + |t^* + \ell|)^q} \right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\left(\int_{\tau' - \frac{\ell+1}{\omega_0 |k|}}^{\tau' - \frac{\ell}{\omega_0 |k|}} \left| \partial_{s^*} (\Psi_\sharp)_k (s^*, \cdot) \right|^2 ds^* \right)^{1/2} \right)^2 dt^* \, d\theta \\
&\leq B_q \omega_0^{-2} \sum_{|k| \geq 1} \int_0^\pi \tilde{\nu}(\theta) \int_0^\infty \left(\sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell)||}{(1 + |t^* + \ell|)^q} \right) \\
&\quad \times \left(\sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell)||}{(1 + |t^* + \ell|)^q} \int_{\tau' - \frac{\ell+1}{\omega_0 |k|}}^{\tau' - \frac{\ell}{\omega_0 |k|}} \left| \partial_{s^*} (\Psi_\sharp)_k (s^*, \cdot) \right|^2 ds^* \right) dt^* \, d\theta \\
&\leq B_q \omega_0^{-2} \int_0^\infty \left(\sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell)||}{(1 + |t^* + \ell|)^q} \right) \left(\sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell)||}{(1 + |t^* + \ell|)^q} \right. \\
&\quad \left. \times \sum_{|k| \geq 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau' - \frac{\ell+1}{\omega_0 |k|}}^{\tau' - \frac{\ell}{\omega_0 |k|}} \left| \partial_{s^*} (\Psi_\sharp)_k (s^*, \cdot) \right|^2 ds^* \, d\theta^* \right) dt^* \\
&\leq B_q \omega_0^{-2} \int_0^\infty \frac{1 + |\log |t^*||}{(1 + |t^*|)^{q-1}} \left(\sum_{\ell=0}^\infty \frac{|\log |t^* + \ell||}{(1 + |t^* + \ell|)^q} \right. \\
&\quad \left. \times \sum_{|k| \geq 1} \int_0^\pi \tilde{\nu}(\theta) \int_{\tau' - \frac{\ell+1}{\omega_0}}^{\tau'} \left| \partial_{s^*} (\Psi_\sharp)_k (s^*, \cdot) \right|^2 ds^* \, d\theta^* \right) dt^* \\
&\leq B_q \omega_0^{-2} \int_0^\infty \frac{1 + |\log |t^*||}{(1 + |t^*|)^{q-1}} \left(\sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell||}{(1 + |t^* + \ell|)^q} \right. \\
&\quad \left. \times \int_0^\pi \tilde{\nu}(\theta) \int_0^{2\pi} \int_{\tau' - \frac{\ell+1}{\omega_0}}^{\tau'} \left| \partial_{s^*} (\Psi_\sharp) (s^*, \cdot) \right|^2 ds^* \, d\phi \, d\theta^* \right) dt^* \\
&\leq B_q \omega_0^{-2} \int_0^\infty \frac{1 + |\log |t^*||}{(1 + |t^*|)^{q-1}} \sum_{\ell=0}^\infty \frac{1 + |\log |t^* + \ell||}{(1 + |t^* + \ell|)^q} \frac{\ell + 1}{\omega_0} \\
&\quad \times \left(\sup_{-\infty \leq \bar{\tau} \leq \infty} \int_0^\pi \tilde{\nu}(\theta) \int_0^{2\pi} \int_{\bar{\tau}+1}^{\bar{\tau}} \left| \partial_{s^*} (\Psi_\sharp) (s^*, \cdot) \right|^2 ds^* \, d\phi \, d\theta^* \right) dt^* \\
&\leq B_q \omega_0^{-3} \sup_{-\infty \leq \bar{\tau} \leq \infty} \int_0^\pi \tilde{\nu}(\theta) \int_0^{2\pi} \int_{\bar{\tau}+1}^{\bar{\tau}} \left| \partial_{s^*} (\Psi_\sharp) (s^*, \cdot) \right|^2 ds^* \, d\phi \, d\theta^*.
\end{aligned}
$$

A similar bound holds for $\mathbf{S}_{3,k}$. We obtain the lemma after appropriate fixing of q. $\square$

Lemma 6.4 *Under the assumptions of the previous lemma, if $\omega_0 \le 1$,*

$$\sup_{-\infty\le\bar\tau\le\infty}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}(\partial_{t^*}\Psi_\sharp)^2 \le B\omega_0^{-1}\sup_{-\infty\le\bar\tau\le\infty}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}(\partial_{t^*}\Psi)^2.$$

Proof Since

$$\partial_{t^*}\Psi_\sharp = \partial_{t^*}\Psi - \partial_{t^*}\Psi_\flat,$$

and $\omega_0{}^{-1}\ge 1$, it suffices in fact to prove

$$\sup_{-\infty\le\bar\tau\le\infty}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}(\partial_{t^*}\Psi_\flat)^2 \le B\omega_0^{-1}\sup_{-\infty\le\bar\tau\le\infty}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}(\partial_{t^*}\Psi)^2.$$

Recall from Sect. 6.2 we have

$$|(\partial_{t^*}\Psi_\flat)_k| \le B_q\int_{-\infty}^{\infty}\omega_0|k|(1+\omega_0|k||s^*-t^*|)^{-q}|(\partial_{t^*}\Psi)_k|\,ds^*.$$

We obtain

$$\begin{aligned}
&\int_{\tilde\tau}^{\tilde\tau+1}|(\partial_{t^*}\Psi_\flat)_k|^2(t^*,\cdot)dt^*\\
&\quad\le B_q\int_{\tilde\tau}^{\tilde\tau+1}\left(\int_{-\infty}^{\infty}\frac{\omega_0|k|}{(1+\omega_0|k||s^*-t^*|)^q}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|\,ds^*\right)^2dt^*\\
&\quad\le B_q\int_{\tilde\tau}^{\tilde\tau+1}\left(\sum_{\ell=-\infty}^{\infty}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}\frac{\omega_0|k|}{(1+\omega_0|k||s^*-t^*|)^q}\right.\\
&\qquad\left.\times|(\partial_{t^*}\Psi)_k(s^*,\cdot)|\,ds^*\right)^2dt^*\\
&\quad\le B_q(\omega_0k)^2\int_{\tilde\tau}^{\tilde\tau+1}\left(\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|\,ds^*\right)^2dt^*\\
&\quad\le B_q(\omega_0k)^2\int_{\tilde\tau}^{\tilde\tau+1}\left(\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\right)\\
&\qquad\times\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}(\omega_0|k|)^{-1}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,ds^*\,dt^*
\end{aligned}$$

$$
\begin{aligned}
&\leq B_q(\omega_0|k|)\int_{\tilde{\tau}}^{\tilde{\tau}+1}\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,ds^*\,dt^*\\
&\leq B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tilde{\tau}}^{\tilde{\tau}+1}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,ds^*\,dt^*\\
&\leq B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tilde{\tau}+\frac{\ell}{\omega_0|k|}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0|k|}}\\
&\qquad\times\int_{s^*-\frac{\ell+1}{\omega_0|k|}}^{s^*-\frac{\ell}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,dt^*\,ds^*\\
&\leq B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tilde{\tau}+\frac{\ell}{\omega_0|k|}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0|k|}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,ds^*\\
&\leq B_q\sum_{\ell=0}^{\infty}(1+|\ell|)^{-q}\int_{\tilde{\tau}-\frac{\ell}{\omega_0}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0}}|(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\,ds^*
\end{aligned}
$$

for $q > 1$.

Integrating and summing over k, we obtain

$$
\begin{aligned}
&\int_0^{\pi}\tilde{\nu}(\theta)\int_0^{2\pi}\int_{\tilde{\tau}}^{\tilde{\tau}+1}(\partial_{t^*}\Psi_\flat)^2(t^*,\cdot)dt^*\,d\phi\,d\theta\\
&\quad\leq B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\int_0^{\pi}\tilde{\nu}(\theta)\int_0^{2\pi}\int_{\tilde{\tau}-\frac{\ell}{\omega_0}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0}}|(\partial_{t^*}\Psi)(s^*,\cdot)|^2\,ds^*\,d\phi\,d\theta\\
&\quad\leq B_q\omega_0^{-1}\sum_{\ell=0}^{\infty}(1+\ell)^{-q}(2\ell+2)\sup_{-\infty<\bar{\tau}<\infty}\int_0^{\pi}\tilde{\nu}(\theta)\\
&\qquad\times\int_0^{2\pi}\int_{\bar{\tau}}^{\bar{\tau}+1}|(\partial_{t^*}\Psi)(s^*,\cdot)|^2\,ds^*\,d\phi\,d\theta,
\end{aligned}
$$

where we have used in the last line that $\omega_0 \leq 1$. The lemma follows after fixing $q > 2$. □

6.3.4 Application to $\psi_\sharp^\tau$

We may now easily prove

Proposition 6.2 *Let $0 \le \tau' < \tau'' \le \tau$ and let ψ be as in Theorem 4.1. We have*

$$\int_{\mathcal{H}(\tau',\tau'')} (\partial_{t^*}\psi^{\tau}_{\sharp})^2 \ge B{\omega_0}^2 \int_{\mathcal{H}(\tau',\tau'')} (\partial_{\phi}\psi^{\tau}_{\sharp})^2 \\ - B{\omega_0}^{-2}e^{-1} \sup_{0\le\bar{\tau}\le\tau-1} \int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} \mathbf{J}^{\mathbb{N}_e}_{\mu}(\psi)\mathbb{n}^{\mu}.$$

Proof To prove the proposition from the above lemmas, we first remark that it suffices to prove the inequality under the assumption

$$(\psi^{\tau}_{\sharp})_0 = \int_0^{2\pi} \psi^{\tau}_{\sharp}\, d\phi = 0,$$

for the inequality is trivially true for $(\psi^{\tau}_{\sharp})_0$. By (79), this is equivalent to assuming

$$\int_0^{2\pi} \psi^{\tau}_{\scissors} d\phi = 0,$$

and by (76)

$$\int_0^{2\pi} \psi\, d\phi = 0$$

in the support of $\psi_{\scissors}$. Of course, under this assumption it follows that this holds in all of $\mathcal{R}$. Thus we may assume

$$\int_0^{2\pi} \psi^2\, d\phi \le \int_0^{2\pi} (\partial_{\phi}\psi)^2\, d\phi \tag{86}$$

in the relevant region. From the above lemma, we just notice that on $\mathcal{H}(0,\tau)$

$$\begin{aligned}\int_0^{2\pi} (\partial_{t^*}\psi^{\tau}_{\scissors})^2\, d\phi &\le \int_0^{2\pi} ((\partial_{t^*}\chi^{\tau}_{\scissors})\psi + \chi^{\tau}_{\scissors}\partial_{t^*}\psi)^2\, d\phi \\ &\le \int_0^{2\pi} (B\psi^2 + B(\partial_t\psi)^2)\, d\phi \\ &\le \int_0^{2\pi} (B(\partial_{\phi}\psi)^2 + B(\partial_t\psi)^2)\, d\phi \\ &\le \int_0^{2\pi} Be^{-1}\mathbf{J}^{\mathbb{N}_e}_{\mu} n^{\mu}_{\mathcal{H}},\end{aligned}$$

where we have used (86), (60) and (61). The proposition follows. □

6.4 Comparing $\mathbf{q}_e[\psi_\flat^\tau]$, $\mathbf{q}_e[\psi_\sharp^\tau]$ and $\mathbf{q}_e[\psi]$

In view of (77), we clearly have the pointwise relation

$$\mathbf{q}_e[\psi] \leq 2\left(\mathbf{q}_e[\psi_\flat^\tau] + \mathbf{q}_e[\psi_\sharp^\tau]\right) \tag{87}$$

in $\mathcal{R}(1, \tau - 1)$. It will be necessary, however, to compare also in the opposite direction. We have

Proposition 6.3 *Let $\omega_0 \leq 1 \leq \tau_{\text{step}} \leq \tau' \leq \tau - \tau_{\text{step}}$. Then*

$$\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e[\psi_\flat^\tau] \leq B{\omega_0}^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar{\tau} \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi]$$
$$+ B{\omega_0}^{-7} e^{-1} \tau_{\text{step}}^{-2} \sup_{0 \leq \bar{\tau} \leq \tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi],$$

$$\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e[\psi_\sharp^\tau] \leq B\omega_0^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar{\tau} \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi]$$
$$+ B{\omega_0}^{-7} e^{-1} \tau_{\text{step}}^{-2} \sup_{0 \leq \bar{\tau} \leq \tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].$$

Proof Since $\psi_\sharp^\tau = \psi_{\text{✂}}^\tau - \psi_\flat^\tau$ and $\mathbf{q}_e[\psi_\sharp^\tau] \leq 2(\mathbf{q}_e[\psi^\tau] + \mathbf{q}_e[\psi_\flat^\tau])$ it will be sufficient to prove the first statement of the proposition. We begin with the following

Lemma 6.5 *Let Ψ be smooth of compact support in t^* and $\omega_0 \leq 1$. Then*

$$\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \Psi_\flat^2 \leq B\omega_0^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar{\tau} \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \Psi^2$$
$$+ B \sup_{-\infty \leq \bar{\tau} \leq \tau'-\tau_{\text{step}} \cup \tau'+\tau_{\text{step}} \leq \bar{\tau} \leq \infty} \omega_0^{-7} |\bar{\tau} - \tau_{\text{step}}|^{-6} \int_{\Sigma(\bar{\tau})} \Psi^2.$$

Proof For any $q > 0$, we have

$$\begin{aligned}
|(\Psi_\flat)_k(t^*, \cdot)| &\leq B_q(\omega_0|k|) \int_{-\infty}^{\infty} (1 + |\omega_0 k(t^* - s^*)|)^{-q} |\Psi_k|(s^*, \cdot)\, ds^* \\
&\leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1 + |\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|(s^*, \cdot)\, ds^* \\
&\leq B_q(\omega_0|k|)^{\frac{1}{2}} \sum_{\ell=-\infty}^{\infty} (1 + |\ell|)^{-q} \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*, \cdot)\, ds^* \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\tau'-1}^{\tau'} |(\Psi_\flat)_k(t^*,\cdot)|^2 dt^* \\
&\quad \le B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1}^{\tau'} dt^* \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\,ds^* \\
&\quad \le B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1+\frac{\ell}{\omega_0|k|}}^{\tau'+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\,ds^*.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\int_{\tau'-1}^{\tau'}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)\int_0^{2\pi}|(\Psi_\flat)(t^*,\cdot)|^2 d\phi\, d\theta\, dr\, dt^* \\
&\quad = \sum_{|k|\ge 1}\int_{\tau'-1}^{\tau'}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)|(\Psi_\flat)_k(t^*,\cdot)|^2 d\theta\, dr\, dt^* \\
&\quad \le B_q\sum_{|k|\ge 1}\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1+\frac{\ell}{\omega_0|k|}}^{\tau'+\frac{\ell+1}{\omega_0|k|}}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)|\Psi_k|^2(s^*,\cdot)\,ds^* \\
&\quad \le B_q\sum_{\ell\ge 0}(1+|\ell|)^{-q}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta) \\
&\qquad \times\int_0^{2\pi}\int_{\tau'-1-\frac{\ell}{\omega_0}}^{\tau'+\frac{\ell+1}{\omega_0}}|\Psi|^2(s^*,\cdot)\,ds^*\,d\phi\,d\theta\,dr \\
&\quad \le B\omega_0^{-1}\sup_{\tau'-\tau_{\rm step}\le\bar\tau\le\tau'+\tau_{\rm step}}\int_{\Sigma(\bar\tau)}\Psi^2 \\
&\qquad + B\sup_{-\infty\le\bar\tau\le\tau'-\tau_{\rm step}\cup\tau'+\tau_{\rm step}\le\bar\tau\le\infty}\omega_0^{-7}\left(\tau_{\rm step}+|\bar\tau-\tau_{\rm step}|\right)^{-6}\int_{\Sigma(\bar\tau)}\Psi^2
\end{aligned}$$

for q chosen sufficiently large. $\square$

Note that

$$\left((\partial_v\Psi)^2+(\partial_u\Psi)^2+|\nabla\!\!\!\!/\,\Psi|^2+e\frac{(\partial_u\Psi)^2}{(1-\mu)^2}\right)\sim\mathbf{q}_e[\Psi],$$

and as a consequence,

$$(\partial_v \Psi)^2_\flat + (\partial_u \Psi)^2_\flat + (\partial_{z^A}\Psi)^2_\flat + (\partial_{z^B}\Psi)^2_\flat + e\left(\frac{\partial_u \Psi}{1-\mu}\right)^2_\flat \sim \mathbf{q}_e[\Psi_\flat],$$

where z^A denote alternative coordinates x^A or $\tilde{x}^A$ of our atlas (26). Thus, we obtain from the above lemma applied to $\Psi = \partial_v \psi^\tau_{\text{✂}}$, $\Psi = \partial_u \psi^\tau_{\text{✂}}$, $\Psi = \partial_{z^A}\psi^\tau_{\text{✂}}$,[23] $\Psi = \sqrt{e}\frac{\partial_u \psi^\tau_{\text{✂}}}{1-\mu}$, the statement

$$\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e[\psi^\tau_\flat] \le B\omega_0^{-1} \sup_{\tau'-\tau_{\text{step}}\le\bar{\tau}\le\tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi^\tau_{\text{✂}}]$$
$$+ B \sup_{-\infty\le\bar{\tau}\le\tau'-\tau_{\text{step}}\cup\tau'+\tau_{\text{step}}\le\bar{\tau}\le\infty} \omega_0{}^{-7}\left(\tau_{\text{step}} + |\bar{\tau}-\tau_{\text{step}}|\right)^{-6} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi^\tau_{\text{✂}}]. \tag{88}$$

Note that it is sufficient to prove the inequality under the assumption $\psi_0 = 0$, and thus we may assume (86). Note the inequality

$$\mathbf{q}_e[\psi^\tau_{\text{✂}}](t,r,\cdot) \le \mathbf{q}_e[\psi](t,r,\cdot) + B\left(\sum_i (\partial_i(\chi(t^+ + 1 - \tau)\chi(-t^- + 1)))^2\right)\psi^2. \tag{89}$$

Now, B can be chosen such that in the support of the first term on the right hand side of (88) either $r \le B$ or $\psi^\tau_{\text{✂}} = \psi$. In view of (86), it follows thus that we may there replace $\mathbf{q}_e[\psi^\tau_{\text{✂}}]$ with $\mathbf{q}_e[\psi]$.

Turning to the second supremum term of (88) and applying

$$|t^* - \tau_{\text{step}}|^{-4}({}^+\chi^\tau_{\text{✂}} + {}^-\chi^\tau_{\text{✂}}) \le Br^{-2},$$

the statement of the proposition follows immediately in view of the restriction on τ_{step} and (86). □

6.5 Estimates for $\mathbf{Er}[\Psi]$

In view of the cutoffs, $\psi^\tau_\flat$ and $\psi^\tau_\sharp$ no longer satisfy (2).

Define

$$F^\tau_{\text{✂}} = \psi\,\Box_g \chi^\tau_{\text{✂}} + g^{\mu\nu}\partial_\mu(\chi^\tau_{\text{✂}})\partial_\nu\psi. \tag{90}$$

Note that $F^\tau_{\text{✂}}$ is supported in $\mathcal{R}^+(\tau-1,\tau) \cup \mathcal{R}^-(0,1)$.

[23] Of course, one needs to multiply this by a cutoff on the sphere to make it a well defined smooth function.

We may write

$$\Box_g \psi_\flat^\tau = F_\flat^\tau, \tag{91}$$

$$\Box_g \psi_\sharp^\tau = F_\sharp^\tau, \tag{92}$$

where $F_\flat^\tau$ and $F_\sharp^\tau$ are defined from $F_{\text{✂}}^\tau$ as in Sect. 6.2.

The right hand sides of (91) and (92) generate error terms in applying (45) with our various currents. We have the following

Proposition 6.4 *Let $\omega_0 \le 1 \le \tau_{\text{step}} \le \tau' \le \tau'' \le \tau - \tau_{\text{step}}$ and consider $\mathbb{V} = \mathbb{X}$, $\mathbb{N}_e$, or $\mathbb{T}$. Then the following holds*

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{Er}^{\mathbb{V}}[\psi_\flat^\tau] \le B\omega_0^{-8}\tau_{\text{step}}^{-2}e^{-1} \sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi], \tag{93}$$

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{Er}^{\mathbb{V}}[\psi_\sharp^\tau] \le B\omega_0^{-8}\tau_{\text{step}}^{-2}e^{-1} \sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi]. \tag{94}$$

Proof Decompose

$$F_{\text{✂}}^\tau = {}^1F_{\text{✂}}^\tau + {}^2F_{\text{✂}}^\tau$$

where

$${}^1F_{\text{✂}}^\tau = {}^-\chi_{\text{✂}}^\tau F_{\text{✂}}^\tau, \qquad {}^2F_{\text{✂}}^\tau = {}^+\chi_{\text{✂}}^\tau F_{\text{✂}}^\tau,$$

and consider ${}^jF_\flat^\tau$, ${}^jF_\sharp^\tau$, defined in Sect. 6.2, for $j = 1, 2$.

Recall the definitions (44), (47) of $\mathbf{Er}^{\mathbb{V}}$ and $\mathbf{Er}^{\mathbb{V},w}$. Since $(F_\flat^\tau)_0 = 0$ and $(F_\sharp^\tau)_0 = \int_0^{2\pi} F_{\text{✂}}^\tau\, d\phi = 0$ in $\mathcal{R}(\tau',\tau'')$, it follows that $\mathbf{Er}^{\mathbb{V}}[(\psi_\flat^\tau)_0] = \mathbf{Er}^{\mathbb{V}}[(\psi_\sharp^\tau)_0] = 0$ in $\mathcal{R}(\tau',\tau'')$ and thus equations (93) and (94) are trivially satisfied. By subtraction, we may thus assume in what follows that

$$\psi_0 = \int_0^{2\pi} \psi\, d\phi = 0,$$

and thus

$$r^{-2}\int_0^{2\pi} \psi^2 d\phi \le r^{-2}\int_0^{2\pi} (\partial_\phi\psi)^2\, d\phi \le Be^{-1}\int_0^{2\pi} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}_{\Sigma^+}^\mu\, d\phi \tag{95}$$

and similarly with $\mathrm{n}_{\Sigma^-}^\mu$.[24]

[24] In fact, the e^{-1} factor above is not necessary, but, this factor will be lost by another term to which this will be added.

Lemma 6.6 *For any* $q \geq 0$, $\tau_0 \leq \tau - 1$, *there exists a* B_q *such that*

$$|({}^2F^\tau_\flat)_k|(t^+ = \tau_0, \cdot) \leq B_q {\omega_0}^{1-q}(\tau - \tau_0)^{-q}|k|^{1-q} \int_{\tau-1}^{\tau} |({}^2F^\tau_{\text{✂}})_k(t^+, \cdot)|\, dt^+,$$
$$|({}^2F^\tau_\sharp)_k|(t^+ = \tau_0, \cdot) \leq B_q {\omega_0}^{1-q}(\tau - \tau_0)^{-q}|k|^{1-q} \int_{\tau-1}^{\tau} |({}^2F^\tau_{\text{✂}})_k(t^+, \cdot)|\, dt^+.$$

Proof This is standard. □

It follows from the above lemma applied to $q = 6$, the restriction on τ', τ'', and the relation between t^* and t^+ that

$$\begin{aligned}
&\int_{\tau'}^{\tau''} (1 + (\tau - t^*))^3 |({}^2F^\tau_\flat)_k|^2\, dt^* \\
&\quad \leq B\tau_{\mathrm{step}}^{-5}{\omega_0}^{-8} \int_{\tau-1}^{\tau} r^{-2} |({}^2F^\tau_{\text{✂}})_k|^2\, dt^+ \\
&\quad \leq B\tau_{\mathrm{step}}^{-5}{\omega_0}^{-8} \int_{\tau-1}^{\tau} r^{-2}(|\psi_k|^2 + e^{-1}\, \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_k]\mathbb{n}^\mu_{\Sigma^+})\, dt^+.
\end{aligned}$$

We remark that the powers of $\tau_{\mathrm{step}}^{-1}$ and r^{-1} can be chosen arbitrarily above, at the expense of the constant B and powers of ${\omega_0}^{-1}$, but this would give no advantage in what follows. Thus,

$$\begin{aligned}
&\int_{\mathcal{R}(\tau', \tau'')} (1 + (\tau - t^*))^3 ({}^2F^\tau_\flat)^2 \\
&\quad \leq \sum_k B\tau_{\mathrm{step}}^{-5}{\omega_0}^{-8} \int_{\mathcal{R}^+(\tau-1, \tau)} (r^{-2}|\psi_k|^2 + r^{-2}e^{-1}\, \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_k]\mathbb{n}^\mu_{\Sigma^+}) \\
&\quad \leq B\tau_{\mathrm{step}}^{-5}{\omega_0}^{-8} \sup_{\tau-1 \leq \bar\tau \leq \tau} \int_{\Sigma^+(\bar\tau)} r^{-2}(\partial_\phi \psi)^2 + r^{-2}e^{-1}\, \mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathbb{n}^\mu_{\Sigma^+} \\
&\quad \leq B\tau_{\mathrm{step}}^{-5}e^{-1}{\omega_0}^{-8} \sup_{\tau-1 \leq \bar\tau \leq \tau} \int_{\Sigma^+(\bar\tau)} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathbb{n}^\mu_{\Sigma^+},
\end{aligned}$$

where we have used (95). On the other hand, by conservation of energy we have that

$$\sup_{\tau-1 \leq \bar\tau \leq \tau} \int_{\Sigma^+(\bar\tau)} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathbb{n}^\mu_{\Sigma^+} \leq 2 \sup_{\tau-1 \leq \bar\tau \leq \tau} \int_{\Sigma(\bar\tau)} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathbb{n}^\mu_{\Sigma},$$

and thus,

$$\begin{aligned}
&\int_{\mathcal{R}(\tau', \tau'')} (1 + (\tau - t^*))^3 ({}^2F^\tau_\flat)^2\, dt^* \\
&\quad \leq B\tau_{\mathrm{step}}^{-5}e^{-1}{\omega_0}^{-8} \sup_{\tau-1 \leq \bar\tau \leq \tau} \int_{\Sigma(\bar\tau)} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathbb{n}^\mu_{\Sigma}
\end{aligned}$$

$$\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{\tau-1\leq\bar{\tau}\leq\tau}\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi]$$
$$\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{0\leq\bar{\tau}\leq\tau}\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi]. \tag{96}$$

Clearly, an identical bound holds for

$$\int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^3({}^1F_\flat^\tau)^2\,dt^*.$$

Let us consider first the cases where $\mathbb{V}\neq\mathbb{X}$. For $\mathbb{V}=\mathbb{T},\mathbb{N}_e$ we have

$$\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')}\mathbf{Er}^{\mathbb{V}}[\psi_\flat^\tau] &= \int_{\mathcal{R}(\tau',\tau'')}{}^1F_\flat^\tau\mathbb{V}^\nu(\psi_\flat^\tau)_\nu+{}^2F_\flat^\tau\mathbb{V}^\nu(\psi_\flat^\tau)_\nu\\
&\leq \int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^3\,({}^1F_\flat^\tau)^2\\
&\quad+\int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^{-3}(\mathbb{V}^\nu(\psi_\flat^\tau)_\nu)^2\\
&\quad+\int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^3({}^2F_\flat^\tau)^2\\
&\quad+\int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^{-3}(\mathbb{V}^\nu(\psi_\flat^\tau)_\nu)^2\\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{0\leq\bar{\tau}\leq\tau}\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi]\\
&\quad+B\int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^{-3}\,\mathbf{q}_e[\psi_\flat^\tau]\\
&\quad+B\int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^{-3}\,\mathbf{q}_e[\psi_\flat^\tau]\\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{0\leq\bar{\tau}\leq\tau}\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi]\\
&\quad+B\tau_{\mathrm{step}}^{-2}\sup_{\tau_{\mathrm{step}}\leq\bar{\tau}\leq\tau-\tau_{\mathrm{step}}}\int_{\bar{\tau}-1}^{\bar{\tau}}\left(\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi_\flat^\tau]\right)d\tilde{\tau}\\
&\leq (B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}+\tau_{\mathrm{step}}^{-2}B(\omega_0^{-1}+\omega_0^{-7}e^{-1}\tau_{\mathrm{step}}^{-2}))\\
&\quad\times\sup_{0\leq\bar{\tau}\leq\tau}\int_{\Sigma(\bar{\tau})}\mathbf{q}_e[\psi]
\end{aligned}$$

where for the last inequality we have used Proposition 6.3. We argue similarly for $\mathbf{Er}^{\mathbb{V}}[\psi_\sharp^\tau]$.

For the case of $\mathbf{Er}^{\mathbb{V}}$ where $\mathbb{V}=\mathbb{X}$, we have an additional error term

$$\widetilde{\mathbf{Er}}^{\mathbb{X}}[\psi_\flat^\tau] = -\frac{1}{4}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)f_b}{r^2}\right)\psi_\flat^\tau F_\flat^\tau.$$

Recall that $|f_b|\leq B\chi$, and $|f_b'|\leq Br^{-2}\chi$, where χ is a cutoff function such that $\chi=0$ in $r^*\leq 0$. Arguing as in the previous bound we obtain

$$\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')}\widetilde{\mathbf{Er}}^{\mathbb{X}}[\psi_\flat^\tau] &\leq \int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^3\,({}^1F_\flat^\tau)^2\\
&\quad + B\int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^{-3}\chi^2 r^{-2}(\psi_\flat^\tau)^2\\
&\quad + \int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^3({}^2F_\flat^\tau)^2\\
&\quad + B\int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^{-3}\chi^2 r^{-2}(\psi_\flat^\tau)^2\\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi]\\
&\quad + \int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^{-3}\,\mathbf{q}_e[\psi_\flat^\tau]\\
&\quad + \int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^{-3}\,\mathbf{q}_e[\psi_\flat^\tau]\\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi]\\
&\quad + B\tau_{\mathrm{step}}^{-2}\sup_{\tau_{\mathrm{step}}\leq\bar\tau\leq\tau-\tau_{\mathrm{step}}}\int_{\bar\tau-1}^{\bar\tau}\left(\int_{\Sigma(\tilde\tau)}\mathbf{q}_e[\psi_\flat^\tau]\right)d\tilde\tau\\
&\leq (B\tau_{\mathrm{step}}^{-5}e^{-1}\omega_0{}^{-8}+\tau_{\mathrm{step}}^{-2}B(\omega_0^{-1}+\omega_0^{-7}e^{-1}\tau_{\mathrm{step}}^{-2}))\\
&\quad \times\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi].
\end{aligned}$$

In the above, we have used again Proposition 6.3 as well as the inequality

$$r^{-2}\int_0^{2\pi}(\psi_\flat^\tau)^2 d\phi \leq r^{-2}\int_0^{2\pi}(\partial_\phi\psi_\flat^\tau)^2 d\phi \leq \mathbf{q}_e[\psi_\flat^\tau]$$

in the support of χ. The other terms of $\mathbf{Er}^{\mathbb{X}}$ can be handled as in the argument for $\mathbf{Er}^{\mathbb{T}}$, $\mathbf{Er}^{\mathbb{N}_e}$. Again, the argument for $\psi_\sharp^\tau$ is identical. □

6.6 Revisiting the relation between $\mathbf{q}_e[\psi_\flat^\tau]$, $\mathbf{q}_e[\psi_\sharp^\tau]$ and $\mathbf{q}_e[\psi]$

With the Proposition of the previous section, we may now refine Proposition 6.3 to a pointwise-in-time bound:

Proposition 6.5 *Let $\omega_0 \le 1 \le \tau_{\mathrm{step}} \le \tau' \le \tau - \tau_{\mathrm{step}}$. Then*

$$\int_{\Sigma(\tau')} \mathbf{q}_e[\psi_\flat^\tau] \le B \sup_{\tau'-\tau_{\mathrm{step}}\le\bar{\tau}\le\tau'+\tau_{\mathrm{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] + B{\omega_0}^{-8} e^{-1} \tau_{\mathrm{step}}^{-2} \sup_{0\le\bar{\tau}\le\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi],$$

$$\int_{\Sigma(\tau')} \mathbf{q}_e[\psi_\sharp^\tau] \le B \sup_{\tau'-\tau_{\mathrm{step}}\le\bar{\tau}\le\tau'+\tau_{\mathrm{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] + B{\omega_0}^{-8} e^{-1} \tau_{\mathrm{step}}^{-2} \sup_{0\le\bar{\tau}\le\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].$$

Proof Once again it is sufficient to establish this for $\psi_\flat^\tau$.

We write the energy identity (45) for the vector field $\mathbb{N}_e$ to obtain

$$\int_{\mathcal{H}(\tau_0,\tau')} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi_\flat^\tau] \mathrm{n}_{\mathcal{H}}^\mu + \int_{\Sigma(\tau')} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi_\flat^\tau] \mathrm{n}_\Sigma^\mu + \int_{\mathcal{R}(\tau_0,\tau')} \mathbf{K}^{\mathbb{N}_e}[\psi_\flat^\tau]$$
$$= \int_{\mathcal{R}(\tau_0,\tau')} \mathbf{Er}^{\mathbb{N}_e}[\psi_\flat^\tau] + \int_{\Sigma(\tau_0)} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi_\flat^\tau] \mathrm{n}_\Sigma^\mu.$$

By (57), (59), and the nonnegativity of the first term on the left hand side above, we obtain

$$\int_{\Sigma(\tau')} \mathbf{q}_e[\psi_\flat^\tau] \le eB \int_{\tau_0}^{\tau'} \int_{\Sigma(t^*)} \mathbf{q}_e[\psi_\flat^\tau] dt^* + B \left| \int_{\mathcal{R}(\tau_0,\tau')} \mathbf{Er}^{\mathbb{N}_e}[\psi_\flat^\tau] \right| + B \int_{\Sigma(\tau_0)} \mathbf{q}_e[\psi_\flat^\tau].$$

We integrate the above inequality with respect to τ_0 between $\tau' - 1$ and τ' and use Propositions 6.3, 6.4 to obtain the desired estimate. □

7 The main estimates

7.1 Estimates for the superradiant part $\psi_\flat^\tau$

Let us assume always

$$\tau_{\text{step}} \le \tau' \le \tau'' \le \tau - \tau_{\text{step}}. \tag{97}$$

Proposition 7.1 *For $\psi_\flat^\tau$ we have*

$$\int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}_e^\bigstar[\psi_\flat^\tau] \right) d\bar\tau \le B \int_{\mathcal{R}(\tau'.\tau'')} \left(\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e} \right) [\psi_\flat^\tau] + B \sup_{0 \le \bar\tau \le \tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi].$$

Proof Recall that $\int_0^{2\pi} \psi_\flat^\tau \, d\phi = 0$.

In the region $r \le r_{\mathbb{Y}}^-$, we have immediately from (74) that

$$\int_0^{2\pi} \mathbf{q}_e^\bigstar[\psi_\flat^\tau]\, d\phi \le B \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\psi_\flat^\tau]\, d\phi.$$

Similarly, in the region $r \ge R$, we have from (72) that

$$\int_0^{2\pi} \mathbf{q}_e^\bigstar[\psi_\flat^\tau]\, d\phi \le B \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\psi_\flat^\tau]\, d\phi.$$

For $r_{\mathbb{Y}}^- \le r \le R$, we have from (73) that

$$\int_0^{2\pi} \mathbf{q}_e^\bigstar[\psi_\flat^\tau]\, d\phi \le B \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\psi_\flat^\tau]\, d\phi - \int_0^{2\pi} \left(b |\nabla\!\!\!\!/ \psi_\flat^\tau|^2 - B(\partial_t \psi_\flat^\tau)^2 \right) d\phi.$$

Note also that

$$\int_0^{2\pi} |\nabla\!\!\!\!/ \psi_\flat^\tau|^2\, d\phi \ge b \int_0^{2\pi} |\partial_\phi \psi_\flat^\tau|^2\, d\phi$$

for constant (r, θ, t) curves in the region $r_{\mathbb{Y}}^- \le r \le R$. We have thus

$$\int_0^{2\pi} \mathbf{q}_e^\bigstar[\psi_\flat^\tau]\, d\phi \le B \int_0^{2\pi} (\mathbf{K}^{\mathbb{X}} + \mathbf{K}^{\mathbb{N}_e})[\psi_\flat^\tau]\, d\phi - \int_0^{2\pi} \left(b(\partial_\phi \psi_\flat^\tau)^2 - B(\partial_t \psi_\flat^\tau)^2 \right) d\phi.$$

The Proposition follows now from Proposition 6.1 for ω_0 chosen appropriately, in view also of our remarks on the measure of integration. □

In what follows we shall consider ω_0 to have been chosen and absorb such factors into the constants B.

Proposition 7.2 *For $\psi_\flat^\tau$, we have*

$$\int_{\tau'}^{\tau''}\left(\int_{\Sigma(\bar\tau)}\mathbf{q}_e^\star[\psi_\flat^\tau]\right)d\bar\tau \le B\left(\int_{\Sigma(\tau')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma+\int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma\right.$$
$$\left.+\int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_{\mathcal{H}}\right)+B\sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi].$$

Proof To prove Proposition 7.2 from Proposition 7.1, note that from (45) applied to the current $\mathbf{J}^{\mathbb{X}}+\mathbf{J}^{\mathbb{N}_e}$ we have

$$\begin{aligned}\int_{\mathcal{R}(\tau',\tau'')}(\mathbf{K}^{\mathbb{X}}+\mathbf{K}^{\mathbb{N}_e})[\psi_\flat^\tau] &\le \left|\int_{\Sigma(\tau')}(\mathbf{J}^{\mathbb{X}}_\mu+\mathbf{J}^{\mathbb{N}_e}_\mu)[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma\right|\\
&\quad+\left|\int_{\Sigma(\tau'')}(\mathbf{J}^{\mathbb{X}}_\mu+\mathbf{J}^{\mathbb{N}_e}_\mu)[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma\right.\\
&\quad\left.+\int_{\mathcal{H}(\tau',\tau'')}(\mathbf{J}^{\mathbb{X}}_\mu+\mathbf{J}^{\mathbb{N}_e}_\mu)[\psi_\flat^\tau]\mathrm{n}^\mu_{\mathcal{H}}\right|\\
&\quad+\left|\int_{\mathcal{R}(\tau',\tau'')}\mathbf{Er}^{\mathbb{X}+\mathbb{N}_e}[\psi_\flat^\tau]\right|\\
&\le B\left(\int_{\Sigma(\tau')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma+\int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma\right.\\
&\quad\left.+\int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_{\mathcal{H}}\right)\\
&\quad+B\tau_{\mathrm{step}}^{-2}e^{-1}\sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi].\end{aligned}$$

Above we have used (75) and Proposition 6.4. It will be important for later that $eB\ll 1$. The Proposition now follows immediately. □

Proposition 7.3

$$\int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_{\mathcal{H}}+\int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\flat^\tau]\mathrm{n}^\mu_\Sigma$$
$$\le B\int_{\Sigma(\tau')}\mathbf{q}_e[\psi_\flat]+B\tau_{\mathrm{step}}^{-2}e^{-1}\sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi].$$

Proof This follows from the divergence identity (45) for the current $\mathbf{J}^{\mathbb{T}}[\psi_\flat^\tau]$ and the fact that $\mathbf{K}^{\mathbb{T}} = 0$ and the inequality

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{Er}^{\mathbb{T}}[\psi_\flat^\tau] \leq B\tau_{\text{step}}^{-2} e^{-1} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi]$$

of Proposition 6.4. □

Proposition 7.4

$$\begin{aligned}
&\int_{\mathcal{H}(\tau',\tau'')} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_{\mathcal{H}} + \int_{\Sigma(\tau'')} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_\Sigma \\
&\quad \leq \int_{\Sigma(\tau')} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_\Sigma + Be \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar{\tau})} \mathbf{q}_e^\bigstar[\psi_\flat^\tau] \right) d\bar{\tau} \\
&\qquad + B\tau_{\text{step}}^{-2} e^{-1} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].
\end{aligned}$$

Proof This follows just from the divergence identity (45) for $\mathbf{J}^{\mathbb{N}_e}$ together with the bounds (57) and (93). □

Proposition 7.5

$$\begin{aligned}
&\int_{\Sigma(\tau'')} \mathbf{q}_e[\psi_\flat^\tau] + \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar{\tau})} \mathbf{q}_e^\bigstar[\psi_\flat^\tau] \right) d\bar{\tau} \\
&\quad \leq B \int_{\Sigma(\tau')} \mathbf{q}_e[\psi_\flat^\tau] + B \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].
\end{aligned}$$

Proof This follows immediately from Propositions 7.2 and 7.4 in view of (59) and the fact that for e small we have $Be \ll 1$. □

Proposition 7.6

$$\begin{aligned}
&\int_{\Sigma(\tau'')} \mathbf{J}^{\mathbb{T}}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_\Sigma \\
&\quad \leq B \int_{\Sigma(\tau')} \mathbf{q}_e[\psi_\flat^\tau] + (B\tau_{\text{step}}^{-2} e^{-1} + B\epsilon_{\text{close}} e^{-1}) \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].
\end{aligned}$$

Proof This follows from Propositions 7.3, 7.4 and 7.5 together with the one-sided bound

$$-\int_{\mathcal{H}(\tau',\tau'')} \mathbf{J}^{\mathbb{T}}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_{\mathcal{H}} \leq B\epsilon_{\text{close}} e^{-1} \int_{\mathcal{H}(\tau',\tau'')} \mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\flat^\tau] \mathrm{n}^\mu_{\mathcal{H}}. \qquad \square$$

7.2 Estimates for the non-superradiant part $\psi_\sharp^\tau$

We assume always (97).

Proposition 7.7 *For* $\psi_\sharp^\tau$,

$$\frac{1}{2}\int_{\mathcal{H}(\tau',\tau'')}\left|\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_{\mathcal{H}}\right| + \int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu$$
$$\leq \int_{\Sigma(\tau')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]n^\mu + B\tau_{\mathrm{step}}^{-2}\epsilon^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi]$$
$$+ B\epsilon_{\mathrm{close}}e^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathrm{n}^\mu_{\mathcal{H}}.$$

Proof From (45) applied to $\psi_\sharp^\tau$ with $\mathbb{V}=\mathbb{T}$ we have

$$\int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_\Sigma = \int_{\Sigma(\tau')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_\Sigma - \int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{T}}_\mu(\psi_\sharp^\tau)\mathrm{n}^\mu_{\mathcal{H}}$$
$$+\int_{\mathcal{R}(\tau',\tau'')}\mathbf{Er}^{\mathbb{T}}[\psi_\sharp^\tau].$$

On the other hand, by (30), we have the one-sided bound

$$-\int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_{\mathcal{H}} \leq B\epsilon_{\mathrm{close}}\int_{\mathcal{H}(\tau',\tau'')}\partial_t\psi_\sharp^\tau\partial_\phi\psi_\sharp^\tau - b\int_{\mathcal{H}(\tau',\tau'')}(\partial_t\psi_\sharp^\tau)^2$$
$$\leq B\epsilon_{\mathrm{close}}\int_{\mathcal{H}(\tau',\tau'')}(\partial_\phi\psi_\sharp^\tau)^2 - b\int_{\mathcal{H}(\tau',\tau'')}(\partial_t\psi_\sharp^\tau)^2$$

and thus by Proposition 6.2 we have

$$-\int_{\mathcal{H}(\tau',\tau'')}\mathbf{J}^{\mathbb{T}}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_{\mathcal{H}} \leq B\epsilon_{\mathrm{close}}e^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi]\mathrm{n}^\mu_{\mathcal{H}}.$$

The desired result now follows from Proposition 6.4. □

Proposition 7.8

$$\int_{\Sigma(\tau'')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_\Sigma + \int_{\mathcal{R}(\tau',\tau'')}\mathbf{K}^{\mathbb{N}_e}[\psi_\sharp^\tau]$$
$$\leq \int_{\Sigma(\tau')}\mathbf{J}^{\mathbb{N}_e}_\mu[\psi_\sharp^\tau]\mathrm{n}^\mu_\Sigma + B\tau_{\mathrm{step}}^{-2}e^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi].$$

Proof This is the energy identity (45) for $\mathbb{N}_e$ in view of the nonnegativity of the flux on the horizon and the estimate (94). □

Proposition 7.9

$$
\begin{aligned}
b\int_{\Sigma(\tau'')} \mathbf{q}_e[\psi_\sharp^\tau] + b\int_{\tau'}^{\tau''}\left(\int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi_\sharp^\tau]\right)d\bar\tau
&\le (\tau''-\tau')\int_{\Sigma(\tau_{\mathrm{step}})} \mathbf{J}_\mu^{\mathbb{T}}[\psi_\sharp^\tau]\mathrm{n}_\Sigma^\mu + B\int_{\Sigma(\tau')}\mathbf{q}_e[\psi_\sharp^\tau]\\
&\quad + (\tau''-\tau'+1)\left(B\tau_{\mathrm{step}}^{-2}e^{-1}\sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi]\right.\\
&\quad \left. + B\epsilon_{\mathrm{close}}e^{-1}\sup_{0\le\bar\tau\le\tau}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}\mathbf{J}_\mu^{\mathbb{N}_e}[\psi]\mathrm{n}_{\mathcal{H}}^\mu\right).
\end{aligned}
$$

Proof The proof follows from Propositions 7.7 (applied with $\tau'=\tau_{\mathrm{step}}$ and $\tau''=\bar\tau$), Proposition 7.8 applied to the given τ' and τ'', (58) and (59). □

8 The bootstrap

Given a solution ψ as in the assumptions of Theorem 4.1 and a parameter $e>0$, then for each $\hat C>0$, consider the set $\mathcal{T}_{\hat C,e}\subset[0,\infty)$ consisting of all τ such that for $0\le\bar\tau\le\tau$, we have

$$\int_{\Sigma(\bar\tau)}\mathbf{q}_e[\psi]\le \hat C\int_{\Sigma(0)}\mathbf{q}_e[\psi]. \tag{98}$$

In view of (40), which holds for $\epsilon_{\mathrm{close}}\ll e$, the statement of Theorem 4.1 would follow from

Proposition 8.1 *There exist constants $\epsilon_{\mathrm{close}}>0$, $e>0$, $\hat C>0$, depending only M, such that* (40) *holds, and such that for all ψ as in the statement of Theorem* 4.1, *then*

$$\mathcal{T}_{\hat C,e}=[0,\infty),$$

i.e. (98) *holds for all $\tau\ge0$.*

For this it suffices to show that $\mathcal{T}_{\hat C,e}$ is non-empty, open and closed. The non-emptyness is clear for sufficiently large $\hat C$. It thus suffices to show that $\hat C$ can be chosen depending only on M such that for all $\tau \in \mathcal{T}_{\hat{C},e}$, then

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq \frac{\hat{C}}{2} \int_{\Sigma(0)} \mathbf{q}_e[\psi] \tag{99}$$

for $0 \leq \bar{\tau} \leq \tau$.

8.1 Evolution for time τ_{step}

We will need the following proposition

Proposition 8.2 *Let τ_{step} be given. For small enough e depending on τ_{step}, $\epsilon_{\text{close}} \ll e$, it follows that for all τ_0 and $\bar{\tau} \in [\tau_0, \tau_0 + \tau_{\text{step}}]$,*

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq 2 \int_{\Sigma(\tau_0)} \mathbf{q}_e[\psi]. \tag{100}$$

Proof We write the energy identity (45) for the vector field $\mathbb{N}_e$ to obtain

$$\int_{\mathcal{H}(\tau_0,\bar{\tau})} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}_{\mathcal{H}}^\mu + \int_{\Sigma(\bar{\tau})} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}_\Sigma^\mu + \int_{\mathcal{R}(\tau_0,\bar{\tau})} \mathbf{K}^{\mathbb{N}_e}[\psi] = \int_{\Sigma(\tau_0)} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}_\Sigma^\mu.$$

By (57), (59), and the nonnegativity of the first term on the left hand side above, we obtain

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq eB \int_{\tau_0}^{\bar{\tau}} \int_{\Sigma(\hat{\tau})} \mathbf{q}_e[\psi] d\hat{\tau} + \int_{\Sigma(\tau_0)} \mathbf{q}_e[\psi]$$

and thus

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq \exp(eB(\bar{\tau} - \tau_0)) \int_{\Sigma(\tau_0)} \mathbf{q}_e[\psi].$$

The result follows thus if e is chosen so that

$$\exp(eB\tau_{\text{step}}) \leq 2.$$ □

8.2 Estimate for the local horizon flux of $\mathbf{J}_\mu^{\mathbb{N}_e}[\psi]$

A corollary of the proof of the previous Proposition is the following

Proposition 8.3

$$\int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}^\mu \leq B \int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi].$$

Of course, if we choose e to be sufficiently small as in the previous Proposition, we may replace B with 2.

8.3 Bounds for $\psi_\flat^\tau$

From Proposition 7.5 applied for $\tau' = n\tau_{\rm step}$, $\tau'' = (n+1)\tau_{\rm step}$, $n = 1, 2, \ldots, n_f$ where n_f is the largest integer such that $(n_f+1)\tau_{\rm step} \leq \tau - \tau_{\rm step}$, Proposition 6.5 and the bootstrap assumption (98), it follows that in each interval $[n\tau_{\rm step}, (n+1)\tau_{\rm step}]$, we can find a τ_n such that

$$\begin{aligned}\int_{\Sigma(\tau_n)} \mathbf{q}_e^\bigstar[\psi_\flat^\tau] &\leq \frac{B}{\tau_{\rm step}} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] + \frac{B}{\tau_{\rm step}} \int_{\Sigma(n\tau_{\rm step})} \mathbf{q}_e[\psi_\flat^\tau] \\ &\leq B\tau_{\rm step}^{-1}\hat{C} \int_{\Sigma(0)} \mathbf{q}_e[\psi]\end{aligned}$$

for appropriate choice of $\tau_{\rm step}$. On the other hand, by Proposition 7.6 applied with $\tau' = \tau_{\rm step}$, $\tau'' = \tau_n$, Proposition 6.5, (100) applied to $\tau_0 = 0$ and again to $\tau_0 = \tau_{\rm step}$, and the bootstrap assumption (98), we have

$$\begin{aligned}\int_{\Sigma(\tau_n)} \mathbf{J}_\mu^{\mathbb{T}}[\psi_\flat^\tau] \mathrm{n}_\Sigma^\mu &\leq B \int_{\Sigma(\tau_{\rm step})} \mathbf{q}_e[\psi_\flat^\tau] \\ &\quad + (B\tau_{\rm step}^{-2} e^{-1} + B\epsilon_{\rm close} e^{-1}) \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] \\ &\leq B \sup_{0\leq\bar\tau\leq 2\tau_{\rm step}} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] \\ &\quad + (B\tau_{\rm step}^{-2} e^{-1} + B\epsilon_{\rm close} e^{-1}) \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] \\ &\leq (B + B\tau_{\rm step}^{-2} e^{-1}\hat{C} + B\epsilon_{\rm close} e^{-1}\hat{C}) \cdot \int_{\Sigma(0)} \mathbf{q}_e[\psi]. \quad (101)\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\Sigma(\tau_n)} \mathbf{q}_e[\psi_\flat^\tau] &\leq B \int_{\Sigma(\tau_n)} \mathbf{q}_e^\bigstar[\psi_\flat^\tau] + \int_{\Sigma(\tau_n)} \mathbf{J}_\mu^{\mathbb{T}}[\psi_\flat^\tau] \mathrm{n}_\Sigma^\mu \\ &\leq (B + B\epsilon_{\rm close} e^{-1}\hat{C} + B\tau_{\rm step}^{-2} e^{-1}\hat{C} + B\tau_{\rm step}^{-1}\hat{C}) \int_{\Sigma(0)} \mathbf{q}_e[\psi]. \end{aligned} \quad (102)$$

8.4 Bounds for $\psi_\sharp^\tau$

Since

$$\int_{\Sigma(\bar\tau)} \mathbf{J}_\mu^{\mathbb{T}}[\psi_\sharp^\tau] \leq B \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi_\sharp^\tau],$$

it follows from Proposition 7.9 applied to $\tau' = n\tau_{\mathrm{step}}$, $\tau'' = (n+1)\tau_{\mathrm{step}}$, $n = 1, 2, \ldots, n_f$, Proposition 8.3 and (100) applied (twice) with $\tau_0 = 0$ and $\tau_0 = \tau_{\mathrm{step}}$, and Proposition 6.5 that in each interval $[n\tau_{\mathrm{step}}, (n+1)\tau_{\mathrm{step}}]$, we can find an τ_n such that

$$\begin{aligned}
b\int_{\Sigma(\tau_n)} \mathbf{q}_e[\psi_\sharp^\tau] &\leq \int_{\Sigma(\tau_{\mathrm{step}})} \mathbf{J}_\mu^{\mathbb{T}}[\psi_\sharp^\tau] \mathrm{n}_\Sigma^\mu + \tau_{\mathrm{step}}^{-1} B \int_{\Sigma(n\tau_{\mathrm{step}})} \mathbf{q}_e[\psi_\sharp^\tau] \\
&\quad + B\tau_{\mathrm{step}}^{-2} e^{-1}\hat{C} \int_{\Sigma(0)} \mathbf{q}_e[\psi] \\
&\quad + B\epsilon_{\mathrm{close}} \sup_{0\leq\bar\tau\leq\tau-1} \int_{\mathcal{H}(\bar\tau,\bar\tau+1)} \mathbf{J}_\mu^{\mathbb{N}_e}[\psi] \mathrm{n}_{\mathcal{H}}^\mu \\
&\leq B\int_{\Sigma(\tau_{\mathrm{step}})} \mathbf{q}_e[\psi_\sharp^\tau] + \tau_{\mathrm{step}}^{-1} B \int_{\Sigma(n\tau_{\mathrm{step}})} \mathbf{q}_e[\psi_\sharp^\tau] \\
&\quad + (B\tau_{\mathrm{step}}^{-2} e^{-1}\hat{C} + B\epsilon_{\mathrm{close}}\hat{C}) \int_{\Sigma(0)} \mathbf{q}_e[\psi] \\
&\leq B \sup_{0\leq\bar\tau\leq 2\tau_{\mathrm{step}}} \int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] \\
&\quad + (\tau_{\mathrm{step}}^{-1} B\hat{C} + \tau_{\mathrm{step}}^{-2} e^{-1} B\hat{C} + B\epsilon_{\mathrm{close}}\hat{C}) \int_{\Sigma(0)} \mathbf{q}_e[\psi] \\
&\leq (B + B\tau_{\mathrm{step}}^{-1}\hat{C} + B\tau_{\mathrm{step}}^{-2} e^{-1}\hat{C} + B\epsilon_{\mathrm{close}}\hat{C}) \cdot \int_{\Sigma(0)} \mathbf{q}_e[\psi].
\end{aligned} \tag{103}$$

8.5 Bounds for ψ

Choosing $\hat{C}$ sufficiently large, τ_{step} sufficiently large, e sufficiently small so that Proposition 8.2 holds, and $\epsilon_{\mathrm{close}} \ll e$ sufficiently small, from (87), (102), and (103) it follows that

$$\int_{\Sigma(\tau_n)} \mathbf{q}_e[\psi] \leq \frac{\hat{C}}{8} \int_{\Sigma(0)} \mathbf{q}_e[\psi].$$

From Proposition 8.2 it follows that for $\bar\tau \in [\tau_n, \tau_{n+1}]$

$$\int_{\Sigma(\bar\tau)} \mathbf{q}_e[\psi] \leq 2\int_{\Sigma(\tau_n)} \mathbf{q}_e[\psi] \leq \frac{\hat{C}}{4} \int_{\Sigma(0)} \mathbf{q}_e[\psi].$$

For $\bar{\tau} \in [0, \tau_1]$ we apply twice Proposition 8.2

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq 2 \int_{\Sigma(\tau_{\text{step}})} \mathbf{q}_e[\psi] \leq 4 \int_{\Sigma(0)} \mathbf{q}_e[\psi] \leq \frac{\hat{C}}{2} \int_{\Sigma(0)} \mathbf{q}_e[\psi],$$

as long as we assume $\hat{C} \geq 8$. Similarly, for $\bar{\tau} \in [\tau_{n_f}, \tau]$, we apply Proposition 8.2 twice to obtain

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e[\psi] \leq 2 \int_{\Sigma(\tau-\tau_{\text{step}})} \mathbf{q}_e[\psi] \leq 4 \int_{\Sigma(\tau_{n_f})} \mathbf{q}_e[\psi] \leq \frac{\hat{C}}{2} \int_{\Sigma(0)} \mathbf{q}_e[\psi].$$

We have shown (99), thus Proposition 8.1, and thus Theorem 4.1.

For the rest of this paper, all small quantities can be considered fixed. In particular, we may now write $\mathbf{q}_e[\psi] \sim \mathbf{q}[\psi]$. We note, however, that once Proposition 8.1 has been obtained for some value $e > 0$, it holds for any $\tilde{e} \geq e$, for appropriate $\hat{C} = \hat{C}(\tilde{e})$. On the other hand, for $0 < \tilde{e} < e$, the statement can only be inferred for choices of $\tilde{e}$ for which (40) holds, and in principle, $\hat{C}(\tilde{e})$ grows as $\tilde{e}$ is taken smaller. This is in sharp distinction with the Schwarzschild case, where the Proposition holds with $\tilde{e} = 0$ and $\hat{C} = 1$.

9 Estimate for the total horizon and null-infinity flux of $\mathbf{J}^{\mathbb{T}}_{\mu}[\psi]$

An immediate corollary of Theorem 4.1 is the statement

$$\int_{\mathcal{H}(0,\tau)} \mathbf{J}^{\mathbb{T}}_{\mu}[\psi] \mathrm{n}^{\mu}_{\mathcal{H}} \leq B \int_{\Sigma(0)} \mathbf{q}[\psi]. \tag{104}$$

Note, however, that the left hand of (104) is not necessarily a nonnegative quantity. As it is the 'superradiant' part of the solution for which this expression fails to be nonnegative, and we have better control for this, it turns out that from the estimates derived in the course of the proof of Theorem 4.1, one can in fact strengthen (104) to obtain

Theorem 9.1 *Let ϵ_{close} be the constant of Theorem 4.1. Then there exists a positive constant $\tilde{C}$ depending only on M such that the following holds. Let ψ be as in Sect. 4.1 where ψ satisfies (2). Then, for all $\tau \geq 0$ we have*

$$\int_{\mathcal{H}(0,\tau)} \left| \mathbf{J}^{\mathbb{T}}_{\mu}[\psi] \mathrm{n}^{\mu}_{\mathcal{H}} \right| \leq \tilde{C} \int_{\Sigma(0)} \mathbf{q}_e[\psi]. \tag{105}$$

Proof For (105), in view of the relations

$$\mathbf{J}^{\mathbb{T}}_{\mu}[\psi] \leq B \left(\left| \mathbf{J}^{\mathbb{T}}_{\mu}[\psi^{\tau}_{\sharp}] \right| + \left| \mathbf{J}^{\mathbb{T}}_{\mu}[\psi^{\tau}_{\flat}] \right| \right)$$

valid in $\mathcal{R}(1, \tau - 1)$, and

$$\left|\mathbf{J}^{\mathbb{T}}_\mu[\Psi]\mathrm{n}^\mu_{\mathcal{H}}\right| \leq B\mathbf{J}^{\mathbb{N}_e}_\mu[\Psi]\mathrm{n}^\mu_{\mathcal{H}} \tag{106}$$

on $\mathcal{H}^+$, it follows from Proposition 8.3 and Proposition 8.1, that it suffices to show

$$\int_{\mathcal{H}(\tau_{\mathrm{step}}, \tau-\tau_{\mathrm{step}})} \left|\mathbf{J}^{\mathbb{T}}_\mu[\psi^\tau_\flat]\mathrm{n}^\mu\right| \leq B\int_{\Sigma(0)} \mathbf{q}_e[\psi], \tag{107}$$

$$\int_{\mathcal{H}(\tau_{\mathrm{step}}, \tau-\tau_{\mathrm{step}})} \left|\mathbf{J}^{\mathbb{T}}_\mu[\psi^\tau_\sharp]\mathrm{n}^\mu\right| \leq B\int_{\Sigma(0)} \mathbf{q}_e[\psi]. \tag{108}$$

Inequality (107) follows immediately from (106) applied to $\psi^\tau_\flat$, and Propositions 7.4, 7.5, 6.5 and 8.1.

For (108), in view of the bound

$$\int_{\Sigma(\tau')} \mathbf{J}^{\mathbb{T}}_\mu[\psi^\tau_\sharp]\mathrm{n}^\mu_\Sigma \leq B\int_{\Sigma(\tau')} \mathbf{q}_e[\psi^\tau_\sharp]$$

we need only apply Propositions 6.5, 7.7, 8.3, and 8.1. □

Corollary 9.1 *Under the assumptions of the above theorem,*

$$\int_{\mathcal{I}^+} \mathbf{J}^{\mathbb{T}}_\mu[\psi]\mathrm{n}^\mu_{\mathcal{I}^+} \leq \tilde{C}\int_{\Sigma(0)} \mathbf{q}[\psi]. \tag{109}$$

Proof This follows immediately (after redefinition of $\tilde{C}$) from Theorems 4.1 and 9.1 and the statement $\mathbf{K}^{\mathbb{T}}[\psi] = 0$. The definition of the left hand side of the above inequality is as in immediately following the statement of Theorem 1.1, with r replacing $\hat{r}$. □

When specialised to the Kerr or Kerr-Newman case, Theorems 9.1 and Corollary 9.1 give in particular (7) and (8). The complete statement of Theorem 1.1 is thus proven.

10 Higher order energies and pointwise bounds

A limitation of previous understanding of boundedness, even in the Schwarzschild case, is that it relied on commuting the wave equation (2) with a full basis of angular momentum operators Ω_i, $i = 1, \ldots, 3$. In view of the loss of symmetry when passing from Schwarzschild to Kerr, this approach is no longer tenable. A much more robust approach to boundedness is via commutation with n_Σ, or equivalently, the vector field $\hat{\mathbb{Y}}$ to be discussed below. It turns out that the dangerous extra terms arising have a good sign. This can be viewed of as yet another manifestation of the redshift effect.

In Sect. 10.1 below, we will first derive L^2 estimates for higher order energies. These will rely on certain elliptic estimates derived in Sect. 10.2. Pointwise estimates will then follow in Sect. 10.3 from standard Sobolev inequalities.

10.1 Higher order energies

Let us consider now the quantity

$$\mathbf{q}^j[\Psi] \doteq \sum_{i=0}^{j} \mathbf{J}_\mu^{\mathrm{n}_\Sigma}[\mathrm{n}_\Sigma^i(\Psi)]\mathrm{n}_\Sigma^\mu$$

where $\mathrm{n}^i\Psi$ denotes $\mathrm{n}(\mathrm{n}(\mathrm{n}\ldots\psi))$ where i n's appear. Under our smoothness assumptions, coupled with our assumptions about the support, we have that

$$\int_{\Sigma(\tau)} \mathbf{q}^j[\psi] < \infty.$$

We have the following

Theorem 10.1 *There exists a positive constant $\epsilon_{\mathrm{close}}$ depending only on M, such that the following holds.*

Let g, $\Sigma(\tau)$ be as in Sect. 3.2, such that in addition $g \in C^{j+1}$ for an integer $j \geq 0$. Then there exists a constant C_j depending on M, j and g, such that the following holds. Let ψ, ψ', ψ be as in Sect. 4.1 where ψ satisfies (2). Then, for $\tau \geq 0$,

$$\int_{\Sigma(\tau)} \mathbf{q}^j[\psi] \leq C_j \int_{\Sigma(0)} \mathbf{q}^j[\psi].$$

Proof We shall give the proof only for the case $j = 1$, as this will be sufficient for deriving pointwise bounds for ψ. The dependence of C_1 on g can be expressed in terms of the C^2 norm of the components of g with respect to the atlas (26), restricted say to $r \leq r_{\mathbb{Y}}^+$.

Commute (2) with $\mathbb{T}$. One obtains from (39) that for $\tau \geq 0$

$$\int_{\Sigma(\tau)} \mathbf{q}[\mathbb{T}\psi] \leq B \int_{\Sigma(0)} \mathbf{q}[\mathbb{T}\psi]. \tag{110}$$

Note that from (39) we have for $\tau'' > \tau' \geq 0$,

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}[\psi] \leq B(\tau'' - \tau') \int_{\Sigma(0)} \mathbf{q}[\psi], \tag{111}$$

and from (110),

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}[\mathbb{T}\psi] \le B(\tau''-\tau') \int_{\Sigma(0)} \mathbf{q}[\mathbb{T}\psi]. \tag{112}$$

Now commute (2) with the vector field

$$\hat{\mathbb{Y}} \doteq \frac{1}{1-\mu}\partial_u$$

where ∂_u refers to the coordinate system described in Sect. 5.2. We obtain

Lemma 10.1 *Let ψ satisfy* (2). *Then we may write*

$$\Box_g(\hat{\mathbb{Y}}\psi) = \frac{2}{r}\hat{\mathbb{Y}}(\hat{\mathbb{Y}}[\psi]) - \frac{4}{r}(\hat{\mathbb{Y}}(\mathbb{T}\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{\mathbb{Y}}, P_2]\psi$$

where P_1 is the first order operator $P_1\psi \doteq \frac{2}{r^2}(\mathbb{T}\psi - \hat{\mathbb{Y}}\psi)$, and P_2 is the second order operator $P_2 = \Box_{g_M} - \Box_g$.

Now apply the basic identity (45) to $\Psi = \hat{\mathbb{Y}}\psi$ with $\mathbb{V} = \mathbb{Y}$. We have that for $r \le r^-_{\mathbb{Y}}$,

$$\mathbf{K}^{\mathbb{Y}}[\hat{\mathbb{Y}}\psi] \ge b\mathbf{q}[\hat{\mathbb{Y}}\psi]$$

while for $r \ge r^-_{\mathbb{Y}}$,

$$\mathbf{K}^{\mathbb{Y}}[\hat{\mathbb{Y}}\psi] \le B\mathbf{q}[\hat{\mathbb{Y}}\psi].$$

On the other hand,

$$\begin{aligned}\mathbf{Er}^{\mathbb{Y}}[\hat{\mathbb{Y}}\psi] &= -\left(\frac{2}{r}\hat{\mathbb{Y}}(\hat{\mathbb{Y}}\psi) - \frac{4}{r}(\hat{\mathbb{Y}}(\mathbb{T}\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{\mathbb{Y}}, P_2]\psi\right)\mathbb{Y}(\hat{\mathbb{Y}}\psi)\\ &\le -\left(\frac{2}{r}(\hat{\mathbb{Y}} - \mathbb{Y})(\hat{\mathbb{Y}}\psi) - \frac{4}{r}(\hat{\mathbb{Y}}(\mathbb{T}\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{\mathbb{Y}}, P_2]\psi\right)\\ &\quad \times \mathbb{Y}(\hat{\mathbb{Y}}\psi).\end{aligned}$$

Recall also that the above expression is supported in $\{r \le r^+_{\mathbb{Y}}\}$.

The following lemmas are immediate:

Lemma 10.2

$$\int_{\mathcal{R}(\tau',\tau'')} \frac{16}{r^2}(\hat{\mathbb{Y}}(\mathbb{T}\psi))^2 \le B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}[\mathbb{T}\psi],$$

$$\int_{\mathcal{R}(\tau',\tau'')} (P_1\psi)^2 \le B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}[\psi].$$

Lemma 10.3

$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^+_{\mathbb{Y}}\}} 4\frac{(P_2\psi)^2}{r^2} + ([P_2,\hat{\mathbb{Y}}]\psi)^2 \le B\epsilon^2_{\mathrm{close}}\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}^1[\psi] + B_2(g)\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi].$$

(Here $B_2(g)$ is a constant depending on the C^2 norm of g_{ij} in the atlas (26) restricted to $r \le r^+_{\mathbb{Y}}$.)

Lemma 10.4 *Given $r_{\hat{\mathbb{Y}}} > 2M$, we may choose a $\delta_{\hat{\mathbb{Y}}}$ (with $\delta_{\hat{\mathbb{Y}}} \to 0$ as $r_{\hat{\mathbb{Y}}} \to 2M$) such that*

$$\int_{\mathcal{R}(\tau',\tau'')} \frac{4}{r^2}((\hat{\mathbb{Y}}-\mathbb{Y})\hat{\mathbb{Y}}\psi)^2 \le B\delta_{\hat{\mathbb{Y}}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi] + B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi].$$

For convenience, let us require in what follows that $r_{\hat{\mathbb{Y}}} \le r^-_{\mathbb{Y}}$.

It follows from (45), the above lemmas and Cauchy-Schwarz (applied with a small parameter λ) that

$$\begin{aligned}
&\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]\\
&\quad\le B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi] + B\lambda^{-1}\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi]\\
&\qquad+B\lambda^{-1}\epsilon^2_{\mathrm{close}}\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}^1[\psi] + B\lambda^{-1}\delta_{\hat{\mathbb{Y}}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]\\
&\qquad+B_2(g)\lambda^{-1}\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi] + B\lambda^{-1}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]\\
&\qquad+B\lambda\int_{\mathcal{R}(\tau',\tau'')}(\mathbb{Y}(\hat{\mathbb{Y}}\psi))^2 + B\int_{\Sigma(\tau')}\mathbf{q}[\hat{\mathbb{Y}}\psi].
\end{aligned}$$

Since $(\mathbb{Y}(\hat{\mathbb{Y}}\psi))^2 \le B\mathbf{q}[\hat{\mathbb{Y}}\psi]$, it follows that λ can be chosen so that

$$\begin{aligned}
&\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]\\
&\quad\le B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi] + B\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]
\end{aligned}$$

$$
\begin{aligned}
&+(B+B_2(g))\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi]+B\epsilon_{\rm close}^2\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}^1[\psi]\\
&+B\delta_{\hat{\mathbb{Y}}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]+B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]\\
&+B\int_{\Sigma(\tau')}\mathbf{q}[\hat{\mathbb{Y}}\psi]\\
\le{}& B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]+B\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]\\
&+(B+B_2(g))\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi]+B\epsilon_{\rm close}^2\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}^1[\psi]\\
&+B\delta_{\hat{\mathbb{Y}}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]+B\epsilon_{\rm close}^2\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r^-_{\mathbb{Y}}\}}\mathbf{q}^1[\psi]\\
&+B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\ge r_{\hat{\mathbb{Y}}}\}}\mathbf{q}^1[\psi]+B\int_{\Sigma(\tau')}\mathbf{q}[\hat{\mathbb{Y}}\psi].
\end{aligned}
$$

From the Propositions of Sect. 10.2 below, we obtain

$$
\begin{aligned}
&\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]\\
&\quad\le B\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]+(B+B_2(g))\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi]\\
&\qquad+B(\epsilon_{\rm close}^2+\delta_{\hat{\mathbb{Y}}})\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]+\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi]\\
&\qquad+B(r_{\hat{\mathbb{Y}}})\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi]+B\int_{\Sigma(\tau')}\mathbf{q}[\hat{\mathbb{Y}}\psi],
\end{aligned}
$$

and thus, for small enough $\epsilon_{\rm close}$, and choosing $r_{\hat{\mathbb{Y}}}$ close enough to $2M$ (and thus small enough $\delta_{\hat{\mathbb{Y}}}$), we obtain

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r^-_{\mathbb{Y}}\}}\mathbf{q}[\hat{\mathbb{Y}}\psi]\le B\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\mathbb{T}\psi]+(B+B_2(g))\int_{\mathcal{R}(\tau',\tau'')}\mathbf{q}[\psi]\\
+B\int_{\Sigma(\tau')}\mathbf{q}[\hat{\mathbb{Y}}\psi].
\end{aligned}
$$

(The choice of $r_{\hat{\mathbb{Y}}}$ having been made, we have written above $B(r_{\hat{\mathbb{Y}}})$ as B following our convention.) From (111) and (112), it now follows that

$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_{\mathbb{Y}}^-\}} \mathbf{q}[\hat{\mathbb{Y}}\psi] \le (B+B_2(g))(\tau''-\tau')\int_{\Sigma(0)} (\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi])) + B\int_{\Sigma(\tau')} \mathbf{q}[\hat{\mathbb{Y}}\psi].$$

It follows immediately that there exists a sequence $0=\tau_0<\tau_i<\tau_{i+1}$ such that

$$|\tau_i-\tau_j|\le B, \quad \tau_i\to\infty \tag{113}$$

with

$$\int_{\Sigma(\tau_i)\cap\{r\le r_{\mathbb{Y}}^-\}} \mathbf{q}[\hat{\mathbb{Y}}\psi] \le (B+B_2(g))\int_{\Sigma(0)} (\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi]) + B\int_{\Sigma(0)} \mathbf{q}[\hat{\mathbb{Y}}\psi].$$

From (110), we have on the other hand

$$\int_{\Sigma(\tau_i)} \mathbf{q}[\mathbb{T}\psi] \le B\int_{\Sigma(0)} \mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi],$$

and from (39)

$$\int_{\Sigma(\tau_i)} \mathbf{q}[\psi] \le B\int_{\Sigma(0)} \mathbf{q}[\psi].$$

From Proposition 10.1 below it follows that

$$\int_{\Sigma(\tau_i)\cap\{r\le r_{\mathbb{Y}}^-\}} \mathbf{q}^1(\psi) \le (B+B_2(g))\int_{\Sigma(0)} \mathbf{q}[\hat{\mathbb{Y}}\psi]+\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi],$$

while from Proposition 10.2 below, it follows that

$$\int_{\Sigma(\tau_i)\cap\{r\ge r_{\mathbb{Y}}^-\}} \mathbf{q}^1[\psi] \le (B+B_2(g))\int_{\Sigma(0)} \mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi].$$

Thus in fact,

$$\int_{\Sigma(\tau_i)} \mathbf{q}^1[\psi] \le (B+B_2(g))\int_{\Sigma(0)} \mathbf{q}[\hat{\mathbb{Y}}\psi]+\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi].$$

In view of (113), we obtain now easily

$$\int_{\Sigma(\tau)} \mathbf{q}^1(\psi) \le (B+B_2(g))\int_{\Sigma(0)} \mathbf{q}[\hat{\mathbb{Y}}\psi]+\mathbf{q}[\mathbb{T}\psi]+\mathbf{q}[\psi] \le (B+B_2(g))\int_{\Sigma(0)} \mathbf{q}^1[\psi].$$

□

10.2 Elliptic estimates

We have the following elliptic estimate on spheres:

Proposition 10.1 *Let S_r denote a set of constant r in a t^*, r, x^A, x^B coordinate system. For ψ a solution of $\Box_g\psi = 0$, we have*

$$\int_{S_r} \mathbf{q}^1[\psi] \leq B \int_{S_r} \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\hat{\mathbb{Y}}\psi] + \mathbf{q}[\psi].$$

Proof Note first that

$$\begin{aligned} \mathbf{q}^1[\psi] &\leq B\left(|\nabla\!\!\!/^2\psi|^2 + |\nabla\!\!\!/(\mathbb{T}\psi)|^2 + |\nabla\!\!\!/(\hat{\mathbb{Y}}\psi)|^2 + |\mathbb{T}\mathbb{T}\psi|^2 \right. \\ &\quad \left. + |\hat{\mathbb{Y}}\hat{\mathbb{Y}}\psi|^2 + |\mathbb{T}\hat{\mathbb{Y}}\psi|^2 + \mathbf{q}[\psi]\right) \\ &\leq B\left(|\nabla\!\!\!/^2\psi|^2 + \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\hat{\mathbb{Y}}\psi] + \mathbf{q}[\psi]\right). \end{aligned} \tag{114}$$

Let $\triangle_{\mathbb{S}^2}$ denote the standard Laplacian on the unit sphere. In the coordinates of the first paragraph of Sect. 5.2, we may write

$$\frac{1}{r^2}\triangle_{\mathbb{S}^2}\psi = \partial_v(\hat{\mathbb{Y}}\psi) - \frac{2}{r}(\mathbb{T}\psi - \hat{\mathbb{Y}}\psi) - P_2\psi.$$

Integrating over S_r endowed with metric of the standard unit sphere, we obtain the elliptic estimate

$$\frac{1}{r^2}\int_{S_r} |\nabla^2_{\mathbb{S}^2}\psi|^2 \, dA_{\mathbb{S}^2} \leq B\int_{S_r} \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\hat{\mathbb{Y}}\psi] + \mathbf{q}[\psi] + (P_2\psi)^2 \, dA_{\mathbb{S}^2},$$

i.e., in view of the assumptions (28) on the metric,

$$\int_{S_r} |\nabla\!\!\!/^2\psi|^2 \leq B\int_{S_r} \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\hat{\mathbb{Y}}\psi] + \mathbf{q}[\psi] + (P_2\psi)^2. \tag{115}$$

On the other hand, from (28), (29) we have

$$(P_2\psi)^2 \leq B\epsilon_{\mathrm{close}}\mathbf{q}^1[\psi].$$

The above, (115) and (114) yield the proposition, for $\epsilon_{\mathrm{close}}$ sufficiently small. □

In addition, we have the following elliptic estimates away from the event horizon.

Proposition 10.2 *For ψ a solution of $\Box_g \psi = 0$, and r_0 a parameter with $r_0 > 2M$, then, for ϵ_{close} sufficiently small, we have*

$$\int_{\Sigma(\tau')\cap\{r\geq r_0\}} \mathbf{q}^1[\psi] \leq B(r_0) \int_{\Sigma(\tau')} \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\psi],$$
$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_0\}} \mathbf{q}^1[\psi] \leq B(r_0) \int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}[\mathbb{T}\psi] + \mathbf{q}[\psi].$$

Proof The proof of this straightforward elliptic estimate is left to the reader. □

10.3 Pointwise bounds

We have the following Sobolev-type estimate on Schwarzschild

Proposition 10.3 *Let Ψ be a smooth function on Σ_τ of compact support. Then*

$$\sup_{\Sigma_\tau} \Psi^2 \leq B \int_{\Sigma_\tau} |\nabla^2_{\Sigma_\tau} \Psi|^2_{(g_M)_{\Sigma_\tau}} + |\nabla_{\Sigma_\tau} \Psi|^2_{(g_M)_{\Sigma_\tau}}.$$

This in turn follows from the following Euclidean space estimate:

Proposition 10.4 *There exists a constant K such that the following holds. Let Ψ be a smooth function on $\mathbb{R}^3 \cap \{r \geq 1\}$ of compact support. Then*

$$\sup_{r\geq 1} \Psi^2 \leq K \int_{\{r\geq 1\}} (|\nabla^2 \Psi|^2 + |\nabla \Psi|^2)\, dx^1\, dx^2\, dx^3.$$

Proof Omitted. □

We obtain

Theorem 10.2 *There exists a positive constant ϵ_{close}, depending only on M, such that the following holds.*

Let g, $\Sigma(\tau)$ be as in Sect. 3.2, such that in addition, $g \in C^{n+2}$. There exists a positive constant C_n, depending on M, n and g, such that the following holds. Let ψ, ψ', ψ be as in Sect. 4.1 where ψ satisfies (2). Then, for $\tau \geq 0$,

$$|\nabla^{(n)} \psi|^2_{g_{\Sigma_\tau}} \leq \lim_{r\to\infty} \psi^2 + C_n \int_{\Sigma(0)} \mathbf{q}^{n+1}[\psi].$$

Here, $\nabla^{(n)}\Psi$ denotes the n'th order spacetime covariant derivative tensor and $|\cdot|_{g_{\Sigma_\tau}}$ denotes the induced Riemannian supremum norm. The dependence of C_n on g is inherited from the dependence in Theorem 10.1.

Theorem 1.2 in particular follows from the above.

11 Further notes

11.1 The Schwarzschild case

In the Schwarzschild case, we may apply the estimates proven here for $\psi_\sharp$ in Sects. 7.2 and 8.4 directly to the whole ψ. Since no frequency decomposition need be made, no associated error terms arise and the whole argument can be reduced to a few pages. The resulting energy estimate, coupled with the higher order and pointwise estimates of Sect. 10, yield a new proof for uniform boundedness of solutions to the wave equation on Schwarzschild which is in some sense the simplest one yet—using neither the discrete isometry exploited by Kay-Wald [35], nor the vector field X of our [19] or [20], nor commuting with angular momentum operators. Moreover, one shows the uniform boundedness of all derivatives on the event horizon up to all order, whereas previous results could control only tangential derivatives.

In fact, one can obtain a much more general statement applying to all static spherically symmetric non-extremal black holes. We have

Theorem 11.1 *Let $(\mathcal{D}, g)$ be a static spherically symmetric asymptotically flat exterior black hole spacetime bounded by a non-extremal event horizon $\mathcal{H}^+$. Then the estimates of Theorems 1.1 and 1.2 hold.*

11.2 Kerr-de Sitter

Our argument is easily adapted to spacetimes which are small perturbations of non-extremal Schwarzschild-de Sitter, in particular to slowly rotating non-extremal Kerr-de Sitter, or Kerr-Newman-de Sitter. See [21] for the setting. One fixes the manifold structure on a subregion $\mathcal{D} \cap J^+(\Sigma_0)$ where $\mathcal{D}$ is here the region between a set of black/white hole and cosmological horizons and Σ_0 is a Cauchy surface crossing both horizons to the future of the bifurcate spheres (see Fig. 3).

One continues as in the Schwarzschild case. The argument is in fact easier at several points. Because r is bounded in $\mathcal{D}$, the zero-order terms pose no difficulty. In particular, one need not introduce the Σ^+ and Σ^- surfaces, nor must one modify $\mathbf{J}^{\mathbb{X}_a}$ by the addition of $\mathbf{J}^{\mathbb{X}_b, w_b}$. We leave the details for a subsequent paper.

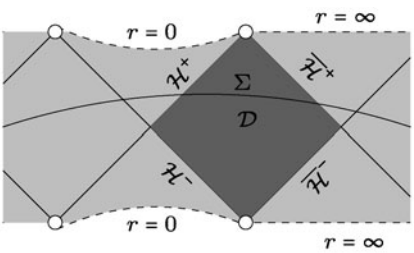


Fig. 3 The Schwarzschild-de Sitter metric

11.3 Non-quantitative decay

As a final application, we note that uniform boundedness is sufficient to translate non-quantitative results for fixed angular frequency into non-quantitative results for ψ itself. For instance, Theorem 10.2 immediately gives:

Corollary 11.3.1 *Under the assumptions of Theorem 10.2 with $n = 0$, suppose in addition that (r, θ, ϕ) is such that for each k we have*

$$\lim_{t^* \to \infty} \mathfrak{P}_k(t^*, r, \theta, \phi) = 0, \tag{116}$$

where $\mathfrak{P}_k$ denotes the projection of ψ to the k'th azimuthal mode. Then

$$\lim_{t^* \to \infty} \psi(t^*, r, \theta, \phi) = 0.$$

The statement (116) is contained in [27] for g exactly Kerr, with r restricted however to $r > 2M$ (i.e. Boyer-Lindquist $\hat{r} > M + \sqrt{M^2 - a^2}$), and where ψ must be assumed to admit an extension to $\mathcal{D}$ of Sect. 1.1 (as a solution of (2)) vanishing in a neighborhood of the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$.

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