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A characterization of hyperbolic rational maps

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Abstract We give a topological characterization of rational maps with disconnected Julia sets. Our results extend Thurston's characterization of postcritically finite rational maps. In place of iteration on Teichmüller space, we use quasiconformal surgery and Thurston's original result.

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1 Introduction

A rational map $f(z) = P(z)/Q(z)$ acts on the Riemann sphere $\overline{\mathbb{C}}$ as a branched covering. The iteration of f yields a dynamical system on $\overline{\mathbb{C}}$. The degree of f is defined to be the maximum of the degrees of P and Q. It is also the degree of f as a branched covering of $\overline{\mathbb{C}}$. We will always assume that the degree is at least two (a degree one map is a Möbius transformation and generates an uninteresting dynamical system). Thus the covering feature of f forces it to be globally expanding, whereas the presence of critical points (i.e. points z such that $f'(z) = 0$) makes it locally strongly contracting. The overall behavior of f depends therefore very much on the interplay of these two opposite forces.

The iteration of a rational map f decomposes $\overline{\mathbb{C}}$ naturally into the Fatou set F_f (the stable locus) and the Julia set J_f (the chaotic locus), with J_f being defined as the set of initial values $z \in \overline{\mathbb{C}}$ such that the iterated sequence $\{f^n\}$ does not form a normal family on any neighborhood of z.

A rational map f is *hyperbolic* if it is uniformly expanding near its Julia set. These are the natural analogs of Smale's Axiom A maps in this setting. If in addition the Julia set is connected, the dynamics of f on J_f is equivalent to the dynamics of a map f_0 in which all critical points are eventually periodic under iteration; such maps are called *postcritically finite*.

Extending works of Milnor, Sullivan and Thurston on the dynamics of unimodal interval maps, Thurston gave a complete topological characterization of postcritically finite rational maps f_0 (see [13, 35]), which can be stated roughly as follows: The set of postcritically finite rational maps (except the Lattès examples) are in one-to-one correspondence with the homotopy classes of postcritically finite branched self-coverings of the two sphere with no Thurston obstructions (which will be defined in Sect. 3.1).

Thus in order to build a rational map with desired combinatorial properties one may first construct a branched covering F as a topological model (this is a lot more flexible than building holomorphic objects, for example one may freely cut, paste and interpolate various holomorphic objects), and then check whether F has Thurston obstructions (this is not always easy). If not then Thurston's theorem ensures the existence of a rational map with the same combinatorial properties.

There are many applications of Thurston's theorem. These include Douady's proof of monotonicity of entropy for unimodel maps [11], Rees' descriptions of parameter spaces [30], Kiwi's characterization of polynomial laminations [23] (using previous work of Bielefield-Fisher-Hubbard [2] and Poirier [28]), Rees, Shishikura and Tan's studies on matings of polynomials ([29, 31, 33, 34]), and Pilgrim and Tan's cut-and-paste surgery along arcs ([26]). Furthermore, one of the two main outstanding questions in the field, namely, the density of hyperbolicity in the quadratic polynomial family, can be reduced to the assertion that every (infinitely renormalizable) quadratic polynomial p is a limit of certain postcritically finite ones p_n obtained via Thurston's theorem and McMullen's quotienting process ([24]). The detailed knowledge of the combinatorics of the parameter space of quadratic polynomials (which follows from a special case of Thurston's theorem) was used by Sørensen ([32]) to construct highly non-hyperbolic quadratic polynomials with non-locally connected Julia sets, and this in turn was used by Henriksen ([18]) to show that McMullen's combinatorial rigidity property fails for cubic polynomials.

We mention here an interesting application of Thurston's theorem beyond the field of complex dynamics. Khavinson and Swiatek ([22]) proved that harmonic polynomials $z-\overline{p(z)}$, where p is a holomorphic polynomial of degree $n>1$, have at most $3n-2$ roots, and the bound is sharp for $n=2,3$. Bshouty and Lyzzaik ([5]) extended the sharpness of the bound to the cases $n=4,5,6$ and 8, using purely algebraic methods. Finally L. Geyer ([16]) settled the sharpness for all n at once, by constructing 'à la Thurston' a polynomial p of degree n with real coefficients and with mutually distinct critical points $z_1, z_2, \ldots, z_{n-1}$ such that $\overline{p(z_j)}=z_j$.

There are, however, two drawbacks of Thurston's theorem.

Problem 1 *Up to now it can only be applied to postcritically finite rational maps. On one hand, these maps all have a connected Julia set; on the other hand, they form a totally disconnected subset in the parameter space (except the Lattès examples). Therefore the theorem can not characterize the combinatorics of disconnected Julia sets, nor the dynamical bifurcations through continuous parameter perturbations.*

Problem 2 *In general it is difficult to apply the theorem effectively, namely to check whether a specific branched covering has Thurston obstructions or not. Each successful application is usually a theorem in its own right.*

Over the years, Problem 1 has been addressed by several people through various attempts. For example, David Brown ([4]), supported by previous work of Hubbard and Schleicher ([19]), has succeeded in extending the theory to the uni-critical polynomials with an infinite postcritical set (but always with a connected Julia set), and pushed it even further to the infinite degree case, namely the exponential maps. We would also like to mention a recent work of Hubbard-Schleicher-Shishikura ([20]) extending Thurston's theorem to postcritically finite exponential maps.

Regarding Problem 2 in the postcritically finite setting, many methods have been developed (see [31] and the references therein for a rather complete survey of such techniques), although it remains a difficult problem in general.

In the present work, supported by previous works of Cui, Jiang and Sullivan ([7]), as well as unpublished manuscripts of the first author, we extend Thurston's theorem to the set of non-postcritically finite hyperbolic or sub-hyperbolic rational maps (they are conjecturally dense in the parameter space). In other words we will prove that such maps are in one-to-one correspondence with the isotopy classes of branched coverings with similar properties and with no Thurston obstructions. To be more accurate we require also the branched coverings to record both the global combinatorial data and the local analytic data around the cluster points of the postcritical set.

At the same time, we will provide effective criteria for the absence of obstructions. More precisely, we will decompose the dynamics of a candidate branched covering F into several sub-systems that are postcritically finite, together with a transition matrix that records the gluing data. We then show that in order to verify the absence of obstructions for F it suffices to check the property for the sub-systems (thus reducing the problem to the postcritically finite case) and for the gluing data, which is only one more eigenvalue to calculate.

We remark that the decomposition above is in some sense canonical and presents some interest even for rational maps. The spirit is close to previous works of Pilgrim-Tan and Cui on the topology of disconnected Julia sets (see for example [27]).

Our analysis here leads naturally to the concept of repelling systems over a disjoint union of Riemann surfaces of finite type and allows us to establish an analog of Thurston's theorem for such systems as well. This result is of independent interest.

Our argument leads also to a ready-to-use combination result. We will show (in Theorem 9.1) that for any finite collection of rational maps f_i with connected Julia set J_i (postcritically finite or not), together with compatible (unobstructed) gluing data D, one can glue together the f_i's on neighborhoods of J_i following D in order to obtain a rational map g, so that each f_i appears as a renormalization of g (this should be compared to Pilgrim's work [25]).

At first sight one may be surprised how it is possible to achieve such a degree of freedom in the 'interpolation' of objects as rigid as rational maps. Here is a quick justification. Alongside Thurston's theorem, there is another, equally powerful, tool for constructing rational map dynamical systems, namely the Measurable Riemann Mapping Theorem. Both results allow one to perform surgery on holomorphic objects. But they are only applicable in somewhat transverse settings. The former does not care about the Julia set, but requires postcritical finiteness. The latter does not care much about the postcritical set, but often requires the surgery to touch only a very small part of the Julia set (e.g. finitely many points). There exist also extensions of both results by simply composing one after the other (such as replacing a periodic orbit containing a critical point by an attracting periodic orbit). In order to achieve our combination result and at the same time extend Thurston's theorem to postcritically infinite maps, we had to merge the power of both theorems in a more substantial way. Each of them must be applied at the right place and at the right moment, with just the right estimates for the pieces to be fitted together.

The flexibility of our combination result is better illustrated in the Ph.D. thesis of Godillon ([17]), where he uses piecewise linear tree maps to model the dynamical system of a hyperbolic rational map f on the set of connected components of its Julia set J_f.

The present work consists of the first step of a long program established by the first author. This program concerns the study of deformations and bifurcations of rational maps ([9]).

In our forthcoming paper [10], we will extend our characterization to the setting of geometrically finite rational maps (i.e. maps having non-hyperbolic periodic points with a rational rotation number), and then give a detailed study of their relations to hyperbolic rational maps. The hyperbolic rational maps of a given degree form an open set in the parameter space, and maps within a common connected component have similar dynamics (they are structurally stable). A geometrically finite map g often sits on the boundary of several hyperbolic components, and does so in quite a subtle way: if you approach it algebraically, you may or may not get a different geometric limit, depending very much upon how you approach it. This subtlety makes the study of the deformations of g very difficult. However, it is relatively easy to describe combinatorially all possible bifurcations. Then, equipped with our Thurston-like realization result, we will be able to prove the existence of such bifurcations. For example we will classify all hyperbolic components H that contain a path converging to g such that along the path the algebraic and geometric limits coincide. Conversely, given a hyperbolic component H, we will apply our technique to determine all geometrically finite maps g that are path accessible from H with similar dynamical properties.

Statements All branched coverings and homeomorphisms in this paper are orientation preserving. We will assume that the reader has already some basic knowledge on holomorphic dynamical systems.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a branched covering with degree $\deg f \geq 2$. Its **postcritical set** is defined to be

$$P_f := \text{closure}\{f^n(c) \mid n > 0, \text{ for all critical points } c \text{ of } f\}.$$

Denote by P'_f the accumulation set of P_f.

We say that the map f is **postcritically finite** if every critical point has a finite orbit (i.e. $P'_f = \emptyset$). We say that the map f is a **(sub-hyperbolic) semi-rational map** if P'_f is finite (or empty); and in case $P'_f \neq \emptyset$, the map f is holomorphic in a neighborhood of P'_f and every periodic point in P'_f is either attracting or super-attracting.

Two semi-rational maps f_1 and f_2 are called **c-equivalent**, if there is a pair (ϕ, ψ) of homeomorphisms of $\overline{\mathbb{C}}$ and a neighborhood U_0 of P'_{f_1} such that:

(a) $\phi \circ f_1 = f_2 \circ \psi$;
(b) ϕ is holomorphic in U_0;
(c) the two maps ϕ and ψ are equal on P_{f_1}, thus on a neighborhood of P'_{f_1} (by the isolated zero theorem for holomorphic maps);
(d) the two maps ϕ and ψ are isotopic to each other rel $P_{f_1} \cup \overline{U_0}$ (i.e. there is a continuous map $H : [0, 1] \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $H(t, \cdot)$ is a homeomorphism of $\overline{\mathbb{C}}$ for all $t \in [0, 1]$, $H(0, \cdot) = \phi$, $H(1, \cdot) = \psi$, and $H(t, z) = \phi(z)$ for all $t \in [0, 1]$ and any $z \in P_{f_1} \cup \overline{U_0}$).

Given a semi-rational map f, we consider the problem of whether there is a rational map c-equivalent to it.

Thurston gave a combinatorial criterion of the same problem for postcritically finite branched coverings (see Sect. 3.1 and Theorem 3.2 below). We prove here:

Theorem 1.1 *Let f be a semi-rational map with $P'_f \neq \emptyset$. Then the map f is c-equivalent to a rational map R if and only if f has no Thurston obstructions. In this case the rational map R is unique up to Möbius conjugations.*

See the definition of Thurston obstructions in Sect. 3.1. The case $P'_f = \emptyset$ is already addressed by Thurston's original theorem. The necessity of having no Thurston obstructions and the unicity of the rational map R are known for a wider class of maps. See [24] (or Theorem 3.3 below) and [8]. Thus it remains only to prove the existence part here: i.e. to show that if the map f has no Thurston obstructions then it is c-equivalent to a rational map.

In the process of proving the theorem, we introduce the concept of repelling systems *with constant complexity*. We develop a corresponding Thurston-like theory, including the notions of c-equivalence, boundary multicurve, Thurston obstructions, renormalizations etc., and then a theorem saying that such a system with no obstructions is c-equivalent to a holomorphic repelling system (see Theorems 3.5 and 5.4 for detailed statements). This leads naturally to a combination result for rational maps (Theorem 9.1).

The general strategy of the proof of Theorem 1.1 can then be described as follows: we define K_f, its **filled-in Julia set** relative to P'_f, to be the set of points not attracted by the cycles in P'_f, i.e.

$$K_f := \left\{ z \in \overline{\mathbb{C}} \,\middle|\, \overline{\bigcup_{n>0} \{f^n(z)\}} \cap P'_f = \emptyset \right\}. \tag{1}$$

Step 0. We show that up to a c-equivalence, we may assume that the map f is quasiregular (Lemma 2.1).

Now let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasiregular semi-rational map with no Thurston obstructions. We will show that:

1. There is a restriction of f to a neighborhood L_1 of the filled-in Julia set K_f so that $f|_{L_1} : L_1 \to f(L_1)$ is a postcritically finite repelling system with no Thurston obstructions (Lemma 3.6 and Definition 2).
2. There is a further restriction of f near K_f, which is a marked repelling system with no Thurston obstructions and with constant complexity (Definition 5, Theorem 4.1(2) and Theorem 5.1).
3. This marked repelling system with constant complexity has no boundary obstruction nor renormalization obstructions (Definitions 6 and 7, Lemma 5.3).
4. Any marked repelling system with constant complexity and being unobstructed as in Step 3 is c-equivalent to a holomorphic marked repelling system (Theorem 5.4).
5. $f|_{L_1} : L_1 \to f(L_1)$ is c-equivalent to a holomorphic marked repelling system (Theorem 4.1(1)).
6. $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is c-equivalent to a rational map (Proposition 2.4).

Steps 1–3 consist of a detailed study of Thurston obstructions for repelling systems, as well as combinatorics of puzzle neighborhoods of the filled-in Julia set K_f.

Step 4 (Theorem 5.4) is the core part of this work. It is proved by Grötzsch inequality, Thurston's original theorem, the Measurable Riemann Mapping theorem, together with a reversed form of Grötzsch inequality.

Steps 5–6 are standard applications of the Measurable Riemann Mapping Theorem.

Steps 2–5 together lead to a Thurston-like theorem for repelling systems (see Theorem 3.5 for a precise statement), which is of independent interest.

Remark Theorem 1.1 was already announced in [7], together with a sketch of the main ideas of the proof. However numerous details were either missing or erroneous. The presentation here will be totally different. In particular the concept of repelling systems and the related Thurston-like theory are new. This in turn leads to several effective criteria for the absence of obstructions, as well as an easy-to-use combination result: Theorem 9.1.

While this paper was circulating as an arXiv preprint, another proof of Theorem 1.1 was posted by Jiang and Zhang ([21]). Their proof is closer to the original proof of Thurston. But they do not address the problem of verifying absence of obstructions. In particular they do not provide a combination result, nor a detailed description of the structure of disconnected Julia sets.

Organization The paper is organized as follows: In Sect. 2 we prove Step 0 and Step 6 above. We introduce the concept of a repelling system and show how it appears as a restriction near K_f of a global map f.

In Sect. 3 we first recall the definition of Thurston obstructions and state Thurston's original theorem. We then develop the corresponding concepts for repelling systems and state a Thurston-like theorem in this setting (Theorem 3.5). Finally we use this result to prove Theorem 1.1 (this amounts to prove Step 1 above).

In Sects. 4–5 we introduce the concepts of constant complexity repelling systems and the specific obstructions associated to them. We state our Thurston-like theorem, Theorem 5.4, in this setting. Assuming this we complete Steps 2–5 above and prove Theorem 3.5.

In Sects. 6–8 we give the proof of Theorem 5.4.

In the final section Sect. 9 we state Theorem 9.1.

Along the way we will provide numerous supporting diagrams and pertinent examples.

2 Reduction to a restriction near the filled-in Julia set

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a semi-rational map with $P'_f \neq \emptyset$, i.e. the map f is a branched covering such that the cluster set P'_f of its postcritical set is finite and non-empty, f is holomorphic in a neighborhood of P'_f, and every periodic cycle in P'_f is either attracting or superattracting. We will give some criteria here for f to be c-equivalent to a rational map.

2.1 Making the map quasiregular

Lemma 2.1 *Let f be a semi-rational map with $P'_f \neq \emptyset$. Then the map f is c-equivalent to a quasiregular semi-rational map.*

Proof Consider f as a branched covering from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. There is a unique complex structure $\mathcal{X}'$ on $\overline{\mathbb{C}}$ such that $f : (\overline{\mathbb{C}}, \mathcal{X}') \to \overline{\mathbb{C}}$ is holomorphic.[1] The uniformization theorem provides a conformal homeomorphism $\xi : (\overline{\mathbb{C}}, \mathcal{X}') \to \overline{\mathbb{C}}$. Set $R := f \circ \xi^{-1}$. Then $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering, holomorphic with respect to the standard complex structure, and therefore a rational map.

Let $U \subset \overline{\mathbb{C}}$ be a finite union of quasidiscs with pairwise disjoint closures, such that $P'_f \subset U$, $f^{-1}(U) \supset \overline{U}$, ∂U does not contain critical points of f, and f is holomorphic in a neighborhood of $\overline{U}$. Using the relation $f = R \circ \xi$ one sees that the homeomorphism ξ is holomorphic in a neighborhood of ∂U with respect to the standard complex structure. It follows that $\xi(\partial U)$ consists of finitely many pairwise disjoint quasi-circles.

Set $L := \overline{\mathbb{C}} \smallsetminus U$. Then there is a quasiconformal homeomorphism $\eta : L \to \xi(L)$ such that $\eta = \xi$ on $\partial L \cup (P_f \cap L)$ and η is isotopic to ξ rel $\partial L \cup (P_f \cap L)$ (see Lemma C.2 in the appendix). Set $\zeta = \eta^{-1} \circ \xi$ on L and $\zeta = \mathrm{id}$ on U. Then ζ is isotopic to the identity rel $\overline{U} \cup P_f$, so $f \circ \zeta^{-1}$ is c-equivalent to f. But $f \circ \zeta^{-1} = R \circ \eta$ on L, with η quasiconformal and R holomorphic. One sees that $f \circ \zeta^{-1}$ is quasiregular in L. But on $\overline{\mathbb{C}} \smallsetminus L = U$, the map $f \circ \zeta^{-1}$ is equal to f and therefore is holomorphic. Thus $f \circ \zeta^{-1}$ is quasiregular on the entire space $\overline{\mathbb{C}}$. □

2.2 Repelling system as a restriction

Notation For two subsets E_1, E_2 of $\overline{\mathbb{C}}$, we write $E_1 \Subset E_2$ if the closure of E_1 is contained in the interior of E_2.

We will cover the filled-in Julia set K_f by a surface puzzle $\mathcal{L}$ such that $f^{-1}(\mathcal{L}) \Subset \mathcal{L}$, just as in Branner and Hubbard's study of cubic polynomials with disconnected Julia sets ([3]). The restriction $f|_{f^{-1}(\mathcal{L})}$, considered as a dynamical system, leads naturally to the concept of repelling systems. Such dynamical systems can also be considered as a generalization of Douady and Hubbard's polynomial-like mappings ([14]) in three aspects: The domain of definition will have finitely many (instead of just one) connected components, every component will have finitely many (instead of just one) boundary curves (this is necessary as we are dealing with rational maps), and the dynamics will be quasiregular branched coverings (instead of holomorphic proper maps).

Definition 1 A **(quasidisc) bordered surface** is the Riemann sphere minus finitely many (or zero) open quasidiscs whose closures are mutually disjoint. A **surface puzzle** $\mathcal{L} = S_1 \sqcup \cdots \sqcup S_k$ is a finite disjoint union of bordered

[1] The charts about non critical points are defined using directly f. But to get a chart about a critical point c say of local degree δ one should take a lift of f^{-1} to $z^\delta + f(c)$. See [12], Sect. 6.1.10 for details.

surfaces S_i. Each S_i is also called an **$\mathcal{L}$-piece**. Note that two $\mathcal{L}$-pieces might be contained in different copies of the Riemann sphere.

We say that a map $F:\mathcal{E}\to\mathcal{L}$ is a **(quasiregular) repelling system**, if

(I) The two sets $\mathcal{E}$ and $\mathcal{L}$ are surface puzzles, and satisfy $\mathcal{E}\Subset\mathcal{L}$;
(II) the map F maps every $\mathcal{E}$-piece E properly onto an $\mathcal{L}$-piece S as a quasiregular map, i.e. there is a quasiconformal homeomorphism $\phi: E\to\phi(E)\subset\overline{\mathbb{C}}$ such that $F\circ\phi^{-1}:\phi(E)\to S$ is a holomorphic proper map;
(III) the orbit $\{F^n(c)\}_{n>0}$ of every critical point c of F is disjoint from the boundary of $\mathcal{E}$.

In this paper, a quasiconformal (quasiregular, holomorphic) map between a pair of surface puzzles means that it is quasiconformal (quasiregular, holomorphic) in the interior. From (III), we see that for any $k\ge 1$ and $n\ge 0$, the map $F^k: F^{-n-k}(\mathcal{L})\to F^{-n}(\mathcal{L})$ is also a repelling system. Notice that due to the disconnectedness of the domains the number of preimages $\#F^{-1}(b)$ needs not be constant when we let b vary in $\mathcal{L}$.

A repelling system $F:\mathcal{E}\to\mathcal{L}$ is **holomorphic** if F is holomorphic in $\mathcal{E}$. To a repelling system $F:\mathcal{E}\to\mathcal{L}$ we associate its **postcritical set** P_F and its **filled-in Julia set** K_F as follows:

$$P_F := \text{closure}\{F^n(c)\mid \text{for all critical points } c \text{ of } F,\ n>0\},$$
$$K_F := \{z\in\mathcal{E}\mid F^n(z)\in\mathcal{E},\ \text{for all } n>0\} = \bigcap_{n>0} F^{-n}(\mathcal{L}).$$

Note that F has only finitely many critical points. The set P_F might be empty. One may construct examples for which K_F is empty (for example, $\mathcal{L}=S_1\sqcup S_2$, $\mathcal{E}\Subset S_1$ and $F(\mathcal{E})=S_2$), although we will be only interested in the case that $K_F\neq\emptyset$, with either P_F empty or not empty. We have $F^{-1}(K_F)=K_F=F(K_F)$, $P_F\Subset\mathcal{L}$ and $F(P_F\cap\mathcal{E})\subset P_F$.

We say furthermore that $F:\mathcal{E}\to\mathcal{L}$ is **postcritically finite** if P_F is finite or empty. In particular we say that $F:\mathcal{E}\to\mathcal{L}$ is an **annuli-covering** if every $\mathcal{L}$-piece is a closed annulus and every $\mathcal{E}$-piece is a closed essential sub-annulus in $\mathcal{L}$. An **essential sub-annulus** E in an annulus S means that the sub-annulus E separates the two boundary components of S. Obviously, $P_F=\emptyset$ for an annuli-covering F.

The following *restriction principle* provides the most fundamental examples of the above concepts:

Lemma 2.2 *Let f be a quasiregular semi-rational map with $P'_f\neq\emptyset$. Then there exists a surface puzzle neighborhood L_0 of K_f such that $L_1 := f^{-1}(L_0)\Subset L_0$ and $f|_{L_1}: L_1\to L_0$ is a postcritically finite repelling system.*

Proof One can find an open set U_0 which is the union of finitely many quasidiscs with the following properties: Each disc contains exactly one point of P'_f, the discs have pairwise disjoint closures, the boundary ∂U_0 is disjoint from P_f, the map f is holomorphic in a neighborhood of $\overline{U_0}$, and finally $f(U_0) \Subset U_0$.

Set $L_0 = \overline{\mathbb{C}} \setminus U_0$. Then $K_f \Subset f^{-1}(L_0) \Subset L_0$, the intersection $P_f \cap L_0$ is finite and P_f is disjoint from the boundary of $f^{-1}(L_0)$. This surface puzzle L_0 satisfies the requirement of the lemma. □

Examples of repelling systems

1. Let $\mathcal{E} \Subset \mathcal{L}$ be two closed quasidiscs in $\overline{\mathbb{C}}$, and $F : \mathcal{E} \to \mathcal{L}$ a holomorphic proper map with degree $\deg F > 1$. Then F is a polynomial-like map in the sense of Douady and Hubbard, K_F is simply the filled-in Julia set, and P_F is the postcritical set.
2. The set $\mathcal{L}$ consists of a single closed quasidisc, the set $\mathcal{E}$ is the union of finitely many disjoint closed quasidiscs contained in the interior of $\mathcal{L}$, and F maps each $\mathcal{E}$-piece quasiconformally onto the larger disc $\mathcal{L}$. In this case $P_F = \emptyset$ and K_F is the non-escaping set of F. If F is also holomorphic, the filled-in Julia set K_F is a Cantor set. This happens when $F(z) = z^2 + c$ for large c.
3. By convention we may consider $\mathcal{E} = \mathcal{L} = \overline{\mathbb{C}}$ and a quasiregular postcritically finite branched covering $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as a repelling system.

Definition 2 By a **marked repelling system** $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ (denoted also by (F, P) for simplicity) we mean that we have a postcritically finite repelling system $F : \mathcal{E} \to \mathcal{L}$, together with a finite marked set P, such that $P \Subset \mathcal{L}$, $P_F \subset P$ and $F(P \cap \mathcal{E}) \subset P$.

If it is not explicitly mentioned, we will consider F to be marked by its postcritical set P_F.

Motivated by Thurston's theory, we say that two marked repelling systems $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ and $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ are **c-equivalent**, if there is a pair of quasiconformal homeomorphisms $\Phi, \Psi : \mathcal{L} \to \mathcal{M}$ such that

$$\begin{cases} \Psi(\mathcal{E}) = \mathcal{B} \text{ and } \Psi(P) = Q, \\ \Psi \text{ is isotopic to } \Phi \text{ rel } \partial\mathcal{L} \cup P, \\ \text{and } \Phi \circ F = G \circ \Psi. \end{cases} \tag{2}$$

Remark This definition matches with McMullen's definition of combinatorial equivalence among Riemann surface self-coverings ([24], p. 288).

Lemma 2.3 *A marked repelling system $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ is c-equivalent to a holomorphic marked repelling system iff there is a pair (Θ, μ) such that*:

(a) $\Theta : \mathcal{L} \to \mathcal{L}$ *is a quasiconformal map isotopic to the identity rel* $\partial\mathcal{L} \cup P$.
(b) μ *is a Beltrami differential on* $\mathcal{L}$ *with* $\|\mu\|_\infty < 1$ *and* $(F \circ \Theta^{-1})^*(\mu) = \mu|_{\Theta(\mathcal{E})}$.

Proof Assume that $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ is c-equivalent to a holomorphic marked repelling system $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$. Let (Φ, Ψ) be the pair of quasiconformal maps given by Definition 2. Set $\Theta = \Phi^{-1} \circ \Psi$ and let μ be the Beltrami coefficient of Φ. Then Θ satisfies (a) and $(F \circ \Theta^{-1})^*(\mu) = \mu|_{\Theta(\mathcal{E})}$.

Conversely, assume the existence of the pair (Θ, μ). For each $\mathcal{L}$-piece S, denote by $\overline{\mathbb{C}}(S)$ the Riemann sphere containing S (we consider each such S to be contained in a distinct copy of the Riemann sphere). By the Measurable Riemann Mapping Theorem, there is a quasiconformal map $\Phi_S : \overline{\mathbb{C}}(S) \to \overline{\mathbb{C}}$ whose Beltrami coefficient is equal to μ on S and equal to zero elsewhere. Set $\mathcal{M} = \bigsqcup \Phi_S(S)$, $\mathcal{B} = \bigsqcup \Phi_S(\mathcal{E} \cap S)$ and $Q = \bigsqcup \Phi_S(P \cap S)$. Define $\Phi, \Psi : \mathcal{L} \to \mathcal{M}$ by $\Phi|_S = \Phi_S$ and $\Psi|_S = \Phi_S \circ \Theta|_S$, and $G := \Phi \circ F \circ \Psi^{-1}$. Then $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ is a holomorphic marked repelling system c-equivalent to $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$. □

The following result relates repelling systems to our main theorem of interest (Theorem 1.1) through a restriction:

Proposition 2.4 *Let* f *be a quasiregular semi-rational map with* $P'_f \neq \emptyset$. *Let* $\mathcal{L}$ *be a surface puzzle neighborhood of* K_f *such that* $\partial\mathcal{L} \cap P_f = \emptyset$, *the set* $P := P_f \cap \mathcal{L}$ *is a finite set, and the set* $\mathcal{E} := f^{-1}(\mathcal{L})$ *satisfies* $\mathcal{E} \Subset \mathcal{L}$. *If the restriction* $f|_\mathcal{E} : \mathcal{E} \to \mathcal{L}$ *as a repelling system is c-equivalent to a holomorphic repelling system, then* f *is c-equivalent to a rational map.*

Proof By Lemma 2.3 there is a pair (Θ, μ), with Θ a quasiconformal map of $\mathcal{L}$ isotopic to the identity rel $\partial\mathcal{L} \cup P$, with μ a Beltrami differential on $\mathcal{L}$ such that $\|\mu\|_\infty < 1$ and $(f \circ \Theta^{-1})^*\mu = \mu|_{\Theta(\mathcal{E})}$.

Choose U_0, an open neighborhood of P'_f disjoint from $\mathcal{L}$, so that $f^{-1}(U_0) \supset \overline{U_0}$ and f is holomorphic on $f^{-1}(U_0)$. Set $L_0 = \overline{\mathbb{C}} \setminus U_0$ and $L_n = f^{-n}(L_0)$, $n \geq 1$. The sequence $(L_n)_{n \geq 1}$ is decreasing and shrinks down to K_f. There is therefore an integer $N \geq 0$ such that $L_N \subset \mathcal{L}$ and $L_{N+1} \subset \mathcal{E}$. So every orbit passing through the set $L_0 \setminus \mathcal{E}$ stays there for at most $N + 1$ times before being trapped by U_0.

Extend the map Θ to a quasiconformal map of $\overline{\mathbb{C}}$ by setting $\Theta :=$ id on $\overline{\mathbb{C}} \setminus \mathcal{L}$, then Θ is quasiconformal and isotopic to the identity rel P_f. Set $f_1 = f \circ \Theta^{-1}$. Then f_1 is again quasiregular and is holomorphic on $\Theta(U_0) = U_0$. Clearly, every f_1-orbit passes through $L_0 \setminus \Theta(\mathcal{E})$ at most $N + 1$ times.

Now extend the Beltrami differential μ to $\overline{\mathbb{C}}$ by setting $\mu = 0$ outside $\mathcal{L}$. Let $\Phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a global integrating map of this extended μ. Set $f_2 := \Phi_1 \circ f_1 \circ \Phi_1^{-1}$. Then f_2 is again quasiregular and is holomorphic in the interior of $\Phi_1 \circ \Theta(\mathcal{E})$ and in $\Phi_1(U_0)$. Every f_2-orbit passes through at most $N+1$ times $\overline{\mathbb{C}} \setminus \Phi_1(\Theta(\mathcal{E}) \cup U_0)$.

We can now apply Shishikura's principle (see [1], Lemma 15, p. 130): we spread out the Beltrami differential $\nu_0 \equiv 0$ using the iterations of f_2 to get an f_2-invariant Beltrami differential ν. Note that $\nu = 0$ on $\Phi_1(U_0)$, and $\|\nu\|_\infty < 1$. Integrating ν by a quasiconformal map Φ_2 (necessarily holomorphic on $\Phi_1(U_0)$), we get a new map $R := \Phi_2 \circ f_2 \circ \Phi_2^{-1}$ which is a rational map and is c-equivalent to f_2, therefore also to f. □

The above constructions are illustrated by the following commutative diagram:

$$
\begin{array}{ccccccc}
(\overline{\mathbb{C}}, \mathcal{E}) & \xrightarrow{\Theta} & \overline{\mathbb{C}} & \xrightarrow{\Phi_1} & \overline{\mathbb{C}} & \xrightarrow{\Phi_2} & \overline{\mathbb{C}} \\
f \downarrow & & f_1 \downarrow & & \downarrow f_2 & & \downarrow R \\
(\overline{\mathbb{C}}, \mathcal{L}) & \xrightarrow[\text{id}]{} & \overline{\mathbb{C}} & \xrightarrow[\Phi_1]{} & \overline{\mathbb{C}} & \xrightarrow[\Phi_2]{} & \overline{\mathbb{C}}
\end{array}
$$

3 Thurston-like theory for repelling systems

Proposition 2.4 leads us to consider whether a given marked repelling system is c-equivalent to a holomorphic marked repelling system. We will see that, similar to Thurston's theory, the answer is yes if the map is unobstructed. And the obstructions that can arise are very similar to Thurston's original ones.

3.1 Grötzsch's inequality and Thurston obstructions

Thurston obstructions are in fact closely related to Grötzsch's inequality on moduli of annuli. We will illustrate this by starting from real models.

3.1.1 Slope obstructions

Suppose we want to make a tent map f on the interval $[0, 1]$ with folding point c and with $f(c) > 1$, with left slope $d_1 > 0$ and right slope $-d_2$ ($d_2 > 0$). This is possible if and only if $d_1^{-1} + d_2^{-1} < 1$.

More generally, suppose that I and J are two closed subsets of $\mathbb{R}$ such that each of them is a union of finitely many disjoint closed intervals, and $J \Subset I$. Let $f : J \to I$ be a map such that for each component K of J, the restriction $f|_K : K \to f(K)$ is a homeomorphism and $f(K)$ is a component of I. The question we ask is: given a real number $d(K) > 0$ for each component K of J, are there homeomorphisms $h : I \to I'$ and $\theta : I \to I$ with $\theta|_{\partial I} = \text{id}$ such that for each component K of J, the new map $h \circ f \circ \theta \circ h^{-1}$ is affine on $h(K)$ with slope (in absolute value) $d(K)$?

To fix our ideas we may at first assume that f is itself piece-wise affine. Then $|K| = |f(K)|/d(K)$ (where $|\cdot|$ denotes the length of the interval). Hence for each interval I_i of I,

$$\sum_j \left(\sum \frac{1}{d(K)} \right) |I_j| < |I_i|, \tag{3}$$

where the second sum is taken over the components K of J satisfying $K \subset I_i$ and $f(K) = I_j$. Denote by a_{ij} this second sum. Collecting the a_{ij}'s together we get the **transition matrix** $W = (a_{ij})$. It is a non-negative matrix. Then the inequality (3) can be reformulated as $Wv < v$ with $v := (|I_i|)$. Note that v is a vector with strictly positive entries.

It is then conceivable that a necessary and sufficient condition for a positive answer of the above question could be: The transition matrix W must admit a vector v with strictly positive entries so that $Wv < v$. Notice that this condition on W is equivalent to the one that the leading eigenvalue of W from Perron-Frobenius theorem is strictly less than 1 (see Lemma A.1).

Rather than proving this statement in detail, we will give a complexified version of it and provide a complete proof.

3.1.2 Grötzsch obstructions for annuli-coverings

Now we replace the intervals I_j above by thin tubes. More precisely, let $\mathcal{A} = A_1 \sqcup \cdots \sqcup A_k$ be a surface puzzle with each piece A_i a closed annulus. Let $\mathcal{E} \Subset \mathcal{A}$ be a surface puzzle such that each $\mathcal{E}$-piece is a closed sub-annulus essentially contained in some A_i. Consider $f : \mathcal{E} \to \mathcal{A}$ as a repelling system (an annuli covering). The question is: Is $f : \mathcal{E} \to \mathcal{A}$ c-equivalent to a holomorphic repelling system?

Define the transition matrix $W = (a_{ij})$ by

$$a_{ij} = \sum \frac{1}{\deg(f : E \to A_j)}$$

where the sum is taken over the $\mathcal{E}$-pieces E satisfying $E \subset A_i$ and $f(E) = A_j$.

Assume that f is already holomorphic. Denote by $|\cdot|$ the modulus of the interior of an annulus. Then, due to Grötzsch's inequality, for each piece A_i,

$$\sum_j a_{ij} |A_j| < |A_i|.$$

Therefore, as above, the leading eigenvalue $\lambda(W)$ of the transition matrix W is less than 1. We have, naturally:

Lemma 3.1 *An annuli-covering $f : \mathcal{E} \to \mathcal{A}$ is c-equivalent to a holomorphic repelling system if and only if $\lambda(W) < 1$.*

Proof Assume that $f : \mathcal{E} \to \mathcal{A}$ is c-equivalent to a holomorphic repelling system. Then the two maps have the same transition matrix W. By the argument above $\lambda(W) < 1$.

The remaining part will be done in Lemma 6.2. □

This lemma is not really needed in the proof of our main result. But it helps understanding Thurston obstructions, and its proof will shed light on our more complicated situation.

3.1.3 Thurston obstructions

By a **marked branched covering** (f, P) we mean that $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering and the **marked set** $P \subset \overline{\mathbb{C}}$ is a closed set such that $P_f \subset P$ and $f(P) \subset P$.

A Jordan curve γ in $\overline{\mathbb{C}} \setminus P$ is called **null-homotopic** (resp. **peripheral**) in $\overline{\mathbb{C}} \setminus P$ if one of its complementary components contains zero (resp. one) points of P; and is otherwise called **non-peripheral** in $\overline{\mathbb{C}} \setminus P$, i.e. if each of its two complementary components contains at least two points of P.

We say that $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ is a **multicurve in** $\overline{\mathbb{C}} \setminus P$, if each γ_i is a non-peripheral Jordan curve in $\overline{\mathbb{C}} \setminus P$, these curves are pairwise disjoint and pairwise non homotopic in $\overline{\mathbb{C}} \setminus P$. Its (f, P)-**transition matrix** $W_\Gamma = (a_{ij})$ is defined by:

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(f : \alpha \to \gamma_j)},$$

where the summation is taken over the components α of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\overline{\mathbb{C}} \setminus P$.

We say that a multicurve Γ in $\overline{\mathbb{C}} \setminus P$ is f-**stable** if every curve of $f^{-1}(\gamma)$, $\gamma \in \Gamma$ is either null-homotopic or peripheral in $\overline{\mathbb{C}} \setminus P$ or is homotopic in $\overline{\mathbb{C}} \setminus P$ to a curve in Γ. This implies that for every curve $\gamma \in \Gamma$ and any $m > 0$, every curve of $f^{-m}(\gamma)$ is either null-homotopic or peripheral in $\overline{\mathbb{C}} \setminus P$ or is homotopic in $\overline{\mathbb{C}} \setminus P$ to a curve in Γ.

We say that a multicurve Γ in $\overline{\mathbb{C}} \setminus P$ is a **Thurston obstruction** for the marked branched covering (f, P) if it is f-stable and the leading eigenvalue of its transition matrix is greater than or equal to 1. In the particular case $P = P_f$ we say that Γ is a Thurston obstruction for f.

In the case that P is finite (in particular f is postcritically finite) we say that two such marked branched coverings (f, P) and (g, Q) are **c-equivalent** if there is a pair of homeomorphisms $(\phi, \psi) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that ϕ is isotopic to ψ rel P (in particular $\phi|_P = \psi|_P$) and $\phi \circ f \circ \psi^{-1} = g$.

Theorem 3.2 (Marked Thurston's theorem) *Let f be a postcritically finite branched covering of $\overline{\mathbb{C}}$ with* $\deg f \geq 2$. *Assume that the signature of its orbifold is not* $(2, 2, 2, 2)$ (*see the remark below*). *Let P be a finite set containing P_f such that $f(P) \subset P$. If (f, P) has no Thurston obstructions, then (f, P) is c-equivalent to a unique marked rational map. More precisely, there are homeomorphisms $(\phi, \psi) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that ϕ is isotopic to ψ rel P and $R := \phi \circ f \circ \psi^{-1}$ is a rational map. The conformal conjugacy class of the marked rational map $(R, \phi(P))$ is unique.*

Furthermore, when f is quasiregular, both ϕ and ψ can be taken to be quasiconformal.

For the definition of orbifolds and their signatures, see [13] or [24]. We only mention here that if P_f contains periodic critical points or at least 5 points, then the signature of the orbifold of f is not $(2, 2, 2, 2)$. This will be enough for our purpose here.

Remark Our statement is slightly stronger than Thurston's original theorem (see [35] or [13]), where $P = P_f$. But the arguments in [13] can easily be adapted to prove this more general form. See for example the survey paper [6] for details. In the case that f is quasiregular, we may replace ϕ by a quasiconformal map ϕ_1 isotopic to ϕ rel P. This is possible since P is finite (see Lemma C.2). Lifting this isotopy will give us a quasiconformal map ψ_1 isotopic to ψ rel P such that $\phi \circ f \circ \psi^{-1} = \phi_1 \circ f \circ \psi_1^{-1}$.

Conversely, we have the following result of McMullen [24]:

Theorem 3.3 *Let f be a rational map with* $\deg f \geq 2$, *and let P be a closed subset such that $f(P) \subset P$ and $P_f \subset P$. Let Γ be a multicurve in $\overline{\mathbb{C}} \setminus P$ whose transition matrix is denoted by W. Then $\lambda(W) \leq 1$. If $\lambda(W) = 1$, then either f is a postcritically finite map whose orbifold has signature* $(2, 2, 2, 2)$ *or Γ includes a curve that is contained in a Siegel disc or a Herman ring of f.*

Again this form is slightly stronger than McMullen's original version. But the proof goes through without any problem.

3.2 Thurston obstructions for repelling systems.

Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. In other words $F : \mathcal{E} \to \mathcal{L}$ is a postcritically finite quasiregular branched covering between a pair of nested surface puzzles $\mathcal{E} \Subset \mathcal{L}$, and $P \Subset \mathcal{L}$ is a finite set containing P_F such that $F(P \cap \mathcal{E}) \subset P$. (In the case $\mathcal{L} = \overline{\mathbb{C}}$ we are back to Thurston's setting.)

Definition 3 Two Jordan curves in $\mathcal{L} \setminus P$ are **homotopic** if they are both contained in a common $\mathcal{L}$-piece S and are homotopic to each other in $S \setminus P$.

A Jordan curve $\gamma \subset \mathcal{L} \setminus P$ is called **null-homotopic** (resp. **peripheral**) in $\mathcal{L} \setminus P$ if it bounds an open disc $D \subset \mathcal{L}$ and $D \cap P = \emptyset$ (resp. $\#(D \cap P) = 1$); it is called **non-peripheral** in $\mathcal{L} \setminus P$ otherwise (this is equivalent to say that γ is contained in $S \setminus P$ for some $\mathcal{L}$-piece S, and if γ bounds a disc D that is entirely contained in S, then D contains at least two points of P).

Remark The above definition of peripheral curves is not standard. In the literature, a boundary curve γ of $\mathcal{L}$ is considered to be peripheral. However, in our definition, the curve γ is non-peripheral if either the $\mathcal{L}$-piece S that contains γ is not a closed disc or the piece S is a closed disc satisfying $\#(S \cap P) \geq 2$.

We say that $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a **multicurve** in $\mathcal{L} \setminus P$ if each γ_i is a non-peripheral Jordan curve in $\mathcal{L} \setminus P$ and these curves are pairwise disjoint and pairwise non homotopic in $\mathcal{L} \setminus P$. Its **(F, P)-transition matrix** $W_\Gamma = (a_{ij})$ is defined by:

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)},$$

where the summation is taken over the components α of $F^{-1}(\gamma_j)$ homotopic to γ_i in $\mathcal{L} \setminus P$.

We say that a multicurve Γ in $\mathcal{L} \setminus P$ is F-**stable** if every curve of $F^{-1}(\gamma)$, $\gamma \in \Gamma$ is either null-homotopic or peripheral in $\mathcal{L} \setminus P$ or is homotopic in $\mathcal{L} \setminus P$ to a curve in Γ.

We say that a multicurve Γ in $\mathcal{L} \setminus P$ is a **Thurston obstruction** for the marked repelling system $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ if it is F-stable and the leading eigenvalue of its transition matrix satisfies $\lambda(W_\Gamma) \geq 1$.

The following principle will be used frequently, and is a direct consequence of the facts that $F(P \cap \mathcal{E}) \subset P$ and F is a covering over $\mathcal{L} \setminus P$:

Basic Pullback Principle.

1. Let D be a Jordan disc contained in $\mathcal{L} \setminus P$ with $\partial D \cap P = \emptyset$. Then every component of $F^{-1}(D)$ is again a disc and is contained in $\mathcal{E} \setminus P$. Each curve in $F^{-1}(\partial D)$ is the boundary of a component of $F^{-1}(D)$.
2. Let A be an annulus contained in $\mathcal{L} \setminus P$. Then every component of $F^{-1}(A)$ is again an annulus and is contained in $\mathcal{E} \setminus P$.
3. Let D be a Jordan disc contained in $\mathcal{L}$ with $\partial D \cap P = \emptyset$ and $\#D \cap P = 1$. Then every component of $F^{-1}(D)$ is again a Jordan disc in $\mathcal{E}$ containing at most one point of P. Each curve in $F^{-1}(\partial D)$ is the boundary of a component of $F^{-1}(D)$.

The following is an easy consequence:

Lemma 3.4 *Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. For any peripheral (resp. null-homotopic) curve $\gamma \subset \mathcal{L} \setminus P$, every curve in $F^{-1}(\gamma)$ is either peripheral or null-homotopic (resp. is null-homotopic) in $\mathcal{L} \setminus P$.*

We will prove the following Thurston-like theorem for marked repelling systems:

Theorem 3.5 *Let $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ be a marked repelling system such that every $\mathcal{M}$-piece has a non-empty boundary. If (G, Q) has no Thurston obstructions, then it is c-equivalent to a holomorphic marked repelling system.*

Remark By definition the map G must map the interior of every $\mathcal{B}$-piece E properly onto the interior of an $\mathcal{M}$-piece S. In particular $S = \overline{\mathbb{C}}$ if and only if $E = \overline{\mathbb{C}}$. Therefore if some $\mathcal{M}$-pieces are the whole spheres, the dynamical system G can be decomposed into two sub-systems as follows:

First we decompose the surface puzzle $\mathcal{M}$ into the disjoint union $\mathcal{M}_0 \sqcup \mathcal{M}_1$ so that $\mathcal{M}_0$ is the union of the spherical $\mathcal{M}$-pieces. We then decompose the surface puzzle $\mathcal{B}$ into $\mathcal{B}_0 \sqcup \mathcal{B}_1 \sqcup \mathcal{B}_{01}$, with $\mathcal{B}_0$ the union of the spherical $\mathcal{B}$-pieces (hence it is contained in $\mathcal{M}_0$), $\mathcal{B}_1$ the union of the $\mathcal{B}$-pieces that are contained in $\mathcal{M}_1$ (they are necessarily non-spherical), and $\mathcal{B}_{01}$ the union of the non-spherical $\mathcal{B}$-pieces that are contained in $\mathcal{M}_0$. Then $G(\mathcal{B}_0) \subset \mathcal{M}_0$ and $G(\mathcal{B}_1 \sqcup \mathcal{B}_{01}) \subset \mathcal{M}_1$.

Denote by N the number of $\mathcal{M}$-pieces. One can show that for any point $z \in K_G$, the tail of its forward orbit $\{G^n(z)\}_{n>N}$ is either totally contained in $\mathcal{B}_1$ or totally contained in $\mathcal{B}_0$. Thus the dynamics of G splits into two parts according to the above two cases. The former case satisfies our assumption. In the latter case, for each cycle of the spheres, if the degree of the return map is greater than one, then it is reduced to Thurston's setting; otherwise the return map is a homeomorphism and the corresponding problem is to find an invariant complex structure on a punctured sphere for a modular transformation (this is related to another topic of research, namely Thurston's classification of surface homeomorphisms).

3.3 Proof of Theorem 1.1 using Theorem 3.5

Lemma 3.6 *Assume that $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiregular semi-rational map with $P'_f \neq \emptyset$ and with no Thurston obstructions. Then there is a surface puzzle $L_0 \subset \overline{\mathbb{C}}$ such that $K_f \Subset L_1 := f^{-1}(L_0) \Subset L_0$ and the restriction $f|_{L_1} : L_1 \to L_0$ is a postcritically finite repelling system with no Thurston obstructions.*

Proof As in Lemma 2.2 one can find an open set U_0 which is the union of finitely many quasidiscs with the following properties: Each disc contains exactly one point of P'_f, the discs have pairwise disjoint closures, the boundary ∂U_0 is disjoint from P_f, the map f is holomorphic in a neighborhood of $\overline{U_0}$, and finally $f(U_0) \Subset U_0$.

Set $L_0 = \overline{\mathbb{C}} \setminus U_0$. Topologically L_0 is the sphere minus finitely many (open) holes. Set $L_1 = f^{-1}(L_0)$, $F = f|_{L_1}$. Then $F : L_1 \to L_0$ is a postcritically finite repelling system. Note that $P_F = P_f \cap L_0$.

We will now show: if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has no Thurston obstructions, then the postcritically finite repelling system $F : L_1 \to L_0$ has none either.

Assume at first that L_0 is a closed disc containing at most one point of P_f. In this case ∂L_0 is a single curve and is either null-homotopic or peripheral in $L_0 \setminus P_F$, and there is no multicurve in $L_0 \setminus P_F$. Consequently $F : L_1 \to L_0$ has no Thurston obstructions.

Next assume that L_0 is a closed annulus disjoint from P_f. Then there is only one homotopy class of non-peripheral Jordan curves in L_0, namely that of a boundary curve γ of L_0. Such a γ is necessarily non-peripheral in $\overline{\mathbb{C}} \setminus P_f$ as well, for each of the two disc-components of $\overline{\mathbb{C}} \setminus L_0$ contains points of P'_f. The curves in $f^{-1}(\gamma)$ are all contained in L_0, and are either null-homotopic in L_0 or homotopic to γ in L_0. Therefore $\{\gamma\}$ is stable for both f and F, and the corresponding transition matrices are identical. By the assumption that f has no Thurston obstructions, the corresponding leading eigenvalue is less than one. Therefore $\{\gamma\}$ is not a Thurston obstruction for $F : L_1 \to L_0$ either. And $F : L_1 \to L_0$ has no obstructions.

In the remaining case, L_0 is a connected surface puzzle with

$$\#P_F + \#\{\text{boundary curves of } L_0\} \geq 3.$$

In particular each of its boundary curves is non-peripheral in $L_0 \setminus P_F$.

Let Γ be a multicurve in $L_0 \setminus P_F$. In other words,

(a) every curve in Γ is non-peripheral in $L_0 \setminus P_F$;
(b) the curves in Γ are mutually disjoint;
(c) no two curves in Γ are homotopic in $L_0 \setminus P_F$.

We want to show that Γ is also a multicurve in $\overline{\mathbb{C}} \setminus P_f$, i.e. Γ satisfies (a), (b) and (c) with $L_0 \setminus P_F$ replaced by $\overline{\mathbb{C}} \setminus P_f$. By (a), for every curve γ in Γ, either each of the two disc-components of $\overline{\mathbb{C}} \setminus \gamma$ contains a component of $\overline{\mathbb{C}} \setminus L_0 = U_0$ (and therefore infinitely many points of P_f); or one of them is contained in L_0 and contains at least two points of $P_F \subset P_f$, while the other one contains all components of U_0 (therefore infinitely many points of P_f). In both cases each component of $\overline{\mathbb{C}} \setminus \gamma$ contains at least two points of P_f, so γ is non-peripheral in $\overline{\mathbb{C}} \setminus P_f$.

By (b) the curves in Γ are mutually disjoint.

By (c), for any two curves γ and β in Γ, the open annulus $A(\gamma, \beta)$ bounded by γ and β intersects either U_0 or P_F (or both). In the former case $A(\gamma, \beta)$ contains a component of U_0. Therefore in both cases $A(\gamma, \beta)$ intersects P_f, so γ and β are not homotopic in $\overline{\mathbb{C}} \setminus P_f$.

These arguments imply that Γ is also a multicurve in $\overline{\mathbb{C}} \setminus P_f$.

Now assume that the multicurve Γ is F-stable. Then, for any $\gamma \in \Gamma$ and any curve δ in $F^{-1}(\gamma) = f^{-1}(\gamma)$, either δ bounds a disc that is contained in L_0 and contains at most one point of P_F or δ is homotopic in $L_0 \setminus P_F$ to a curve β in Γ. Thus either δ bounds a disc that contains at most one point of P_f or it is homotopic in $\overline{\mathbb{C}} \setminus P_f$ to β. This shows that Γ is also f-stable. The two transition matrices (for F and for f) are identical and therefore have the same leading eigenvalue λ.

By the assumption that f has no Thurston obstructions, we know that $\lambda < 1$, so Γ is not a Thurston obstruction for $F : L_1 \to L_0$. Therefore $F : L_1 \to L_0$ has no obstructions. □

Assuming Theorem 3.5, we may now give the

Proof of Theorem 1.1 (the existence part). Let f be a sub-hyperbolic semi-rational map with $P'_f \neq \emptyset$ and with no Thurston obstructions. We may assume in addition that f is globally quasiregular, up to a change of representatives in its c-equivalence class (by Lemma 2.1).

We may then apply Lemma 3.6 to f to show that it has a restriction near K_f which is a postcritically finite repelling system with no Thurston obstructions and is therefore c-equivalent to a holomorphic repelling system by Theorem 3.5. We may then apply Proposition 2.4 to conclude that f is c-equivalent to a rational map. □

4 Admissible restriction

In the following two sections we will reduce a marked repelling system to a restriction of it which is furthermore of constant complexity (we refer to Definition 5 in Sect. 5). Such a restriction should also satisfy some specific properties in order to not create extra obstructions.

Definition 4 Let $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ be a marked repelling system such that every $\mathcal{M}$-piece has a non-empty boundary. Let $\mathcal{E}$ and $\mathcal{L}$ be a pair of surface-puzzle-neighborhoods of K_G such that

$$K_G \Subset \mathcal{E} \Subset \mathcal{L} \Subset \mathcal{M}, \quad \mathcal{E} = G^{-1}(\mathcal{L}) \quad \text{and} \quad \partial\mathcal{L} \cap Q = \emptyset.$$

Then the repelling system $F = G|_{\mathcal{E}} : \mathcal{E} \to \mathcal{L}$ marked by $P := Q \cap \mathcal{L}$ is called an **admissible restriction** of (G, Q).

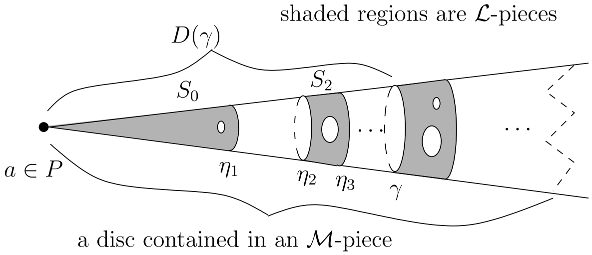


Fig. 1 The curve γ is not peripheral in $\mathcal{L} \setminus P$, but $\iota(\gamma)$ is peripheral in $\mathcal{M} \setminus Q$ since it bounds a disc $D(\gamma)$ that is contained in $\mathcal{M}$ and contains a unique point of Q, which is also a point of P

Theorem 4.1 *Let $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ be a marked repelling system such that every $\mathcal{M}$-piece has a non-empty boundary. Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be an admissible restriction of (G, Q). Then,*

(1) *If (F, P) is c-equivalent to a holomorphic marked repelling system, so is (G, Q).*
(2) *If (G, Q) has no Thurston obstructions, then neither does (F, P).*

The condition $K_G \Subset \mathcal{E}$ is necessary for the theorem to be true. See Example 4 in Sect. 7.1 for a counter-example.

Consider the **inclusion map** $\iota : \mathcal{L} \setminus P \hookrightarrow \mathcal{M} \setminus Q$.

Let Γ be a multicurve in $\mathcal{L} \setminus P$. When we consider the homotopy class of $\iota(\gamma)$ in $\mathcal{M} \setminus Q$ for $\gamma \in \Gamma$, the following cases may happen:

- Some $\iota(\gamma)$ may now become null-homotopic in $\mathcal{M} \setminus Q$. This means that there are artificial holes in $\mathcal{L}$.
- Some $\iota(\gamma)$ may now become peripheral in $\mathcal{M} \setminus Q$ (Fig. 1 shows how this may happen).
- Some pair of curves $\iota(\gamma_1)$ and $\iota(\gamma_2)$ may now become homotopic to each other in $\mathcal{M} \setminus Q$.

Lemma 4.2 *There is a positive integer $n_0 \geq 1$ such that for every non-peripheral curve γ in $\mathcal{L} \setminus P$ with $\iota(\gamma)$ null-homotopic in $\mathcal{M} \setminus Q$, every curve in $F^{-n_0}(\gamma)$ is null-homotopic in $\mathcal{L} \setminus P$.*

Proof We define a *hole* D to be a connected component of $\mathcal{M} \setminus \mathcal{L}$ that is a disc, is disjoint from Q, and so that the boundary ∂D is contained in $\mathcal{L}$.

Let D be a hole. Then there is an $\mathcal{L}$-piece S and an $\mathcal{M}$-piece $\hat{S}$ such that D is a component of $\hat{S} \setminus S$. By Basic Pullback Principle, every component of $G^{-n}(D)$ for $n \geq 1$ is also a disc in $\mathcal{M}$ and is disjoint from Q. Since $\partial D \subset \partial \mathcal{L}$, a component D_i of $G^{-i}(D)$ and a component D_j of $G^{-j}(D)$ with $i > j \geq 0$ are either disjoint or one contains the other.

Denote by $n_0 \geq 1$ the number of Jordan curves in $\partial \mathcal{L}$. We claim that for any hole D, its n_0-th pullback $G^{-n_0}(D)$ is contained in $\mathcal{L}$.

Assume by contradiction that $G^{-n_0}(D_0)$ is not contained in $\mathcal{L}$ for some hole D_0. Then there is a component D_{n_0} of $G^{-n_0}(D_0)$ that is not entirely contained in $\mathcal{L}$.

Fix any integer k with $1 \le k \le n_0$. Set $D_k = G^{n_0-k}(D_{n_0})$. Then $G^k(D_k) = D_0$. Since $G^{-(n_0-k)}(\mathcal{L}) \subset \mathcal{L}$, $D_{n_0} \subset G^{-(n_0-k)}(D_k)$ and $D_{n_0} \not\subset \mathcal{L}$, we see that the disc D_k is not contained in $\mathcal{L}$. But $\partial D_k \subset G^{-k}(\partial D_0) \subset \mathcal{E} \Subset \mathcal{L}$, so ∂D_k is contained in the interior of an $\mathcal{L}$-piece. It follows that the disc D_k contains at least one boundary component of $\mathcal{L}$.

At first we want to show that for any pair of integers i, j with $n_0 \ge i > j \ge 0$, either D_i is disjoint from D_j, or $D_i \Subset D_j$. Otherwise we have $D_j \Subset D_i$; and hence $G^{i-j}(D_i) = D_j \Subset D_i$. This implies that points in $\overline{D_i}$ never escape and hence are contained in K_G. So are points in $\overline{D}$. This contradicts the facts that $K_G \Subset \mathcal{L}$ and $\partial D \subset \mathcal{L}$.

Since the number of Jordan curves in $\partial\mathcal{L}$ is n_0, and every disc D_k for $n_0 \ge k \ge 1$ contains a component of $\partial\mathcal{L}$ which is not the specific component ∂D_0 of $\partial\mathcal{L}$, we see that these discs cannot be pairwise disjoint. There exist integers $n_0 \ge i > j \ge 1$ such that $D_i \Subset D_j$ and $D_j \smallsetminus D_i$ is disjoint from $\partial\mathcal{L}$. This implies that the $\mathcal{L}$-piece S' which contains ∂D_j in its interior must also contain ∂D_i. Hence $D_j \smallsetminus D_i \Subset S' \subset \mathcal{L}$, and D_i contains a boundary curve γ' of S'. Set $g = G^{i-j}|_{D_i}$ and $A = D_j \smallsetminus D_i$ (it is a half open annulus). Then $g : D_i \to D_j$ is a homeomorphism, and the set $\bigcup_{m\ge 0} g^{-m}(A)$ is connected and is contained in $\mathcal{L}$, therefore in S'. Furthermore

$$\gamma' \subset D_j \smallsetminus \bigcup_{m\ge 0} g^{-m}(A) \subset K_G \Subset \mathcal{L}\ .$$

This leads to a contradiction since γ' is a boundary curve of $\mathcal{L}$.

The claim is proved.

Now let γ be a non-peripheral curve in $\mathcal{L} \smallsetminus P$ such that for the inclusion map $\iota : \mathcal{L} \smallsetminus P \to \mathcal{M} \smallsetminus Q$, the image $\iota(\gamma)$ is null-homotopic in $\mathcal{M} \smallsetminus Q$. Denote by $D(\gamma) \subset \mathcal{M}$ the disc bounded by $\iota(\gamma)$. Denote also by S (resp. $\hat{S}$) the $\mathcal{L}$-piece (resp. the $\mathcal{M}$-piece) containing γ. Then $D(\gamma)$ is disjoint from Q and is entirely contained in $\hat{S}$. Now $D(\gamma)$ can be decomposed into $D(\gamma) \cap S$ together with a union of finitely many holes (it may happen that $D(\gamma)$ is itself a hole). For every hole D inside $D(\gamma)$ we have $G^{-n_0}(D) \subset \mathcal{L}$ and

$$G^{-n_0}(D(\gamma) \cap S) \subset G^{-n_0}(\mathcal{L}) \subset \mathcal{L},$$

so $G^{-n_0}(D(\gamma)) \subset \mathcal{L}$. Clearly $G^{-n_0}(D(\gamma))$ is disjoint from Q and hence from P (as $P \subset Q$). Therefore every curve in $F^{-n_0}(\gamma) = G^{-n_0}(D(\gamma))$ is null-homotopic in $\mathcal{L} \smallsetminus P$. □

Lemma 4.3 *There is an integer $n_1 \geq n_0$ such that for every non-peripheral curve γ in $\mathcal{L} \setminus P$ with $\iota(\gamma)$ peripheral in $\mathcal{M} \setminus Q$, every curve in $F^{-n_1}(\gamma)$ is either null-homotopic or peripheral in $\mathcal{L} \setminus P$.*

Proof Since $\gamma = \iota(\gamma)$ is peripheral in $\mathcal{M} \setminus Q$, it bounds a disc $D(\gamma) \subset \mathcal{M}$ which contains a unique point of Q, denoted by a.

Assume that a is not a periodic point for G. Denote by k_1 the number of non-periodic points in Q. Then $G^{-k_1}(a)$ is disjoint from Q. Therefore $G^{-k_1}(D(\gamma))$ is also disjoint from Q, so the curves in $G^{-k_1}(\gamma)$ are null-homotopic in $\mathcal{M} \setminus Q$. It follows by Lemma 4.2 that for $k \geq k_1 + n_0$ the curves in $G^{-k}(\gamma)$ are null-homotopic in $\mathcal{L} \setminus P$.

Now assume that a is a periodic point for G. In particular the orbit of a does not escape $\mathcal{M}$, so $a \in K_G = K_F \subset \mathcal{L}$.

Denote by $\{\eta_1, \dots, \eta_m\}$ the set of curves in $\partial\mathcal{L}$ which are homotopic to γ in $\mathcal{M} \setminus Q$, i.e. each η_j bounds a disc $D(\eta_j) \subset \mathcal{M}$ with $D(\eta_j) \cap Q = \{a\}$. We enumerate the η_j's so that $D(\eta_{j-1}) \subset D(\eta_j)$. See Fig. 1.

Since there are no Jordan curves in $\partial\mathcal{L}$ separating η_1 from $a \in \mathcal{L}$, we know that both η_1 and a are contained in a common $\mathcal{L}$-piece, denoted by S_0. Recursively, for every even number $2 \leq i < m$, the two curves η_i and η_{i+1} are boundary curves of a common $\mathcal{L}$-piece, denoted by S_i. Clearly for every even number $2 \leq i < m$, we have $S_i \neq S_{i+2}$, and any $\mathcal{L}$-piece contained in the annulus between S_i and S_{i+2} does not separate S_i from S_{i+2}.

Let p be the period of a. Fix $j \in \{1, \dots, m\}$. For any $k \geq 1$, the components of $G^{-kp}(D(\eta_j))$ are all discs, with one of them containing a, and with each of the others containing at most one point of Q (this point is necessarily non-periodic). Therefore $G^{-kp}(\eta_j)$ has a unique component, denoted by β_j^k, homotopic to γ in $\mathcal{M} \setminus Q$. Denote by $D(\beta_j^k)$ the disc bounded by β_j^k in $\mathcal{M}$.

First, consider $j = 1$. Since both β_1^1 and the point a are contained in a common $\mathcal{E}$-piece, they must be compactly contained in a common $\mathcal{L}$-piece as well, that is $\beta_1^1 \Subset S_0$. Hence $a \in D(\beta_1^1) \Subset D(\eta_1)$. Set then $g = G^p|_{D(\beta_1^1)} : D(\beta_1^1) \to D(\eta_1)$ and $\beta_1^k = g^{-k}(\eta_1)$ $(k \geq 1)$.

Consider the sequence of (half-open) annuli $D(\beta_1^l) \setminus D(\beta_1^{l-1})$, $l \geq 1$ (setting $\beta_1^0 = \eta_1$). Since the set $\partial S_0 \cap D(\eta_1)$ contains at most $n_0 - 1$ Jordan curves, there must be an integer $k \leq n_0$, such that the annulus $A := D(\beta_1^k) \setminus D(\beta_1^{k-1})$ does not contain any boundary point of S_0. Therefore $A \subset S_0 \subset \mathcal{L}$. It follows that for any $n \geq 1$, the set $g^{-n}(A)$ as a subset of $G^{-pn}(A)$, is contained in $\mathcal{L}$. Notice that $\bigcap_{l \geq 1} D(\beta_1^l) \subset K_G \Subset \mathcal{L}$, so

$$D(\beta_1^{n_0}) \subset D(\beta_1^k) = \left(\bigcup_{n \geq 0} g^{-n}(A)\right) \cup \left(\bigcap_{l \geq 1} D(\beta_1^l)\right) \subset \mathcal{L}.$$

It follows that $\beta_1^{n_0}$ is peripheral in $\mathcal{L} \setminus P$.

We claim that $D(\beta_j^1) \subset D(\eta_j)$ for $j = 2, \ldots, m$.

Assume by contradiction that there is a minimal integer $j \geq 2$ such that $\beta_j^1 \not\subset D(\eta_j)$. Then $\eta_j \subset D(\beta_j^1)$ (recall that $\eta_j \subset \partial\mathcal{L}$ and $\beta_j^1 \Subset \mathcal{L}$ so $\eta_j \cap \beta_j^1 = \emptyset$). Denote by $A(\eta_j, \eta_{j-1})$ the annulus enclosed by η_j and η_{j-1}. Then $A(\eta_j, \eta_{j-1}) \subset D(\eta_j) \subset \mathcal{M}$. By Basic Pullback Principle the components of $G^{-p}(A(\eta_j, \eta_{j-1}))$ are all annuli. One of them must be $A(\beta_j^1, \beta_{j-1}^1)$, the annulus enclosed by β_j^1 and β_{j-1}^1. Since $\beta_{j-1}^1 \subset D(\eta_{j-1})$ and $\eta_j \subset D(\beta_j^1)$, we have

$$G^p(A(\beta_j^1, \beta_{j-1}^1)) = A(\eta_j, \eta_{j-1}) \subset A(\beta_j^1, \beta_{j-1}^1).$$

Therefore $G^k(\overline{A(\eta_j, \eta_{j-1})})$, $(k = 0, \ldots, p-1)$ are all contained in $\mathcal{B} = G^{-1}(\mathcal{M})$. Set $\overline{U} = \bigcup_{k=0}^{p-1} G^k(\overline{A(\eta_j, \eta_{j-1})})$. Then $G(\overline{U}) \subset \overline{U}$. This implies that $\overline{U} \subset K_G = K_F$, contradicting the fact that $K_F \Subset \mathcal{L}$ since the curve η_j lies on the boundary of $\mathcal{L}$. The claim is proved.

Let i be an even number with $2 \leq i < m$. Then the two curves η_i and η_{i+1} are both contained in the $\mathcal{L}$-piece S_i. And any other $\mathcal{L}$-piece lying in $D(\eta_i) \setminus D(\eta_{i-1})$, which is the annulus between S_i and S_{i-2}, does not separate S_i from S_{i-2}. Denote by E_i^1 the $G^{-p}(\mathcal{L})$-piece containing β_i^1. Then $E_i^1 = E_{i+1}^1$ (i.e. both β_i^1 and β_{i+1}^1 are contained in a common $G^{-p}(\mathcal{L})$-piece). By the above claim E_i^1 is contained in $D(\eta_i)$. Therefore $E_{i+1}^1 \subset S_{i-2t}$ for some integer $t > 0$. Inductively, let E_i^k be the $G^{-kp}(\mathcal{L})$-piece containing β_i^k. Then $E_i^k \subset S_0$ for $k > i$. Thus both β_i^k and β_{i+1}^k are peripheral in $\mathcal{L} \setminus P$ for $k \geq n_0 + m$.

Since $\gamma = \iota(\gamma)$ is peripheral around a in $\mathcal{M} \setminus Q$, it must be contained in an S_i for some even number $2 \leq i < m$. Let k_2 be the number of periodic points in Q. Set $n_1 = k_1 + k_2 n_0$ (note that $m \leq n_0$). Combining the above arguments, we know that for $n \geq n_1$, every curve in $F^{-n}(\gamma) = G^{-n}(\gamma)$ is either null-homotopic or peripheral in $\mathcal{L} \setminus P$. □

Proof of Theorem 4.1 (1) This part can be proved similarly as Proposition 2.4. We will omit the details here.

(2) Now assume that (G, Q) has no Thurston obstructions. We will prove that (F, P) has no obstructions either. Let Γ be a multicurve in $\mathcal{L} \setminus P$. We decompose it into $\Gamma = \Gamma_1 \sqcup \Gamma_2$ with Γ_1 (resp. Γ_2) being the set of $\gamma \in \Gamma$ such that $\iota(\gamma)$ is null-homotopic or peripheral (resp. non-peripheral) in $\mathcal{M} \setminus Q$. Clearly, Γ_1 is F-stable. Equivalently, the F-transition matrix W_Γ of Γ has the following block decomposition:

$$W_\Gamma = \begin{pmatrix} W_2 & O \\ * & W_1 \end{pmatrix},$$

where W_1 and W_2 denote the F-transition matrices of Γ_1 and Γ_2, respectively. The symbol O denotes a rectangular zero-matrix of appropriate size.

By Lemmas 4.2 and 4.3, there is a positive integer $n \geq 1$ such that $(W_1)^n =: O'$ is a square zero-matrix. Notice that

$$(W_\Gamma)^n = \begin{pmatrix} (W_2)^n & O \\ * & (W_1)^n \end{pmatrix} = \begin{pmatrix} (W_2)^n & O \\ * & O' \end{pmatrix}.$$

By Theorem A.4 in the appendix, we have $(\lambda(W_\Gamma))^n = \lambda(((W_\Gamma)^n)^t) = \lambda(((W_2)^n)^t) = (\lambda(W_2))^n$, where W^t is the transpose of W. Hence $\lambda(W_2) = \lambda(W_\Gamma)$.

Now we decompose Γ_2 into the equivalence classes $\Upsilon_1 \sqcup \cdots \sqcup \Upsilon_k$ with the equivalence relation: $\gamma_1 \sim \gamma_2$ if $\iota(\gamma_1)$ and $\iota(\gamma_2)$ are homotopic in $\mathcal{M} \smallsetminus Q$. Pick one representative γ_i in each class Υ_i and set $\Xi := \{\gamma_1, \ldots, \gamma_k\} = \{\iota(\gamma_1), \ldots, \iota(\gamma_k)\}$. Clearly Ξ is a G-stable multicurve in $\mathcal{M} \smallsetminus Q$.

Let $W_\Xi := (b_{ij})$ be the G-transition matrix of Ξ. Set $W_2 = (a_{\delta\beta})$, where W_2 is the F-transition matrix of Γ_2. By definition:

$$b_{ij} = \sum_{\alpha \in \Lambda_{ij}} \frac{1}{\deg(G : \alpha \to \gamma_j)} \quad \text{and} \quad a_{\delta\beta} = \sum_{\alpha \in \Omega_{\delta\beta}} \frac{1}{\deg(F : \alpha \to \beta)},$$

where Λ_{ij} is the collection of curves in $G^{-1}(\gamma_j)$ which are homotopic to γ_i in $\mathcal{M} \smallsetminus Q$; and $\Omega_{\delta\beta}$ is the collection of curves in $F^{-1}(\beta)$ which are homotopic to δ in $\mathcal{L} \smallsetminus P$. We claim that for every pair (i, j) $(1 \leq i, j \leq k)$ and every $\beta \in \Upsilon_j$,

$$\sum_{\delta \in \Upsilon_i} a_{\delta\beta} \leq b_{ij} \ .$$

Assume at first $\beta = \gamma_j$. We have

$$\begin{aligned}
&\bigcup_{\delta \in \Upsilon_i} \Omega_{\delta\gamma_j} \\
&\quad = \{\eta \in F^{-1}(\gamma_j) \mid \eta \text{ is homotopic in } \mathcal{L} \smallsetminus P \text{ to a curve } \delta \in \Upsilon_i\} \\
&\quad \subset \{\eta \in F^{-1}(\gamma_j) \mid \iota(\eta) \text{ is homotopic to } \iota(\delta) \text{ in } \mathcal{M} \smallsetminus Q \text{ for a curve } \delta \in \Upsilon_i\} \\
&\quad = \{\eta \in G^{-1}(\gamma_j) \mid \iota(\eta) \text{ is homotopic to } \iota(\delta) \text{ in } \mathcal{M} \smallsetminus Q \text{ for a curve } \delta \in \Upsilon_i\} \\
&\quad = \{\eta \in G^{-1}(\gamma_j) \mid \iota(\eta) \text{ and } \iota(\gamma_i) \text{ are homotopic in } \mathcal{M} \smallsetminus Q\} \\
&\quad = \Lambda_{ij}.
\end{aligned}$$

Therefore

$$\sum_{\delta\in\Upsilon_i} a_{\delta\gamma_j} = \sum_{\delta\in\Upsilon_i}\sum_{\alpha\in\Omega_{\delta\gamma_j}} \frac{1}{\deg(F:\alpha\to\gamma_j)} = \sum_{\alpha\in\bigcup_{\delta\in\Upsilon_i}\Omega_{\delta\gamma_j}} \frac{1}{\deg(F:\alpha\to\gamma_j)}$$

$$\le \sum_{\alpha\in\Lambda_{ij}} \frac{1}{\deg(G:\alpha\to\gamma_j)} = b_{ij}.$$

This implies the claim for $\beta=\gamma_j$.

If $\beta\neq\gamma_j$, replace γ_j by β in Ξ. The replacement does not change the transition matrix W_Ξ, so the claim still holds.

Applying Corollary A.7, we see that $\lambda(W_2)\le\lambda(W_\Xi)$.

But $\lambda(W_\Xi)<1$ as (G,Q) has no Thurston obstructions. Consequently $\lambda(W_\Gamma)=\lambda(W_2)<1$, so (F,P) has no Thurston obstructions. □

Remark One can further show that under the assumption of Theorem 4.1, if (F,P) has no Thurston obstructions then (G,Q) has none either. More generally, one can show that if two marked repelling systems (F,P) and (G,Q) satisfy: $K_F=K_G$, $P\cap K_F=Q\cap K_G$ and $F|_{K_F}=G|_{K_G}$, then they are either both obstructed or both unobstructed. This statement is not needed for our purpose here, so we omit the proof.

5 Constant complexity

We introduce here a special class of repelling systems, namely those of *constant complexity*. We show that such a system appears as an admissible restriction of any repelling system. We then introduce two particular types of obstructions for this class of maps and state a corresponding Thurston-like theorem, Theorem 5.4. The proof of which will occupy the following three sections. We conclude the present section with a proof of Theorem 3.5 using Theorem 5.4.

Definition 5 Let $\mathcal{L}$ be a surface puzzle such that every $\mathcal{L}$-piece has a non-empty boundary. Let $P\Subset\mathcal{L}$ be a finite marked set.

(a) Let S be an $\mathcal{L}$-piece, and E a bordered surface with $E\Subset S$.
- We say that S is a **complex** piece if

$$\#\{\text{curves in } \partial S\}+\#(S\cap P)\ge 3. \tag{4}$$

Otherwise S is a **simple** piece.
- We say that E is **of complex type rel.** $(\mathcal{L},P)$ if

$$\#\{\text{comp. of } S\smallsetminus E \text{ containing comp. of } P\cup\partial S\}+\#(E\cap P)\ge 3. \tag{5}$$

Otherwise E is **of simple type rel.** $(\mathcal{L}, P)$.
• In the case that S is a complex piece we say that E is **parallel** to S if $E \cap P = S \cap P$ and the interior of every component M of $S \setminus E$ is either a disc or an annulus (in the latter case, one of the boundary curves of M is necessarily in ∂S and the other is in ∂E).[2]
• In the case that S is a simple piece, we say that it is a **disc piece** if it is a closed disc (and containing at most one point of P) and an **annular piece** if it is a closed annulus (and disjoint from P).

(b) Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. We say that it is **of constant complexity** if $P \subset K_F$ and every complex $\mathcal{L}$-piece S (if any) contains an $\mathcal{E}$-piece E parallel to S. It is easy to see that such an $\mathcal{E}$-piece for a given S is necessarily unique (see Lemma 5.2 below). For this reason we will denote it by E_S.

5.1 Achieving constant complexity via an admissible restriction

Theorem 5.1 *Let $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ be a marked repelling system such that every $\mathcal{M}$-piece has a non-empty boundary. Then (G, Q) has an admissible restriction $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ which is of constant complexity.*

Recall that a marked repelling system $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ is an admissible restriction of (G, Q) if $K_G \Subset \mathcal{E} \Subset \mathcal{L} \Subset \mathcal{M}$, $\mathcal{E} = G^{-1}(\mathcal{L})$, $P = Q \cap \mathcal{E}$ and $F = G|_{\mathcal{E}}$. To prove the theorem, we need the following process together with its two properties:

Hole-filling process Let $E \Subset \mathcal{M}$ be a continuum. Let S_0 be the $\mathcal{M}$-piece containing E. Then each component of $S_0 \setminus E$ is either a disc (which may contain points of Q) or intersects ∂S_0. The **filling** of E in $(\mathcal{M}, Q)$, denoted by $\widehat{E}$, is defined to be the union of E with the components of $S_0 \setminus E$ which are disjoint from $\partial S_0 \cup Q$ (and thus they are open discs disjoint from Q). Clearly, $\widehat{E} \subset S_0$ and $\partial \widehat{E} \subset \partial E$.

Monotonicity For two continua E_1, E_2 with $E_1 \Subset E_2 \Subset \mathcal{M}$, we have $\widehat{E}_1 \Subset \widehat{E}_2$.

Proof This property is easier to understand by looking at the complements. There is an $\mathcal{M}$-piece S_0 containing both E_1 and E_2. Note that $S_0 \setminus \widehat{E}_2$ is the union of the components of $S_0 \setminus E_2$ which intersect $\partial S_0 \cup Q$. Since

[2]One way to obtain a parallel subsurface of S is as follows: first thicken the boundary of S (without touching P) to reduce S to a sub-surface E', then dig finitely many open holes (whose closures are mutually disjoint and contained in the interior of $E' \setminus P$). The result is a surface E parallel to S.

$S_0 \smallsetminus E_1 \supset S_0 \smallsetminus E_2$, these components are contained in $S_0 \smallsetminus E_1$, which can not be thrown away under hole-filling process of E_1 since they intersect $\partial S_0 \cup Q$. So $S_0 \smallsetminus \widehat{E}_1 \supset S_0 \smallsetminus \widehat{E}_2$. Combining the fact that $\partial \widehat{E}_i \subset \partial E_i$, we have $\widehat{E}_1 \Subset \widehat{E}_2$. □

Pullback property Let $E \Subset \mathcal{M}$ be a continuum, E_1 a component of $G^{-1}(E)$ and $\breve{E}_1$ the component of $G^{-1}(\widehat{E})$ containing E_1. Then $\breve{E}_1 \subset \widehat{E}_1$.

Proof Denote by $\breve{E}_i$ $(1 \le i \le k)$ the components of $G^{-1}(\widehat{E})$. Note that $\widehat{E} \smallsetminus E$ is a disjoint union of open discs disjoint from Q. Set $V = G^{-1}(\widehat{E}) \smallsetminus G^{-1}(E) = G^{-1}(\widehat{E} \smallsetminus E)$. Then V is also a disjoint union of open discs disjoint from Q. These discs are contained in $G^{-1}(\widehat{E}) = \bigcup \breve{E}_i$ and hence $\breve{E}_i \smallsetminus V$ is connected for $1 \le i \le k$. Noting that $\breve{E}_1 \smallsetminus V \subset G^{-1}(\widehat{E}) \smallsetminus V = G^{-1}(E)$ and $\breve{E}_1 \smallsetminus V \supset E_1$, we see that $E_1 = \breve{E}_1 \smallsetminus V$ since E_1 is a component of $G^{-1}(E)$. Since $\breve{E}_1 \cap V$ is the union of some components of V which are discs disjoint from Q, we have $\breve{E}_1 = E_1 \cup (\breve{E}_1 \cap V) \subset \widehat{E}_1$. □

Proof of Theorem 5.1 Set $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_1 = \mathcal{B}$ and $\mathcal{M}_n = G^{-n}(\mathcal{M})$ for $n > 1$.

Choice of the iterate N_0 to stabilize the postcritical set. Clearly there is an integer $N_0 \ge 1$ such that for all $n \ge N_0$, we have $\mathcal{M}_n \cap Q = K_G \cap Q$. In other words every critical point of G in $\mathcal{M}_n$ is actually in K_G and is eventually periodic.

Choice of the iterate N_1 to stabilize the homotopy classes of the boundary curves. For any integer $m \ge 1$, we consider the homotopy classes in $\mathcal{M} \smallsetminus Q$ of the Jordan curves in $\bigcup_{k=0}^{m} \partial \mathcal{M}_k$. The number of these homotopy classes is weakly increasing with respect to m,but is uniformly bounded from above, as $Q \cup \partial \mathcal{M}$ has only finitely many connected components. Hence there is an integer $N_1 \ge N_0$ such that for any $n \ge N_1$, every boundary curve of $\mathcal{M}_n$ is either null homotopic or homotopic in $\mathcal{M} \smallsetminus Q$ to a curve in $\bigcup_{k=0}^{N_1-1} \partial \mathcal{M}_k$.

Hole-filling for $\mathcal{M}_n$. Fix any $n \ge N_0$. Let S be an $\mathcal{M}_n$-piece. Let E be a component of $G^{-1}(S)$ and $\breve{E}$ the component of $G^{-1}(\widehat{S})$ containing E. Then $\breve{E} \subset \widehat{E}$ by Pullback Property. From $E \subset \mathcal{M}_{n+1} \Subset \mathcal{M}_n$, we know that $E \Subset S'$ for some $\mathcal{M}_n$-piece S'. By the monotonicity of filling, we have $\breve{E} \subset \widehat{E} \Subset \widehat{S'}$.

Set $\widehat{\mathcal{M}_n} = \bigcup_{S \text{ an } \mathcal{M}_n\text{-piece}} \widehat{S}$.[3] Then every $\widehat{\mathcal{M}_n}$-piece is the filling of an $\mathcal{M}_n$-piece. Therefore every $G^{-1}(\widehat{\mathcal{M}_n})$-component is realized by some $\breve{E}$ above. It follows that

$$G^{-1}(\widehat{\mathcal{M}_n}) \subset \widehat{\mathcal{M}}_{n+1} \Subset \widehat{\mathcal{M}}_n.$$

[3]The total number of $\widehat{\mathcal{M}}_n$-pieces might be less than that for $\mathcal{M}_n$, since some $\mathcal{M}_n$-pieces might be hidden in the holes of others and disappear in the hole-filling process.

Choice of the iterate N to stabilize the number and the shape of the complex pieces. From now on we assume $n \geq N_1$.

All homotopies here are within $\mathcal{M} \setminus Q$.

We claim that for $k \geq 1$, every non-null-homotopic curve γ on $G^{-k}(\partial\widehat{\mathcal{M}}_n)$ is homotopic to a curve in $\partial\widehat{\mathcal{M}}_n$. Note that

$$\gamma \subset G^{-k}(\partial\widehat{\mathcal{M}}_n) \subset G^{-k}(\partial\mathcal{M}_n) = \partial\mathcal{M}_{n+k}.$$

By the choice of N_1, the curve γ is homotopic to a curve β in $\partial\mathcal{M}_m$ for some $m < N_1$. Since $\mathcal{M}_{n+k} \Subset \mathcal{M}_n \Subset \mathcal{M}_m$, there exists a curve α in $\partial\mathcal{M}_n$ separating β from γ. So α and γ are also homotopic. Let S be the $\mathcal{M}_n$-piece containing α. Since α is non-null-homotopic (as γ), the filling $\widehat{S}$ can not be hidden in the hole of some other $\mathcal{M}_n$-piece. So $\widehat{S}$ is an $\widehat{\mathcal{M}}_n$-piece with $\alpha \subset \partial\widehat{S}$. The claim is proved.

Set $P = \mathcal{M}_n \cap Q$ which is equal to $\widehat{\mathcal{M}}_n \cap Q$ and $K_G \cap Q$. Consider $(\widehat{\mathcal{M}}_n, P)$ as a surface puzzle. Let $\widehat{S}$ be a complex $\widehat{\mathcal{M}}_n$-piece in the sense of (4). The pieces of simple or complex types in what follows are rel. $(\mathcal{M}, Q)$, in the sense of (5).

First we show that $\widehat{S}$ contains at most one $G^{-1}(\widehat{\mathcal{M}}_n)$-piece of complex type. Assume that E_1 and E_2 are $G^{-1}(\widehat{\mathcal{M}}_n)$-pieces in $\widehat{S}$ with E_1 of complex type. There is a boundary curve γ of E_1 separating $E_1 \setminus \gamma$ from E_2. If γ is null-homotopic, then E_2 is of simple type and $E_2 \cap Q = \emptyset$. Assume that γ is non-null-homotopic. From the above claim, there is a curve α in $\partial\widehat{\mathcal{M}}_n$ homotopic to γ. Moreover, α can be taken in $\partial\widehat{S}$ since $E_1 \subset \widehat{S}$. Now the closed annulus enclosed by γ and α, denoted by $A(\gamma, \alpha)$, is disjoint from $\partial\mathcal{M} \cup Q$. It must contain either E_1 or E_2 since γ separates $E_1 \setminus \gamma$ from E_2. But it can not contain E_1 as E_1 is of complex type. So $E_2 \subset A(\gamma, \alpha)$. We see that E_2 is of simple type with $E_2 \cap Q = \emptyset$.

Next we show that if $\widehat{S}$ contains a $G^{-1}(\widehat{\mathcal{M}}_n)$-piece E of complex type then E is parallel to S. By above the other $G^{-1}(\widehat{\mathcal{M}}_n)$-pieces in $\widehat{S}$ are of simple type and are disjoint from Q. Combining this with the fact that $G^{-1}(\widehat{\mathcal{M}}_n) \cap Q = \widehat{\mathcal{M}}_n \cap Q$, we see that $E \cap Q = \widehat{S} \cap Q$ and $E \cap P = S \cap P$. It remains to show that each component of $\widehat{S} \setminus E$ contains at most one component of $\partial\widehat{S}$.

Let D be a component of $\widehat{S} \setminus E$ and $\gamma = \partial D \cap \partial E$. If γ is null-homotopic, then D is disjoint from $\partial\widehat{S}$ since every closed curve in $\partial\widehat{S}$ is non-null-homotopic. Now assume that γ is non-null-homotopic. Then there is a curve β in $\partial\widehat{S}$ homotopic to γ. Therefore the closed annulus $A(\gamma, \beta)$ is contained in $\mathcal{M} \setminus Q$.

If β is disjoint from D, then $E \subset A(\gamma, \beta)$. This contradicts the fact that E is of complex type, so $\beta \subset D$. Thus $A(\gamma, \beta) \subset \widehat{S}$ since $\widehat{S}$ is the filling of an $\mathcal{M}_n$-piece. Therefore no other component of $\partial\widehat{S}$ is contained in D. This implies that the interior of every component of $\widehat{S} \setminus E$ is either a disc or an annulus. Consequently, E is parallel to $\widehat{S}$.

Now let s_n be the number of complex $\widehat{\mathcal{M}}_n$-pieces and t_{n+1} the number of $G^{-1}(\widehat{\mathcal{M}}_n)$-pieces of complex type. We want to show that $t_{N+1} = s_N$ for some N.

We have $t_{n+1} \le s_n$ since every complex $\widehat{\mathcal{M}}_n$-piece contains at most one $G^{-1}(\widehat{\mathcal{M}}_n)$-piece of complex type.

We will now prove $s_{n+1} \le t_{n+1}$, by proving that every complex $\widehat{\mathcal{M}}_{n+1}$-piece contains at least one $G^{-1}(\widehat{\mathcal{M}}_n)$-piece of complex type. Let $\widehat{S}'$ be a complex $\widehat{\mathcal{M}}_{n+1}$-piece. It is the filling of an $\mathcal{M}_{n+1}$-piece S'. The component $\breve{S}'$ of $G^{-1}(\widehat{G(S')})$ containing S' is a $G^{-1}(\widehat{\mathcal{M}}_n)$-piece. By Pullback Principle we have $S' \subset \breve{S}' \subset \widehat{S}'$. By (4)

$$\#\{\text{curves in } \partial\widehat{S}'\} + \#(\widehat{S}' \cap P) \ge 3.$$

On the other hand, for S_0 the $\mathcal{M}$-piece containing $\widehat{S}'$, every component of $S_0 \smallsetminus \widehat{S}'$ contains a component of $Q \cup \partial S_0$. As the number of the components of $S_0 \smallsetminus \widehat{S}'$ is equal to that of the boundary curves of $\widehat{S}'$, we know that $\widehat{S}'$ is also of complex type rel. $(\mathcal{M}, Q)$ by (5). It follows that both S' and $\breve{S}'$ are of complex type.

Hence the sequence of non-negative integers $(s_n)_{n \ge N_1}$ is weakly decreasing, so $s_n \equiv s_N$ for some $N \ge N_1$ and for all $n \ge N$, in particular $t_{N+1} = s_N$.

Finally setting $\mathcal{L} = \widehat{\mathcal{M}}_N$, $\mathcal{E} = G^{-1}(\mathcal{L})$ and $F = G|_{\mathcal{E}}$, we see that $P = Q \cap \mathcal{L}$, $P \subset K_F = K_G$, and $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ is a marked repelling system of constant complexity and an admissible restriction of (G, Q). □

5.2 Boundary curves and complex pieces

We will now study the properties of constant complexity maps.

Lemma 5.2 *Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity.*

(1) *For every complex $\mathcal{L}$-piece S, there is a unique $\mathcal{E}$-piece E_S parallel to S. And $F(E_S)$ is again a complex $\mathcal{L}$-piece.*
(2) *For any $n \ge 1$, every curve in $F^{-n}(\partial\mathcal{L})$ is either null-homotopic or homotopic in $\mathcal{L} \smallsetminus \mathcal{P}$ to a boundary curve of $\mathcal{L}$.*
(3) *For every complex $\mathcal{L}$-piece S and any integer $m \ge 1$, there is a unique $F^{-m}(\mathcal{L})$-piece E in S parallel to S. Moreover, $F^m(E)$ is again a complex $\mathcal{L}$-piece.*

Definition of F_* By (1), we can define a map F_* from the set of complex pieces of $\mathcal{L}$ into itself by $F_*(S_1) = S_2$ if $F(E_{S_1}) = S_2$. Since $\mathcal{L}$ has only finitely many complex pieces, every complex piece is eventually periodic under F_*.

Proof of Lemma 5.2 (1) The existence of E_S is given by the definition of constant complexity. Its uniqueness follows from the fact that the interior of every component of $S \smallsetminus (E_S \cup P)$ is either an annulus or a disc. We know that $F(E_S)$ is again an $\mathcal{L}$-piece. It must also be a complex piece since E_S is of complex type and every component of the F-preimage of a simple piece is of simple type.

(2) Let S_1 be a complex piece. From (1) we know that every component M of $S_1 \smallsetminus E_{S_1}$ is disjoint from P and the interior of M is either a disc or an annulus, and in the latter case one boundary curve of M is also a boundary curve of $\mathcal{L}$.

Due to Basic Pullback Principle we just need to prove Point (2) for $n = 1$.

Let δ be a curve in $F^{-1}(\partial\mathcal{L})$, E the $\mathcal{E}$-piece containing δ as a boundary curve, and S_1 the $\mathcal{L}$-piece containing E.

Assume at first that E is an $\mathcal{E}$-piece of simple type. Set $M = S_1$ if S_1 is a simple piece. Otherwise let M be the component of $S_1 \smallsetminus E_{S_1}$ containing E. In any case every boundary curve of E, in particular δ, is either null-homotopic or homotopic to a boundary curve of M therefore to a curve in $\partial\mathcal{L}$.

Now assume that E is an $\mathcal{E}$-piece of complex type. Then S_1 is necessarily a complex piece and $E = E_{S_1}$. Thus every boundary curve of E, in particular δ, is homotopic to a curve in $\partial S_1 \subset \partial\mathcal{L}$.

(3) Set $S_0 = S$ and $S_{i+1} = F(S_i^1)$ $(i \geq 0)$, where $S_i^1 = E_{S_i}$ is the unique $\mathcal{E}$-piece parallel to S_i. Since the interior of every component of $S_i \smallsetminus (S_i^1 \cup P)$ is either a disc or an annulus with one boundary curve in ∂S_i, the interior of every component of $F^{-1}(S_{i+1} \smallsetminus (S_{i+1}^1 \cup P))$ in S_i^1 is either a disc or an annulus with one boundary curve in ∂S_i^1. Hence there is a unique component of $F^{-1}(S_{i+1}^1)$ contained in S_i^1, denoted by S_i^2 (note that $\deg F|_{S_i^1} = \deg F|_{S_i^2}$). Moreover, the interior of every component of $S_i \smallsetminus (S_i^2 \cup P)$ is either a disc or an annulus with one boundary curve in ∂S_i. This shows that as an $F^{-2}(\mathcal{L})$-piece, S_i^2 is parallel to S_i.

Define recursively S_i^{j+1} to be the unique component of $F^{-1}(S_{i+1}^j)$ contained in S_i^j $(j > 1)$. Then S_i^j is an $F^{-j}(\mathcal{L})$-piece parallel to S_i. □

5.3 The boundary multicurve

Definition 6 The **boundary multicurve** Υ of a marked repelling system $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ is a multicurve in $\mathcal{L}$ representing the set of non-peripheral homotopy classes (in $\mathcal{L} \smallsetminus P$) of the boundary curves of $\mathcal{L}$. Its transition matrix $W_\Upsilon = (a_{ij})$ is defined by

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}, \tag{6}$$

where the sum is taken over the components of $F^{-1}(\gamma_j)$ homotopic to γ_i in $\mathcal{L} \setminus P$.

We will say that (F, P) has a **boundary obstruction** if the boundary multicurve Υ is not empty and $\lambda(W_\Upsilon) \geq 1$.

In general Υ is not F-stable. However, if (F, P) has constant complexity, then Υ is F-stable by Lemma 5.2. It can be described more explicitly using the above classification of $\mathcal{L}$-pieces, as follows:

The boundary curve of a disk piece is either null-homotopic or peripheral. The two boundary curves of an annular piece are non-peripheral and homotopic to each other. Every boundary curve of a complex piece is non-peripheral and is not homotopic to other boundary curves. Therefore the boundary multicurve Υ can be represented by the collection of all boundary curves of all complex pieces together with one of the two boundary curves of every annular piece.

Remark The boundary multicurve is canonical in the following sense: If (F, P) is an admissible restriction of a marked repelling system $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ and both of them have constant complexity, then the homotopy class (in $(\mathcal{M} \setminus Q)$) of the boundary multicurve of (F, P) represents exactly the boundary multicurve of (G, Q). Let f be a quasiregular semi-rational map and (F, P) a marked repelling system of constant complexity as a restriction of f with $K_F = K_f$. Then the above statement shows that the homotopy class (in $\overline{\mathbb{C}} \setminus P_f$) of the boundary multicurve of (F, P) is an f-stable multicurve in $\overline{\mathbb{C}} \setminus P_f$ which is independent of the choice of (F, P). This multicurve also describes the combinatorics of the components of the Julia set when f is a rational map.

5.4 Renormalizations

A marked repelling system of constant complexity has another, somewhat more important property: It can be decomposed into sub-systems which behave like postcritically finite branched coverings of $\overline{\mathbb{C}}$.

Definition 7 A marked repelling system $H : (E, Z) \to (S, Z)$ of constant complexity is of **Thurston type** if both E and S are connected and S is a complex piece, i.e.

$$\#Z + \#\{\text{boundary curves of } S\} \geq 3.$$

In other words, $E, S \subset \overline{\mathbb{C}}$ are bordered surfaces with $E \Subset S$. The interior of every component of $S \setminus E$ is either a disc or an annulus. The map $H : E \to S$ is a quasiregular branched covering. The postcritical set P_H is contained in E

(or empty). The set $Z \subset E$ is a finite (or empty) set containing P_H such that $H(Z) \subset Z$. And finally $\#Z + \#\{\text{boundary curves of } S\} \geq 3$.

Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity. Assume that there is at least one complex $\mathcal{L}$-piece. Recall that we have a map F_* defined on the set of complex pieces by $F_*(S_1) = S_2$ if $F(E_{S_1}) = S_2$. Assume that S is a complex piece that is p-periodic under F_*. Let E be the unique $F^{-p}(\mathcal{L})$-piece in S parallel to S. Then $F^p(E) = S$.

Set $H = F^p|_E$ and $Z = P \cap E$. By Definition 7 the map $H : (E, Z) \to (S, Z)$ is a Thurston type marked repelling system. We will say that H is a **renormalization** of (F, P). We say that (F, P) has a **renormalization obstruction** if it has a renormalization which has a Thurston obstruction.

Remark Let f be a sub-hyperbolic rational map, and (F, P) a marked repelling system as a restriction of f with $K_F = K_f$ and with constant complexity. Suppose that H is a renormalization of (F, P) with period $p \geq 1$. Then there is a pair of connected and finitely-connected domains $U, V \subset \overline{\mathbb{C}}$, with $K_H \subset U \Subset V$, such that $f^p : U \to V$ is a proper holomorphic map. This proper map can be considered as a generalized polynomial-like map, and hence as a generalization of a renormalization in the polynomial setting.

Lemma 5.3 *Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity. If (F, P) has no Thurston obstructions, then it has no boundary obstruction nor renormalization obstructions.*

Proof As (F, P) has no Thurston obstructions, we have $\lambda(W_\Gamma) < 1$ for the transition matrix W_Γ of every F-stable multicurve Γ, in particular the boundary multicurve. Therefore (F, P) has no boundary obstruction.

It remains to show that any renormalization $H : E \to S$ marked by $Z = P \cap S$ has no Thurston obstructions. Assume by contradiction that $\lambda(W_\Gamma) \geq 1$ for the transition matrix W_Γ of some H-stable multicurve Γ.

Note that Γ is also a multicurve in $\mathcal{L} \smallsetminus P$. Set $\Gamma_0 = \Gamma$. For every $k \geq 1$, let Γ_k be a multicurve in $\mathcal{L} \smallsetminus P$ representing the set of homotopy classes of the non-peripheral curves in $\bigcup_{\gamma \in \Gamma} F^{-k}(\gamma)$.

Claim *For $0 \leq i < j$, pick any pair of curves $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$. Then they are homotopically disjoint, that is, there exists a curve β_i in $\mathcal{L} \smallsetminus P$ homotopic to γ_i and disjoint from γ_j.*

Proof Obviously, if one of γ_i and γ_j is homotopic to a boundary curve in $\partial \mathcal{L}$, then they are homotopically disjoint. This implies that if one of them is contained in an annular piece, or more generally in an $F^{-k}(\mathcal{L})$-piece of simple type, then they are homotopically disjoint.

Now we assume that neither of them is homotopic to a boundary curve. Assume by contradiction that they intersect homotopically. Then they are contained in a complex piece S_0. If $S_0 = S$, then $i, j \equiv 0 \pmod p$, where $p \geq 1$ is the F_*-period of the complex $\mathcal{L}$-piece S. This is due to the fact that $F^{-k}(S)$ has exactly one piece of complex type contained in S for $k \equiv 0 \pmod p$ and it has no pieces of complex type contained in S for other cases by Theorem 5.1. As Γ is H-stable, both γ_i and γ_j are homotopic to a curve in Γ. This is a contradiction.

In the case that $S_0 \neq S$, both $F^i(\gamma_i)$ and $F^i(\gamma_j)$ are contained in S by the assumption $\gamma_i \cap \gamma_j \neq \emptyset$. They are homotopically disjoint by the above argument. Hence γ_i and γ_j are homotopically disjoint. The claim is proved. □

By the claim, for every $k \geq 1$, there is a multicurve Ξ_k in $\mathcal{L} \setminus P$ which represents all non-peripheral curves in $F^{-i}(\gamma)$ for all $\gamma \in \Gamma$ and $0 \leq i \leq k$. Obviously, every curve in Ξ_k is homotopic to a curve in Ξ_{k+1}, and every non-peripheral curve in $F^{-1}(\gamma)$ for $\gamma \in \Xi_k$ is homotopic to a curve in Ξ_{k+1}. Since the set $\mathcal{L} \setminus P$ has only finitely many boundary components, $\#\Xi_k$ is uniformly bounded from above. Hence there is an integer $k_0 \geq 1$ such that $\#\Xi_{k_0} = \#\Xi_{k_0+1}$ since $\#\Xi_k$ is increasing. This implies that Ξ_{k_0}, denoted also by Ξ for simplicity, is F-stable.

By definition of Ξ, every curve in Γ is homotopic to a curve in Ξ. Thus their transition matrices satisfy the following relations:

$$(W_\Xi)^p \geq \begin{pmatrix} W_\Gamma & * \\ * & * \end{pmatrix} \geq \begin{pmatrix} W_\Gamma & * \\ O_1 & O_2 \end{pmatrix},$$

where O_1 and O_2 are zero-matrices of appropriate sizes. Hence their leading eigenvalues satisfy $(\lambda(W_\Xi))^p \geq \lambda(W_\Gamma) \geq 1$ by Theorem A.4 and Corollary A.3. This contradicts the assumption that (F, P) has no Thurston obstructions.

We can now state our Thurston-like result in this setting. Its proof will occupy Sects. 6–8.

Theorem 5.4 *Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity. Assume that (F, P) has no boundary obstruction nor renormalization obstructions. Then (F, P) is c-equivalent to a holomorphic marked repelling system.*

5.5 Proof of Theorem 3.5 using Theorem 5.4

Proof of Theorem 3.5 Let $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ be a marked repelling system with no Thurston obstructions. We will prove that (G, Q) is c-equivalent to a holomorphic marked repelling system.

At first we apply Theorem 5.1 to (G, Q) to show that it has a restriction $F : \mathcal{E} \to \mathcal{L}$ near K_G which, marked by $P := Q \cap \mathcal{L}$, is a marked repelling system of constant complexity, and satisfies the conditions in Theorem 4.1. So we may apply Theorem 4.1(2) to show that (F, P) has no Thurston obstructions, and subsequently apply Lemma 5.3 to show that (F, P) has no boundary obstruction nor renormalization obstructions. Now we may apply Theorem 5.4 to conclude that (F, P) is c-equivalent to a holomorphic marked repelling system. Finally we conclude for (G, Q) using Theorem 4.1(1). □

Note that it could be more difficult to check the absence of obstructions required by Theorems 1.1 and 3.5. Whereas Theorem 5.4 turns the problem into the one of checking the leading eigenvalue of W_Υ for a single multicurve Υ, and then the absence of obstructions for postcritically finite branched coverings (arising from the renormalizations). This form is particularly suitable for combinations of rational maps, i.e. starting with postcritically finite rational maps (thus already holomorphic) as the renormalizations and then gluing them suitably together. See for example Sect. 9 below, or [17].

6 Proof of Theorem 5.4 for simple pieces

From now on we concentrate on the proof of Theorem 5.4: A marked repelling system of constant complexity having no boundary obstruction nor renormalization obstructions is c-equivalent to a holomorphic system. In this section we will prove the theorem in the case that there are no complex pieces. In this case we only need to apply Grötzsch's inequality, but not Thurston's original theorem.

6.1 A criterion

The following criterion makes it easier to check if two repelling systems are c-equivalent (note that constant complexity is not needed here).

Lemma 6.1 *Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. Then (F, P) is c-equivalent to a holomorphic marked repelling system if and only if for every $\mathcal{L}$-piece S_i, there is a pair of quasiconformal maps $\theta_i : S_i \to S_i$ and $\phi_i : S_i \to \phi_i(S_i) \subset \overline{\mathbb{C}}$ such that*:

(a) *θ_i is isotopic to the identity rel $\partial S_i \cup (S_i \cap P)$;*
(b) *for every $\mathcal{E}$-piece E contained in an $\mathcal{L}$-piece S_i, and for S_j the $\mathcal{L}$-piece equal to $F(E)$, the composition $R_E := \phi_j \circ F \circ \theta_i^{-1} \circ \phi_i^{-1}$ is holomorphic in $\phi_i \circ \theta_i(E)$.*

Proof The proof is a straight forward consequence of Lemma 2.3. One just needs to set $\theta_i = \theta|_{S_i}$ and $\mu|_{S_i} = \mu_{\phi_i}$, the Beltrami differential of ϕ_i. $\square$

Strategy to prove Theorem 5.4. According to the above lemma, establishing the c-equivalence of (F, P) to a holomorphic system amounts to constructing the maps ϕ_i, θ_i and R_E satisfying (a) and (b) for every $\mathcal{L}$-piece S_i and every $\mathcal{E}$-piece E in S_i. In practice the maps ϕ_i and R_E will be constructed first. Then we construct θ_E and finally we glue the various θ_E's together to get θ_i. See the following schema:

Order of the construction

$$\begin{array}{ccccc} S_i & \xrightarrow[\mathbf{4.}]{\theta_i} & S_i & \xrightarrow[\mathbf{1.}]{\phi_i} & \phi_i(S_i) \subset \overline{\mathbb{C}} \\ \cup & & \cup & & \cup \\ E & \xrightarrow[\mathbf{3.}]{\theta_E} & \underset{\mathbf{2.}}{\tilde{E}} & \xrightarrow{\phi_i} & R_E^{-1}(\phi_j(S_j)) \\ F \downarrow & & & & \downarrow \mathbf{1.}\ R_E \text{ holomorphic} \\ S_j & \xrightarrow{\text{id}} & S_j & \xrightarrow[\mathbf{1.}]{\phi_j} & \phi_j(S_j) \subset \overline{\mathbb{C}} \end{array} \tag{7}$$

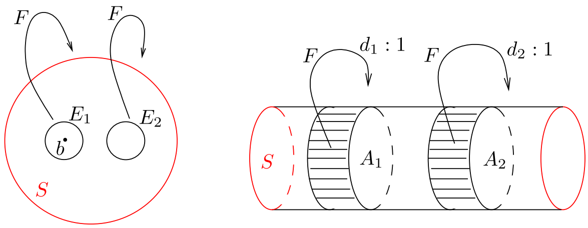
Example 1 Let $S \subset \overline{\mathbb{C}}$ be a closed quasidisc with a marked point $b \in S$. Let E_1 and E_2 be two disjoint closed quasidiscs in the interior of S with $b \in E_1$. Let $F : E = E_1 \cup E_2 \to S$ be a map such that $F|_{E_i} : E_i \to S$ is a quasiregular branched covering with a unique critical value at b (see Fig. 2). Assume $F(b) = b$. Then the marked repelling system $F : (E, \{b\}) \to (S, \{b\})$ is always c-equivalent to a holomorphic marked repelling system by the following constructions.

1. First construct a holomorphic system R as follows: For $i = 1, 2$, let $R|_{E_i} : E_i \to S$ be a holomorphic branched covering with the same degree as $F|_{E_i}$, and with a unique critical value at b, such that $R(b) = b$.
2. Take a lift $\theta : E \to E$ of the identity map via the branched covering F and R. Then $R \circ \theta = F$ on E and $\theta(b) = b$.
3. Extend θ to a quasiconformal map of S with $\theta = \text{id}$ on the boundary ∂S. It is automatically isotopic to the identity rel $\partial S \cup \{b\}$.

6.2 Annuli-coverings

Example 2 Let $S \subset \overline{\mathbb{C}}$ be a closed annulus with quasicircle boundaries. Let A_1 and A_2 be two disjoint closed essential sub-annuli contained in the interior of S. Let $F : A_1 \cup A_2 \to S$ so that $F|_{A_i}$ is a quasiregular covering of degree $d_i \geq 1$, $i = 1, 2$ (see Fig. 2).

Fig. 2 A repelling system with only disc pieces, and an annuli-covering



There is a unique multicurve Γ in S, up to homotopy, which consists of a boundary curve of S. Its transition matrix has only one entry $d_1^{-1} + d_2^{-1}$. Therefore F has a Thurston obstruction if and only if $d_1^{-1} + d_2^{-1} \geq 1$. In the following we will establish the c-equivalence of an annuli-covering F to a holomorphic system, under the assumption $d_1^{-1} + d_2^{-1} < 1$.

1. First construct a round annulus M, say of modulus $v > 0$. Let $\phi : S \to M$ be a quasiconformal homeomorphism.
2. Construct then two disjoint essential closed and round sub-annuli B_1 and B_2 in M with respective moduli v/d_1 and v/d_2. This can be done due to the inequality $d_1^{-1} + d_2^{-1} < 1$. We also require that the order of displacement of B_1 and B_2 in M is so that there is an automorphism H of S preserving each boundary component and mapping A_i onto $\phi^{-1}(B_i)$, $i = 1, 2$.
3. Now fix an $i \in \{1, 2\}$. Choose $R|_{B_i} : B_i \to M$ to be a holomorphic covering of degree d_i, so that for a boundary curve γ of A_i, we have $R \circ \phi \circ H(\gamma) = \phi \circ F(\gamma)$. Lift ϕ to a pair of quasiconformal maps $\psi_i : A_i \to B_i$ such that $R \circ \psi_i = \phi \circ F$.
4. Set $\begin{cases} \theta|_{A_i} = \phi^{-1} \circ \psi_i : A_i \to \phi^{-1}(B_i), \ i = 1, 2, \\ \theta|_{\partial S} = \text{id}. \end{cases}$
5. Extend θ to a quasiconformal map of S. Then $R \circ \phi \circ \theta|_{A_1 \cup A_2} = \phi \circ F$.
6. If necessary modify the extension by postcomposing with a quasiconformal repeated Dehn twist on $S \setminus (A_1 \cup A_2)$ so that θ is isotopic to the identity rel ∂S.

Using the same idea as in the above example, we will show Theorem 5.4 for annuli-coverings.

Let $F : \mathcal{E} \to \mathcal{A}$ be an annuli-covering. More precisely, $\mathcal{A} = A_1 \sqcup \cdots \sqcup A_n$ is a surface puzzle with every A_i being a closed annulus, $\mathcal{E} = E_1 \sqcup \cdots \sqcup E_m \Subset \mathcal{A}$ is a surface puzzle in which every E_k is a closed annulus essentially contained in some A_i, and $F : E_k \to F(E_k)$ is a quasiregular covering of degree $d_k \geq 1$, with $F(E_k)$ equal to some A_j.

We decompose the index set $\{1, \ldots, m\}$ into $\bigsqcup_{(i,j)} I_{ij}$, with

$$I_{ij} := \{k \mid E_k \subset A_i \text{ and } F(E_k) = A_j\}.$$

In this case the transition matrix $W = (a_{ij})$ takes the following form:

$$a_{ij} = \sum_{k \in I_{ij}} \frac{1}{d_k} \quad (\text{with } a_{ij} = 0 \text{ if } I_{ij} = \emptyset).$$

We will prove the following more concrete form of Lemma 3.1:

Lemma 6.2 *For the annuli-covering $F : \mathcal{E} \to \mathcal{A}$ defined as above, assume that there is a vector $v = (v_1, \dots, v_n)$ with positive entries such that $Wv < v$, i.e. for every $1 \le i \le n$,*

$$\sum_j \sum_{k \in I_{ij}} \frac{v_j}{d_k} < v_i \ . \tag{8}$$

Then $F : \mathcal{E} \to \mathcal{A}$ is c-equivalent to a holomorphic annuli-covering $R : \mathcal{B} \to \mathcal{M} = M_1 \sqcup \dots \sqcup M_n$ with $\mathrm{mod}(M_i) = v_i$.

Here the modulus of a closed annulus means the modulus of its interior as an open annulus. Now Lemma A.1 relates $\lambda(W) < 1$ to the existence of such a vector v. And Lemma 3.1 follows.

Proof of Lemma 6.2 1. *Definition of ϕ_i and R_k.* For every $i \in \{1, \dots, n\}$, choose $M_i \subset \overline{\mathbb{C}}$ a closed round annulus with modulus v_i. Let $\phi_i : A_i \to M_i$ be a quasiconformal homeomorphism.

For every $k \in I_{ij}$, choose a closed round essential sub-annulus $B_k \Subset M_i$ such that mod $(B_k) = v_j/d_k$ and the B_k's are mutually disjoint (this can be done due to (8)) and are displaced in the same order as the E_k's in A_i.

Now choose $R_k : B_k \to M_j$ a holomorphic covering of degree d_k, so that it maps the boundary curves in the same way as $F : E_k \to A_j$. This can be done through *boundary labeling*: for every A_i choose a labeling by $+$ and $-$ for its two boundary curves. This induces a labeling by $\pm$ on the boundary curves of each essential sub-annulus E_k so that $\partial_- E_k$ separates $\partial_- A_i$ from $\partial_+ E_k$. Now use each ϕ_i to transport these labellings to ∂M_i which then induce a labeling on each ∂B_k. The covering $F : E_k \to A_j$ maps $\partial_- E_k$ to one of $\partial_\pm A_j$. We choose R_k so that it sends $\partial_- B_k$ to $\phi_j(F(\partial_- E_k))$, the corresponding boundary component of M_j.

2. *Definition of $\widetilde{E}_k$.* For every $1 \le k \le m$, set $\widetilde{E}_k := \phi_i^{-1}(B_k)$ (there are a priori two ways to label its boundary curves: as an essential sub-annulus of A_i, or the labeling of ∂B_k transported by ϕ_i^{-1}, but these two labellings actually coincide).

3. *Definition of θ'_k.* For every $1 \le k \le m$, let $\psi_k : E_k \to B_k$ be a (choice of a) lift of the quasiconformal map $\phi_j : A_j \to M_j$ via the two quasiregular coverings of the same degree: $F|_{E_k}$ and R_k. Set $\theta'_k = \phi_i^{-1} \circ \psi_k$. It is a

quasiconformal map that preserves the boundary labeling.

$$\begin{array}{ccccc} E_k & \overset{\theta'_k}{\dashrightarrow} & \widetilde{E}_k & \overset{\phi_i}{\longrightarrow} & B_k \\ {\scriptstyle F}\downarrow & & & & \downarrow{\scriptstyle R_k} \\ A_j & & \underset{\phi_j}{\xrightarrow{\qquad\qquad}} & & M_j \end{array}$$

4. *Definition of* θ_i. For every $1 \le i \le n$, define $\theta_i : A_i \to A_i$ to be a quasiconformal map such that $\theta_i|_{E_k} = \theta'_k$ and $\theta_i|_{\partial A_i} = \text{id}$. It always exists, due to the facts that all boundary curves are quasi-circles and all θ'_k are quasiconformal maps preserving the boundary labeling (see Lemma C.2).

The map θ_i satisfies all required properties, except possibly the one about its homotopy class.

4′. *Adjustment of the homotopy class of* θ_i. We will modify every θ_i without changing its value on the set $\mathcal{E}$. For every $1 \le i \le n$, choose an arc $\beta \Subset A_i$ connecting the two boundary curves of A_i. Then $\theta_i(\beta)$ is again an arc in A_i with the same end points. If necessary we postcompose θ_i with a quasiconformal repeated Dehn twist supported in the interior of $A_i \smallsetminus \mathcal{E}$, to ensure that $\theta_i(\beta)$ is homotopic to β (rel ∂A_i). After this adjustment, θ_i is isotopic to the identity rel ∂A_i. □

6.3 Proof of Theorem 5.4 for simple pieces

Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. An $\mathcal{E}$-piece E is said to be of **disc type** if some Jordan curve in ∂E bounds a closed disc Δ_E such that $E \subset \Delta_E \subset \mathcal{L}$ and Δ_E contains at most one point of P. The disc Δ_E is called the **hull** of E. Denote by $\mathcal{E}^o$ the union of the $\mathcal{E}$-pieces of disc type.

Lemma 6.3 *If either* $\mathcal{E} \smallsetminus \mathcal{E}^o = \emptyset$ *or* F *restricted to* $\mathcal{E} \smallsetminus \mathcal{E}^o$ *is holomorphic, then the marked repelling system* (F, P) *is c-equivalent to a holomorphic marked repelling system.*

Proof Let U be the union of the hulls of all $\mathcal{E}^o$-pieces. Then it is a finite disjoint union of closed quasidiscs each containing at most one point of P. Define a Beltrami differential μ on U by

$$\mu = \begin{cases} \text{the Beltrami differential of } F & \text{on } \mathcal{E}^o, \\ 0 & \text{on } U \smallsetminus \mathcal{E}^o. \end{cases}$$

By the Measurable Riemann Mapping Theorem and the Riemann Mapping Theorem, there is a quasiconformal map θ defined on U with Beltrami differential μ, such that θ maps every (disk-)component U_0 of U onto itself and at the same time fixes the eventual point of P in U_0. Since $U \Subset \mathcal{L}$ and U is a finite disjoint union of closed quasidiscs, there is an open set $V \Subset \mathcal{L}$ which is a finite disjoint union of open quasidiscs such that $U \subset V$. Therefore we can extend θ to a quasiconformal automorphism θ_0 of $\mathcal{L}$ such that $\theta_0 = \theta$ on U and $\theta_0 = \mathrm{id}$ on $\mathcal{L} \smallsetminus V$. Clearly, θ_0 is isotopic to the identity rel $\partial\mathcal{L} \cup P$ and $F \circ \theta_0^{-1}$ is holomorphic in $\theta_0(\mathcal{E})$. □

Proof of Theorem 5.4 for simple pieces Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system. Assume that every $\mathcal{L}$-piece is simple, i.e. every $\mathcal{L}$-piece is either a closed disc containing at most one point of P or a closed annulus disjoint from P. Then (F, P) is automatically of constant complexity. The boundary multicurve is simply the collection of one boundary curve in every annular piece of $\mathcal{L}$. Now assume that the leading eigenvalue of its transition matrix W satisfies $\lambda(W) < 1$. We want to prove that (F, P) is c-equivalent to a holomorphic marked repelling system.

Let $\mathcal{A}$ be the union of the annular $\mathcal{L}$-pieces, and $\mathcal{O} = \mathcal{L} \smallsetminus \mathcal{A}$ the union of the disc pieces. Then $P \subset \mathcal{O}$. Recall that $\mathcal{E}^o$ is the union of the $\mathcal{E}$-pieces of disc type. Set $\mathcal{E}^a = \mathcal{E} \smallsetminus \mathcal{E}^o$. If $\mathcal{E}^a = \emptyset$, then (F, P) is c-equivalent to a holomorphic repelling system by Lemma 6.3.

Note that every $\mathcal{E}^a$-piece is a closed essential sub-annulus in $\mathcal{A}$, and $F(\mathcal{E}^a) \subset \mathcal{A}$. The restriction $F|_{\mathcal{E}^a} : \mathcal{E}^a \to \mathcal{A}$ is an annular covering with the same transition matrix W as F and hence $\lambda(W) < 1$. By Lemma 6.2, it is c-equivalent to a holomorphic annuli-covering $R : \mathcal{B} \to \mathcal{M}$ through a pair of quasiconformal maps $(\Phi, \Psi) : \mathcal{A} \to \mathcal{M}$, i.e. Ψ is isotopic to Φ rel $\partial\mathcal{A}$, $\Psi(\mathcal{E}^a) = \mathcal{B}$ and $\Phi \circ F = R \circ \Psi$ on $\mathcal{E}^a$. Set subsequently

$$\mathcal{M}_1 = \mathcal{M} \sqcup \mathcal{O}, \qquad \mathcal{B}_1 = \mathcal{B} \cup \Psi(\mathcal{E}^o \cap \mathcal{A}) \sqcup (\mathcal{E}^o \cap \mathcal{O}),$$

$\Phi_1 = \Psi_1 = \mathrm{id}$ on $\mathcal{O}$, $\Phi_1 = \Phi$ on $\mathcal{M}$, $\Psi_1 = \Psi$ on $\mathcal{M}$, and $G = \Phi_1 \circ F \circ \Psi_1^{-1}$. Then (F, P) is c-equivalent to the marked repelling system $G : (\mathcal{B}_1, P) \to (\mathcal{M}_1, P)$ through the pair (Φ_1, Ψ_1).

Note that $G|_B = R$ and hence is already holomorphic. Every piece in $\mathcal{B}_1 \smallsetminus \mathcal{B}$ is of disc type. By Lemma 6.3, we see that (G, P), and hence (F, P) is c-equivalent to a holomorphic marked repelling system. □

7 Proof of Theorem 5.4 for a cycle of complex pieces

In this section we assume that (F, P) is a marked repelling system with only complex pieces, and furthermore these pieces form a single periodic cycle under F_* (see Sect. 5.2 for the definition). We will apply Thurston's theorem to prove Theorem 5.4 in this particular setting. Our resulting holomorphic marked repelling system will also satisfy some prescribed moduli properties. These properties will ensure crucial Grötzsch's spaces in the next section.

We denote by $\mathbb{D}$ the unit disc. A **marked disc** is a pair (Δ, a) with Δ an open hyperbolic disc in $\overline{\mathbb{C}}$ and $a \in \Delta$ a marked point. An **equipotential** γ of (Δ, a) is a Jordan curve that is mapped to a round circle with center zero under a conformal homeomorphism $\chi : \Delta \to \mathbb{D}$ with $\chi(a) = 0$. The **potential** of an equipotential γ is defined to be $\kappa(\gamma) := \mathrm{mod}(A(\partial\Delta, \gamma))$, the modulus of the annulus between $\partial\Delta$ and γ. These definitions do not depend on the choice of χ. For example in the marked disc $(\mathbb{D}, 0)$, the equipotential with potential $v > 0$ is the circle $\{|z| = e^{-v}\}$ (we define $\mathrm{mod}\{r < |z| < 1\} := -\log r$).

Let f be a postcritically finite rational map with non-empty Fatou set. Then its Julia set is connected and every Fatou domain Δ is canonically a marked disc marked by the unique eventually periodic point $a \in \Delta$. We call (Δ, a) **a marked Fatou domain** of f. An equipotential of a marked Fatou domain will be called **an equipotential** of f. Notice that equipotentials in a periodic Fatou domain correspond to round circles in Böttcher coordinates. We will also use $\kappa(*)$ to denote the potential of such an equipotential.

Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity having no boundary obstruction nor renormalization obstructions. Assume

- $\mathcal{L} = S_1 \sqcup \cdots \sqcup S_p$ with every S_i being a complex piece,
- $\mathcal{E} = E_1 \sqcup \cdots \sqcup E_p$ with $E_i = E_{S_i}$, $i = 1, \ldots, p$,
- and $F(E_i) = S_{i+1}$ ($S_{p+1} := S_1$) for $1 \le i \le p$.

For every $F^{-p}(\mathcal{L})$-piece E, the map $F^p|_E : E \mapsto F^p(E)$ marked by $E \cap P$ is a Thurston type system by Definition 7. Note that the boundary multicurve of $\mathcal{L}$ consists of all boundary curves of $\mathcal{L}$.

Denote by $\overline{\mathbb{C}}(S_i)$ the Riemann sphere containing S_i (we consider each piece S_i to be embedded in a distinct copy of the Riemann sphere). In this section we will prove the following theorem.

Theorem 7.1 *Let W be the transition matrix of the boundary multicurve of $\mathcal{L}$. Let $u > 0$ be a positive vector such that $Wu < u$. Then for every $1 \le i \le p$, there is a pair of quasiconformal maps $(\phi_i, \psi_i) : \overline{\mathbb{C}}(S_i) \to \overline{\mathbb{C}}$ and a holomorphic map $R_i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*:

(a) $\phi_i(S_i) = \psi_i(S_i)$ *and ϕ_i is isotopic to ψ_i rel $\partial S_i \cup (\mathcal{P} \cap S_i)$*;
(b) $\phi_{i+1} \circ F \circ \psi_i^{-1}|_{\psi_i(E_i)} = R_i|_{\psi_i(E_i)}$;
(c) *the return map $f_i := R_{i-1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i$ is a postcritically finite rational map whose conformal conjugacy class depends only on (F, P)*;
(d) *for every $i \in \{1, \ldots, p\}$ and every Jordan curve $\gamma \subset \partial S_i$, let β_γ be the curve in ∂E_i homotopic to γ in $S_i \smallsetminus P$, then both $\phi_i(\gamma)$ and $\psi_i(\beta_\gamma)$ are equipotentials in the same marked Fatou domain of f_i with potentials*

$$\kappa(\phi_i(\gamma)) = u(\gamma) \quad \textit{and} \quad \kappa(\psi_i(\beta_\gamma)) = \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}. \tag{9}$$

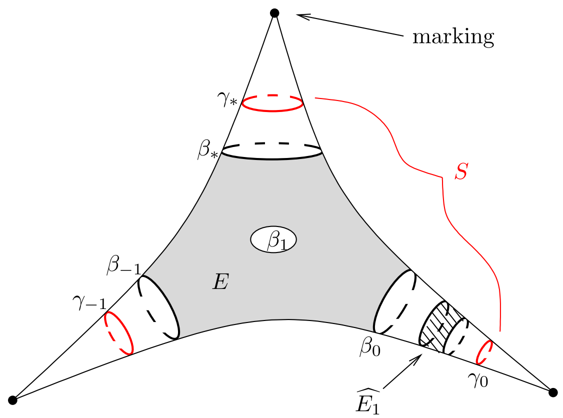


Fig. 3 A repelling system with a complex piece

Note that (a) and (b) together assert that (F, P) is c-equivalent to a holomorphic marked repelling system by Lemma 6.1.

7.1 Examples

We begin with some examples to illustrate the ideas of the proof.

Example 3 (See Fig. 3) Let S be the closure of a pair of pants bounded by quasicircles γ_0, γ_{-1} and γ_*. Let $E \Subset S$ be a bordered surface bounded by quasicircles $\beta_0, \beta_{-1}, \beta_*$ and β_1 such that β_1 bounds a disc in S, and $A(\gamma_i, \beta_i)$ $(i = 0, -1, *)$, the annulus between γ_i and β_i, are components of the interior of $S \smallsetminus E$. Let $H : E \to S$ be a quasiregular covering of degree 2. Then H is a Thurston type repelling system with $P_H = \emptyset$. The boundary multicurve is $\Upsilon = \{\gamma_0, \gamma_{-1}, \gamma_*\}$.

Now we require $H : \beta_* \to \gamma_*$ to be of degree 2, $H : \beta_{\pm 1} \to \gamma_0$ of degree 1 and $H : \beta_0 \to \gamma_{-1}$ of degree 2. Then the H-transition matrix of the boundary multicurve is

$$W = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

It is easy to check that $\lambda(W) = 1/\sqrt{2}$.

Example 4 The bordered surfaces S and E are defined as above. But this time we require $H_1 : \beta_* \to \gamma_*$ to be of degree 2, $H_1 : \beta_{\pm 1} \to \gamma_{-1}$ of degree 1 and $H_1 : \beta_0 \to \gamma_0$ of degree 2. The H_1-transition matrix of Υ is

$$W_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Its leading eigenvalue is $\lambda(W_1) = 1$.

The map H_1 can be constructed more explicitly as follows (suggested by X. Buff): Let $g(z) = -z^2$. Let S be $\overline{\mathbb{C}}$ minus a sufficient small round disc of radius ε (in spherical metric) around each of the three points $0, -1$ and ∞. Let $E' = g^{-1}(S)$. As -1 is a repelling fixed point of g, E' is not contained in S. Now let $\eta : D(-1, 2\varepsilon) \to D(-1, 2\varepsilon) = \{z : |z+1| \le 2\varepsilon\}$ be a homeomorphism fixing pointwise the boundary and the center, mapping the boundary curve of E' into the interior of S. Extend η elsewhere by identity. Set $E = \eta(E')$ and $H_1 = g \circ \eta^{-1} : E \to S$. Then $E \Subset S$ and H_1 satisfies the above requirement.

Define $G = g \circ \eta^{-1} : A(\beta_0, \beta_*) \to A(\gamma_0, \gamma_*)$. Then the marked repelling system $(G, \{-1\})$ is c-equivalent to the restriction of $(g, \{-1\})$ on $A(\beta_0, \beta_*)$. Thus it has no Thurston obstructions. Note that H_1 is a restriction of G. This gives an example of a non-admissible restriction (see Definition 4 in Sect. 4).

Example 5 The bordered surfaces S and E are defined as above. Choose a closed essential sub-annulus $\widehat{E}_1$ (with a quasicircle boundary) in the annulus $A(\gamma_0, \beta_0)$. Let G be a quasiregular covering from $\widehat{E}_1$ to the closure of $A(\gamma_{-1}, \gamma_*)$ with degree $d \ge 1$. Set $E_1 = G^{-1}(S)$. It is $\widehat{E}_1$ minus d holes. Define $F = H$ on E and $F = G$ on E_1. Then $F : E \cup E_1 \to S$ is a repelling system of constant complexity. The multicurve Υ is F-stable, with the following transition matrix:

$$W_2 = \begin{pmatrix} 0 & \frac{1}{2} + \frac{1}{d} & \frac{1}{d} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Its leading eigenvalue is $\lambda(W_2) = \sqrt{1/2 + 1/d}$. Note that $\lambda(W_2) < 1$ if and only if $d > 2$.

Note that every multicurve in S is contained in Υ (up to homotopy). Therefore none of H and F (for $d > 2$) has Thurston obstructions, whereas H_1 and F (for $d \le 2$) both have obstructions.

Now we want to construct the required maps in order to establish the c-equivalence of the Thurston type repelling system $H : E \to S$ in Example 3 to a holomorphic repelling system.

Mark one point in each component of $\overline{\mathbb{C}} \setminus S$. Denote the marked set by P. Extend H to a quasiregular branched covering h of $\overline{\mathbb{C}}$ such that the critical values of h are contained in P and $h(P) \subset P$. Then the global map h is postcritically finite.

The fact that $\#P$ equals to 3 implies that (h, P) has no Thurston obstructions. By Theorem 3.2 there are a pair of quasiconformal maps (Φ, Ψ) of $\overline{\mathbb{C}}$ and a rational map f such that $f \circ \Psi = \Phi \circ h$ and Ψ is isotopic to Φ rel P.

(One may prove that f is in fact $z \mapsto z^2 - 1$ if Φ is normalized appropriately so that $\Phi(P) = \{0, -1, \infty\}$.)

Modify the quasiconformal map Φ in its homotopy class such that the set $B := \Psi(E) = f^{-1}(\Phi(S))$ is contained in the interior of $M := \Phi(S)$. This can be done by choosing every boundary curve of M to be a suitable Jordan curve in the corresponding Fatou component of f. Then $\tilde{E} := \Phi^{-1}(B)$ is contained in the interior of S. Set $\theta_1 = \Phi^{-1} \circ \Psi$. Then θ_1 is isotopic to the identity rel P and $\theta_1(E) = \tilde{E}$. Since both E and $\tilde{E}$ are contained in S, we have a quasiconformal map θ of S such that $\theta|_E = \theta_1|_E$ and θ is isotopic to the identity rel ∂S.

Set $\phi = \Phi|_S$. Then H is c-equivalent to $f : B \to M$ through the pair $(\phi, \phi \circ \theta)$.

Let us consider the map F in Example 5 (for $d > 2$), with H as a subsystem. To construct maps realizing a global c-equivalence of F to a holomorphic system, we need to find an annulus $\widehat{B}_1$ in the corresponding component of $M \setminus B$, and a holomorphic covering from $B := \widehat{B}_1 \setminus \{d \text{ holes}\}$ to M. Therefore we have to make an estimate on the moduli of the annular components of $M \setminus B$ and on the modulus of $\widehat{B}_1$. For this purpose, it will be convenient to choose the boundary curves of M to be equipotentials in the Fatou set of f, i.e. round circles in the Böttcher coordinates. See Sect. 8.4 for details.

Note that the map $H : E \to S$ in Example 3 is a renormalization of F with period $p = 1$. Later on we will consider the more general case with $p \geq 2$. For example, we may have two complex pieces S_1 and S_2 of some repelling system F with $F_*(S_1) = S_2$ and $F_*(S_2) = S_1$. Assume we have constructed the map ϕ_1 for S_1. Then the value of ϕ_2 on E_{S_2} has to be the pullback of ϕ_1. But we still have to define ϕ_2 on the remaining part $S_2 \setminus E_{S_2}$. A good way to do this is to make an additional requirement for the marked extension h and for the quasiconformal map ϕ_1 on the free part $\overline{\mathbb{C}} \setminus S_1$, so that the global pullback of ϕ_1 automatically satisfies the required properties. This consideration will be dealt with in the next section.

7.2 Disc-marked extension for complex pieces

Let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity.

Let S be a complex $\mathcal{L}$-piece. Recall that $\overline{\mathbb{C}}(S)$ denotes the copy of the Riemann sphere containing S. Mark one point in each component of $\overline{\mathbb{C}}(S) \setminus S$. Set $P(S)$ to be the union of $P \cap S$ with these marked points. We call $(\overline{\mathbb{C}}(S), P(S))$ a **marked sphere** of S. We use κ_S to denote the potential function of the complementary marked discs of S.

For two complex $\mathcal{L}$-pieces S_1 and S_2 with $F(E_{S_1}) = S_2$, there are many ways to extend $F|_{E_{S_1}}$ to a branched covering from $\overline{\mathbb{C}}(S_1)$ to $\overline{\mathbb{C}}(S_2)$. We choose the following one in order to rigidify the extension.

Lemma 7.2 *Let S_1 and S_2 be complex $\mathcal{L}$-pieces with $F(E_{S_1}) = S_2$. Let ρ be a positive function defined on the set of Jordan curves in ∂S_1. Then there is a quasiregular branched covering $h : (\overline{\mathbb{C}}(S_1), P(S_1)) \to (\overline{\mathbb{C}}(S_2), P(S_2))$ as an extension of $F|_{E_{S_1}}$ such that:*

(a) $h(\overline{\mathbb{C}}(S_1)\smallsetminus E_{S_1}) = \overline{\mathbb{C}}(S_2)\smallsetminus S_2$.
(b) *$h(P(S_1)) \subset P(S_2)$ and the critical values of h are contained in $P(S_2)$.*
(c) *For any Jordan curve $\gamma \subset \partial S_1$, the curve $h(\gamma)$ is an equipotential in a complementary marked disc of S_2, with potential $\kappa_{S_2}(h(\gamma)) = \rho(\gamma)$.*
(d) *h is holomorphic on $\overline{\mathbb{C}}(S_1)\smallsetminus S_1$.*

Such a map $h : (\overline{\mathbb{C}}(S_1), P(S_1)) \to (\overline{\mathbb{C}}(S_2), P(S_2))$ will be called a **disc-marked extension of $F|_{E_{S_1}}$ associated to the function** ρ.

Proof Let α be a boundary component of E_{S_1}, bounding a unique complementary component Δ_α of E_{S_1}. Then $\eta := F(\alpha)$ is a boundary curve of S_2 and bounds a unique complementary marked disc (Δ_η, b) of S_2. Set $d := \deg(F : \alpha \to \eta)$.

Note that Δ_α may contain zero or one complementary component of S_1. In the former case, define $h_\alpha : \Delta_\alpha \to \Delta_\eta$ to be a quasiconformal homeomorphism if $d = 1$ or a quasiregular branched covering with a unique critical value b if $d > 1$, such that $h_\alpha|_\alpha = F|_\alpha$.

In the latter case α is homotopic in $S_1\smallsetminus P$ to a unique boundary curve γ of S_1 since (F, P) is of constant complexity. Let Δ_γ be the component of $\overline{\mathbb{C}}(S_1)\smallsetminus S_1$ enclosed by γ. Then $\Delta_\gamma \Subset \Delta_\alpha$, and Δ_γ together with the marked point $a \in \Delta_\gamma$ is a complementary marked disc of S_1.

Let η_1 be the equipotential in the marked disc (Δ_η, b) with potential $\kappa_{S_2}(\eta_1) = \rho(\gamma)$. Denote by Δ_1 the disc enclosed by η_1 and contained in Δ_η. Define $h_\gamma : \Delta_\gamma \to \Delta_1$ by $h_\gamma(z) = \varphi_1^{-1} \circ (\varphi(z))^d$, where φ (resp. φ_1) is a conformal map from the marked disc (Δ_γ, a) (resp. (Δ_1, b)) onto the unit disc $\mathbb{D}$ with $\varphi(a) = 0$ (resp. $\varphi_1(b) = 0$). Then there is a quasiregular covering $h_{\alpha\gamma}$ from $\Delta_\alpha\smallsetminus\overline{\Delta_\gamma}$ onto $\Delta_\eta\smallsetminus\overline{\Delta_1}$ so that $h_{\alpha\gamma}|_\alpha = F|_\alpha$ and $h_{\alpha\gamma}|_\gamma = h_\gamma|_\gamma$. Set $h_\alpha := h_\gamma$ on Δ_γ and $h_\alpha := h_{\alpha\gamma}$ on $\Delta_\alpha\smallsetminus\overline{\Delta_\gamma}$. Then $h_\alpha : \Delta_\alpha \to \Delta_\eta$ is quasiregular in Δ_α, and is more particularly holomorphic in Δ_γ.

The map $F|_{E_{S_1}}$ together with the collection of h_α for all possible α form a quasiregular branched covering $h : \overline{\mathbb{C}}(S_1) \to \overline{\mathbb{C}}(S_2)$ satisfying the properties (a)–(d). □

7.3 Applying Thurston's theorem

Now consider a marked repelling system $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ of constant complexity, and having no boundary obstruction nor renormalization obstructions. Assume $\mathcal{L} = S_1 \sqcup \cdots \sqcup S_p$ with each S_i being a complex piece,

$\mathcal{E} = E_1 \sqcup \cdots \sqcup E_p$ with $E_i = E_{S_i}\ \forall i$, and $F(E_i) = S_{i+1}$ ($S_{p+1} := S_1$) for $1 \le i \le p$.

Let ρ be a positive function defined on the set of Jordan curves in $\partial\mathcal{L}$. Let $(\overline{\mathbb{C}}(S_i), P(S_i))$ be a marked sphere of S_i, and let $h_i : (\overline{\mathbb{C}}(S_i), P(S_i)) \to (\overline{\mathbb{C}}(S_{i+1}), P(S_{i+1}))$ be a disc-marked extension of $F : E_i \to S_{i+1}$ associated to the function ρ for each $1 \le i \le p$.

Lemma 7.3 *Let σ be a positive function defined on the set of Jordan curves in ∂S_1. Then there are pairs of quasiconformal maps (Φ_i, Ψ_i) from $\overline{\mathbb{C}}(S_i)$ to $\overline{\mathbb{C}}$ and holomorphic maps $R_i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ $(i = 1, \ldots, p)$ satisfying the following properties*:

(1) *Ψ_i is isotopic to Φ_i rel $P(S_i)$, and Φ_i is holomorphic on $\overline{\mathbb{C}}(S_i) \setminus S_i$ $(i = 1, \ldots, p)$.*
(2) *$R_i \equiv \Phi_{i+1} \circ h_i \circ \Phi_i^{-1}$ for $2 \le i \le p$ (with $\Phi_{p+1} := \Phi_1$), and $R_1 \equiv \Phi_2 \circ h_1 \circ \Psi_1^{-1}$.*
(3) *The return maps $f_i := R_{i-1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i$ $(i = 1, \ldots, p)$ are postcritically finite rational maps whose conformal conjugacy classes depend only on (F, P).*
(4) *For each Jordan curve $\gamma \subset \partial S_1$, let β_γ be the unique curve in ∂E_1 homotopic to γ in $S_1 \setminus P$; then both $\Phi_1(\gamma)$ and $\Psi_1(\beta_\gamma)$ are equipotentials in the same marked Fatou domain of f_1 with potentials*

$$\kappa(\Phi_1(\gamma)) = \sigma(\gamma) \quad \textit{and} \quad \kappa(\Psi_1(\beta_\gamma)) = \frac{\kappa(\Phi_2 \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}. \tag{10}$$

(5) *Fix $2 \le i \le p$. For each Jordan curve $\gamma \subset \partial S_i$, let β_γ be the unique curve in ∂E_i homotopic to γ in $S_i \setminus P$. Then both $\Phi_i(\gamma)$ and $\Phi_i(\beta_\gamma)$ are equipotentials in the same marked Fatou domain of f_i and their potentials are related as follows*:

$$\kappa(\Phi_i(\gamma)) = \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \kappa(\Phi_i(\beta_\gamma)) \tag{11}$$

and

$$\kappa(\Phi_i(\beta_\gamma)) = \frac{\kappa(\Phi_{i+1} \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}. \tag{12}$$

See the following commutative diagram.

$$
\begin{array}{ccccccccc}
E_1 & & E_2 & & E_3 & & E_p & & E_1 \\
& \overset{F}{\searrow} & & \overset{F}{\searrow} & & & & \overset{F}{\searrow} & \\
\cap & & \cap & & \cap & & \cap & & \cap \\
S_1 & & S_2 & & S_3 & & S_p & & S_1 \\
\cap & & \cap & & \cap & & \cap & & \cap \\
\overline{\mathbb{C}}(S_1) & \xrightarrow{h_1} & \overline{\mathbb{C}}(S_2) & \xrightarrow{h_2} & \overline{\mathbb{C}}(S_3) & \dashrightarrow & \overline{\mathbb{C}}(S_p) & \xrightarrow{h_p} & \overline{\mathbb{C}}(S_1) \\
* \downarrow \Psi_1 & & \Psi_2 \downarrow \Phi_2 & & \Psi_3 \downarrow \Phi_3 & & \Psi_p \downarrow \Phi_p & & \Psi_1 \downarrow \Phi_1 \\
\overline{\mathbb{C}} & \xrightarrow{R_1} & \overline{\mathbb{C}} & \xrightarrow{R_2} & \overline{\mathbb{C}} & \dashrightarrow & \overline{\mathbb{C}} & \xrightarrow{R_p} & \overline{\mathbb{C}}
\end{array}
$$

Proof Denote by $H : E \to S_1$ the renormalization of F relative to S_1. Set

$$h := h_p \circ \cdots \circ h_2 \circ h_1 : \overline{\mathbb{C}}(S_1) \to \overline{\mathbb{C}}(S_1).$$

Then $h(P(S_1)) \subset P(S_1)$ and $P_h \subset P(S_1)$. Clearly, $(h, P(S_1))$ is an extension of the renormalization $H : E \to S_1$.

It is easy to see that the c-equivalence class of $(h, P(S_1))$ does not depend on the choice of the extensions.

Now, the assumption that (F, P) has no renormalization obstructions implies that $(H, P \cap S_1)$, as a marked repelling system, has no Thurston obstructions. This in turn will imply that the signature of the orbifold of h is not $(2, 2, 2, 2)$ and $(h, P(S_1))$ has no Thurston obstructions. The argument goes as follows:

Since the set of marked points in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ is mapped into itself by h, these points are eventually h-periodic. Let b be a periodic marked point in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ with period $k \geq 1$. Denote by Δ_b the component of $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ containing the marked point b and let $\gamma_b := \partial \Delta_b$. Then there is a unique component β of $h^{-k}(\gamma)$ homotopic to γ rel $P(S_1)$. Note that γ is contained in the boundary multicurve Υ of $\mathcal{L}$ and β is a component of $F^{-kp}(\gamma)$ in S_1. Thus the assumption $\lambda(W_\Upsilon) < 1$ implies that

$$\deg(F^{kp} : \beta \to \gamma) = \deg(h^k : \beta \to \gamma) = \deg_b h^k > 1.$$

This implies that h has a periodic critical point (in the cycle of b). Therefore the signature of the orbifold of h is not $(2, 2, 2, 2)$. Now any multicurve in $\overline{\mathbb{C}}(S_1) \smallsetminus P(S_1)$ can be represented by a multicurve in $S_1 \smallsetminus P(S_1) = S_1 \smallsetminus (P \cap S_1)$. So its h-transition matrix and H-transition matrix are equal, hence have the same leading eigenvalue, which is less than one. This implies that $(h, P(S_1))$ has no Thurston obstructions.

We can then apply Theorem 3.2 to obtain a pair of quasiconformal maps (ϕ, ψ) from $\overline{\mathbb{C}}(S_1)$ to $\overline{\mathbb{C}}$ and a rational map f_1 on $\overline{\mathbb{C}}$, whose conformal conjugate class depends only on the c-equivalence class of $(h, P(S_1))$ (and hence depends only on (F, P)), such that ψ is isotopic to ϕ rel $P(S_1)$ and $f_1 = \phi \circ h \circ \psi^{-1}$.

As any periodic cycle of the marked points in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ contains a critical point of h, its ϕ-image is a superattracting periodic cycle for f_1. Consequently, for every marked point a in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$, the point $\phi(a)$ is an eventually superattracting periodic point of f_1.

From (ϕ, ψ) *to* (Φ_1, Ψ_1). For every marked point a in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$, denote by Δ_a the component of $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ that contains the point a and set $\gamma_a = \partial \Delta_a$. Denote by η_a the equipotential in the marked Fatou domain of f_1 containing $\phi(a)$ (with $\phi(a)$ as a marked point) with potential $\kappa(\eta_a) = \sigma(\gamma_a)$. Then there is a quasiconformal map Φ_1 in the isotopy class (rel $P(S_1)$) of ϕ such that $\Phi_1(\gamma_a) = \eta_a$ for every marked point a in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$ (this is because γ_a, resp. η_a, is peripheral around the point $a \in P(S_1)$, resp. the point $\phi(a) \in \phi(P(S_1))$). Moreover, Φ_1 can be taken to be holomorphic on $\bigcup_a \Delta_a = \overline{\mathbb{C}}(S_1) \smallsetminus S_1$.

As Φ_1 is isotopic to ϕ rel $P(S_1)$, there is a quasiconformal map $\Psi_1 : \overline{\mathbb{C}}(S_1) \to \overline{\mathbb{C}}$ such that it is isotopic to ψ rel $P(S_1)$ and $\Phi_1 \circ h \circ \Psi_1^{-1} = f_1$.

Getting (*in order*) $\Phi_p, R_p, \Phi_{p-1}, R_{p-1}, \ldots, \Phi_2, R_2$ *and then* R_1. This is illustrated in the following diagrams:

$$
\begin{array}{ccccccc}
\overline{\mathbb{C}}(S_2) & \xrightarrow{h_2} & \overline{\mathbb{C}}(S_3) & \cdots\rightarrow & \overline{\mathbb{C}}(S_p) & \xrightarrow{h_p} & \overline{\mathbb{C}}(S_1) \\
\downarrow \Phi_2 & & \downarrow \Phi_3 & & \Phi_p \downarrow & & \downarrow \Phi_1 \\
\overline{\mathbb{C}} & \xrightarrow[R_2]{} & \overline{\mathbb{C}} & \cdots\rightarrow & \overline{\mathbb{C}} & \xrightarrow[R_p]{} & \overline{\mathbb{C}}
\end{array}
\tag{13}
$$

and

$$
\begin{array}{ccc}
\overline{\mathbb{C}}(S_1) & \xrightarrow{h_1} & \overline{\mathbb{C}}(S_2) \\
\Psi_1 \downarrow & & \downarrow \Phi_2 \\
\overline{\mathbb{C}} & \xrightarrow[R_1]{} & \overline{\mathbb{C}}
\end{array}
\tag{14}
$$

More precisely we pullback the complex structure of $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}(S_p)$ by $\Phi_1 \circ h_p$ and then integrate it to obtain a quasiconformal map $\Phi_p : \overline{\mathbb{C}}(S_p) \to \overline{\mathbb{C}}$ such that $R_p := \Phi_1 \circ h_p \circ \Phi_p^{-1}$ is holomorphic.

As a disc-marked extension, we know that h_p is holomorphic in $\overline{\mathbb{C}}(S_p) \smallsetminus S_p$ whose h_p-image is contained in $\overline{\mathbb{C}}(S_1) \smallsetminus S_1$. Combining with the result that

Φ_1 is holomorphic in $\overline{\mathbb{C}}(S_1)\smallsetminus S_1$ and the equation $R_p \circ \Phi_p = \Phi_1 \circ h_p$, we see that Φ_p is holomorphic in $\overline{\mathbb{C}}(S_p)\smallsetminus S_p$.

Inductively, for $i = p-1, \dots, 2$, we have a quasiconformal map $\Phi_i : \overline{\mathbb{C}}(S_i) \to \overline{\mathbb{C}}$ such that $R_i := \Phi_{i+1} \circ h_i \circ \Phi_i^{-1}$ is holomorphic and Φ_i is holomorphic in $\overline{\mathbb{C}}(S_i)\smallsetminus S_i$.

Finally, set $R_1 := \Phi_2 \circ h_1 \circ \Psi_1^{-1}$. Then $R_p \circ \cdots \circ R_2 \circ R_1 = f_1$. Therefore R_1 is also holomorphic and Ψ_1 is holomorphic in $\overline{\mathbb{C}}(S_1)\smallsetminus S_1$.

Getting Ψ_i and f_i. As a disc-marked extension, we know that the critical values of h_i are contained in $P(S_{i+1})$ and $h_i(P(S_i)) \subset P(S_{i+1})$ for $1 \le i \le p$. Since Ψ_1 is isotopic to Φ_1 rel $P(S_1)$, there is a quasiconformal map $\Psi_p : \overline{\mathbb{C}}(S_p) \to \overline{\mathbb{C}}$ such that Ψ_p is isotopic to Φ_p rel $P(S_p)$ and $\Psi_1 \circ h_p = R_p \circ \Psi_p$. Inductively, there is a quasiconformal map $\Psi_i : \overline{\mathbb{C}}(S_i) \to \overline{\mathbb{C}}$ for $i = p-1, \dots, 2$, such that Ψ_i is isotopic to Φ_i rel $P(S_i)$ and $\Psi_{i+1} \circ h_i = R_i \circ \Psi_i$. Set then $f_i := R_{i-1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i$. Now we have the following commutative diagrams:

$$\begin{array}{ccccccccccc}
\overline{\mathbb{C}}(S_i) & \xrightarrow{h_i} & \cdots & \xrightarrow{h_p} & \overline{\mathbb{C}}(S_1) & \xrightarrow{h_1} & \overline{\mathbb{C}}(S_2) & \cdots & \xrightarrow{h_{i-1}} & \overline{\mathbb{C}}(S_i) \\
\Psi_i \downarrow & & \Psi_p \downarrow & & \Psi_1 \downarrow & & \downarrow \Phi_2 & & & \downarrow \Phi_i \\
\overline{\mathbb{C}} & \xrightarrow[R_i]{} & \cdots & \xrightarrow[R_p]{} & \overline{\mathbb{C}} & \xrightarrow[R_1]{} & \overline{\mathbb{C}} & \cdots & \xrightarrow[R_{i-1}]{} & \overline{\mathbb{C}}
\end{array}$$

and

$$\begin{array}{ccc}
\overline{\mathbb{C}}(S_i) & \xrightarrow{*} & \overline{\mathbb{C}}(S_i) \\
\Psi_i \downarrow & & \downarrow \Phi_i \\
\overline{\mathbb{C}} & \xrightarrow[f_i]{} & \overline{\mathbb{C}}
\end{array}$$

It is easy to see that f_i is c-equivalent to $h_{i-1} \circ \cdots \circ h_1 \circ h_p \circ \cdots \circ h_i$, which is postcritically finite, so f_i is also postcritically finite. Clearly its conformal conjugate class depends only on (F, P).

Potentials. Notice that $f_{i+1} \circ R_i = R_i \circ f_i$, i.e. R_i is a holomorphic (semi-)conjugacy from f_i to f_{i+1} (set $f_{p+1} = f_1$). It is a classical result that their Julia sets are related by $J(f_i) = R_i^{-1}(J(f_{i+1}))$. Note that the critical values of R_i are contained in $\Phi_{i+1}(P(S_{i+1}))$, which is eventually periodic under f_{i+1}. We see that R_i maps equipotentials of f_i to equipotentials of f_{i+1}.

As a disc-marked extension, for each Jordan curve $\gamma \subset \partial S_p$, the curve $h_p(\gamma)$ lies on an equipotential in a complementary marked disc of S_1. Since every Jordan curve in $\partial\Phi_1(S_1)$ lies on an equipotential of f_1 and Φ_1 is holomorphic in $\overline{\mathbb{C}}(S_1)\smallsetminus S_1$, the curve $h_p(\gamma)$ goes to an equipotential of f_1 by Φ_1. This equipotential of f_1 is pulled back by R_p to some equipotentials of f_p. Thus $\Phi_p(\gamma)$ lies on an equipotential of f_p. Inductively, we have that each Jordan curve in $\partial\Phi_i(S_i)$ lies on an equipotential of f_i for $i=1,\dots,p$.

Similarly, each curve in $\Phi_i(\partial E_i)$ lies on an equipotential of f_i for $i\ge 2$ and each curve in $\Psi_1(\partial E_1)$ lies on an equipotential of f_1.

Fix $i\in\{1,\dots,p\}$. For each Jordan curve $\gamma\subset\partial S_i$, and for β_γ the curve in ∂E_i homotopic to γ in $S_i\smallsetminus P$, we have that $h_i(\beta_\gamma)=F(\beta_\gamma)$ is a curve in ∂S_{i+1}. Note that $\Phi_{i+1}\circ h_i(\beta_\gamma)=R_i\circ\Phi_i(\beta_\gamma)$ if $i\ne 1$ (with $\Phi_{p+1}=\Phi_1$) and $\Phi_2\circ h_1(\beta_\gamma)=R_1\circ\Psi_1(\beta_\gamma)$ if $i=1$. Their potentials are related by:

$$\kappa(\Phi_i(\beta_\gamma))=\frac{\kappa(\Phi_{i+1}\circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}\quad \text{if } i\ne 1$$

or

$$\kappa(\Psi_1(\beta_\gamma))=\frac{\kappa(\Phi_2\circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}\quad \text{if } i=1.$$

Now fix $2\le i\le p$. By the construction of h_i in Lemma 7.2, the curve $h_i(\gamma)$ is an equipotential with potential $\rho(\gamma)$ in a complementary marked disc of S_{i+1}. We have $\mathrm{mod}(h_i(A(\gamma,\beta_\gamma)))=\rho(\gamma)$, where $A(\gamma,\beta_\gamma)$ is the annulus between them. Notice that Φ_{i+1} is conformal in $\overline{\mathbb{C}}(S_{i+1})\smallsetminus S_{i+1}$. We also have $\mathrm{mod}(\Phi_{i+1}\circ h_i(A(\gamma,\beta_\gamma)))=\rho(\gamma)$. From the equation $R_i\circ\Phi_i=\Phi_{i+1}\circ h_i$, we get

$$\kappa(\Phi_i(\gamma))-\kappa(\Phi_i(\beta_\gamma))=\mathrm{mod}(\Phi_i(A(\gamma,\beta_\gamma)))=\frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})}. \qquad \square$$

Remark 1 For every i, if we make a normalization by requiring that three given distinct points in $P(S_i)$ (note that $\#P(S_i)\ge 3$ since S_i is a complex piece) go to $(0,-1,\infty)$ under the action of Φ_i, then f_i and the homotopy class (rel $P(S_i)$) of Φ_i are uniquely determined.

Remark 2 For a F_*-periodic cycle $(S_1,\dots,S_p)$, we have p renormalizations (one for each S_i). Lemma 7.3 shows that none of them has a Thurston obstruction if at least one of them has no Thurston obstructions.

7.4 Proof of Theorem 7.1

Now fix the positive functions σ and ρ as follows:

$$\forall\gamma\subset\partial S_1,\quad \sigma(\gamma):=u(\gamma);$$

$$\forall \gamma \subset \bigcup_{i=1}^{p} \partial S_i, \quad \rho(\gamma) := \left(u(\gamma) - \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \right) \deg(F|_{\beta_\gamma}), \tag{15}$$

where β_γ is the curve in $\bigcup_{i=1}^{p} \partial E_i$ homotopic to γ in $\mathcal{L} \smallsetminus P$. Note that $\rho(\gamma) > 0$ for every γ by the assumption $Wu < u$.

Let $(\Phi_i, \Psi_i, R_i, f_i)_{i=1,\dots,p}$ be the collection of maps derived from Lemma 7.3 with the functions ρ and σ defined above.

Let γ be a Jordan curve in ∂S_1. Then $\kappa(\Phi_1(\gamma)) = \sigma(\gamma) = u(\gamma)$ by Lemma 7.3(4).

Let γ be a Jordan curve in ∂S_p and β_γ the curve in ∂E_p homotopic to γ in $S_p \smallsetminus P$. By (15) and Lemma 7.3(4)–(5) we have

$$\begin{aligned}\kappa(\Phi_p(\gamma)) &= \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \frac{\kappa(\Phi_1 \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \\ &= \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} = u(\gamma).\end{aligned}$$

Inductively, for $i = p-1, \dots, 2$, we have $\kappa(\Phi_i(\gamma)) = u(\gamma)$ for any Jordan curve $\gamma \subset \partial S_i$. Therefore $\kappa(\Phi_i(\gamma)) = u(\gamma)$ for any i and any $\gamma \subset \partial S_i$.

Fix any $i \in \{1, \dots, p\}$. Let β be a curve in ∂E_i which is non-peripheral in $S_i \smallsetminus P$. By (11) and (10), we have

$$\kappa(\Phi_i(\beta)) = \frac{\kappa(\Phi_{i+1} \circ F(\beta))}{\deg(F|_\beta)} = \frac{u(F(\beta))}{\deg(F|_\beta)} \quad \text{if } i \neq 1$$

and

$$\kappa(\Psi_1(\beta)) = \frac{u(F(\beta))}{\deg(F|_\beta)} \quad \text{if } i = 1.$$

Let γ be a Jordan curve in ∂S_1 and β_γ the Jordan curve in ∂E_1 homotopic to γ in $S_1 \smallsetminus P$. From the above formula and the fact that $Wu < u$, we deduce that $\kappa(\Psi_1(\beta_\gamma)) < \kappa(\Phi_1(\gamma))$. This implies that $\Psi_1(E_1) \Subset \Phi_1(S_1)$.

For $i = 2, \dots, p$, set $\phi_i = \psi_i = \Phi_i$. Set also $\phi_1 = \Phi_1$. Obviously, (a)–(d) hold for $i \geq 2$ by the above computation. Now we want to define ψ_1.

Notice that $\Psi_1(E_1) \Subset \phi_1(S_1)$ and Ψ_1 is isotopic to ϕ_1 rel $P(S_1)$. Set $\theta = \Psi_1 \circ \phi_1^{-1}$. Then θ is isotopic to the identity rel $\phi_1(P(S_1))$. Both $\phi_1(E_1)$ and $\theta \circ \phi_1(E_1) = \Psi_1(E_1)$ are disjoint from the closure of $\phi_1(\overline{\mathbb{C}}(S_1) \smallsetminus S_1)$. By Corollary D.2, there is a homeomorphism ξ of $\overline{\mathbb{C}}$ isotopic to the identity rel $\phi_1(P(S_1)) \cup \phi_1(\overline{\mathbb{C}}(S_1) \smallsetminus S_1)$, such that $\xi|_{\phi_1(E_1)} = \theta$. Set $\psi_1 = \xi \circ \phi_1$. Then ψ_1 is isotopic to ϕ_1 rel $P(S_1) \cup (\overline{\mathbb{C}}(S_1) \smallsetminus S_1)$, and $\psi|_{E_1} = \Psi|_{E_1}$. □

8 Proof of Theorem 5.4

Now let $F : (\mathcal{E}, P) \to (\mathcal{L}, P)$ be a marked repelling system of constant complexity having no boundary obstruction no renormalization obstructions. We prove in this section that (F, P) is c-equivalent to a holomorphic marked repelling system.

8.1 Choice of the positive vector v, and the constants C and M

Let Υ be the boundary multicurve of (F, P). By assumption, $\lambda(W_\Upsilon) < 1$ for its transition matrix W_Υ. Applying Lemma A.1, we have a positive vector $v \in \mathbb{R}^\Upsilon$ so that $W_\Upsilon v < v$, i.e. there is a positive function $v : \Upsilon \to \mathbb{R}^+$ such that

$$(W_\Upsilon v)_\gamma = \sum_{\beta\in\Upsilon}\sum_{\alpha\sim\gamma} \frac{v(\beta)}{\deg(F : \alpha \to \beta)} < v(\gamma), \tag{16}$$

where the last sum is taken over the curves α in $F^{-1}(\beta)$ homotopic to γ in $\mathcal{L} \smallsetminus P$. Let $C > 0$ be a constant to be determined later (Sect. 8.3.2). Denote by $\mathbf{1}$ the vector whose every entry is 1. Choose $M > 0$ to be a large number so that $W_\Upsilon(Mv) + C\mathbf{1} < Mv$, i.e. for each $\gamma \in \Upsilon$,

$$\sum_{\beta\in\Upsilon}\sum_{\alpha\sim\gamma} \frac{Mv(\beta)}{\deg(F : \alpha \to \beta)} + C < Mv(\gamma). \tag{17}$$

For any $\gamma \in \Upsilon$, the quantity $Mv(\gamma)$ will be the prescribed potential for $\phi_S(\gamma)$, with S the $\mathcal{L}$-piece admitting γ as a boundary curve.

8.2 Definition of (ϕ_S, ψ_S) and R_S for complex pieces

Let $(\overline{\mathbb{C}}(S), P(S))$ be a marked sphere for each complex $\mathcal{L}$-piece S. Let $h_S : (\overline{\mathbb{C}}(S), P(S)) \to (\overline{\mathbb{C}}(F(E_S)), P(F(E_S)))$ be a disc-marked extension, associated to the function

$$\rho(\gamma) := \left(Mv(\gamma) - \frac{Mv(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}\right)\deg(F|_{\beta_\gamma}),$$

where γ is a Jordan curve in ∂S and β_γ is the curve in ∂E_S homotopic to γ in $S \smallsetminus P$.

For each F_*-cycle of complex $\mathcal{L}$-pieces $S_1, \ldots, S_p$, set $\mathcal{L}_0 = \bigsqcup_{i=1}^p S_i$, $\mathcal{E}_0 = \bigsqcup_{i=1}^p E_{S_i}$, $P_0 = P \cap \mathcal{L}_0$ and $F_0 = F|_{\mathcal{E}_0}$. Then the marked repelling subsystem $F_0 : (\mathcal{E}_0, P_0) \to (\mathcal{L}_0, P_0)$ satisfies the conditions of Theorem 7.1. Let Υ_0 be the boundary multicurve of $\mathcal{L}_0$. It consists of Jordan curves in $\partial\mathcal{L}_0$. Denote by W_0 the F_0-transition matrix of Υ_0. Set $u(\gamma) := Mv(\gamma)$ for every

$\gamma \in \Upsilon_0$. It is easy to check that $W_0 u < u$. We construct ϕ_{S_i}, ψ_{S_i}, R_{S_i} and f_{S_i} according to Theorem 7.1 for $i = 1, \dots, p$.

Now assume that S is a complex $\mathcal{L}$-piece and is not F_*-periodic. Then there are complex $\mathcal{L}$-pieces $S_{-k} := S, S_{-k+1}, \dots, S_0$ $(k > 0)$ such that S_i is not F_*-periodic for $i < 0$ but S_0 is F_*-periodic. Since S_0 is F_*-periodic, we have already constructed a quasiconformal map $\phi_{S_0} : \overline{\mathbb{C}}(S_0) \to \overline{\mathbb{C}}$ and a post-critically finite rational map f_{S_0}.

As before, there are quasiconformal maps $\phi_{S_i} : \overline{\mathbb{C}}(S_i) \to \overline{\mathbb{C}}$ and holomorphic maps $R_{S_i} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\overline{\mathbb{C}}(S_{-k}) & \xrightarrow{h_{S_{-k}}} & \overline{\mathbb{C}}(S_{-k+1}) & \xrightarrow{h_{S_{-k+1}}} & \cdots & \xrightarrow{h_{S_{-1}}} & \overline{\mathbb{C}}(S_0) \\
\Phi_{S_{-k}} \downarrow & & \downarrow \Phi_{S_{-k+1}} & & & & \downarrow \Phi_{S_0} \\
\overline{\mathbb{C}} & \xrightarrow[R_{S_{-k}}]{} & \overline{\mathbb{C}} & \xrightarrow[R_{S_{-k+1}}]{} & \cdots & \xrightarrow[R_{S_{-1}}]{} & \overline{\mathbb{C}}
\end{array}$$

Since $h_{S_i}(P(S_i)) \subset P(S_{i+1})$ and every critical value (if any) of h_{S_i} lies in $P(S_{i+1})$, we have $R_i(\phi_{S_i}(P(S_i))) \subset \phi_{S_{i+1}}(P(S_{i+1}))$, and every critical value of $R_{S_{-1}} \circ \cdots \circ R_{S_i}$ lies in $\phi_{S_0}(P(S_0))$.

Set $f_{S_i} := R_{S_{-1}} \circ \cdots \circ R_{S_i}$. Let $b \in \overline{\mathbb{C}}(S_i) \smallsetminus S_i$ be a marked point. Then $f_{S_i} \circ \phi_i(b)$ is the center of a marked Fatou domain Δ of f_{S_0}. The component $\Delta_{\phi_i(b)}$ of $f_{S_i}^{-1}(\Delta)$ that contains $\phi_i(b)$ is a disc. We will call $(\Delta_{\phi_i(b)}, \phi_i(b))$ a *canonical marked disc*.

The name 'canonical' means that up to a Möbius transformation, the configuration formed by these marked discs is uniquely determined. Note that when a disc-marked extension h_{S_i} is chosen, the map ϕ_{S_i} is uniquely determined by $\phi_{S_{i-1}}$ up to a Möbius transformation. As $\phi_{S_{i-1}}$ varies in its homotopy class, ϕ_{S_i} varies simultaneously in its homotopy class while R_{S_i} remains unchanged. On the other hand various choices of disc-marked extensions are related by quasiconformal maps. More precisely, if $\tilde{h}_{S_i}$ is another choice of the disc-marked extension, then there is a quasiconformal map ξ of $\overline{\mathbb{C}}(S_i)$ such that $\tilde{h}_{S_i} = h_{S_i} \circ \xi$. Setting $\tilde{\phi}_{S_i} = \phi_{S_i} \circ \xi$, we get the same holomorphic map R_{S_i} as before. This implies that the maps f_{S_i} are independent of the extensions. In particular, the canonical marked discs are independent of the large number M involved in the function ρ (therefore involved in the extensions h_{S_i}).

Lemma 8.1 *With the assumption above, for any* $i = -k, \dots, -1$, *there are quasiconformal maps* $\psi_{S_i} = \phi_{S_i} : \overline{\mathbb{C}}_{S_i} \to \overline{\mathbb{C}}$ *such that*:

(1) $R_{S_i} := \phi_{S_{i+1}} \circ h_{S_i} \circ \phi_{S_i}^{-1}$ *is holomorphic and depends only on* (F, P).

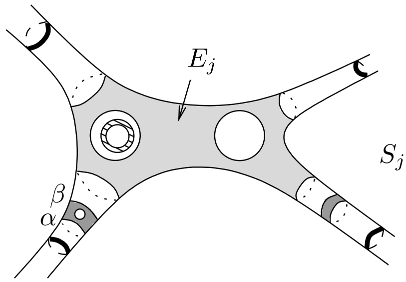


Fig. 4 The $\mathcal{L}$-piece S_j is bounded by the *thick curves*. The *light grey piece* E_j is an $\mathcal{E}^c$-piece. *The darker-grey pieces* are $\mathcal{E}^a$-pieces

(2) *For any marked point $b \in \mathcal{P}(S_i) \smallsetminus S_i$, denote by γ_b the component of ∂S_i that separates b from $S_i \smallsetminus \gamma_b$ and by β_b the component of ∂E_{S_i} that separates b from $E_{S_i} \smallsetminus \beta_b$. Then both $\phi_{S_i}(\gamma_b)$ and $\phi_{S_i}(\beta_b)$ are equipotentials in the canonical marked disc $(\Delta_{\phi_{S_i}(b)}, \phi_{S_i}(b))$ with potentials*

$$\begin{aligned} \kappa(\phi_{S_i}(\gamma_b)) &= Mv(\gamma_b), \\ \kappa(\phi_{S_i}(\beta_b)) &= \kappa(\psi_{S_i}(\beta_b)) = \frac{Mv(F(\beta_b))}{\deg(F|_{\alpha_b})}. \end{aligned} \tag{18}$$

Proof (1) is obvious. The proof of (2) is quite easy by following the same argument as before. □

8.3 Definition of ϕ_S for simple $\mathcal{L}$-pieces

For each disc $\mathcal{L}$-piece S, define ϕ_S to be a quasiconformal map from S onto a closed quasidisc in $\overline{\mathbb{C}}$. Assume that S is an annular piece. Then it is a closed annulus and one of its boundary curves, say γ, is contained in the boundary multicurve Υ. We define ϕ_S to be a quasiconformal map from S onto a closed round annulus in $\overline{\mathbb{C}}$ with modulus

$$\operatorname{mod} \phi_S(S) = Mv(\gamma). \tag{19}$$

8.4 Definition of ψ_S for $\mathcal{E}$-pieces of simple type

Decompose $\mathcal{E}$ into $\mathcal{E}^c \sqcup \mathcal{E}^a \sqcup \mathcal{E}^o$ as follows (see Fig. 4):

- $\mathcal{E}^c$ is the union of the $\mathcal{E}$-pieces of complex type;
- $\mathcal{E}^o$ is the union of the $\mathcal{E}$-pieces of disc type (see Sect. 6.3);
- $\mathcal{E}^a$ is the union of the remaining $\mathcal{E}$-pieces.

Clearly, the above three sets are mutually disjoint. Each $\mathcal{E}^a$-piece E is contained essentially in either an annular $\mathcal{L}$-piece or an annular component of $\mathcal{L} \smallsetminus \mathcal{E}^c$.

8.4.1 Definition of an auxiliary map φ_E for $\mathcal{E}^a$-pieces

Let E be an $\mathcal{E}^a$-piece, S the $\mathcal{L}$-piece containing E, and $S_0 = F(E)$.

Then S_0 is either an annular or a complex $\mathcal{L}$-piece. We say $E \in \mathcal{E}^{(a,a)}$ in the former case and $E \in \mathcal{E}^{(a,c)}$ in the latter. This decomposes $\mathcal{E}^a$ into $\mathcal{E}^{(a,a)} \sqcup \mathcal{E}^{(a,c)}$.

If S_0 is an annular piece, then there is a quasiconformal map φ_E from E onto a closed round annulus in $\overline{\mathbb{C}}$ such that $\phi_{S_0} \circ F \circ \varphi_E^{-1}$ is holomorphic in the interior of $\varphi_E(E)$.

Let γ be one of the two boundary curves in ∂S_0 with $\gamma \in \Upsilon$. Then there is a Jordan curve β in ∂E so that $F(\beta) = \gamma$. From (19), we have:

$$\operatorname{mod} \varphi_E(E) = \frac{\operatorname{mod}(\phi_{S_0}(S_0))}{\deg F|_E} = \frac{Mv(F(\beta))}{\deg(F|_\beta)}. \tag{20}$$

Now assume S_0 is a complex piece. Then there is a quasiregular branched covering $h_E : \overline{\mathbb{C}} \to (\overline{\mathbb{C}}(S_0), P(S_0))$ such that $h_E|_E = F|_E$, $\deg h_E = \deg F|_E$ and every critical value of h_E is contained in $P(S_0)$. As before, we have a quasiconformal map φ_E of $\overline{\mathbb{C}}$ such that $R_E := \phi_{S_0} \circ h_E \circ \varphi_E^{-1}$ is a holomorphic map, which depends only on (F, P) (up to a Möbius transformation) since ϕ_{S_0} depends only on (F, P).

$$\begin{array}{ccccc}
\overline{\mathbb{C}} & \supset E & \xrightarrow{\varphi_E} & \varphi_E(E) \subset \overline{\mathbb{C}} \\
\downarrow{\scriptstyle h_E} & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle R_E} \\
\overline{\mathbb{C}}(S_0) & \supset S_0 & \xrightarrow[\phi_{S_0}]{} & M_0 \subset \overline{\mathbb{C}}
\end{array}
\qquad
\begin{array}{ccc}
\alpha, \beta & \xrightarrow{\varphi_E} & \subset \Delta_1, \Delta_2 \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle R_E} \\
 & \xrightarrow[\phi_{S_0}]{} & \subset \Delta_a, \Delta_b
\end{array}$$

Note that ∂E has exactly two boundary curves α and β which are non-peripheral and homotopic to each other in $S \smallsetminus P$.

We apply Theorem 7.1 and Lemma 8.1 to the curve $\phi_{S_0} \circ F(\alpha)$ (resp. $\phi_{S_0} \circ F(\beta)$):

- if S_0 is F_*-periodic, the curve is an equipotential in a marked disc (Δ_a, a) (resp. (Δ_b, b)) of the postcritically finite rational map f_{S_0};
- if S_0 is not F_*-periodic, the curve is an equipotential in a canonical marked disc, denoted also by (Δ_a, a) (resp. (Δ_b, b)).

The respective potentials of these curves are

$$\kappa(\phi_{S_0} \circ F(\alpha)) = Mv(F(\alpha)), \qquad \kappa(\phi_{S_0} \circ F(\beta)) = Mv(F(\beta)).$$

Let Δ_1 and Δ_2 be the component of $R_E^{-1}(\Delta_a)$ and $R_E^{-1}(\Delta_b)$ that contains $\varphi_E(\alpha)$ and $\varphi_E(\beta)$, respectively. Then they are disjoint discs since neither

$\Delta_a \smallsetminus \{a\}$ nor $\Delta_b \smallsetminus \{b\}$ contains critical values of R_E. Set $z_1 := \Delta_1 \cap R_E^{-1}(a)$ and $z_2 := \Delta_2 \cap R_E^{-1}(b)$. Then (Δ_1, z_1) and (Δ_2, z_2) are disjoint marked discs in $\overline{\mathbb{C}}$. Moreover they are independent of the choice of M, since (Δ_a, a) and (Δ_b, b) are independent of the choice of M.

Clearly, $\varphi_E(\alpha)$ and $\varphi_E(\beta)$ are equipotentials with potentials

$$\kappa(\varphi_E(\alpha)) = \frac{Mv(F(\alpha))}{\deg F|_\alpha} \quad \text{and} \quad \kappa(\varphi_E(\beta)) = \frac{Mv(F(\beta))}{\deg F|_\beta}.$$

Let $A(E) = A(\alpha, \beta)$ denote the annulus bounded by α and β. Applying Lemma B.1, there is a constant $C(E) > 0$ which is independent of the choice of M, such that

$$\begin{aligned} &\frac{Mv(F(\alpha))}{\deg F|_\alpha} + \frac{Mv(F(\beta))}{\deg F|_\beta} \\ &\quad \le \operatorname{mod} \varphi_E(A(\alpha, \beta)) \\ &\quad \le \frac{Mv(F(\alpha))}{\deg F|_\alpha} + \frac{Mv(F(\beta))}{\deg F|_\beta} + C(E). \end{aligned} \tag{21}$$

8.4.2 The constant C

The set $\mathcal{E}^{(a,c)}$ has only finitely many pieces E with $C(E)$ independent of the choice of the number M. Set $C := \sum_E C(E)$. It is also independent of M.

8.4.3 Embedding of $\varphi_E(E)$ and construction of ψ_A

Every $\mathcal{E}^a$-piece E is contained in either an annular piece S or an annular component A of $S \smallsetminus E_S$ for some complex piece S.

Assume that S is an annular piece. Let γ be a boundary curve of S with $\gamma \in \Upsilon$. From (20) and (21), we have

$$\sum_{E \subset S \cap \mathcal{E}^{(a,a)}} \operatorname{mod} \varphi_E(E) + \sum_{E \subset S \cap \mathcal{E}^{(a,c)}} \operatorname{mod} \varphi_E(A(E)) \le \sum_\beta \frac{Mv(F(\beta))}{\deg(F|_\beta)} + C,$$

where the last sum is taken over the curves β in $\bigcup_{\eta \in \Upsilon} F^{-1}(\eta)$ homotopic to γ in $S \smallsetminus P$.

The right hand side term is less than $Mv(\gamma) = \operatorname{mod}(\phi_S(S))$ by (17). Therefore, as in Sect. 6, one can embed holomorphically $\varphi_E(E)$ essentially into the interior of $\phi_S(S)$ for every $\mathcal{E}^a$-piece $E \subset S$ according to the original order of their non-peripheral boundary curves so that the embedded $\varphi_E(E)$'s are mutually disjoint. In other words, we have a quasiconformal map ψ_S from S onto $\phi_S(S)$ such that

- $\psi_S|_{\partial S} = \phi_S|_{\partial S}$ and ψ_S is isotopic to ϕ_S rel ∂S;
- for every $\mathcal{E}^a$-piece $E \subset S$, the map $\varphi_E \circ \psi_S^{-1}$ is holomorphic in $\psi_S(E)$.

Consequently,

- for every $\mathcal{E}^a$-piece $E \subset S$ and for $S_0 := F(E)$ the map $\phi_{S_0} \circ F \circ \psi_S^{-1} = \phi_{S_0} \circ F \circ \varphi_E^{-1} \circ \varphi_E \circ \psi_S^{-1}$ is holomorphic in $\psi_S(E)$.

Now assume that S is a complex piece and A is an annular component of $S \smallsetminus E_S$. Following an argument similar to the one above, we have a quasiconformal map ψ_A from A onto $\psi_S(A)$, such that

- $\psi_A|_{\partial A} = \psi_S|_{\partial A}$ and ψ_A is isotopic to $\psi_S|_A$ rel ∂A;
- for every $\mathcal{E}^a$-piece $E \subset A$ and for $S_0 := F(E)$ the map $\phi_{S_0} \circ F \circ \psi_A^{-1}$ is holomorphic in $\psi_A(E)$.

8.4.4 Definition of θ_S

Define $\theta_S = \phi_S^{-1} \circ \psi_S$ for every annular piece S. If S is a complex piece, define

$$\theta_S = \begin{cases} \phi_S^{-1} \circ \psi_A & \text{on every annular component } A \text{ of } S \smallsetminus \mathcal{E}^c; \\ \phi_S^{-1} \circ \psi_S & \text{otherwise.} \end{cases}$$

Then $\theta_S|_{\partial S} = \text{id}$ and θ_S is isotopic to the identity rel $\partial S \cup (S \cap \mathcal{P})$. Moreover, for every $\mathcal{E}^a \cup \mathcal{E}^c$-piece $E \subset S$ and for $S_0 = F(E)$, the map $\phi_{S_0} \circ F \circ \theta_S^{-1} \circ \phi_S^{-1}$ is holomorphic in $\phi_S \theta_S(E)$.

Now if $\mathcal{E}^o = \emptyset$, the proof of Theorem 5.4 is already completed. Otherwise, as in Sect. 6.3, we can define a new marked repelling system $G : (\mathcal{B}, Q) \to (\mathcal{M}, Q)$ c-equivalent to (F, P) and holomorphic everywhere except on the $\mathcal{B}$-pieces of disc type. Now applying Lemma 6.3, we see that (G, Q), and hence (F, P), is c-equivalent to a holomorphic marked repelling system. This ends the proof of Theorem 5.4.

9 A combination result

A **regular open set** is the complement of a surface puzzle in $\overline{\mathbb{C}}$. Let U, V be regular open sets in $\overline{\mathbb{C}}$ with $V \Subset U$. Let $G : U \to V$ be a quasiregular branched covering. We say that (G, U, V) is a **locally holomorphic attracting system** if there is a finite set $P' \subset U$ such that:

- $G(P') = P'$;
- G is holomorphic in a neighborhood of P' and each cycle in P' is (super)attracting;
- for any $z \in V$ the limit set of $\{G^n(z)\}$ is contained in P'.

Let $F:\mathcal{E}\to\mathcal{L}$ be a quasiregular repelling system of constant complexity, in the sense that for every $\mathcal{L}$-piece S with

$$\#(S\cap P_F)+\#\{\text{boundary components of } S\}\geq 3,$$

there is an $\mathcal{E}$-piece $E_S\subset S$ such that $E_S\cap P_F=S\cap P_F$ and the interior of each component of $S\setminus E_S$ is either a disc or an annulus.

We say that $H:E\to S$ is a renormalization of F, if S is a complex $\mathcal{L}$-piece, E is a component of $F^{-p}(S)$ (for some integer $p\geq 1$) that is contained S and is parallel to S, and $H=F^p|_E$. Note that H needs not to be postcritically finite.

We say that F is **unobstructed** if it has no boundary obstruction and, for each renormalization $H:E\to S$, either
(1) $\#(P_F\cap S)<\infty$ and $(H, P_F\cap S)$ as a marked repelling system has no Thurston obstructions; or
(2) for the integer $p\geq 1$ such that $H=F^p|_E$, each step of the composition

$$E\xrightarrow{F}F(E)\xrightarrow{F}F^2(E)\xrightarrow{F}\cdots\xrightarrow{F}F^{p-1}(E)\xrightarrow{F}S$$

is holomorphic.

What we have proved in this paper can be reformulated in the following stronger form:

Theorem 9.1 *Let G be a quasiregular branched covering of $\overline{\mathbb{C}}$ of degree at least* 2. *Assume that $\overline{\mathbb{C}}=V\sqcup\mathcal{L}$ is a splitting with $\mathcal{L}$ a surface puzzle such that*:

(a) $G^{-1}(V)\Supset V$;
(b) $(G, G^{-1}(V), V)$ *is a locally holomorphic attracting system*;
(c) $G:G^{-1}(\mathcal{L})\to\mathcal{L}$ *is a repelling system of constant complexity and is unobstructed.*

Let K be the union of the filled-in Julia set K_H of all renormalizations H satisfying the condition (2). *Then there is a rational map g and a pair of qc-homeomorphisms ϕ,ψ of $\overline{\mathbb{C}}$ such that*

- $\phi\circ G=g\circ\psi$;
- ψ *is isotopic to ϕ rel $P_G\cup K$*;
- *the Beltrami coefficient of ϕ is equal to* 0 *almost everywhere on K.*

Appendix A: Non-negative matrices

We say that a square matrix A is *non-negative* if every entry of it is a non-negative real number. By Perron-Frobenius theory (refer to [15]), the spectral radius of A is an eigenvalue of A, named the **leading eigenvalue**. Moreover this eigenvalue has a non-negative eigenvector. Clearly, the norm of any other eigenvalue of A is at most equal to the leading eigenvalue.

For a vector $v = (v_i) \in \mathbb{R}^n$ we write $v > 0$ if every coordinate v_i is strictly positive.

Lemma A.1 *Let $A = (a_{ij})$ be a non-negative square matrix. Denote by λ its leading eigenvalue. Then $\lambda < 1$ iff there is a vector $v > 0$ such that $Av < v$.*

Proof The following proof is provided by H.H. Rugh. Necessity: Assume $v > 0$ and $Av < v$. Then $Av \leq av$ for some $0 \leq a < 1$. Define a norm on the underlying vector space by $\|x\| = \sum_i (v_i \cdot |x_i|)$. Then, writing $|x|$ as the vector whose i-th entry is $|x_i|$, we have

$$\|A^t x\| = v^t A^t |x| = (Av)^t |x| \leq a v^t |x| = a\|x\|,$$

where A^t and v^t denote the transposes. Therefore, $\lambda = \|A^t\| \leq a < 1$.

Sufficiency: Now assume $\lambda < 1$. By the continuity of the spectral radius, there is $\epsilon > 0$ such that the spectral radius λ_ϵ of $A + \epsilon := (a_{ij} + \epsilon)$ satisfies $\lambda_\epsilon < 1$. Now the Perron-Frobenius Theorem assures that λ_ϵ is the leading eigenvalue and it has a strictly positive eigenvector $v > 0$. So $Av \leq (A + \epsilon)v = \lambda_\epsilon v < v$. □

Lemma A.1 actually gives an equivalent definition of the leading eigenvalue.

Corollary A.2 *Let $\lambda(A)$ be the leading eigenvalue of a non-negative square matrix A. Then*

$$\lambda(A) = \inf\{\lambda | \ \exists\ v > 0 \textit{ such that } Av < \lambda v\}.$$

Corollary A.3 *Assume that A and B are non-negative square matrices with $A \leq B$ (i.e. every entry of A is less than or equal to the corresponding entry of B), then $\lambda(A) \leq \lambda(B)$.*

Proof From Lemma A.1, we see that for any $\lambda_0 > \lambda(B)$, there is a vector $v > 0$ so that $Bv < \lambda_0 v$. Thus $Av \leq Bv < \lambda_0 v$. Again by Lemma A.1, we have $\lambda(A) < \lambda_0$. So $\lambda(A) \leq \lambda(B)$. □

Theorem A.4 *Let A be a non-negative $n \times n$ matrix with a block decomposition*

$$A = \begin{pmatrix} B & * \\ O_1 & O_2 \end{pmatrix}$$

where B is a square $k \times k$ matrix, and O_1 and O_2 are zero-matrices of appropriate sizes. Then $\lambda(A) = \lambda(B)$.

Proof Either $k = n$ or the set of eigenvalues of A is equal to the set of eigenvalues of B union $\{0\}$. The theorem follows. □

Let A be an $n \times n$ matrix with a block decomposition

$$\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

where A_{ij} is an $n_i \times n_j$ matrix (in particular each A_{ii} is a square matrix). We say that the block decomposition is **projected** if for each A_{ij}, there is a number b_{ij} such that the summation of each column of A_{ij} is equal to b_{ij}.

This property could be understood as the following: An $n \times n$ matrix can be considered as a linear map of $\mathbb{R}^n$ defined by the left action:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

According to the block decomposition of A, there is a corresponding decomposition of the index set $I = \{1, \dots, n\}$ by $I = I_1 \sqcup \cdots \sqcup I_k$ with $\#I_i = n_i$. Define a linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ by $(\pi v)_i = \sum_{\delta \in I_i} v_\delta$.

Lemma A.5 *There is a $k \times k$ matrix B such that $\pi \circ A = B \circ \pi$ if and only if the block decomposition $A = (A_{ij})$ is projected. In this case, $B = (b_{ij})$.*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R}^k & \xrightarrow[B]{} & \mathbb{R}^k \end{array}$$

Proof Set $A = (a_{\delta\beta})$. For any $v \in \mathbb{R}^n$,

$$(\pi \circ Av)_i = \sum_j \sum_{\beta \in I_j} \left(\sum_{\delta \in I_i} a_{\delta\beta} \right) v_\beta, \quad \text{and} \quad (B \circ \pi(v))_i = \sum_j \sum_{\beta \in I_j} b_{ij} v_\beta \ .$$

If the block decomposition is projected, then for $\beta \in I_j$, $\sum_{\delta \in I_i} a_{\delta\beta} = b_{ij}$. Therefore $\pi \circ Av = B \circ \pi(v)$. Conversely, assume that $\pi \circ A = B \circ \pi$. For

$\beta \in I_j$, let $e_\beta \in \mathbb{R}^n$ be a vector whose βth entry is 1 and whose other entries are all 0. Then $(\pi \circ Ae_\beta)_i = b_{ij}$, and $(B \circ \pi(e_\beta))_i = \sum_{\delta \in I_i} a_{\delta\beta}$. So for $\beta \in I_j$, $\sum_{\delta \in I_i} a_{\delta\beta} = b_{ij}$, i.e. the block decomposition is projected. □

Theorem A.6 *Assume that A is a non-negative square matrix with a projected block decomposition $A = (A_{ij})$, i.e. the summation of each column of A_{ij} is equal to b_{ij}. Set $B = (b_{ij})$. Then $\lambda(A) = \lambda(B)$.*

Proof Let $v \neq 0$ be an eigenvector of A for the leading eigenvalue $\lambda(A)$, i.e. $Av = \lambda(A)v$. Set $u = \pi(v)$. Then $Bu = \pi \circ Av = \pi(\lambda(A)v) = \lambda(A)\pi(v) = \lambda(A)u$ by the above lemma. So $\lambda(A)$ is an eigenvalue of B and hence $\lambda(A) \leq \lambda(B)$ since the leading eigenvalue is the maximum of the eigenvalues.

Conversely, let $u \neq 0$ be an eigenvector of B^t, the transpose of B, for the leading eigenvalue $\lambda(B)$ (note that B and B^t have the same leading eigenvalue), i.e. $B^t u = \lambda(B)u$. Set $v = (v_\beta) \in \mathbb{R}^n$ by $v_\beta := u_j$ for $\beta \in I_j$. Then for $\delta \in I_i$,

$$(A^t v)_\delta = \sum_j \sum_{\beta \in I_j} a_{\beta\delta} v_\beta = \sum_j b_{ji} v_j = (B^t u)_i = \lambda(B) u_i = \lambda(B) v_\delta.$$

So $\lambda(B)$ is an eigenvalue of A^t. We again have $\lambda(B) \leq \lambda(A)$. □

Corollary A.7 *Let A' be a non-negative square matrix with a block decomposition (A'_{ij}). Assume that the summation of each column of A'_{ij} is less than or equal to b_{ij}. Set $B = (b_{ij})$. Then $\lambda(A') \leq \lambda(B)$.*

Proof For each pair (i, j), we just need to replace one entry of each column of A'_{ij} by a larger number so that the summation of the column becomes exactly b_{ij}. Denote by A_{ij} the modified matrix. Set $A = (A_{ij})$. Then $\lambda(A) = \lambda(B)$ by Theorem A.6 and $\lambda(A') \leq \lambda(A)$ by Corollary A.3. □

Appendix B: Reversing Grötzsch's inequality

Let $\Delta \subset \overline{\mathbb{C}}$ be a hyperbolic disc with a marked point $a \in \Delta$. Then there is a conformal homeomorphism $\varphi : \Delta \to \{z : |z| < r\}$ for some $r > 0$, with $\varphi(a) = 0$ and $\varphi'(a) = 1$ if $a \in \mathbb{C}$ or $(\varphi(1/w))'|_{w=0} = 1$ if $a = \infty$. The **conformal radius** of the marked disc (Δ, a) is defined to be the radius r. Recall that an **equipotential** γ in the marked disc (Δ, a) is a curve mapped onto a round circle with center 0 under the conformal map φ, and the **potential** of γ is defined to be the modulus of the annulus between $\partial\Delta$ and γ (we define $\mathrm{mod}(\{z : t < |z| < 1\}) = |\log t|$).

Lemma B.1 *Let* (Δ_i, a_i), $i=1,2$ *be a pair of disjoint marked hyperbolic discs. Then there is a constant* $C>0$ *which depends only on the conformal radii of the marked discs* (Δ_i, a_i), *such that for any* $v_1>0, v_2>0$,

$$v_1+v_2 \le \mathrm{mod}(A(v_1,v_2)) \le v_1+v_2+C,$$

where $A(v_1,v_2)$ *is the annulus bounded by the equipotential in* Δ_1 *of potential* v_1 *and the equipotential in* Δ_2 *of potential* v_2.

Proof The left hand side is just Grötzsch's inequality.

Let ξ be a Möbius transformation of $\overline{\mathbb{C}}$ with $\xi(a_1)=0$ and $\xi(a_2)=\infty$. Any two such maps differ by a multiplicative constant. So the product of the conformal radius R_1 of $(\xi(\Delta_1),0)$ and the conformal radius R_2 of $(\xi(\Delta_2),\infty)$ is equal to the product of the conformal radii of (Δ_1,a_1) and (Δ_2,a_2). Denote by W_i the component of $\overline{\mathbb{C}}\smallsetminus A(v_1,v_2)$ containing a_i $(i=1,2)$. By Koebe's 1/4-Theorem, $\xi(W_1)$ contains $\{z:|z|\le R_1r_1/4\}$ and $\xi(W_2)$ contains $\{z:|z|\ge 4/(R_2r_2)\}$, where $r_i=\exp(-v_i)$. Therefore by Grötzsch's inequality,

$$\mathrm{mod}(A(v_1,v_2)) \le \log\left(\frac{4}{R_2r_2}\cdot\frac{4}{R_1r_1}\right) = \log\frac{16}{R_1R_2}+v_1+v_2. \qquad \square$$

Appendix C: Quasiconformal extensions

We state here several results about quasiconformal maps that have been frequently used in the paper. For the general theory, we refer to [1].

Lemma C.1 *Let* $h:\gamma_1\to\gamma_2$ *be a homeomorphism between two quasicircles* γ_1 *and* γ_2 *in* $\overline{\mathbb{C}}$. *If* h *can be extended to a quasiconformal map on a one-side neighborhood of* γ_1, *then* h *can be extended to a global quasiconformal map of* $\overline{\mathbb{C}}$. *Moreover the extension can be chosen to be a diffeomorphism from* $\overline{\mathbb{C}}\smallsetminus\gamma_1$ *onto* $\overline{\mathbb{C}}\smallsetminus\gamma_2$.

Lemma C.2 *Let* $U_i\subset\overline{\mathbb{C}}$ $(i=1,2)$ *be a pair of domains such that each of* $\partial U_i, i=1,2$ *consists of* $p\ge 0$ *disjoint quasicircles. Let* $P\subset U_1$ *be a finite* (*or empty*) *set. Let* $f:\overline{U_1}\to\overline{U_2}$ *be an orientation preserving homeomorphism. If* $f|_{\partial U_1}$ *can be extended to a quasiconformal map on a one-side neighborhood of each curve of* ∂U_1 (*or* $p=0$), *then there is a quasiconformal map in the isotopy class of* f *rel* $\partial U_1\cup P$.

Lemma C.3 *Let* $h:S^1\to S^1$ *be an orientation preserving homeomorphism of the unit circle. Assume that* h *can be extended to a quasiconformal map* f *on an inner neighborhood* B *of* S^1 (*i.e.* $B\supset\{z:1-\varepsilon<|z|<1\}$ *for some* $\varepsilon>0$), *then* h *is quasisymmetric*.

Proof Denote by μ the Beltrami coefficient of f. Denote by $\mathbb{D}$ the unit disc. Let $\nu = \mu$ on B and $\nu = 0$ on $\mathbb{D}\setminus B$. By the Measurable Riemann Mapping Theorem, there is a quasiconformal map g of $\mathbb{D}$ whose Beltrami differential is ν. So $g|_{S^1}$ is quasisymmetric. On the other hand, $f \circ g^{-1}$ is holomorphic on $g(B)$. Therefore $f \circ g^{-1}$ is real-analytic on S^1, in particular quasisymmetric. So $h = (f \circ g^{-1}) \circ g|_{S^1}$ is also quasisymmetric. □

Proof of Lemma C.1 Fix $i = 1, 2$. By the definition of quasicircles, there is a quasiconformal map ϕ_i of $\overline{\mathbb{C}}$ such that $\phi_i(\gamma_i) = S^1$. Furthermore ϕ_i can be chosen to be a diffeomorphism on $\overline{\mathbb{C}}\setminus\gamma_i$ as follows: Set $\Delta = \phi_i^{-1}(\mathbb{D})$. Let $\psi : \Delta \to \mathbb{D}$ be a conformal map. Then $\phi_i \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$ is a quasiconformal map. Thus its boundary map is quasisymmetric. Let η be the Beurling-Ahlfors extension of this boundary map, it is a diffeomorphism of $\mathbb{D}$. Now $\eta \circ \psi|_\Delta$ is again a diffeomorphism, whose boundary map is $\phi_i|_{S^1}$.

Set $h_1 = \phi_2 \circ h \circ \phi_1^{-1}$. Then h_1 is quasisymmetric by Lemma C.3, and thus having a quasiconformal extension to $\overline{\mathbb{C}}$. Moreover its extension can be chosen to be a diffeomorphism outside S^1. Thus $h = \phi_2^{-1} \circ h_1 \circ \phi_1$ can be extended to a quasiconformal map of $\overline{\mathbb{C}}$ and a diffeomorphism outside γ_1. □

Proof of Lemma C.2 By Lemma C.1, $f|_{\partial U_1}$ can be extended to a quasiconformal map g on a small neighborhood W of ∂U_1 such that g is differentiable on $W\setminus\partial U_1$. Let $V_1 \subset U_1$ be a domain such that its boundary consists of disjoint smooth quasicircles in W and $U\setminus V \subset W$. Then $g|_{\partial V_1}$ is a diffeomorphism and hence can be extended to a diffeomorphism such that $g|_{U_1}$ is isotopic to f rel $\partial U_1 \cup P$. □

Appendix D: A lemma about isotopy

Let $P \subset \overline{\mathbb{C}}$ be a finite set. We say U is a *disc neighborhood* of P if U is the union of Jordan domains which have disjoint closures and each of them contains exactly one point of P.

Lemma D.1 *Let $P \subset P_0 \subset \overline{\mathbb{C}}$ be two finite sets. Let $E \subset \overline{\mathbb{C}}$ be a compact set disjoint from P. Assume that $h : I \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ ($I = [0, 1]$) is an isotopy rel P_0 with $h(0, \cdot) = \mathrm{id}$, i.e. h is continuous and $h(t, \cdot)$ is a homeomorphism of $\overline{\mathbb{C}}$ with every point in P_0 fixed for all $t \in I$. Then there is an isotopy (rel P_0) $H : I \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $H(0, \cdot) = id$ such that:*

(1) *$H(\cdot, z) = h(\cdot, z)$ for $z \in E$; and*
(2) *there is a disc neighborhood U of P such that $H(t, \cdot) = id$ on U for all $t \in I$.*

Proof Pick a disc neighborhood V of P such that its closure is disjoint from E. Set $K = \overline{\mathbb{C}} \smallsetminus V$. The compact set $h(I \times K)$ is disjoint from P. Thus the spherical distance $d(h(I \times K), P) =: \epsilon$ is strictly positive. Let U be another disc neighborhood of P such that the radius of each disc is less than $\epsilon/2$. Then $U \Subset h(t, V)$ for all $t \in I$. In particular, $d(h(I \times \partial V), U) > \epsilon/2$. Set

$$H(\cdot, z) = \begin{cases} h(\cdot, z) & \text{for } z \in \overline{\mathbb{C}} \smallsetminus V, \\ \text{interpolation} & \text{for } z \in V \smallsetminus U, \\ \text{id} & \text{for } z \in U. \end{cases}$$

It satisfies (1) and (2). □

Corollary D.2 *With the assumption as in Lemma D.1, suppose furthermore that W is a disc neighborhood of P such that both E and $h(1, E)$ are disjoint from the closure of W. Then there is an isotopy (rel P_0) $H : I \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with* $H(0, \cdot) = \text{id}$ *such that*:

(a) $H(1, \cdot) = h(1, \cdot)$ *on* E; *and*
(b) $H(t, \cdot) = id$ *on* W *for all* $t \in I$.

Proof By Lemma D.1, there is a disc neighborhood V of P with $V \Subset W$ such that $H(\cdot, z) = h(\cdot, z)$ for $z \in E$ and $H(t, \cdot) = \text{id}$ on V. Pick another disc neighborhood U_0 of P such that $W \Subset U_0$ and both E and $h(1, E)$ are disjoint from U_0. Then there is a homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\phi(V) = W$ and $\phi = \text{id}$ on $P_0 \cup (\overline{\mathbb{C}} \smallsetminus U_0)$. In particular $\phi = \text{id}$ on $E \cup h(1, E)$. The isotopy $\phi \circ H_t \circ \phi^{-1}$ (where $H_t = H(t, \cdot)$) satisfies the properties (a) and (b). □

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