

# Liouville quantum gravity and KPZ

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**Abstract** Consider a bounded planar domain  $D$ , an instance  $h$  of the Gaussian free field on  $D$ , with Dirichlet energy  $(2\pi)^{-1} \int_D \nabla h(z) \cdot \nabla h(z) dz$ , and a constant  $0 \leq \gamma < 2$ . The *Liouville quantum gravity measure* on  $D$  is the weak limit as  $\varepsilon \rightarrow 0$  of the measures

$$\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz,$$

where  $dz$  is Lebesgue measure on  $D$  and  $h_\varepsilon(z)$  denotes the mean value of  $h$  on the circle of radius  $\varepsilon$  centered at  $z$ . Given a random (or deterministic) subset  $X$  of  $D$  one can define the scaling dimension of  $X$  using either Lebesgue measure or this random measure. We derive a general quadratic relation between these two dimensions, which we view as a probabilistic formulation of the Knizhnik, Polyakov, Zamolodchikov (Mod. Phys. Lett. A, 3:819–826, 1988) relation from conformal field theory. We also present a boundary analog of KPZ (for subsets of  $\partial D$ ). We discuss the connection between discrete and continuum quantum gravity and provide a framework for understanding Euclidean scaling exponents via quantum gravity.

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*“There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude[s] are given by the sums over all possible surfaces with fixed boundary.”* (A.M. Polyakov, Moscow, 1981) [[121](#)]

## 1 Introduction

### 1.1 Overview

The study of certain natural probability measures on the space of two dimensional Riemannian manifolds (and singular limits of these manifolds) is often called “two-dimensional quantum gravity.” These models have been very thoroughly studied in the physics literature, in part because of connections to string theory and conformal field theory [[7](#), [11](#), [13](#), [42](#), [43](#), [49](#), [64](#), [71](#), [79](#), [90](#), [93](#), [121–123](#), [125](#), [130](#)], and to random matrix theory and geometrical models; see, e.g., the references [[1](#), [2](#), [5](#), [6](#), [20](#), [29–31](#), [39](#), [40](#), [44](#), [46](#), [48](#), [50](#), [53](#), [58–63](#), [65](#), [67](#), [68](#), [73](#), [77](#), [86](#), [88](#), [89](#), [91](#), [92](#), [94–98](#), [100](#), [101](#), [105](#), [116](#)]. More recently, a purely combinatorial approach to discretized quantum gravity has been successful [[12](#), [15–19](#), [21–28](#), [33](#), [47](#), [75](#), [106–108](#), [113–115](#), [117](#), [128](#)], as well as the so-called topological expansion involving higher-genus random surfaces [[35](#), [36](#), [38](#), [69](#), [70](#), [72](#)].

One of the most influential papers in this field is a 1988 work of Knizhnik, Polyakov, and Zamolodchikov [[103](#)]. Building on a 1987 work of Polyakov [[124](#)], the authors derive a relationship (the *KPZ formula*) between scaling dimensions of fields defined using Euclidean geometry and analogous dimensions defined via Liouville quantum gravity (as described earlier in [[121](#), [122](#)]; see [[120](#)] for a recent historical recount). An alternative heuristic derivation using Liouville field theory in the so-called conformal gauge was proposed shortly after [[41](#), [52](#)] (see also [[135](#)]). The original work by KPZ has been cited roughly a thousand times in a variety of contexts, which we will not attempt to survey here, though we mention that there have been a number of explicit calculations in Liouville field theory with matching results in the random matrix theory approach, e.g., [[55](#), [76](#), [78](#), [84](#), [99](#), [102](#), [126](#), [136–139](#), [141](#), [142](#)]; for a review, see [[118](#)].

The relationship in [103] has never been proved or even precisely formulated mathematically. The main goal of this work is to formulate and prove the KPZ scaling dimension relationship in a probabilistic setting.

### 1.2 Critical Liouville quantum gravity

The study of two dimensional random surfaces makes frequent use of the Riemann uniformization theorem, which states that every smooth simply connected Riemannian manifold  $\mathcal{M}$  can be conformally mapped to either the unit disc  $\mathbb{D}$ , the complex plane  $\mathbb{C}$ , or the complex sphere  $\mathbb{C} \cup \{\infty\}$ . (If a manifold is not simply connected then its universal cover can be conformally mapped to one of these spaces. See, e.g., Chap. 4 of [74] for more exposition; see also [54, 80, 81, 87, 140] for approximation algorithms and beautiful computer illustrations of these maps.) Another way to say this is that  $\mathcal{M}$  can be parameterized by points  $z = x + iy$  in one of these spaces in such a way that the metric takes the form  $e^{\lambda(z)}(dx^2 + dy^2)$  for some real-valued function  $\lambda$ . The  $(x, y)$  are called *isothermal coordinates* or *isothermal parameters* for  $\mathcal{M}$ . In most of this paper we let the parameter space be a general simply connected proper subdomain  $D$  of the plane (which, of course, is conformally equivalent to  $\mathbb{D}$ ).

We remark that the existence of isothermal coordinates does not require that  $\mathcal{M}$  be smooth; for example, it can be deduced whenever  $\mathcal{M}$  can be parameterized by a simply connected planar domain in which the metric has the form  $E(x, y)dx^2 + 2F(x, y)dx dy + G(x, y)dy^2$  where  $EG - F^2 > 0$ ,  $E > 0$ , and  $E, F$ , and  $G$  are  $\beta$ -Hölder continuous for some  $0 < \beta < 1$  [37].

Length, area, and curvature are easy to express in isothermal coordinates. The length of a path in  $\mathcal{M}$  parameterized by a smooth path  $P$  in  $D$  is given by

$$\int_P e^{\lambda(s)/2} ds,$$

where  $ds$  is the Euclidean length measure on  $D$ . Given a measurable subset  $A$  of  $D$ , the integral  $\int_A e^{\lambda(z)} dz$  (where  $dz$  denotes Lebesgue measure on  $D$ ) is the area of the portion of  $\mathcal{M}$  parameterized by  $A$ . The function  $K = -e^{-\lambda} \Delta \lambda$  (where  $\Delta \lambda = \lambda_{xx} + \lambda_{yy}$  is the Laplacian operator) is called the *Gaussian curvature* of  $\mathcal{M}$ . If  $A$  is a measurable subset of the  $(x, y)$  parameter space, then the integral of the Gaussian curvature with respect to the portion of  $\mathcal{M}$  parameterized by  $A$  can be written  $\int_A e^{\lambda(z)} K(z) dz = \int_A -\Delta \lambda(z) dz$  where  $dz$  denotes Lebesgue measure on  $D$ . In other words,  $-\Delta \lambda$  gives the density of Gaussian curvature in the isothermal coordinate space. In particular,  $\mathcal{M}$  is flat if and only if  $\lambda$  is harmonic.

The above suggests that one can study random simply connected Riemannian manifolds by studying random functions  $\lambda$  on  $\mathbb{C}$  or  $\mathbb{C} \cup \{\infty\}$  or any

fixed simply connected subdomain  $D$  of  $\mathbb{C}$ . In the probabilistic formulation of the so-called critical Liouville quantum gravity,  $\lambda$  is taken to be a multiple of the Gaussian free field (GFF), although some care will be required to make sense of this construction, since the GFF is a distribution and not a function. (The relationship between our probabilistic formulation and the original formulation of Polyakov will be discussed in Sect. 2.)

For concreteness, let  $h$  be an instance of a centered GFF on a bounded simply connected domain  $D$  with zero boundary conditions. This means that  $h = \sum_n \alpha_n f_n$  where the  $\alpha_n$  are i.i.d. zero mean unit variance normal random variables and the  $f_n$  are an orthonormal basis, with respect to the inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

of the Hilbert space closure  $H(D)$  of the space  $H_s(D)$  of  $C^\infty$  real-valued functions compactly supported on  $D$ . Although this sum diverges pointwise almost surely, it does converge almost surely in the space of distributions on  $D$ , and one can also make sense of the mean value of  $h$  on various sets. (See [132] for a detailed account of this construction of the GFF; see Sect. 3.1 for a quick overview. Note that the  $(2\pi)^{-1}$  in the definition above does not appear, e.g., in [132]; including this factor in the definition, as is common in the physics literature, is equivalent to multiplying the corresponding  $h$  by  $\sqrt{2\pi}$ . This will simplify some of our formulas later on. In particular, in this formulation the two point covariance scales like  $-\log(|z - w|)$  instead of  $-(2\pi)^{-1} \log(|z - w|)$ ; see Sect. 3.1.)

Given an instance  $h$  of the Gaussian free field on  $D$ , let  $h_\varepsilon(z)$  denote the mean value of  $h$  on  $\partial B_\varepsilon(z)$ , the circle of radius  $\varepsilon$  centered at  $z$  (where  $h(z)$  is defined to be zero for  $z \in \mathbb{C} \setminus D$ ). This is almost surely a locally Hölder continuous function of  $(\varepsilon, z)$  on  $(0, \infty) \times D$  (see Sect. 3.1). For each fixed  $\varepsilon$ , consider the surface  $\mathcal{M}_\varepsilon$  parameterized by  $D$  with metric  $e^{\gamma h_\varepsilon(z)}(dx^2 + dy^2)$ . We would like to define a surface  $\mathcal{M}$  parameterized by  $D$  to be some sort of limit as  $\varepsilon \rightarrow 0$  of these surfaces. Since we would not expect the limit to be a Riemannian manifold in any classical sense, we have to state carefully what we mean by this. There are many ways we could attempt to make sense of this limit, depending on what quantities we focus on. For example, we could consider

1. The length of the shortest path connecting a fixed pair of points in  $D$ .
2. The area of a fixed subset of  $D$ .
3. The length of a fixed smooth curve in  $D$ .
4. The length of a smooth boundary arc of  $D$  (which becomes interesting when  $h$  is an instance of the GFF with free boundary conditions).

Intuitively, we might expect each quantity above to scale like a random constant times a (possibly different) power of  $\varepsilon$  as  $\varepsilon$  tends to zero—i.e., we would

expect that if the  $\mathcal{M}_\varepsilon$  were rescaled by the appropriate powers of  $\varepsilon$ , the above quantities would have limits as  $\varepsilon \rightarrow 0$ . Focusing on lengths of shortest paths, one might guess that the random surfaces  $\mathcal{M}_\varepsilon$  (rescaled by some power of  $\varepsilon$ ) would almost surely converge (in some natural topology on the set of metric spaces) to a non-trivial random metric space parameterized by  $D$ . However, this is not something we are currently able to prove. Focusing on areas, one might expect that for some  $\alpha$  the renormalized area measures  $\varepsilon^\alpha e^{\gamma h_\varepsilon(z)} dz$  would almost surely converge weakly to a random measure on  $D$ . This is the limit we will construct and work with in this paper. We will also address the lengths of fixed curves and boundary curves; see Sect. 6. Although the constructions are quite similar, we will not use the so-called Wick normal ordering terminology in this paper (see e.g., [133]). We present a self-contained proof of the following (although similar measures have appeared much earlier, and are called the *Høegh-Krohn model* [82]—see also [8, 9] for a discussion on the level of Schwinger functions, and a more recent survey [10]):

**Proposition 1.1** *Fix  $\gamma \in [0, 2)$  and define the zero boundary GFF  $h$  and  $D$  as above. Then it is almost surely the case that as  $\varepsilon \rightarrow 0$  along powers of two, the measures  $\mu_\varepsilon := \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$  converge weakly inside  $D$  to a limiting measure, which we denote by  $\mu = \mu_h = e^{\gamma h(z)} dz$ . This remains true if we replace  $h$  with a non-centered GFF on  $D$ —i.e., if we set  $h = \bar{h} + h^0$  where  $\bar{h}$  is the zero boundary GFF on  $D$  and  $h^0$  is a deterministic, non-zero continuous function on  $D$ .*

For each  $z \in D$ , denote by  $C(z; D)$  the conformal radius of  $D$  viewed from  $z$ . That is,  $C(z; D) = |\phi'(z)|^{-1}$  where  $\phi : D \rightarrow \mathbb{D}$  is a conformal map to the unit disc with  $\phi(z) = 0$ . The following gives an equivalent definition of  $\mu$ .

**Proposition 1.2** *Write  $h = \bar{h} + h^0$  where  $\bar{h}$  is the zero boundary GFF on  $D$  and  $h^0$  is a deterministic continuous function on  $D$ . Let  $f_1, f_2, \dots$  be an orthonormal basis for  $H(D)$  comprised of continuous functions on  $D$  and let  $h^n$  be the expectation of  $h$  given its projection onto the span of  $\{f_1, f_2, \dots, f_n\}$ . (In other words,  $h^n$  is  $h^0$  plus the projection of  $\bar{h}$  onto the span of  $\{f_1, f_2, \dots, f_n\}$ .) Then  $\mu = \mu_h$  (as defined in Proposition 1.1) is almost surely the weak limit for  $n \rightarrow +\infty$  of the measures*

$$\mu^n = \exp \left( \gamma h^n(z) - \frac{\gamma^2}{2} \text{Var } h^n(z) + \frac{\gamma^2}{2} \log C(z; D) \right) dz. \tag{1}$$

For each measurable  $A \subset D$ , we have

$$\mathbb{E}[\mu(A)|h^n] = \mu^n(A). \tag{2}$$

In particular,

$$\mathbb{E}\mu(A) = \int_A C(z; D)^{\frac{\gamma^2}{2}} e^{\gamma h^0(z)} dz.$$

Intuitively, we interpret the pair  $(D, \mu)$  as describing a “random surface”  $\mathcal{M}$  parameterized conformally by  $D$ , with area measure given by  $\mu$ . In the physics literature, the more commonly used term is “random metric”; however, we stress that we have not endowed  $D$  with a two point distance function, so we cannot mathematically interpret “random metric” to mean “random metric space.”

In the Liouville quantum gravity literature, the term “metric” is used to mean alternately a two-point distance function, a measure of areas and lengths of curves, or a Riemannian metric tensor (usually the latter). The first maps pairs of points to  $\mathbb{R}^+$ , the second maps sets/curves to  $\mathbb{R}^+$ , and the third maps pairs of tangent vectors to  $\mathbb{R}$ . A smooth manifold can be equivalently characterized by any one of these objects; however, the relationships between these notions are less obvious for the limiting (and highly non-smooth) “random surfaces”  $\mathcal{M}$  we deal with here. The pair  $(D, \mu)$  represents a conformal parameterization of  $\mathcal{M}$ , with area measure  $\mu$ . However, further work would be required to use this structure to construct a two-point distance function on  $\mathcal{M}$ , or vice versa. To avoid ambiguity arising from the multiple definitions of the term “metric”, we will use the term “random surface” instead of “random metric” in this paper to describe the pair  $(D, \mu)$ .

### 1.3 Scaling exponents and KPZ

**Definition 1.3** For any fixed measure  $\mu$  on  $D$  (which we call the “quantum” measure), we let  $B^\delta(z)$  be the Euclidean ball centered at  $z$  whose radius is chosen so that  $\mu(B^\delta(z)) = \delta$ . (If there does not exist a unique  $\delta$  with this property, take the radius to be  $\sup\{\varepsilon : \mu(B_\varepsilon(z)) \leq \delta\}$ .) We refer to  $B^\delta(z)$  as the *isothermal quantum ball* of area  $\delta$  centered at  $z$ . In particular, if  $\gamma = 0$  then  $\mu$  is Lebesgue measure and  $B^\delta(z)$  is  $B_\varepsilon(z)$  where  $\delta = \pi\varepsilon^2$ .

Given a subset  $X \subset D$ , we denote the  $\varepsilon$  neighborhood of  $X$  by

$$B_\varepsilon(X) = \{z : B_\varepsilon(z) \cap X \neq \emptyset\}.$$

We also define the *isothermal quantum  $\delta$  neighborhood* of  $X$  by

$$B^\delta(X) = \{z : B^\delta(z) \cap X \neq \emptyset\}.$$

Translated into probability language, the so-called *KPZ formula* is a quadratic relationship between the expectation fractal dimension of a random

subset of  $D$  defined in terms of Euclidean measure (which is the Liouville gravity measure with  $\gamma = 0$ ) and the corresponding expectation fractal dimension of  $X$  defined in terms of Liouville gravity with  $\gamma \neq 0$ .

Fix  $\gamma \in [0, 2)$  and let  $\mu_0$  denote Lebesgue measure on  $D$ . We say that a (deterministic or random) fractal subset  $X$  of  $D$  has *Euclidean expectation dimension*  $2 - 2x$  and *Euclidean scaling exponent*  $x$  if the expected area of  $B_\varepsilon(X)$  decays like  $\varepsilon^{2x} = (\varepsilon^2)^x$ , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0(B_\varepsilon(X))}{\log \varepsilon^2} = x.$$

We say that  $X$  has *quantum scaling exponent*  $\Delta$  if when  $X$  and  $\mu$  (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu(B^\delta(X))}{\log \delta} = \Delta,$$

where here  $\mathbb{E}$  is with respect to both random variables  $X$  and  $\mu$ . (Section 7 will provide some discrete quantum gravity heuristics that motivate the idea of taking  $X$  and  $\mu$  to be independent of one another, as well as our particular definition of scaling exponent.)

The following is the KPZ scaling exponent relation. To avoid boundary technicalities, we restrict attention here to a compact subset of  $D$ . The case of boundary exponents will be dealt with in Sect. 6.

**Theorem 1.4** *Fix  $\gamma \in [0, 2)$  and a compact subset  $\tilde{D}$  of  $D$ . If  $X \cap \tilde{D}$  has Euclidean scaling exponent  $x \geq 0$  then it has quantum scaling exponent  $\Delta$ , where  $\Delta$  is the non-negative solution to*

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta. \tag{3}$$

It also turns out that Theorem 1.4 admits the following straightforward generalization:

**Theorem 1.5** *Let  $\mathcal{X}$  be any random measurable subset of the set of all balls of the form  $B_\varepsilon(z)$  for  $\varepsilon > 0$  and  $z$  in a fixed compact subset  $\tilde{D}$  of  $D$ . Fix  $\gamma \in [0, 2)$ . Then if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0\{z : B_\varepsilon(z) \in \mathcal{X}\}}{\log \varepsilon^2} = x,$$

then it follows that, when  $\mathcal{X}$  and  $\mu$  (as defined above) are chosen independently, we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu \{z : B^\delta(z) \in \mathcal{X}\}}{\log \delta} = \Delta,$$

where  $\Delta$  is the non-negative solution to

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.$$

(Again, expectation in the above theorem is with respect to both random variables,  $X$  and  $\mu$ .) We obtain Theorem 1.4 as a special case of Theorem 1.5 by writing  $\mathcal{X} = \{B_\varepsilon(z) : B_\varepsilon(z) \cap X \neq \emptyset\}$ . Theorem 1.5 allows us to consider  $x$  that are greater than 1 (in which case the “dimension”  $2 - 2x$  would be negative). If one considers, for example, a conformal loop ensemble on  $D$  with  $\kappa = 6$  (corresponding to a scaling limit of the cluster-boundary loops in site percolation on the triangular lattice) one could let  $\mathcal{X}$  be the set of balls contained in  $\tilde{D}$  that intersect  $\ell$  distinct “macroscopic” loops (where “macroscopic” means that their diameters are greater than some fixed constant). In this case, the value  $x$  depends on  $\ell$  and is called a *multi-arm exponent* [4, 59, 129, 134] and we may view the corresponding  $\Delta$  as a quantum analog of such an exponent.

As another example, for some integer  $L$  fix distinct points  $z_1, z_2, \dots, z_L$  in  $D \setminus \tilde{D}$  and run  $L$  independent Brownian motions started at the points  $z_1, \dots, z_L$ . Then let  $\mathcal{X}$  be the set of balls  $B_\varepsilon(z)$  contained in  $\tilde{D}$  with the property that the Brownian motions—stopped at the first time they intersect  $\partial B_\varepsilon(z)$ —do not intersect one another.

In this case, the Euclidean scaling exponent  $x = x_L$  is called a *Brownian intersection exponent*. It was conjectured in [51] and rigorously derived in a celebrated series of papers by Lawler, Schramm, and Werner using the Schramm-Loewner evolution with  $\kappa = 6$  [109–111]:

$$x_L = \frac{1}{24}(4L^2 - 1).$$

Although we will not fully explain this in this paper, there is a close connection between  $\text{SLE}_\kappa$  and Liouville quantum gravity models with  $\gamma = \sqrt{\min\{\kappa, 16/\kappa\}}$  (see Sect. 7), in agreement with the relationship between CFT central charge  $c$  and parameter  $\gamma$  in Liouville quantum gravity [41, 52, 79, 103, 130]. Taking  $\gamma = \sqrt{16/6} = \sqrt{8/3}$  and  $x_L$  as above, the KPZ formula gives

$$\Delta_L = \frac{1}{2} \left( L - \frac{1}{2} \right),$$



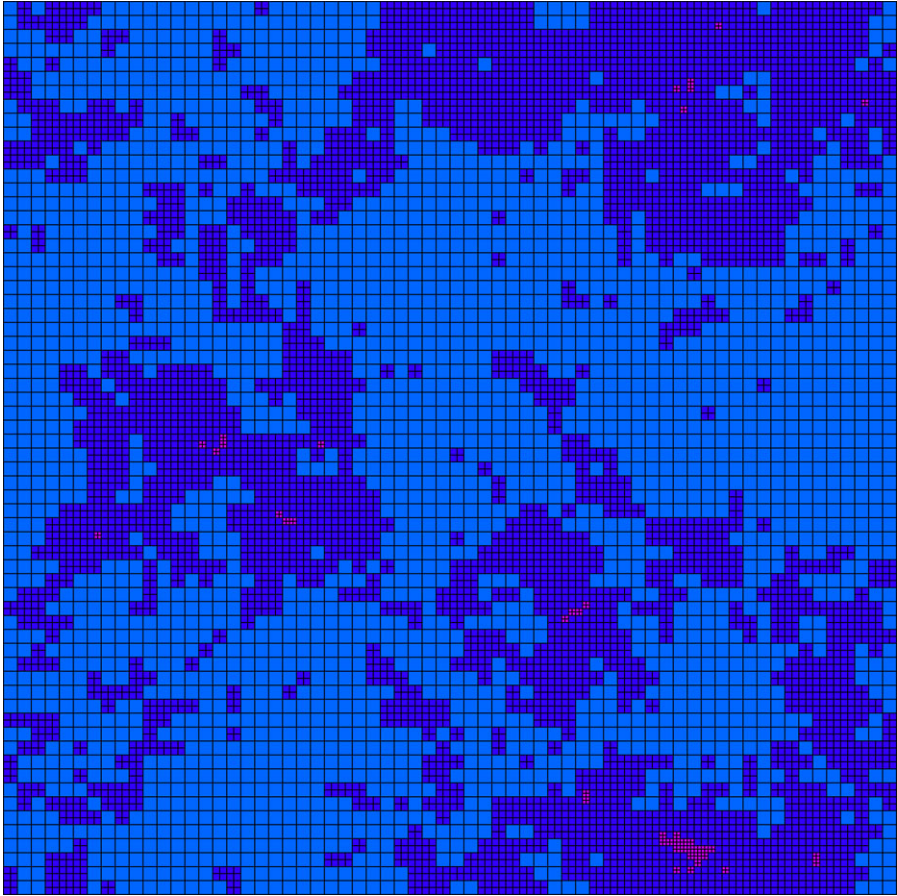
which is an affine function of  $L$ . The first co-author predicted several years ago, based on an approach via discrete quantum gravity models, that this  $\Delta$  would be an affine function of  $L$  (see [58, 60, 61, 63] and the discussion in Sect. 7). The derivation is based on a simple and general geometric argument that discrete quantum gravity exponents should be in a certain sense additive together with a heuristic connection between the discrete and the continuous models. A direct calculation via discrete graphs appears in [58]. This is related to the cascade relations given earlier by Lawler and Werner using different techniques [112].

Three papers that build on our work (as announced and presented in talks and minicourses beginning in 2007, and later in the Letter [57]) have already been posted online: Benjamini and Schramm cited the ideas of our paper to produce an analog of Theorem 1.4 in a one dimensional cascade model; their proof uses a Frostman measure construction in place of the large deviations construction used here, and almost sure Hausdorff dimension in place of expectation dimension [34]. A follow up paper [127] adapts the arguments of [34] to a class of cascade models, which was expanded to include (in a revised version) a measure based on the exponential of the Gaussian free field, like the measures we construct here. Another paper provides a heuristic heat kernel based derivation of the KPZ relation [45].

Intuitively, one reason to expect Hausdorff-like variants of KPZ to be accessible is that the second moments (and higher moments) of the random measures are essentially trivial to compute (see Sect. 3.2). It might be interesting to try to derive other variants of KPZ: for example, one could try to relate the actual Minkowski or Hausdorff measure of a set, in the Euclidean sense, with some kind of expected Minkowski or Hausdorff measure in the quantum sense. We will not address these alternative formulations here. However, we will present below a picturesque formulation of KPZ in terms of box decompositions.

#### 1.4 Statement of box formulation of KPZ

Define a *diadic square* to be a closed square (including its interior) of one of the grids  $2^{-k}\mathbb{Z}^2$  for some integer  $k$ . Let  $\mu$  be any measure on  $\mathbb{C}$ . For  $\delta > 0$ , we define a  $(\mu, \delta)$  *box*  $S$  to be a diadic square  $S$  with  $\mu(S) < \delta$  and  $\mu(S') \geq \delta$  where  $S'$  is the diadic parent of  $S$ . Clearly, if a point  $z \in \mathbb{C}$  does not lie on a boundary of a diadic square—and it satisfies  $\mu(\{z\}) < \delta < \mu(\mathbb{C})$ —then there is a unique  $(\mu, \delta)$  box containing  $z$ , which we denote by  $S^\delta(z)$ . Let  $\mathcal{S}_\mu^\delta$  be the set of all  $(\mu, \delta)$  boxes. The boxes in  $\mathcal{S}_\mu^\delta$  do not overlap one another except at their boundaries. Thus, they form a tiling of  $\mathbb{R}^2$  (see Figs. 1, 2, and 3 for an illustration of this construction on a torus).



**Fig. 1**  $(\mu, \delta)$  boxes of the random measure  $\mu = e^{\gamma h} dz$ , where  $\gamma = .5$  and  $h$  is the (discrete) Gaussian free field on a very fine  $(1024 \times 1024)$  grid on the torus,  $dz$  is counting measure on the vertices of that grid, and  $\delta$  is  $2^{-12}$  times the total mass of  $\mu$ . (We view  $\mu$  as an approximation of the continuum Liouville quantum gravity measure.) One way to construct this figure is to view the entire torus as a square; then subdivide each square whose  $\mu$  measure is at least  $\delta$  into four smaller squares, and repeat until all squares have  $\mu$  measure less than  $\delta$ . The squares shown have roughly the same  $\mu$  size—in the sense that each square has  $\mu$  measure less than  $\delta$  but each square's diadic parent has  $\mu$  measure greater than  $\delta$

We remark that the  $(\mu, \delta)$  boxes should not be confused with the diadic boxes in the so-called  $\delta$ -Calderón Zygmund decomposition of  $\mu$ . Readers familiar with that decomposition may recall that while the  $(\mu, \delta)$  boxes are diadic squares  $S$  with  $\mu(S) < \delta \leq \mu(S')$ , the  $\delta$ -Calderón Zygmund boxes are diadic squares  $S$  with  $\mu(S)/\mu_0(S) > \delta \geq \mu(S')/\mu_0(S')$ , where  $\mu_0$  is Lebesgue measure. Roughly speaking, the  $\mu$  measure on each  $(\mu, \delta)$  box approximates  $\delta$ , while the  $\mu$  density on each Calderón Zygmund box approximates  $\delta$ .

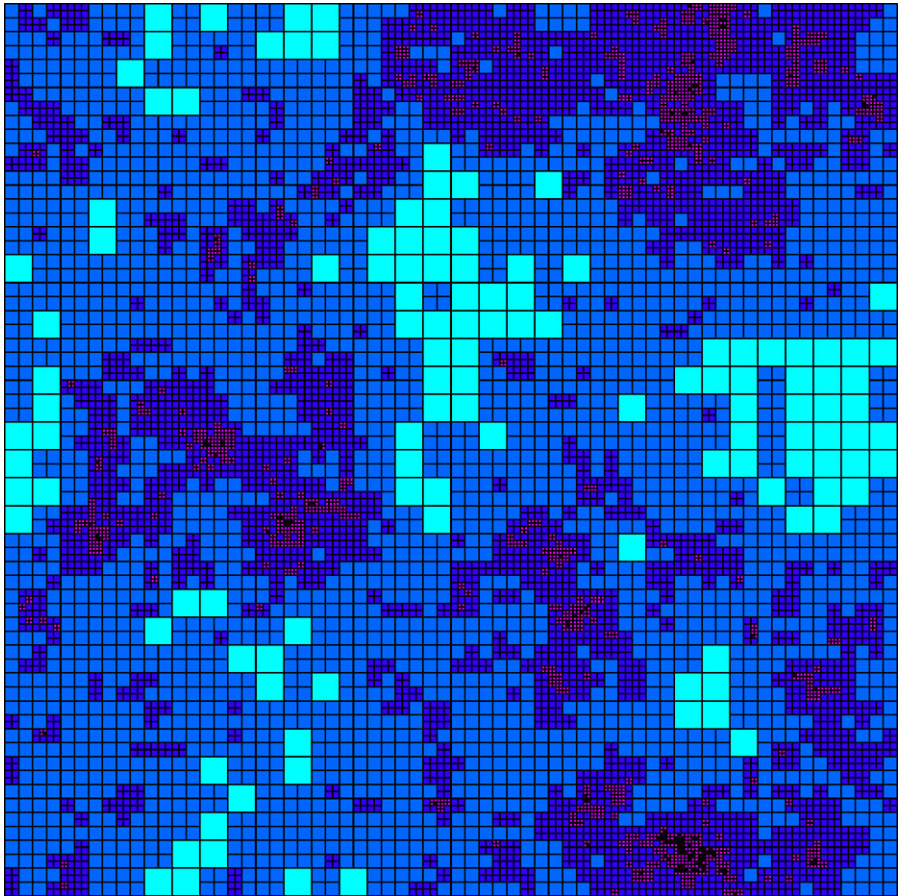


Fig. 2 Analog of Fig. 1 with  $\gamma = 1$ , using the same instance  $h$  of the GFF

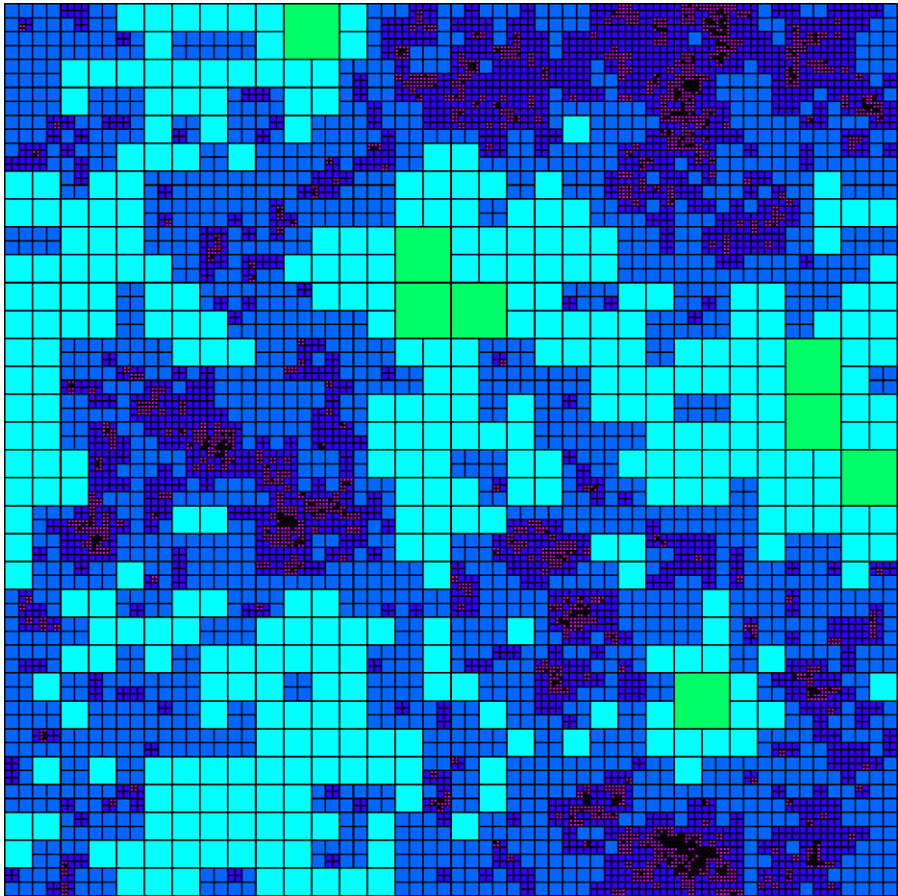
When  $\varepsilon$  is a power of 2, analogously define  $S_\varepsilon(z)$  to be the diadic square containing  $z$  with edge length  $\varepsilon$ . Likewise, define

$$S_\varepsilon(X) = \{z : S_\varepsilon(z) \cap X \neq \emptyset\},$$

$$S^\delta(X) = \{z : S^\delta(z) \cap X \neq \emptyset\}.$$

The following gives the equivalence of the scaling dimension definition when boxes are used instead of balls. (The first half is well known and easy to verify.)

**Proposition 1.6** Fix  $\gamma \in [0, 2)$  and let  $X$  be a random subset of a deterministic compact subset  $\tilde{D}$  of  $D$ . Let  $N(\mu, \delta, X)$  be the number of  $(\mu, \delta)$  boxes intersected by  $X$  and  $N(\varepsilon, X)$  the number of diadic squares intersecting  $X$  that have edge length  $\varepsilon$  (a power of 2). Then  $X$  has Euclidean scaling expo-



**Fig. 3** Analog of Fig. 1 with  $\gamma = 1.5$ , using the same instance  $h$  of the GFF

ment  $x \geq 0$  if and only if

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[\mu_0(S_\varepsilon(X))]}{\log \varepsilon^2} = \lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[\varepsilon^2 N(\varepsilon, X)]}{\log \varepsilon^2} = x,$$

or equivalently,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[N(\varepsilon, X)]}{\log \varepsilon^2} = x - 1.$$

Similarly, the following are equivalent

1.  $X$  has quantum scaling exponent  $\Delta$ .

2. When  $X$  and  $\mu$  (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\mu(S^\delta(X))]}{\log \delta} = \Delta. \tag{4}$$

3. When  $X$  and  $\mu$  (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[N(\mu, \delta, X)]}{\log \delta} = \Delta - 1. \tag{5}$$

Of course, this immediately implies the following restatement of Theorem 1.4 in terms of boxes instead of balls:

**Corollary 1.7** Fix  $\gamma \in [0, 2)$  and a compact subset  $\tilde{D}$  of  $D$  and  $X$  and  $\mu$  as above. Then if

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E}[N(\varepsilon, X)]}{\log \varepsilon^2} = x - 1$$

for some  $x > 0$  then

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[N(\mu, \delta, X)]}{\log \delta} = \Delta - 1,$$

where  $\Delta$  is the non-negative solution to (3).

One could also phrase Theorem 1.5 in terms of boxes instead of balls, but for simplicity we will refrain from doing this here.

## 2 Coordinate changes and the physical Liouville action

Polyakov understood early on that the Liouville quantum gravity action becomes a free field action in the conformal gauge, but he did not construct the random area measure the way we do. In [124], where Polyakov begins the KPZ derivation, he refers to the Liouville quantum gravity action and writes

*“The most simple form this formula takes is in the conformal gauge, where  $g_{ab} = e^\phi \delta_{ab}$  where it becomes a free field action. Unfortunately this simplicity is an illusion. We have to set a cut-off in quantizing this theory, such that it is compatible with general covariance. Generally, it is not clear how to do this. For that reason, we take a different approach.”*

Indeed, the actual derivation given in [124] and subsequently in Knizhnik, Polyakov, and Zamolodchikov [103] is more complicated than ours and is not based on the Gaussian free field. It does not give precise mathematical

meaning to the random surfaces. We feel that the Gaussian free field based random measure we construct is the correct one, at least in the sense that it is likely to arise as a scaling limit of the discrete quantum gravity models mentioned in [103] (see Sect. 7). In a way our approach is more similar to the work of David [41] and of Distler and Kawai [52], which heuristically derived KPZ from Liouville field theory in the so-called conformal gauge.

In this section, we describe how the Liouville quantum gravity measure we construct transforms covariantly under coordinate changes and use this to explain the connection between the Gaussian free field and the more familiar and more general curvature-based definition of the Liouville action that is conventional in the physics literature. The covariance properties of the random measures in our point of view are very simple and agree with those postulated in the physics literature.

If  $\phi$  is a conformal map from  $D$  to a domain  $\tilde{D}$  and  $h$  is a distribution on  $D$ , then we define the pullback  $h \circ \phi^{-1}$  of  $h$  to be a distribution on  $\tilde{D}$  defined by  $(h \circ \phi^{-1}, \tilde{\rho}) = (h, \rho)$  whenever  $\rho$  is smooth and compactly supported on  $D$  and  $\tilde{\rho} = |\phi'|^{-2} \rho \circ \phi^{-1}$ . (Here  $\phi'$  is the complex derivative of  $\phi$ , and  $(h, \rho)$  is the value of the distribution  $h$  integrated against  $\rho$ .) Note that if  $h$  is a continuous function (viewed as a distribution via the map  $\rho \rightarrow \int_D \rho(z)h(z)dz$ ), then the distribution  $h \circ \phi^{-1}$  thus defined is the ordinary composition of  $h$  and  $\phi^{-1}$  (viewed as a distribution).

The following transformation rule is a simple consequence of Proposition 1.2 and the definitions above.

**Proposition 2.1** *Let  $h$  be an instance of the GFF on  $D$  and  $\psi$  a conformal map from a domain  $\tilde{D}$  to  $D$ . Write  $\tilde{h}$  for the distribution on  $\tilde{D}$  given by  $h \circ \psi + Q \log |\psi'|$  where*

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

*Then  $\mu_h$  is almost surely the image under  $\psi$  of the measure  $\mu_{\tilde{h}}$  on  $\tilde{D}$ . That is,  $\mu_{\tilde{h}}(A) = \mu_h(\psi(A))$  for each Borel measurable  $A \subset \tilde{D}$ .*

*Proof* Using the notation of Proposition 1.2, if  $f_1, f_2, \dots$  are an orthonormal basis for  $H(D)$ , then the conformal invariance of  $(\cdot, \cdot)_\nabla$  implies that  $f_1 \circ \psi, f_2 \circ \psi, \dots$  are an orthonormal basis for  $H(\tilde{D})$ , and as  $n \rightarrow \infty$  the functions  $h^n \circ \psi$  converge in law to the GFF on  $\tilde{D}$ , and the functions  $\tilde{h}^n = h^n \circ \psi + Q \log |\psi'|$  converge in law to  $\tilde{h}$ . If we define  $\tilde{\mu}^n$  analogously to  $\mu^n$  in (1) but with  $h^n$  replaced by  $\tilde{h}^n$ , then the  $\tilde{\mu}^n$  converge weakly to the random distribution  $\tilde{\mu} := \mu_{\tilde{h}}$ .

To see that  $\mu$  is the image of  $\tilde{\mu}$  under  $\psi$ , we will observe that  $\mu^n$  is the image of  $\tilde{\mu}^n$  under  $\psi$  for each  $n$ . To see this, consider the term  $Q \log |\psi'| = (2/\gamma) \log |\psi'| + (\gamma/2) \log |\psi'|$  in the definition of  $\tilde{h}$ . Adding  $(2/\gamma) \log |\psi'|$

to  $h^n \circ \psi$  corresponds to multiplying (1) by a factor of  $|\psi'|^2$ . This compensates for the fact that the Radon-Nikodym derivative of a measure on  $\tilde{D}$  at a point  $z$  and the Radon-Nikodym derivative of the same measure pushed forward on  $D$  at  $\psi(z)$  differ by a factor of  $|\psi'(z)|^2$ . Adding  $(\gamma/2) \log |\psi'(z)|$  to  $h^n \circ \psi$  compensates the expression (1) for the change in conformal radius:  $\log C(\psi(z); D) - \log C(z; \tilde{D}) = \log |\psi'(z)|$ . □

We interpret Proposition 2.1 as a rule for changing the parametrization of a “random surface.” For example, consider the random surface one constructs from the Gaussian free field on a fixed domain  $D$ . Then if we are given any other domain  $\tilde{D}$  and a conformal map  $\psi : \tilde{D} \rightarrow D$ , we may wish to consider the same random surface parameterized by  $\tilde{D}$  instead of  $D$ . In this case, the transformation rule tells us that on  $\tilde{D}$  we should consider the Liouville quantum gravity measure defined using  $\tilde{h} = h \circ \psi + Q \log |\psi'|$ , where  $h \circ \psi$  is the GFF on  $\tilde{D}$  with zero boundary conditions.

We remark that one can make a similar argument when  $\tilde{D}$  is a curved simply connected manifold and  $\psi : \tilde{D} \rightarrow D$  a conformal map; the metric on  $\tilde{D}$ —when parameterized by  $D$  using the map  $\psi^{-1}$ —takes the form  $e^{\lambda(z)}(dx^2 + dy^2)$ , for  $z \in D$ , where we write  $\lambda(\psi(w)) = -2 \log |\psi'(w)|$  for  $w \in \tilde{D}$ . Although we will not prove it here, the analog of Proposition 1.1 for smooth curved surfaces is straightforward, and the transformation rule Proposition 2.1 remains the same in this case; as in the flat case, the law of the Liouville quantum gravity measure on  $D$  pulled back to  $\tilde{D}$  is that of  $\tilde{h} = h \circ \psi + Q \log |\psi'|$  where  $h \circ \psi$  is the GFF on  $\tilde{D}$  with zero boundary conditions. (Alternatively, we may take this as a definition of the Liouville quantum gravity measure on curved  $\tilde{D}$  with zero boundary conditions.)

The remainder of this subsection describes the connection between our notation and the common physics literature Liouville gravity notation. (This discussion can be skipped, on a first read, by readers with no prior familiarity with the latter.) What we call the GFF on  $D$  (with the  $1/2\pi$  normalization, as discussed in the introduction) is often written (sometimes without a rigorous definition) as the measure  $e^{-S(h)} dh$ , where

$$S(h) = \frac{1}{4\pi} \int_D \nabla h(z) \cdot \nabla h(z) dz$$

is called the *action* and  $dh$  is defined heuristically as a “uniform measure on the space of all functions.” (Of course, the latter makes perfect sense if one considers only a finite dimensional vector space of functions, such as real-valued functions defined on the vertices of a lattice, or functions whose Fourier coefficients beyond a certain frequency threshold are identically zero—in this case  $dh$  would be the Lebesgue measure on the vector

space.) In this paper, we will write

$$(h, h)_\nabla := \frac{1}{2\pi} \int_D \nabla h(z) \cdot \nabla h(z) dz,$$

so that the above becomes  $S(h) = \frac{1}{2}(h, h)_\nabla$ .

In the following, let  $D$  be a subdomain of  $\mathbb{C}$  and  $\tilde{D}$  a possibly curved surface for which there is a conformal map  $\psi : \tilde{D} \rightarrow D$ . Write  $\tilde{h}^0 = \log |\psi'|$ . Now, if we switch parametrization to  $\tilde{D}$ , we are adding  $Q\tilde{h}^0$  deterministically to  $h \circ \psi$  to get  $\tilde{h}$ , so we may rewrite the action as

$$S = \frac{1}{2}(\tilde{h} - Q\tilde{h}^0, \tilde{h} - Q\tilde{h}^0)_\nabla,$$

which (at least when  $\tilde{h}^0$  is smooth and compactly supported) is seen by integrating by parts to be equivalent (up to the additive constant  $\frac{1}{2}\|Q\tilde{h}^0\|_{\tilde{D}}^2$ ) to

$$S = \frac{1}{4\pi} \int_{\tilde{D}} dw \left( \nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w) + 2\tilde{h}(w)Q\Delta \tilde{h}^0(w) \right), \tag{6}$$

where the pairing  $\nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w)$  and the Laplacian  $\Delta \tilde{h}^0(w)$  are now defined using the metric on  $\tilde{D}$  and where now  $dw$  represents the measure on  $\tilde{D}$  instead of  $D$ . This can also be written

$$S = \frac{1}{4\pi} \int_{\tilde{D}} dw \left( \nabla \tilde{h}(w) \cdot \nabla \tilde{h}(w) + Q\tilde{h}(w)K(w) \right), \tag{7}$$

where  $K$  is the Gaussian curvature of  $\tilde{D}$  and  $dw$  is integration with respect to the curved metric. (When  $\tilde{h}^0$  is not compactly supported, the formula can be modified to include a term for boundary curvature, but we will not discuss this here.)

Adding in one additional term which is a constant  $\mu_{\mathcal{L}}$  times the total area of  $\tilde{D}$  (and making the following symbol substitutions:  $b = \gamma/2$ ,  $\varphi = \tilde{h}$ ,  $g$  is the underlying metric of  $\tilde{D}$ , and  $j$  and  $k$  are summed-over indices ranging over the two tangent space directions), we obtain the more familiar formula for the Liouville action:

$$S_{\mathcal{L}} = \frac{1}{4\pi} \int_{\tilde{D}} dw \sqrt{g} \left( g^{jk} \partial_j \varphi \partial_k \varphi + QK\varphi + 4\pi \mu_{\mathcal{L}} e^{2b\varphi} \right),$$

where  $Q = b + b^{-1}$ . The action is defined similarly when free boundary conditions are used instead of zero boundary conditions—or when  $\tilde{D}$  is a compact Riemann surface of some genus. (In this case,  $e^{-S_{\mathcal{L}}(\varphi)} d\varphi$  is an infinite measure, although it can be “localized,” e.g., by requiring the mean value of  $\varphi$  to be zero.)



This paper will focus exclusively on the case  $\gamma \in [0, 2)$  (which is said to correspond to physical models below the central charge  $c = 1$  threshold) and  $\mu_{\mathcal{L}} = 0$  (the so-called *critical* Liouville quantum gravity). The string theory and quantum gravity literatures deal with other parameter choices as well—including non-zero  $\mu_{\mathcal{L}}$  and complex values for  $\gamma$  and  $Q$ —but these appear to be beyond the scope of our methodology, in part because, when  $S_{\mathcal{L}}$  is complex valued, the expression  $e^{-S_{\mathcal{L}}(\varphi)} d\varphi$  is no longer a probability measure in even a heuristic sense.

### 3 Constructing the random measures

#### 3.1 GFF definition and normalization

Let  $D$  be a bounded planar domain and let  $dz$  denote Lebesgue measure on  $D$ . We assume the reader is familiar with the Gaussian free field, as defined, e.g., in [132], but we briefly review the definition here. As described earlier, to make our formulas consistent with the physics literature, the definitions of Green’s function and the Dirichlet form will differ from the ones in [132] by factors of  $2\pi$ .

Again, let  $H_s(D)$  be the space of  $C^\infty$  real-valued functions compactly supported on  $D$ . We define the Dirichlet inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

on  $H_s(D)$ . Then an instance  $h$  of the Gaussian free field (GFF) may be viewed as a standard Gaussian on the Hilbert space closure  $H(D)$  of  $H_s(D)$  (i.e., as a sum of the form  $\sum_n \alpha_n f_n$  where  $f_n$  are any orthonormal basis for  $H(D)$ )—the sum converges almost surely in the space of distributions on  $D$ , see [132]. In fact, we may define  $(h, f)_\nabla$  as random variables for non-smooth  $f$  as well; these are zero mean Gaussian random variables for each  $f \in H(D)$ , and

$$\text{Cov}((h, f_1)_\nabla, (h, f_2)_\nabla) = (f_1, f_2)_\nabla. \tag{8}$$

The collection of random variables  $(h, f)_\nabla$  for  $f \in H(D)$  is thus a Hilbert space (isomorphic to  $H(D)$ ) under the covariance inner product.

When  $x \in D$  is fixed, we let  $\tilde{G}_x(y)$  be the harmonic extension to  $y \in D$  of the function of  $y$  on  $\partial D$  given by  $-\log|y - x|$ . Then *Green’s function in the domain  $D$*  is defined by

$$G(x, y) = -\log|y - x| - \tilde{G}_x(y).$$

When  $x \in D$  is fixed, Green’s function may be viewed as a distributional solution of  $\Delta G(x, \cdot) = -2\pi \delta_x(\cdot)$  with zero boundary conditions [132]. It is non-negative for all  $x, y \in D$ .

For any function  $\rho$  on  $H_s(D)$ , we define a function  $\Delta^{-1}\rho$  on  $D$  by

$$\Delta^{-1}\rho(\cdot) := -\frac{1}{2\pi} \int_D G(\cdot, y) \rho(y) dy.$$

This is a  $C^\infty$  (though not necessarily compactly supported) function in  $D$  whose Laplacian is  $\rho$ . We use the same notation for more general measurable functions  $\rho$ , as well as the case that  $\rho$  is a measure. (For example, we will sometimes speak of the inverse Laplacian of uniform measure on a particular circle or disc contained in  $D$ .)

If  $f_1 = -\Delta^{-1}\rho_1$  and  $f_2 = -\Delta^{-1}\rho_2$ , then integration by parts implies that  $(f_1, f_2)_\nabla = (2\pi)^{-1}(\rho_1, -\Delta^{-1}\rho_2)$ , where  $(\cdot, \cdot)$  denotes the standard inner product on  $L^2(D)$ . We next observe that every  $h \in H(D)$  is naturally a distribution, since we may define the map  $(h, \cdot)$  by  $(h, \rho) := 2\pi(h, -\Delta^{-1}\rho)_\nabla$ . (It is not hard to see that  $-\Delta^{-1}\rho \in H(D)$ , since its Dirichlet energy is given explicitly by (9).) When  $-\Delta f = \rho$ , we may write  $(h, \rho) = 2\pi(h, f)_\nabla$ , and hence

$$\text{Cov}((h, \rho_1), (h, \rho_2)) = (2\pi)^2(f_1, f_2)_\nabla.$$

We claim that the latter expression may be rewritten to give

$$\text{Cov}((h, \rho_1), (h, \rho_2)) = \int_{D \times D} \rho_1(x)G(x, y)\rho_2(y) dx dy \tag{9}$$

where  $G(x, y)$  is Green’s function in  $D$ . Since  $\Delta G(x, \cdot) = -2\pi\delta_x(\cdot)$  and

$$\int_D G(x, y) \rho_2(y) dy = -2\pi \Delta^{-1}\rho_2(x),$$

we obtain (9) by multiplying each side by  $-\Delta f_1(x) = \rho_1(x)$  and integrating by parts with respect to  $x$ .

Denote by  $h_\varepsilon(z)$  the average value of  $h$  on the circle of radius  $\varepsilon$  centered at  $z$ . Similar averages were considered in [14]. (For this definition, we assume  $h$  is identically zero outside of  $D$ .) Then  $h_\varepsilon(z)$  is a Gaussian process with covariances defined by

$$G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) := \text{Cov}(h_{\varepsilon_1}(z_1), h_{\varepsilon_2}(z_2)) \tag{10}$$

given by

$$\int G(x, y)\rho_{\varepsilon_1}^{z_1}(x)\rho_{\varepsilon_2}^{z_2}(y)dxdy$$

where  $\rho_\varepsilon^z(x)dx$  is the uniform measure (of total mass one) on  $\partial B_\varepsilon(z)$ . In fact (like Brownian motion) the process  $h_\varepsilon(z)$  determines a random continuous function (of  $z$  and  $\varepsilon$ ):

**Proposition 3.1** *The process  $h_\varepsilon(z)$  has a modification which is almost surely locally  $\eta$ -Hölder continuous in the pair  $(z, \varepsilon) \in \mathbb{C} \times (0, \infty)$  for every  $\eta < 1/2$ .*

In other words, the Hölder regularity enjoyed by  $h_\varepsilon(z)$ —as a function of the pair  $(z, \varepsilon)$ —is the same as that of Brownian motion or the Brownian sheet. In fact, as we observe below (Proposition 3.3), when  $z$  is fixed,  $h_\varepsilon$  is a Brownian motion with respect to the parameter  $t = -\log \varepsilon$ . We may view  $h_\varepsilon$  as an approximation to  $h$  that gets better as  $\varepsilon \rightarrow 0$ . Before we prove Proposition 3.1, let us make some observations about the covariance function  $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$  defined in (10). (We will also sometimes write  $G_\varepsilon(z_1, z_2) := G_{\varepsilon, \varepsilon}(z_1, z_2)$ .)

First we define the function  $\xi_\varepsilon^z(y)$ , for  $y \in D$ , by

$$\xi_\varepsilon^z(y) = -\log \max(\varepsilon, |z - y|) - \tilde{G}_{z, \varepsilon}(y), \tag{11}$$

where  $\tilde{G}_{z, \varepsilon}(y)$  is the harmonic extension to  $D$  of the restriction of  $-\log \max(\varepsilon, |z - y|)$  to  $\partial D$ . Note that  $\tilde{G}_{z, \varepsilon} = \tilde{G}_z$  provided that  $B_\varepsilon(z) \subset D$ . Observe that this  $\xi_\varepsilon^z(y)$  tends to zero as  $y \rightarrow \partial D$  and that as a distribution  $-\Delta \xi_\varepsilon^z$  (restricted to  $D$ ) is equal to  $2\pi \rho_\varepsilon^z$ , where as before  $\rho_\varepsilon^z$  is a uniform measure on  $D \cap \partial B_\varepsilon(z)$ . Integrating by parts, we immediately have

$$h_\varepsilon(z) = (h, \xi_\varepsilon^z)_\nabla,$$

and from (8) and (10) the following:

**Proposition 3.2** *The function  $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$  is equal to the Dirichlet inner product  $(\xi_{\varepsilon_1}^{z_1}, \xi_{\varepsilon_2}^{z_2})_\nabla$  and to the mean value of  $\xi_{\varepsilon_1}^{z_1}$  on the circle  $\partial B_{\varepsilon_2}(z_2)$ . In particular, if  $B_{\varepsilon_1}(z_1)$  and  $B_{\varepsilon_2}(z_2)$  are disjoint and contained in  $D$  then  $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) = G(z_1, z_2)$ . If  $B_{\varepsilon_1}(z) \subset D$  and  $\varepsilon_1 \geq \varepsilon_2$  then*

$$G_{\varepsilon_1, \varepsilon_2}(z, z) = -\log \varepsilon_1 + \log C(z; D).$$

It then follows that

$$G_{\varepsilon, \varepsilon}(z, z) = \text{Var } h_\varepsilon(z) = (\xi_\varepsilon^z, \xi_\varepsilon^z)_\nabla = \xi_\varepsilon^z(z) = -\log \varepsilon + \log C(z; D). \tag{12}$$

*Proof of Proposition 3.1* We first claim that for each  $\varepsilon_0$  and  $D$  there exists a constant  $K$  such that

$$\text{Var} (h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) \leq K [ |z_1 - z_2| + |\varepsilon_1 - \varepsilon_2| ]$$

for all  $z_1, z_2 \in D$  and  $\varepsilon_1, \varepsilon_2 \in [\varepsilon_0, \infty)$ . Since the variance can only increase if  $D$  is replaced with a larger domain, it suffices to show this holds when  $D$  is replaced by a sufficiently large disc  $D'$  (say, centered in  $D$  with 10 times the

diameter  $r$  of  $D$ ), and  $\varepsilon$  is restricted to values in  $[\varepsilon_0, 5r]$ . (For larger values of  $\varepsilon$ , the set  $\partial B_\varepsilon(z)$  cannot intersect  $D$  when  $z \in D$ .) Since

$$\text{Var}(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) = G_{\varepsilon_1, \varepsilon_1}(z_1, z_1) - 2G_{\varepsilon_1, \varepsilon_2}(z_1, z_2) + G_{\varepsilon_2, \varepsilon_2}(z_2, z_2),$$

it suffices to show that  $G_{\varepsilon_1, \varepsilon_2}(z_1, z_2)$  is a Lipschitz function of  $(\varepsilon_1, \varepsilon_2, z_1, z_2)$  for the range of  $(\varepsilon_1, \varepsilon_2, z_1, z_2)$  values indicated above. This follows from Proposition 3.2 and the fact (whose proof we leave to the reader) that  $\xi_\varepsilon^z$  is a Lipschitz function when  $z \in D$  and  $\varepsilon > \varepsilon_0$ , with a Lipschitz constant that holds uniformly over these  $\varepsilon$  and  $z$  values.

The claim implies that for all  $\alpha > 0$  there is some  $K = K(\alpha) > 0$  such that

$$\mathbb{E}[|h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)|^\alpha] \leq K[|z_1 - z_2| + |\varepsilon_1 - \varepsilon_2|]^\alpha.$$

This puts us in the setting of the multiparameter Kolmogorov-Čentsov theorem [104, 119], which states the following: Suppose that the random field  $X(a)$ ,  $a \in \prod_{i=1}^n [0, t_i]$  satisfies  $\mathbb{E}[|X(a) - X(b)|^\alpha] \leq K|a - b|^{n+\beta}$  for all  $a, b$ , for some fixed constants  $\alpha, \beta, K$ . Then there exists an almost surely continuous modification of the random field and this process is  $\eta$ -Hölder continuous for every  $\eta < \beta/\alpha$ . Applying this for  $n = 3$  and  $\beta = \alpha/2 - 3$  and large  $\alpha$  allows us to deduce that  $h_\varepsilon(z)$ , as a function of  $\varepsilon$  and  $z$ , is locally  $\eta$ -Hölder continuous for all  $\eta < 1/2$ . □

**Proposition 3.3** Write  $\mathcal{V}_t = h_{e^{-t}}(z)$ , and  $t_0^z = \inf\{t : B_{e^{-t}}(z) \subset D\}$ . If  $z \in D$  is fixed, then the law of

$$V_t := \mathcal{V}_{t_0^z+t} - \mathcal{V}_{t_0^z}$$

is a standard Brownian motion in  $t$ .

*Proof* Since we already know that the  $h_\varepsilon(z)$  are jointly Gaussian random variables, it is enough to compute the variances of  $h_\varepsilon(z)$  and  $h_{\varepsilon'}(z)$  for fixed  $\varepsilon, \varepsilon'$ , and these are given in Proposition 3.2. □

### 3.2 Random measure: Liouville quantum gravity

The remainder of the paper makes frequent use of the following simple fact, which the reader may recall (or verify): if  $N$  is a Gaussian random variable with mean  $a$  and variance  $b$  then

$$\mathbb{E} e^N = e^{a+b/2}. \tag{13}$$

Since  $\mathbb{E}h_\varepsilon(z) = 0$  when  $h$  is an instance of the GFF with zero boundary conditions, we have

$$\mathbb{E}e^{\gamma h_\varepsilon(z)} = e^{\text{Var}[\gamma h_\varepsilon(z)]/2}.$$

Recall

$$\text{Var}(h_\varepsilon(z)) = G_\varepsilon(z, z) = \log C(z; D) - \log \varepsilon \tag{14}$$

when  $B_\varepsilon(z) \subset D$ . Then we have

$$\mathbb{E}e^{\gamma h_\varepsilon(z)} = \exp\left(\gamma^2/2(-\log \varepsilon + \log C(z; D))\right) = \left(\frac{C(z; D)}{\varepsilon}\right)^{\gamma^2/2}. \tag{15}$$

More general moments of the random variables  $e^{\gamma h_\varepsilon(z)}$  are also easy to calculate. For example, we have

$$\begin{aligned} \mathbb{E}e^{\gamma h_\varepsilon(y)} e^{\gamma h_\varepsilon(z)} &= \exp(\text{Var}[\gamma(h_\varepsilon(y) + h_\varepsilon(z))]/2) \\ &= \exp\left(\frac{\gamma^2}{2}(G_\varepsilon(y, y) + G_\varepsilon(z, z) + 2G_\varepsilon(y, z))\right). \end{aligned} \tag{16}$$

By Proposition 3.2 we have  $G_\varepsilon(y, z) = G(y, z)$  whenever  $|y - z| \geq 2\varepsilon$  and  $B_\varepsilon(y) \cup B_\varepsilon(z) \subset D$ . In this case we have

$$\mathbb{E}e^{\gamma h_\varepsilon(y)} e^{\gamma h_\varepsilon(z)} = \left(\frac{C(y; D)C(z; D)}{\varepsilon^2}\right)^{\gamma^2/2} e^{\gamma^2 G(y, z)}.$$

Write

$$\bar{h}_\varepsilon := \gamma h_\varepsilon + \frac{\gamma^2}{2} \log \varepsilon.$$

Then we have

$$\mathbb{E}e^{\bar{h}_\varepsilon(z)} = C(z; D)\gamma^2/2 \asymp 1$$

and when  $|y - z| > 2\varepsilon$  we have

$$\begin{aligned} \mathbb{E}e^{\bar{h}_\varepsilon(y)} e^{\bar{h}_\varepsilon(z)} &= (C(y; D) C(z; D))^{\gamma^2/2} e^{\gamma^2 G(y, z)} \\ &\asymp (C(y; D) C(z; D))^{\gamma^2/2} |y - z|^{-\gamma^2} \\ &\asymp |y - z|^{-\gamma^2} \end{aligned}$$

where  $\asymp$  indicates that equality holds up to a constant factor when  $y$  and  $z$  are restricted to any compact subset of  $D$ .

Now, for each fixed  $\varepsilon$ , write  $\mu_\varepsilon := e^{\bar{h}_\varepsilon(z)} dz$  (which in essence corresponds to the ‘‘Wick normal ordering’’ of the original measure [133]). We now argue that these converge weakly to a limiting random measure on  $D$ .

*Proof of Proposition 1.1* Fix  $\gamma \in [0, 2)$ . It is easy to see that if for each diadic square  $S$  compactly supported in  $D$  the random variables  $\mu_{2^{-k}}(S)$  con-

verge to a finite limit as  $k \rightarrow \infty$ , almost surely, then  $\mu_{2^{-k}}$  almost surely converges weakly to a limiting measure. We will prove convergence of  $\mu_{2^{-k}}(S)$  by showing that the expectation of  $|\mu_{2^{-k}}(S) - \mu_{2^{-k-1}}(S)|$  decays exponentially in  $k$ . (Convergence follows, e.g., by using the Borel-Cantelli lemma to show that a.s.  $|\mu_{2^{-k}}(S) - \mu_{2^{-k-1}}(S)|$  is greater than some exponentially decaying function of  $k$  for at most finitely many  $k$ .) Without loss of generality, we may assume  $S$  is the unit square  $[0, 1]^2$ , so that  $\mu_\varepsilon(S)$  is precisely the mean value of  $e^{\bar{h}_\varepsilon(z)}$  on  $S$ .

As shown above, we have

$$\mathbb{E}e^{\bar{h}_\varepsilon(z)} = C(z; D)^{\gamma^2/2}$$

(which is bounded between positive constants) when  $z \in S$  and  $\varepsilon$  is sufficiently small.

For  $y = (y_1, y_2) \in (0, 1)^2$  and  $k \geq 1$ , let  $S_k^y$  be the discrete set of  $2^{2k}$  points  $(a, b) \in S$  with the property that  $(2^k a - 2^k y_1, 2^k b - 2^k y_2) \in \mathbb{Z}^2$ . Let  $A_k^y$  be the mean value of  $\exp \bar{h}_{2^{-k-1}}(z)$  on the set  $S_k^y$ , and  $B_k^y$  the mean value of  $\exp \bar{h}_{2^{-k-2}}(z)$  over the same set:

$$A_k^y := 2^{-2k} \sum_{z \in S_k^y} \exp \bar{h}_{2^{-k-1}}(z), \quad B_k^y := 2^{-2k} \sum_{z \in S_k^y} \exp \bar{h}_{2^{-k-2}}(z). \tag{17}$$

Clearly,  $\mu_{2^{-k-1}}(S)$  is the mean value of  $A_k^y$  over  $y \in [0, 1]^2$  and  $\mu_{2^{-k-2}}(S)$  the mean value of  $B_k^y$  over  $y \in [0, 1]^2$ . Applying Jensen’s inequality to the convex function  $|\cdot|$ , it now suffices for us to prove that  $\mathbb{E}|A_k^y - B_k^y|$  tends to zero exponentially in  $k$  (uniformly in  $y$ ). Since the balls of radius  $2^{-k-1}$  centered at points in  $S_k^y$  do not overlap, and by the *Markov property* of the GFF (see, e.g., [132]), we have that conditioned on the values of  $h_{2^{-k-1}}(z)$  for  $z \in S_k^y$ , the random variables  $h_{2^{-k-2}}(z)$  for  $z \in S_k^y$  are independent of one another; thanks to Propositions 3.2 and 3.3, each is a Gaussian of variance  $\log 2$  and mean  $h_{2^{-k-1}}(z)$ .

Hence, given the values of  $h_{2^{-k-1}}(z)$  for  $z \in S_k^y$ , the value of the conditional expectation of  $|A_k^y - B_k^y|^2$  is given by

$$\begin{aligned} & \mathbb{E} \left( |A_k^y - B_k^y|^2 | h_{2^{-k-1}}(z) \right) \\ &= 2^{-4k} \sum_{z \in S_k^y} \mathbb{E} \left( |e^{\bar{h}_{2^{-k-1}}(z)} - e^{\bar{h}_{2^{-k-2}}(z)}|^2 | h_{2^{-k-1}}(z) \right) \\ &= 2^{-4k} C \sum_{z \in S_k^y} \left( e^{\bar{h}_{2^{-k-1}}(z)} \right)^2, \end{aligned} \tag{18}$$

where

$$C = \mathbb{E} \left( |1 - 2^{-\gamma^2/2} e^{\gamma h_{2^{-k-2}}(z)}|^2 |h_{2^{-k-1}}(z) = 0 \right) = 2^{\gamma^2} - 1,$$

is a constant independent on  $k$  and  $z$ . The unconditional expectation of  $|A_k^y - B_k^y|^2$  is given by the expectation of (18). It is tempting to argue that this expectation tends to zero exponentially in  $k$  (which would in turn imply that  $\mathbb{E}|A_k^y - B_k^y|$  tends to zero exponentially in  $k$ ), but this turns out to be true only for  $0 \leq \gamma^2 < 2$  and not for  $2 \leq \gamma^2 < 4$ . To see this, set  $\varepsilon := 2^{-k-1}$ , and note that

$$\mathbb{E} \left[ (\varepsilon^2 e^{\bar{h}_\varepsilon(z)})^2 \right] = \varepsilon^{4+\gamma^2} \mathbb{E}[e^{2\gamma h_\varepsilon}] \asymp \varepsilon^{4+\gamma^2} e^{-\frac{4\gamma^2 \log \varepsilon}{2}} = \varepsilon^{4-\gamma^2}. \tag{19}$$

Summing over the  $2^{2k} = (2\varepsilon)^{-2}$  points  $z$  in  $S_k^y$  in (18), yields for the expectation of the latter (up to an  $\varepsilon$ -independent constant factor)

$$\mathbb{E} |A_k^y - B_k^y|^2 \asymp \varepsilon^{2-\gamma^2},$$

which does not tend to zero when  $\gamma^2 \geq 2$ .

However, we can deal with the case  $\gamma^2 \geq 2$  by breaking the sum over  $z \in S_k^y$  in (17) defining  $A_k^y - B_k^y$  into two parts and dealing with them separately. The idea is that the estimate in (18) and (19) of the expectation of  $|A_k^y - B_k^y|^2$  is dominated (and can only be made large) by rare occurrences at points  $z$  where  $h_\varepsilon(z)$  is much larger than typical. Their contribution to the expectations of  $A_k^y$  and  $B_k^y$ , hence to  $\mathbb{E}|A_k^y - B_k^y|$ , is exponentially small in  $k$ .

To make this precise, fix some  $\alpha$  with  $\gamma < \alpha < 2\gamma$  and let  $\tilde{S}_k^y$  denote set of points  $z \in S_k^y$  with the property that  $h_\varepsilon(z) > -\alpha \log[\varepsilon/C(z; D)]$ , where  $\varepsilon = 2^{-k-1}$ . Let  $\tilde{A}_k^y$  denote the mean value of  $1_{\tilde{S}_k^y} \exp \bar{h}_{2^{-k-1}}(z)$  over  $S_k^y$  and  $\tilde{B}_k^y$  the mean value of  $1_{\tilde{S}_k^y} \exp \bar{h}_{2^{-k-2}}(z)$  over  $S_k^y$ , so that

$$\begin{aligned} A_k^y &= \tilde{A}_k^y + 2^{-2k} \sum_{z \in S_k^y \setminus \tilde{S}_k^y} \exp \bar{h}_{2^{-k-1}}(z), \\ B_k^y &= \tilde{B}_k^y + 2^{-2k} \sum_{z \in S_k^y \setminus \tilde{S}_k^y} \exp \bar{h}_{2^{-k-2}}(z). \end{aligned} \tag{20}$$

We claim that  $\mathbb{E}\tilde{A}_k^y$  tends to zero exponentially in  $k$ . To see this, observe that for fixed  $z \in S$ , the random variable  $h_\varepsilon(z)$  is a centered Gaussian with variance  $\sigma^2 = -\log[\varepsilon/C(z; D)]$ ; and the expectation of  $e^{\bar{h}_\varepsilon(z)}$ —which we know to be constant for all  $\varepsilon$  small enough so that  $z$  is distance at least  $\varepsilon$  from

the boundary of  $D$ —takes the form

$$\mathbb{E}e^{\bar{h}_\varepsilon(z)} = \frac{\varepsilon^{\gamma^2/2}}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta = C(z; D)^{\gamma^2/2}. \tag{21}$$

We can therefore simply write for points  $z \in \tilde{S}_k^y$ :

$$\mathbb{E} 1_{\tilde{S}_k^y} e^{\bar{h}_\varepsilon(z)} = \frac{\int_{\mathbb{R}} 1_{\eta > \alpha\sigma^2} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta}{\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{\gamma\eta} d\eta} \times \mathbb{E} e^{\bar{h}_\varepsilon(z)}. \tag{22}$$

The ratio of integrals in (22) is the probability that a normal random variable  $\eta$  of mean  $\gamma\sigma^2$  and variance  $\sigma^2$  satisfies  $\eta > \alpha\sigma^2$ , with  $\alpha > \gamma$ , and this clearly tends to zero exponentially in  $\sigma^2$  with rate  $\frac{1}{2}(\alpha - \gamma)^2$ , from which the claim easily follows.

Note that (21) remains unchanged if we replace  $\varepsilon = 2^{-k-1}$  with  $2^{-k-2}$ , which implies that  $\mathbb{E}\tilde{B}_k^y = \mathbb{E}\tilde{A}_k^y$ , and in particular  $\mathbb{E}\tilde{B}_k^y$  also tends to zero exponentially in  $k$ . Since  $\tilde{B}_k^y$  and  $\tilde{A}_k^y$  are non-negative, applying the triangle inequality shows that  $\mathbb{E}|\tilde{B}_k^y - \tilde{A}_k^y|$  tends to zero exponentially in  $k$ .

For the next step, we wish to bound  $\mathbb{E}|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)|^2$ , which requires us to consider the set of points  $z \in S_k^y \setminus \tilde{S}_k^y$  in (20). Applying formula (18), we are led to estimate the expectation

$$\varepsilon^4 \mathbb{E} \left[ 1_{S_k^y \setminus \tilde{S}_k^y} \left( e^{\bar{h}_\varepsilon(z)} \right)^2 \right] = \varepsilon^{4+\gamma^2} \mathbb{E} \left[ 1_{S_k^y \setminus \tilde{S}_k^y} e^{2\gamma h_\varepsilon(z)} \right]. \tag{23}$$

Using the explicit relation (compare to (22))

$$\mathbb{E} \left[ 1_{S_k^y \setminus \tilde{S}_k^y} e^{2\gamma h_\varepsilon(z)} \right] = \frac{\int_{\mathbb{R}} 1_{\eta < \alpha\sigma^2} e^{-\frac{\eta^2}{2\sigma^2}} e^{2\gamma\eta} d\eta}{\int_{\mathbb{R}} e^{-\frac{\eta^2}{2\sigma^2}} e^{2\gamma\eta} d\eta} \times \mathbb{E} e^{2\gamma h_\varepsilon(z)}, \tag{24}$$

we find that (23) differs from (19) by a factor that represents the probability that a Gaussian random variable with variance  $-\log \varepsilon$  (plus a constant term) and mean  $-2\gamma \log \varepsilon$  is less than  $-\alpha \log \varepsilon$ . Since  $\alpha < 2\gamma$ , this probability decays exponentially in  $-\log \varepsilon$  at rate  $(2\gamma - \alpha)^2/2$ . Thus (23) becomes, up to a constant factor (universal in  $\varepsilon$  and  $z \in S$ ),

$$\varepsilon^{4-\gamma^2} \varepsilon^{(2\gamma-\alpha)^2/2}.$$

Summing over the  $2^{2k} = (2\varepsilon)^{-2}$  points  $z \in S_k^y$ , we obtain the estimate

$$\mathbb{E}|(B_k^y - \tilde{B}_k^y) - (A_k^y - \tilde{A}_k^y)|^2 \asymp \varepsilon^{2-\gamma^2+(2\gamma-\alpha)^2/2}.$$



To conclude, we only need to make sure we chose  $\alpha \in (\gamma, 2\gamma)$  small enough so that the sum in the exponent is positive, and this is clearly possible. In fact, taking  $\alpha$  close to  $\gamma$ , the exponent becomes close to  $2 - \gamma^2 + \gamma^2/2 = 2 - \frac{\gamma^2}{2}$ , which is positive when  $\gamma < 2$ . □

### 3.3 Rooted random measures

Before proving Proposition 1.2, we introduce a notion of rooted random measure and use it to prove a uniform integrability result for the random variables  $\mu_\varepsilon(S)$  discussed above.

Write  $\Theta_\varepsilon := Z_\varepsilon^{-1} e^{\gamma h_\varepsilon(z)} dz dh$ , where  $Z_\varepsilon$  is a constant chosen to make  $\Theta_\varepsilon$  a probability measure. Note that  $dz dh$  is a probability measure on a standard Borel space and that  $Z_\varepsilon^{-1} e^{\gamma h_\varepsilon(z)}$  is a measurable function on that space with expectation one. There is therefore no difficulty in defining the  $\Theta_\varepsilon$  as the measure whose Radon-Nikodym derivative with respect to  $dz dh$  is  $Z_\varepsilon^{-1} e^{\gamma h_\varepsilon(z)}$ . Integrating, we see that the  $\Theta_\varepsilon$  marginal distribution of  $z$  is given by  $f(z) dz$  where  $f(z) = Z_\varepsilon^{-1} \mathbb{E}_h e^{\gamma h_\varepsilon(z)}$ . Thus  $f(z)$  is proportional to  $[C(z; D)]^{\gamma^2/2}$  by (15) (provided  $B_\varepsilon(z) \subset D$ ). Similarly, the  $\Theta_\varepsilon$  marginal law of  $h$  is  $Z_\varepsilon^{-1} (\int_D e^{\gamma h_\varepsilon(z)} dz) dh$ . Given  $z$ , a regular conditional probability distribution for  $h$  is given by  $(\mathbb{E}_h e^{\gamma h_\varepsilon(z)})^{-1} e^{\gamma h_\varepsilon(z)} dh$ .

In other words, sampling from  $\Theta_\varepsilon$  may be described as a two step procedure. First sample  $z$  from its marginal distribution. Then sample  $h$  from the distribution of the Gaussian free field *weighted* by  $e^{\gamma h_\varepsilon(z)}$ . Since  $dh$  is a Gaussian measure, we find that given  $z$ ,  $h$  has the law of the original GFF *plus*  $\gamma \xi_\varepsilon^z$  where  $\xi_\varepsilon^z$  satisfies a Dirichlet problem:  $-\Delta \xi_\varepsilon^z$  is the multiple of the uniform measure on  $\partial B_\varepsilon(z)$  with total mass  $2\pi$  (because  $h$  is  $\sqrt{2\pi}$  times the standard GFF; if  $h$  were the standard GFF the total mass would be 1). As noted in Sect. 3.1, this  $\xi_\varepsilon^z$  has been computed explicitly:

$$\xi_\varepsilon^z(y) = -\log \max\{|z - y|, \varepsilon\} - \tilde{G}_z(y), \tag{25}$$

where  $\tilde{G}_z$  is the harmonic interpolation to  $D$  of the first term on  $\partial D$ , as long as  $B_\varepsilon(z) \subset D$ .

Let  $\Theta$  be the limit of the measures  $\Theta_\varepsilon$  as  $\varepsilon \rightarrow 0$ : that is,  $\Theta$  is the measure on pairs  $(z, h)$  for which the marginal distribution of  $z$  is proportional to  $[C(z; D)]^{\gamma^2/2} dz$  and, given  $z$ , the  $\Theta$  conditional law of  $h$  is that of the original standard GFF plus the deterministic function  $\gamma \xi_0^z$  (viewed as a distribution). For any  $\tilde{D}$  compactly supported on  $D$ , we will also write  $\Theta^{\tilde{D}}$  for the measure  $\Theta$  conditioned on the event  $z \in \tilde{D}$ . We similarly define  $\Theta_\varepsilon^{\tilde{D}}$  to be  $\Theta_\varepsilon$  conditioned on  $z \in \tilde{D}$  (where  $\varepsilon$  is less than the distance from  $\tilde{D}$  to  $\partial D$ ). By the above construction and Proposition 3.3, we have that conditioned on

$z$ , the  $\Theta$  law of  $h_{e^{-t}}$  is essentially that of a Brownian motion plus a drift term of  $\gamma t$ . This in particular implies the following:

**Proposition 3.4** *With  $\Theta$  probability one,  $z$  is a  $\gamma$ -thick point of  $h$  by the definition in [83]. That is,*

$$\liminf_{\varepsilon \rightarrow 0} h_\varepsilon(z) / \log \varepsilon^{-1} \geq \gamma.$$

*In fact, the limit exists and equality holds almost surely.*

Since the  $\Theta$  marginal law of  $h$  is absolutely continuous with respect to the law of  $h$  (with Radon-Nikodym derivative  $\mu_h(D)$ ), this implies that  $\mu_h$  is almost surely supported on  $\gamma$ -thick points. It was shown by Hu, Miller, and Peres that the set of  $\gamma$ -thick points has Hausdorff dimension  $2 - \frac{\gamma^2}{2}$  almost surely [83].

*Proof of Proposition 1.2* The almost sure weak convergence of the  $\mu^n$  to a limit  $\tilde{\mu}$  is immediate from the martingale convergence theorem. Recall the expression (1)

$$\mu^n = \exp \left( \gamma h^n(z) - \frac{\gamma^2}{2} \text{Var } h^n(z) + \frac{\gamma^2}{2} \log C(z; D) \right) dz,$$

and observe that for each  $z$ , the exponential term

$$\exp \left( \gamma h^n(z) - \frac{\gamma^2}{2} \text{Var } h^n(z) + \frac{\gamma^2}{2} \log C(z; D) \right)$$

is a non-negative martingale with respect to the filtration of  $h^n$ . (This is a consequence of (13).) Fubini’s theorem implies that  $\mu^n(A)$  is a martingale for any Borel measurable set  $A \subset D$ , and the martingale convergence theorem implies that the limit  $\tilde{\mu}(A) := \lim \mu^n(A)$  exists almost surely. In particular, this holds whenever  $A$  is a diadic square contained in  $D$  and from this easily follows the desired weak convergence.

We still need to show that  $\mu = \tilde{\mu}$  almost surely, where  $\mu$  is as constructed in Proposition 1.1. It is enough to show that  $\mu(S) = \tilde{\mu}(S)$  almost surely for each diadic square  $S$  compactly supported on  $D$ , and since both  $\mu$  and  $\tilde{\mu}$  are functions of  $h$ , this is equivalent to showing that  $\mathbb{E}[\mu(S)|h] = \mathbb{E}[\tilde{\mu}(S)|h]$ . This in turn follows if we can show  $\mathbb{E}[\mu(S)|h^n] = \mathbb{E}[\tilde{\mu}(S)|h^n]$  for all  $n$ ; the latter expectation is just  $\mu^n(S)$  by definition, so it remains only to show

$$\mathbb{E}[\mu(S)|h^n] = \mu^n(S). \tag{26}$$

To this end, let  $h_\varepsilon^n$  denote the mean value of  $h^n$  on  $\partial B_\varepsilon(z)$ . For each particular choice of  $z$ , and  $\varepsilon$  small enough so that  $B_\varepsilon(z) \subset D$ , and for each  $n$ ,

we recall from (14) that  $h_\varepsilon(z)$  is a Gaussian random variable with variance  $\log C(z; D) - \log \varepsilon$  and that  $h_\varepsilon^n(z)$  is the conditional expectation of  $h_\varepsilon(z)$  given the projection of  $h$  to the span of  $f_1, f_2, \dots, f_n$ . Hence

$$\mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} | h^n] = \exp\left(\gamma h_\varepsilon^n(z) - \frac{\gamma^2}{2} \text{Var} h_\varepsilon^n(z) + \frac{\gamma^2}{2} \log C(z; D)\right).$$

Taking the limit as  $\varepsilon \rightarrow 0$  and using the continuity of  $h^n(z)$  and  $\text{Var} h^n(z)$  and the expression (1), we have

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n] = \mu^n(S) \tag{27}$$

for each diadic square  $S$ . Using (27) we will have established (26) once we show that

$$\mathbb{E}[\mu(S) | h^n] := \mathbb{E}[\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(S) | h^n] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n], \tag{28}$$

provided that  $0 \leq \gamma < 2$ .

We first argue this in the case  $n = 0$  and  $h^0 = 0$ , that is

$$\mathbb{E}[\mu(S)] := \mathbb{E}[\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(S)] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S)]. \tag{29}$$

Since Proposition 1.1 implies the existence of the limit of  $\mu_\varepsilon(S)$ , it is enough to show that the random variables  $M_\varepsilon = \mu_\varepsilon(S)$  are uniformly integrable as  $\varepsilon \rightarrow 0$ . Let  $M = \mathbb{E}M_\varepsilon$  for  $\varepsilon$  small enough so that  $B_\varepsilon(S) \subset D$ . (By (15) this expectation is the same for all sufficiently small  $\varepsilon$ .) The uniform integrability is equivalent to the statement that the probability measures  $\eta_\varepsilon := M^{-1}M_\varepsilon dM_\varepsilon$  are tight, i.e., for all  $\delta$  there exists a constant  $C > 0$  such that  $\eta_\varepsilon([C, \infty)) < \delta$  for all  $\varepsilon$ . (Here  $M^{-1}M_\varepsilon dM_\varepsilon$  denotes the probability measure on  $\mathbb{R}$  whose Radon-Nikodym derivative with respect to the law of  $M_\varepsilon$  is given by  $M^{-1}M_\varepsilon$ .) Since  $M_\varepsilon$  is a function of  $h$ , this is equivalent to the statement that with respect to the measure  $M^{-1}M_\varepsilon(h)dh$  the random variables  $M_\varepsilon(h)$  are tight. Recalling that  $M_\varepsilon$  is (up to a constant factor) the Radon-Nikodym derivative of the  $h$  marginal of  $\Theta_\varepsilon^S$  with respect to  $dh$ , this in turn can be rewritten as the statement that for each  $\delta$  we can find a  $C$  such that

$$\Theta_\varepsilon^S\{M_\varepsilon(h) > C\} < \delta \tag{30}$$

for all  $\varepsilon$ .

Throughout the remainder of the proof of (29), all probabilities and expectations will be taken with respect to  $\Theta_\varepsilon^S$ . Let  $\varepsilon_0 = \sup\{\varepsilon' : B_{\varepsilon'}(S) \subset D\}$ . It suffices to prove the above statement, that for each  $\delta$  we can find a  $C$  such that (30) holds, for small  $S$ —precisely, for avoiding boundary effects, we may

restrict attention to  $S$  such that  $\varepsilon_0$  is larger than the diameter of  $S$ , in which case  $S \subset B_{\varepsilon_0}(z)$  for any  $z \in S$ .

In order to obtain (30), we will describe a procedure for sampling from  $\Theta_\varepsilon^S$  in multiple stages. We will then show that (30) holds when  $M_\varepsilon(h)$  is replaced by its conditional expectation given the observations from the first two stages, and the statement we require will follow easily from this.

Precisely, we may sample the pair  $(z, h)$  from  $\Theta_\varepsilon^S$  in the following steps. In the first step, we sample  $z$  from its marginal law (which is independent of  $\varepsilon$  for  $\varepsilon$  sufficiently small). Write  $\tilde{h} = h - \gamma \xi_\varepsilon^z$ . Given  $z$ , the  $\Theta_\varepsilon^S$  law of  $\tilde{h}$  is that of a GFF on  $D$ . In the second step, we sample  $\mathcal{B}_t = \tilde{h}_{e^{-t}\varepsilon_0}(z) - \tilde{h}_{\varepsilon_0}(z)$  for all  $t \in [0, -\log(\varepsilon/\varepsilon_0)]$ . By Proposition 3.3,  $\mathcal{B}_t$  is a Brownian motion on this interval independent of  $z$ . In the third and final step, we choose  $h$  conditioned on the results of the first two steps.

The  $\Theta_\varepsilon^S$  conditional expectation of  $\tilde{h}$  given the whole process  $\mathcal{B}_t$  (which we have defined only for  $t \in [0, -\log(\varepsilon/\varepsilon_0)]$ ) and  $z$  is given by the function (viewed as a distribution)

$$\tilde{h}^*(w) := \mathbb{E}[\tilde{h}(w)|z, \mathcal{B}_t] = \begin{cases} \mathcal{B}_{u(w)}, & (|z - w| < \varepsilon_0), \\ 0, & (|z - w| \geq \varepsilon_0), \end{cases}$$

where

$$u(w) := -\log \frac{|z - w| \vee \varepsilon}{\varepsilon_0}.$$

(We will discuss similar conditional expectations in detail in Sect. 4.1.) Note that once  $z$  is fixed, for each  $w$  the mean value of  $\tilde{h}^*(\cdot)$  on  $\partial B_\varepsilon(w)$  (which we denote by  $\tilde{h}_\varepsilon^*(w)$ ) is a weighted average of  $\mathcal{B}_t$  over values of  $t$  between  $u_1(w) := -\log(\varepsilon_1(w)/\varepsilon_0)$  and  $u_2(w) := -\log(\varepsilon_2(w)/\varepsilon_0)$ , where

$$\varepsilon_1(w) := \varepsilon_0 \wedge (|w - z| + \varepsilon), \quad \varepsilon_2(w) := (|w - z| - \varepsilon) \vee \varepsilon,$$

with, for  $|z - w| \leq \varepsilon_0$ ,  $\varepsilon_1(w) \geq \varepsilon_2(w)$ , hence  $u_1(w) \leq u_2(w)$ . From this it is not hard to see that given  $z$  the variance of  $\tilde{h}_\varepsilon^*(w)$  is between the two values  $\{u_1(w), u_2(w)\}$  of  $t$ . We claim that each of these bounds differs from the intermediate value  $u(w)$  above, with  $u_1(w) \leq u(w) \leq u_2(w)$ , by at most an additive constant  $\log 2$ . This is equivalent to the statement that  $\varepsilon_1(w)$  and  $\varepsilon_2(w)$  differ from  $|z - w| \vee \varepsilon$  by a multiplicative or inverse factor of at most two, which is easily checked under the further mild assumption that  $2\varepsilon \leq \varepsilon_0$ . Thus the variance of  $\tilde{h}_\varepsilon^*(w)$  is within  $\log 2$  of the value

$$u(w) = -\log \frac{|z - w|}{\varepsilon_0} \wedge -\log \frac{\varepsilon}{\varepsilon_0}.$$

Since  $\mathbb{E}[\tilde{h}_\varepsilon(w)|\tilde{h}_\varepsilon^\bullet(w)] = \tilde{h}_\varepsilon^\bullet(w)$ , we have

$$\text{Var}(\tilde{h}_\varepsilon(w)) = \mathbb{E} \text{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}_\varepsilon^\bullet(w)) + \text{Var}(\tilde{h}_\varepsilon^\bullet(w)). \tag{31}$$

Since  $\tilde{h}_\varepsilon(w)$  and  $\tilde{h}_\varepsilon^\bullet(w)$  (both linear functionals of  $h$ ) are jointly Gaussian, the quantity  $\text{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}_\varepsilon^\bullet(w))$  is in fact independent of  $\tilde{h}_\varepsilon^\bullet(w)$ . Since (as observed above)  $|\text{Var}(\tilde{h}_\varepsilon^\bullet(w)) - u(w)| < \log 2$ , we conclude that

$$|\text{Var}(\tilde{h}_\varepsilon(w)|\tilde{h}_\varepsilon^\bullet(w)) - \text{Var}(\tilde{h}_\varepsilon(w)) + u(w)| < \log 2,$$

almost surely.

Thus, with respect to  $\Theta_\varepsilon^S$ , we have

$$\begin{aligned} \mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(w)} |z, \mathcal{B}_t] &\asymp \exp\left(\gamma \tilde{h}_\varepsilon^\bullet(w) + \gamma^2 \xi_\varepsilon^z(w) - \gamma^2 u(w)/2\right) \\ &\asymp \exp\left(\gamma \tilde{h}_\varepsilon^\bullet(w) + \gamma^2 u(w)/2\right), \end{aligned}$$

where we recall that, thanks to (25),  $\xi_\varepsilon^z(w) = u(w) - \log \varepsilon_0 - \tilde{G}_z(w)$ , and where  $\asymp$  indicates equality up to a multiplicative factor bounded between positive constants uniformly in  $\varepsilon$  and  $z$ .

Now, given any positive constants  $a$  and  $b$ , there is a positive probability that a Brownian motion  $\mathcal{B}_t$  run for an infinite amount of time will satisfy  $\gamma \mathcal{B}_t < a + bt$  for all  $t \geq 0$ . In fact, for each fixed  $b$ , this probability can be made as close to one as possible by taking  $a$  sufficiently large. Since  $0 \leq \gamma < 2$  we can choose a value of  $b$  with  $0 < b < 2 - \gamma^2/2$ . Then note that conditioned on the event  $\mathcal{A} : \gamma \mathcal{B}_t < a + bt$  for all  $t$ , and since the mean value  $\tilde{h}_\varepsilon^\bullet(w)$  is a weighted average of  $\mathcal{B}_t$  over values of  $t \in [u_1(w), u_2(w)]$ , we have  $\gamma \tilde{h}_\varepsilon^\bullet(w) < a + bu_2(w) \leq a + b \log 2 + bu(w)$ . We therefore have, for some constant  $C_0$

$$\begin{aligned} \mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(w)} |z, \mathcal{B}_t, \mathcal{A}] &\leq C_0 e^a \exp\left[(b + \gamma^2/2)u(w)\right] \\ &\leq C_0 e^a |z - w|^{-b-\gamma^2/2} \end{aligned}$$

for  $|z - w| < \varepsilon_0$ . Since  $S \subset B_{\varepsilon_0}(z)$  for  $z \in S$ , this in turn implies that

$$\mathbb{E}[\mu_\varepsilon(S) |z, \mathcal{B}_t, \mathcal{A}] \leq \int_{B_{\varepsilon_0}(z)} C_0 e^a |z - w|^{-b-\gamma^2/2} dw,$$

and since  $b + \gamma^2/2 < 2$ , the right hand side is at most a finite constant  $C_1 = C_1(a)$  that is independent of  $\varepsilon$ . Now, given  $b$  and a constant  $\delta > 0$  we can choose  $a$  large enough so that the probability of the event  $\mathcal{A}$  (that  $\gamma \mathcal{B}_t <$

$a + bt$  for all  $t$ ) is at least  $1 - \delta/2$ . Then we take  $C = \frac{C_1(a)}{\delta/2}$ . If there were probability at least  $\delta$  that  $\mu_\varepsilon(S) > C$  then there would have to be probability at least  $\delta/2$  that event  $\mathcal{A}$  and  $\mu_\varepsilon(S) > C$  simultaneously happen, which would contradict our bound on the conditional expectation of  $\mu_\varepsilon(S)$  given  $\mathcal{A}$ . This implies that the probability measures  $\eta_\varepsilon$  are tight, which in turn completes the proof of (29), which is the  $n = 0$  and  $h^0 = 0$  case of (28).

As a tool, we used heavily within the proof of (29) the probability measure  $\Theta_\varepsilon^S$  on  $(z, h)$  pairs. Extending (28) to the case  $n \neq 0$  does not require this tool; in the discussion below we will use only the original  $dzdh$  measure. Note that since the random variables  $\mu_\varepsilon(S)$  converge  $dzdh$  almost surely to a limit (with expectation  $\lim_{\varepsilon \rightarrow 0} \mathbb{E}\mu_\varepsilon(S)$ ), it must be the case that conditioned on  $h^n$  (for almost all values of  $h^n$ ), we still have that  $\mu_\varepsilon(S)$  converges  $dzdh$  almost surely to a limit. The fact that

$$\mathbb{E}[\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(S) | h^n] \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E}[\mu_\varepsilon(S) | h^n] \tag{32}$$

for almost all  $h^n$  is immediate from Fatou’s lemma. From the unconditional result, we know that equality holds when we integrate over possible values of  $h^n$ —hence equality must hold in (32) for almost all  $h^n$ . The extension of (28) to non-zero  $h^0$  is trivial for functions that are piecewise constant on diadic squares, and the more general case follows easily by approximation by piecewise constant functions.

Proposition 1.2 is an immediate consequence of (27) and (28). □

## 4 KPZ proofs

### 4.1 Circle average KPZ

For fixed  $z \in D$ , choose some radius  $\varepsilon_0$  such that  $B_{\varepsilon_0}(z) \subset D$ . As a first step, we estimate the expectation of the quantum measure  $\mu_h(B_\varepsilon(z))$ , given the difference of circle averages  $h_\varepsilon(z) - h_{\varepsilon_0}(z)$  for  $\varepsilon \leq \varepsilon_0$ . Recalling the notation of Proposition 1.2, we take  $h^0 = 0$ ,  $n = 1$ , and

$$f_1 = (\xi_\varepsilon^z - \xi_{\varepsilon_0}^z) / \|\xi_\varepsilon^z - \xi_{\varepsilon_0}^z\|_{\nabla}. \tag{33}$$

Recall from (12) that the square Dirichlet norm of function  $\xi_\varepsilon^z$  (11) is such that  $\|\xi_\varepsilon^z\|_{\nabla}^2 = (\xi_\varepsilon^z, \xi_\varepsilon^z)_{\nabla} = \xi_\varepsilon^z(z)$ , and from Proposition 3.2 that  $(\xi_\varepsilon^z, \xi_{\varepsilon_0}^z)_{\nabla} = \xi_{\varepsilon_0}^z(z)$ . One thus finds  $\|\xi_\varepsilon^z - \xi_{\varepsilon_0}^z\|_{\nabla}^2 = -\log(\varepsilon/\varepsilon_0)$  and

$$(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)(y) = \begin{cases} -\log(\varepsilon/\varepsilon_0), & 0 \leq |y - z| \leq \varepsilon, \\ -\log(|y - z|/\varepsilon_0), & \varepsilon \leq |y - z| \leq \varepsilon_0, \\ 0, & \varepsilon_0 \leq |y - z|. \end{cases} \tag{34}$$

The projection  $h^1$  of  $h$  onto the span of  $f_1$  and its variance are then

$$h^1(y) = [h_\varepsilon(z) - h_{\varepsilon_0}(z)] \frac{(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)(y)}{-\log(\varepsilon/\varepsilon_0)}, \tag{35}$$

$$\text{Var } h^1(y) = \frac{(\xi_\varepsilon^z - \xi_{\varepsilon_0}^z)^2(y)}{-\log(\varepsilon/\varepsilon_0)}, \tag{36}$$

where we recall that  $\text{Var}[h_\varepsilon(z) - h_{\varepsilon_0}(z)] = -\log(\varepsilon/\varepsilon_0)$ .

Recalling the notation of Proposition 1.1, the conditional expectation formula (2) for  $\mu$  in Proposition 1.2 gives

$$\begin{aligned} & \mathbb{E}_h \left[ \int_{B_\varepsilon(z)} e^{\gamma h} dz | h_\varepsilon(z) - h_{\varepsilon_0}(z) \right] \\ &= \mathbb{E}_h [\mu_h(B_\varepsilon(z)) | h_\varepsilon(z) - h_{\varepsilon_0}(z)] = \mu^1(B_\varepsilon(z)), \end{aligned} \tag{37}$$

where  $\mu^1$  is the projected measure (1)

$$\mu^1(dy) = \exp \left( \gamma h^1(y) - \frac{\gamma^2}{2} \text{Var } h^1(y) + \frac{\gamma^2}{2} \log C(y; D) \right) dy. \tag{38}$$

Note that by (34),  $h^1(y)$  does not depend on  $y$  for  $y \in B_\varepsilon(z)$

$$\begin{aligned} h^1(y) &= h_\varepsilon(z) - h_{\varepsilon_0}(z), \quad y \in B_\varepsilon(z), \\ \text{Var } h^1(y) &= -\log(\varepsilon/\varepsilon_0). \end{aligned}$$

We therefore have

$$\mu^1(dy) = \mu^0(dy) \left( \frac{\varepsilon}{\varepsilon_0} \right)^{\gamma^2/2} \exp[h_\varepsilon(z) - h_{\varepsilon_0}(z)], \quad y \in B_\varepsilon(z), \tag{39}$$

$$\mu^0(dy) := [C(y; D)]^{\gamma^2/2} dy. \tag{40}$$

Define the ( $\gamma$ -dependent) average  $C_\varepsilon(z; D)$  of the conformal radius over the ball  $B_\varepsilon(z)$  via the average moment

$$[C_\varepsilon(z; D)]^{\gamma^2/2} := \frac{\mu^0(B_\varepsilon(z))}{\mu_0(B_\varepsilon(z))} = \frac{1}{\pi \varepsilon^2} \int_{B_\varepsilon(z)} [C(y; D)]^{\gamma^2/2} dy, \tag{41}$$

so that for  $\varepsilon \rightarrow 0$

$$\lim_{\varepsilon \rightarrow 0} C_\varepsilon(z; D) = C(z; D).$$

We then have the simple expression

$$\mu^1(B_\varepsilon(z)) = \pi \varepsilon^\gamma Q \left( \frac{C_\varepsilon(z; D)}{\varepsilon_0} \right)^{\gamma^2/2} \exp[h_\varepsilon(z) - h_{\varepsilon_0}(z)], \tag{42}$$

where, as above,  $Q = 2/\gamma + \gamma/2$ .

As an alternative, one may wish to estimate the expectation of the quantum measure  $\mu_h(B_\varepsilon(z))$ , given only the circle average  $h_\varepsilon(z)$ . In the notation of Proposition 1.2, we take in that case  $h^0 = 0$ ,  $n = 1$ , and  $\tilde{f}_1 = \xi_\varepsilon^z / \|\xi_\varepsilon^z\|_\nabla$ , with the square Dirichlet norm  $\|\xi_\varepsilon^z\|_\nabla^2 = (\xi_\varepsilon^z, \xi_\varepsilon^z)_\nabla = \xi_\varepsilon^z(z)$ . The projection  $\tilde{h}^1$  of  $h$  onto the span of  $f_1$  and its variance are then

$$\tilde{h}^1(y) = h_\varepsilon(z) \frac{\xi_\varepsilon^z(y)}{\xi_\varepsilon^z(z)}, \tag{43}$$

$$\text{Var } \tilde{h}^1(y) = \text{Var } h_\varepsilon(z) \left( \frac{\xi_\varepsilon^z(y)}{\xi_\varepsilon^z(z)} \right)^2 = \frac{(\xi_\varepsilon^z(y))^2}{\xi_\varepsilon^z(z)}, \tag{44}$$

where we recall that  $\text{Var } h_\varepsilon(z) = \xi_\varepsilon^z(z) = -\log(\varepsilon/C(z; D))$ .

The conditional expectation formula (2) for  $\mu$  in Proposition 1.2 gives in this case

$$\mathbb{E}_h \left[ \int_{B_\varepsilon(z)} e^{\gamma h} dz | h_\varepsilon(z) \right] = \mathbb{E}_h [\mu_h(B_\varepsilon(z)) | h_\varepsilon(z)] = \tilde{\mu}^1(B_\varepsilon(z)), \tag{45}$$

where  $\tilde{\mu}^1$  is the projected measure (1)

$$\tilde{\mu}^1(dy) = \exp \left( \gamma \tilde{h}^1(y) - \frac{\gamma^2}{2} \text{Var } \tilde{h}^1(y) + \frac{\gamma^2}{2} \log C(y; D) \right) dy. \tag{46}$$

Note that when  $y \in B_\varepsilon(z)$ ,  $\xi_\varepsilon^z(y) = -\log \varepsilon - \tilde{G}_z(y)$ , so that the difference  $\xi_\varepsilon^z(z) - \xi_\varepsilon^z(y) = \log C(z; D) + \tilde{G}_z(y)$  is harmonic in  $y$  and its modulus is equivalent to  $\varepsilon |\tilde{G}'_z(z)|$  for  $\varepsilon$  small, where  $\tilde{G}'_z(z)$  is the derivative at  $z$  of the harmonic extension  $\tilde{G}_z$ . It follows that in ball  $B_\varepsilon(z)$ ,  $\xi_\varepsilon^z(y)/\xi_\varepsilon^z(z) = 1 + O(\varepsilon/\log \varepsilon)$ . Lastly, the function  $C(y; D)$  is real analytic. Hence from (43), (44) and (46) above, it follows from (45) that for  $\varepsilon \rightarrow 0$

$$\mathbb{E}[\mu(B_\varepsilon(z)) | h_\varepsilon(z)] = \tilde{\mu}^1(B_\varepsilon(z)) \simeq \mu_\odot(B_\varepsilon(z)), \tag{47}$$

where  $\mu_\odot$  is defined as

$$\mu_\odot(B_\varepsilon(z)) := \pi \varepsilon^\gamma Q e^{\gamma h_\varepsilon(z)}, \quad Q = 2/\gamma + \gamma/2, \tag{48}$$

in the sense that the ratio of the two quantities tends to 1 as  $\varepsilon \rightarrow 0$ . Note that  $\mu_\odot$  is not a measure, but simply a quantity defined on balls of the form



$B_\varepsilon(z)$ . Notice then that the first conditional measure  $\mu^1(B_\varepsilon(z))$  (42) can also be written as

$$\mu^1(B_\varepsilon(z)) = \pi \varepsilon_0^2 C_\varepsilon(z; D)^{\gamma^2/2} \frac{\mu_\odot(B_\varepsilon(z))}{\mu_\odot(B_{\varepsilon_0}(z))}. \tag{49}$$

For any  $\varepsilon \leq \varepsilon_0$  define then

$$t := -\log(\varepsilon/\varepsilon_0), \tag{50}$$

$$V_t := h_\varepsilon(z) - h_{\varepsilon_0}(z). \tag{51}$$

The law of  $V_t$  is that of a Brownian motion with  $V_0 = 0$  (by Proposition 3.3). We can then rewrite (42) as

$$\mu^1(B_\varepsilon(z)) = \pi \varepsilon_0^2 C_\varepsilon(z; D)^{\gamma^2/2} e^{\gamma V_t - \gamma Q t}. \tag{52}$$

Similarly, we can rewrite (48) identically as

$$\mu_\odot(B_\varepsilon(z)) = \mu_\odot(B_{\varepsilon_0}(z)) e^{\gamma V_t - \gamma Q t}, \tag{53}$$

in accordance with (49). In the expression (52) for the measure  $\mu^1$ , the first non constant factor is the same as (41), which is a slowly varying, deterministic function of  $z$  (and of  $\varepsilon$ ), whereas in the expression (53) for  $\mu_\odot$ , the first factor is the quantity  $\mu_\odot(B_{\varepsilon_0}(z))$ , which is the exponential of a centered Gaussian variable,  $h_{\varepsilon_0}(z)$ , whose variance,  $-\log(\varepsilon_0/C(z; D))$ , varies slowly with  $z$ . In both expressions, the latter factor is the exponential of a simple Brownian motion with drift, and is *independent* of  $z$ .

**Definition 4.1** Let  $\tilde{B}^\delta(z)$  be the largest Euclidean ball in  $D$  centered at  $z$  for which  $e^{\gamma V_t - \gamma Q t}$  is equal to  $\delta$ . The radius of this ball is  $e^{-T_A}$  where

$$T_A := \inf\{t : -V_t + Q t = A\},$$

and  $A := -(\log \delta)/\gamma$ .

As a step towards Theorem 1.5 we prove the following in this section, which is perhaps the most straightforward form of KPZ to prove:

**Theorem 4.2** *Theorem 1.5 holds with  $B^\delta(z)$  replaced with  $\tilde{B}^\delta(z)$ . That is, in the setting of Theorem 1.5, if*

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0\{z : B_\varepsilon(z) \in \mathcal{X}\}}{\log \varepsilon^2} = x,$$

then it follows that, when  $\mathcal{X}$  and  $\mu$  are chosen independently, we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu \{z : \tilde{B}^\delta(z) \in \mathcal{X}\}}{\log \delta} = \Delta,$$

where  $\Delta$  is the non-negative solution to

$$x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.$$

We present two proofs: the first based on exponential martingales, the second based on large deviations theory and Schilder’s theorem. (The first proof is shorter, but readers familiar with large deviations of Brownian motion will recognize that it is essentially the second proof in the disguise.)

Both proofs use the fact that

$$\mathbb{E}_h \mu \{z : \tilde{B}^\delta(z) \in \mathcal{X}\}$$

is proportional to

$$\Theta \{(z, h) : \tilde{B}^\delta(z) \in \mathcal{X}\},$$

to replace an expectation computation with a probability computation. (Recall the definition of  $\Theta$  from Sect. 3.3.) While this rephrasing is not strictly necessary for the expectation computation below, it is conceptually quite natural.

We use the definitions (50) and (51) of  $V_t$  given above, and assume that the fixed  $\varepsilon_0$  is smaller than the distance from  $\tilde{D}$  (recall that this was the compact subset of  $D$  in Theorem 1.5) to  $\partial D$ .

As mentioned in Sect. 3.3, the  $\Theta$  conditional law of  $h$  given  $z \in D$  is that of the original GFF plus the deterministic function  $-\gamma \log |z - y| - \gamma \tilde{G}_z(y)$ . The  $\Theta$  conditional law of the circular average  $h_\varepsilon(z)$  is then that of the original GFF circular average plus  $-\gamma \log \varepsilon + \gamma \log C(z; D)$ . Thus (for  $z$  restricted to points of distance at least  $\varepsilon_0$  from  $\partial D$ ) the  $\Theta$  conditional law of (51)  $V_t = h_{\varepsilon_0 e^{-t}}(z) - h_{\varepsilon_0}(z)$  given  $z$  is that of  $\mathcal{B}_t + \gamma t$ , with  $t = -\log(\varepsilon/\varepsilon_0)$ , and where  $\mathcal{B}_t$  evolves as a standard Brownian motion—in particular,  $z$  is independent of the process  $V_t$ .

*Proof* The  $\Theta$  law of  $T_A$  is that of

$$\inf\{t : \mathcal{B}_t + at = A = -(\log \delta)/\gamma\}, \quad a := Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2} > 0, \quad (54)$$

where  $(\pm)\mathcal{B}_t$  is standard Brownian motion with  $\mathcal{B}_0 = 0$ . Let  $q_A$  be the  $\Theta$  probability that the ball of radius  $e^{-T_A}$  centered at  $z$  is in  $\mathcal{X}$ . Since  $z$  is independent of  $T_A$ , the theorem hypothesis implies that conditioned on  $T_A$ , the

probability that the ball of radius  $e^{-T_A}$  centered at  $z$  is in  $\mathcal{X}$  is approximately  $\exp(-2xT_A)$ , in the sense that the ratio of the logs of these two quantities tends to 1 as  $T_A \rightarrow \infty$ . Computing the expectation

$$\mathbb{E}[\exp(-2xT_A)], \tag{55}$$

with respect to a random  $T_A$  will give us upper and lower bounds on  $q_A$  since it easily follows that

$$\mathbb{E}[\exp(-2x_1T_A)] \leq q_A \leq \mathbb{E}[\exp(-2x_2T_A)], \tag{56}$$

for any fixed  $0 < x_2 < x < x_1$  and sufficiently large  $A$ .

To compute (55), consider for any  $\beta$  the exponential martingale  $\exp(\beta\mathcal{B}_t - \beta^2t/2)$ . Since  $a > 0$ , the stopping time  $T_A$  is finite a.s. Since  $\mathcal{B}_t + at \leq A$  for  $t \in [0, T_A]$ , the argument of the exponential,  $\beta\mathcal{B}_t - \beta^2t/2$ , stays bounded from above, for  $\beta \geq 0$ , by  $\beta A - (\beta a + \beta^2/2)t \leq \beta A$ , hence by a fixed constant. One can thus apply the exponential martingale at the stopping time  $T_A < \infty$

$$\mathbb{E}[\exp(\beta\mathcal{B}_{T_A} - \beta^2T_A/2)] = 1.$$

By definition  $\mathcal{B}_{T_A} = A - aT_A$ . Thus,

$$\mathbb{E} \exp[-(\beta a + \beta^2/2)T_A] = \exp(-\beta A).$$

Setting  $2x := \beta a + \beta^2/2$ , we obtain

$$\mathbb{E} \exp(-2xT_A) = \exp(-\beta A) = \delta^{\beta/\gamma}. \tag{57}$$

Now if we set  $\Delta = \beta/\gamma$ , and  $a = Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2}$ , we find that the equation  $2x := \beta a + \beta^2/2$ , with  $\beta \geq 0$ , is equivalent to the KPZ formula. The continuity of this expression and (56) together yield the theorem.  $\square$

We remark that the above yields the explicit probability distribution  $P_A(t)$ . The inverse Laplace transform  $P_A(t)$  of  $f_A(x) := \mathbb{E} \exp(-2xT_A)$ , with respect to  $2x$ , is the probability density such that  $P_A(t)dt := \text{Prob}(T_A \in [t, t + dt])$ . Its explicit expression is [32]

$$P_A(t) = (2\pi)^{-1/2} A t^{-3/2} \exp\left[-(1/2) \left(At^{-1/2} - at^{1/2}\right)^2\right], \tag{58}$$

where as above we have  $A = -(\log \delta)/\gamma$ ,  $t = -\log(\varepsilon/\varepsilon_0)$  and  $a = Q - \gamma$ .

### 4.2 Large deviations proof of circle average KPZ

In this section, we present an alternative proof of Theorem 4.2, using Schilder’s theorem.

**Lemma 4.3** *Fix a constant  $a > 0$ . Let  $\mathcal{B}_t$  be a standard Brownian motion. For each  $A > 0$ , write*

$$T_A = \inf\{t : \mathcal{B}_t + at = A\}. \tag{59}$$

*Then the family of random variables  $A^{-1}T_A$  satisfies a large deviations principle with speed  $A$  and rate function*

$$I(\eta) = \frac{\eta}{2} \left( \frac{1}{\eta} - a \right)^2 = \frac{\eta^{-1}}{2} - a + a^2 \frac{\eta}{2}.$$

*Proof* Schilder’s Theorem (see Theorem 5.3.2 of [66]) gives an LDP for the sample path of  $\alpha^{-1}\mathcal{B}_t$  (where  $\mathcal{B}_t$  is standard Brownian motion) with speed  $\alpha^2$  and rate function given by the Dirichlet energy. The variable  $A^{-1}T_A$  can be written as  $\inf\{t : W_t + at = 1\}$  where  $W_t = \mathcal{B}_{At}/A$ , which has the same law as  $\sqrt{A^{-1}}\mathcal{B}_t$ . Clearly, among all functions  $\phi \in H_1([0, \infty))$  satisfying  $\phi(0) = 0$  and  $\inf\{t : \phi(t) + at = 1\} \leq \eta$ , the one with minimal Dirichlet energy is given by

$$\phi(t) = \begin{cases} (\frac{1}{\eta} - a)t, & t < \eta, \\ (\frac{1}{\eta} - a)\eta, & t \geq \eta. \end{cases}$$

By the contraction principle (Theorem 4.2.1 of [66]), the rate function desired in Lemma 4.3 is given by this minimal Dirichlet energy, i.e.,  $I(\eta) = \eta(\frac{1}{\eta} - a)^2/2$ . □

**Lemma 4.4** *Consider the following two part experiment. First choose  $T_A$  as above. Then toss a coin that comes up heads with probability*

$$e^{-2xT_A}.$$

*Then the probability that the coin comes up heads decays exponentially in  $A$  at rate  $\beta$  where  $\beta$  and  $x$  are related by*

$$\beta = \inf_{\eta} \{I(\eta) + 2x\eta\}, \tag{60}$$

*or equivalently by*

$$4x = \beta^2 + 2a\beta. \tag{61}$$

*Proof* The exponential decay with the exponent given in (60) is an immediate consequence of Varadhan’s integral lemma (Theorem 4.3.1 of [66]). To derive (61) from (60), we set the derivative of  $I(\eta) + 2x\eta$  to zero and find  $-\eta^{-2}/2 + a^2/2 + 2x = 0$ . Hence the minimum is achieved at

$$\eta_0 = (a^2 + 4x)^{-1/2}. \tag{62}$$

We then compute  $\beta = I(\eta_0) + 2x\eta_0$  to be

$$(a^2 + 4x)^{1/2}/2 - a + a^2(a^2 + 4x)^{-1/2}/2 + 2x(a^2 + 4x)^{-1/2}.$$

Simplifying, we have  $\beta = (a^2 + 4x)^{1/2} - a$ , which is equivalent to (61).  $\square$

*Proof of Theorem 4.2* As above, we aim to show that  $P\{\tilde{B}^\delta(z) \in \mathcal{X}\}$  scales as  $e^{-\beta A} = \delta^{\beta/\gamma} = \delta^\Delta$  where  $\Delta = \beta/\gamma$ , where  $\delta$  and  $\varepsilon$  are related via the stopping time  $T_A$  (54). Rescaling  $T_A$  by  $A^{-1}$  as in (59) puts us in the framework of large deviations Lemma 4.3. As above, to describe the probability  $P\{\tilde{B}^\delta(z) \in \mathcal{X}\}$  we can imagine that we first choose the radius  $\varepsilon$  of  $\tilde{B}^\delta(z)$  and then toss a coin that comes up heads with probability  $\varepsilon^{2x}$  to decide whether the ball is in  $\mathcal{X}$ . This puts us in the framework of the second large deviations Lemma 4.4. Using (61), we have

$$4x = \beta^2 + 2a\beta = (\gamma \Delta)^2 + 2a\gamma \Delta,$$

where  $a = Q - \gamma$ . Plugging in this value of  $a$  and simplifying, we obtain the KPZ relation

$$x = \frac{1}{4} \left( \gamma^2 \Delta^2 + 2\gamma(Q - \gamma)\Delta \right) = \frac{\gamma^2}{4} \Delta^2 + \left( 1 - \frac{\gamma^2}{4} \right) \Delta.$$

As in the previous proof, if the probability given  $\varepsilon$  is not exactly  $\varepsilon^{2x}$ , but the ratio of the log of this probability to the log of  $\varepsilon^{2x}$  tends to 1 as  $\varepsilon \rightarrow 0$ , we obtain the same theorem by using alternate values of  $x$  to give upper and lower bounds.  $\square$

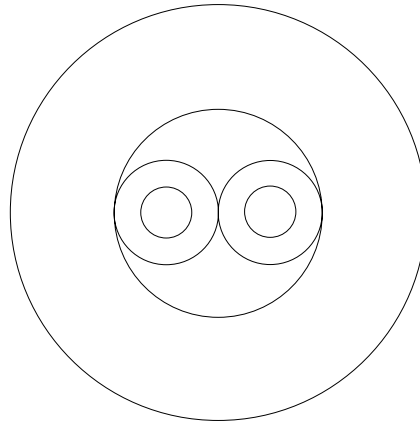
The optimum  $\eta_0 = (a^2 + 4x)^{-1/2}$  obtained in (62) has a natural interpretation—it suggests that (in the large deviations sense described above)  $T_A/A$  is concentrated near  $\eta_0$ .

Equivalently, since

$$\Delta = \frac{\beta}{\gamma} = \frac{(a^2 + 4x)^{1/2} - a}{\gamma},$$

we can say that  $A/T_A$  is concentrated near  $\gamma \Delta + a = \gamma \Delta + Q - \gamma$ , which implies that  $\frac{\log \delta}{\log \varepsilon}$  is concentrated near  $\gamma(\gamma \Delta + Q - \gamma)$ . Note that the same

**Fig. 4** The balls  $B_1(0)$ ,  $B_{1/2}(0)$ ,  $B_{1/4}(\pm 1/4)$ , and  $B_{\pm} := B_{1/8}(\pm 1/4)$



result can also be obtained directly from the explicit probability density (58). This is the concentration one obtains at an  $\alpha$ -thick point of the GFF  $h$ , where

$$\alpha = \gamma - \gamma \Delta. \tag{63}$$

Very informally, this suggests that the quantum support of a quantum fractal of dimension  $\Delta$  is made up of  $\alpha$ -thick points of  $h$ . This generalizes the idea of Proposition 3.4, which concerns the case  $\Delta = 0$ .

### 4.3 Tail estimates for quantum measure

**Lemma 4.5** *Let  $D = \mathbb{D} = B_1(0)$  be the unit disc and fix  $\gamma \in [0, 2)$  and take  $\mu = e^{\gamma h(z)} dz$  as defined previously. Then the random variable  $A = \log \mu(B_{1/2}(0))$  satisfies  $p_A(\eta) := \mathbb{P}[A < \eta] < e^{-C\eta^2}$  for some fixed constant  $C > 0$  and all sufficiently negative values of  $\eta$ .*

*Proof* Let  $h'$  be the projection of  $h$  onto the space of functions in  $H(\mathbb{D})$  that are harmonic inside the two discs  $B_{1/4}(1/4)$  and  $B_{1/4}(-1/4)$ . (See Fig. 4.) Recall that the orthogonal complement of this space is the space of functions supported on these discs, or more precisely, the space  $H[B_{1/4}(1/4) \cup B_{1/4}(-1/4)]$ . Hence, the law of  $h - h'$  is that of a sum of independent Gaussian free fields on  $B_{1/4}(1/4)$  and  $B_{1/4}(-1/4)$  with zero boundary conditions (see, e.g., [132]).

Let  $\underline{h}$  be the infimum of  $h'$  over the union of the two smaller discs  $B_- := B_{1/8}(-1/4)$  and  $B_+ := B_{1/8}(1/4)$ . Write  $A_- = \log \mu_{h-h'}(B_-)$  and  $A_+ = \log \mu_{h-h'}(B_+)$ . By Proposition 2.1 the law of each of  $A_+$  and  $A_-$  is the same as the law of  $A + \gamma Q \log(1/4) = A - \gamma Q \log 4$ ; clearly  $A_+$  and  $A_-$  are independent of one another. Also,  $\mu_h(B_+) \geq e^{\gamma \underline{h}} \mu_{h-h'}(B_+)$  (and similarly for  $B_-$ ), which implies

$$A \geq \max\{A_-, A_+\} + \gamma \underline{h}. \tag{64}$$

First we will show that the probability distribution of  $\underline{h}$  has superexponential decay. Since  $h'$  is harmonic on  $B_+$  (with  $h'(1/4) = h'_{1/8}(1/4)$ ) this  $h'$  is the real part of an analytic function on  $B_+$ . In particular,  $h'$  restricted to  $B_+$  can be expanded as  $h'(1/4) + \sum_{n=1}^\infty \text{Re} [a_n 4^n (z - 1/4)^n]$  for some complex  $a_n$ . Since each of the random variables  $\text{Re} a_n$  and  $\text{Im} a_n$  is a real-valued linear functional of  $h'$ , it is a Gaussian random variable. The variance of the latter can be estimated as follows.

Under the conformal map  $\varphi$  such that  $\varphi(1/4) = 0$  and  $\varphi(B_{1/4}(1/4)) = B_1(0)$ , the original domain  $\mathbb{D}$  is mapped onto a new domain  $D = \varphi(\mathbb{D})$ . Let us define on  $\mathbb{C}$  the set of real functions

$$\phi_n(z) := \text{Re} [\Phi_n(z)], \quad \psi_n(z) := \text{Im} [\Phi_n(z)], \tag{65}$$

$$\Phi_n(z) := \begin{cases} \bar{z}^n / (\pi n)^{1/2} & (|z| \leq 1), \\ z^{-n} / (\pi n)^{1/2} & (|z| \geq 1). \end{cases} \tag{66}$$

The functions  $\text{Re} z^n$  and  $\text{Im} z^n$  have on  $\mathbb{D}$  the Dirichlet energy

$$\int_{\mathbb{D}} n^2 |z^{n-1}|^2 dz = n^2 \int_0^1 r^{2n-2} 2\pi r dr = \pi n, \tag{67}$$

so that the set  $\{\phi_n, \psi_n\}$  obeys the orthogonality relations in  $\mathbb{D}$

$$(\phi_m, \phi_n)_{\mathbb{D}}^{\mathbb{D}} = (\psi_m, \psi_n)_{\mathbb{D}}^{\mathbb{D}} = \delta_{m,n}; \quad (\phi_m, \psi_n)_{\mathbb{D}}^{\mathbb{D}} = 0.$$

From the conformal invariance of the Dirichlet inner product, and from the expansion

$$h'(z) = h'(0) + \sum_{n=1}^\infty \text{Re} a_n \phi_n(z) + \text{Im} a_n \psi_n(z)$$

we obtain the explicit form of the coefficients  $a_n$

$$\text{Re} a_n = (h', \phi_n)_{\mathbb{D}}^{\mathbb{D}}, \quad \text{Im} a_n = (h', \psi_n)_{\mathbb{D}}^{\mathbb{D}},$$

where after the conformal map  $\varphi$ ,  $h'$  is now understood as the projection on  $\text{Harm } \mathbb{D}$  of the GFF  $h$  with zero boundary conditions on  $\partial D$ . This can be rewritten in the complex form

$$a_n = (h', \Phi_n)_{\mathbb{D}}^{\mathbb{D}} = (h, \Phi_n)_{\mathbb{D}}^{\mathbb{D}},$$

where use was made of the orthogonality of  $h - h'$  and  $\Phi_n$ . By inversion with respect to the unit circle  $\partial \mathbb{D}$ , this is also  $(h, \Phi_n)_{\mathbb{D}}^{\mathbb{C} \setminus \mathbb{D}}$ , so that

$$a_n = \frac{1}{2} (h, \Phi_n)_{\mathbb{D}}^{\mathbb{C}},$$

where now the Dirichlet inner product extends to the whole plane. Since  $h$  vanishes outside of  $D$ , we can also write

$$(h, \Phi_n)_{\nabla}^{\mathbb{C}} = (h, \tilde{\Phi}_n)_{\nabla}^D$$

where  $\tilde{\Phi}_n := \Phi_n - \Phi_n^H$ ,  $\Phi_n^H$  being the harmonic extension to  $D$  of  $\Phi_n$  restricted to  $\partial D$ . Specifying this separately for the real and imaginary parts of  $a_n$  and  $\Phi_n$ , we have

$$\operatorname{Re} a_n = \frac{1}{2}(h, \tilde{\phi}_n)_{\nabla}^D, \quad \operatorname{Im} a_n = \frac{1}{2}(h, \tilde{\psi}_n)_{\nabla}^D,$$

where  $\tilde{\phi}_n := \phi_n - \phi_n^H$ , with a similar definition for the imaginary component. Since  $\tilde{\phi}_n$  is in  $H(D)$ , we have

$$\operatorname{Var} \operatorname{Re} a_n = \frac{1}{4} \operatorname{Var}(h, \tilde{\phi}_n)_{\nabla}^D = \frac{1}{4}(\tilde{\phi}_n, \tilde{\phi}_n)_{\nabla}^D,$$

together with an entirely similar expression for  $\operatorname{Var} \operatorname{Im} a_n$ . We now wish to argue that

$$(\|\tilde{\phi}_n\|_{\nabla}^D)^2 \leq (\|\tilde{\phi}_n\|_{\nabla}^{\mathbb{C}})^2 \leq (\|\phi_n\|_{\nabla}^{\mathbb{C}})^2 = \frac{2}{\pi n}.$$

The first inequality is obvious, while the second one is a consequence of the orthogonal decomposition  $\phi_n = \tilde{\phi}_n + \phi_n^H$  on  $\mathbb{C}$ . We thus conclude that the variances  $\operatorname{Var} \operatorname{Re} a_n$  and  $\operatorname{Var} \operatorname{Im} a_n$  are at most  $1/(2\pi n)$ .

In particular, the variance of  $|a_n|r^n$ , for any fixed  $r < 1$ , will decay exponentially in  $n$ . Thus, the probability that even one of the  $a_n$  satisfies  $|a_n|r^n > c$ , where  $c$  is a fixed constant, decays quadratic-exponentially in  $c$ . It follows that the probability distribution function  $\underline{p}$  of  $\underline{h}$  satisfies  $\underline{p}(\eta) := \mathbb{P}(\underline{h} < \eta) < e^{-\underline{C}\eta^2}$  for some  $\underline{C} > 0$  and all sufficiently negative  $\eta$ .

Now, let  $P_1(\eta)$  be the probability that  $\underline{h} < .1\eta/\gamma$  and  $A < \eta$ . Let  $P_2(\eta)$  be the probability that  $A < \eta$  and  $\underline{h} \geq .1\eta/\gamma$ . Then  $p_A(\eta) = \mathbb{P}[A < \eta] = P_1 + P_2$ . From the above discussion, we have  $P_1(\eta) \leq e^{-\underline{C}\eta^2}$  for all sufficiently negative values of  $\eta$ . Note from (64) that

$$P_2(\eta) \leq [p_A(.9\eta + \gamma Q \log 4)]^2$$

and

$$\begin{aligned} P_2(\eta) &\leq [P_1(.9\eta + \gamma Q \log 4) + P_2(.9\eta + \gamma Q \log 4)]^2 \\ &\leq [e^{-C'\eta^2} + P_2(.9\eta + \gamma Q \log 4)]^2, \end{aligned}$$



for some  $C'$ . Fix a sufficiently negative  $\eta_0$  and inductively determine  $\eta_k$  via  $\eta_{k-1} = .9\eta_k + \gamma Q \log 4$ . The above can be stated as

$$P_2(\eta_k) \leq \left( e^{-C'\eta_k^2} + P_2(\eta_{k-1}) \right)^2.$$

If we write  $p_k = \frac{P_2(\eta_k)}{e^{-2C'\eta_k^2}}$ , then this can be restated as  $p_k \leq (1 + p_{k-1}e^{-C'(2\eta_{k-1}^2 - \eta_k^2)})^2$ . It is easy to see that we can have  $p_k > 2$  for only finitely many  $k$ , which implies that the lemma holds for  $C < \min\{C, 2C'\}$ , when restricted to the sequence  $\eta_k$ . Because of the monotonicity of  $p_A(\eta)$ , this implies the lemma for all  $\eta$ .  $\square$

**Lemma 4.6** Fix  $z$  and  $\varepsilon$  so that  $B_\varepsilon(z) \subset D$ . Then

$$\mathbb{E}[\mu(B_\varepsilon(z)) | h_\varepsilon(z)] \simeq \pi \varepsilon^\gamma Q e^{\gamma h_\varepsilon(z)} = \mu_\odot(B_\varepsilon(z)),$$

where

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2},$$

as in Proposition 2.1. Moreover, conditioned on  $h_{\varepsilon'}(z)$ , for all  $\varepsilon' \geq \varepsilon$ , we have that

$$\mathbb{P} \left[ \frac{\mu(B_\varepsilon(z))}{\mu_\odot(B_\varepsilon(z))} < e^\eta \right] \leq C_1 e^{-C_2 \eta^2}, \tag{68}$$

for some positive constants  $C_1$  and  $C_2$  independent of  $\eta \leq 0$ ,  $z$ ,  $D$ , and the values  $h_{\varepsilon'}(z)$  for  $\varepsilon' \geq \varepsilon$ .

*Proof* The first sentence is a restatement of (47) and (48). It remains to prove the second half. For a fixed  $\varepsilon$ , we want to show that the probability that

$$A := \log \frac{\mu(B_\varepsilon(z))}{\pi \varepsilon^\gamma Q e^{\gamma h_\varepsilon(z)}} \leq \eta, \quad \eta \leq 0,$$

decays quadratic-exponentially in  $\eta$ . Let  $\underline{A}$  be the infimum of  $\gamma(\tilde{h}(\cdot) - h_\varepsilon(z))$  on  $B_{\varepsilon/2}(z)$ , where  $\tilde{h}$  is  $h$  projected onto  $\text{Harm } B_\varepsilon(z)$ . With these definitions, one easily sees that

$$A' \leq A - \underline{A},$$

where  $A' := \log[\mu_{h-\tilde{h}}(B_{\varepsilon/2}(z))/\pi \varepsilon^\gamma Q]$ . In this proof, we let  $\mathbb{P}'$  denote probability conditioned on  $z$  and on the map  $\chi : [\varepsilon, \varepsilon_0^z] \rightarrow \mathbb{R} : \chi(\varepsilon') := h_{\varepsilon'}(z)$ .

One may use the techniques in the proof of Lemma 4.5 to show that  $\mathbb{P}'[\underline{A} < \eta]$  (which is a priori a function of  $\chi$ ) decays quadratic-exponentially

in  $\eta$ , uniformly in  $\chi$  (and hence in  $h_\varepsilon(z)$ ). Then we have by the above construction  $\mathbb{P}'[A - \underline{A} < \eta] \leq \mathbb{P}[A' < \eta]$ , so that Lemma 4.5 applied to  $p_{A'}(\eta) = \mathbb{P}[A' < \eta]$  implies that  $\mathbb{P}'[A - \underline{A} < \eta]$  decays quadratic-exponentially in  $\eta$ , also uniformly in  $\chi$ . We conclude that the probability that either  $\underline{A} < \eta/2$  or  $A - \underline{A} < \eta/2$  decays quadratic-exponentially in  $\eta$ , and the claim follows.  $\square$

Roughly speaking, the above lemma says that the total quantum area in a ball is unlikely to be a lot smaller than the area we would predict given the average value of  $h$  on the boundary of that ball; the following says that (even when we use the  $\Theta$  measure), the total quantum area has some constant probability to be (at least a little bit) smaller than this prediction.

**Lemma 4.7** *Let  $z$  and  $h$  be chosen from  $\Theta^{\tilde{D}}$  (as defined in Sect. 3.3) for a fixed compact subset  $\tilde{D}$  of  $D$ , and fix a  $\delta > 0$ , with quantum balls  $B^\delta(z)$  and  $\tilde{B}^\delta(z)$  defined as in Definitions 1.3 and 4.1. (Definition 4.1 implicitly makes use of a constant  $\varepsilon_0$ , which we take here to be  $\sup\{\varepsilon' : B_{\varepsilon'}(\tilde{D}) \subset D\}$ .)*

*Conditioned on the radius of  $\tilde{B}^\delta(z)$ , the conditional probability that  $\tilde{B}^\delta(z) \subset B^\delta(z)$  is bounded below by a positive constant  $c$  independent of  $D$ ,  $\tilde{D}$ , and  $\delta$ .*

*Proof* Define  $\varepsilon$  to be the radius of  $\tilde{B}^\delta(z)$ . Let  $\mathbb{P}'$  be  $\Theta^{\tilde{D}}$  probability conditioned on  $z$  and on the map  $\chi : [\varepsilon, \varepsilon_0] \rightarrow \mathbb{R} : \chi(\varepsilon') := h_{\varepsilon'}(z)$ . As before we assume  $\varepsilon$  is less than the distance  $\varepsilon_0$  from  $\tilde{D}$  to  $\partial D$ . It now suffices for us to show that

$$\mathbb{P}'\left(\tilde{B}^\delta(z) \subset B^\delta(z)\right) = \mathbb{P}'\left(\mu(\tilde{B}^\delta(z)) < \delta\right)$$

is bounded below independently of  $\chi$  and  $z$ .

Consider now the map  $\bar{\chi} : (0, \varepsilon_0] \rightarrow \mathbb{R}$  given by  $\bar{\chi}(\varepsilon') = h_{\varepsilon'}(z)$  for all  $\varepsilon' \in (0, \varepsilon_0]$ . Let  $h^{\bar{\chi}}$  denote the conditional expectation of  $h$  (in the standard GFF probability measure) given  $\bar{\chi}$ . Clearly  $h^{\bar{\chi}}(y)$  is a.s. radially symmetric, with center  $z$ , and equals zero for  $\varepsilon_0 < |y - z|$ . It corresponds to the projection of  $h$  onto the space of functions with these properties. (We similarly define  $h^\chi$ , so that  $h^\chi$  is constant in  $B_\varepsilon(z)$  and coincides with  $h^{\bar{\chi}}$  outside.) Note that, given  $z$ , the  $\Theta^{\tilde{D}}$  law of  $h^{\bar{\chi}}$  is that of  $\tilde{h}^{\bar{\chi}}$  (where  $\tilde{h}$  is a standard GFF) plus a deterministic function with the same properties: the function

$$\zeta(y) := \gamma[\xi_0^z(y) - \tilde{\xi}_{\varepsilon_0}^z(y)] = \begin{cases} -\gamma \log \frac{|y-z|}{\varepsilon_0}, & y \in B_{\varepsilon_0}(z), \\ 0, & \text{otherwise.} \end{cases}$$

Although  $h^{\bar{\chi}}$  is a projection onto an infinite dimensional space, it is not hard to see (e.g., by approximating with finite dimensional spaces) that the obvious

analog of (2) in Proposition 1.2 still holds, i.e., taking expectation with respect to  $\Theta^{\tilde{D}}$  we have

$$\mathbb{E}[\mu(A)|\bar{\chi}] = \mu^{\bar{\chi}}(A),$$

where

$$\mu^{\bar{\chi}} := \exp\left(\gamma h^{\bar{\chi}}(y) + \frac{\gamma^2}{2} \log \frac{|y-z|}{\varepsilon_0} + \frac{\gamma^2}{2} \log C(y; D)\right) dy,$$

for  $|y-z| \leq \varepsilon_0$ , and in this range we have for some  $C_0 \geq 1$  (depending on  $\tilde{D}$  and  $D$ ) that

$$C_0^{-1} \mu_0^{\bar{\chi}} \leq \mu^{\bar{\chi}} \leq C_0 \mu_0^{\bar{\chi}}, \tag{69}$$

where

$$\mu_0^{\bar{\chi}} := \exp\left(\gamma h^{\bar{\chi}}(y) + \frac{\gamma^2}{2} \log \frac{|y-z|}{\varepsilon_0}\right) dy. \tag{70}$$

The fact that  $\mu^{\bar{\chi}}$  and hence  $\mu_0^{\bar{\chi}}$  is almost surely finite follows from the fact that it is a conditional expectation of  $\mu = \mu_h$  with respect to  $\Theta^{\tilde{D}}$ , and  $\mu$  is  $\Theta^{\tilde{D}}$  almost surely finite. It can also be seen directly from (70), using the same argument as in the proof of Proposition 1.2: note first that (70) becomes integrable for  $\gamma \in [0, 2)$  if  $h^{\bar{\chi}}(y)$  is replaced with its expectation  $\zeta(y)$ , and second that  $|h^{\bar{\chi}}(y) - \zeta(y)|$  a.s. does not grow too quickly as  $y \rightarrow z$ .

Note that  $h_\varepsilon^{\bar{\chi}}(z) = h_\varepsilon^\chi(z) = h^\chi(z) = h_\varepsilon(z) - h_{\varepsilon_0}(z)$ . From definition (48) and from (53) and the definition (4.1) of  $\tilde{B}^\delta(z) = B_\varepsilon(z)$ , we have that

$$\left(\frac{\varepsilon}{\varepsilon_0}\right)^2 \exp\left(\gamma h_\varepsilon^\chi(z) + \frac{\gamma^2}{2} \log \frac{\varepsilon}{\varepsilon_0}\right) = \delta$$

and that

$$\delta^{-1} \mu_0^{\bar{\chi}}(\tilde{B}^\delta(z)) \tag{71}$$

is a random variable independent of  $\varepsilon$  and  $\chi$ . (It depends only on the Brownian process given by  $\tilde{B}_s := \mathcal{B}_{s+t} - \mathcal{B}_t$  defined for  $s \geq 0$ , where  $\mathcal{B}_{s'} := h_{e^{-s'}}^{\bar{\chi}}(z) - \gamma s'$  and  $t = -\log(\varepsilon/\varepsilon_0)$ . Note that  $\tilde{B}_s$  is a standard Brownian motion in  $s \geq 0$ , independent of  $\varepsilon, z$ , and  $\chi$ .)

It is not hard to see that this random variable is not bounded below by any number greater than zero; thus there is an event—call it  $\mathcal{A}$ —independent of  $\chi$ , and occurring with a probability  $\mathbb{P}'(\mathcal{A})$  bounded below by some  $c' > 0$ , on which (71) is less than a small number, say 1/100 (indeed, we may assume the same holds with  $\mu^{\bar{\chi}}$  replacing  $\mu_0^{\bar{\chi}}$ , because of (69)). Given this claim, it follows that on the event  $\mathcal{A}$ , one has  $\mathbb{E}[\mu(\tilde{B}^\delta(z))|\bar{\chi}] = \mu^{\bar{\chi}}(\tilde{B}^\delta(z)) < \delta/100$ , so

that the conditional probability that  $\mu(\tilde{B}^\delta(z)) < \delta$  is at least .99. The lemma follows using the constant  $c = .99c'$ .  $\square$

#### 4.4 Proof of interior KPZ

In this section we derive Theorem 1.5 as a consequence of Lemmas 4.6, 4.7, and the arguments in Theorem 4.2.

*Proof of Theorem 1.5* We use the same notation as in Theorem 4.2, but we write  $\bar{T}_A = -\log(\bar{\varepsilon}/\varepsilon_0)$  where  $\bar{\varepsilon}$  is the radius of  $B^\delta(z)$ . In this proof, we use the probability measure  $\Theta^{\tilde{D}}$  and  $\mathbb{E}$  denotes expectation with respect to  $\Theta^{\tilde{D}}$ . The proof of Theorem 4.2 carries through exactly once we show that (when  $h$  is chosen from  $\Theta^{\tilde{D}}$ )

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E}[\exp(-2x\bar{T}_A)]}{\log \mathbb{E}[\exp(-2xT_A)]} = 1, \tag{72}$$

since this implies the analog of (57) with  $T_A$  replaced by  $\bar{T}_A$ .

Note that the numerator of (68) is related to  $\bar{T}_A$  while the denominator is related to  $T_A$ ; if the numerator and denominator were precisely equal for all  $\varepsilon$ , we would have  $\bar{T}_A = T_A$ .

For any  $a, 0 < a < 1$ , let  $\varepsilon_a$  be the value for which  $B_{\varepsilon_a}(z) = \tilde{B}^{\delta^a}(z)$ . Then  $\mu_\odot(B_{\varepsilon_a}(z)) = \delta^a \mu_\odot(B_{\varepsilon_0}(z))$ . This corresponds to a stopping time  $T_{aA} = -\log(\varepsilon_a/\varepsilon_0)$ .

On the event  $\bar{T}_A < T_{aA}$ , we have  $\varepsilon_a < \bar{\varepsilon}$  so that  $\mu(B_{\varepsilon_a}(z)) \leq \mu(B_{\bar{\varepsilon}}(z)) = \delta$ . It follows that

$$\mu(B_{\varepsilon_a}(z)) / \mu_\odot(B_{\varepsilon_a}(z)) \leq \delta^{1-a} / \mu_\odot(B_{\varepsilon_0}(z)).$$

Thanks to definition (48), the probability that  $\mu_\odot(B_{\varepsilon_0}(z)) \leq \delta^{(1-a)/2}$  decays quadratic-exponentially in  $A = -\log(\delta/\gamma)$  when  $\delta \rightarrow 0$ . On the event of the contrary,  $\mu_\odot(B_{\varepsilon_0}(z)) > \delta^{(1-a)/2}$ , one then has  $\mu(B_{\varepsilon_a}(z)) / \mu_\odot(B_{\varepsilon_a}(z)) < \delta^{(1-a)/2}$ , whose probability, by Lemma 4.6 applied for  $B_{\varepsilon_a}(z)$  and  $\eta = -\gamma A(1-a)/2$ , also decays quadratic-exponentially in  $A$ . This implies that the probability that  $\bar{T}_A < T_{aA}$  decays superexponentially in  $A$ . This implies that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E}[\exp(-2x\bar{T}_A)]}{\log \mathbb{E}[\exp(-2xT_{aA})]} \leq 1.$$

Since this holds for all  $a < 1$ , it follows immediately from the continuity of the coefficient of  $A$  in the exponent in (57) that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E}[\exp(-2x\bar{T}_A)]}{\log \mathbb{E}[\exp(-2xT_A)]} \leq 1.$$

From Lemma 4.7, it follows that, conditioned on  $T_A$ , the  $\Theta^{\tilde{D}}$  probability that  $\overline{T}_A < T_A$  is at least  $c > 0$ , which implies

$$c \mathbb{E}[\exp(-2xT_A)] \leq \mathbb{E}[\exp(-2x\overline{T}_A)]$$

for any  $x \geq 0$ , which in turn implies the equivalence of logarithms in (72).  $\square$

### 5 Box formulation of KPZ

In this section we prove Proposition 1.6.

*Proof of Proposition 1.6* The first fact is standard; simply observe that if  $\varepsilon$  is a power of 2 then  $S_\varepsilon(X) \subset B_{2\varepsilon}(X)$ , hence  $\mu_0(S_\varepsilon(X)) \leq \mu_0(B_{2\varepsilon}(X))$ , since the ball of radius  $2\varepsilon$  about a point contains any diadic box of width  $\varepsilon$  that contains the same point. Similarly,  $B_{2\varepsilon}(z)$  is contained in the union of a diadic box—of width  $2\varepsilon$ , containing  $z$ —with the eight diadic boxes of the same size whose boundaries touch its boundary. This implies that  $B_{2\varepsilon}(X)$  is contained in the union of  $S_{2\varepsilon}(X)$  and corresponding 8 translations of  $S_{2\varepsilon}(X)$ , so  $\mu_0(B_{2\varepsilon}(X)) \leq 9\mu_0(S_{2\varepsilon}(X))$ .

For the second part, we first argue that  $X$  has quantum scaling exponent  $\Delta$  if and only if (4) holds. We use the notation in the proof Theorem 4.2 but set  $\tilde{T}_A$  to be  $-\log(\tilde{\varepsilon}/\varepsilon_0)$ , where  $\tilde{\varepsilon}$  is the largest value of  $\varepsilon$  for which the diadic box  $S_\varepsilon(z)$  with edge length  $\varepsilon$  has  $\mu$  area at most  $\delta$ . The remainder of the argument is essentially the same as the proof of Theorem 1.5. Just as (72) was sufficient in that case, it is enough for us to verify that when  $h$  is chosen from  $\Theta^{\tilde{D}}$  and  $X$  is chosen independently, we have the following analog of (72) (where  $\overline{T}_A$  is replaced by  $\tilde{T}_A$ ):

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E}[\exp(-2x\tilde{T}_A)]}{\log \mathbb{E}[\exp(-2xT_A)]} = 1. \tag{73}$$

The proof is essentially the same as the proof of (72), but we will sketch the differences here. As in the proof of (72) one argues first that the probability that  $\tilde{T}_A < T_{aA}$ , with  $0 < a < 1$ , decays superexponentially in  $A$  and by continuity when  $a \rightarrow 1$  concludes that

$$\lim_{A \rightarrow \infty} \frac{\log \mathbb{E}[\exp(-2x\tilde{T}_A)]}{\log \mathbb{E}[\exp(-2xT_A)]} \leq 1.$$

The only difference is that one has to first obtain a modified Lemma 4.6, in which the  $\mu(B_\varepsilon(z))$  in (68) is replaced with  $\mu(S_{\varepsilon/2}(z))$ ; this straightforward exercise is left to the reader. Then, using the same notation as in the proof of

Theorem 1.5, one chooses  $\varepsilon_a$  so that  $B_{\varepsilon_a}(z) = \tilde{B}^{\delta^a}(z)$ . Then  $\mu_{\odot}(B_{\varepsilon_a}(z)) = \delta^a \mu_{\odot}(B_{\varepsilon_0}(z))$ . On the event  $\tilde{T}_A < T_{aA}$ , one has  $\varepsilon_a < \tilde{\varepsilon}$ , so that  $\mu(S_{\varepsilon_a/2}(z)) \leq \mu(S_{\tilde{\varepsilon}}(z)) < \delta$ , from which it follows that

$$\mu(S_{\varepsilon_a/2}(z)) / \mu_{\odot}(B_{\varepsilon_a}(z)) \leq \delta^{1-a} / \mu_{\odot}(B_{\varepsilon_0}(z)).$$

The discussion then continues identically, depending on whether  $\mu_{\odot}(B_{\varepsilon_0}(z)) \leq \delta^{(1-a)/2}$  holds, the probability of which has superexponential decay in  $A = -(\log \delta) / \gamma$ , or the contrary, which also has superexponential decay by application of the modified Lemma 4.6 to the resulting inequality

$$\mu(S_{\varepsilon_a/2}(z)) / \mu_{\odot}(B_{\varepsilon_a}(z)) \leq \delta^{(1-a)/2}.$$

Next, as in the proof of (72), one argues that  $\mathbb{P}'[\tilde{T}_A < T_A + \log \rho] \geq c > 0$ , for some fixed constant  $\rho \geq 4$ , which implies

$$c \rho^{-2x} \mathbb{E}[\exp(-2xT_A)] \leq \mathbb{E}[\exp(-2x\tilde{T}_A)]$$

for any  $x \geq 0$ , which in turn implies (73). The difference here is that one must first obtain a modified version of Lemma 4.7 in which the event  $\tilde{B}^{\delta}(z) \subset B^{\delta}(z)$  is replaced with the event that  $S^{\delta}(z) = S_{\tilde{\varepsilon}}(z)$  has a width  $\tilde{\varepsilon}$  larger than a fixed constant  $\rho^{-1}$  times the radius  $\varepsilon$  of  $\tilde{B}^{\delta}(z) = B_{\varepsilon}(z)$ , which can be easily proven as follows. First, recall that by definition of  $\tilde{\varepsilon}$ ,  $\mu(S_{\tilde{\varepsilon}}(z)) < \delta \leq \mu(S_{2\tilde{\varepsilon}}(z))$ . We thus have

$$\mu(B_{\tilde{\varepsilon}}(z)) = \delta \leq \mu(S_{2\tilde{\varepsilon}}(z)) \leq \mu(B_{4\tilde{\varepsilon}}(z)),$$

hence  $\bar{\varepsilon} \leq 4\tilde{\varepsilon}$ . From Lemma 4.7 there is a finite probability  $c$  that  $\tilde{B}^{\delta}(z) \subset B^{\delta}(z)$ , i.e., that  $\varepsilon \leq \bar{\varepsilon}$ , hence that  $\varepsilon \leq 4\tilde{\varepsilon}$ , which proves the modified version of Lemma 4.7 for  $\rho \geq 4$ .

Next, we observe that the above arguments still work if we replace the  $S^{\delta}(z)$  in (4) with  $\hat{S}^{\delta}(z)$ , defined to be the diadic parent of  $S^{\delta}(z)$ —this only changes  $\tilde{T}_A$  by an additive constant. Thus (4) is equivalent to the analog of (4) in which  $S^{\delta}(z)$  is replaced with  $\hat{S}^{\delta}(z)$ . Now define  $\hat{N}$  analogously to  $N$  (counting  $\hat{S}^{\delta}(z)$  squares instead of  $S^{\delta}(z)$  squares). We obtain the equivalence of (4) and (5) by observing that

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\mu(S^{\delta}(X))]}{\log \delta} &\leq \lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\delta N(\mu, \delta, X)]}{\log \delta} \\ &\leq \lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\delta \hat{N}(\mu, \delta, X)]}{\log \delta} \\ &\leq \lim_{\delta \rightarrow 0} \frac{\log \mathbb{E}[\mu(\hat{S}^{\delta}(X))]}{\log \delta}. \end{aligned}$$

The first and last inequalities are true because, by definition,  $\mu(S^\delta(z)) \leq \delta$  and  $\mu(\hat{S}^\delta(z)) \geq \delta$ . The middle inequality is true because  $N(\mu, \delta, X) \leq 4\hat{N}(\mu, \delta, X)$ . □

## 6 Boundary KPZ

### 6.1 Boundary semi-circle average

Most of the results in this paper about random measures on  $D$  have straightforward analogs about random measures on  $\partial D$ . The proofs are essentially identical, but we will sketch the differences in the arguments here.

Suppose that  $D$  is a domain with piecewise linear boundary or a domain with a smooth boundary containing a linear piece  $\partial D \subset \partial D$  and that  $h$  is an instance of the GFF on  $D$  with free boundary conditions, normalized to have mean zero.

This means that  $h = \sum_n \alpha_n f_n$  where the  $\alpha_n$  are i.i.d. zero mean unit variance normal random variables and the  $f_n$  are an orthonormal basis, with respect to the inner product

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

of the Hilbert space closure  $H(D)$  of the space of  $C^\infty$  bounded real-valued (but *not* necessarily compactly supported) functions on  $D$  with mean zero.

Note that if  $f$  is a compactly supported smooth function on  $D$  for which  $-\Delta f = \rho$ , then integration by parts implies that the variance of  $(h, \rho)$  is the Dirichlet energy of  $f$ —same as in the zero boundary case. Similarly, suppose that  $f$  is a smooth function that is *not* compactly supported but has a gradient that vanishes in the normal direction to  $\partial D$ , and we write  $\rho = -\Delta f$ . Then integration by parts implies that the variance of  $(h, \rho)$  is  $2\pi(f, f)_\nabla$ .

We can also make sense of  $h_\varepsilon(z)$ , for a point  $z$  on a linear part  $\partial D$  of the boundary of  $D$ , to be the mean value of  $h$  on the semicircle of radius  $\varepsilon$  centered at  $z$  and contained in the domain  $D$ . For  $z$  fixed, let  $\varepsilon_0$  be chosen small enough so that  $B_{\varepsilon_0}(z) \cap \partial D \subset \partial D$  and exactly one semi-disc of  $B_{\varepsilon_0}(z)$  lies in  $D$ . Define for any  $\varepsilon \leq \varepsilon_0$

$$h_\varepsilon(z) = (h, \hat{\rho}_\varepsilon^z),$$

where  $\hat{\rho}_\varepsilon^z(y) dy$  is the uniform measure (of total mass one) localized on the semicircle  $\partial B_\varepsilon(z) \cap D$ . Let us introduce the function  $\hat{\xi}_\varepsilon^z(y)$ , for  $y \in D$ ,

such that

$$-\Delta \hat{\xi}_\varepsilon^z = 2\pi (\hat{\rho}_\varepsilon^z - 1/|D|), \tag{74}$$

$$n \cdot \nabla \hat{\xi}_\varepsilon^z|_{\partial D} = 0, \quad \int_D \hat{\xi}_\varepsilon^z dy = 0, \tag{75}$$

with  $n$  the current normal to  $\partial D$ , and  $|D| := \int_D dy$  the area of  $D$ . Hence,  $\hat{\xi}_\varepsilon^z$  satisfies *Neumann* boundary conditions and has zero mean, and integration by parts shows that

$$h_\varepsilon(z) = (h, \hat{\xi}_\varepsilon^z)_\nabla.$$

Let us introduce the auxiliary function

$$\zeta_\varepsilon^z(y) := -2 \log(|y - z| \vee \varepsilon) + \frac{\pi}{2|D|} (|y - z|^2 + \varepsilon^2), \tag{76}$$

such that  $-\Delta \zeta_\varepsilon^z(\cdot) = 2\pi(\hat{\rho}_\varepsilon^z(\cdot) - 1/|D|)$ . The  $2 \log(|\cdot - z| \vee \varepsilon)$  in place of  $\log(|\cdot - z| \vee \varepsilon)$  comes from the fact that  $\hat{\rho}_\varepsilon^z$  is a unit mass measure over half a circle.

The solution  $\hat{\xi}_\varepsilon^z$  to (74) and (75) is then given by

$$\hat{\xi}_\varepsilon^z = \zeta_\varepsilon^z - \hat{G}_z, \tag{77}$$

where  $\hat{G}_z$  is the harmonic function in  $D$ , solution to the Neumann problem (75) on  $\partial D$ . Note that the function  $\zeta_\varepsilon^z$  (76) has been chosen such that both the boundary normal derivative  $n \cdot \nabla \zeta_\varepsilon^z|_{\partial D}$  and the integral  $\int_D \zeta_\varepsilon^z dy$  are actually *independent* of  $\varepsilon$  for  $\varepsilon \leq \varepsilon_0$ . The normal derivative vanishes on the linear boundary component  $\underline{\partial D}$ :  $n \cdot \nabla \zeta_\varepsilon^z|_{\underline{\partial D}} = 0$ . Therefore  $\hat{G}_z$  is independent of  $\varepsilon$ , and satisfies the Neumann condition on  $\underline{\partial D}$ :  $n \cdot \nabla \hat{G}_z|_{\underline{\partial D}} = 0$ . By the Schwarz reflection principle, this allows extending  $\hat{G}_z$  to a harmonic function in the domain  $\bar{D}$ , complex conjugate and symmetrical of  $D$  with respect to  $\underline{\partial D} \subset \mathbb{R}$ , through  $\hat{G}_z(\bar{y}) = \hat{G}_z(y)$ .

When considering the reference radius  $\varepsilon_0$ , we then have that

$$h_\varepsilon(z) - h_{\varepsilon_0}(z) = (h, \hat{\xi}_\varepsilon^z - \hat{\xi}_{\varepsilon_0}^z)_\nabla = (h, \zeta_\varepsilon^z - \zeta_{\varepsilon_0}^z)_\nabla. \tag{78}$$

Thus  $h_\varepsilon(z) - h_{\varepsilon_0}(z)$  is equal to  $(h, \hat{\zeta})_\nabla$ , where  $\hat{\zeta} := \zeta_\varepsilon^z - \zeta_{\varepsilon_0}^z$  (up to a constant) is the continuous function to  $-2 \log|\cdot - z|$  on the half-annulus  $\mathbb{H} \cap \{y : \varepsilon \leq |y - z| \leq \varepsilon_0\}$  and is constant outside of the half-annulus. The variance of  $h_\varepsilon(z) - h_{\varepsilon_0}(z)$  is then given by the Dirichlet energy  $(\hat{\zeta}, \hat{\zeta})_\nabla = -2 \log(\varepsilon/\varepsilon_0)$ . We thus have that the Gaussian random variable  $h_\varepsilon(z) - h_{\varepsilon_0}(z)$  is a standard Brownian motion  $\mathcal{B}_{2t}$  in time  $2t = -2 \log(\varepsilon/\varepsilon_0)$ , with boundary condition  $\mathcal{B}_0 = 0$ , as in Proposition 3.3.



Thanks to (74) to (77), the set of functions  $\hat{\xi}_\varepsilon^z$  has Dirichlet inner products

$$(\hat{\xi}_\varepsilon^z, \hat{\xi}_{\varepsilon'}^z)_\nabla = -2 \log(\varepsilon \vee \varepsilon') + \frac{\pi}{2|D|}(\varepsilon^2 + \varepsilon'^2) - \hat{G}_z(z); \tag{79}$$

one finds in particular for  $\varepsilon' = 0$  that  $(\hat{\xi}_\varepsilon^z, \hat{\xi}_0^z)_\nabla = \hat{\xi}_\varepsilon^z(z)$ . At a boundary point  $z \in \partial D$ , the variance of  $h_\varepsilon(z)$  is

$$\text{Var } h_\varepsilon(z) = (\hat{\xi}_\varepsilon^z, \hat{\xi}_\varepsilon^z)_\nabla = -2 \log \varepsilon + \frac{\pi}{|D|}\varepsilon^2 - \hat{G}_z(z); \tag{80}$$

this variance thus scales for  $\varepsilon$  small like  $-2 \log \varepsilon$  instead of  $-\log \varepsilon$ , because of the free boundary conditions on  $\partial D$ .

### 6.2 Mixed boundary conditions

Notice that one can also consider other types of boundary conditions for the Gaussian free field  $h$ , like *mixed* boundary conditions in domain  $D$ , with *free* boundary conditions on a linear component  $\partial D \subset \mathbb{R}$ , and *Dirichlet* ones on its complement  $\partial D \setminus \partial D$ . In this case, one uses a reflection principle and considers the whole domain  $D^\dagger := D \cup \bar{D}$ , where  $\bar{D}$  is the complex conjugate of  $D$ , symmetrical of  $D$  with respect to the real axis, and takes Dirichlet boundary conditions on  $\partial D^\dagger$ . The Hilbert space closure  $H(D^\dagger)$  of the space of  $C^\infty$  real-valued functions compactly supported on  $D^\dagger$  can be written as the direct sum  $H_e(D^\dagger) \oplus H_o(D^\dagger)$  of the Hilbert space closures corresponding to *even* and *odd* functions on  $D^\dagger$  with respect to the real line supporting  $\partial D$ . The Gaussian free field  $h$  in  $D$ , with mixed boundary conditions on  $\partial D$ , is then simply obtained by projecting the GFF in  $D^\dagger$  onto the even space  $H_e(D^\dagger)$ , and restricting the result to  $D$ .

It is not hard to see that the semi-circle average  $h_\varepsilon(z)$  of  $h$ , for  $z \in \partial D$  and  $\varepsilon \leq \varepsilon_0$ , is then given by  $(h, \tilde{\xi}_\varepsilon^z)_\nabla$ , where  $\tilde{\xi}_\varepsilon^z(y) = -2 \log(|y - z| \vee \varepsilon) - \tilde{G}_z(y)$ , with now  $\tilde{G}_z(y)$  the harmonic extension to  $D^\dagger$  (here restricted to  $y \in D$ ) of the restriction of the function  $-2 \log |y - z|$  to  $y \in \partial D^\dagger$ . These functions have Dirichlet inner products

$$(\tilde{\xi}_\varepsilon^z, \tilde{\xi}_{\varepsilon'}^z)_\nabla = -2 \log(\varepsilon \vee \varepsilon') - \tilde{G}_z(z), \tag{81}$$

in place of (79). Similarly, at a boundary point  $z \in \partial D$ , the variance of  $h_\varepsilon(z)$  is

$$\text{Var } h_\varepsilon(z) = (\tilde{\xi}_\varepsilon^z, \tilde{\xi}_\varepsilon^z)_\nabla = -2 \log \varepsilon - \tilde{G}_z(z), \tag{82}$$

instead of (80). Lastly, exactly as in the case of free boundary conditions, the Gaussian random variable  $h_\varepsilon(z) - h_{\varepsilon_0}(z)$  has variance  $-2 \log(\varepsilon/\varepsilon_0)$ , and is a

standard Brownian motion  $\mathcal{B}_{2t}$  in time  $2t = -2\log(\varepsilon/\varepsilon_0)$ , with initial value  $\mathcal{B}_0 = 0$ .

In the following section, we shall consider equally well free or mixed boundary conditions, up to some minor differences that are mentioned in each case.

### 6.3 Boundary measure and KPZ

We define the boundary measure  $\mu_\varepsilon^B := \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(z)/2} dz$ , where in this case  $dz$  is Lebesgue measure on the boundary component  $\partial D$ . Here we use  $e^{\gamma h_\varepsilon(z)/2}$  instead of  $e^{\gamma h_\varepsilon(z)}$  because we are integrating a length instead of an area; as before, the power of  $\varepsilon$  that we chose makes the expectation of the factor preceding  $dz$ ,  $\varepsilon^{\gamma^2/4} \mathbb{E} e^{\gamma h_\varepsilon(z)/2} = \varepsilon^{\gamma^2/4} e^{\gamma^2 \text{Var } h_\varepsilon(z)/8}$ , have a finite limit when  $\varepsilon \rightarrow 0$ .

We define  $\mu^B$  to be the weak limit as  $\varepsilon \rightarrow 0$  of the measures  $\mu_\varepsilon^B$  (see the theorem below for existence of this limit when  $0 \leq \gamma < 2$ ). For  $z \in \partial D$  we write  $\hat{B}_\varepsilon(z) := B_\varepsilon(z) \cap \partial D$  and we define  $\hat{B}^\delta(z)$  to be the (largest) set  $\hat{B}_\varepsilon(z)$  whose  $\mu^B$  measure is  $\delta$ .

Likewise define

$$\hat{B}_\varepsilon(X) = \{z \in \partial D : \hat{B}_\varepsilon(z) \cap X \neq \emptyset\}$$

and

$$\hat{B}^\delta(X) = \{z \in \partial D : \hat{B}^\delta(z) \cap X \neq \emptyset\}.$$

We say that a (deterministic or random) fractal subset  $X$  of the boundary component  $\partial D$  has *Euclidean expectation dimension*  $1 - \tilde{x}$  and *Euclidean scaling exponent*  $\tilde{x}$  in the boundary sense if the expected measure of  $\hat{B}_\varepsilon(X)$  decays like  $\varepsilon^{\tilde{x}}$ , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{\log \mathbb{E} \mu_0(\hat{B}_\varepsilon(X))}{\log \varepsilon} = \tilde{x}.$$

We say that  $X$  has *boundary quantum scaling exponent*  $\tilde{\Delta}$  if when  $X$  and  $\mu^B$  (as defined above) are chosen independently we have

$$\lim_{\delta \rightarrow 0} \frac{\log \mathbb{E} \mu^B(\hat{B}^\delta(X))}{\log \delta} = \tilde{\Delta}.$$

**Theorem 6.1** *Given the assumptions above, Proposition 1.1 and Theorems 1.5 and 4.2 hold, precisely as stated, when  $\mu_\varepsilon$  is replaced by  $\mu_\varepsilon^B$ ,  $\mu$  is replaced by  $\mu^B$ ;  $\mu_0$  (Lebesgue measure on  $D$ ) is replaced by Lebesgue measure on one of the boundary line segments  $\partial D$  of  $D$ ;  $B_\varepsilon$  and  $B^\delta$  are replaced with  $\hat{B}_\varepsilon$  and  $\hat{B}^\delta$ , respectively; and the compact subset of  $D$  is replaced with a closed subinterval of  $\partial D$ .*

*Proof* The proofs in the boundary case proceed exactly the same as in the interior point case, up to factors of 2 in various places. We sketch the proof of an analog of Theorem 4.2 in order to indicate where those factors of 2 appear.

Write  $t := -\log(\varepsilon/\varepsilon_0)$ , and let  $V_t := h_\varepsilon(z) - h_{\varepsilon_0}(z)$ . It is not hard to see that the expectation of the boundary line integral

$$\mathbb{E}_h \left[ \int_{\hat{B}_\varepsilon(z)} e^{\gamma h/2} dy \middle| V_t \right] = \mathbb{E}_h \left[ \mu_h^B(\hat{B}_\varepsilon(z)) |h_\varepsilon(z) - h_{\varepsilon_0}(z)| \right]$$

has approximately the form (which replaces (37) and (52))

$$\exp \left( \frac{\gamma}{2} V_t - \frac{\gamma}{2} Q t \right), \tag{83}$$

in the sense that the ratio of the logarithms of the two quantities tends to 1 when  $\varepsilon \rightarrow 0$  and  $t \rightarrow \infty$ . Let  $\tilde{B}^\delta(z)$  now be the largest Euclidean ball  $B_\varepsilon(z)$  in  $D$  centered at  $z \in \partial D$  for which (83) is equal to the quantum length  $\delta$ , and  $\tilde{B}^\delta(X) := \{z \in \partial D : \tilde{B}^\delta(z) \cap X \neq \emptyset\}$ .

As before, we use the fact that  $\mathbb{E}_h \mu_h^B(\tilde{B}^\delta(X))$  is proportional to  $\hat{\Theta}\{(z, h) : z \in \partial D, \tilde{B}^\delta(z) \cap X \neq \emptyset\}$ , where  $\hat{\Theta}$  is the boundary rooted measure such that, given  $z \in \partial D$ ,  $h$  is sampled from the Gaussian free field distribution *weighted* by  $e^{\gamma h(z)/2}$ . For *free* boundary conditions, the  $\hat{\Theta}$  conditional law of  $h$  is then that of the original GFF *plus* the deterministic function  $\frac{\gamma}{2} \hat{\xi}_0^z(\cdot) = -\gamma \log|\cdot - z| + \frac{\gamma\pi}{4|D|} |\cdot - z|^2$ . Then given  $z \in \partial D$ , the  $\hat{\Theta}$  conditional law of the semi-circular average  $h_\varepsilon(z) = (h, \hat{\xi}_\varepsilon^z)_\nabla$  is that of the original GFF semi-circular average, plus the Dirichlet inner product  $\frac{\gamma}{2} (\hat{\xi}_0^z, \hat{\xi}_\varepsilon^z)_\nabla = \frac{\gamma}{2} \hat{\xi}_\varepsilon^z(z)$ .

Then given  $z \in \partial D$ , the  $\hat{\Theta}$  conditional law of  $V_t = h_\varepsilon(z) - h_{\varepsilon_0}(z)$  is that of

$$\begin{aligned} \mathcal{B}_{2t} + \frac{\gamma}{2} (\hat{\xi}_\varepsilon^z(z) - \hat{\xi}_{\varepsilon_0}^z(z)) &= \mathcal{B}_{2t} - \gamma \log(\varepsilon/\varepsilon_0) + b_\varepsilon - b_{\varepsilon_0} \\ &= \mathcal{B}_{2t} + \gamma t + b_{\varepsilon_0 e^{-t}} - b_{\varepsilon_0}, \end{aligned}$$

where  $b_\varepsilon := \frac{\gamma}{2} \frac{\pi \varepsilon^2}{2|D|}$ ; thus  $V_t$  evolves *independently* of  $z$ , as a Brownian motion  $\mathcal{B}_{2t}$  with twice the variance of standard Brownian motion, because of the free boundary conditions on  $\partial D$ , plus a drift term  $\gamma t$ , and up to a constant and an exponentially small term when  $t \rightarrow \infty$ .

In the case of *mixed* boundary conditions, the same line of arguments (recall (81)) shows that the  $\hat{\Theta}$  conditional law of  $V_t = h_\varepsilon(z) - h_{\varepsilon_0}(z)$  is simply that of

$$\mathcal{B}_{2t} + \frac{\gamma}{2} (\tilde{\xi}_\varepsilon^z(z) - \tilde{\xi}_{\varepsilon_0}^z(z)) = \mathcal{B}_{2t} + \gamma t.$$

Using (83), we have both for free and mixed boundary conditions

$$\mathbb{E} \left[ \int_{\hat{B}_\varepsilon(z)} e^{\gamma h/2} d\gamma | V_t \right] \asymp \exp \left( \frac{\gamma}{2} \mathcal{B}_{2t} + \frac{1}{2} \gamma^2 t - \frac{\gamma}{2} Q t \right). \tag{84}$$

This will be equal to the quantum boundary length  $\delta$  at the smallest  $t$  for which  $\gamma \mathcal{B}_{2t} + \gamma^2 t - \gamma Q t = 2 \log \delta$ , with  $\mathcal{B}_0 = 0$ . If we set  $A := -(\log \delta)/\gamma$ , this smallest time is a stopping time  $T_A$  such that

$$\begin{aligned} T_A &= \inf\{t : \mathcal{B}_{2t} + at = 2A = -2(\log \delta)/\gamma\}, \\ a &= Q - \gamma = \frac{2}{\gamma} - \frac{\gamma}{2} > 0. \end{aligned} \tag{85}$$

As above, we consider the two part experiment in which we first sample  $T_A$  and then sample  $z$  and check to see whether the ball of radius  $\varepsilon = \varepsilon_0 e^{-T_A}$  intersects  $X$  on the boundary. Given  $T_A$ , the ratio of the logarithms of this probability and

$$\mathbb{E} \left[ \exp(-\tilde{x} T_A) \right]$$

tends to 1 as  $A \rightarrow \infty$ .

Consider next for any  $\beta$  the exponential martingale  $\exp(\frac{\beta}{2} \mathcal{B}_{2t} - \frac{\beta^2}{4} t)$ , such that

$$\mathbb{E} \left[ \exp \left( \frac{\beta}{2} \mathcal{B}_{2t} - \frac{\beta^2}{4} t \right) \right] = \mathbb{E} \left[ \exp \left( \frac{\beta}{2} \mathcal{B}_0 \right) \right] = 1.$$

As before, for  $\beta \geq 0$ , the martingale stays bounded from above by a fixed constant for  $t \in [0, T_A]$  with  $T_A < \infty$  a.s. One thus applies this exponential martingale at the stopping time  $T_A$ :

$$\mathbb{E} \left[ \exp \left( \frac{\beta}{2} \mathcal{B}_{2T_A} - \frac{\beta^2}{4} T_A \right) \right] = 1.$$

By definition  $\mathcal{B}_{2T_A} = 2A - aT_A$ . One thus gets the identity

$$\mathbb{E} \exp[-(\beta a/2 + \beta^2/4)T_A] = \exp(-\beta A),$$

and it now suffices to identify  $2\tilde{x} := \beta a + \beta^2/2$ , with  $\beta \geq 0$ , to obtain the boundary KPZ with  $\tilde{\Delta} := \beta/\gamma$ , and

$$\mathbb{E} \exp(-\tilde{x} T_A) = \delta^{\tilde{\Delta}} = \exp(-\beta A) = \exp \left\{ -A[(a^2 + 4\tilde{x})^{1/2} - a] \right\},$$

in complete analogy to (57). □

The reader may observe that the boundary measures described above are preserved under the transformations described in Proposition 2.1. One can use

this to define the boundary measure on more general domains, which may not have piecewise linear boundary conditions.

We also remark that a similar procedure to that above allows us to make sense of measure restricted to lines in the interior of the domain.

## 7 Discrete random surface dimensions and heuristics

Historically, one of the uses of the KPZ formula has been to make heuristic predictions about the scaling exponents of random fractal subsets of the plane (see, e.g., [48, 60, 62–64], and the references surveyed therein for much more detail).

In this subsection, we give a very rough and very brief sketch of what such a heuristic might entail in a simple example. Readers familiar with discrete quantum gravity models (a.k.a. random planar map models, random quadrangulation models, etc.) should note that these models have natural interpretations as continuum random metric spaces as well. For example, a random planar quadrangulation  $M_n$  on the sphere—chosen uniformly from the set of all simply connected planar quadrangulations with  $n$  quadrilaterals—can be viewed as a manifold by endowing each quadrilateral with the metric of a unit square. (Of course, the resulting manifold will have singularities: negative curvature point masses at vertices where more than four unit squares coincide and positive curvature point masses at vertices where fewer than four unit squares coincide.) We may then choose a uniform square from among this set. Taking an “infinite volume limit” (as  $n \rightarrow \infty$ ) one obtains an infinite random quadrangulation  $M_\infty$  with a distinguished square. (See, e.g., [12] for a precise description of this construction for triangulations.) This infinite random surface can be conformally mapped to the plane in such a way that the center of the distinguished square is mapped to the origin and the volume of the image of the distinguished square is a constant  $\delta$  (with a rotation chosen uniformly at random). The images of the unit squares of  $M_\infty$  form a tiling of  $\mathbb{C}$  by “conformally distorted” unit squares. Different squares have different sizes with respect to the Euclidean metric on the plane; intuitively, one would expect such a tiling to look something vaguely like the tilings in Figs. 1, 2, and 3 except that the “squares” would be randomly oriented and distorted. The pullback of the intrinsic metric of  $M_\infty$  to the plane via this map takes the form  $e^\lambda(dx^2 + dy^2)$  for some function random  $\lambda$  (which has logarithmic singularities at the images of the vertices of the squares). Although the equivalence of Liouville quantum gravity and discrete quantum gravity is taken as an Ansatz throughout much of the literature, to our knowledge the following is the first precise conjecture for the complete scaling limit of a discrete quantum gravity model:

**Conjecture 7.1** *As  $\delta \rightarrow 0$ , the function  $\lambda$  converges in law (e.g., w.r.t. to the weak topology on the space of distributions on the plane modulo additive constants) to  $\gamma(h(\cdot) - \gamma \log|\cdot|)$  where  $h$  is an instance of the whole plane Gaussian free field (defined up to additive constants) and  $\gamma^2 = \kappa = 8/3$ .*

We further conjecture that other values of  $\gamma$  are obtained by choosing a random quadrangulation together with a statistical physical model on the quadrangulation (FK cluster model, percolation,  $O(N)$  model, uniform spanning tree); in this case, the probability of a given quadrangulation is proportional to the partition function of the statistical physics model on that quadrangulation. (See the references on random matrix theory and geometrical models cited in the introduction for much more detail; see [64] for a review with additional references.) One can also consider scaling limits on spheres or higher genus surfaces, as well as different kinds of marked points (corresponding to different logarithmic singularities in the scaling limit); however, these are a bit more complicated to describe, so we limit attention to the infinite volume case for now.

By the usual conformal invariance Ansatz, it is natural to expect that if one conditions on the infinite quadrangulation, and then samples the loops or trees in these models (as mapped into the plane), their law (in the scaling limit) will be *independent* of the metric.

Now suppose that for each  $n$  we define a random subset  $X_n$  of  $M_n$  (for example,  $X_n$  could be the set of the squares hit by a simple random walk started at the root square and stopped the first time that the walk hits a square on the boundary of the quadrangulation). Then one can define a discrete scaling exponent (analogous to the box counting exponent in (4), with  $\delta$  replaced by  $n^{-1}$ ) as follows:

$$\Delta_D = \lim_{n \rightarrow \infty} \frac{\log \mathbb{E}(n^{-1} | X_n|)}{\log n^{-1}}.$$

Identifying  $X_n$  with its image in a conformal map to, say,  $\mathbb{D}$ , one might guess that the random pair  $(X_n, \lambda_n)$ —where  $e^{\lambda_n(z)} dz$  is uniform measure on the discrete surface, mapped to  $\mathbb{D}$ —has a scaling limit  $(X, \lambda)$ , where  $X$  is a random subset of  $\mathbb{D}$  (in our example, it might be a Brownian motion) and  $\lambda$  is some form of the Gaussian free field.

If this is the case, then on a heuristic level, one would expect that the quantum scaling exponent of  $X$  is  $\Delta = \Delta_D$ , since, in the notation of Corollary 1.7, if we write  $\delta = n^{-1}$ , we would expect that  $\mathbb{E}[\delta N(\mu, \delta, X)]$  scales like  $\mathbb{E}(n^{-1} | X_n|)$ .

In discrete quantum gravity models, it is often possible to compute  $\Delta_D$  explicitly (and rigorously) using random matrix techniques or tree bijections; it is also often possible to compute  $\gamma$  directly using discrete quantum gravity machinery and so heuristically obtain its value in the continuum limit.

Assuming values for  $\Delta_D$  and  $\gamma$ —and assuming  $\Delta = \Delta_D$ —the KPZ formula gives the Euclidean scaling dimension of  $X$ . In many interesting examples,  $X$  is a random fractal (a Schramm-Loewner evolution, for example, or the outer boundary of a planar Brownian motion) whose Euclidean scaling dimension might not be immediately obvious otherwise.

Finally, we mention that, in the standard realm of conformal field theory, there exists a precise relation between the central charge  $c \leq 1$  of the statistical model coupled to quantum gravity and the value of Liouville parameter,  $\gamma = (\sqrt{25 - c} - \sqrt{1 - c})/\sqrt{6}$ , [41, 52, 79, 103, 130], as well as a corresponding connection between  $\text{SLE}_\kappa$  and Liouville quantum gravity models with  $\gamma = \sqrt{\min\{\kappa, 16/\kappa\}}$ .

Our result extends the validity of the KPZ relation outside that CFT framework to any value of Liouville parameter  $\gamma < 2$ , with the Ansatz that the fractal set  $X$  and the GFF are sampled independently. A possible interpretation of the KPZ relation in that case would be that it describes the quantum geometry of the given fractal in the *quenched* random surface generated by random graphs, equilibrated with a conformally invariant system with a value of  $c$  or  $\kappa$  corresponding to the chosen value of  $\gamma$ . For example, one could first choose a random graph weighted by the critical Ising model partition function; and then perform a loop erased random walk on that graph, ignoring Ising clusters. In this case, one would expect the Euclidean dimension of the path to be that of  $\text{SLE}_2$  (which corresponds to loop erased random walk), while the value of  $\gamma$  describing the metric would be  $\sqrt{3}$  (which corresponds to the critical Ising model), and one could use KPZ to predict the quantum scaling dimension.

Similar ideas appeared in previous numerical work [3, 85], but the data so far appear as inconclusive.

Finally, we remark that the original (still accessible) arXiv version of this paper contained an additional section: a three-page sketch of some work in progress, including some results about the conformal welding of quantum random surfaces and about the scaling limits of discrete quantum gravity models. Many of these results will appear in [56, 131].

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