Invent math (2011) 184: 1–46
DOI 10.1007/s00222-010-0284-5

# On the weights of mod $p$ Hilbert modular forms

**Toby Gee**

Received: 22 October 2008 / Accepted: 3 August 2010 / Published online: 28 September 2010


**Abstract** We prove many cases of a conjecture of Buzzard, Diamond and Jarvis on the possible weights of mod $p$ Hilbert modular forms, by making use of modularity lifting theorems and computations in $p$-adic Hodge theory.

## Contents



## 1 Introduction

If a representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is continuous, odd, and irreducible, then a conjecture of Serre (now a theorem of Khare-Wintenberger and Kisin) predicts that $\overline{\rho}$ is modular. More precisely, Serre predicted a minimal weight $k(\overline{\rho})$ and a minimal level $N(\overline{\rho})$ for a modular form giving rise to $\overline{\rho}$.

T. Gee (✉)
Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA
e-mail: gee@math.northwestern.edu

It is natural to try to extend these results to totally real fields $F$. The natural generalisation of Serre's conjecture is to conjecture that if

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is continuous, irreducible and totally odd, then it is modular (in the sense that it arises from a Hilbert modular form). It is straightforward to generalise the definition of $N(\overline{\rho})$ to this setting, and there has been much progress on "level-lowering" for Hilbert modular forms. It is, however, much harder to generalise the definition of $k(\overline{\rho})$. For example, there is no longer a total ordering on the weights, and the $p$-adic Hodge theory is much more complicated than in the classical case.

Suppose that $p$ is unramified in $F$. Recently (see [3]), Buzzard, Diamond and Jarvis have proposed a conjectural set $W(\overline{\rho})$ of weights attached to $\overline{\rho}$, from which in the classical case one can deduce the weight part of Serre's conjecture (see [3] for more details). In this paper we prove many cases of a closely related conjecture (we work with a definite, rather than indefinite quaternion algebra; as we discuss below, it should be straightforward to prove the corresponding results in the setting of [3]). To be precise, a weight is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathcal{O}_F/p)$, and such a representation factors as a tensor product

$$\bigotimes_{v|p} \sigma_{\vec{a},\vec{b}}$$

where $\vec{a}$, $\vec{b}$ are $[k_v : \mathbb{F}_p]$-tuples indexed by embeddings $\tau : k_v \hookrightarrow \overline{\mathbb{F}}_p$, and $0 \le a_\tau \le p-1$, $1 \le b_\tau \le p$. Then we say that a weight is *regular* if in fact $2 \le b_\tau \le p-2$ for all $\tau$. Our main theorem requires a technical condition which we prefer to define later, that of a weight being partially ordinary of type $I$ for $\overline{\rho}$, $I$ a set of places of $F$ dividing $p$; see Sect. 2.

**Theorem** *Suppose that $\overline{\rho}$ is modular, that $p \ge 5$, and that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Then if $\sigma$ is a regular weight and $\overline{\rho}$ is modular of weight $\sigma$ then $\sigma \in W(\overline{\rho})$. Conversely, if $\sigma \in W(\overline{\rho})$ and $\sigma$ is non-ordinary for $\overline{\rho}$, then $\overline{\rho}$ is modular of weight $\sigma$. If $\sigma$ is partially ordinary of type $I$ for $\overline{\rho}$ and $\overline{\rho}$ has a partially ordinary modular lift of type $I$ then $\overline{\rho}$ is modular of weight $\sigma$.*

Before we discuss the proof, we make some remarks about the assumptions in the theorem. The assumption that $\overline{\rho}$ is modular is essential to our methods. The assumption that $p \ge 5$ is needed in order for there to be any regular weights at all; it is possible that this could be relaxed in future work, as there is no essential obstruction to the application of the techniques that we employ in characteristic 3. In characteristic 2 the results from 2-adic Hodge theory that we would require have not yet been developed in sufficient generality, but this too does not appear to be an insurmountable difficulty. The assumption that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and the assumption on partial ordinarity, are needed in order to apply $R = T$ theorems.

The main idea of our proof is the same as that for our proof of a companion forms theorem for totally real fields (see [14]), namely that we use a lifting theorem to construct lifts of $\overline{\rho}$ satisfying certain local properties at places $v|p$, and then use a modularity lifting theorem of Kisin to prove that these representations are modular. In fact, Kisin's theorem is not general enough for our applications, and we need to use the main theorem of [13]. The arguments are much more complicated than those in [14] because we need to construct liftings with more delicate local properties; rather than just considering ordinary lifts, we must consider potentially Barsotti-Tate lifts of specified type.

The other complication which intervenes is that the connection between being modular of a certain weight and having a lift of a certain type is rather subtle, and this is the reason for our hypothesis that the weight be regular. One needs to consider many liftings for each weight, and we have only obtained the necessary combinatorial results in the case where the weight is regular. However, while these results appear to hold for most non-regular weights, there are cases where they do not hold, so it seems that it is not possible to give a general proof that the list of weights is correct by simply considering the types of potentially Barsotti-Tate lifts. It is possible to give a complete proof in the case where $p$ splits completely in $F$, and we do this in [15].

We now outline the structure of the paper. Rather than working with the "geometric" conventions of [3], we prefer to work with more "arithmetic" ones. In particular, we work with automorphic forms on definite quaternion algebras. We set out our conventions in Sect. 2, and we state the appropriate reformulation of the conjectures of [3] here. In Sect. 3 we carry out the required local analysis in the case where the local representation is reducible. Sections 3.1 and 3.2 use Breuil modules and strongly divisible modules to determine when reducible representations arise as the generic fibres of certain finite flat group schemes. In Sect. 3.4 we relate these finite flat group schemes to certain crystalline representations considered in [3], and in Sect. 3.5 we prove the necessary combinatorial results relating types and regular weights.

We then repeat this analysis in the irreducible case in Sect. 4, and finally in Sect. 5 we combine these results with the lifting theorems mentioned above to deduce our main results. Firstly, we use our local results to show that if $\overline{\rho}$ is modular of weight $\sigma$ with $\sigma$ regular, then $\sigma \in W(\overline{\rho})$. For each regular weight $\sigma \in W(\overline{\rho})$ we then produce a modular lift of $\overline{\rho}$ which is potentially Barsotti-Tate of a specific type, so that $\overline{\rho}$ must be modular of some weight occurring in the mod $p$ reduction of this type. We then check that $\sigma$ is the only element of $W(\overline{\rho})$ occurring in this reduction, so that $\overline{\rho}$ is modular of weight $\sigma$, as required. In fact, we do not quite do this; the combinatorics is slightly more involved, and we are forced to make use of a notion of a "weakly regular" weight. See Sect. 5 for the details.

It is a pleasure to thank Fred Diamond for numerous helpful discussions regarding this work; without his patient advice this paper could never have been written. We would like to thank David Savitt for pointing out several errors and omissions in an earlier version of this paper, and for writing [21]. We would like to thank Florian Herzig for pointing out an inconsistency between our conventions and those of [3], which led to the writing of Sect. 2. We are extremely grateful to Xavier Caruso and Christophe Breuil for their many helpful comments and corrections; in particular, the material in Sect. 3.4 owes a considerable debt to Caruso's efforts to correct a number of inaccuracies, and the proof of Lemma 3.6 is based on an argument of his. We would also like to thank the anonymous referee for a careful reading, and for pointing out a number of serious errors in an earlier version of the paper.

## 2 Definitions

2.1

Rather than use the conventions of [3], we choose to state a closely related variant of their conjectures by working on totally definite quaternion algebras. This formulation is more suited to applications to modularity lifting theorems, and indeed to the application of modularity lifting theorems to proving cases of the conjecture.

We begin by recalling some standard facts from the theory of quaternionic modular forms; see either [23], Sect. 3 of [17] or Sect. 2 of [18] for more details, and in particular the proofs of the results claimed below. We will follow Kisin's approach closely. We fix throughout this paper an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and regard all algebraic extensions of $\mathbb{Q}$ as subfields of $\overline{\mathbb{Q}}$. For each prime $p$ we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. In this way, if $v$ is a finite place of a number field $F$, we have a homomorphism $G_{F_v} \hookrightarrow G_F$.

Let $F$ be a totally real field in which $p > 2$ is unramified, and let $D$ be a quaternion algebra with center $F$ which is ramified at all infinite places of $F$ and at a set $\Sigma$ of finite places, which contains no places above $p$. Fix a maximal order $\mathcal{O}_D$ of $D$ and for each finite place $v \notin \Sigma$ fix an isomorphism $(\mathcal{O}_D)_v \stackrel{\sim}{\longrightarrow} M_2(\mathcal{O}_{F_v})$. For any finite place $v$ let $\pi_v$ denote a uniformiser of $F_v$.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact subgroup, with each $U_v \subset (\mathcal{O}_D)_v^\times$. Furthermore, assume that $U_v = (\mathcal{O}_D)_v^\times$ for all $v \in \Sigma$, and that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if $v|p$.

Take $A$ a topological $\mathbb{Z}_p$-algebra. For each $v|p$, fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_{\sigma_v})$ with $W_{\sigma_v}$ a finite free $A$-module. Write $W_\sigma = \bigotimes_{v|p,A} W_{\sigma_v}$ and let $\sigma = \prod_{v|p} \sigma_v$. We regard $\sigma$ as a representation of $U$ in the obvious way (that is, we let $U_v$ act trivially if $v \nmid p$). Fix also a character $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \to A^\times$ such that for any place $v$ of $F$, $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times}$ is multiplication by $\psi^{-1}$. Then we can think of $W_\sigma$ as a $U(\mathbb{A}_F^f)^\times$-module by letting $(\mathbb{A}_F^f)^\times$ act via $\psi^{-1}$.

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma$$

such that for all $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have

$$f(gu) = \sigma(u)^{-1} f(g) \quad \text{for all } u \in U,$$
$$f(gz) = \psi(z) f(g) \quad \text{for all } z \in (\mathbb{A}_F^f)^\times.$$

We can write $(D \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$ for some finite index set $I$ and some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$. Then we have

$$S_{\sigma,\psi}(U, A) \xrightarrow{\sim} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i)/F^\times},$$

the isomorphism being given by the direct sum of the maps $f \mapsto \{f(t_i)\}$. From now on we make the following assumption:

$$\text{For all } t \in (D \otimes_F \mathbb{A}_F^f)^\times \text{ the group } (U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times = 1.$$

One can always replace $U$ by a subgroup (obeying the assumptions above) for which this holds (cf. Sect. 3.1.1 of [19]). Under this assumption, which we make from now on, $S_{\sigma,\psi}(U, A)$ is a finite free $A$-module, and the functor $W_\sigma \mapsto S_{\sigma,\psi}(U, A)$ is exact in $W_\sigma$.

We now define some Hecke algebras. Let $S$ be a set of finite places containing $\Sigma$, the places dividing $p$, and the primes of $F$ such that $U_v$ is not a maximal compact subgroup of $D_v^\times$. Let $\mathbb{T}_{S,A}^{\mathrm{univ}} = A[T_v]_{v \notin S}$ be the commutative polynomial ring in the formal variables $T_v$. Consider the left action of $(D \otimes_F \mathbb{A}_F^f)^\times$ on $W_\sigma$-valued functions on $(D \otimes_F \mathbb{A}_F^f)^\times$ given by $(gf)(z) = f(zg)$. For each finite place $v$ of $F$ we fix a uniformiser $\pi_v$ of $F_v$. Then we make $S_{\sigma,\psi}(U, A)$ a $\mathbb{T}_{S,A}^{\mathrm{univ}}$-module by letting $T_v$ act via the double coset $U\left(\begin{smallmatrix}\pi_v & 0\\ 0 & 1\end{smallmatrix}\right)U$. These are independent of the choices of $\pi_v$. We will write $\mathbb{T}_{\sigma,\psi}(U, A)$ or $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}_{S,A}^{\mathrm{univ}}$ in $\mathrm{End}\, S_{\sigma,\psi}(U, A)$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\text{univ}}_{S,A}$. We say that $\mathfrak{m}$ is in the support of $(\sigma, \psi)$ if $S_{\sigma,\psi}(U, A)_{\mathfrak{m}} \neq 0$. Now let $\mathcal{O}$ be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$, and suppose that $A = \mathcal{O}$ in the above discussion, and that $\sigma$ has open kernel. Consider a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{\text{univ}}_{S,\mathcal{O}}$ which is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U, \mathcal{O})$. Then there is a semisimple Galois representation $\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbb{F})$ associated to $\mathfrak{m}$ which is characterised up to equivalence by the property that if $v \notin S$ and $\mathrm{Frob}_v$ is an arithmetic Frobenius at $v$, then the trace of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is the image of $T_v$ in $\mathbb{F}$.

We are now in a position to define what it means for a Galois representation to be modular of some weight. Let $v|p$ be a place of $F$, let $F_v$ have ring of integers $\mathcal{O}_v$ and residue field $k_v$, and let $\sigma$ be an irreducible $\overline{\mathbb{F}}_p$-representation of $G := \prod_{v|p} \mathrm{GL}_2(k_v)$. We also denote by $\sigma$ the representation of $\prod_{v|p} \mathrm{GL}_2(\mathcal{O}_v)$ induced by the surjections $\mathcal{O}_v \twoheadrightarrow k_v$.

**Definition 2.1** We say that an irreducible representation $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight $\sigma$ if for some $D$, $S$, $U$, $\psi$, and $\mathfrak{m}$ as above we have $S_{\sigma,\psi}(U, \mathbb{F})_{\mathfrak{m}} \neq 0$ and $\overline{\rho}_{\mathfrak{m}} \cong \overline{\rho}$.

We now show how one can gain information about the weights associated to a particular Galois representation by considering lifts to characteristic zero.

**Lemma 2.2** *Let $\psi : F^\times \backslash (\mathbb{A}_F^f)^\times \to \mathcal{O}^\times$ be a continuous character, and write $\overline{\psi}$ for the composite of $\psi$ with the projection $\mathcal{O}^\times \to \mathbb{F}^\times$. Fix a representation $\sigma$ of $\prod_{v|p} U_v$ on a finite free $\mathcal{O}$-module $W_\sigma$, and an irreducible representation $\sigma'$ on a finite free $\mathbb{F}$-module $W_{\sigma'}$. Suppose that for each $v|p$ we have $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ and $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \overline{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$.*

*Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{univ}_{S,\mathcal{O}}$.*

*Suppose that $W_{\sigma'}$ occurs as a $\prod_{v|p} U_v$-module subquotient of $W_{\overline{\sigma}} := W_\sigma \otimes \mathbb{F}$. If $\mathfrak{m}$ is in the support of $(\sigma', \overline{\psi})$, then $\mathfrak{m}$ is in the support of $(\sigma, \psi)$.*

*Conversely, if $\mathfrak{m}$ is in the support of $(\sigma, \psi)$, then $\mathfrak{m}$ is in the support of $(\sigma', \overline{\psi})$ for some irreducible $\prod_{v|p} U_v$-module subquotient $W_{\sigma'}$ of $W_{\overline{\sigma}}$.*

*Proof* The first part is proved just as in Lemma 3.1.4 of [17], and the second part follows from Proposition 1.2.3 of [1]. □

We note a special case of this result, relating the existence of potentially Barsotti-Tate lifts of a particular tame type to information about Serre weights. Firstly, we recall some particular representations of $\mathrm{GL}_2(k_v)$. For any pair of distinct characters $\chi_1, \chi_2 : k_v^\times \to \mathcal{O}^\times$ we let $I(\chi_1, \chi_2)$ denote the irreducible $(q+1)$-dimensional $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(k_v)$ induced from the character of $B$ (the upper triangular matrices in $\mathrm{GL}_2(k_v)$) given by

$$\begin{pmatrix} x & w \\ 0 & y \end{pmatrix} \mapsto \chi_1(x)\chi_2(y).$$

We let $\sigma_{\chi_1,\chi_2}$ denote the representation of $\mathrm{GL}_2(k_v)$ on an $\mathcal{O}$-lattice in $I(\chi_1,\chi_2)$; we also regard this as a representation of $\mathrm{GL}_2(\mathcal{O}_v)$ via the natural projection. Let $\tau(\sigma_{\chi_1,\chi_2})$ be the inertial type $\chi_1 \oplus \chi_2$ (regarded as a representation of $I_{F_v}$ via local class field theory, normalised so that a uniformiser corresponds to a geometric Frobenius element).

Let $k'_v$ be the quadratic extension of $k_v$. For any character $\theta : k'^{\times}_v \to \mathcal{O}^{\times}$ which does not factor through the norm $k'^{\times}_v \to k_v^{\times}$, there is an irreducible $(q-1)$-dimensional cuspidal representation $\Theta(\theta)$ of $\mathrm{GL}_2(k_v)$ (see Sect. 1 of [11] for the definition of $\Theta(\theta)$). Let $\sigma_{\Theta(\theta)}$ denote the representation of $\mathrm{GL}_2(k_v)$ on an $\mathcal{O}$-lattice in $\Theta(\theta)$; we also regard this as a representation of $\mathrm{GL}_2(\mathcal{O}_v)$ via the natural projection. Let $q_v$ be the cardinality of $k_v$, and let $\tau(\sigma_{\Theta(\theta)})$ be the inertial type $\theta \oplus \theta^{q_v}$ (again regarded as a representation of $I_{F_v}$ via local class field theory).

**Definition 2.3** Let $\tau$ be an inertial type, and let $v|p$ be a place of $F$. We say that a lift $\rho$ of $\overline{\rho}|_{G_{F_v}}$ is *potentially Barsotti-Tate of type* $\tau$ if $\rho$ is potentially Barsotti-Tate, has determinant a finite order character of order prime to $p$ times the cyclotomic character, and the corresponding Weil-Deligne representation (see Appendix B of [9]), when restricted to $I_{F_v}$, is isomorphic to $\tau$.

**Lemma 2.4** *For each $v|p$, fix a representation $\sigma_v$ of the type just considered (that is, isomorphic to $\sigma_{\chi_1,\chi_2}$ or to $\sigma_{\Theta(\theta)}$). Let $\tau_v = \tau(\sigma_v)$ be the corresponding inertial type. Suppose that $\overline{\rho}$ is modular of weight $\sigma$, and that $\sigma$ is a $\prod_{v|p} \mathrm{GL}_2(k_v)$-subquotient of $\bigotimes_{v|p} \sigma_v \otimes_{\mathcal{O}} \mathbb{F}$. Then $\overline{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type $\tau_v$ for each $v|p$.*

*Conversely, suppose that $\overline{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type $\tau_v$ for each $v|p$. Then $\overline{\rho}$ is modular of weight $\sigma$ for some $\prod_{v|p} \mathrm{GL}_2(k_v)$-subquotient $\sigma$ of $\bigotimes_{v|p} \sigma_v \otimes_{\mathcal{O}} \mathbb{F}$.*

*Proof* This follows from Lemma 2.2, the Jacquet-Langlands correspondence, and the compatibility of the local and global Langlands correspondences at places dividing $p$ (see [18]). □

We now state a conjecture on Serre weights, following [3]. Note that our conjecture is only valid for regular weights (a notion which we will define shortly); there are some additional complications when dealing with non-regular weights. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be modular. We propose a conjectural set of regular weights $W(\overline{\rho})$ for $\overline{\rho}$.

In fact, for each place $v|p$ we propose a set of weights $W(\overline{\rho}|_{G_{F_v}})$, and we define

$$W(\overline{\rho}) := \left\{ \bigotimes_{v|p} \sigma_v | \sigma_v \in W(\overline{\rho}|_{G_{F_v}}) \right\}.$$

Let $S_v$ be the set of embeddings $k_v \hookrightarrow \overline{\mathbb{F}}_p$. A weight for $\mathrm{GL}_2(k_v)$ is an isomorphism class of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_v)$, which automatically contains one of the form

$$\sigma_{\vec{a},\vec{b}} = \bigotimes_{\tau \in S_v} \det{}^{a_\tau} \mathrm{Sym}^{b_\tau - 1} k_v^2 \otimes_\tau \overline{\mathbb{F}}_p,$$

with $0 \le a_\tau \le p-1$ and $1 \le b_\tau \le p$ for each $\tau \in S_v$. We demand further that some $a_\tau < p-1$, in which case the representations $\sigma_{\vec{a},\vec{b}}$ are pairwise non-isomorphic.

**Definition 2.5** We say that a weight $\sigma_{\vec{a},\vec{b}}$ is *regular* if $2 \le b_\tau \le p-2$ for all $\tau$. We say that it is *weakly regular* if $1 \le b_\tau \le p-1$ for all $\tau$.

For each $\tau \in S_v$ we have the fundamental character $\omega_\tau$ of $I_{F_v}$ given by composing $\tau$ with the homomorphism $I_{F_v} \to k_v^\times$ given by local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements. Let $k_v'$ denote the quadratic extension of $k_v$. Let $S_v'$ denote the set of embeddings $\sigma : k_v' \hookrightarrow \overline{\mathbb{F}}_p$, and let $\omega_\sigma$ denote the fundamental character corresponding to $\sigma$.

Suppose firstly that $\overline{\rho}|_{G_{F_v}}$ is irreducible. There is a natural $2-1$ map $\pi : S_v' \to S_v$ given by restriction to $k_v$, and we say that a subset $J \subset S_v'$ is a *full subset* if $|J| = |\pi(J)| = |S_v|$. Then we have

**Definition 2.6** Let $\sigma_{\vec{a},\vec{b}}$ be a regular weight for $\mathrm{GL}_2(k_v)$. Then $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})$ if and only if there exists a full subset $J \subset S_v'$ such that

$$\overline{\rho}|_{I_{F_v}} \sim \prod_{\tau \in S_v} \omega_\tau^{a_\tau} \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{b_{\sigma|_{k_v}}} & 0 \\ 0 & \prod_{\sigma \notin J} \omega_\sigma^{b_{\sigma|_{k_v}}} \end{pmatrix}.$$

Suppose now that $\overline{\rho}|_{G_{F_v}}$ is reducible, say $\overline{\rho}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \psi_1 & * \\ 0 & \psi_2 \end{smallmatrix}\right)$. We define the set $W(\overline{\rho}|_{G_{F_v}})$ in two stages. Firstly, define a set $W(\overline{\rho}|_{G_{F_v}})'$ of regular weights as follows.

**Definition 2.7** Let $\sigma_{\vec{a},\vec{b}}$ be a regular weight for $\mathrm{GL}_2(k_v)$. Then $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$ if and only if there exists $J \subset S_v$ such that $\psi_1|_{I_{F_v}} = \prod_{\tau \in S_v} \omega_\tau^{a_\tau} \times \prod_{\tau \in J} \omega_\tau^{b_\tau}$ and $\psi_2|_{I_{F_v}} = \prod_{\tau \in S_v} \omega_\tau^{a_\tau} \prod_{\tau \notin J} \omega_\tau^{b_\tau}$. We say that $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$

is *ordinary for* $\overline{\rho}$ if furthermore $J = S_v$ or $J = \emptyset$ (note that the set $J$ is uniquely determined, because $\sigma_{\vec{a},\vec{b}}$ is regular).

Suppose that we have a regular weight $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$ and a corresponding subset $J \subset S_v$. We now define crystalline lifts $\widetilde{\psi}_1$, $\widetilde{\psi}_2$ of $\psi_1$, $\psi_2$. If $\psi : G_{F_v} \to \overline{\mathbb{Q}}_p^{\times}$ is a crystalline character, and $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$, we say that the Hodge-Tate weight of $\psi$ with respect to $\tau$ is the $i$ for which $gr^{-i}((\psi \otimes_{\mathbb{Q}_p} B_{dR})^{G_{F_v}} \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} F_v, 1\otimes\tau} \overline{\mathbb{Q}}_p) \neq 0$. Then we demand that for some fixed Frobenius element $\mathrm{Frob}_v$ of $G_{F_v}$, $\widetilde{\psi}_i(\mathrm{Frob}_v)$ is the Teichmüller lift of $\psi_i(\mathrm{Frob}_v)$, and that:

- $\widetilde{\psi}_1$ is crystalline, and the Hodge-Tate weight of $\widetilde{\psi}_1$ with respect to $\tau$ is $a_\tau + b_\tau$ if $\tau \in J$, and $a_\tau$ if $\tau \notin J$.
- $\widetilde{\psi}_2$ is crystalline, and the Hodge-Tate weight of $\widetilde{\psi}_2$ with respect to $\tau$ is $a_\tau + b_\tau$ if $\tau \notin J$, and $a_\tau$ if $\tau \in J$.

The existence and uniqueness (for our fixed choice of $\mathrm{Frob}_v$) of $\widetilde{\psi}_1$, $\widetilde{\psi}_2$ is straightforward (see [3]). Then we have

**Definition 2.8** $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})$ if and only if $\overline{\rho}|_{G_{F_v}}$ has a lift to a crystalline representation $\left(\begin{smallmatrix} \widetilde{\psi}_1 & * \\ 0 & \widetilde{\psi}_2 \end{smallmatrix}\right)$.

Note that by Remark 3.10 of [3], and the regularity of $\sigma_{\vec{a},\vec{b}}$, this definition is independent of the choice of $\mathrm{Frob}_v$.

For future reference, we say that a weight $\sigma$ is partially ordinary of type $I$ for $\overline{\rho}$ if $I$ is the set of places $v|p$ for which $\sigma_v$ is ordinary for $\overline{\rho}$. We say that $\overline{\rho}$ has a partially ordinary modular lift of type $I$ if it has a potentially Barsotti-Tate modular lift which is potentially ordinary at precisely the places in $I$.

### 2.2 Relation to the Buzzard-Diamond-Jarvis conjecture

Our conjectured sets of regular weights are exactly the same as the regular weights predicted in [3]. However, they work with indefinite quaternion algebras rather than the definite ones of this paper, and in the absence of a mod $p$ Jacquet-Langlands correspondence our results do not automatically prove cases of their conjectures. That said, our arguments are for the most part purely local, with the only global input being in characteristic zero, where one does have a Jacquet-Langlands correspondence. In particular, given the analogue of Lemma 2.4 in the setting of [3] (cf. Proposition 2.10 of [3]) our arguments will go over unchanged to their setting.

## 3 Local analysis—the reducible case

### 3.1 Breuil modules

Let $p > 2$ be prime, let $k$ be a finite extension of $\mathbb{F}_p$, let $K_0 = W(k)[1/p]$, and let $K$ be a finite Galois totally tamely ramified extension of $K_0$, of degree $e$. Fix a subfield $M$ of $K_0$, and assume that there is a uniformiser $\pi$ of $\mathcal{O}_K$ such that $\pi^e \in M$, and fix such a $\pi$. Since $K/M$ is tamely ramified (and automatically Galois), the category of Breuil modules with coefficients and descent data is easy to describe (see [21]). Let $k \in [2, p-1]$ be an integer (there will never be any ambiguity in our two uses of the symbol $k$, one being a finite field and the other a positive integer). Let $E$ be a finite extension field of $\mathbb{F}_p$. The category $\mathrm{BrMod}_{dd,M}^{k-1}$ consists of quintuples $(\mathcal{M}, \mathcal{M}_{k-1}, \phi_{k-1}, \hat{g}, N)$ where:

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module, free over $k[u]/u^{ep}$.
- $\mathcal{M}_{k-1}$ is a $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^{e(k-1)}\mathcal{M}$.
- $\phi_{k-1} : \mathcal{M}_{k-1} \to \mathcal{M}$ is $E$-linear and $\phi$-semilinear (where $\phi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the $p$-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module.
- $N : \mathcal{M} \to u\mathcal{M}$ is $(k \otimes_{\mathbb{F}_p} E)$-linear and satisfies $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$, $u^e N(\mathcal{M}_{k-1}) \subset \mathcal{M}_{k-1}$, and $\phi_{k-1}(u^e N(x)) = (-\pi^e/p) \times N(\phi_{k-1}(x))$ for all $x \in \mathcal{M}_{k-1}$.
- $\hat{g} : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in \mathrm{Gal}(K/M)$, preserving $\mathcal{M}_{k-1}$, commuting with the $\phi_{k-1}$-, $E$-, and $N$-actions, and satisfying $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/M)$, and $\hat{1}$ is the identity. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} E$, $m \in \mathcal{M}$ then $\hat{g}(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \hat{g}(m)$.

We will omit $M$ from the notation in the case $M = K_0$. We write $\mathrm{BrMod}_{dd,M} = \mathrm{BrMod}_{dd,M}^1$. The category $\mathrm{BrMod}_{dd,M}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with an $E$-action and descent data on the generic fibre from $K$ to $M$ (this equivalence depends on $\pi$). In this case it follows from the other axioms that there is always a unique $N$ which satisfies the required properties, and we will frequently omit the details of this operator when we are working in the case $k = 2$. In Sect. 3.4 we will also use the case $k = p-1$, and here we will make the operators $N$ explicit.

We choose in this paper (except in Sect. 3.4) to adopt the conventions of [4] and [20], rather than those of [2]; thus rather than working with the usual contravariant equivalence of categories, we work with a covariant version of it, so that our formulae for generic fibres will differ by duality and a twist from those following the conventions of [2]. To be precise, we obtain the associated $G_M$-representation (which we will refer to as the generic fibre) of an object of $\mathrm{BrMod}_{dd}$ via the functor $T_{st,2}^M$, which is defined in Sect. 4 of [20].

Let $\rho : G_{K_0} \to \mathrm{GL}_2(E)$ be a continuous representation. We assume from now on that $E$ contains $k$. Suppose for the rest of this section that $\rho$ is reducible but not scalar, say $\rho \sim \left(\begin{smallmatrix}\psi_1 & * \\ 0 & \psi_2\end{smallmatrix}\right)$. Fix $\pi = (-p)^{1/(p^r-1)}$, where $r = [k : \mathbb{F}_p]$, and fix $K = K_0(\pi)$, so that $\pi$ is a uniformiser of $\mathcal{O}_K$, the ring of integers of $K$. By class field theory $\psi_1|_{I_K}$ and $\psi_2|_{I_K}$ are trivial.

We fix some general notation for elements of $\mathrm{BrMod}_{dd}$. Let $S$ denote the set of embeddings $\tau : k \hookrightarrow E$. We have an isomorphism $k \otimes_{\mathbb{F}_p} E \xrightarrow{\sim} \oplus_S E_\tau$, where $E_\tau := k \otimes_{k,\tau} E$, and we let $\epsilon_\tau$ denote the idempotent corresponding to the embedding $\tau$. Then any element $\mathcal{M}$ of $\mathrm{BrMod}_{dd}$ can be decomposed into $E[u]/u^{ep}$-modules $\mathcal{M}^\tau := \epsilon_\tau \mathcal{M}$, $\tau \in S$, so that $\hat{g} : \mathcal{M}^\tau \to \mathcal{M}^\tau$, and $\phi_1 : \mathcal{M}_1^\tau \to \mathcal{M}^{\tau\circ\phi^{-1}}$, so that $\mathcal{M}$ is free over $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$. We now write $S = \{\tau_1, \ldots, \tau_r\}$, numbered so that $\tau_{i+1} = \tau_i \circ \phi^{-1}$, where we identify $\tau_{r+1}$ with $\tau_1$. In fact, it will often be useful to consider the indexing set of $S$ to be $\mathbb{Z}/r\mathbb{Z}$, and we will do so without further comment.

Fix $J \subset S$. We wish to single out particular representations $\rho$ depending on $J$. Firstly, we need some notation. Recall that (as in Appendix B of [9]) if $\rho' : G_{K_0} \to \mathrm{GL}_2(\mathcal{O}_L)$ is potentially Barsotti-Tate, where $L$ is a finite extension of $W(E)[1/p]$, then there is a Weil-Deligne representation $WD(\rho') : W_{K_0} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ associated to $\rho'$, and we say that $\rho'$ has type $WD(\rho')|_{I_{K_0}}$.

**Definition 3.1** We say that $\rho$ has a *lift of type* $J$ if there is a representation $\rho' : G_{K_0} \to \mathrm{GL}_2(\mathcal{O}_L)$ lifting $\rho$, where $L$ is a finite extension of $W(E)[1/p]$, such that $\rho'$ becomes Barsotti-Tate over $K$, with $\varepsilon^{-1}\det\rho'$ equal to the Teichmüller lift of $\varepsilon^{-1}\det\rho$ (with $\varepsilon$ denoting the cyclotomic character) and $\rho'$ has type $\widetilde{\psi}_1|_{I_{K_0}} \prod_{\tau\in J} \widetilde{\omega}_\tau^{-p} \oplus \widetilde{\psi}_2|_{I_{K_0}} \prod_{\tau\notin J} \widetilde{\omega}_\tau^{-p}$. Here a tilde denotes the Teichmüller lift.

**Definition 3.2** For any subset $H \subset S$, we say that an element $\mathcal{M}$ of $\mathrm{BrMod}_{dd}$ is of class $H$ if it is of rank one, and for all $\tau \in S$ we can choose a basis $e_\tau$ of $\mathcal{M}^\tau$ such that $\mathcal{M}_1^\tau$ is generated by $u^{j_\tau} e_\tau$, where

$$j_\tau = \begin{cases} 0 & \text{if } \tau\circ\phi^{-1} \notin H, \\ e & \text{if } \tau\circ\phi^{-1} \in H. \end{cases}$$

**Definition 3.3** We say that an element $\mathcal{M}$ of $\mathrm{BrMod}_{dd}$ is of type $J$ if $\mathcal{M}$ is an extension of an element of class $J^c$ by an element of class $J$, and we say that $\rho$ has a model of type $J$ if there is an element of $\mathrm{BrMod}_{dd}$ of type $J$ with generic fibre $\rho$.

We will also refer to finite flat group schemes with descent data as being of class $J$ or of type $J$ if they correspond to Breuil modules with descent data of this kind. The notions of having a model of type $J$ and having a lift of type $J$ are closely related, although not in general equivalent. We will see in Sect. 3.3 that in sufficiently generic cases, if $\rho$ has a model of type $J$ then it has a lift of type $J$, and in Sect. 3.5 we prove a partial converse (see Proposition 3.13).

## 3.2 Strongly divisible modules

In this section we prove that if $\rho$ has a model of type $J$ then it has a lift of type $J$. We begin by recalling the definition and basic properties of strongly divisible modules from [20]. For the purpose of giving these definitions we return briefly to the general setting of $K_0$ an unramified finite extension of $\mathbb{Q}_p$ and $K$ a totally tamely ramified Galois extension of $K_0$ of degree $e$, with uniformiser $\pi$, satisfying $\pi^e \in M$ for some subfield $M$ of $K_0$.

Let $L$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and residue field $E$. Let $S_K$ be the ring

$$\left\{\sum_{j=0}^{\infty} r_j \frac{u^j}{\lfloor j/e \rfloor !},\ r_j \in W(k),\ r_j \to 0 \ p\text{-adically as } j \to \infty\right\},$$

and let $S_{K,\mathcal{O}_L} = S_K \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. Let $\mathrm{Fil}^1 S_{K,\mathcal{O}_L}$ be the $p$-adic completion of the ideal generated by $E(u)^j/j!$, $j \geq 1$, where $E(u)$ is the minimal polynomial of $\pi$ over $K_0$. Let $\phi : S_{K,\mathcal{O}_L} \to S_{K,\mathcal{O}_L}$ be the unique $\mathcal{O}$-linear, $W(k)$-semilinear ring homomorphism with $\phi(u) = u^p$, and let $N$ be the unique $W(k) \otimes \mathcal{O}_L$-linear derivation such that $N(u) = -u$ (so that $N\phi = p\phi N$). One can check that $\phi(\mathrm{Fil}^1 S_{K,\mathcal{O}_L}) \subset pS_{K,\mathcal{O}_L}$, and we define $\phi_1 : \mathrm{Fil}^1 S_{K,\mathcal{O}_L} \to S_{K,\mathcal{O}_L}$ by $\phi_1 = (\phi|_{\mathrm{Fil}^1 S_{K,\mathcal{O}_L}})/p$. One can check (see Sect. 4 of [20]) that if $I$ is an ideal of $\mathcal{O}_L$, then $IS_{K,\mathcal{O}_L} \cap \mathrm{Fil}^1 S_{K,\mathcal{O}_L} = I\,\mathrm{Fil}^1 S_{K,\mathcal{O}_L}$. We give $S_K$ an action of $\mathrm{Gal}(K/M)$ via ring isomorphisms via the usual action on $W(k)$, and by letting $\hat{g}(u) = (g(\pi)/\pi)u$. We extend this action $\mathcal{O}_L$-linearly to $S_{K,\mathcal{O}_L}$.

We now define the category $\mathcal{O}_L - \mathrm{Mod}^1_{cris,dd,M}$, the category of strongly divisible $\mathcal{O}_L$-modules with descent data from $K$ to $M$.

**Definition 3.4** A strongly divisible $\mathcal{O}_L$-module with descent data from $K$ to $M$ is a finitely generated free $S_{K,\mathcal{O}_L}$-module $\mathcal{M}$, together with a sub-$S_{K,\mathcal{O}_L}$-module $\mathrm{Fil}^1 \mathcal{M}$ and a map $\phi : \mathcal{M} \to \mathcal{M}$, and additive bijections $\hat{g} : \mathcal{M} \to \mathcal{M}$ for each $g \in \mathrm{Gal}(K/M)$, satisfying the following conditions:

(1) $\mathrm{Fil}^1 \mathcal{M}$ contains $(\mathrm{Fil}^1 S_{K,\mathcal{O}_L})\mathcal{M}$,
(2) $\mathrm{Fil}^1 \mathcal{M} \cap I\mathcal{M} = I\,\mathrm{Fil}^1 \mathcal{M}$ for all ideals $I$ in $\mathcal{O}_L$,
(3) $\phi(sx) = \phi(s)\phi(x)$ for $s \in S_{K,\mathcal{O}_L}$ and $x \in \mathcal{M}$,
(4) $\phi(\mathrm{Fil}^1 \mathcal{M})$ is contained in $p\mathcal{M}$ and generates it over $S_{K,\mathcal{O}_L}$,
(5) $\hat{g}(sx) = \hat{g}(s)\hat{g}(x)$ for all $s \in S_{K,\mathcal{O}_L}$, $x \in \mathcal{M}$, $g \in \mathrm{Gal}(K/M)$,

(6) $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/M)$,
(7) $\hat{g}(\mathrm{Fil}^1 \mathcal{M}) \subset \mathrm{Fil}^1 \mathcal{M}$ for all $g \in \mathrm{Gal}(K/M)$, and
(8) $\phi$ commutes with $\hat{g}$ for all $g \in \mathrm{Gal}(K/M)$.

Note that it is not immediately obvious that this definition is equivalent to Definition 4.1 of [20], as we have made no mention of the operator $N$ of *loc. cit.* However, since $\mathcal{O}_L$ is finite over $\mathbb{Z}_p$, it follows from part (1) of Proposition 5.1.3 of [7] that any such operator $N$ is unique. The existence of an operator $N$ satisfying all of the conditions of Definition 4.1 of [20] except possibly for $\mathcal{O}_L$-linearity follows from the argument at the beginning of Sect. 3.5 of [20]. To check $\mathcal{O}_L$-linearity it is enough (by $\mathbb{Z}_p$-linearity) to check that $N$ is compatible with the action of the units in $\mathcal{O}_L$, but this is clear from the uniqueness of $N$.

By Proposition 4.13 of [20] (and the remarks immediately preceding it), there is a functor $T_{st,2}^M$ from the category $\mathcal{O}_L - \mathrm{Mod}^1_{cris,dd,M}$ to the category of $G_M$-stable $\mathcal{O}_L$-lattices in representations of $G_M$ which become Barsotti-Tate on restriction to $G_K$. This functor preserves dimensions in the obvious sense.

Recall also from Sect. 4.1 of [20] that there is a functor $T_0$, compatible with $T_{st,2}^M$, from $\mathcal{O}_L - \mathrm{Mod}^1_{cris,dd,M}$ to $\mathrm{BrMod}_{dd,M}$. The functor $T_0$ is given by $\mathcal{M} \mapsto (\mathcal{M}/\mathfrak{m}_L\mathcal{M}) \otimes_{S_K} k[u]/u^{ep}$.

### 3.3 Models of type $J$

We now wish to discuss the relationships between models of type $J$ and lifts of type $J$. With an eye to our future applications, we will often make a simplifying assumption.

**Definition 3.5** Say that $\rho$ is *$J$-regular* if $\psi_1\psi_2^{-1}|_{I_{K_0}} = \prod_{\tau\in J} \omega_\tau^{b_\tau} \prod_{\tau \in J^c} \omega_\tau^{-b_\tau}$ for some $2 \leq b_\tau \leq p-2$.

Suppose now that $\rho$ has a model of type $J$. Recall that this means that, with the notation of Sect. 3.1, we can write down a Breuil module $\mathcal{M}$ with descent data whose generic fibre is $\rho$, which is an extension of a Breuil module with descent data $\mathcal{B}$ by a Breuil module with descent data $\mathcal{A}$, where $\mathcal{A}$ is of class $J$ and $\mathcal{B}$ is of class $J^c$. Let $\psi_i'$ denote $\psi_i|_{I_{K_0}}$ regarded as a character of $\mathrm{Gal}(K/K_0)$. By Theorem 3.5 and Example 3.7 of [21] we see that we can choose bases for $\mathcal{A}$ and $\mathcal{B}$ so that they take the following form:

$$\mathcal{A}^{\tau_i} = E[u]/u^{ep} \cdot e_{\tau_i},$$

$$\mathcal{A}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{j_{\tau_i}} e_{\tau_i},$$

$$\phi_1(u^{j_{\tau_i}} e_{\tau_i}) = (a^{-1})_i e_{\tau_{i+1}},$$

$$\hat{g}(e_{\tau_i}) = \left(\left(\psi_1' \prod_{\sigma\in J} \omega_\sigma^{-p}\right)(g)\right) e_{\tau_i},$$

$$\mathcal{B}^{\tau_i} = E[u]/u^{ep} \cdot \overline{f}_{\tau_i},$$

$$\mathcal{B}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{e-j_{\tau_i}} \overline{f}_{\tau_i},$$

$$\phi_1(u^{e-j_{\tau_i}} \overline{f}_{\tau_i}) = (b^{-1})_i \overline{f}_{\tau_{i+1}},$$

$$\hat{g}(\overline{f}_{\tau_i}) = \left(\left(\psi_2' \prod_{\sigma\notin J} \omega_\sigma^{-p}\right)(g)\right) \overline{f}_{\tau_i}$$

where $a, b \in E^\times$, the notation $(x)_i$ means $x$ if $i = 1$ and 1 otherwise, and

$$j_{\tau_i} = \begin{cases} e & \text{if } \tau_{i+1} \in J, \\ 0 & \text{if } \tau_{i+1} \notin J. \end{cases}$$

We now seek to choose a basis for $\mathcal{M}$ extending the basis $\{e_\tau\}$ for $\mathcal{A}$. Such a basis will be given by lifting the $\overline{f}_\tau$ to elements $f_\tau$ (where we mean lifting under the map $e_\tau \mapsto 0$).

**Lemma 3.6** *Assume that $\rho$ is $J$-regular and has a model $\mathcal{M}$ of type $J$. Then for some choice of basis, we can write*

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot e_{\tau_i} + E[u]/u^{ep} \cdot f_{\tau_i},$$

$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{j_{\tau_i}} e_{\tau_i} + E[u]/u^{ep} \cdot (u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}),$$

$$\phi_1(u^{j_{\tau_i}} e_{\tau_i}) = (a^{-1})_i e_{\tau_{i+1}},$$

$$\phi_1(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}) = (b^{-1})_i f_{\tau_{i+1}},$$

$$\hat{g}(e_{\tau_i}) = \left(\left(\psi_1' \prod_{\sigma\in J} \omega_\sigma^{-p}\right)(g)\right) e_{\tau_i},$$

$$\hat{g}(f_{\tau_i}) = \left(\left(\psi_2' \prod_{\sigma\notin J} \omega_\sigma^{-p}\right)(g)\right) f_{\tau_i}$$

*where* $\lambda_{\tau_i} \in E$, *with* $\lambda_{\tau_i} = 0$ *if* $\tau_{i+1} \notin J$, *the* $i_{\tau_i}$ *are such that* $\mathcal{M}_1$ *is Galois-stable and* $0 \le i_{\tau_i} \le e-1$, *and*

$$j_{\tau_i} = \begin{cases} e & \text{if } \tau_{i+1} \in J, \\ 0 & \text{if } \tau_{i+1} \notin J. \end{cases}$$

*Proof* Assume firstly that $J \neq S$, and choose $k$ so that $\tau_{k+1} \notin J$. One can lift $\overline{f}_{\tau_k}$ to an element $f_{\tau_k}$ of $\phi_1(\mathcal{M}^{\tau_{k-1}})$, and in fact one can choose $f_{\tau_k}$ so that for all $g \in \mathrm{Gal}(K/K_0)$ we have

$$\hat{g}(f_{\tau_k}) = \left( \left( \psi_2' \prod_{\sigma \notin J} \omega_\sigma^{-p} \right)(g) \right) f_{\tau_k}$$

(the obstruction to doing this is easily checked to vanish, as the degree of $K/K_0$ is prime to $p$). As $\tau_{k+1} \notin J$, we have $j_{\tau_k} = 0$, so that $e_{\tau_k}$ and $u^e f_{\tau_k}$ must generate $\mathcal{M}_1^{\tau_k}$.

Now, suppose inductively that for some $i$ we have chosen $f_{\tau_i}$ and $\lambda_{\tau_i}$ so that $\mathcal{M}_1^{\tau_i}$ is generated by $u^{j_{\tau_i}} e_{\tau_i}$ and $(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})$. Then we put $f_{\tau_{i+1}} = \phi_1(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})/(b^{-1})_i$. Then $f_{\tau_{i+1}}$ is a lift of $\overline{f}_{\tau_{i+1}}$, and the commutativity of $\phi_1$ and the action of $\mathrm{Gal}(K/K_0)$ ensures that

$$\hat{g}(f_{\tau_{i+1}}) = \left( \left( \psi_2' \prod_{\sigma \notin J} \omega_\sigma^{-p} \right)(g) \right) f_{\tau_{i+1}}.$$

Then the fact that $\mathcal{M}_1$ is $\mathrm{Gal}(K/K_0)$-stable ensures that for some $\lambda_{t_{i+1}} \in E$ we must have that $u^{j_{\tau_{i+1}}} e_{\tau_{i+1}}$ and $(u^{e-j_{\tau_{i+1}}} f_{\tau_{i+1}} + \lambda_{\tau_{i+1}} u^{i_{\tau_{i+1}}} e_{\tau_{i+1}})$ generate $\mathcal{M}_1^{\tau_{i+1}}$, and of course if $\tau_{i+2} \notin J$ we can take $\lambda_{\tau_{i+1}} = 0$.

So, beginning at $k$ we inductively define $f_{\tau_i}$ and $\lambda_{\tau_i}$ for all $i$, which automatically satisfy all the required properties, except that we do not know that

$$\phi_1(u^{e-j_{\tau_{k-1}}} f_{\tau_{k-1}} + \lambda_{\tau_{k-1}} u^{i_{\tau_{k-1}}} e_{\tau_{k-1}}) = (b^{-1})_{k-1} f_{\tau_k}.$$

However, because $k+1 \notin J$, we may replace $f_{\tau_k}$ with $\phi_1(u^{e-j_{\tau_{k-1}}} f_{\tau_{k-1}} + \lambda_{\tau_{k-1}} u^{i_{\tau_{k-1}}} e_{\tau_{k-1}})/(b^{-1})_{k-1}$ without altering the fact that

$$\phi_1(u^e f_{\tau_k}) = (b^{-1})_k f_{\tau_{k+1}},$$

so we are done.

Suppose now that $J = S$. Then we may carry out a similar inductive procedure starting with $\tau_1$, and we again define $f_{\tau_i}$ and $\lambda_{\tau_i}$ for all $i$, satisfying all the required properties, except that we do not know that

$$\phi_1(f_{\tau_r} + \lambda_{\tau_r} u^{i_{\tau_r}} e_{\tau_r}) = f_{\tau_1}.$$

We wish to redefine $f_{\tau_1}$ to be $\phi_1(f_{\tau_r}+\lambda_{\tau_r}e_{\tau_r})$, and we claim that doing so does not affect the relation

$$\phi_1(f_{\tau_1}+\lambda_{\tau_1}u^{i_{\tau_1}}e_{\tau_1})=b^{-1}f_{\tau_2}.$$

To see this, note that we are modifying $f_{\tau_1}$ by a multiple of $e_{\tau_1}$ which is in the image of $\phi_1$, which by considering the action of $\mathrm{Gal}(K/K_0)$ must in fact be of the form $\theta u^{pi_{\tau_r}}e_{\tau_1}$, with $\theta\in E$ and $pi_{\tau_r}\equiv i_{\tau_1} \bmod e$. Now, the assumption that $\rho$ is $S$-regular means that $i_{\tau_1}=e-\sum_{l=1}^{r}p^{r-l}(b_{\tau_{l+1}}-1)\equiv -b_{\tau_1} \bmod p$, with $2\leq b_{\tau_l}\leq p-2$. Now, if we write $pi_{\tau_r}=i_{\tau_1}+me$, we see that $m\equiv i_{\tau_1}\equiv -b_{\tau_1} \bmod p$, and since $2\leq b_{\tau_1}\leq p-2$ we see that $m\geq 2$. But then $\phi_1(\theta u^{pi_{\tau_r}}e_{\tau_1})=\phi_1(\theta u^{i_{\tau_1}+(m-1)e}u^e e_{\tau_1})$ is divisible by $u^{p(m-1)e}$ and is thus 0, as required. □

**Theorem 3.7** *Assume that $\rho$ is $J$-regular and has a model of type $J$. Then $\rho$ has a lift of type $J$, which is potentially ordinary if and only if $J=S$ or $J=\emptyset$.*

*Proof* We will write down an element $\mathcal{M}_J$ of $W(E)-\mathrm{Mod}_{cris,dd,K_0}$ such that $T_0(\mathcal{M}_J)=\mathcal{M}$, where $\mathcal{M}$ is as in Lemma 3.6. We can write $S_{K,W(E)}$ as $\bigoplus_{\tau\in S}S_K$, and we then define

$$\mathcal{M}_J^{\tau_i}=S_{K,W(E)}\cdot e_{\tau_i}+S_{K,W(E)}\cdot f_{\tau_i},$$

$$\hat{g}(e_{\tau_i})=\left(\left(\widetilde{\psi}_1'\prod_{\sigma\in J}\widetilde{\omega}_\sigma^{-p}\right)(g)\right)e_{\tau_i},$$

$$\hat{g}(f_{\tau_i})=\left(\left(\widetilde{\psi}_2'\prod_{\sigma\notin J}\widetilde{\omega}_\sigma^{-p}\right)(g)\right)f_{\tau_i}.$$

If $\tau_{i+1}\in J$,

$$\begin{aligned}\mathrm{Fil}^1\mathcal{M}_J^{\tau_i}&=\mathrm{Fil}^1 S_{K,W(E)}\cdot\mathcal{M}_J^{\tau_i}+S_{K,W(E)}\cdot(f_{\tau_i}+\tilde{\lambda}_{\tau_i}u^{i_{\tau_i}}e_{\tau_i}),\\ \phi(e_{\tau_i})&=(\tilde{a}^{-1})_i e_{\tau_{i+1}},\\ \phi(f_{\tau_i}+\tilde{\lambda}_{\tau_i}u^{i_{\tau_i}}e_{\tau_i})&=(\tilde{b}^{-1})_i p f_{\tau_{i+1}}.\end{aligned}$$

If $\tau_{i+1}\notin J$,

$$\begin{aligned}\mathrm{Fil}^1\mathcal{M}_J^{\tau_i}&=\mathrm{Fil}^1 S_{K,W(E)}\cdot\mathcal{M}_J^{\tau_i}+S_{K,W(E)}\cdot e_{\tau_i},\\ \phi(e_{\tau_i})&=(\tilde{a}^{-1})_i p e_{\tau_{i+1}},\\ \phi(f_{\tau_i})&=(\tilde{b}^{-1})_i f_{\tau_{i+1}}.\end{aligned}$$

Here a tilde denotes a Teichmüller lift.

Firstly we verify that this really is an element of $W(E)-\mathrm{Mod}^1_{cris,dd,K_0}$. Of the properties in Definition 3.4, the only non-obvious points are that $\mathrm{Fil}^1\mathcal{M}_J\cap I\mathcal{M}_J=I\,\mathrm{Fil}^1\mathcal{M}_J$ for all ideals $I$ of $\mathcal{O}_L$, and that $\phi(\mathrm{Fil}^1\mathcal{M}_J)$ is contained in $p\mathcal{M}_J$ and generates it over $S_{K,W(E)}$. But these are both straightforward; that $\mathrm{Fil}^1\mathcal{M}_J\cap I\mathcal{M}_J=I\,\mathrm{Fil}^1\mathcal{M}_J$ follows at once from the definition of $\mathrm{Fil}^1\mathcal{M}_J$ and the corresponding assertion for $S_K$, and that $\phi(\mathrm{Fil}^1\mathcal{M}_J)$ is contained in $p\mathcal{M}_J$ and generates it over $S_{K,W(E)}$ follows by inspection and the corresponding assertions for $S_K$.

It is immediate from the definition of $T_0$ that $T_0(\mathcal{M}_J)\simeq\mathcal{M}$. To see that $T^{K_0}_{st,2}(\mathcal{M}_J)$ is a lift of $\rho$ of type $J$, note firstly that the Hodge-Tate weights of $T^{K_0}_{st,2}(\mathcal{M}_J)$ can be read off from the form of the filtration, exactly as in the last two paragraphs of the proof of Theorem 6.1 of [16]. This shows that the determinant is a finite order character times the cyclotomic character, and it also shows that the representation is potentially ordinary if and only if $J=S$ or $J=\emptyset$. That the lift is of type $J$ is then immediate from the form of the $\mathrm{Gal}(K/K_0)$-action and Proposition 5.1 of [16]. □

## 3.4 Breuil modules and Fontaine-Laffaille theory

In this section we relate the notion of having a model of type $J$ to that of possessing a certain crystalline lift. Suppose as usual that $\rho\sim\begin{pmatrix}\psi_1 & *\\ 0 & \psi_2\end{pmatrix}$, and that we can write $\psi_1|_{I_{K_0}}=\prod_{\tau\in J}\omega_\tau^{b_\tau}$, $\psi_2|_{I_{K_0}}=\prod_{\tau\notin J}\omega_\tau^{b_\tau}$ with $2\le b_\tau\le p-2$ (note that for a fixed $J$ it is *not* always possible to do this, even after twisting - indeed, up to twisting it is equivalent to $\rho$ being $J$-regular). In this case we define canonical crystalline lifts $\psi_{1,J}$, $\psi_{2,J}$ of $\psi_1$, $\psi_2$, as in Sect. 2. That is, we demand that for some choice of a Frobenius element $\mathrm{Frob}_{K_0}\in G_{K_0}$, $\psi_{i,J}(\mathrm{Frob}_{K_0})$ is the Teichmüller lift of $\psi_i(\mathrm{Frob}_{K_0})$, and that:

- $\psi_{1,J}$ is crystalline, and the Hodge-Tate weight of $\psi_{1,J}$ with respect to $\tau$ is $b_\tau$ if $\tau\in J$, and 0 if $\tau\notin J$.
- $\psi_{2,J}$ is crystalline, and the Hodge-Tate weight of $\psi_{2,J}$ with respect to $\tau$ is $b_\tau$ if $\tau\notin J$, and 0 if $\tau\in J$.

The main result of this section is

**Proposition 3.8** *Under the above hypotheses, $\rho$ has a model of type $J$ if and only if $\rho$ has a lift to a crystalline representation* $\begin{pmatrix}\psi_{1,J} & *\\ 0 & \psi_{2,J}\end{pmatrix}$.

*Proof* The idea of the proof is to express both the condition of having a model of type $J$ and the condition of having a crystalline lift of the prescribed type in terms of conditions on strongly divisible modules. In fact, we already have a description of the general model of type $J$ in terms of Breuil modules with descent data, and it is easy to write down the general crystalline representation $\begin{pmatrix} \psi_{1,J} & * \\ 0 & \psi_{2,J} \end{pmatrix}$ in terms of Fontaine-Laffaille theory. The only difficulty comes in relating the generic fibres of the Breuil modules to the generic fibres of the Fontaine-Laffaille modules, as the image of the functors describing passage to the generic fibre is in general too complicated to describe directly. Fortunately, it is relatively easy to compare the two generic fibres we obtain, without explicitly determining either.

Let $\mathcal{M} \in \mathrm{BrMod}_{dd}^{k-1}$ for some $k \in [2, p-1]$. Let $\hat{A}$ be the filtered ring defined in Sect. 2.1 of [8]. There is a contravariant functor $T_{st}^*$ from $\mathrm{BrMod}_{dd}^{k-1}$ to the category of $E$-representations of $G_{K_0}$ given by

$$T_{st}^*(\mathcal{M}) := \mathrm{Hom}_{k[u]/u^{ep},\phi_{k-1},N,\mathrm{Fil}^{\cdot}}(\mathcal{M}, \hat{A})$$

(where compatibility with $\mathrm{Fil}^{\cdot}$ means that the image of $\mathcal{M}_{k-1}$ is contained in $\mathrm{Fil}^{k-1}\hat{A}$). The action of $G_{K_0}$ on $T_{st}^*(\mathcal{M})$ is given by

$$(gf)(x) := gf(\widehat{\overline{g}}^{-1}(x)),$$

where $\overline{g}$ is the image of $g$ in $\mathrm{Gal}(K/K_0)$, and the action of $\mathrm{Gal}(K/K_0)$ on $\hat{A} = \hat{A}_K$ is defined in Sect. 4.2 of [8]. For the compatibility of this definition with those used in [6], [10] and [20], see Lemma 3.3.1.2 of [5]. This functor is exact and faithful, and preserves dimension in the obvious sense. To see these properties, it is enough to work with the category $\mathrm{BrMod}^{k-1}$ without descent data, and it is also straightforward to see that it suffices to consider the case $E = \mathbb{F}_p$. In this case, the fact that $T_{st}^*$ is faithful is Corollary 2.3.3 of [10], and exactness follows from Proposition 2.3.1 of [8] and the duality explained in Sect. 2.1 of [8]. The preservation of dimension is Lemma 2.3.1.2 of [6].

We will see below that by Breuil's generalisation of Fontaine-Laffaille theory (see [5]) there are objects of $\mathrm{BrMod}_{dd}^{p-2}$ which correspond via $T_{st}^*$ to the reductions mod $\pi$ of crystalline representations with Hodge-Tate weights in $[0, p-2]$. In order to compare the generic fibres of these Breuil modules to those of finite flat group schemes with descent data, we need to be able to compare elements of $\mathrm{BrMod}_{dd}^1$ and $\mathrm{BrMod}_{dd}^{p-2}$. This is straightforward: it is easy to check that there is a fully faithful functor from $\mathrm{BrMod}_{dd}^1$ to $\mathrm{BrMod}_{dd}^{p-2}$, given by defining (for $\mathcal{M} \in \mathrm{BrMod}_{dd}^1$) $\mathcal{M}_{p-2} := u^{e(p-3)}\mathcal{M}_1$, $\phi_{p-2}(u^{e(p-3)}x) = \phi_1(x)$ for all $x \in \mathcal{M}_1$, and leaving the other structures unchanged. This functor commutes with $T_{st}^*$.

Because we are now using the functor $T_{st}^*$ rather than $T_{st,2}^{K_0}$, the form of the Breuil modules (and in particular their descent data) corresponding to models of type $J$ under $T_{st}^*$ is slightly different. We will simultaneously write it as an element of $\mathrm{BrMod}_{dd}^1$ and $\mathrm{BrMod}_{dd}^{p-2}$ (making use of the fully faithful functor of the previous paragraph), by specifying $\mathcal{M}_1$, $\mathcal{M}_{p-2}$, $\phi_1$ and $\phi_{p-2}$.

Explicitly, we see (recalling that the operator $N$ is uniquely determined for an element of $\mathrm{BrMod}^1_{dd}$, so it suffices to check that it satisfies $N(\mathcal{M}) \subset u\mathcal{M}$ and the commutation relations with $\phi_1$ and $\hat{g}$, which we will check below) from Lemma 3.6 that $\rho$ has a model of type $J$ if and only if there are $\lambda_{\tau_i} \in E$ with $\lambda_{\tau_i} = 0$ if $\tau_{i+1} \notin J$, and elements $a, b \in E^\times$ such that $\rho \cong T^*_{st}(\mathcal{M})$, where

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot e_{\tau_i} + E[u]/u^{ep} \cdot f_{\tau_i},$$

$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{j_{\tau_i}} e_{\tau_i} + E[u]/u^{ep} \cdot (u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}),$$

$$\begin{aligned}\mathcal{M}_{p-2}^{\tau_i} = {} & E[u]/u^{ep} \cdot u^{(p-3)e+j_{\tau_i}} e_{\tau_i} \\ & + E[u]/u^{ep} \cdot (u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i}),\end{aligned}$$

$$\phi_1(u^{j_{\tau_i}} e_{\tau_i}) = (a^{-1})_i e_{\tau_{i+1}},$$

$$\phi_1(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}) = (b^{-1})_i f_{\tau_{i+1}},$$

$$\phi_{p-2}(u^{(p-3)e+j_{\tau_i}} e_{\tau_i}) = (a^{-1})_i e_{\tau_{i+1}},$$

$$\phi_{p-2}(u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i}) = (b^{-1})_i f_{\tau_{i+1}},$$

$$\hat{g}(e_{\tau_i}) = \left(\left(\prod_{\sigma \notin J} \omega_\sigma^{p-b_\sigma}\right)(g)\right) e_{\tau_i},$$

$$\hat{g}(f_{\tau_i}) = \left(\left(\prod_{\sigma \in J} \omega_\sigma^{p-b_\sigma}\right)(g)\right) f_{\tau_i},$$

$$N(e_{\tau_i}) = 0,$$

$$N(f_{\tau_i}) = -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}}} e_{\tau_i}$$

where $\lambda_{\tau_i} \in E$, with $\lambda_{\tau_i} = 0$ if $\tau_{i+1} \notin J$, the $i_{\tau_i}$ are such that $\mathcal{M}_{p-2}$ is Galois-stable and $0 \le i_{\tau_i} \le e-1$, and

$$j_{\tau_i} = \begin{cases} e & \text{if } \tau_{i+1} \in J, \\ 0 & \text{if } \tau_{i+1} \notin J. \end{cases}$$

To see that $N(\mathcal{M}) \subset u\mathcal{M}$, it is enough to check that $i_{\tau_i} > 0$ for all $i$. In fact, we claim that we have $p i_{\tau_i} \ge e$ for all $i$. To see this, note that by definition we have $i_{\tau_{i+1}} \equiv pi_{\tau_i} \pmod{e}$. If $pi_{\tau_i} < e$ for some $i$, then this congruence forces $i_{\tau_{i+1}} = pi_{\tau_i}$. However, it is easy to check that since $2 \le b_{\tau_i} \le p-2$ for all $i$, no $i_{\tau_i}$ is divisible by $p$ (for example, by (6) below we have

$$i_{\tau_i} \equiv b_{\tau_i}(\delta_{J^c}(\tau_i) - \delta_J(\tau_i)) - \delta_{J^c}(\tau_{i+1}) \pmod{p},$$

which is never $0 \pmod{p}$), so the claim follows. The compatibility of $N$ with $\hat{g}$ is evident from the definition of $i_{\tau_i}$.

To see that $u^e N(\mathcal{M}_1) \subset \mathcal{M}_1$, and that $\phi_1(u^e N(x)) = N(\phi_1(x))$ for all $x \in \mathcal{M}_1$, we compute as follows (recalling that the Leibniz rule implies that $N(u^i x) = u^i N(x) - iu^i x$):

$$N(u^{j_{\tau_i}} e_{\tau_i}) = -j_{\tau_i} u^{j_{\tau_i}} e_{\tau_i} \in \mathcal{M}_1,$$

so that

$$\phi_1(u^e N(u^{j_{\tau_i}} e_{\tau_i})) = 0 = N((a^{-1})_i e_{\tau_{i+1}}) = N(\phi_1(u^{j_{\tau_i}} e_{\tau_i})).$$

Similarly, we have

$$\begin{aligned} N(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}) = {} & -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{e-j_{\tau_i}+pi_{\tau_{i-1}}} e_{\tau_i} \\ & - \Big((e - j_{\tau_i}) u^{e-j_{\tau_i}} f_{\tau_i} + i_{\tau_i} \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}\Big). \end{aligned}$$

Recalling that if $y \in \mathcal{M}_1$ then $\phi_1(u^e y) = 0$, we see that it is enough to compute the right hand side modulo $\mathcal{M}_1$. Since $pi_{\tau_{i-1}} \ge e$, the exponent of $u$ in the first term on the right hand side is at least $2e - j_{\tau_i} \ge j_{\tau_i}$, so this term is contained in $\mathcal{M}_1$. We thus see that modulo $\mathcal{M}_1$, the right hand side is congruent to

$$(e - j_{\tau_i} - i_{\tau_i}) \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i} = -i_{\tau_i} \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}$$

(since if $\lambda_{\tau_i} \ne 0$, we have $\tau_{i+1} \in J$, so that $j_{\tau_i} = e$). Then

$$\begin{aligned} \phi_1(-u^e i_{\tau_i} \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}) &= \phi_1(-i_{\tau_i} \lambda_{\tau_i} u^{i_{\tau_i}} u^{j_{\tau_i}} e_{\tau_i}) \\ &= -i_{\tau_i} \lambda_{\tau_i} u^{pi_{\tau_i}} (a^{-1})_i e_{\tau_{i+1}} \\ &= (b^{-1})_i N(f_{\tau_{i+1}}) \\ &= N(\phi_1(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})) \end{aligned}$$

as required.

It is an easy exercise to write down the reductions mod $p$ of the strongly divisible modules corresponding to crystalline representations $\begin{pmatrix} \psi_{1,J} & * \\ 0 & \psi_{2,J} \end{pmatrix}$, as we now explain. Firstly, we must recall one of the main results of [12]. Let $L$ be a finite extension of $\mathbb{Q}_p$ with residue field $E$. We say that an admissible $\mathcal{O}_L$-lattice is a finite free $(\mathcal{O}_{K_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_L)$-module $M$ together with a decreasing filtration $\mathrm{Fil}^i M$ by $\mathcal{O}_{K_0}$-direct summands and $\phi$-linear, $\mathcal{O}_L$-linear maps $\phi_i : \mathrm{Fil}^i M \to M$ for all $0 \le i \le p-2$ such that

- $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} M = 0$.
- For all $0 \le i \le p-3$, $\phi_i|_{\mathrm{Fil}^{i+1} M} = p\phi_{i+1}$.
- $\sum_{i=0}^{p-2} \phi_i(\mathrm{Fil}^i M) = M$.

There is an exact functor $T^*_{cris}$ from the category of admissible $\mathcal{O}_L$-lattices to the category of $G_{K_0}$-representations on free $\mathcal{O}_L$-lattices defined by

$$T^*_{cris}(M) = \mathrm{Hom}_{\mathcal{O}_{K_0},\mathrm{Fil}^{\cdot},\phi}(M, A_{cris}).$$

This gives an equivalence of categories between the category of admissible $\mathcal{O}_L$-lattices and the category of $G_{K_0}$-stable $\mathcal{O}_L$-lattices in crystalline $L$-representations in $G_{K_0}$ with all Hodge-Tate weights in $[0, p-2]$.

In particular, one can easily write down the form of the rank one $\mathcal{O}_L$-lattices corresponding to the characters $\psi_{1,J}$ and $\psi_{2,J}$, and we must then compute the possible form of extensions of these two lattices. As usual, we decompose $M$ as a direct sum of $\mathcal{O}_L$-modules $M^{\tau_i}$. We obtain the following general form:

$$M^{\tau_i} = \mathcal{O}_L E_{\tau_i} + \mathcal{O}_L F_{\tau_i},$$

$$\mathrm{Fil}^0 M^{\tau_i} = M^{\tau_i},$$

$$\mathrm{Fil}^{b_{\tau_i}+1} M^{\tau_i} = 0,$$

$$\text{if } \tau_i \in J, \quad \mathrm{Fil}^j M^{\tau_i} = \mathcal{O}_L F_{\tau_i} \quad \text{for all } 1 \le j \le b_{\tau_i},$$

$$\text{if } \tau_i \in J^c, \quad \mathrm{Fil}^j M^{\tau_i} = \mathcal{O}_L E_{\tau_i} \quad \text{for all } 1 \le j \le b_{\tau_i},$$

$$\text{if } \tau_i \in J, \quad \phi_0(E_{\tau_i}) = (\tilde{a}^{-1})_i E_{\tau_{i+1}} \quad \text{and}$$
$$\phi_{b_{\tau_i}}(F_{\tau_i}) = (\tilde{b}^{-1})_i (F_{\tau_{i+1}} - \lambda'_{\tau_i} E_{\tau_{i+1}}),$$

$$\text{if } \tau_i \notin J, \quad \phi_{b_{\tau_i}}(E_{\tau_i}) = (\tilde{a}^{-1})_i E_{\tau_{i+1}} \quad \text{and}$$
$$\phi_0(F_{\tau_i}) = (\tilde{b}^{-1})_i (F_{\tau_{i+1}} - \lambda'_{\tau_i} E_{\tau_{i+1}})$$

where $\tilde{a}, \tilde{b} \in \mathcal{O}_L^\times$, $\lambda'_{\tau_i} \in \mathcal{O}_L$, and $\lambda'_{\tau_i} = 0$ if $\tau_{i+1} \notin J$. To see this, note that the form of the filtration is easily deduced from the relationship between the filtration of a Fontaine-Laffaille module, and the Hodge-Tate weights of the corresponding Galois representation, and the form of the Frobenius action on the $E_{\tau_i}$ is also determined. To see that we can arrange the Frobenius action as claimed, suppose firstly that $J \neq \emptyset$, and choose $\tau_i \in J$. Then $F_{\tau_i}$ is determined (by the form of the filtration) up to an element of $\mathcal{O}_L^\times$, and we fix a choice of $F_{\tau_i}$. If $\tau_{i+1} \notin J$, we can simply define $F_{\tau_{i+1}} = (\tilde{b})_i \phi_{b_{\tau_i}}(F_{\tau_i})$. If $\tau_{i+1} \in J$, then there is a unique $\lambda'_{\tau_i} \in \mathcal{O}_L$ such that

$$\mathrm{Fil}^{b_{\tau_{i+1}}} M^{\tau_{i+1}} = \mathcal{O}_L((\tilde{b})_i \phi_{b_{\tau_i}}(F_{\tau_i}) + \lambda'_{\tau_i} E_{\tau_{i+1}}),$$

and we set

$$F_{\tau_{i+1}} = (\tilde{b})_i \phi_{b_{\tau_i}}(F_{\tau_i}) + \lambda'_{\tau_i} E_{\tau_{i+1}}.$$

We can then continue in the same fashion, defining $F_{\tau_{i+2}}$ and so on, and the fact that $\tau_i \in J$ gives us the freedom to choose $\lambda'_{\tau_{i-1}}$ so that

$$\phi_{\delta_J(\tau_{i-1}) b_{\tau_{i-1}}}(F_{\tau_{i-1}}) = (\tilde{b}^{-1})_{i-1}(F_{\tau_i} - \lambda'_{\tau_{i-1}} E_{\tau_i}).$$

The case $J = \emptyset$ is similar, except that one may need to modify the initial choice of $F_{\tau_i}$; the argument is very similar to that used in the case $J = S$ in the proof of Lemma 3.6. In this case one also needs to use the fact that $\tilde{b}^{-1} - \tilde{a}^{-1} p^{\sum_{\tau \in S} b_\tau} \in \mathcal{O}_L^\times$, which holds as $\sum_{\tau \in S} b_\tau > 0$.

Breuil's generalisation of Fontaine-Laffaille theory ([5]) allows us to reduce these modules mod $\pi_L$ and obtain the corresponding elements of the category $K_0 - \mathrm{BrMod}_{dd,K_0}^{p-2}$ of Breuil modules with descent data for the case $K = K_0$ (so that the descent data is trivial, as the group $\mathrm{Gal}(K/K_0)$ is trivial). We find that they are of the form:

$$\mathcal{Q}^{\tau_i} = E[u]/u^p \cdot E_{\tau_i} + E[u]/u^p \cdot F_{\tau_i},$$

$$\mathcal{Q}_{p-2}^{\tau_i} = E[u]/u^p \cdot u^{(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i} + E[u]/u^p \cdot u^{(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i},$$

$$\phi_{p-2}(u^{(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i}) = (a^{-1})_i E_{\tau_{i+1}},$$

$$\phi_{p-2}(u^{(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i}) = (b^{-1})_i (F_{\tau_{i+1}} - \lambda'_{\tau_i} E_{\tau_{i+1}}),$$

$$N(E_{\tau_i}) = 0,$$

$$N(F_{\tau_i}) = 0$$

where $\lambda'_{\tau_i} \in E$, with $\lambda'_{\tau_i} = 0$ if $\tau_{i+1} \notin J$.

Of course, we wish to know the corresponding objects of $\mathrm{BrMod}_{dd}^{p-2}$. This is straightforward: by Proposition 4.2.2 of [8], and the discussion preceding and following it, we see that we can obtain the requisite modules by simply taking the extension of scalars $k[u]/u^p \to k[u]/u^{ep}$ given by $u \mapsto u^e$, and allowing $\mathrm{Gal}(K/K_0)$ to act via its action on $k[u]/u^{ep}$. We obtain the following general form:

$$\mathcal{N}^{\tau_i} = E[u]/u^{ep} \cdot E_{\tau_i} + E[u]/u^{ep} \cdot F_{\tau_i},$$

$$\mathcal{N}^{\tau_i}_{p-2} = E[u]/u^{ep} \cdot u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i} + E[u]/u^{ep} \cdot u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i},$$

$$\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i}) = (a^{-1})_i E_{\tau_{i+1}},$$

$$\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i}) = (b^{-1})_i (F_{\tau_{i+1}} - \lambda'_{\tau_i} E_{\tau_{i+1}}),$$

$$\hat{g}(E_{\tau_i}) = E_{\tau_i},$$

$$\hat{g}(F_{\tau_i}) = F_{\tau_i},$$

$$N(E_{\tau_i}) = 0,$$

$$N(F_{\tau_i}) = 0$$

where $\lambda'_{\tau_i} \in E$, with $\lambda'_{\tau_i} = 0$ if $\tau_{i+1} \notin J$. We claim that if for each $i$ we have

$$\lambda_{\tau_i}(b)_i = \lambda'_{\tau_i}(a)_i \tag{3.1}$$

then $T^*_{st}(\mathcal{M}) \cong T^*_{st}(\mathcal{N})$. This is of course enough to demonstrate the proposition, as given any set of $\lambda_{\tau_i}$ (respectively $\lambda'_{\tau_i}$) such that $\lambda_{\tau_i} = 0$ (respectively $\lambda'_{\tau_i} = 0$) if $\tau_{i+1} \notin J$, we may choose a set of $\lambda'_{\tau_i}$ (respectively $\lambda_{\tau_i}$) so that (3.1) holds.

Assume now that (3.1) holds. Note that we may write both $\mathcal{M}$ and $\mathcal{N}$ as extensions

$$0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to 0,$$

$$0 \to \mathcal{N}'' \to \mathcal{N} \to \mathcal{N}' \to 0$$

with $T^*_{st}(\mathcal{M}'') \cong T^*_{st}(\mathcal{N}'') \cong \psi_2$, $T^*_{st}(\mathcal{M}') \cong T^*_{st}(\mathcal{N}') \cong \psi_1$.

To prove that $T^*_{st}(\mathcal{M}) \cong T^*_{st}(\mathcal{N})$, we will construct a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}'' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}' & \longrightarrow & 0 \\
 & & \downarrow f_{\mathcal{M}''} & & \downarrow f_{\mathcal{M}} & & \downarrow f_{\mathcal{M}'} & & \\
0 & \longrightarrow & \mathcal{P}'' & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}' & \longrightarrow & 0 \\
 & & \uparrow f_{\mathcal{N}''} & & \uparrow f_{\mathcal{N}} & & \uparrow f_{\mathcal{N}'} & & \\
0 & \longrightarrow & \mathcal{N}'' & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{N}' & \longrightarrow & 0
\end{array}$$

such that each of $T^*_{st}(f_{\mathcal{M}''})$, $T^*_{st}(f_{\mathcal{M}'})$, $T^*_{st}(f_{\mathcal{N}''})$ and $T^*_{st}(f_{\mathcal{N}'})$ are isomorphisms. From the five lemma it then follows that $T^*_{st}(f_{\mathcal{M}})$ and $T^*_{st}(f_{\mathcal{N}})$ are isomorphisms, and we will be done.

In fact, we take

$$\mathcal{P}^{\tau_i} = E[u]/u^{ep} \cdot e'_{\tau_i} + E[u]/u^{ep} \cdot f'_{\tau_i},$$

$$\mathcal{P}^{\tau_i}_{p-2} = E[u]/u^{ep} \cdot u^{n_{\tau_i}} e'_{\tau_i} + E[u]/u^{ep} \cdot (u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i} - \beta_{\tau_{i+1}}} e'_{\tau_i}),$$

$$\phi_{p-2}(u^{n_{\tau_i}} e'_{\tau_i}) = (a^{-1})_i e'_{\tau_{i+1}},$$

$$\phi_{p-2}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i} - \beta_{\tau_{i+1}}} e'_{\tau_i}) = (b^{-1})_i f'_{\tau_{i+1}},$$

$$\hat{g}(e'_{\tau_i}) = \nu_{1,\tau_i}(g) e'_{\tau_i},$$

$$\hat{g}(f'_{\tau_i}) = \nu_{2,\tau_i}(g) f'_{\tau_i},$$

$$N(e'_{\tau_i}) = 0,$$

$$N(f'_{\tau_i}) = -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}} - p\alpha_{\tau_i}} e'_{\tau_i}$$

where

$$\alpha_{\tau_i} = \sum_{j=0}^{r-1} p^{r-1-j} \left( b_{\tau_{i+j}} \delta_{J^c}(\tau_{i+j}) - \delta_{J^c}(\tau_{i+j+1}) \right),$$

$$\beta_{\tau_i} = \sum_{j=0}^{r-1} p^{r-1-j} \left( b_{\tau_{i+j}} \delta_{J}(\tau_{i+j}) - \delta_{J}(\tau_{i+j+1}) \right),$$

$$\nu_{1,\tau_i}(g) = \begin{cases} \prod_{\sigma \notin J} \omega_\sigma^{p-b_\sigma}(g) & \text{if } \tau_i \notin J, \\ 1 & \text{if } \tau_i \in J, \end{cases}$$

$$\nu_{2,\tau_i}(g) = \begin{cases} \prod_{\sigma \in J} \omega_\sigma^{p-b_\sigma}(g) & \text{if } \tau_i \in J, \\ 1 & \text{if } \tau_i \notin J, \end{cases}$$

$$n_{\tau_i} = (p-2-\delta_{J^c}(\tau_i)b_{\tau_i})e + p\delta_{J^c}(\tau_i)\alpha_{\tau_i} - \delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}},$$

$$n'_{\tau_i} = (p-2-\delta_J(\tau_i)b_{\tau_i})e + p\delta_J(\tau_i)\beta_{\tau_i} - \delta_J(\tau_{i+1})\beta_{\tau_{i+1}}.$$

We then define $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ by

$$f_{\mathcal{M}}(e_{\tau_i}) = u^{-p\alpha_{\tau_i}\delta_J(\tau_i)} e'_{\tau_i},$$

$$f_{\mathcal{M}}(f_{\tau_i}) = u^{-p\beta_{\tau_i}\delta_{J^c}(\tau_i)} f'_{\tau_i},$$

$$f_{\mathcal{N}}(E_{\tau_i}) = u^{p\alpha_{\tau_i}\delta_{J^c}(\tau_i)} e'_{\tau_i},$$

$$f_{\mathcal{N}}(F_{\tau_i}) = u^{p\beta_{\tau_i}\delta_J(\tau_i)} f'_{\tau_i}.$$

We define $\mathcal{P}'$ to be the submodule generated by the $e'_{\tau_i}$, and $\mathcal{P}''$ to be the quotient obtained by $e'_{\tau_i} \mapsto 0$. The remaining maps are then defined by the commutativity of the diagram.

Before we verify that this construction behaves as claimed, we pause to record a number of useful identities and inequalities.

(1) If $\tau_{i+1} \notin J$, then $\lambda_{\tau_i} = \lambda'_{\tau_i} = 0$ by definition.

(2)

$$p\alpha_{\tau_i} - \alpha_{\tau_{i+1}} = e(b_{\tau_i}\delta_{J^c}(\tau_i) - \delta_{J^c}(\tau_{i+1})),$$

$$p\beta_{\tau_i} - \beta_{\tau_{i+1}} = e(b_{\tau_i}\delta_J(\tau_i) - \delta_J(\tau_{i+1})).$$

These both follow immediately from the definitions of $\alpha_{\tau_i}, \beta_{\tau_i}$.

(3)

$$n_{\tau_i} = \alpha_{\tau_{i+1}}\delta_J(\tau_{i+1}) - p\alpha_{\tau_i}\delta_J(\tau_i) + e(p-3) + e\delta_J(\tau_{i+1}),$$

$$n'_{\tau_i} = \beta_{\tau_{i+1}}\delta_{J^c}(\tau_{i+1}) - p\beta_{\tau_i}\delta_{J^c}(\tau_i) + e(p-3) + e\delta_{J^c}(\tau_{i+1}).$$

These both follow from the definitions of $n_{\tau_i}, n'_{\tau_i}$ and property (2) above.

(4) We have $\tau_i \in J$ if and only if $\beta_{\tau_i} > 0$ if and only if $\alpha_{\tau_i} \leq 0$. To see this, note that from the definition, the sign of $\alpha_{\tau_i}$ is determined by the sign of the first non-zero term in the sum (this uses that $2 \leq b_{\tau_j} \leq p-2$). If $\tau_i \notin J$ then the first term is positive, and thus so is the whole sum.

If $\tau_i \in J$ then either every term in the sum is zero, or the first non-zero term must be negative. A similar analysis applies to the sign of $\beta_{\tau_i}$.

(5)

$$-e/(p-1) < \alpha_{\tau_i}, \qquad \beta_{\tau_i} < e(p-2)/(p-1).$$

This is immediate from the definitions, and the fact that $2 \le b_{\tau_j} \le p-2$ for all $j$.

(6)

$$i_{\tau_{i-1}} = \alpha_{\tau_i} - \beta_{\tau_i} + e\delta_J(\tau_i).$$

It follows straightforwardly from the forms of the $\hat{g}$-actions that the two side are congruent modulo $e$, so it suffices to check that the right hand side is an element of $[0, e-1]$. This follows from points (4) and (5).

(7)

$$(p-2)e \ge n_{\tau_i}, \quad n_{\tau_i} \ge 0.$$

We demonstrate these inequalities for $n_{\tau_i}$, the argument for $n'_{\tau_i}$ being formally identical after exchanging $\alpha_{\tau_j}$ and $\beta_{\tau_j}$, $J$ and $J^c$. We examine 4 cases in turn. If $\tau_i \in J$ and $\tau_{i+1} \in J$, then $n_{\tau_i} = (p-2)e$ and there is nothing to prove. If $\tau_i \in J$ and $\tau_{i+1} \notin J$, then $n_{\tau_i} = (p-2)e - \alpha_{\tau_{i+1}}$, and the inequalities follow from points (4) and (5) above. If $\tau_i \notin J$ and $\tau_{i+1} \in J$ then by point (3) above we have $n_{\tau_i} = (p-2)e + \alpha_{\tau_{i+1}}$, and the inequalities follow from (4) and (5). Finally, if $\tau_i \notin J$ and $\tau_{i+1} \notin J$, then $n_{\tau_i} = (p-2-b_{\tau_i})e + p\alpha_{\tau_i} - \alpha_{\tau_{i+1}} = e(p-3)$ by (2).

(8) If $\tau_{i+1} \in J$, we have

$$n_{\tau_i} - \beta_{\tau_{i+1}} = e(p-3) - p\alpha_{\tau_i}\delta_J(\tau_i) + i_{\tau_i}.$$

This follows from (3) and (6) above.

(9) If $\tau_{i+1} \in J$, then

$$n_{\tau_i} - \beta_{\tau_{i+1}} \equiv n'_{\tau_i} + i_{\tau_i} \pmod{p}.$$

This follows from (3) and (8).

(10) If $\tau_{i+1} \in J$, then $n_{\tau_i} \ge \beta_{\tau_{i+1}}$. This follows from (4) and (8).

(11) If $\tau_{i+1} \in J$, then

$$n'_{\tau_i} + \beta_{\tau_{i+1}} \le e(p-2).$$

From (2) and (3), we obtain

$$n'_{\tau_i} + \beta_{\tau_{i+1}} = e(p-2) + \delta_J(\tau_i)(p\beta_{\tau_i} - eb_{\tau_i}),$$

so we must check that if $\tau_i \in J$, then $p\beta_{\tau_i} - eb_{\tau_i} \leq 0$. But by the definition of $\beta_{\tau_i}$, if $\tau_i \in J$ then we have

$$p\beta_{\tau_i} - eb_{\tau_i} = -p^r + b_{\tau_i} + \sum_{j=1}^{r-1} p^{r-j} \left(b_{\tau_{i+j}} \delta_J(\tau_{i+j}) - \delta_J(\tau_{i+j+1})\right)$$

and the result follows as $2 \leq b_{\tau_j} \leq p-2$ for all $j$.

(12) If $\tau_i \in J$, then

$$n'_{\tau_i} + pi_{\tau_{i-1}} - p\alpha_{\tau_i} \geq n_{\tau_i}.$$

To see this, by (2), (3) and (6) we have that if $\tau_{i+1} \in J$, then

$$\begin{aligned} n'_{\tau_i} + pi_{\tau_{i-1}} - p\alpha_{\tau_i} - n_{\tau_i} &= (p-1)e - p\beta_{\tau_i} \\ &\geq \left((p-1) - \frac{p(p-2)}{p-1}\right) e \\ &\geq 0 \end{aligned}$$

by (5). If on the other hand $\tau_{i+1} \notin J$, we find that

$$\begin{aligned} n'_{\tau_i} + pi_{\tau_{i-1}} - p\alpha_{\tau_i} - n_{\tau_i} &= (p+1-b_{\tau_i})e + p\alpha_{\tau_i} \\ &\geq 3e + p\alpha_{\tau_i} \\ &\geq \left(3 - \frac{p}{p-1}\right) e \\ &\geq 0 \end{aligned}$$

by (5).

We now verify that $\mathcal{P}$ is indeed an object of $\mathrm{BrMod}_{dd}^{p-2}$.

- To see that we have defined a $(k \otimes_{\mathbb{F}_p} E)[u])/u^{ep}$-module, we must check that all of the exponents of $u$ in the definition are nonnegative; so, we need to check the inequalities $n_{\tau_i} \geq 0$, $n'_{\tau_i} \geq 0$, and if $\lambda_{\tau_i} \neq 0$ we need to verify that $n_{\tau_i} \geq \beta_{\tau_{i+1}}$. These follow from (1), (7) and (10) above.
- To see that $u^{e(p-2)}\mathcal{P} \subset \mathcal{P}_{p-2}$, we need to verify that $(p-2)e \geq n_{\tau_i}$, that $(p-2)e \geq n'_{\tau_i}$, and if $\lambda_{\tau_i} \neq 0$ then $(p-2)e \geq n'_{\tau_i} + \beta_{\tau_{i+1}}$. These follow from (1), (7) and (11).
- To see that $N(\mathcal{P}) \subset u\mathcal{P}$, we need to check that if $\lambda_{\tau_{i-1}} \neq 0$ then $i_{\tau_{i-1}} > \alpha_{\tau_i}$. This follows from (1), (5) and (6).
- To see that $u^e N(\mathcal{P}_{p-2}) \subset \mathcal{P}_{p-2}$, we note by the Leibniz rule we have

$$N(u^{n_{\tau_i}} e'_{\tau_i}) = -n_{\tau_i} u^{n_{\tau_i}} e'_{\tau_i} \in \mathcal{P}_{p-2}$$

and

$$\begin{aligned}
&N(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}) \\
&\quad = -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{n'_{\tau_i}+pi_{\tau_{i-1}}-p\alpha_{\tau_i}} e'_{\tau_i} \\
&\qquad - n'_{\tau_i} u^{n'_{\tau_i}} f'_{\tau_i} - \lambda_{\tau_i}(n_{\tau_i} - \beta_{\tau_{i+1}}) u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}.
\end{aligned}$$

By (1) and (12), the first time on the right hand side is contained in $\mathcal{P}_{p-2}$. By (9), the remaining terms are equal to

$$-n'_{\tau_i}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}) - \lambda_{\tau_i} i_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}.$$

The first term is contained in $\mathcal{P}_{p-2}$ by definition, and if we multiply the second term by $u^e$, we obtain

$$-\lambda_{\tau_i} i_{\tau_i} u^{n_{\tau_i}+e-\beta_{\tau_{i+1}}} e'_{\tau_i},$$

which is contained in $\mathcal{P}_{p-2}$ by (5).

- To see that $\phi_{p-2}(u^e N(x)) = N(\phi_{p-2}(x))$ for all $x \in \mathcal{P}_{p-2}$, we recall that $\phi_{p-2}(u^e y) = 0$ if $y \in \mathcal{P}_{p-2}$. Thus

$$\phi_{p-2}(u^e N(u^{n_{\tau_i}} e'_{\tau_i})) = 0 = N((a^{-1})_i e'_{\tau_{i+1}}) = N(\phi_{p-2}(u^{n_{\tau_i}} e'_{\tau_i})).$$

We also have, using (1), (6) and the calculation of the previous bullet point,

$$\begin{aligned}
&\phi_{p-2}(u^e N(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i})) \\
&\quad = \phi_{p-2}(-\lambda_{\tau_i} i_{\tau_i} u^{n_{\tau_i}+e-\beta_{\tau_{i+1}}} e'_{\tau_i}) \\
&\quad = -\lambda_{\tau_i} i_{\tau_i} u^{p(e-\beta_{i+1})} (a^{-1})_i e'_{\tau_{i+1}} \\
&\quad = -\lambda_{\tau_i} i_{\tau_i} u^{p(i_{\tau_i}-\alpha_{\tau_{i+1}})} (a^{-1})_i e'_{\tau_{i+1}} \\
&\quad = (b^{-1})_i N(f'_{\tau_{i+1}}) \\
&\quad = N(\phi_{p-2}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i})).
\end{aligned}$$

- That $\mathcal{P}_{p-2}$ is $\hat{g}$-stable follows directly from the definitions of $\beta_{\tau_i}$, $n_{\tau_i}$ and $n'_{\tau_i}$.
- That the action of $\hat{g}$ commutes with $\phi_{p-2}$ follows from the definition of $n_{\tau_i}$ and $n'_{\tau_i}$.
- That the action of $\hat{g}$ commutes with $N$ follows from (6).

We now verify the claimed properties of $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$.

- In order that the maps $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ be defined, it is necessary that the exponents of $u$ in their definition be non-negative. This follows from (4).
- To see that $f_{\mathcal{M}}(\mathcal{M}_{p-2}) \subset \mathcal{P}_{p-2}$ and $f_{\mathcal{N}}(\mathcal{N}_{p-2}) \subset \mathcal{P}_{p-2}$, we compute as follows.

$$\begin{aligned} f_{\mathcal{M}}(u^{(p-3)e+j_{\tau_i}} e_{\tau_i}) &= u^{(p-3)e+j_{\tau_i}-n_{\tau_i}-p\alpha_{\tau_i}\delta_J(\tau_i)} u^{n_{\tau_i}} e'_{\tau_i} \\ &= u^{-\delta_J(\tau_{i+1})\alpha_{\tau_{i+1}}}(u^{n_{\tau_i}} e'_{\tau_i}) \end{aligned}$$

by (3) and the definition of $j_{\tau_i}$. Similarly, by using (1), (3) and (6), we find that

$$\begin{aligned} & f_{\mathcal{M}}(u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i}) \\ &\quad = u^{-\beta_{\tau_{i+1}}\delta_{J^c}(\tau_{i+1})}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}). \end{aligned}$$

In the same way, using (1) and the definitions of $n_{\tau_i}$ and $n'_{\tau_i}$,

$$f_{\mathcal{N}}(u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i}) = u^{\delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}}}(u^{n_{\tau_i}} e'_{\tau_i}),$$

$$\begin{aligned} & f_{\mathcal{N}}(u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i}) \\ &\quad = u^{\delta_J(\tau_{i+1})\beta_{\tau_{i+1}}}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}) - \lambda_{\tau_i} u^{n_{\tau_i}} e'_{\tau_i}. \end{aligned}$$

The result then follows from (4).
- To check that $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ commute with $\phi_{p-2}$, we again compute directly. We have

$$\begin{aligned} f_{\mathcal{M}}(\phi_{p-2}(u^{(p-3)e+j_{\tau_i}} e_{\tau_i})) &= f_{\mathcal{M}}((a^{-1})_i e_{\tau_{i+1}}) \\ &= (a^{-1})_i u^{-p\alpha_{\tau_{i+1}}\delta_J(\tau_{i+1})} e'_{\tau_{i+1}}, \end{aligned}$$

while

$$\begin{aligned} \phi_{p-2}(f_{\mathcal{M}}(u^{(p-3)e+j_{\tau_i}} e_{\tau_i})) &= \phi_{p-2}(u^{-\delta_J(\tau_{i+1})\alpha_{\tau_{i+1}}}(u^{n_{\tau_i}} e'_{\tau_i})) \\ &= (a^{-1})_i u^{-p\alpha_{\tau_{i+1}}\delta_J(\tau_{i+1})} e'_{\tau_{i+1}}. \end{aligned}$$

Similarly, we find

$$\begin{aligned} & f_{\mathcal{M}}(\phi_{p-2}(u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i})) \\ &\quad = f_{\mathcal{M}}((b^{-1})_i f_{\tau_{i+1}}) \\ &\quad = (b^{-1})_i u^{-p\beta_{\tau_{i+1}}\delta_{J^c}(\tau_{i+1})} f'_{\tau_{i+1}}, \end{aligned}$$

$$\begin{aligned}&\phi_{p-2}(f_{\mathcal{M}}(u^{(p-2)e-j_{\tau_i}}f_{\tau_i}+\lambda_{\tau_i}u^{(p-3)e+i_{\tau_i}}e_{\tau_i})))\\&\quad=\phi_{p-2}(u^{-\beta_{\tau_{i+1}}\delta_{J^c}(\tau_{i+1})}(u^{n'_{\tau_i}}f'_{\tau_i}+\lambda_{\tau_i}u^{n_{\tau_i}-\beta_{\tau_{i+1}}}e'_{\tau_i})))\\&\quad=(b^{-1})_i u^{-p\beta_{\tau_{i+1}}\delta_{J^c}(\tau_{i+1})}f'_{\tau_{i+1}},\end{aligned}$$

$$\begin{aligned}f_{\mathcal{N}}(\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))}E_{\tau_i}))&=f_{\mathcal{N}}((a^{-1})_i E_{\tau_{i+1}})\\&=(a^{-1})_i u^{p\delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}}}e'_{\tau_{i+1}},\end{aligned}$$

$$\begin{aligned}\phi_{p-2}(f_{\mathcal{N}}(u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))}E_{\tau_i}))&=\phi_{p-2}(u^{\delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}}}(u^{n_{\tau_i}}e'_{\tau_i})\\&=(a^{-1})_i u^{p\delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}}}e'_{\tau_{i+1}},\end{aligned}$$

$$\begin{aligned}&f_{\mathcal{N}}(\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))}F_{\tau_i}))\\&\quad=f_{\mathcal{N}}((b^{-1})_i(F_{\tau_{i+1}}-\lambda'_{\tau_i}E_{\tau_{i+1}}))\\&\quad=(b^{-1})_i(u^{p\delta_J(\tau_{i+1})\beta_{\tau_{i+1}}}f'_{\tau_{i+1}}-\lambda'_{\tau_i}u^{p\delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}}}e'_{\tau_{i+1}}),\end{aligned}$$

$$\begin{aligned}&\phi_{p-2}(f_{\mathcal{N}}(u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))}F_{\tau_i}))\\&\quad=\phi_{p-2}(u^{\delta_J(\tau_{i+1})\beta_{\tau_{i+1}}}(u^{n'_{\tau_i}}f'_{\tau_i}+\lambda_{\tau_i}u^{n_{\tau_i}-\beta_{\tau_{i+1}}}e'_{\tau_i})-\lambda_{\tau_i}u^{n_{\tau_i}}e'_{\tau_i})\\&\quad=u^{p\delta_J(\tau_{i+1})\beta_{\tau_{i+1}}}(b^{-1})_i f'_{\tau_{i+1}}-\lambda_{\tau_i}(a^{-1})_i e'_{\tau_{i+1}}.\end{aligned}$$

The result follows, because $\lambda_{\tau_i}(a^{-1})_i=\lambda'_{\tau_i}(b^{-1})_i$ by ((3.1)), and if $\lambda_{\tau_i}\neq 0$ then $\delta_{J^c}(\tau_{i+1})=0$ by (1).

- To check that $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ commute with $N$, we again compute directly. We have

$$\begin{aligned}N(f_{\mathcal{M}}(e_{\tau_i})&=N(u^{-p\alpha_{\tau_i}\delta_J(\tau_i)}e'_{\tau_i})\\&=-p\alpha_{\tau_i}\delta_J(\tau_i)u^{-p\alpha_{\tau_i}\delta_J(\tau_i)}e'_{\tau_i}\\&=0\\&=f_{\mathcal{M}}(N(e_{\tau_i})).\end{aligned}$$

Similarly,

$$\begin{aligned}N(f_{\mathcal{M}}(f_{\tau_i}))&=N(u^{-p\beta_{\tau_i}\delta_{J^c}(\tau_i)}f'_{\tau_i})\\&=u^{-p\beta_{\tau_i}\delta_{J^c}(\tau_i)}N(f'_{\tau_i})\end{aligned}$$

$$= -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}} - p\alpha_{\tau_i} - p\beta_{\tau_i}\delta_{J^c}(\tau_i)} e'_{\tau_i},$$

while

$$f_{\mathcal{M}}(N(f_{\tau_i})) = f_{\mathcal{M}}\left(-\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}}} e_{\tau_i}\right)$$
$$= -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}} - p\alpha_{\tau_i}\delta_J(\tau_i)} e'_{\tau_i},$$

and these two expressions are equal by (1). In the same fashion, we find that

$$N(f_{\mathcal{N}}(E_{\tau_i})) = f_{\mathcal{N}}(N(E_{\tau_i})) = 0,$$

while

$$f_{\mathcal{N}}(N(F_{\tau_i})) = f_{\mathcal{N}}(0) = 0,$$

and

$$N(f_{\mathcal{N}}(F_{\tau_i})) = N(u^{p\beta_{\tau_i}\delta_J(\tau_i)} f'_{\tau_i})$$
$$= -\frac{(b)_{i-1}}{(a)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}} - p\alpha_{\tau_i} + p\beta_{\tau_i}\delta_J(\tau_i)} e'_{\tau_i}.$$

If $\tau_i \notin J$, this expression is 0 by (1). On the other hand, if $\tau_i \in J$, then the exponent of $u$ in this expression is $p(i_{\tau_{i-1}} - \alpha_{\tau_i} + \beta_{\tau_i}) = pe$ by (6), so the expression is again 0, as required.

- Finally, that $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ commute with $\hat{g}$ follows directly from the definitions of $\alpha_{\tau_i}$ and $\beta_{\tau_i}$.

It is clear from the construction that the maps $f_{\mathcal{M}''}$, $f_{\mathcal{M}'}$, $f_{\mathcal{N}''}$ and $f_{\mathcal{N}'}$ are nonzero. Since $T^*_{st}$ is faithful, the maps $T^*_{st}(f_{\mathcal{M}''})$, $T^*_{st}(f_{\mathcal{M}'})$, $T^*_{st}(f_{\mathcal{N}''})$ and $T^*_{st}(f_{\mathcal{N}'})$ are all nonzero, and are thus isomorphisms (as they are maps between one-dimensional $E$-vector spaces). The result follows. □

## 3.5 Weights and types

We recall some definitions and results from [11]. Fix, as ever, $\rho \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$. We make the following definitions:

**Definition 3.9** A weight $\sigma_{\vec{a},\vec{b}}$ is *compatible* with $\rho$ (via $J$) if and only if there exists a subset $J \in S$ so that

$$\psi_1|_{I_{K_0}} = \prod_{\tau\in S} \omega_\tau^{a_\tau} \prod_{\tau\in J} \omega_\tau^{b_\tau},\ \psi_2|_{I_{K_0}} = \prod_{\tau\in S} \omega_\tau^{a_\tau} \prod_{\tau\notin J} \omega_\tau^{b_\tau}.$$

Suppose that these equations hold. We define

$$c_{\tau_i}=\begin{cases} b_{\tau_i}-\delta_J(\tau_{i+1}) & \text{if } \tau_i\in J,\\ p-b_{\tau_i}-\delta_J(\tau_{i+1}) & \text{if } \tau_i\notin J\end{cases}$$

where $\delta_J$ is the characteristic function of $J$. Define a character $\chi_{\vec{a},\vec{b},J}$ by

$$\chi_{\vec{a},\vec{b},J}=\prod_{\tau_i\in S}\omega_{\tau_i}^{a_{\tau_i}}\prod_{\tau_i\notin J}\omega_{\tau_i}^{b_{\tau_i}-p}.$$

Suppose that the $c_\tau$ are not all equal to either 0 or $p-1$. Then we define a representation $I_{\vec{a},\vec{b},J}$ of $\mathrm{GL}_2(k)$ and a type $\tau_{\vec{a},\vec{b},J}$ by

$$I_{\vec{a},\vec{b},J}=I\left(\tilde{\chi}_{\vec{a},\vec{b},J},\tilde{\chi}_{\vec{a},\vec{b},J}\prod_{\tau\in S}\tilde{\omega}_\tau^{c_\tau}\right),$$

$$\tau_{\vec{a},\vec{b},J}=\tilde{\chi}_{\vec{a},\vec{b},J}\oplus\tilde{\chi}_{\vec{a},\vec{b},J}\prod_{\tau\in S}\tilde{\omega}_\tau^{c_\tau}.$$

Note that if $\overline{\rho}$ is compatible with $\sigma_{\vec{a},\vec{b}}$, then a lift of type $J$ is precisely a lift of type $\tau_{\vec{a},\vec{b},J}$ with specified determinant.

**Proposition 3.10** *Suppose that $\sigma_{\vec{a},\vec{b}}$ is regular. If $\rho$ is compatible with $\sigma_{\vec{a},\vec{b}}$ via $J$, then $\rho$ is compatible with precisely one of the Jordan-Hölder factors of the reduction mod $p$ of $I_{\vec{a},\vec{b},J}$, and that factor is isomorphic to $\sigma_{\vec{a},\vec{b}}$.*

*Proof* We use the explicit computations of [11]. Firstly, note that reduction mod $p$ and the notion of compatibility both commute with twisting, so we may replace $\rho$ by $\rho\otimes\chi^{-1}_{\vec{a},\vec{b},J}$. By Proposition 1.1 of [11], we have $\overline{I}_{\vec{a},\vec{b},J}\sim\bigoplus_{K\subset S}\sigma_{\vec{a}_K,\vec{b}_K}$ where $a_K$ and $b_K$ are defined as follows:

$$a_{K,\tau_i}=\begin{cases}0 & \text{if } \tau_i\in K,\\ c_{\tau_i}+\delta_K(\tau_{i+1}) & \text{if } \tau_i\notin K,\end{cases}$$

$$b_{K,\tau_i}=\begin{cases}c_{\tau_i}+\delta_K(\tau_{i+1}) & \text{if } \tau_i\in K,\\ p-c_{\tau_i}-\delta_K(\tau_{i+1}) & \text{if } \tau_i\notin K.\end{cases}$$

By the definition of the $c_\tau$, we see at once that $\sigma_{\vec{a}_J,\vec{b}_J}=\sigma_{\vec{a},\vec{b}}$, and in fact

$$\psi_1|_{I_{K_0}}=\prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\in J}\omega_\tau^{b_{J,\tau}},\qquad \psi_2|_{I_{K_0}}=\prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\notin J}\omega_\tau^{b_{J,\tau}}.$$

If $\rho$ is compatible with another Jordan-Hölder factor, there are subsets $J'$, $K' \subset S$, $J' \neq J$ such that

$$\psi_1|_{I_{K_0}} = \prod_{\tau \in S} \omega_\tau^{a_{J,\tau}} \prod_{\tau \in J} \omega_\tau^{b_{J,\tau}} = \prod_{\tau \in S} \omega_\tau^{a_{J',\tau}} \prod_{\tau \in K'} \omega_\tau^{b_{J',\tau}},$$

$$\psi_2|_{I_{K_0}} = \prod_{\tau \in S} \omega_\tau^{a_{J,\tau}} \prod_{\tau \notin J} \omega_\tau^{b_{J,\tau}} = \prod_{\tau \in S} \omega_\tau^{a_{J',\tau}} \prod_{\tau \notin K'} \omega_\tau^{b_{J',\tau}}.$$

Using the formulae above, the first equation simplifies to

$$\prod_{\tau_i \in S} \omega_{\tau_i}^{c_{\tau_i} + \delta_J(\tau_{i+1})} = \prod_{\tau_i \in (J' \cap K') \cup (J'^c \cap K'^c)} \omega_{\tau_i}^{c_{\tau_i} + \delta_{J'}(\tau_{i+1})} \prod_{\tau_{i+1} \in K' \cap J'^c} \omega_{\tau_i}.$$

By the assumption that $\sigma_{\vec{a},\vec{b}}$ is regular, we have $1 \leq c_{\tau_i} \leq p-2$ and $2 \leq c_{\tau_i} + \delta_J(\tau_{i+1}) \leq p-2$ for each $i$. Then we see that we can equate the exponents of $\omega_{\tau_i}$ on each side of each equation, and we easily obtain $(J' \cap K') \cup (J'^c \cap K'^c) = S$, whence $J' = K'$. But then the equation becomes

$$\prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_J(\tau_{i+1})} = \prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_{J'}(\tau_{i+1})},$$

whence $J = J'$, a contradiction. □

*Remark 3.11* Note that it follows from the formulae in the proof of Proposition 3.10 that if $\sigma_{\vec{a},\vec{b}}$ is regular, then all the Jordan-Hölder factors of the reduction mod $p$ of $I_{\vec{a},\vec{b},J}$ are weakly regular.

**Proposition 3.12** *Let $\theta_1$, $\theta_2$ be two tamely ramified characters of $I_{K_0}$ which extend to $G_{K_0}$. If $\rho$ has a potentially Barsotti-Tate lift* (*with determinant equal to a finite order character times the $p$-adic cyclotomic character*) *of type $\theta_1 \oplus \theta_2$, then $\rho$ is compatible with some weight occurring in the mod $p$ reduction of $I(\theta_1, \theta_2)$.*

*Proof* This follows easily from consideration of the possible Breuil modules corresponding to the $\pi_L$-torsion in the $p$-divisible group of such a lift (where the corresponding Galois representation is valued in $\mathcal{O}_L$, and $\pi_L$ is a uniformiser of $\mathcal{O}_L$). The case $\theta_1 = \theta_2$ is easier, so from now on we assume that $\theta_1 \neq \theta_2$. The $\pi_L$-torsion must contain a closed sub-group-scheme (with descent data) with generic fibre $\psi_1$. Suppose that this group scheme corresponds to a Breuil module with descent data $\mathcal{M}$. Then we can choose a basis so that $\mathcal{M}$ takes the following form:

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot x_{\tau_i},$$

$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{r_i} x_{\tau_i},$$

$$\phi_1(u^{r_i} x_{\tau_i}) = (a^{-1})_i x_{\tau_{i+1}},$$

$$\hat{g}(x_{\tau_i}) = \theta^i(g) x_{\tau_i}.$$

Here $0 \leq r_i \leq e$ is an integer, and $\theta^i : \mathrm{Gal}(K/K_0) \to E^\times$ is a character. Now, by Corollary 5.2 of [16], because the lift is of type $\theta_1 \oplus \theta_2$, we must have $\theta^i = \theta_1$ or $\theta_2$ for each $i$ (here and below we denote the reduction mod $p$ of the $\theta_i$ by the same symbol). Define subsets $Y$, $Z$ by

$$Y = \{\tau_i \in S | \theta^i \neq \theta^{i+1}\},$$

$$Z = \{\tau_i \in S | \theta^i = \theta_1\}.$$

Because $\theta_1 \neq \theta_2$, if $i \in Y$ then the compatibility of the $\phi_1$- and $\mathrm{Gal}(K/K_0)$-actions determines $r_i$ uniquely, and if $i \in Y^c$ then we can take either $r_i = 0$ or $r_i = e$. Having written down all possible $\mathcal{M}$, we now need to determine their generic fibres. This is a straightforward calculation using Example 3.7 of [21]. Without loss of generality, we may twist and assume that $\theta_1 = \prod_{\tau_i \in S} \omega_{\tau_i}^{c_i}$, $\theta_2 = 1$, with $0 \leq c_i \leq p-1$. Then one easily obtains

$$\psi_1|_{I_{K_0}} = \omega_{\tau_1}^{m_1+n_1} \prod_{\tau_i \in \{Y^c | r_i = e\}} \omega_{\tau_i} \prod_{\tau_i \in Y \cap Z} \omega_{\tau_i},$$

where

$$m_1 = \begin{cases} 0 & \text{if } \tau_1 \notin Z, \\ c_1 + pc_r + \cdots + p^{r-1}c_2 & \text{if } \tau_1 \in Z, \end{cases}$$

$$n_1 = \frac{1}{e} \sum_{\tau_i \in Y \cap Z^c} p^{r-i}(p^i c_1 + p^{i+1} c_r + \cdots + p^r c_i + c_{i+1} + \cdots + p^{i-1} c_2)$$
$$- \frac{1}{e} \sum_{\tau_i \in Y \cap Z} p^{r-i}(p^i c_1 + p^{i+1} c_r + \cdots + p^r c_i + c_{i+1} + \cdots + p^{i-1} c_2).$$

Now, consider the coefficient of $c_1$ in $n_1$. The sets $Y \cap Z^c$ and $Y \cap Z$ have equal cardinality, so this coefficient is in fact zero. Thus the coefficient of $c_1$ in $m_1 + n_1$ is 1 if $\tau_1 \in Z$, and 0 otherwise. By cyclic symmetry, we obtain

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in Z} \omega_{\tau_i}^{c_i} \prod_{\tau_i \in X} \omega_{\tau_i},$$

where

$$X = \{\tau_i \in Y^c | r_i = e\} \cup (Y \cap Z).$$

We wish to show that $\rho$ is compatible with some weight in the reduction mod $p$ of $I(\theta_1,\theta_2)$. It is easy to check that the determinant of $\rho$ is correct, so it suffices to examine $\psi_1$; in the notation of Proposition 3.10, we see that $\rho$ is compatible with $\sigma_{\vec{a}_K,\vec{b}_K}$ via $L$ if and only if

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i\in(K^c\cap L)\cup(K\cap L^c)} \omega_{\tau_i}^{c_i+\delta_{K^c}(\tau_{i+1})} \prod_{\tau_i\in S} \omega_{\tau_i}^{\delta_{K\cap L}(\tau_{i+1})}$$

(note that our convention that $\theta_2 = 1$ causes $K^c$ to appear in this formula rather than $K$).

The result now follows upon taking, for example,

$$K = \{\tau_i|\tau_{i-1} \in (X^c \cap Y^c \cap Z) \cup (X \cap Y^c \cap Z^c)\}$$

and

$$L = (K^c \cap Z) \cup (K \cap Z^c).$$

□

**Proposition 3.13** *Suppose that $\sigma_{\vec{a},\vec{b}}$ is regular. If $\rho$ is compatible with $\sigma_{\vec{a},\vec{b}}$ via $J$, and $\rho$ has a lift of type $J$, then $\rho$ has a model of type $J$.*

*Proof* This follows from similar considerations to those involved in the proof of Proposition 3.12. Consider the $\pi_L$-torsion in the $p$-divisible group corresponding to the lift of type $J$. It contains a closed sub-group-scheme (with descent data) with generic fibre $\psi_1$. Suppose that this group scheme corresponds to a Breuil module with descent data $\mathcal{M}$. Then we can choose a basis so that $\mathcal{M}$ takes the following form:

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep}\cdot x_{\tau_i},$$

$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep}\cdot u^{r_i}x_{\tau_i},$$

$$\phi_1(u^{r_i}x_{\tau_i}) = (a^{-1})_i x_{\tau_{i+1}},$$

$$\hat{g}(x_{\tau_i}) = \theta^i(g)x_{\tau_i}.$$

Again, by Corollary 5.2 of [16] and the definition of a lift of type $J$, for each $i$ we must have $\theta^i = \theta_1$ or $\theta^i = \theta_2$ where

$$\theta_1 = \prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\tau\in J}\omega_\tau^{b_\tau-p},$$

$$\theta_2 = \prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\tau\in J^c}\omega_\tau^{b_\tau-p}.$$

Note that $\psi_1|_{I_{K_0}} = \theta_1 \prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_J(\tau_{i+1})}$. Without loss of generality, we can twist so that $\theta_1 = \prod_{\tau_i\in S}\omega_{\tau_i}^{c_i}$, $\theta_2 = 1$, with $0 \le c_i \le p-1$. Then we obtain

$$\theta_1 = \theta_1\theta_2^{-1} = \prod_{\tau_i\in J}\omega_{\tau_i}^{b_{\tau_j}-\delta_J(\tau_{i+1})}\prod_{\tau_i\in J^c}\omega_{\tau_i}^{p-b_{\tau_i}-\delta_J(\tau_{i+1})}.$$

Since $0 \le c_i \le p-1$ and $2 \le b_{\tau_i} \le p-2$, we obtain

$$c_i = \begin{cases} b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \in J, \\ p - b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \notin J. \end{cases}$$

Note that this implies that $2 \le c_i + \delta_J(\tau_{i+1}) \le p-2$. Now, using the same definitions of $X$, $Y$ and $Z$ as in the proof of Proposition 3.12, we can compare the two expressions we have for $\psi_1|_{I_{K_0}}$ to obtain

$$\prod_{\tau_i\in S}\omega_{\tau_i}^{c_i+\delta_J(\tau_{i+1})} = \prod_{\tau_i\in Z}\omega_{\tau_i}^{c_i}\prod_{\tau_i\in X}\omega_{\tau_i}.$$

Since $2 \le c_i + \delta_J(\tau_{i+1}) \le p-2$, this gives $Z = S$, and $X = \{\tau_i|\tau_{i+1} \in J\}$. Since $Z = S$, we have $Y = \emptyset$, and thus the fact that $X = \{\tau_i|\tau_{i+1} \in J\}$ means that $\mathcal{M}$ is in fact of class $J$. It is then clear that the $\pi_L$-torsion is a model of $\rho$ of type $J$, as required. □

## 4 Local analysis—the irreducible case

4.1

We now prove the analogues of some of the results of Sect. 3 in the case where $\rho$ is irreducible.

We assume that $\rho$ is irreducible from now on. In addition to the assumptions made at the beginning of Sect. 3, we now also assume that $\mathbb{F}_{p^2} \subset E$, where $\rho : G_{K_0} \to \mathrm{GL}_2(E)$. Let $k'$ be the (unique) quadratic extension of $k$.

Label the embeddings $k' \hookrightarrow \overline{\mathbb{F}}_p$ as $S' = \{\sigma_i\}$, $0 \le i \le 2r-1$, so that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$, and $\sigma_i|_k = \tau_{\pi(i)}$, where $\pi : \mathbb{Z}/2r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$ is the natural surjection. For simplicity of notation we will sometimes refer to the elements of $S'$ as elements of $\mathbb{Z}/2r\mathbb{Z}$, and the elements of $S$ as elements of $\mathbb{Z}/r\mathbb{Z}$.

Recall that we say that a subset $H \subset S'$ is a *full subset* if $|H| = |\pi(H)| = r$.

**Definition 4.1** We say that $\rho$ is *compatible* with a weight $\sigma_{\vec{a},\vec{b}}$ (via $J$) if there exists a full subset $J \subset S'$ such that

$$\rho|_{I_{K_0'}} \sim \prod_{\sigma\in S'}\omega_\sigma^{a_\sigma}\begin{pmatrix}\prod_{\sigma\in J}\omega_\sigma^{b_\sigma} & 0 \\ 0 & \prod_{\sigma\notin J}\omega_\sigma^{b_\sigma}\end{pmatrix},$$

where we write $a_\sigma$, $b_\sigma$ for $a_{\pi(\sigma)}$, $b_{\pi(\sigma)}$ respectively.

Note that the predicted set of weights $W(\overline{\rho})$ is just the set of compatible weights; this is one way in which the irreducible case is simpler than the reducible one.

Given a regular weight $\sigma_{\vec{a},\vec{b}}$ and a full subset $J \subset S'$, we wish to define a representation and a type. Let $K_J = \pi(J \cap \{1, \dots, r\})$. Then let

$$c_i = \begin{cases} b_i + \delta_{K_J}(1) - 1 & \text{if } 0 = i \in K_J, \\ p - b_i + \delta_{K_J}(1) - 1 & \text{if } 0 = i \notin K_J, \\ b_i - \delta_{K_J}(i+1) & \text{if } 0 \neq i \in K_J, \\ p - b_i - \delta_{K_J}(i+1) & \text{if } 0 \neq i \notin K_J. \end{cases}$$

Define a character

$$\psi_{\vec{a},\vec{b},J} = \omega_{\tau_0}^{-\delta_{K_J}(1)} \prod_{\tau \in S} \omega_\tau^{a_\tau} \prod_{\tau \notin K_J} \omega_\tau^{b_\tau - p}.$$

Then we define

$$I'_{\vec{a},\vec{b},J} = \Theta\left(\tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i}\right),$$

$$\tau'_{\vec{a},\vec{b},J} = \tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i} \oplus \left(\tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i}\right)^{p^r}.$$

**Proposition 4.2** *Recall that $\sigma_{\vec{a},\vec{b}}$ is regular. If $\rho$ is compatible with $\sigma_{\vec{a},\vec{b}}$ via $J$, then $\rho$ is compatible with precisely one of the Jordan-Hölder factors of the reduction mod $p$ of $I'_{\vec{a},\vec{b},J}$, and that factor is isomorphic to $\sigma_{\vec{a},\vec{b}}$.*

*Proof* We may twist and assume without loss of generality that $\psi_{\vec{a},\vec{b},J} = 1$. Then by Proposition 1.3 of [11] (note here that Diamond's conventions on the numbering of the elements of $S'$ are the opposite of ours, so that his $\sigma_i$ is our $\sigma_{-i}$), the Jordan-Hölder factors of the reduction mod $p$ of $I'_{\vec{a},\vec{b},J}$ are $\{\sigma_{\vec{a}_K,\vec{b}_K}\}_{K \subset S}$, where

$$a_{K,\tau_i} = \begin{cases} \delta_K(1) & \text{if } 0 = i \in K, \\ c_i + 1 & \text{if } 0 = i \notin K, \\ 0 & \text{if } 0 \neq i \in K, \\ c_i + \delta_K(i+1) & \text{if } 0 \neq i \notin K, \end{cases}$$

$$b_{K,\tau_i}=\begin{cases} c_i+1-\delta_K(1) & \text{if } 0=i\in K,\\ p-c_i+\delta_K(1)-1 & \text{if } 0=i\notin K,\\ c_i+\delta_K(i+1) & \text{if } 0\neq i\in K,\\ p-c_i-\delta_K(i+1) & \text{if } 0\neq i\notin K.\end{cases}$$

From the definition of the $c_i$ and of $\psi_{\vec{a},\vec{b},J}$, we have $\sigma_{\vec{a}_{K_J},\vec{b}_{K_J}}=\sigma_{\vec{a},\vec{b}}$. Suppose that $\rho$ is compatible with $\sigma_{\vec{a}_{K'},\vec{b}_{K'}}$ via $J'$. Then, replacing $J'$ by $(J')^c$ if necessary, we must have

$$\prod_{i\in S'}\omega_{\sigma_i}^{a_{K_J,i}}\prod_{i\in J}\omega_{\sigma_i}^{b_{K_J,i}}=\prod_{i\in S'}\omega_{\sigma_i}^{a_{K',i}}\prod_{i\in J'}\omega_{\sigma_i}^{b_{K',i}}.$$

Using the formulae above, this becomes

$$\begin{aligned}&\omega_{\sigma_0}^{\delta_{J',K'}(1)}\omega_{\sigma_r}^{\delta_{J',K'}(r+1)}\prod_{i\in T'}\omega_{\sigma_i}^{c_i+\delta_{K'}(i+1)}\prod_{i\in S'}\omega_{\sigma_i}^{\delta_{J'\cap\pi^{-1}((K')^c)}(i+1)}\\ &\quad=\omega_{\sigma_0}^{\delta_{J,K_J}(1)}\omega_{\sigma_r}^{\delta_{J,K_J}(r+1)}\prod_{i\in T}\omega_{\sigma_i}^{c_i+\delta_{K_J}(i+1)}\prod_{i\in S'}\omega_{\sigma_i}^{\delta_{J\cap\pi^{-1}(K_J^c)}(i+1)},\end{aligned}\tag{4.1}$$

where

$$T=(J\cap\pi^{-1}(K_J))\cup(J^c\cap\pi^{-1}(K_J^c))=\{1,\dots,r\},$$

$$T'=(J'\cap\pi^{-1}(K'))\cup((J')^c\cap\pi^{-1}((K')^c)),$$

$$\delta_{J,K_J}(i+1)=\begin{cases}1-\delta_{K_J}(i+1) & \text{if } i\in T,\\ \delta_{K_J}(i+1) & \text{if } i\notin T,\end{cases}$$

$$\delta_{J',K'}(i+1)=\begin{cases}1-\delta_{K'}(i+1) & \text{if } i\in T',\\ \delta_{K'}(i+1) & \text{if } i\notin T'.\end{cases}$$

Note that (since $\sigma_{\vec{a},\vec{b}}$ is regular) all the exponents on the right hand side of (4.1) are in the range $[0,p-1]$. On the left hand side, this is true except possibly for the exponents of $\omega_{\sigma_0},\omega_{\sigma_r}$. Since $T=\{1,\dots,r\}$, it is easy to see that the only opportunity for this not to hold is for the exponent of $\omega_{\sigma_0}$ to be $p$ on the left hand side and 0 on the right hand side. However, in order for the exponent of $\omega_{\sigma_0}$ to be $p$ on the left hand side we require $c_0=p-2$, which requires that $1\in K_J$. But then the exponent of $\omega_{\sigma_0}$ on the right hand side is 1, a contradiction.

Thus we may equate exponents on each side of (4.1). In particular, if $i\neq 0$, we have (again because $\sigma_{\vec{a},\vec{b}}$ is regular) $c_i+\delta_{K_J}(i+1)\in[2,p-2]$, so that we must have $\{1,\dots,r-1\}\subset T'$. We also have $c_0\in[1,p-2]$. If $0\in T'$, we see that the exponent of $\omega_{\sigma_0}$ on the left hand side of (4.1) is $c_0+1+$

$\delta_{J'\cap\pi^{-1}((K')^c)}(1) = c_0 + 1$ (because $1 \in T'$), which is at least 2. However the exponent of $\omega_{\sigma_0}$ on the right hand side of (4.1) is 0 or 1, as $0 \notin T$, which is a contradiction. Thus $T' = T = \{1, \ldots, r\}$.

Then (4.1) simplifies to

$$\prod_{i=0}^{r-1} \omega_{\sigma_i}^{\delta_{K'}(i+1)} \prod_{i=r}^{2r-1} \omega_{\sigma_i}^{\delta_{(K')^c}(i+1)} = \prod_{i=0}^{r-1} \omega_{\sigma_i}^{\delta_{K_J}(i+1)} \prod_{i=r}^{2r-1} \omega_{\sigma_i}^{\delta_{K_J^c}(i+1)},$$

whence $K' = K_J$, as required. □

*Remark 4.3* Note that it follows easily from the formulae in the proof of Proposition 4.2 that if $\sigma_{\vec{a},\vec{b}}$ is regular, then all the Jordan-Hölder factors of the reduction mod $p$ of $I'_{\vec{a},\vec{b},J}$ are weakly regular.

**Theorem 4.4** *Assume that $\sigma_{\vec{a},\vec{b}}$ is regular and that $\rho$ is compatible with $\sigma_{\vec{a},\vec{b}}$ via $J$. Then $\rho$ has a lift of type $\tau'_{\vec{a},\vec{b},J}$ which is not potentially ordinary.*

*Proof* A simple computation shows that we in fact have

$$\tau'_{\vec{a},\vec{b},J} = \prod_{\tau\in S} \omega_\tau^{a_\tau} \prod_{\sigma\in J} \omega_\sigma^{b_\sigma - p} \oplus \prod_{\tau\in S} \omega_\tau^{a_\tau} \prod_{\sigma\notin J} \omega_\sigma^{b_\sigma - p}.$$

This means that we only need to make a very minor modification to the proof of Theorem 3.7. Let $K'_0 = W(k')[1/p]$. Fix $\pi' = (-p)^{1/(p^{2r}-1)}$, and let $K' = K'_0(\pi')$. Let $g_\phi$ be the nontrivial element of $\mathrm{Gal}(K'/K_0)$ which fixes $\pi'$. It is clear from the proof of Theorem 3.7 that for some choice of $a \in W(E)^\times$ the following object of $W(E) - \mathrm{Mod}^1_{cris,dd,K_0}$ provides us with the required lift.

$$\mathcal{M}_J^{\sigma_i} = S_K \cdot e_{\sigma_i} + S_K \cdot f_{\sigma_i},$$

$$\hat{g}_\phi(e_{\sigma_i}) = f_{\sigma_{i+r}},$$

$$\hat{g}_\phi(f_{\sigma_i}) = e_{\sigma_{i+r}}.$$

If $g \in \mathrm{Gal}(K'/K'_0)$,

$$\hat{g}(e_{\sigma_i}) = \left(\left(\prod_{\tau\in S} \widetilde{\omega}_\tau^{a_\tau} \prod_{\sigma\in J} \widetilde{\omega}_\sigma^{b_\sigma - p}\right)(g)\right) e_{\sigma_i},$$

$$\hat{g}(f_{\sigma_i}) = \left(\left(\prod_{\tau\in S} \widetilde{\omega}_\tau^{a_\tau} \prod_{\sigma\notin J} \widetilde{\omega}_\sigma^{b_\sigma - p}\right)(g)\right) f_{\sigma_i}.$$

If $\sigma_{i+1} \in J$,

$$\mathrm{Fil}^1 \mathcal{M}_J^{\sigma_i} = \mathrm{Fil}^1 S_K \cdot \mathcal{M}_J^{\sigma_i} + S_K \cdot f_{\sigma_i},$$

$$\phi(e_{\sigma_i}) = (a^{-1})_i e_{\sigma_{i+1}},$$

$$\phi(f_{\sigma_i}) = (a^{-1})'_i p f_{\sigma_{i+1}}.$$

If $\sigma_{i+1} \notin J$,

$$\mathrm{Fil}^1 \mathcal{M}_J^{\sigma_i} = \mathrm{Fil}^1 S_K \cdot \mathcal{M}_J^{\sigma_i} + S_K \cdot e_{\sigma_i},$$

$$\phi(e_{\sigma_i}) = (a^{-1})_i p e_{\sigma_{i+1}},$$

$$\phi(f_{\sigma_i}) = (a^{-1})'_i f_{\sigma_{i+1}}.$$

Here the notation $(x)'_i$ means $x$ if $i = r + 1$ and 1 otherwise. □

## 5 Global results

5.1

We now show how the local results obtained in the previous sections can be combined with lifting theorems to prove results about the possible weights of mod $p$ Hilbert modular forms. Firstly, we show that if $\overline{\rho}$ is modular of some regular weight, then $\overline{\rho}$ is compatible with that weight, by making use of Lemma 2.4 and Proposition 3.12. We then turn this analysis around. We take a conjectural regular weight $\sigma$ for $\overline{\rho}$, and using modularity lifting theorems we produce a modular lift of $\overline{\rho}$ of a specific type, which is enough to prove that $\overline{\rho}$ is modular of weight $\sigma$ by Propositions 3.10 and 4.2.

Assume now that $F$ is a totally real field in which $p > 2$ is unramified, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(E)$ is a continuous representation, known to be modular, where $E$ is a finite extension of $\mathbb{F}_p$.

Let $W(\overline{\rho})$ be the conjectural set of Serre weights for $\overline{\rho}$, as defined in Sect. 2. Recall that the elements of $W(\overline{\rho})$ are just the tensor products of elements of $W_v(\overline{\rho})$, for $v|p$, and that such elements are of the form $\sigma_{\vec{a},\vec{b}}$, as described above. We say that a weight is (weakly) regular if and only if it is a tensor product of (weakly) regular weights.

The following argument is based on an argument of Michael Schein (cf. Proposition 5.11 of [22]), and is due to him in the case that $\overline{\rho}|_{G_{F_v}}$ is irreducible.

**Lemma 5.1** *Suppose that $p \geq 3$, that $\overline{\rho}$ is modular of weight $\sigma = \bigotimes_v \sigma^v_{\vec{a},\vec{b}}$, and that $\sigma$ is weakly regular. Then for each $v$, either $\overline{\rho}|_{G_{F_v}}$ is compatible* *with $\sigma^v_{\vec{a},\vec{b}}$, or $\sigma^v_{\vec{a},\vec{b}}$ is not regular and $\overline{\rho}|_{G_{F_v}}$ is not compatible with any regular weight.*

*Proof* Suppose firstly that $\overline{\rho}|_{G_{F_v}}$ is reducible. We will assume for the rest of this proof that $F_v \neq \mathbb{Q}_p$; the argument needed when $F_v = \mathbb{Q}_p$ is slightly different, although much simpler, and the result follows from Lemma 4.4.6 of [15]. We will also assume that there is at least one $b_{\tau_i} \neq 1$; the case where all $b_{\tau_i} = 1$ is much easier, and we leave it to the reader. Then for any type $\tau = \chi_1 \oplus \chi_2$ (with $\chi_1 \neq \chi_2$ tame characters of $I_{F_v}$ which extend to $G_{F_v}$) such that $\sigma^v_{\vec{a},\vec{b}}$ occurs in the reduction of $I(\chi_1, \chi_2)$, it follows from Lemma 2.4 and Proposition 3.12 that there must be a weight $\sigma^v_{\vec{a}',\vec{b}'}$ in the reduction of $I(\chi_1, \chi_2)$ which is compatible with $\overline{\rho}|_{G_{F_v}}$. Since we are working purely locally, we return to the notation of Sect. 3.5.

Twisting, we may without loss of generality suppose that $a_\tau = 0$ for all $\tau$. By Proposition 1.1 of [11] (and the fact that $\sigma$ is weakly regular, with at least one $b_{\tau_i} \neq 1$) there is for each $J \subset S$ a unique pair of characters $\prod_{\tau\in S} \tilde{\omega}^{c^J_\tau}$, $\prod_{\tau\in S} \tilde{\omega}^{d^J_\tau}$ (with $0 \le c^J_\tau, d^J_\tau \le p-1$) such that if we define

$$\sigma^J = I\left(1, \prod_{\tau\in S} \tilde{\omega}^{d^J_\tau}\right) \otimes \prod_{\tau\in S} \tilde{\omega}_\tau^{c^J_\tau} \circ \det$$

then, with the same notation for reductions as in [11], extended to be compatible with twisting, $\sigma^J_J \sim \sigma_{\vec{a},\vec{b}}$. Then there must (by the argument above) be some subset $K_J \subset S$, such that $\sigma^J_{K_J}$ is compatible with $\rho$. If $\sigma^J_{K_J} \sim \sigma_{\vec{m}^J_{K_J},\vec{n}^J_{K_J}}$ this means that there must be a subset $L_J \subset S$ such that

$$\psi_1|_{I_{K_0}} = \prod_{\tau\in S} \omega_\tau^{m^J_{K_J,\tau}} \prod_{\tau\in L_J} \omega_\tau^{n^J_{K_J,\tau}}.$$

By Proposition 1.1 of [11], this is equal to

$$\prod_{\tau_i\in S} \omega_{\tau_i}^{c^J_{\tau_i}} \prod_{\tau_i\in L_J\cap K^c_J} \omega^p_{\tau_i} \prod_{\tau_i\in(L_J\cap K_J)\cup(L^c_J\cap K^c_J)} \omega_{\tau_i}^{d^J_{\tau_i}+\delta_{K_J}(\tau_{i+1})}.$$

Now, since $\sigma^J_J \sim \sigma_{\vec{a},\vec{b}}$, we have

$$\prod_{\tau_i\in S} \omega_{\tau_i}^{c^J_{\tau_i}} \prod_{\tau_i\notin J} \omega_{\tau_i}^{d^J_{\tau_i}+\delta_J(\tau_{i+1})} = \prod_{\tau_i\in S} \omega_{\tau_i}^{a_{\tau_i}} = 1,$$

by the assumption that $a_\tau = 0$ for all $\tau$, so that in fact

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in J^c} \omega_{\tau_i}^{-(d^J_{\tau_i}+\delta_J(\tau_{i+1}))} \prod_{\tau_i \in L_J \cap K_J^c} \omega_{\tau_i}^p \\ \times \prod_{\tau_i \in (L_J \cap K_J)\cup(L_J^c \cap K_J^c)} \omega_{\tau_i}^{d^J_{\tau_i}+\delta_{K_J}(\tau_{i+1})}.$$

Since $\sigma_J^J \sim \sigma_{\vec{a},\vec{b}}$, we have

$$d^J_{\tau_i} = \begin{cases} b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \in J, \\ p - b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \notin J. \end{cases}$$

Substituting, we see that

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in (T_J \cap J)\cup(T_J^c \cap J^c)} \omega_{\tau_i}^{b_{\tau_i}} \prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_{L_J \cap K_J^c}(\tau_{i+1}) - \delta_{T_J^c \cap J^c}(\tau_{i+1})} \\ \times \prod_{\tau_i \in T_J} \omega_{\tau_i}^{\delta_{K_J}(\tau_{i+1}) - \delta_J(\tau_{i+1})},$$

where we write $T_J = (K_J \cap L_J) \cup (K_J^c \cap L_J^c)$.

Putting $J = S$, we obtain

$$\begin{aligned} \psi_1|_{I_{K_0}} &= \prod_{\tau_i \in T_S} \omega_{\tau_i}^{b_{\tau_i}} \prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_{L_S \cap K_S^c}(\tau_{i+1})} \prod_{\tau_i \in T_S} \omega_{\tau_i}^{\delta_{K_S}(\tau_{i+1})-1} \\ &= \prod_{\tau_i \in T_S} \omega_{\tau_i}^{b_{\tau_i} - \delta_{K_S^c \cap L_S^c}(\tau_{i+1})} \prod_{\tau_i \in T_S^c} \omega_{\tau_i}^{\delta_{L_S \cap K_S^c}(\tau_{i+1})}. \end{aligned} \tag{5.1}$$

Now, suppose that $\sigma_{\vec{a},\vec{b}}$ is *not* compatible with $\rho$, so that for all $J$ we have $K_J \neq J$. We can uniquely write

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in S} \omega_{\tau_i}^{c_{\tau_i}}$$

with $0 \le c_{\tau_i} \le p-1$ not all equal to $p-1$ (in fact, an examination of the product just written shows that the exponents are already in this range). Examining the formula just established, we see that after possibly exchanging $\psi_1$ and $\psi_2$ (which we can do, as the definition of "compatible" is unchanged by this exchange), there must be some $j$ such that $b_{\tau_j} \neq 1$, $c_{\tau_j} = b_{\tau_j} - 1$, $\tau_j \in T_S$, and $\tau_{j+1} \in K_S^c \cap L_S^c \subset T_S$ (else $\rho$ would be compatible with $\sigma_{\vec{a},\vec{b}}$).

Now take $J=\{\tau_j\}$, so that

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i\in(T_{\{\tau_j\}}\cap\{\tau_j\})\cup(T^c_{\{\tau_j\}}\cap\{\tau_j\}^c)} \omega_{\tau_i}^{b_{\tau_i}} \times \prod_{\tau_i\in S} \omega_{\tau_i}^{\delta_{L_{\{\tau_j\}}\cap K^c_{\{\tau_j\}}}(\tau_{i+1})-\delta_{T^c_{\{\tau_j\}}\cap\{\tau_j\}^c}(\tau_{i+1})} \times \prod_{\tau_i\in T_{\{\tau_j\}}} \omega_{\tau_i}^{\delta_{K_{\{\tau_j\}}}(\tau_{i+1})-\delta_{\{\tau_j\}}(\tau_{i+1})}. \tag{5.2}$$

It is easy to see that the exponent of $\omega_{\tau_i}$ in this product is always between 0 and $p-1$, unless $i=j-1$ or $i=j$. If the exponent is always between 0 and $p-1$, then we have a contradiction, because we already know that $c_{\tau_j}=b_{\tau_j}-1$, but from (5.2) we see that the exponent of $\omega_{\tau_j}$ can only be 0, $b_{\tau_j}$ or $b_{\tau_j}+1$.

So, at least one of the exponents of $\omega_{\tau_{j-1}}$ and $\omega_{\tau_j}$ must be $-1$ or $p$. We now analyse when this can occur. It's easy to see that the exponent of $\omega_{\tau_j}$ is $-1$ if and only if $\tau_j\notin T_{\{\tau_j\}}$ and $\tau_{j+1}\in L^c_{\{\tau_j\}}\cap K_{\{\tau_j\}}$, and it is $p$ if and only if $b_{\tau_j}=p-1$, $\tau_j\in T_{\{\tau_j\}}$ and $\tau_{j+1}\in L_{\{\tau_j\}}\cap K_{\{\tau_j\}}$. Similarly, the exponent of $\omega_{\tau_{j-1}}$ is $-1$ if and only if $\tau_{j-1}\in T_{\{\tau_j\}}$ and $\tau_j\in L^c_{\{\tau_j\}}\cap K^c_{\{\tau_j\}}$, and it is $p$ if and only if $b_{\tau_{j-1}}=p-1$, $\tau_{j-1}\in T^c_{\{\tau_j\}}$ and $\tau_j\in L_{\{\tau_j\}}\cap K^c_{\{\tau_j\}}$. Thus it is impossible for both exponents to be $p$, or both to be $-1$.

Suppose now that the exponent of $\omega_{\tau_j}$ in (5.2) is $-1$. If we multiply each of the expressions (5.1), (5.2) by $\omega_{\tau_j}$, write each side as a product $\prod_\tau \omega_\tau^{n_\tau}$ with $0\le n_\tau\le p-1$ and equate coefficients of $\omega_{\tau_j}$ in the resulting expression, we obtain $b_{\tau_j}=0$ or 1 (the second case only a possibility when the exponent of $\omega_{\tau_{j-1}}$ in (5.2) is $p$), a contradiction.

Suppose that the exponent of $\omega_{\tau_j}$ in (5.2) is $p$. Then we again easily see that $p-2=b_{\tau_j}-1=0$ or 1. Thus $p-2=1$, and we additionally need to have $(T_{\{\tau_j\}}\cap\{\tau_j\}\cup(T^c_{\{\tau_j\}}\cap\{\tau_j\}^c=S$, so that $T_{\{\tau_j\}}=\{\tau_j\}$. But for the exponent of $\omega_{\tau_j}$ to be $p$ we need that $\tau_{j+1}\in L_{\{\tau_j\}}\cap K_{\{\tau_j\}}\subset T_{\{\tau_j\}}$, a contradiction.

Suppose that the exponent of $\omega_{\tau_{j-1}}$ in (5.2) is $p$. Then in the same fashion we obtain $b_{\tau_j}-1$=0, or 1. The only possibility is that $b_{\tau_j}=2$, when we in addition (in order that the necessary carrying should occur) require that $b_{\tau_i}=p-1$ for all $i\ne j$.

Finally, suppose that the exponent of $\omega_{\tau_{j-1}}$ in (5.2) is $-1$. Multiply each of (5.1), (5.2) by $\omega_{\tau_{j-1}}$. Then we see that the only way for equality to hold is again if $b_{\tau_i}=p-1$ for all $i\ne j$.

So, we have deduced that $b_{\tau_i}=p-1$ for all $i\ne j$, so that $\sigma_{\vec{a},\vec{b}}$ is certainly not regular. It now remains to show that $\rho$ is not compatible with any regular weight. Examining the above argument, we see that we have in fact deduced that (again, after possibly exchanging $\psi_1, \psi_2$)

$$\psi_1|_{I_K} = \omega_{\tau_j}^{b_{\tau_j}-1} \prod_{i \neq j} \omega_{\tau_i}^{p-1},$$

$$\psi_2|_{I_K} = \omega_{\tau_j},$$

with $2 \leq b_{\tau_j} \leq p-1$.

If $\rho$ is compatible with some regular weight, then we have by definition that

$$\psi_1|_{I_K} \psi_2|_{I_K}^{-1} = \prod_{\tau \in J} \omega_\tau^{n_\tau} \prod_{\tau \in J^c} \omega_\tau^{-n_\tau}$$

for some $J \subset S$ and $2 \leq n_\tau \leq p-2$. Substituting, we obtain

$$\omega_{\tau_{j-1}} \prod_{\tau \in J} \omega_\tau^{n_\tau} = \omega_{\tau_j}^{b_{\tau_j}-1} \prod_{\tau \in J^c} \omega_\tau^{n_\tau}.$$

If $\tau_j \in J$ then we can immediately equate coefficients of $\omega_{\tau_{j-1}}$ and deduce a contradiction. If not, then because $n_{\tau_j} + b_{\tau_j} < 2p$ we see that we can still equate coefficients of $\omega_{\tau_{j-1}}$ to obtain a contradiction.

The proof in the irreducible case is very similar, and rather simpler, as less "carrying" is possible. In fact, the argument gives the stronger result that $\overline{\rho}|_{G_{F_v}}$ is compatible with $\sigma^v_{\vec{a},\vec{b}}$ for all $v$. A proof is given in the proof of Proposition 5.11 of [22]; note that [22] works in the setting of [3] (using indefinite quaternion algebras), but the proof of Proposition 5.11 is purely local (using Raynaud's theory of finite flat group schemes of type $(p, \ldots, p)$ in place of the Breuil module calculations used in this paper), and applies equally well in our setting. □

The following theorem is due to Michael Schein in the case that $\overline{\rho}|_{G_{F_v}}$ is irreducible for all places $v|p$ (see [22]).

**Theorem 5.2** *If $\overline{\rho}$ is modular of weight $\sigma$, and $\sigma$ is regular, then $\sigma \in W(\overline{\rho})$.*

*Proof* Suppose that $\sigma = \bigotimes_v \sigma^v_{\vec{a},\vec{b}}$, so that we need to show that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$ for all $v|p$. By Lemma 5.1, $\sigma^v_{\vec{a},\vec{b}}$ is compatible with $\overline{\rho}|_{G_{F_v}}$, via $J$, say. If $\overline{\rho}|_{G_{F_v}}$ is irreducible, we are done, so assume that it is reducible. By Lemma 2.4, $\overline{\rho}|_{G_{F_v}}$ has a lift to a potentially Barsotti-Tate representation of type $\tau_{\vec{a},\vec{b},J}$. By definition, this is, up to an unramified twist, a lift of type $J$. By Proposition 3.13, $\overline{\rho}|_{G_{F_v}}$ has a model of type $J$. Twisting, we may without loss of generality suppose that each $a_\tau = 0$. Then by Proposition 3.8, and the definition of $W_v(\overline{\rho})$, we see that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$, as required. □

**Theorem 5.3** *If $\sigma \in W(\overline{\rho})$ is a regular weight, and $\sigma$ is non-ordinary, then $\overline{\rho}$ is modular of weight $\sigma$. If $\sigma \in W(\overline{\rho})$ is regular, and $\sigma$ is partially ordinary of type $I$ and $\overline{\rho}$ has a partially ordinary modular lift of type $I$ then $\overline{\rho}$ is modular of weight $\sigma$.*

*Proof* Suppose that $\sigma = \bigotimes_v \sigma^v_{\vec{a},\vec{b}}$, so that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$ for all $v|p$. Firstly, we note that (by the definition of $W_v(\overline{\rho})$) $\sigma^v_{\vec{a},\vec{b}}$ is compatible with $\overline{\rho}|_{G_{F_v}}$, via $J_v$, say.

Consider firstly the case where $\overline{\rho}|_{G_{F_v}}$ is reducible. We claim that $\overline{\rho}|_{G_{F_v}}$ has a model of type $J_v$. To see this, we may twist, and without loss of generality suppose that $a_\tau = 0$ for all $\tau$, so that $\overline{\rho}|_{G_{F_v}} \sim \left(\begin{smallmatrix}\psi_1 & * \\ 0 & \psi_2\end{smallmatrix}\right)$, with $\psi_1|_{I_{F_v}} = \prod_{\tau\in J_v} \omega_\tau^{b_\tau}$, $\psi_2|_{I_{F_v}} = \prod_{\tau\notin J_v} \omega_\tau^{b_\tau}$. Now, by Proposition 3.8 (and the definition of $W(\overline{\rho}_v)$) $\overline{\rho}|_{G_{F_v}}$ has a model of type $J_v$, as required. Then Theorem 3.7 shows that $\overline{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate deformation of type $\tau_{\vec{a},\vec{b},J_v}$.

If $\overline{\rho}|_{G_{F_v}}$ is irreducible, then Theorem 4.4 shows that shows that $\overline{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate deformation of type $\tau'_{\vec{a},\vec{b},J_v}$.

By Corollary 3.1.7 of [15] there is a modular lift $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of $\overline{\rho}$ which is potentially Barsotti-Tate of type $\tau_{\vec{a},\vec{b},J_v}$ for each $v|p$ for which $\overline{\rho}|_{G_{F_v}}$ is reducible, and of type $\tau'_{\vec{a},\vec{b},J_v}$ for each $v|p$ for which $\overline{\rho}|_{G_{F_v}}$ is irreducible. Then by Lemma 2.4, $\overline{\rho}$ is modular of weight $\sigma'$ for some Jordan-Hölder constituent $\sigma'$ of the reduction modulo $p$ of $\otimes_v I_v$, where $I_v = I_{\vec{a},\vec{b},J_v}$ if $\overline{\rho}|_{G_{F_v}}$ is reducible, and $I_v = I'_{\vec{a},\vec{b},J_v}$ otherwise. The result then follows from Propositions 3.10 and 4.2, Remarks 3.11 and 4.3, and Lemma 5.1. □

## References


1. Ash, A., Stevens, G.: Modular forms in characteristic $l$ and special values of their $L$-functions. Duke Math. J. **53**(3), 849–868 (1986)
2. Breuil, C., Conrad, B., Diamond, F., Taylor, R.: On the modularity of elliptic curves over **Q**: wild 3-adic exercises. J. Am. Math. Soc. **14**(4), 843–939 (2001) (electronic)
3. Buzzard, K., Diamond, F., Jarvis, F.: On Serre's conjecture for mod $l$ Galois representations over totally real fields. Duke Math. J. (2010, to appear)
4. Breuil, C., Mézard, A.: Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbf{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ en $l = p$. Duke Math. J. **115**(2), 205–310 (2002). With an appendix by Guy Henniart
5. Breuil, C.: Construction de représentations $p$-adiques semi-stables. Ann. Sci. École Norm. Super. (4) **31**(3), 281–327 (1998)
6. Breuil, C.: Représentations semi-stables et modules fortement divisibles. Invent. Math. **136**(1), 89–122 (1999)
7. Breuil, C.: Groupes $p$-divisibles, groupes finis et modules filtrés. Ann. Math. (2) **152**(2), 489–549 (2000)
8. Caruso, X.: $\mathbb{F}_p$-représentations semi-stables. Preprint

9. Conrad, B., Diamond, F., Taylor, R.: Modularity of certain potentially Barsotti-Tate Galois representations. J. Am. Math. Soc. **12**(2), 521–567 (1999)
10. Caruso, X., Liu, T.: Quasi-semi-stable representations. Bull. Soc. Math. Fr. **137**(2), 185–223 (2009)
11. Diamond, F.: A correspondence between representations of local Galois groups and Lie-type groups. In: $L$-functions and Galois representations. London Math. Soc. Lecture Note Ser., vol. 320, pp. 187–206. Cambridge University Press, Cambridge (2007)
12. Fontaine, J.-M., Laffaille, G.: Construction de représentations $p$-adiques. Ann. Sci. École Norm. Super. (4) **15**(4), 547–608 (1982)
13. Gee, T.: A modularity lifting theorem for weight two Hilbert modular forms. Math. Res. Lett. **13**(5–6), 805–811 (2006)
14. Gee, T.: Companion forms over totally real fields, II. Duke Math. J. **136**(2), 275–284 (2007)
15. Gee, T.: Automorphic lifts of prescribed types. Math. Ann. (2010, to appear)
16. Gee, T., Savitt, D.: Serre weights for quaternion algebras. Compositio (2010, to appear)
17. Kisin, M.: Moduli of finite flat group schemes, and modularity. Ann. Math. (2008, to appear)
18. Kisin, M.: Potentially semi-stable deformation rings. J. Am. Math. Soc. **21**(2), 513–546 (2008)
19. Kisin, M.: Modularity of 2-adic Barsotti-Tate representations. Invent. Math. **178**(3), 587–634 (2009)
20. Savitt, D.: On a conjecture of Conrad, Diamond, and Taylor. Duke Math. J. **128**(1), 141–197 (2005)
21. Savitt, D.: Breuil modules for Raynaud schemes. J. Number Theory **128**(11), 2939–2950 (2008)
22. Schein, M.: Weights of Galois representations associated to Hilbert modular forms. J. Reine Angew. Math. **622**, 57–94 (2008)
23. Taylor, R.: On the meromorphic continuation of degree two $L$-functions. Doc. Math. 729–779 (2006) (electronic)