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Geometric invariant theory and the generalized eigenvalue problem

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Abstract Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori and Borel subgroups of G and $\hat{G}$. Consider the cone $\mathcal{LR}(G, \hat{G})$ generated by the pairs $(\nu, \hat{\nu})$ of dominant characters such that V_ν^* is a submodule of $V_{\hat{\nu}}$ (with usual notation). Here we give a minimal set of inequalities describing $\mathcal{LR}(G, \hat{G})$ as a part of the dominant chamber. In other words, we describe the facets of $\mathcal{LR}(G, \hat{G})$ which intersect the interior of the dominant chamber. We also describe smaller faces. Finally, we are interested in some classical redundant inequalities.

Along the way, we obtain results about the faces of the Dolgachev-Hu G-ample cone and variations of this cone.

Contents

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1 Introduction

1.1 The branching cone

In this article, we are mainly interested in the faces of the G-ample cone as defined by Dolgachev-Hu in [9]. In this introduction, we firstly explain the applications to the generalized Horn problem.

Let G be a connected reductive subgroup of a connected reductive group $\hat{G}$ both defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. We consider the following question:

What irreducible representations of G appear in a given irreducible representation of $\hat{G}$?

Once maximal tori ($T \subset \hat{T}$) and Borel subgroups ($B \supset T$ and $\hat{B} \supset \hat{T}$) are fixed, the question is to understand the set $\mathrm{LR}(G, \hat{G})$ of dominant characters $(\nu, \hat{\nu})$ of $T \times \hat{T}$ such that the dual of the G-module associated to ν can be G-equivariantly embedded in the $\hat{G}$-module associated to $\hat{\nu}$. By a result of M. Brion and F. Knop (see [10]), $\mathrm{LR}(G, \hat{G})$ is a finitely generated submonoid of the character group of $T \times \hat{T}$. Our purpose is to study the linear inequalities satisfied by this monoid. More precisely, we consider the convex cone $\mathcal{LR}(G, \hat{G})$ generated by $\mathrm{LR}(G, \hat{G})$: it is a closed rational polyhedral cone. We are interested in the faces of $\mathcal{LR}(G, \hat{G})$.

1.2 Spectral interpretation

In this subsection, we work with complex numbers. Let K (resp. $\hat{K}$) be a maximal compact subgroup of G (resp. $\hat{G}$) such that $K \subset \hat{K}$. Let $\mathfrak{k}$ and $\hat{\mathfrak{k}}$ denote the Lie algebras of K and $\hat{K}$. Consider the restriction map $p : \hat{\mathfrak{k}}^* \longrightarrow \mathfrak{k}^*$ from the dual of $\hat{\mathfrak{k}}$ to that of $\mathfrak{k}$. We are interested in the projections of coadjoint orbits of $\hat{\mathfrak{k}}^*$ in $\mathfrak{k}^*$.

Up to changing T, we may assume that $H = K \cap T$ is a Cartan subgroup of K. Consider the Lie algebra $\mathfrak{h}$ of H. Via the Cartan-Killing form, we embed the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ in the dual $\mathfrak{k}^*$ of $\mathfrak{k}$. Let us recall that the map $\mathfrak{h}^*_+ \longmapsto \mathfrak{k}^*/K$, $\xi \mapsto K.\xi$ is a homeomorphism, where $\mathfrak{h}^*_+$ denotes the dominant Weyl chamber of $\mathfrak{h}^*$. We use similar notation for $\hat{G}$ and $\hat{K}$.

If $\hat{\xi} \in \hat{\mathfrak{k}}^*$ then the K-orbits in $p(\hat{K}\hat{\xi})$ are parameterized by $p(\hat{K}\hat{\xi}) \cap \mathfrak{h}^*_+$. As it was pointed out by Guillemin-Sternberg [13], Heckman's work [14] (see also [20]) implies that the closure of $\mathcal{LR}(G, \hat{G})$ in $\mathfrak{h}^* \times \hat{\mathfrak{h}}^*$ is the set of pairs $(\xi, \hat{\xi}) \in \mathfrak{h}^*_+ \times \hat{\mathfrak{h}}^*_+$ such that $-w_0\xi \in p(\hat{K}\hat{\xi})$. Here w_0 is the longest element of the Weyl group of G.

In the case of the diagonal embedding of GL_n in $\mathrm{GL}_n \times \mathrm{GL}_n$, the problem to describe $\mathcal{LR}(G, \hat{G})$ can be reformulated as follows: what can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands? In 1912, H. Weyl [44] obtained the first nontrivial inequalities for this question. In 1962, Horn [17] gave a conjectural recursive description of a complete list of inequalities to characterize $\mathcal{LR}(\mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_n)$. This cone will be called the Horn cone. In 1998, A. Klyachko [23] made an important step in solving the Horn conjecture: he found a complete list of inequalities which characterize the Horn cone. In 1999, Knutson-Tao [25] shown the saturation conjecture; this implies that the inequalities given by the Horn conjecture and the Klyachko theorem coincide, and so, ends the proof of the Horn conjecture.

In [23], the problem of describing the Horn cone is interpreted in terms of semistability for toric vector bundles on $\mathbb{P}^2$. Following Klyachko, we study here the cones $\mathcal{LR}(G, \hat{G})$ using semistability in Geometric Invariant Theory as in [3, 4].

1.3 A characterization of $\mathcal{LR}(G, \hat{G})$

We assume, from now on that no nonzero ideal of the Lie algebra of G is an ideal of that of $\hat{G}$. In Proposition 12, using a result of [32], we will prove that this assumption implies that the interior of $\mathcal{LR}(G, \hat{G})$ is nonempty.

In [4], Berenstein-Sjamaar gave a list of inequalities which characterizes $\mathcal{LR}(G, \hat{G})$. In [3], Belkale-Kumar obtained a smaller list which also characterizes this cone in the case when G is diagonally embedded in $\hat{G} = G^s$. This case corresponds to the problem of decomposition of the tensor product of s irreducible representations of G. Our first result is a generalization of the Belkale-Kumar result in the Berenstein-Sjamaar setting.

To make our statements precise, we introduce notation. Consider the natural pairing $\langle \cdot, \cdot \rangle$ between the one parameter subgroups and the characters of tori T or $\hat{T}$. Let W (resp. $\hat{W}$) denote the Weyl group of T (resp. $\hat{T}$). If λ is a one parameter subgroup of T (or so of $\hat{T}$), we denote by W_λ (resp. $\hat{W}_\lambda$) the stabilizer of λ for the natural action of the Weyl group. For $w \in W/W_\lambda$ and $\hat{w} \in \hat{W}/\hat{W}_\lambda$, we consider the following linear form on the character group $X(T \times \hat{T})$ of $T \times \hat{T}$:

$$\varphi_{w,\hat{w},\lambda} : (\nu, \hat{\nu}) \mapsto \langle \hat{w}\lambda, \hat{\nu} \rangle + \langle w\lambda, \nu \rangle.$$

In fact, all the inequalities in Theorem A below will have the following form $\varphi_{w,\hat{w},\lambda}(\nu, \hat{\nu}) \geq 0$. We need some notation to explain which triples $(w, \hat{w}, \lambda)$ will appear.

Let $P(\lambda)$ denote the usual parabolic subgroup of G associated to λ (see Sect. 2.4). The cohomology group $\mathrm{H}^*(G/P(\lambda), \mathbb{Z})$ is freely generated by the

Schubert classes σ_w parameterized by the elements $w \in W/W_\lambda$. We will consider $\hat{G}/\hat{P}(\lambda)$, $\sigma_{\hat{w}}$ as above but with $\hat{G}$ in place of G. Consider also the canonical G-equivariant immersion $\iota : G/P(\lambda) \longrightarrow \hat{G}/\hat{P}(\lambda)$; and the corresponding morphism ι^* in cohomology.

Let $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and $\hat{G}$. Let ρ, $\hat{\rho}$ and $\hat{\rho}^\lambda$ be the half-sum of positive roots of G, $\hat{G}$ and of the centralizer $\hat{G}^\lambda$ of λ in $\hat{G}$ respectively. Consider the set $\mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$ of nontrivial weights of T in $\hat{\mathfrak{g}}/\mathfrak{g}$. Let $X(T) \otimes \mathbb{Q}$ denote the rational vector space generated by the characters of T. We consider the set of hyperplanes H of $X(T) \otimes \mathbb{Q}$ spanned by some elements of $\mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$. For each such hyperplane H there exist exactly two opposite indivisible one parameter subgroups $\pm\lambda_H$ which are orthogonal (for the paring $\langle\cdot,\cdot\rangle$) to H. The so obtained one parameter subgroups form a set stable by the action of W. Let $\{\lambda_1, \dots, \lambda_n\}$ be the set of dominant such one parameter subgroups.

We can now give our list of inequalities:

Theorem A *We assume that no nonzero ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Then, $\mathcal{LR}(G, \hat{G})$ has nonempty interior in $X(T \times \hat{T}) \otimes \mathbb{Q}$.*

A dominant weight $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if for any $i = 1, \dots, n$ and for any pair of Schubert classes $(\sigma_w, \sigma_{\hat{w}})$ of $G/P(\lambda_i)$ and $\hat{G}/\hat{P}(\lambda_i)$ associated to $(w, \hat{w}) \in W/W_{\lambda_i} \times \hat{W}/\hat{W}_{\lambda_i}$ such that

(i) $\iota^*(\sigma_{\hat{w}}) \cdot \sigma_w = \sigma_e \in \mathrm{H}^*(G/P(\lambda_i), \mathbb{Z})$, *and*
(ii) $\langle w\lambda_i, \rho\rangle + \langle \hat{w}\lambda_i, \hat{\rho}\rangle = \langle \lambda_i, \rho\rangle + \langle \lambda_i, 2\hat{\rho}^{\lambda_i} - \hat{\rho}\rangle$,

we have

$$\varphi_{w,\hat{w},\lambda_i}(\nu, \hat{\nu}) = \langle w\lambda_i, \nu\rangle + \langle \hat{w}\lambda_i, \hat{\nu}\rangle \geq 0. \tag{1}$$

In [4], Berenstein-Sjamaar showed that $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if $\varphi_{w,\hat{w},\lambda}(\nu, \hat{\nu}) \geq 0$ for a finite list (including $\lambda_1, \dots, \lambda_n$) of one parameter subgroups λ and for any pair of Schubert classes $(\sigma_w, \sigma_{\hat{w}})$ such that $\iota^*(\sigma_{\hat{w}}) \cdot \sigma_w = d.\sigma_e$ for some positive integer d. In the case when G is diagonally embedded in G^s, Kapovich-Leeb-Millson proved that one may assume that $d = 1$. Again in the case when G is diagonally embedded in G^s, Belkale-Kumar obtained in [3] the same inequalities as in Theorem A.

1.4 Irredundancy

Whereas the Berenstein-Sjamaar list is redundant, our list is proved to be irredundant. In some sense, our main result asserts that Theorem A is optimal:

Theorem B *In Theorem A, Inequalities (1) are pairwise distinct and no one can be omitted.*

This result was known is some particular cases. Indeed, Knutson, Tao and Woodward showed in [26] the case when $G = \mathrm{SL}_n$ is diagonally embedded in $\mathrm{SL}_n \times \mathrm{SL}_n$ using combinatorial methods. Using the interpretation of the Littlewood-Richardson coefficients as structure coefficients of the cohomology rings of the Grassmann varieties, Belkale made a geometric proof of the Knutson-Tao-Woodward result (see [2]). Using explicit calculation with the help of a computer, Kapovich, Kumar and Millson proved the case when $G = \mathrm{SO}(8)$ is diagonally embedded in $G \times G$ in [18]. Our proof is different and uses Geometric Invariant Theory.

1.5 Other classical inequalities

Let λ be a one parameter subgroup of G. Let $(\sigma_w, \sigma_{\hat{w}})$ be a pair of Schubert classes of $G/P(\lambda)$ and $\hat{G}/\hat{P}(\lambda)$ such that $\iota^*(\sigma_{\hat{w}}) \cdot \sigma_w \neq 0 \in \mathrm{H}^*(G/P(\lambda), \mathbb{Z})$. The points $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ satisfy $\varphi_{w,\hat{w},\lambda}(\nu, \hat{\nu}) \geq 0$. The inequalities given by the Horn conjecture or by the Berenstein-Sjamaar theorem have this form. For simplicity, we now assume that $\lambda = \lambda_i$ is a one parameter subgroup which appears in Theorem A. Consider the face $\mathcal{F}(w, \hat{w}, \lambda_i)$ of $\mathcal{LR}(G, \hat{G})$ associated to this inequality, that is the set of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\varphi_{w,\hat{w},\lambda_i}(\nu, \hat{\nu}) = 0$. Theorem B shows that if Conditions (i) and (ii) of Theorem A are fulfilled, then $\mathcal{F}(w, \hat{w}, \lambda_i)$ is a facet, else, $\mathcal{F}(w, \hat{w}, \lambda_i)$ is smaller. In fact, we obtain that the extra inequalities are redundant in a stronger way:

Theorem C *Let us fix a λ_i. Let $(\sigma_w, \sigma_{\hat{w}})$ be a pair of Schubert varieties of $G/P(\lambda_i)$ and $\hat{G}/\hat{P}(\lambda_i)$ such that $\iota^*(\sigma_{\hat{w}}) \cdot \sigma_w \neq 0 \in \mathrm{H}^*(G/P(\lambda_i), \mathbb{Z})$. Then,*

(i) *either, $\mathcal{F}(w, \hat{w}, \lambda_i)$ has codimension one,*
(ii) *or, $\mathcal{F}(w, \hat{w}, \lambda_i)$ contains no point $(\nu, \hat{\nu})$ with ν or $\hat{\nu}$ strictly dominant.*

For the Horn cone, this result is due to Knutson-Woodward (see [11, Proposition 10]). Note that the proof of Theorem C explains directly and geometrically why a given redundant inequality is redundant; instead of showing that other inequalities are sufficient to characterize the cone. Note that Theorem 12 in Sect. 7 actually applies to any one parameter subgroup λ, and not only with some λ_i.

1.6 Smaller faces

Theorems A and B can be thought as a description of the facets of $\mathcal{LR}(G, \hat{G})$ which intersect the interior of the dominant chamber. In Theorem 11 below, we study the smaller faces of $\mathcal{LR}(G, \hat{G})$. To avoid some notation in the introduction, we only state our results about these faces in the case when G is diagonally embedded in G^s for some integer $s \geq 2$. Indeed, the beautiful

Belkale-Kumar cohomology product (see [3]) allows a pleasant statement in this case.

If α is a simple root of G, $\omega_{\alpha^\vee}$ denotes the corresponding fundamental one parameter subgroup. If I is a set of simple roots, $P(I)$ denotes the standard parabolic subgroup associated to I and W_I its Weyl group. In [3], Belkale-Kumar defined a new product $\odot_0$ on the cohomology groups $\mathrm{H}^*(G/P(I),\mathbb{Z})$.

Theorem D *We assume that G is semisimple and diagonally embedded in $\hat{G} = G^s$ for some integer $s \geq 2$.*

(i) *Let I be a set of d simple roots and $(w_0,\dots,w_s) \in (W/W_I)^{s+1}$ such that $\sigma_{w_0} \odot_0 \cdots \odot_0 \sigma_{w_s} = \sigma_e \in \mathrm{H}^*(G/P(I),\mathbb{Z})$. Then, the set of $(\nu_0,\dots,\nu_s) \in \mathcal{LR}(G, G^s)$ such that*

$$\forall \alpha \in I \quad \sum_i \langle \omega_{\alpha^\vee}, w_i^{-1}\nu_i \rangle = 0,$$

is a face of codimension d of $\mathcal{LR}(G, G^s)$.

(ii) *Any face of $\mathcal{LR}(G, G^s)$ intersecting the interior of the dominant chamber of G^{s+1} is obtained in this way.*

Even though there is a lot of literature on the facets, the smaller faces have not been studied in detail. Derksen-Weyman (see [8]) obtained results in the quiver setting which can be applied to describe all the faces of the Horn cone. Note that Theorem D gives an application of the Belkale-Kumar product $\odot_0$ for any G/P whereas in [3] only the case when P is maximal was used.

1.7 GIT-cones

The starting point of the proofs of Theorems A to D is the Borel-Weil theorem which gives an interpretation of the cone $\mathcal{LR}(G,\hat{G})$ in terms of Geometric Invariant Theory. This method essentially due to Klyachko for the Horn cone, was already used in [3, 4].

Consider the variety $X = G/B \times \hat{G}/\hat{B}$ endowed with the diagonal action of G. To any character $(\nu,\hat{\nu})$ of $T \times \hat{T}$, one associates a G-linearized line bundle $\mathcal{L}_{(\nu,\hat{\nu})}$ on X such that $\mathrm{H}^0(X, \mathcal{L}_{(\nu,\hat{\nu})}) = V_\nu^* \otimes V_{\hat{\nu}}^*$. Hence, $(\nu,\hat{\nu})$ belongs to $\mathcal{LR}(G,\hat{G})$ if and only if a positive power of $\mathcal{L}_{(\nu,\hat{\nu})}$ admits a nonzero G-invariant section. If $\mathcal{L}_{(\nu,\hat{\nu})}$ is ample this means that X admits semistable points for $\mathcal{L}_{(\nu,\hat{\nu})}$. So, one can use classical results of Geometric Invariant Theory such as the Hilbert-Mumford theorem.

We obtain results in the following more general context. Consider a connected reductive group G acting on a normal projective variety X. To any G-linearized line bundle $\mathcal{L}$ on X we associate the following open subset $X^{\mathrm{ss}}(\mathcal{L})$

of X:

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \tau \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \tau(x) \neq 0\}.$$

The points of $X^{\mathrm{ss}}(\mathcal{L})$ are said to be *semistable* for $\mathcal{L}$. Note that if $\mathcal{L}$ is not ample, this notion of semistability is not the standard one. In particular, the quotient $\pi_{\mathcal{L}} : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$ is a good quotient, if $\mathcal{L}$ is ample; but not in general. In this context, we ask for:

$$\text{What are the } \mathcal{L}\text{'s such that } X^{\mathrm{ss}}(\mathcal{L}) \neq \emptyset?$$

Let us fix a free finitely generated subgroup Λ of the group $\mathrm{Pic}^G(X)$ of G-linearized line bundles on X. Let $\Lambda_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space containing Λ as a lattice. Consider the convex cones $\mathcal{TC}^G_\Lambda(X)$ (resp. $\mathcal{AC}^G_\Lambda(X)$) generated in $\Lambda_{\mathbb{Q}}$ by the $\mathcal{L}$'s (resp. the ample $\mathcal{L}$'s) in Λ which have nonzero G-invariant sections. By [9] (see also [39]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in the ample cone of $\Lambda_{\mathbb{Q}}$ (see Sect. 3.1 for a precise definition). The cones $\mathcal{AC}^G_\Lambda(X)$ and $\mathcal{TC}^G_\Lambda(X)$ (and also $\mathcal{SAC}^G_\Lambda(X)$ which will be defined in Sect. 3.1) will be called the *GIT-cones*. We are interested in the faces of GIT-cones.

Generalizing Levi-movability of [3], we are now going to define the notion of well covering pair. Let λ be a one parameter subgroup of G and C be an irreducible component of its fixed point set X^λ. Consider $C^+ = \{x \in X \mid \lim_{t\to 0} \lambda(t)x \in C\}$ and the natural G-equivariant map $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$ (see Sects. 2.5.1 and 3.2.2 for details). The pair (C, λ) is said to be *well covering* if η induces an isomorphism over an open subset of X intersecting C. The pair is said to be *dominant* if η is dominant.

When $X = (G/B)^3$, C^+ is a $P(\lambda)^3$-orbit corresponding to three Schubert cells X_1, X_2, X_3 in $G/P(\lambda)$. For $i = 1, 2, 3$, let σ_i denote the cycle class in the cohomology $\mathrm{H}^*(G/P(\lambda), \mathbb{Z})$ of the closure of X_i. For any point $x = (g_1B/B, g_2B/B, g_3B/B) \in X$ the fiber $\eta^{-1}(x)$ identifies naturally with the intersection $g_1^{-1}X_1 \cap g_2^{-1}X_2 \cap g_3^{-1}X_3$. By Kleiman's transversality theorem, η is birational if and only if $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 = \sigma_e$. The pair (C, λ) is well covering if in addition $(\sigma_1, \sigma_2, \sigma_3)$ is Levi-movable as defined in [3].

The use of the map η in the context of Geometric Invariant Theory is classical (see for example [21]). This map plays also a central role in Vakil's work about Schubert calculus (see [42]).

For $\mathcal{L} \in \mathrm{Pic}^G(X)$, we denote by $\mu^{\mathcal{L}}(C, \lambda)$ the integer giving the action of λ on the restriction of $\mathcal{L}$ to C. We now give a first description of $\mathcal{AC}^G_\Lambda(X)$ (see Proposition 4 and Theorem 3 for more general statements):

Theorem E *An ample G-linearized line bundle $\mathcal{L} \in \Lambda$ belongs to $\mathcal{AC}^G_\Lambda(X)$ if and only if for any well covering pair (C, λ) with a dominant one parameter subgroup λ of T we have $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.*

If, in addition $X = \hat{G}/\hat{B} \times Y$, for some normal projective G-variety Y, $\Lambda_{\mathbb{Q}} = \mathrm{Pic}^G(X) \otimes \mathbb{Q}$ and $\mathcal{AC}^G_\Lambda(X)$ has nonempty interior, then it is sufficient to keep the pairs (C, λ) such that C contains points with isotropy group of dimension one.

Theorem E is a generalization and clarification of methods used in [1, 3, 4, 9, 23]. The use of optimal destabilizing subgroup in eigenvalue problems actually originates in [1]; the use of the Kempf-Hesselink stratification to study how $X^{\mathrm{ss}}(\mathcal{L})$ depends on $\mathcal{L}$ originates in [9].

If (C, λ) is a dominant pair, then the set of $\mathcal{L} \in \mathcal{TC}^G_\Lambda(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ is a face of $\mathcal{TC}^G_\Lambda(X)$ denoted by $\mathcal{TF}(C, \lambda)$. Note that the centralizer G^λ of λ in G acts on C. To obtain Theorem B, one has to prove that if (C, λ) is a well covering pair as in Theorem E then the face $\mathcal{TF}(C, \lambda)$ is a facet. We prove this by induction using the following:

Theorem F *Let (C, λ) be a well covering pair. We assume that $\Lambda_{\mathbb{Q}} = \mathrm{Pic}^G(X) \otimes \mathbb{Q}$. Consider the linear map ρ induced by the restriction*:

$$\rho : \mathrm{Pic}^G(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}^{G^\lambda}(C) \otimes \mathbb{Q}.$$

Then, $\mathcal{TF}(C, \lambda)$ and the pullback by ρ of $\mathcal{TC}^{G^\lambda}_\Lambda(C)$ span the same subspace.

Theorem C is a consequence of the more general

Theorem G *We assume that $X = \hat{G}/\hat{B} \times Y$, for some normal projective G-variety Y. Let (C, λ) be a dominant pair and $\mathcal{L} \in \mathcal{TF}(C, \lambda)$ be ample. Consider the set of semistable points $C^{\mathrm{ss}}(\mathcal{L}, G^\lambda)$ for the action of G^λ on C.*

Then,

(i) *$X^{\mathrm{ss}}(\mathcal{L})/\!/G$ is isomorphic to $C^{\mathrm{ss}}(\mathcal{L}, G^\lambda)/\!/G^\lambda$; and,*
(ii) *(C, λ) is well covering.*

Theorem D is mainly a consequence of the following

Theorem H *Let $\mathcal{F}$ be a face of $\mathcal{AC}^G_\Lambda(X)$.*

Then, there exists a well covering pair (C, λ) such that $\mathcal{F} = \mathcal{TF}(C, \lambda) \cap \mathcal{AC}^G_\Lambda(X)$.

In this article, we are mainly interested in the cones $\mathcal{LR}(G, \hat{G})$ whereas there are other interesting GIT-cones. For example, if Y is any G-variety endowed with an ample G-linearized line bundle $\mathcal{L}$ the moment polytope $P(Y, \mathcal{L})$ is an affine section of the cone $\mathcal{TC}^G_\Lambda(G/B \times Y)$ for a well chosen Λ. These polytopes were studied from symplectic point of view (see [43]) or from an algebro-geometric point of view in [6, 31].

The Hilbert-Mumford numerical criterion and the Kempf theorem are recalled with some complements in Sect. 2. In Sects. 3 to 6, we prove our general results about GIT-cones. We apply them to the branching cones $\mathcal{LR}(G, \hat{G})$ in Sect. 7. One can move on Sect. 7 after Sects. 3, 4, 5 or 6 to obtain respectively Theorems A, B, D or C.

Convention The ground field $\mathbb{K}$ is assumed to be algebraically closed of characteristic zero. The notation introduced in the environments "**Notation**" are fixed for all the sequence of the article.

2 The Hilbert-Mumford numerical criterion

Notation Let $\mathbb{K}^*$ denote the multiplicative group of $\mathbb{K}$. If G is an affine algebraic group, $X(G)$ denotes the group of characters of G; that is, of algebraic group homomorphisms from G to $\mathbb{K}^*$. If G acts algebraically on a variety X, X is said to be a G*-variety*. If $x \in X$, we will denote by G_x its isotropy subgroup and by $G.x$ its orbit. As in [33], we denote by $\mathrm{Pic}^G(X)$ the group of G-linearized line bundles on X. If $\mathcal{L} \in \mathrm{Pic}^G(X)$, $\mathrm{H}^0(X, \mathcal{L})$ denotes the G-module of regular sections of $\mathcal{L}$.

We will use classical results about the Hilbert-Mumford numerical criterion. In this section, we present these results and give some complements. Let us fix a connected reductive group G and an irreducible projective algebraic G-variety X.

2.1 An ad hoc notion of semistability

Notation If $\mathcal{L}$ is a line bundle on X and x is a point in X, $\mathcal{L}_x$ denotes the fiber in $\mathcal{L}$ over x.

As in the introduction, for any G-linearized line bundle $\mathcal{L}$ on X, we consider the following set of *semistable points* for $\mathcal{L}$:

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \tau \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \tau(x) \neq 0\}.$$

To avoid confusion, we sometimes denote $X^{\mathrm{ss}}(\mathcal{L})$ by $X^{\mathrm{ss}}(\mathcal{L}, G)$. The subset $X^{\mathrm{ss}}(\mathcal{L})$ is open and G-stable. A point x in X is said to be *unstable* for $\mathcal{L}$ if it is not semistable for $\mathcal{L}$.

Remark Note that this definition of $X^{\mathrm{ss}}(\mathcal{L})$ is NOT standard. Indeed, it is usually agreed that the open subset defined by the nonvanishing of τ be affine. This property which is useful to construct a good quotient of $X^{\mathrm{ss}}(\mathcal{L})$ is automatic if $\mathcal{L}$ is ample but not in general; hence, our definition coincides with the usual one if $\mathcal{L}$ is ample.

If $\mathcal{L}$ is ample, there exists a categorical quotient:

$$\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G,$$

such that $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ is a projective variety and π is affine.

The following lemma is easy and well known.

Lemma 1 *Let $\mathcal{L}$ be a G-linearized line bundle on X and $x \in X$ be a semi-stable point.*

Then, the restriction of $\mathcal{L}$ to $G.x$ has finite order in $\mathrm{Pic}^G(G.x)$.

Proof Let us recall that for any $\mathcal{L} \in \mathrm{Pic}^G(G.x)$, the action of G_x on the fiber $\mathcal{L}_x$ determines a character $\mu^{\mathcal{L}}(x, G_x)$ of G_x. Moreover, the map $\mathrm{Pic}^G(G.x) \longrightarrow X(G_x)$, $\mathcal{L} \mapsto \mu^{\mathcal{L}}(x, G_x)$ is an injective homomorphism.

Now, let us assume that the character $\mu^{\mathcal{L}}(x, G_x)$ has infinite order. It remains to prove that x is unstable for $\mathcal{L}$. Let τ be a G-invariant section of $\mathcal{L}^{\otimes n}$ for some $n > 0$. Then $\tau(x)$ is fixed by G_x and belongs to $\mathcal{L}_x^{\otimes n}$. Since, $n.\mu^{\mathcal{L}}(x, G_x)$ is nontrivial, $\tau(x)$ must be zero. So, x is unstable. □

2.2 The functions $\mu^{\bullet}(x, \lambda)$

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. Let x be a point in X and λ be a one parameter subgroup of G. Since X is complete, $\lim_{t\to 0} \lambda(t)x$ exists; let z denote this limit. The image of λ fixes z and so the group $\mathbb{K}^*$ acts via λ on the fiber $\mathcal{L}_z$. There exists an integer denoted by $\mu^{\mathcal{L}}(x, \lambda)$ such that for all $t \in \mathbb{K}^*$ and $\tilde{z} \in \mathcal{L}_z$ we have:

$$\lambda(t).\tilde{z} = t^{-\mu^{\mathcal{L}}(x,\lambda)}\tilde{z}.$$

One can immediately prove that the numbers $\mu^{\mathcal{L}}(x, \lambda)$ satisfy the following properties:

(i) $\mu^{\mathcal{L}}(g \cdot x, g \cdot \lambda \cdot g^{-1}) = \mu^{\mathcal{L}}(x, \lambda)$ for any $g \in G$;
(ii) the map $\mathcal{L} \mapsto \mu^{\mathcal{L}}(x, \lambda)$ is a group homomorphism from $\mathrm{Pic}^G(X)$ to $\mathbb{Z}$;
(iii) for any G-variety Y and for any G-equivariant morphism $f : X \longrightarrow Y$, $\mu^{f^*(\mathcal{L})}(x, \lambda) = \mu^{\mathcal{L}}(f(x), \lambda)$, where $x \in X$ and $\mathcal{L} \in \mathrm{Pic}^G(Y)$.

A less direct property is

Proposition 1 *Let $\mathcal{L}$, x, λ and z be as above. Let $\tilde{x}$ be a nonzero point in $\mathcal{L}_x$. We embed X in $\mathcal{L}$ using the zero section. Then, when t tends to* 0,

(i) $\lambda(t)\tilde{x}$ *tends to z, if* $\mu^{\mathcal{L}}(x, \lambda) < 0$;
(ii) $\lambda(t)\tilde{x}$ *tends to a nonzero point $\tilde{z}$ in $\mathcal{L}_z$, if* $\mu^{\mathcal{L}}(x, \lambda) = 0$;
(iii) $\lambda(t)\tilde{x}$ *has no limit in $\mathcal{L}$, if* $\mu^{\mathcal{L}}(x, \lambda) > 0$.

Proof Set $V = \{\lambda(t).x \mid t \in \mathbb{K}^*\} \cup \{z\}$: it is a locally closed subvariety of X stable by the action of $\mathbb{K}^*$ via λ. Moreover, z is the unique closed $\mathbb{K}^*$-orbit in V. So, [39, Lemma 7] implies that $\mathrm{Pic}^{\mathbb{K}^*}(V)$ is isomorphic to $X(\mathbb{K}^*)$; and finally that the restriction $\mathcal{L}$ to V is the trivial line bundle endowed with the action of $\mathbb{K}^*$ given by $-\mu^{\mathcal{L}}(x,\lambda)$. The proposition follows immediately. □

The integers $\mu^{\mathcal{L}}(x,\lambda)$ are used in [33] to give a numerical criterion (namely the Hilbert-Mumford criterion) for stability with respect to an ample G-linearized line bundle $\mathcal{L}$:

$$x \in X^{\mathrm{ss}}(\mathcal{L}) \Longleftrightarrow \mu^{\mathcal{L}}(x,\lambda) \le 0 \text{ for any one parameter subgroup } \lambda.$$

A line bundle $\mathcal{L}$ over X is said to be *semiample* if a positive power of $\mathcal{L}$ is base point free. With our notion of semistability, the Hilbert-Mumford theorem admits the following direct generalization:

Lemma 2 *Let $\mathcal{L}$ be a G-linearized line bundle over X and x be a point in X. Then,*

(i) *if x is semistable for $\mathcal{L}$, then $\mu^{\mathcal{L}}(x,\lambda) \le 0$ for any one parameter subgroup λ of G;*
(ii) *for any one parameter subgroup λ of G, if x is semistable for $\mathcal{L}$ and $\mu^{\mathcal{L}}(x,\lambda) = 0$, then $\lim_{t\to 0}\lambda(t)x$ is semistable for $\mathcal{L}$;*
(iii) *if in addition $\mathcal{L}$ is semiample, x is semistable for $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x,\lambda) \le 0$ for any one parameter subgroup λ of G.*

Proof Assume that x is semistable for $\mathcal{L}$ and consider a G-invariant section τ of $\mathcal{L}^{\otimes n}$ which does not vanish at x. Since $\lambda(t)\tau(x) = \tau(\lambda(t)x)$ tends to $\tau(z)$ when $t \to 0$, Proposition 1 shows that $\mu^{\mathcal{L}}(x,\lambda) \le 0$. If in addition $\mu^{\mathcal{L}}(x,\lambda) = 0$, Proposition 1 shows that $\tau(z)$ is nonzero; and so that z is semistable for $\mathcal{L}$. This proves the two first assertions.

Assume now that $\mathcal{L}$ is semiample. Let n be a positive integer such that $\mathcal{L}^{\otimes n}$ is base point free. Let V denote the dual of $\mathrm{H}^0(X,\mathcal{L}^{\otimes n})$: V is a finite dimensional G-module. Moreover, the usual map $\phi : X \longrightarrow \mathbb{P}(V)$ is G-equivariant. Let Y denote the image of ϕ and $\mathcal{M}$ denote the restriction of $\mathcal{O}(1)$ to Y.

Then $\mathcal{L}^{\otimes n}$ is the pullback of $\mathcal{M}$ by ϕ. Since X is projective, ϕ induces isomorphisms from $\mathrm{H}^0(Y,\mathcal{M}^{\otimes k})$ onto $\mathrm{H}^0(X,\mathcal{L}^{\otimes nk})$ (for any k). So, $X^{\mathrm{ss}}(\mathcal{L}) = \phi^{-1}(Y^{\mathrm{ss}}(\mathcal{M}))$. We deduce the last assertion of the lemma by applying the Hilbert-Mumford criterion to Y and $\mathcal{M}$ and Property (iii) of the functions $\mu^{\bullet}(x,\lambda)$. □

Remark

(i) If $\mathcal{L}$ is ample, Assertion (ii) of Lemma 2 is [39, Lemma 3].

(ii) The proof of Assertion (iii) shows that a lot of properties of semistability for ample line bundles are also available for semiample line bundles (see Propositions 2 and 7, and Theorems 1 and 2 below).

2.3 Definition of the functions $\mathrm{M}^\bullet(x)$

Notation Let Γ be any affine algebraic group. Its identity component is denoted by Γ°. Let $Y(\Gamma)$ denote the set of one parameter subgroups of Γ. Note that if Γ° is a torus, $Y(\Gamma)$ is a group.

If Λ is an Abelian group, we denote by $\Lambda_{\mathbb{Q}}$ (resp. $\Lambda_{\mathbb{R}}$) the tensor product of Λ with $\mathbb{Q}$ (resp. $\mathbb{R}$) over $\mathbb{Z}$.

2.3.1

Let T be a maximal torus of G. The Weyl group W of T acts linearly on $Y(T)_{\mathbb{R}}$. Since W is finite, there exists a W-invariant Euclidean norm (defined over $\mathbb{Q}$) $\|\cdot\|$ on $Y(T)_{\mathbb{R}}$. On the other hand, for any $\lambda\in Y(G)$ there exists $g\in G$ such that $g\cdot\lambda\cdot g^{-1}\in Y(T)$. We set $\|\lambda\|=\|g\cdot\lambda\cdot g^{-1}\|$. This does not depends on the choice of g since if two elements of $Y(T)$ are conjugate by an element of G, then they are by an element of the normalizer of T (see [33, Lemma 2.8]).

If $\mathcal{L}\in\mathrm{Pic}^G(X)$, we now introduce the following notation:

$$\overline{\mu}^{\mathcal{L}}(x,\lambda)=\frac{\mu^{\mathcal{L}}(x,\lambda)}{\|\lambda\|},\qquad \mathrm{M}^{\mathcal{L}}(x)=\sup_{\substack{\lambda\in Y(G)\\ \lambda\text{ nontrivial}}}\overline{\mu}^{\mathcal{L}}(x,\lambda).$$

In fact, we will see in Corollary 1 that $\mathrm{M}^{\mathcal{L}}(x)$ is finite.

2.3.2 $\mathrm{M}^\bullet(x)$ *for a torus action*

Notation If Y is a variety, and Z is a part of Y, the closure of Z in Y will be denoted by $\overline{Z}$. If Γ is a group acting on Y, Y^Γ denotes the set of fixed points of Γ in Y.

If V is a finite dimensional vector space, and v is a nonzero vector in V, $[v]$ denotes the class of v in the projective space $\mathbb{P}(V)$. If V is a Γ-module, and χ is a character of Γ, we denote by V_χ the set of $v\in V$ such that $g.v=\chi(g)v$ for any $g\in\Gamma$.

In this subsection we consider the action of a torus T on a variety X. Let $\mathcal{L}\in\mathrm{Pic}^T(X)$ and $z\in X^T$. Consider the character $\mu^{\mathcal{L}}(z,T)$ of T such that for all $t\in T$ and $\tilde{z}\in\mathcal{L}_z$, we have:

$$t.\tilde{z}=\mu^{\mathcal{L}}(z,T)(t^{-1})\tilde{z}.$$

We obtain a morphism

$$\mu^\bullet(z,T) : \mathrm{Pic}^T(X) \longrightarrow X(T).$$

For any point x in X, we denote by $\mathcal{P}_T^{\mathcal{L}}(x)$ the convex hull in $X(T)_{\mathbb{R}}$ of characters $\mu^{\mathcal{L}}(z,T)$ for $z \in \overline{T.x}^T$.

The following proposition is an adaptation of a result of L. Ness and gives a pleasant interpretation of the number $\mathrm{M}^{\mathcal{L}}(x)$:

Proposition 2 *We assume that $\mathcal{L}$ is semiample. We have*:

(i) *The point x is unstable if and only if* 0 *does not belong to $\mathcal{P}_T^{\mathcal{L}}(x)$. In this case*, $\mathrm{M}^{\mathcal{L}}(x)$ *is the distance from* 0 *to* $\mathcal{P}_T^{\mathcal{L}}(x)$.

(ii) *There exists a unique indivisible $\lambda \in Y(T)$ such that $\overline{\mu}^{\mathcal{L}}(x,\lambda) = \mathrm{M}^{\mathcal{L}}(x)$.*

Proof Since $\mathcal{L}$ is semiample, there exist a positive integer n, a T-module V, and a T-equivariant morphism $\phi : X \longrightarrow \mathbb{P}(V)$ such that $\mathcal{L}^{\otimes n} = \phi^*(\mathcal{O}(1))$. Since $\mu^\bullet(z,T)$ is a morphism, we have: $\mathcal{P}_T^{\mathcal{L}^{\otimes n}}(x) = n\mathcal{P}_T^{\mathcal{L}}(x)$ for any x. Moreover, $\overline{\mu}^{\mathcal{L}^{\otimes n}}(x,\lambda) = n\overline{\mu}^{\mathcal{L}}(x,\lambda)$, for all x and λ; so, $\mathrm{M}^{\mathcal{L}^{\otimes n}}(x) = n\mathrm{M}^{\mathcal{L}}(x)$. As a consequence, it is sufficient to prove the proposition for $\mathcal{L}^{\otimes n}$; in other words, we may assume that $n = 1$.

Let us recall that:

$$V = \bigoplus_{\chi \in X(T)} V_\chi.$$

Let $x \in X$ and $v \in V$ such that $[v] = \phi(x)$. There exist unique vectors $v_\chi \in V_\chi$ such that $v = \sum_\chi v_\chi$. Let $\mathcal{Q}$ be the convex hull in $X(T)_{\mathbb{R}}$ of the χ's such that $v_\chi \neq 0$. It is well known (see [36]) that the fixed points of T in $\overline{T.[v]}$ are exactly the $[v_\chi]$'s with χ vertex of $\mathcal{Q}$. Moreover, T acts by the character $-\chi$ on the fiber $\mathcal{O}(1)_{[v_\chi]}$ over $[v_\chi]$ in $\mathcal{O}(1)$. One deduces that $\mathcal{Q} = \mathcal{P}_T^{\mathcal{L}}(x)$. So, it is sufficient to prove the proposition with $\mathcal{Q}$ in place of $\mathcal{P}_T^{\mathcal{L}}(x)$ and $[v]$ in place of x (because of our nonstandard definition of semistability): this is a statement of [34]. Note that in [34], an adapted one parameter subgroup λ satisfies $\|\lambda\| = 1$ (in particular, it is a virtual one parameter subgroup). Here, we have chosen a different normalization: an adapted one parameter subgroup is assumed to be undivisible. This is possible, since the quadratic form associated to $\|\cdot\|$ is assumed to be defined on $\mathbb{Q}$. □

2.3.3 $\mathrm{M}^\bullet(x)$ *in general*

An indivisible one parameter subgroup λ of G is said to be *adapted to x and $\mathcal{L}$* if $\overline{\mu}^{\mathcal{L}}(x,\lambda) = \mathrm{M}^{\mathcal{L}}(x)$.

Using the fact that any one parameter subgroup is conjugated to one in a given torus, Proposition 2 implies easily

Corollary 1

(i) *The numbers* $\mathrm{M}^{\mathcal{L}}(x)$ *are finite* (*even if* $\mathcal{L}$ *is not semiample*, *see* [9, *Proposition* 1.1.6]).
(ii) *If* $\mathcal{L}$ *is semiample then there exists an adapted one parameter subgroup to* x *and* $\mathcal{L}$.

We can now reformulate the numerical criterion for stability: if $\mathcal{L}$ is semiample, we have

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X : \mathrm{M}^{\mathcal{L}}(x) \leq 0\}.$$

2.4 Adapted one parameter subgroups

To the one parameter subgroup λ of G, we associate the parabolic subgroup (see [33]):

$$P(\lambda) = \left\{ g \in G : \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

The unipotent radical of $P(\lambda)$ is

$$U(\lambda) = \left\{ g \in G : \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} = e \right\}.$$

Moreover, the centralizer G^{λ} of the image of λ in G is a Levi subgroup of $P(\lambda)$. For $p \in P(\lambda)$, we set $\overline{p} = \lim_{t\to 0} \lambda(t).p.\lambda(t)^{-1}$. Then, we have the following short exact sequence:

$$1 \longrightarrow U(\lambda) \longrightarrow P(\lambda) \xrightarrow{p \mapsto \overline{p}} G^{\lambda} \longrightarrow 1.$$

Note that for $g \in P(\lambda)$, we have $\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, g \cdot \lambda \cdot g^{-1})$. The following theorem due to G. Kempf is a generalization of the last assertion of Proposition 2.

Theorem 1 (see [19]) *Let* x *be an unstable point for a semiample* G-*linearized line bundle* $\mathcal{L}$. *Then*:

(i) *The group* $P(\lambda)$ *does not depend on the one parameter subgroup adapted to* x *and* $\mathcal{L}$. *We denote by* $P^{\mathcal{L}}(x)$ *this subgroup.*
(ii) *Any two one parameter subgroups adapted to* x *and* $\mathcal{L}$ *are conjugate by an element of* $P^{\mathcal{L}}(x)$.

Note that $\mathcal{L}$ is assumed to be ample in [19]; the semiample case follows from the argument used in the proof of Proposition 2.

We will also use the following theorem obtained independently by Ness and Ramanan-Ramanathan.

Theorem 2 ([35, Theorem 9.3] or [38, Proposition 1.9]) *Let x and $\mathcal{L}$ be as in Theorem 1. Let λ be an adapted one parameter subgroup to x and $\mathcal{L}$. We consider $z = \lim_{t\to 0}\lambda(t)\cdot x$. Then, λ is adapted to z and $\mathcal{L}$ and* $\mathrm{M}^{\mathcal{L}}(x) = \mathrm{M}^{\mathcal{L}}(z)$.

2.5 Stratification of X induced from $\mathcal{L}$

2.5.1

Let $\mathcal{L}$ be a semiample G-linearized line bundle on X. If $d > 0$ and $\langle\tau\rangle$ is a conjugacy class of one parameter subgroups of G, we set:

$$S^{\mathcal{L}}_{d,\langle\tau\rangle} = \{x \in X : \mathrm{M}^{\mathcal{L}}(x) = d, \text{ and } \langle\tau\rangle \text{ contains an element adapted to } x \text{ and } \mathcal{L}\}.$$

We now recall the notion of parabolic fiber bundle. It will be used to describe the geometry of $S^{\mathcal{L}}_{d,\langle\tau\rangle}$.

Let us fix a parabolic subgroup P of G and a P-variety Y. Consider over $G\times Y$ the action of $G\times P$ given by the formula (with obvious notation):

$$(g,p).(g',y) = (gg'p^{-1}, py).$$

Since the quotient map $G \longrightarrow G/P$ is a Zariski-locally trivial principal P-bundle, one can easily construct a quotient $G\times_P Y$ of $G\times Y$ by the action of $\{e\}\times P$. The action of $G\times\{e\}$ induces an action of G on $G\times_P Y$. Moreover, the first projection $G\times Y \longrightarrow G$ induces a G-equivariant map $G\times_P Y \longrightarrow G/P$ which is a locally trivial fibration with fiber Y.

The class of a pair $(g,y)\in G\times Y$ in $G\times_P Y$ is denoted by $[g:y]$. If Y is a P-stable locally closed subvariety of a G-variety X, it is well known that the map

$$\begin{aligned} G\times_P Y &\longrightarrow G/P\times X,\\ \big[g:y\big] &\longmapsto (gP, gy) \end{aligned}$$

is an immersion; its image is the set of $(gP,x)\in G/P\times X$ such that $g^{-1}x\in Y$.

2.5.2

If $\mathcal{T}$ is the set of conjugacy classes of one parameter subgroups, Theorem 1 gives us the following partition of X:

$$X = X^{\mathrm{ss}}(\mathcal{L}) \cup \bigcup_{d>0,\ \langle\tau\rangle\in\mathcal{T}} S^{\mathcal{L}}_{d,\langle\tau\rangle}. \tag{2}$$

W. Hesselink showed in [16] that this union is a finite stratification by G-stable locally closed subvarieties of X. We will call it the *stratification induced from* $\mathcal{L}$.

To describe the geometry of these strata, we need additional notation. For $\lambda \in \langle\tau\rangle$, we set:

$$S^{\mathcal{L}}_{d,\lambda} := \{x \in S^{\mathcal{L}}_{d,\langle\tau\rangle} : \lambda \text{ is adapted to } x \text{ and } \mathcal{L}\},$$

and

$$Z^{\mathcal{L}}_{d,\lambda} := \{x \in S^{\mathcal{L}}_{d,\lambda} : \lambda \text{ fixes } x\}.$$

By Theorem 2, we have the map

$$p_\lambda : S^{\mathcal{L}}_{d,\lambda} \longrightarrow Z^{\mathcal{L}}_{d,\lambda},\ \ x \longmapsto \lim_{t\to 0} \lambda(t).x.$$

Proposition 3 *With above notation, if d is positive, we have:*

(i) *$Z^{\mathcal{L}}_{d,\lambda}$ is open in X^λ and stable by G^λ;*
(ii) *$S^{\mathcal{L}}_{d,\lambda} = \{x \in X : \lim_{t\to 0}\lambda(t).x \in Z^{\mathcal{L}}_{d,\lambda}\}$ and is stable by $P(\lambda)$;*
(iii) *there is a bijective morphism $G \times_{P(\lambda)} S^{\mathcal{L}}_{d,\lambda} \longrightarrow S^{\mathcal{L}}_{d,\langle\lambda\rangle}$, which is an isomorphism if $S^{\mathcal{L}}_{d,\langle\lambda\rangle}$ is normal.*

Proof If X is nonsingular, the proof is made in [21, Sect. 13]. Now, X is any projective G-variety. Up to changing $\mathcal{L}$ by a positive power, one may assume that there exist a G-module V and a G-equivariant morphism $\phi : X \longrightarrow \mathbb{P}(V)$ such that $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$. We apply [21] to $\mathbb{P}(V)$, and then, deduce the proposition for X. □

3 First descriptions of the GIT-cones

3.1 Definitions

We fix some definitions about convex cones:

Definition Let E be a finite dimensional rational vector space and $\mathcal{C}$ be a convex cone in E. We will denote by $\langle\mathcal{C}\rangle$ the subspace of E spanned by $\mathcal{C}$. The dimension of $\langle\mathcal{C}\rangle$ will be called the *dimension of* $\mathcal{C}$.

Let U be a subset of E. A *convex cone of* U is the intersection of U with a convex cone of E. The cone $\mathcal{C}$ is said to *polyhedral in* U if it is determined as a part of U by finitely many linear inequalities.

Let $\mathcal{C}$ be a convex cone of U, not necessarily closed or polyhedral. Let φ be a linear form on E which is nonnegative on $\mathcal{C}$. The set of $x\in\mathcal{C}$ such that $\varphi(x)=0$ is called *a face of* $\mathcal{C}$. Note that any face is a convex cone, that it may be empty and that $\mathcal{C}$ is always a face of $\mathcal{C}$. A face different from $\mathcal{C}$ is said to be *strict*.

Let us recall from the introduction that Λ is a free finitely generated subgroup of $\mathrm{Pic}^G(X)$. Since $X^{\mathrm{ss}}(\mathcal{L})=X^{\mathrm{ss}}(\mathcal{L}^{\otimes n})$, for any G-linearized line bundle $\mathcal{L}$ and any positive integer n, we can define $X^{\mathrm{ss}}(\mathcal{L})$ for any $\mathcal{L}\in\Lambda_{\mathbb{Q}}$. The central object of this article is the following *total G-cone*:

$$\mathcal{TC}^G_\Lambda(X)=\{\mathcal{L}\in\Lambda_{\mathbb{Q}}: X^{\mathrm{ss}}(\mathcal{L})\text{ is not empty}\}.$$

Since the tensor product of two nonzero G-invariant sections is a nonzero G-invariant section, $\mathcal{TC}^G_\Lambda(X)$ is a convex cone.

Consider the convex cones $\Lambda^+_{\mathbb{Q}}$ and $\Lambda^{++}_{\mathbb{Q}}$ generated respectively by the semiample and ample elements of Λ. Define the following *semiample and ample G-cones*:

$$\mathcal{SAC}^G_\Lambda(X)=\mathcal{TC}^G_\Lambda(X)\cap\Lambda^+_{\mathbb{Q}}\quad\text{and}\quad\mathcal{AC}^G_\Lambda(X)=\mathcal{TC}^G_\Lambda(X)\cap\Lambda^{++}_{\mathbb{Q}}.$$

By [9] (see also [39]), $\mathcal{AC}^G_\Lambda(X)$ is a closed convex rational polyhedral cone in the open cone $\Lambda^{++}_{\mathbb{Q}}$.

If $\mathrm{Pic}^G(X)$ has finite rank and Λ satisfies $\Lambda_{\mathbb{Q}}=\mathrm{Pic}^G(X)_{\mathbb{Q}}$, we will denote $\mathcal{SAC}^G(X)$, $\mathcal{AC}^G(X)$ and $\mathcal{TC}^G(X)$ without Λ.

3.2 Well covering pairs

3.2.1

Let X be any quasiprojective G-variety. If X is isomorphic as a G-variety to a (quasi)-projective G-stable subvariety of some $\mathbb{P}(V)$ (for some G-module V), we will say that X is G-(quasi)-projective. By [24], if X is normal, then it is G-quasiprojective. >From now on, all the considered G-varieties **will be G-quasiprojective**.

Let λ be a one parameter subgroup of G and C be an irreducible component of X^λ. Since G^λ is connected, C is a closed G^λ-stable subvariety of X.

We set:

$$\tilde{C}^+ := \{x \in X : \lim_{t\to 0} \lambda(t)x \text{ exists and belongs to } C\}. \tag{3}$$

On can easily check that $\tilde{C}^+$ is $P(\lambda)$-stable. Since X is G-quasiprojective, $\tilde{C}^+$ is locally closed in X (one can prove it for $\mathbb{P}(V)$ and then deduce it for X). Moreover, the map $p_\lambda : \tilde{C}^+ \longrightarrow C,\ x \longmapsto \lim_{t\to 0} \lambda(t)x$ is a morphism satisfying:

$$\forall (l,u) \in G^\lambda \times U(\lambda) \quad p_\lambda(lu.x) = l\,p_\lambda(x).$$

We do not know if $\tilde{C}^+$ is always locally closed (see [27] for results about this question). If X is smooth, a Białynicki-Birula result (see Theorem 5 below) shows that $\tilde{C}^+$ is a locally closed irreducible and smooth subvariety of X. Note that for the application to $\mathcal{LR}(G,\hat{G})$, X will always be a product of flag manifolds of Levi subgroups of G and $\hat{G}$.

3.2.2 Definition

Here comes a central definition in this work:

Definition For any irreducible component C^+ of $\tilde{C}^+$, consider the following G-equivariant map

$$\begin{array}{rcl} \eta\ : G \times_{P(\lambda)} C^+ & \longrightarrow & X, \\ {[g:x]} & \longmapsto & g.x. \end{array}$$

The pair (C,λ) is said to be *covering* (resp. *dominant*) if η is birational (resp. dominant) for some irreducible component C^+ of $\tilde{C}^+$. It is said to be *well covering* if in addition there exists a $P(\lambda)$-stable open subset Ω of C^+ intersecting C such that η induces an isomorphism from $G \times_{P(\lambda)} \Omega$ onto an open subset of X.

3.2.3 A basic lemma

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. Since C is irreducible, $\mu^{\mathcal{L}}(x,\lambda)$ does not depend on $x \in C^+$; we denote by $\mu^{\mathcal{L}}(C,\lambda)$ this integer.

Lemma 3 *Let (C,λ) be a dominant pair and let $\mathcal{L} \in \mathcal{TC}^G_\Lambda(X)$. Then,*

(i) $\mu^{\mathcal{L}}(C,\lambda) \leq 0$;
(ii) $\mu^{\mathcal{L}}(C,\lambda) = 0$ *if and only if $X^{\mathrm{ss}}(\mathcal{L})$ intersects C.*

Proof Since $X^{\mathrm{ss}}(\mathcal{L})$ is a G-stable nonempty open subset of X and (C,λ) is dominant, there exists a point $x \in C^+$ semistable for $\mathcal{L}$. Since $x \in C^+$, $\mu^{\mathcal{L}}(x,\lambda) = \mu^{\mathcal{L}}(C,\lambda)$. Since $x \in X^{\mathrm{ss}}(\mathcal{L})$, Lemma 2 shows that $\mu^{\mathcal{L}}(x,\lambda) \leq 0$.

Now, if $X^{\mathrm{ss}}(\mathcal{L})$ intersects C, Lemma 1 shows that $\mu^{\mathcal{L}}(C,\lambda) = 0$. Conversely, assume that $\mu^{\mathcal{L}}(C,\lambda) = 0$. Let $x \in X^{\mathrm{ss}}(\mathcal{L}) \cap C^+$ and set $z = \lim_{t\to 0} \lambda(t)x$. By Lemma 2, z is semistable for $\mathcal{L}$ and belongs to C. □

The lemma gives linear inequalities satisfied by the cone $\mathcal{TC}^G_\Lambda(X)$. It turns out that all the inequalities considered in this article or in [4] or in the Horn conjecture can be obtained from Lemma 3. Note that the face of $\mathcal{TC}^G_\Lambda(X)$ associated to the inequality $\mu^{\mathcal{L}}(C,\lambda) \leq 0$ only depends on C. We will denote by $\mathcal{TF}_\Lambda(C)$, $\mathcal{SAF}_\Lambda(C)$ and $\mathcal{AF}_\Lambda(C)$ the corresponding faces of $\mathcal{TC}^G_\Lambda(X)$, $\mathcal{SAC}^G_\Lambda(X)$ and $\mathcal{AC}^G_\Lambda(X)$ respectively. If $\Lambda_{\mathbb{Q}} = \mathrm{Pic}^G(X)$ we will forget the Λ and we will denote $\mathcal{TF}(C)$ for example.

3.2.4 A description of GIT-cones

Proposition 3 allows us to give a first description of the cone $\mathcal{AC}^G_\Lambda(X)$:

Proposition 4 *We assume that X is normal. Let T be a maximal torus of G and B be a Borel subgroup containing T.*

Then, the cone $\mathcal{AC}^G_\Lambda(X)$ (resp. $\mathcal{SAC}^G_\Lambda(X)$) is the set of $\mathcal{L} \in \Lambda^{++}_{\mathbb{Q}}$ (resp. of $\mathcal{L} \in \Lambda^{+}_{\mathbb{Q}}$) such that for any well covering pair (C,λ) with a dominant one parameter subgroup λ of T we have $\mu^{\mathcal{L}}(C,\lambda) \leq 0$.

Proof Lemma 3 shows that $\mathcal{SAC}^G_\Lambda(X)$ and $\mathcal{AC}^G_\Lambda(X)$ are respectively contained in the part of $\Lambda^{+}_{\mathbb{Q}}$ and $\Lambda^{++}_{\mathbb{Q}}$ defined by the inequalities $\mu^{\mathcal{L}}(C,\lambda) \leq 0$ of the proposition.

Conversely, let $\mathcal{L} \in \Lambda^{+}_{\mathbb{Q}}$ such that $X^{\mathrm{ss}}(\mathcal{L})$ is empty. It remains to construct a well covering pair (C,λ) such that $\mu^{\mathcal{L}}(C,\mathcal{L}) > 0$. Proposition 3 gives such a pair. More precisely, consider the dense stratum $S^{\mathcal{L}}_{d,\langle\tau\rangle}$ in X. Let λ be a dominant one parameter subgroup of T in the class $\langle\tau\rangle$. Since $S^{\mathcal{L}}_{d,\langle\tau\rangle}$ is open in X, it is normal and irreducible. Now, Proposition 3 implies that $S^{\mathcal{L}}_{d,\lambda}$ and then $Z^{\mathcal{L}}_{d,\lambda}$ are irreducible. Moreover, Proposition 3 implies that $Z^{\mathcal{L}}_{d,\lambda}$ is open in X^λ. It follows that the closure C of $Z^{\mathcal{L}}_{d,\lambda}$ is an irreducible component of X^λ. Still by Proposition 3, the closure C^+ of $S^{\mathcal{L}}_{d,\lambda}$ in $\tilde{C}^+$ is an irreducible component of $\tilde{C}^+$ defined as in (3). Finally, the last assertion of Proposition 3 shows that (C,λ) is well covering. Moreover, $\mu^{\mathcal{L}}(C,\lambda) = d > 0$. □

Proposition 4 characterizes the cone $\mathcal{AC}^G_\Lambda(X)$ by infinitely many linear inequalities. Starting with one of these inequalities, the aim of the following section is to bound from above the dimension of the corresponding face. This will make us able to remove a lot of inequalities in Proposition 4.

3.3 Abundance

3.3.1 The Borel-Weil theorem

Let P be a parabolic subgroup of G. Let ν be a character of P. Let $\mathbb{K}_\nu$ denote the field $\mathbb{K}$ endowed with the action of P defined by $p.\tau = \nu(p^{-1})\tau$ for all $\tau \in \mathbb{K}_\nu$ and $p \in P$. The fiber product $G \times_P \mathbb{K}_\nu$ is a G-linearized line bundle on G/P, denoted by $\mathcal{L}_\nu$. In fact, the map $X(P) \longrightarrow \mathrm{Pic}^G(G/P)$, $\nu \longmapsto \mathcal{L}_\nu$ is an isomorphism.

Let B be a Borel subgroup of G contained in P, and T be a maximal torus contained in B. Then, $X(P)$ identifies with a subgroup of $X(T)$. For $\nu \in X(P)$, $\mathcal{L}_\nu$ is semiample if and only if it has nonzero sections if and only if ν is dominant. Moreover, $\mathrm{H}^0(G/P, \mathcal{L}_\nu)$ is the dual of the simple G-module of highest weight ν. For ν dominant, $\mathcal{L}_\nu$ is ample if and only if ν cannot be extended to a subgroup of G bigger than P.

3.3.2 Diagonizable reductive isotropies

The following assumption about isotropies will be useful:

Definition A G-variety X is said to have *diagonalizable reductive isotropies* if for any x in X such that G_x is reductive, G_x is diagonalizable.

Let $\hat{G}$ be a connected reductive group containing G. Consider the variety $\hat{G}/\hat{B}$ of Borel subgroups of $\hat{G}$ endowed with the natural G-action. Let Y be any G-variety.

Proposition 5 *The G-variety $X = \hat{G}/\hat{B} \times Y$ has diagonalizable reductive isotropies.*

Proof Let x in X such that G_x is reductive. The group G_x is contained in a Borel subgroup of $\hat{G}$; so, it is reductive and solvable; and finally it is diagonalizable. □

3.3.3 Abundance

We call a subgroup Γ' of an Abelian group Γ *cofinite* if Γ/Γ' is finite. The following definition is an adaptation of that of Dolgachev-Hu (see [9]).

Definition The subgroup Λ is said to be *abundant* if for any x in X such that G_x is reductive, the image of the restriction $\Lambda \longrightarrow \mathrm{Pic}^G(G.x)$ is cofinite.

The main example of abundant subgroup comes when $X = \hat{G}/\hat{B} \times Y$:

Proposition 6 *Let* $p_1 : X \longrightarrow \hat{G}/\hat{B}$ *denote the projection map and* $p_1^* : \mathrm{Pic}^G(\hat{G}/\hat{B}) \longrightarrow \mathrm{Pic}^G(X)$ *the associated homomorphism.*

Then, the image of p_1^* *is abundant.*

Proof Let $x \in X$ such that G_x is reductive. Let $\hat{B}$ denote the Borel subgroup of $\hat{G}$ corresponding to $p_1(x)$. Let $\mathcal{L} \in \mathrm{Pic}^G(G.x)$ and χ be the corresponding character of G_x. Note that G_x is a subgroup of $\hat{B}$ and that the restriction map $X(\hat{B}) \longrightarrow X(G_x)$ is surjective. There exists $\hat{\nu} \in X(\hat{B})$ such that $\hat{\nu}_{|G_x} = \chi$. The restriction of $p_1^*(\mathcal{L}_{\hat{\nu}})$ to $G.x$ equals $\mathcal{L}$; the proposition follows. □

3.3.4 A description of GIT-cones

We will denote by $\mathrm{Im}\lambda$ the image of the one parameter subgroup λ. The above assumptions allow to improve Proposition 4:

Theorem 3 *We assume that* X *has diagonalizable reductive isotropies and that* Λ *is abundant. We assume that* $\mathcal{AC}^G_\Lambda(X)$ *has nonempty interior in* $\Lambda_{\mathbb{Q}}$. *We fix* $T \subset B \subset G$. *Let* $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$. *The following are equivalent*:

(i) $\mathcal{L} \in \mathcal{AC}^G_\Lambda(X)$;
(ii) *for all dominant indivisible one parameter subgroups* λ *of* T *and well covering pairs* (C, λ) *such that there exists* $x \in C$ *with* $G_x^\circ = \mathrm{Im}\lambda$, *we have* $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.

Proof By Proposition 4, the first assertion implies the second one. Since, $\mathcal{AC}^G_\Lambda(X)$ has nonempty interior, by this proposition, it is sufficient to prove that for any well covering pair (C, λ) such that $\mathcal{AF}_\Lambda(C)$ has codimension one there exists $x \in C$ with $G_x^\circ = \mathrm{Im}\lambda$.

Recall from [9] that the GIT-class of a point $\mathcal{L} \in \Lambda_{\mathbb{Q}}^{++}$ is the set of $\mathcal{M} \in \Lambda_{\mathbb{Q}}^{++}$ such that $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{M})$. By [9, 39], there are finitely many GIT-classes which are locally closed convex cones. Moreover, the second assertion of Lemma 3 implies that $\mathcal{AF}_\Lambda(C)$ is an union of GIT-classes. We deduce that there exists a GIT-class with nonempty interior in $\mathcal{AF}_\Lambda(C)$; let $\mathcal{L}$ be a point in such a class. Let $x \in X^{\mathrm{ss}}(\mathcal{L}) \cap C$ whose the G-orbit is closed in $X^{\mathrm{ss}}(\mathcal{L})$. Then, G_x is reductive. Since $\mathrm{Im}\lambda$ is contained in G_x, with our assumption about X, it is sufficient to prove that the rank of $X(G_x) \simeq \mathrm{Pic}^G(G.x)$ is one. But, Lemma 1 implies that the GIT-class of $\mathcal{L}$, and so, $\mathcal{AF}_\Lambda(C)$ are contained in the kernel of the restriction $\Lambda_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^G(G.x)_{\mathbb{Q}}$. We conclude using abundance of Λ. □

4 Irredundancy

4.1 From well covering pairs to faces of the total G-cone

Proposition 4 shows that any facet of $\mathcal{AC}^G_\Lambda(X)$ is obtained from a well covering pair. Conversely, each such pair gives (see Lemma 3) a face of $\mathcal{TC}^G_\Lambda(X)$. The point to obtain irredundancy in Theorem B is to show that certain faces coming from well covering pairs have codimension one. This will be made by induction using

Theorem 4 *Let X be a smooth projective G-variety. We assume the rank of* $\mathrm{Pic}^G(X)$ *is finite and consider $\mathcal{TC}^G(X)$. Let (C,λ) be a well covering pair. Consider the linear map ρ induced by the restriction*:

$$\rho : \mathrm{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}.$$

Then, $\mathcal{TF}(C)$ and the pullback by ρ of $\mathcal{TC}^{G^\lambda}(C)$ span the same subspace of $\mathrm{Pic}^G(X)_{\mathbb{Q}}$.

Sketch of proof Let $\mathcal{L}$ be a G-linearized line bundle on X whose the restriction $\mathcal{M}$ to C belongs to $\mathcal{TC}^{G^\lambda}(C)$. Up to changing $\mathcal{L}$ by a positive power, let τ be a nonzero G^λ-invariant section of $\mathcal{M}$. To obtain the theorem, one essentially has to extend τ to a regular G-invariant section of $\mathcal{L}$; indeed, the existence of such a section implies that $\mathcal{L}\in\mathcal{TF}(C)$. Unfortunately, this is not always possible: instead, we firstly extend τ to a rational G-invariant section of $\mathcal{L}$ and then add invariant divisors to kill the polar parts. The fact that (C,λ) is well covering implies that the so constructed G-invariant section is nonidentically zero on C; and so, that we have produced a point on $\mathcal{TF}(C)$. Actually, to prove the theorem we will make the above construction with a family of such $\mathcal{L}$'s which spans the subspace of the statement.

Proof Lemma 2 implies that $\rho(\mathcal{TF}(C))$ is contained in $\mathcal{TC}^{G^\lambda}(C)$.

We denote by F the subspace spanned by $\rho^{-1}(\mathcal{TC}^{G^\lambda}(C))$. Let $\mathcal{L}_1,\dots,\mathcal{L}_n$ in $\mathrm{Pic}^G(X)$ which span F and whose the restrictions to C belong to $\mathcal{TC}^{G^\lambda}(C)$. Denote by $\mathcal{M}_i$ the restriction of $\mathcal{L}_i$ to C. For each i, up to changing $\mathcal{L}_i$ by a positive power, one may assume that there exists a nonzero regular G^λ-invariant section τ_i of $\mathcal{M}_i$.

We first prove that each τ_i can be extended to a rational G-invariant section σ_i of $\mathcal{L}_i$. Consider $p_\lambda : C^+ \longrightarrow C$, defined in Sect. 3.2.1. Consider the G-linearized line bundle $G\times_{P(\lambda)} p_\lambda{}^*(\mathcal{M}_i)$ on $G\times_{P(\lambda)} C^+$. Since $\eta^*(\mathcal{L}_i)$ and $G\times_{P(\lambda)} p_\lambda{}^*(\mathcal{M}_i)$ have the same restriction to C, Lemmas 4 and 5 below show that $\eta^*(\mathcal{L}_i)=G\times_{P(\lambda)} p_\lambda{}^*(\mathcal{M}_i)$. Moreover, since $\mu^{\mathcal{M}_i}(C,\lambda)=0$, Lemma 5 shows that τ_i admits a unique $P(\lambda)$-invariant extension τ_i' which is a section of ${p_\lambda}^*(\mathcal{M}_i)$. On the other hand, Lemma 4 below shows that τ_i' admits a unique G-invariant extension $\tilde{\tau}_i$ from C^+ to $G\times_{P(\lambda)} C^+$. Since η is birational, $\tilde{\tau}_i$ descends to a rational G-invariant section σ_i of $\mathcal{L}_i$. So, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{L}_i \arrow[d, shift right] & \eta^*(\mathcal{L}_i) = G\times_{P(\lambda)} {p_\lambda}^*(\mathcal{M}_i) \arrow[l] \arrow[d, shift right] & {p_\lambda}^*(\mathcal{M}_i) \arrow[l] \arrow[r] \arrow[d, shift right] & \mathcal{M}_i \arrow[d, shift right] \\
X \arrow[u, dashed, shift right, "\sigma_i"'] & G\times_{P(\lambda)} C^+ \arrow[l, "\eta"'] \arrow[u, shift right, "\tilde{\tau}_i"'] & C^+ \arrow[l] \arrow[r, "p_\lambda"] \arrow[u, shift right, "\tau_i'"'] & C. \arrow[u, shift right, "\tau_i"']
\end{tikzcd}$$

We are now going to construct an element $\mathcal{L}_0 \in \mathcal{TF}(C)$ which kills the polar part of each σ_i. More precisely, each σ_i will induce a regular G-invariant section of $\mathcal{L}_i\otimes\mathcal{L}_0$. Let X° be a G-stable open subset of X such that η induces an isomorphism from $\eta^{-1}(X^\circ)$ onto X°. Since (C,λ) is well covering, we may (and shall) assume that X° intersects C. Let E_j be the irreducible components of codimension one of $X - X^\circ$. For any j we denote by a_j the maximum of 0 and the $-\nu_{E_j}(\sigma_i)$'s with $i=1,\dots,n$; where ν_{E_j} denotes the valuation associated to E_j. Consider the line bundle $\mathcal{L}_0=\mathcal{O}(\sum a_jE_j)$ on X. Since the E_j's are stable by the action of G, $\mathcal{L}_0$ is canonically G-linearized. By construction, the σ_i's induce G-invariant regular sections σ_i' of $\mathcal{L}_i' := \mathcal{L}_i\otimes\mathcal{L}_0$. Moreover, since no E_j contains C, the restriction of σ_i' to C is nonzero. In particular, the $\mathcal{L}_i'$'s belong to $\mathcal{TF}(C)$.

Moreover, replacing $\mathcal{L}_0$ by $\mathcal{L}_0^{\otimes 2}$ if necessary, we may (and shall) assume that the $\mathcal{L}_i'$'s span F. This ends the proof of the theorem. □

4.2 Line bundles on parabolic fiber products

Notation If Y is a locally closed subvariety of X, and $\mathcal{L}$ is a line bundle on X, $\mathcal{L}_{|Y}$ will denote the restriction of $\mathcal{L}$ to Y.

Let P be a parabolic subgroup of G and Y be a P-variety. In this subsection, we prove results about G-linearized line bundles on $G\times_P Y$ used in Theorem 4.

Lemma 4 *With above notation, we have*:

(i) *The map $\mathcal{L}\longmapsto G\times_P\mathcal{L}$ defines a morphism*

$$e:\mathrm{Pic}^P(Y)\longrightarrow \mathrm{Pic}^G(G\times_P Y).$$

(ii) *The map* $\iota : Y \longrightarrow G\times_P Y,\ y\longmapsto [e:y]$ *is a* P*-equivariant closed immersion. We denote by* $\iota^* : \mathrm{Pic}^G(G\times_P Y)\longrightarrow \mathrm{Pic}^P(Y)$ *the associated restriction homomorphism.*

(iii) *The morphisms* e *and* ι^* *are the inverse one of each other; in particular, they are isomorphisms.*

(iv) *For any* $\mathcal{L}\in\mathrm{Pic}^G(G\times_P Y)$, *the restriction map from* $\mathrm{H}^0(G\times_P Y,\mathcal{L})$ *to* $\mathrm{H}^0(Y,\iota^*(\mathcal{L}))$ *induces a linear isomorphism*

$$\mathrm{H}^0(G\times_P Y,\mathcal{L})^G\simeq \mathrm{H}^0(Y,\iota^*(\mathcal{L}))^P.$$

Proof Let $\mathcal{M}$ be a P-linearized line bundle on Y. Since the natural map $G\times\mathcal{M}\longrightarrow G\times_P\mathcal{M}$ is a categorical quotient, we have the following commutative diagram:

$$\begin{tikzcd} G\times\mathcal{M} \arrow[r] \arrow[d] & G\times_P\mathcal{M} \arrow[d, "p"] \\ G\times Y \arrow[r] & G\times_P Y. \end{tikzcd}$$

Since $G\longrightarrow G/P$ is locally trivial, the map p endows $G\times_P\mathcal{M}$ with a structure of line bundle on $G\times_P Y$. Moreover, the action of G on $G\times_P\mathcal{M}$ endows this line bundle with a G-linearization. This proves Assertion (i). The second one is obvious.

By construction, the restriction of $G\times_P\mathcal{M}$ to Y is $\mathcal{M}$. So, $\iota^*\circ e$ is the identity map. Conversely, let $\mathcal{L}\in\mathrm{Pic}^G(G\times_P Y)$. Then, we have:

$$e\circ\iota^*(\mathcal{L})\simeq\{(gP,l)\in G/P\times\mathcal{L} : g^{-1}l\in\mathcal{L}_{|Y}\}.$$

The second projection induces an isomorphism from $e\circ\iota^*(\mathcal{L})$ onto $\mathcal{L}$. This ends the proof of Assertion (iii).

The map $\mathrm{H}^0(G\times_P Y,\mathcal{L})^G\longrightarrow\mathrm{H}^0(Y,\iota^*(\mathcal{L}))^P$ is clearly well defined and injective. Let us prove the surjectivity. Let $\tau\in\mathrm{H}^0(Y,\iota^*(\mathcal{L}))^P$. Consider the morphism

$$\begin{aligned}\hat\tau : G\times Y &\longrightarrow G\times_P\mathcal{L},\\ (g,y)&\longmapsto [g:\tau(y)].\end{aligned}$$

Since τ is P-invariant, so is $\hat\tau$; and $\hat\tau$ induces a section of $G\times_P\mathcal{L}$ over $G\times_P Y$ which is G-invariant and extends τ. □

4.3 Line bundles on C^+

In this subsection we prove results about the line bundles on C^+ used in Theorem 4.

Let X be any G-variety, $\lambda\in Y(G)$ and $x\in X^\lambda$. We consider the natural action of $\mathbb{K}^*$ induced by λ on the Zariski tangent space T_xX of X at x. We consider the following $\mathbb{K}^*$-submodules of T_xX:

$$T_xX_{>0}=\{\xi\in T_xX:\lim_{t\to 0}\lambda(t)\xi=0\},$$
$$T_xX_{<0}=\{\xi\in T_xX:\lim_{t\to 0}\lambda(t^{-1})\xi=0\},$$
$$T_xX_0=(T_xX)^\lambda,\ T_xX_{\geq 0}=T_xX_{>0}\oplus T_xX_0 \text{ and } T_xX_{\leq 0}=T_xX_{<0}\oplus T_xX_0.$$

A classical result of Białynicki-Birula (see [5]) is

Theorem 5 *We assume that X is smooth and fix an irreducible component C of X^λ. We have*:

(i) *C is smooth and for any $x\in C$ we have $T_xC=T_xX_0$;*
(ii) *C^+ is smooth and irreducible and for any $x\in C$ we have $T_xC^+=T_xX_{\geq 0}$;*
(iii) *the morphism $p_\lambda:C^+\longrightarrow C$ induces a structure of vector bundle on C with fibers isomorphic to $T_xX_{>0}$, for $x\in C$.*

Let $\mathcal{L}$ be a $P(\lambda)$-linearized line bundle on C^+. Whereas X is not necessarily complete, for $x\in C^+$, $\lim_{t\to 0}\lambda(t).x$ exists and $\mu^{\mathcal{L}}(x,\lambda)$ is well defined. Moreover, it does not depend on x and will be denoted by $\mu^{\mathcal{L}}(C,\lambda)$.

Lemma 5 *We assume that X is smooth. Then, we have*:

(i) *The restriction map* $\mathrm{Pic}^{P(\lambda)}(C^+)\longrightarrow\mathrm{Pic}^{G^\lambda}(C)$ *is an isomorphism.*
 Let $\mathcal{L}\in\mathrm{Pic}^{P(\lambda)}(C^+)$.
(ii) *If* $\mu^{\mathcal{L}}(C,\lambda)\neq 0$, *then* $\mathrm{H}^0(C,\mathcal{L}_{|C})^\lambda=\{0\}$.
(iii) *If* $\mu^{\mathcal{L}}(C,\lambda)=0$, *then the restriction map induces an isomorphism from* $\mathrm{H}^0(C^+,\mathcal{L})^{P(\lambda)}$ *onto* $\mathrm{H}^0(C,\mathcal{L}_{|C})^{G^\lambda}$. *Moreover, for any* $\tau\in\mathrm{H}^0(C^+,\mathcal{L})^{P(\lambda)}$, *we have*:

$$\{x\in C^+:\tau(x)=0\}=p_\lambda{}^{-1}(\{x\in C:\tau(x)=0\}).$$

Proof Since p_λ is $P(\lambda)$-equivariant, for any $\mathcal{M}\in\mathrm{Pic}^{G^\lambda}(C)$, $p_\lambda^*(\mathcal{M})$ is $P(\lambda)$-linearized. Since p_λ is a vector bundle, $p_\lambda^*(\mathcal{L}_{|C})$ and $\mathcal{L}$ are isomorphic as line bundles without linearization. But, $X(P(\lambda))\simeq X(G^\lambda)$, so the $P(\lambda)$-linearizations must coincide; and $p_\lambda^*(\mathcal{L}_{|C})$ and $\mathcal{L}$ are isomorphic as $P(\lambda)$-linearized line bundles. Assertion (i) follows.

Assertion (ii) is a direct application of Lemma 1.

Let us fix $\mathcal{L}\in\mathrm{Pic}^{P(\lambda)}(C^+)$ and denote by $p:\mathcal{L}\longrightarrow C^+$ the projection. We assume that $\mu^{\mathcal{L}}(C,\lambda)=0$. Let $\tau\in\mathrm{H}^0(C^+,\mathcal{L})^{P(\lambda)}$. We just proved that

$$\mathcal{L}\simeq p_\lambda^*(\mathcal{L}_{|C})=\{(x,l)\in C^+\times\mathcal{L}_{|C}:p_\lambda(x)=p(l)\}.$$

Let p_2 denote the projection of $p_\lambda^*(\mathcal{L}_{|C})$ onto $\mathcal{L}_{|C}$.

For all $x \in C^+$ and $t \in \mathbb{K}^*$, we have:

$$\begin{aligned}
\tau(\lambda(t).x) &= \big(\lambda(t).x,\ p_2(\tau(\lambda(t).x))\big) \\
&= \lambda(t).\big(x,\ p_2(\tau(x))\big) \quad \text{since } \tau \text{ is invariant,} \\
&= \big(\lambda(t).x,\ p_2(\tau(x))\big) \quad \text{since } \mu^{\mathcal{L}}(C,\lambda) = 0.
\end{aligned}$$

We deduce that for any $x \in C^+$, $\tau(x) = (x, \tau(p_\lambda(x)))$. Assertion (iii) follows. □

Note that Assertion (iii) of Lemma 5 is a direct generalization of [3, Remark 31].

5 The smaller faces of the G-ample cone

In the proof of Proposition 4, starting with an element $\mathcal{L}$ in $\Lambda_{\mathbb{Q}}^{++}$ without semistable points we have constructed a well covering pair (C,λ) and so a face $\mathcal{AF}_\Lambda(C)$ of $\mathcal{AC}_\Lambda^G(X)$. Proposition 4 implies that any facet of $\mathcal{AC}_\Lambda^G(X)$ is obtained in such a way. The goal of this section is to prove that any nonempty strict face of $\mathcal{AC}_\Lambda^G(X)$ is obtained from such a $\mathcal{L}$. We first give some complements about the Hilbert-Mumford numerical criterion and recall a powerful result of Luna. From now on, X is an irreducible G-projective variety.

5.1 A description of $Z_{d,\lambda}^{\mathcal{L}}$

Let λ be a one parameter subgroup of G. Let Z denote the identity component of the center of G^λ and $G^\lambda_{\rm ss}$ be the maximal semisimple subgroup of G^λ. The product induces an isogeny $Z \times G^\lambda_{\rm ss} \longrightarrow G^\lambda$. Let T_1 be a maximal torus of $G^\lambda_{\rm ss}$. Set $T = Z.T_1$. Note that T is a maximal torus of G^λ and G. Let S be the subtorus of Z such that $Y(S.T_1)_{\mathbb{R}}$ is the hyperplane of $Y(T)_{\mathbb{R}}$ orthogonal to λ. Set $H^\lambda = S.G^\lambda_{\rm ss}$. The map $\mathbb{K}^* \times H^\lambda \longrightarrow G^\lambda$, $(t,h) \longmapsto \lambda(t)h$ is an isogeny.

Theorem 6 (Ness-Kirwan) *Let $\mathcal{L}$ be a semiample G-linearized line bundle on X. The one parameter subgroup λ is assumed to be indivisible. Let $x \in X^\lambda$ be such that $\mu^{\mathcal{L}}(x,\lambda) > 0$.*

Then, λ is adapted to x and $\mathcal{L}$ (that is, $x \in Z^{\mathcal{L}}_{\overline{\mu}(x,\lambda),\lambda}$) if and only if x is semistable for $\mathcal{L}$ and the action of H^λ.

Theorem 6 is a version of [35, Theorem 9.4]. Whereas Ness' proof works without changing (even if $\mathcal{L}$ is semiample), the statement in [35] is not correct. In [21, Remark 12.21], F. Kirwan made the above correction. The fact that if λ is adapted to x then x is semistable for the action of H^λ was independently proved by Ramanan-Ramanathan in [38, Proposition 1.12].

5.2 The dense stratum

5.2.1

Let $\mathcal{L}$ be a semiample G-linearized line bundle on X. We denote by $X^\circ(\mathcal{L})$ the dense stratum of Stratification (2). If $X^{\rm ss}(\mathcal{L})$ is not empty then $X^\circ(\mathcal{L}) = X^{\rm ss}(\mathcal{L})$. If $X^{\rm ss}(\mathcal{L})$ is empty then $X^\circ(\mathcal{L})$ is the stratum used in the proof of Proposition 4.

The following proposition is a result of finiteness for the set of functions $\mathrm{M}^\bullet(x)$. It will be used to understand how $X^{\rm ss}(\mathcal{L})$ depends on $\mathcal{L}$ (see Lemma 7 below).

Proposition 7 *When x varies in X, one obtains only a finite number of functions* $\mathrm{M}^\bullet(x) : \Lambda^+_{\mathbb{Q}} \longrightarrow \mathbb{R}$.

Proof Let T be a maximal torus of G. Consider the partial forgetful map $r^T : \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}^T(X)$. Since $\mathrm{M}^\bullet(x) = \max_{g\in G} \mathrm{M}^{r^T(\bullet)}(g.x)$, it is sufficient to prove the proposition for the torus T.

If z and z' belong to the same irreducible component C of X^T then the morphisms $\mu^\bullet(z, T)$ and $\mu^\bullet(z', T)$ are equal.

By Proposition 2, $\mathrm{M}^{\mathcal{L}}(x)$ only depends on $\mathcal{P}^{\mathcal{L}}_T(x)$, which only depends on the set of irreducible components of X^T which intersect $\overline{T.x}$. Since, X^T has finitely many irreducible components, the proposition follows. □

Remark Proposition 7 implies that the set of open subsets of X which can be realized as $X^{\rm ss}(\mathcal{L})$ for some semiample G-linearized line bundle $\mathcal{L}$ on X is finite. This result is due to Dolgachev-Hu (see [9, Theorem 3.9]; see also [41]) if $\mathcal{L}$ is assumed to be ample.

5.2.2

We have the following characterization of $X^\circ(\mathcal{L})$:

Lemma 6 *Let $\mathcal{L}$ be a semiample G-linearized line bundle on X. If $X^{\rm ss}(\mathcal{L}) \neq \emptyset$, set $d_0 = 0$; else, set $d_0 = \min_{x\in X} \mathrm{M}^{\mathcal{L}}(x)$. Then, $X^\circ(\mathcal{L})$ is the set of $x \in X$ such that $\mathrm{M}^{\mathcal{L}}(x) \leq d_0$.*

Proof If $d_0 = 0$, $\mathrm{M}^{\mathcal{L}}(x) \leq 0$ if and only if $x \in X^{\rm ss}(\mathcal{L}) = X^\circ(\mathcal{L})$. We now assume that $d_0 > 0$.

Up to changing $\mathcal{L}$ by a positive power, one may assume that there exist a G-module V and a morphism $\phi : X \longrightarrow \mathbb{P}(V)$ such that $\mathcal{L} = \phi^*(\mathcal{O}(1))$.

Let us fix the positive real number d and an indivisible one parameter subgroup λ such that $X^\circ(\mathcal{L}) = S^{\mathcal{L}}_{d,\langle\lambda\rangle}$. For $i \in \mathbb{Z}$, set $V_i = \{v \in V \mid \lambda(t)v = t^i v\}$.

Set $V^+ = \oplus_{i>d\|\lambda\|} V_i$, $C = \{x \in X : \phi(x) \in \mathbb{P}(V_{d\|\lambda\|})\}$ and $\overline{C^+} = \{x \in X : \phi(x) \in \mathbb{P}(V_{d\|\lambda\|} \oplus V^+)\}$.

We first prove that $d = d_0$; that is, that $\mathrm{M}^{\mathcal{L}}(x) \geq d$ for any $x \in X$. Consider the morphism $\eta : G \times_{P(\lambda)} \overline{C^+} \longrightarrow X$. Since $G/P(\lambda)$ is projective, η is proper; but, the image of η contains $S^{\mathcal{L}}_{d,\langle\lambda\rangle}$; so, η is surjective. Let $x \in X$. There exists $g \in G$ such that $gx \in \overline{C^+}$. Then, $\mathrm{M}^{\mathcal{L}}(x) \geq \bar{\mu}^{\mathcal{L}}(gx, \lambda) \geq d$.

Conversely, let $x \in X$ such that $\mathrm{M}^{\mathcal{L}}(x) = d_0$. Let $g \in G$ such that $gx \in \overline{C}^+$. Let $v_1 \in V_{d\|\lambda\|}$ and $v_2 \in V^+$ such that $g\phi(x) = [v_1 + v_2]$. Since $\overline{\mu}^{\mathcal{L}}(gx, \lambda) \leq d_0$, v_1 is nonzero; so, $\overline{\mu}^{\mathcal{L}}(gx, \lambda) = d_0 = \mathrm{M}^{\mathcal{L}}(x)$. In particular, λ is adapted to x and $\mathcal{L}$; that is, $x \in X^\circ(\mathcal{L})$. □

5.2.3

We will need the following result of monotonicity for the function $\mathcal{L} \mapsto X^\circ(\mathcal{L})$:

Lemma 7 *With above notation, there exists an open neighborhood U of $\mathcal{L}$ in $\Lambda^+_{\mathbb{Q}}$ such that for any $\mathcal{M} \in U$, $X^\circ(\mathcal{M}) \subset X^\circ(\mathcal{L})$.*

Proof By Proposition 7 and Lemma 6, there exist only finitely many open subsets of X which are of the form $X^\circ(\mathcal{M})$ for $\mathcal{M} \in \Lambda^+_{\mathbb{Q}}$. Let $X^\circ_1, \dots, X^\circ_s$ be those which are not contained in $X^\circ(\mathcal{L})$. For each i, fix $x_i \in X^\circ_i - X^\circ(\mathcal{L})$. It remains to prove that for each i, there exists an open neighborhood U_i of $\mathcal{L}$ in $\Lambda^+_{\mathbb{Q}}$ such that $x_i \notin X^\circ(\mathcal{M})$ for any $\mathcal{M} \in U_i$. Indeed, $U = \cap_i U_i$ will work.

Fix $i \in \{1, \dots, s\}$. Set d_0 be as in Lemma 6 for $\mathcal{L}$. Since $x_i \notin X^\circ(\mathcal{L})$, $\mathrm{M}^{\mathcal{L}}(x_i) > d_0$. Let e be such that $\mathrm{M}^{\mathcal{L}}(x_i) > e > d_0$. By Proposition 7, there exists an open neighborhood U' of $\mathcal{L}$ in $\Lambda^+_{\mathbb{Q}}$ such that $\min_{x\in X} \mathrm{M}^{\mathcal{M}}(x) < e$ for any $\mathcal{M} \in U'$. Moreover, there exists U'', such that $\mathrm{M}^{\mathcal{M}}(x_i) > e$ for any $\mathcal{M} \in U''$. Lemma 6 implies that for any $\mathcal{M} \in U' \cap U''$, $x_i \notin X^\circ(\mathcal{M})$. □

5.3 A theorem of Luna

Notation If H is a subgroup of G, G^H denotes the centralizer of H in G.

We will use the following interpretation of a result of Luna:

Proposition 8 *Let $\mathcal{L}$ be a semiample G-linearized line bundle on an irreducible projective G-variety X. Let H be a reductive subgroup of G. Let C be an irreducible component of X^H. Then, the reductive group $(G^H)^\circ$ acts on C.*

Let x be a point in C. Then, the following are equivalent:

(i) *x is semistable for $\mathcal{L}$*;
(ii) *x is semistable for the action of $(G^H)^\circ$ on C and the restriction of $\mathcal{L}$.*

Proof [29, Lemma 1.1] shows that $(G^H)^\circ$ is reductive. Changing $\mathcal{L}$ by a positive power if necessary, one may assume there exist a G-module V and a G-equivariant morphism $\phi : X \longrightarrow \mathbb{P}(V)$ such that $\mathcal{L} = \phi^*(\mathcal{O}(1))$. Let $v \in V$ such that $[v] = \phi(x)$.

Let χ be the character of H such that $hv = \chi(h)v$ for any $h \in H$.

If χ has infinite order, so is its restriction to the connected center Z of H. Then, $Z.v = \mathbb{K}^* v$ and $0 \in \overline{(G^H)^\circ.v}$. In this case, x is unstable for the action of G or $(G^H)^\circ$.

Let us now assume that χ has finite order. Changing $\mathcal{L}$ by a positive power if necessary, one may assume that χ is trivial, that is H fixes v. In this case, [28, Corollary 2 and Remark 1] shows that

$$0 \in \overline{G.v} \iff 0 \in \overline{(G^H)^\circ.v}.$$

Since x is unstable if and only if $\overline{G.v}$ contains 0, the proposition follows. $\square$

5.4 Face viewed from a $\mathcal{L}$ without semistable points

5.4.1

Let $\mathcal{L}$ be a semiample line bundle in Λ without semistable points. Let d be the positive real number and λ be a one parameter subgroup of G such that $X^\circ(\mathcal{L}) = S^{\mathcal{L}}_{d,\langle\lambda\rangle}$.

Lemma 8 *We have*:

(i) *$Z^{\mathcal{L}}_{d,\lambda}$ is irreducible; and the closure C of $Z^{\mathcal{L}}_{d,\lambda}$ is an irreducible component of X^λ.*
(ii) *The pair (C, λ) is dominant. If in addition X is normal, the pair (C, λ) is well covering.*
(iii) *The conjugacy class of the pair (C, λ) only depends on $\mathcal{L}$ (and not on λ).*

Proof The two first assertions make more explicit the arguments used in the proof of Proposition 4. The proof is the same. Since λ is unique up to conjugacy, the last assertion follows. $\square$

Lemma 8 implies that the linear form $\mu^\bullet(C, \lambda)$ on $\Lambda_{\mathbb{Q}}$ only depends on $\mathcal{L}$. We set:

$$\mathcal{H}(\mathcal{L}) = \{\mathcal{M} \in \Lambda_{\mathbb{Q}} : \mu^{\mathcal{M}}(C, \lambda) = 0\}, \quad \text{and}$$
$$\mathcal{H}(\mathcal{L})^{>0} = \{\mathcal{M} \in \Lambda_{\mathbb{Q}} : \mu^{\mathcal{M}}(C, \lambda) > 0\}.$$

Lemma 3 implies that $\mathcal{H}(\mathcal{L}) \cap \mathcal{AC}^G_\Lambda(X)$ is a face of $\mathcal{AC}^G_\Lambda(X)$: this face is denoted by $\mathcal{AF}_\Lambda(\mathcal{L})$ and called the *face viewed from* $\mathcal{L}$.

The aim of this section is to prove

Theorem 7 *Any nonempty face $\mathcal{AF}$ of $\mathcal{AC}^G_\Lambda(X)$ with empty interior in $\Lambda_\mathbb{Q}$ is viewed from some ample point in Λ without semistable points.*

Sketch of proof The proof works by induction on the codimension of $\mathcal{AF}$ in $\Lambda_\mathbb{Q}$. If $\mathcal{AF}$ has codimension one, Proposition 4 allows to conclude. Otherwise, Proposition 4 gives a face (or a linear subspace if $\mathcal{AC}^G_\Lambda(X)$ has empty interior) $\mathcal{AF}_1$ of codimension one and containing $\mathcal{AF}$. Moreover, one can find an ample $\mathcal{L}$ without semistable points such that $\mathcal{AF}_1 = \mathcal{AF}(\mathcal{L})$. Let (C, λ) be the pair associated to $\mathcal{L}$ as in Lemma 8. Considering the restriction, we obtain a face of $\mathcal{AC}^{G^\lambda}(C)$ to which we can apply the induction. Lemmas 9, 10 and 11 below allow to compare the faces of $\mathcal{AC}^G(X)$ and those of $\mathcal{AC}^{G^\lambda}(C)$.

If X is assumed to be smooth (for example, for the applications to the branching cones) then C is smooth. So, the induction flows through smooth varieties. Now, if X is only assumed to be normal, C may not be. So, we have to work with nonnecessarily normal varieties; instead of, we work with G-projective varieties. Since our varieties are not assumed to be normal, we speak about faces viewed from $\mathcal{L}$ instead of associated to well covering pairs (see Lemma 8).

5.4.2

We start by proving Lemmas 9, 10 and 11 which study how $\mathcal{AF}_\Lambda(\mathcal{L})$ depends on $\mathcal{L}$. Let us fix $\mathcal{L} \in \Lambda^{++}_\mathbb{Q}$ without semistable points. Let (C, λ) be as in Lemma 8. Consider the subgroup H^λ of G^λ defined in Sect. 5.1. Consider the morphism induced by restriction $p : \Lambda_\mathbb{Q} \longrightarrow \mathrm{Pic}^{H^\lambda}(C)_\mathbb{Q}$. We first prove the following improvement of Lemma 3:

Lemma 9 *Let $\mathcal{M} \in \Lambda^{++}_\mathbb{Q}$. The following are equivalent*:

(i) $\mathcal{M} \in \mathcal{AF}_\Lambda(\mathcal{L})$;
(ii) $C^{\mathrm{ss}}(\mathcal{M}, G^\lambda)$ *is not empty*;
(iii) $\mathcal{M} \in \mathcal{H}(\mathcal{L})$ *and* $C^{\mathrm{ss}}(\mathcal{M}, H^\lambda)$ *is not empty.*

Remark Writing $C^{\mathrm{ss}}(\mathcal{M}, G^\lambda)$ for example, we write $\mathcal{M}$ instead of its restriction to C endowed with its G^λ-linearization.

Proof Note that $\mathcal{M} \in \mathcal{H}(\mathcal{L})$ if and only if λ acts trivially on $\mathcal{M}_{|C}$. The equivalence between the two last assertions follows. Let us assume that

$C^{\mathrm{ss}}(\mathcal{M}, G^\lambda)$ is not empty. There exists $x \in C$ which is semistable for the action of G^λ and $\mathcal{M}$. Proposition 8 shows that x is semistable for G and $\mathcal{M}$. So, $\mathcal{M} \in \mathcal{AC}^G_\Lambda(X)$.

Conversely, let $\mathcal{M} \in \mathcal{AF}_\Lambda(\mathcal{L})$. By Lemma 3, C intersects $X^{\mathrm{ss}}(\mathcal{M})$. So, $C^{\mathrm{ss}}(\mathcal{M}, G^\lambda)$ is not empty. □

Lemma 10 *Let $\mathcal{L}' \in \mathcal{H}(\mathcal{L})^{>0} \cap \Lambda^{++}_{\mathbb{Q}}$. We assume that $C^{\mathrm{ss}}(\mathcal{L}', H^\lambda)$ is not empty.*

Then, $\mathcal{AF}_\Lambda(\mathcal{L}') = \mathcal{AF}_\Lambda(\mathcal{L})$. More precisely, the pair (C, λ) satisfies Lemma 8 for $\mathcal{L}'$.

Proof Let $x \in C^+$ such that $\lim_{t\to 0} \lambda(t)x \in C^{\mathrm{ss}}(\mathcal{L}', H^\lambda)$. Note that the set of such x's is open in C^+. By Theorem 6 and Proposition 3, λ is adapted to x and $\mathcal{L}'$. Since (C, λ) is covering, $S^{\mathcal{L}'}_{\overline{\mu}^{\mathcal{L}'}(x,\lambda),\langle\lambda\rangle} = X^\circ(\mathcal{L}')$. The lemma follows. □

Lemma 10 shows that if $p(\mathcal{L})$ belongs to the interior of $\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C)$, the function $\mathcal{L} \mapsto \mathcal{AF}_\Lambda(\mathcal{L})$ is constant around $\mathcal{L}$. The following lemma implies that if $p(\mathcal{L})$ belongs to the boundary of $\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C)$, $\mathcal{AF}_\Lambda(\mathcal{L}')$ is also determined by $p(\mathcal{L}')$ for $\mathcal{L}'$ close to $\mathcal{L}$.

Lemma 11 *There exists an open neighborhood U of $\mathcal{L}$ in $\Lambda^{++}_{\mathbb{Q}}$ such that for any $\mathcal{L}'$ in U, $\mathcal{L}'$ has no semistable points and $\mathcal{AF}_\Lambda(\mathcal{L}') \subset \mathcal{AF}_\Lambda(\mathcal{L})$.*

If in addition the above inclusion is strict then $C^{\mathrm{ss}}(\mathcal{L}', H^\lambda)$ is empty. Moreover, $\mathcal{AF}_\Lambda(\mathcal{L}')$ is the set of $\mathcal{M} \in \mathcal{H}(\mathcal{L}) \cap \Lambda^{++}_{\mathbb{Q}}$ such that $p(\mathcal{M})$ belongs to the face of $\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C)$ viewed from $p(\mathcal{L}')$.

Proof We will first describe $X^\circ(\mathcal{L}')$ in terms of the action of H^λ on C, for $\mathcal{L}'$ sufficiently close to $\mathcal{L}$. Let $x \in C^+ \cap X^\circ(\mathcal{L})$. Set $d = \overline{\mu}^{\mathcal{L}}(C, \lambda)$ and $z = \lim_{t\to 0} \lambda(t).x$. Let $\mathcal{L}' \in \Lambda^{++}_{\mathbb{Q}}$ be without semistable points. We assume that $C^{\mathrm{ss}}(\mathcal{L}', H^\lambda)$ is empty and z belongs to $C^\circ(\mathcal{L}', H^\lambda)$. We claim that for $\mathcal{L}'$ sufficiently close to $\mathcal{L}$, x belongs to $X^\circ(\mathcal{L}')$.

Let T be a maximal torus of G containing the image of λ. By [21, Lemma 12.19], there exists a one parameter subgroup adapted to z and $\mathcal{L}'$ which commutes with λ. So, there exist $h_0 \in H^\lambda$ and a one parameter subgroup ζ of T which is adapted to $h_0 z$ and $\mathcal{L}'$. Set $d' = \overline{\mu}(h_0 z, \zeta)$.

By Theorem 2, λ is adapted to z and $\mathcal{L}$, and so to $h_0 z$ and $\mathcal{L}$. In particular, λ is the unique one parameter subgroup of T adapted to $h_0 z$, $\mathcal{L}$ and the action of T. On the other hand, ζ is the unique one parameter subgroup of T adapted to $h_0 z$, $\mathcal{L}'$ and the action of T. It follows that $\frac{\zeta}{\|\zeta\|} \in Y(T)_{\mathbb{R}}$ tends to $\frac{\lambda}{\|\lambda\|}$ when $\mathcal{L}'$ tends to $\mathcal{L}$. Now, working in $\overline{T.h_0 x}$, we deduce that $\lim_{t\to 0} \zeta(t) h_0 x = \lim \zeta(t) h_0 z =: z'$ for $\mathcal{L}'$ sufficiently close to $\mathcal{L}$.

Let H^{ζ} be the subgroup of G^{ζ} defined in Sect. 5.1. Since ζ is adapted to $h_0 z$ and $\mathcal{L}'$, Theorem 6 implies that z' is semistable for the action H^{ζ} and $\mathcal{L}'$. So, by Theorem 6 again, ζ is adapted to $h_0 x$ and $\mathcal{L}'$. It follows that x belongs to $S^{\mathcal{L}'}_{d',\langle\zeta\rangle}$.

The set of $x \in C^+$ satisfying our assumption ($z \in C^{\circ}(\mathcal{L}', H^{\lambda})$) is open in C^+. Since $G.(C^+ \cap X^{\circ}(\mathcal{L}))$ is dense in X, this implies that $S^{\mathcal{L}'}_{d',\langle\zeta\rangle}$ is $X^{\circ}(\mathcal{L}')$. Remark that we need to find a neighborhood of $\mathcal{L}$ which works for any such $x \in C^+$. This is possible, since the neighborhood constructed above only depends on the set of irreducible components of X^T which intersect $\overline{T.h_0 x}$; in particular, there are only finitely many possibilities. This proves the claim.

Let C_{ζ} denote the closure of $Z^{\mathcal{L}'}_{d',\zeta}$. Since $\frac{\zeta}{\|\zeta\|} \in Y(T)_{\mathbb{R}}$ tends to $\frac{\lambda}{\|\lambda\|}$, one may assume that C_{ζ} is contained in C. It follows that $\overline{\mu}^{\bullet}(C_{\zeta}, \zeta)$ tends to $\overline{\mu}^{\bullet}(C, \lambda)$ when $\mathcal{L}'$ tends to $\mathcal{L}$. By a general argument in convex geometry, this implies that there exists an open neighborhood U of $\mathcal{L}$, such that $\mathcal{AF}_{\Lambda}(\mathcal{L}') \subset \mathcal{AF}_{\Lambda}(\mathcal{L})$ for any $\mathcal{L}' \in U$.

Let us now assume that $\mathcal{AF}_{\Lambda}(\mathcal{L}') \neq \mathcal{AF}_{\Lambda}(\mathcal{L})$ for a given $\mathcal{L}' \in U$.

By Lemma 10, $C^{\mathrm{ss}}(\mathcal{L}', H^{\lambda})$ is empty. Let $\mathcal{M} \in \mathcal{AF}_{\Lambda}(\mathcal{L}')$. By Lemma 9, $C^{\mathrm{ss}}_{\zeta}(\mathcal{M}, G^{\zeta})$ and so $C^{\mathrm{ss}}_{\zeta}(\mathcal{M}, G^{\zeta} \cap H^{\lambda})$ are not empty. By Lemma 9 again, $p(\mathcal{M})$ belongs to the face of $\mathcal{AC}^{H^{\lambda}}_{p(\Lambda)}(C)$ viewed from $p(\mathcal{L}')$. Conversely, let $\mathcal{M} \in \mathcal{H}(\mathcal{L}) \cap \Lambda^{++}_{\mathbb{Q}}$ such that $p(\mathcal{M})$ belongs to the face of $\mathcal{AC}^{H^{\lambda}}_{p(\Lambda)}(C)$ viewed from $p(\mathcal{L}')$. Since $\frac{\zeta}{\|\zeta\|} \in Y(T)_{\mathbb{R}}$ tends to $\frac{\lambda}{\|\lambda\|}$ when $\mathcal{L}'$ tends to $\mathcal{L}$, we may assume that $G^{\zeta} \subset G^{\lambda}$. In this case, $G^{\zeta} = (G^{\zeta} \cap H^{\lambda}).\mathrm{Im}\lambda$. Now, Lemma 9 implies that $\mathcal{M} \in \mathcal{AF}_{\Lambda}(\mathcal{L}')$. $\square$

5.4.3

We can now prove the theorem.

Proof of Theorem 7 We will prove the following assertion, by induction on the codimension $\mathrm{codim}_{\Lambda}(\mathcal{AF})$ of $\mathcal{AF}$ in $\Lambda_{\mathbb{Q}}$ using Lemma 11:

Let X be a not necessarily normal variety. Let G, Λ and $\mathcal{AF}$ be as in the theorem. Let U be an open subset of $\Lambda_{\mathbb{Q}}$ intersecting $\mathcal{AF}$. Then, there exists $\mathcal{L} \in U$ such that $\mathcal{AF}_{\Lambda}(\mathcal{L}) = \mathcal{AF}$.

Let $\mathcal{M}$ be a point in the relative interior of $\mathcal{AF} \cap U$. By Lemma 7, there exists $\mathcal{L} \in U$ without semistable points and such that $X^{\circ}(\mathcal{L}) \subset X^{\mathrm{ss}}(\mathcal{M})$. Let (C, λ) be associated to $\mathcal{L}$ as in Lemma 8. Since C intersects $X^{\circ}(\mathcal{L})$ it intersects $X^{\mathrm{ss}}(\mathcal{M})$; so, $\mu^{\mathcal{M}}(C, \lambda) = 0$ and $\mathcal{M} \in \mathcal{AF}_{\Lambda}(\mathcal{L})$. Since $\mathcal{M}$ belongs to the relative interior of $\mathcal{AF}$, we deduce that $\mathcal{AF} \subset \mathcal{AF}_{\Lambda}(\mathcal{L})$. We may assume that this inclusion is strict; otherwise, we have finished.

We now want to "push" $p(\mathcal{L})$ on the boundary of $\mathcal{AC}^{H^{\lambda}}_{p(\Lambda)}(C)$ without changing $\mathcal{AC}_{\Lambda}(\mathcal{L})$ By Lemma 9, $\mathcal{AF} \cap U$ is a face of $U \cap \mathcal{H}(\mathcal{L}) \cap$

$p^{-1}(\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C))$. So, there exists a face $\tilde{\mathcal{AF}}_1$ of $p^{-1}(\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C))$ such that $\mathcal{AF} = \Lambda^{++}_{\mathbb{Q}} \cap \mathcal{H}(\mathcal{L}) \cap \tilde{\mathcal{AF}}_1$. Let us assume that $\tilde{\mathcal{AF}}_1$ is of maximal dimension among such faces. Since $\mathcal{H}(\mathcal{L})^{>0} \cap p^{-1}(\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C))$ is nonempty (it contains $\mathcal{L}$!), we may assume that $\tilde{\mathcal{AF}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$. Let $\mathcal{AF}_1$ denote the unique face of $\mathcal{AC}^{H^\lambda}_{p(\Lambda)}(C)$ such that $\tilde{\mathcal{AF}}_1 = p^{-1}(\mathcal{AF}_1)$.

Since $\tilde{\mathcal{AF}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$, Lemma 10 allows to move $\mathcal{L}$ on $\tilde{\mathcal{AF}}_1$ without changing $\mathcal{AF}_\Lambda(\mathcal{L})$ nor (C,λ). From now on, we assume that $\mathcal{L} \in \tilde{\mathcal{AF}}_1$.

Let $U' \subset U$ be an open neighborhood of $\mathcal{L}$ contained in $\mathcal{H}(\mathcal{L})^{>0}$ and satisfying Lemma 11. Since $\mathcal{AF} \subset \tilde{\mathcal{AF}}_1 \cap \mathcal{H}(\mathcal{L})$ and $\tilde{\mathcal{AF}}_1$ intersects $\mathcal{H}(\mathcal{L})^{>0}$, we have: $\mathrm{codim}_\Lambda(\tilde{\mathcal{AF}}_1) < \mathrm{codim}_\Lambda \mathcal{AF}$. But, $\mathrm{codim}_\Lambda(\tilde{\mathcal{AF}}_1) = \mathrm{codim}_{p(\Lambda)}(\mathcal{AF}_1)$. Moreover, p is linear and so open. We apply the induction to the action of H^λ on C, the face $\mathcal{AF}_1$ and the open subset $p(U')$ of $p(\Lambda)_{\mathbb{Q}}$: there exists $\mathcal{L}'$ in U' such that $\mathcal{AF}_\Lambda(p(\mathcal{L}')) = \mathcal{AF}_1$. Lemma 11 implies now that $\mathcal{AF}_\Lambda(\mathcal{L}') = \mathcal{AF}$. □

5.5 From faces to well covering pairs

Theorem 7 can be restated in terms of well covering pairs:

Corollary 2 *We assume that X is normal. Let $\mathcal{AF}$ be a nonempty face of $\mathcal{AC}^G_\Lambda(X)$ of codimension $r \geq 1$ in Λ. Then, there exist a r-dimensional torus S in G, a one parameter subgroup λ of S and an irreducible component C of X^S such that*:

(i) *C is an irreducible component of X^λ*;
(ii) *the pair (C,λ) is well covering*;
(iii) $\mathcal{AF} = \mathcal{AF}_\Lambda(C)$.

Proof Let $\mathcal{L}$ be an ample line bundle on X without semistable points and such that $\mathcal{AF}$ is viewed from $\mathcal{L}$. Let d be the positive number and λ be a one parameter subgroup of G such that $S^{\mathcal{L}}_{d,\langle\lambda\rangle}$ is open in X. Let C be the closure of $Z^{\mathcal{L}}_{d,\lambda}$.

By Lemma 8, it remains to prove that there exists a torus S of dimension r containing the image of λ and acting trivially on C. The proof of the existence of such a S can easily be integrated in the induction of the proof of Theorem 7. □

The next statement is a precision of Corollary 2.

Corollary 3 *Let X be a normal projective variety with diagonalizable reductive isotropies. We assume that $\Lambda \subset \mathrm{Pic}^G(X)$ is abundant* (*see Sect.* 3.3.3). *Let $\mathcal{AF}$ be a nonempty face of $\mathcal{AC}^G_\Lambda(X)$ of codimension $r \geq 1$ in Λ. Then,* *there exist a r-dimensional torus S in G, a one parameter subgroup λ of S and an irreducible component C of X^S such that*:

(i) *for general $x \in C$, we have $G_x^\circ = S$*;
(ii) *C is an irreducible component of X^λ*;
(iii) *the pair (C, λ) is well covering*;
(iv) *$\mathcal{AF} = \mathcal{AF}_\Lambda(C)$.*

Proof We just have to prove that if (C, λ) and S satisfy Corollary 2, they also satisfy the first assertion.

Let $\mathcal{L}$ be an ample G-linearized line bundle on X and x be a semistable point for $\mathcal{L}$. We claim that G_x is diagonalizable. Consider $\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})//G$. Let y be a point in the closure of $G.x$ such that $G.y$ is closed in $X^{\mathrm{ss}}(\mathcal{L})$. The isotropy G_y is reductive and so diagonalizable. By the Luna slice theorem, $\pi^{-1}(\pi(y)) \simeq G \times_{G_y} \Sigma$ for an affine G_y variety Σ. Since $x \in \pi^{-1}(\pi(y))$, we deduce that G_x is conjugated to a subgroup of G_y. In particular, G_x is diagonalizable.

Let us fix a point x in the intersection of the finitely many sets $X^{\mathrm{ss}}(\mathcal{L}) \cap C$ for $\mathcal{L} \in \mathcal{AF}$.

Since Λ is abundant and G_x diagonalizable, the rank of $\mu^\bullet(x, G_x)$ equals the dimension of G_x. Since $\mathcal{AF}$ is contained in the kernel of $\mu^\bullet(x, G_x)$, the dimension of G_x is less or equal to r. Since $S \subset G_x$, it follows that $G_x^\circ = S$. □

Remark As in the proof of Theorem 3, let $\mathcal{L}$ be in $\mathcal{AF}$ whose the GIT-class has nonempty interior in $\mathcal{AF}$. In the above proof, x can be replaced by any point whose the G-orbit is closed in $X^{\mathrm{ss}}(\mathcal{L})$. In this way, we do not use the Luna slice theorem.

6 About the faces corresponding to dominant pairs

6.1 Dominant pairs and quotient varieties

The following proposition is a description of the quotient variety associated to a point in $\mathcal{AC}_\Lambda^G(X)$ which belongs to a face associated to a dominant pair.

Proposition 9 *Let (C, λ) be a dominant pair and $\mathcal{L}$ be an ample G-linearized line bundle on X such that $\mu^{\mathcal{L}}(C, \lambda) = 0$ and $X^{\mathrm{ss}}(\mathcal{L}) \neq \emptyset$.*

Consider the morphism θ which makes the following diagram commutative:

$$\begin{tikzcd} C^{\mathrm{ss}}(\mathcal{L}, G^\lambda) \arrow[r, hook] \arrow[d, "\pi_C"'] & X^{\mathrm{ss}}(\mathcal{L}) \arrow[d, "\pi"] \\ C^{\mathrm{ss}}(\mathcal{L}, G^\lambda)/\!/G^\lambda \arrow[r, "\theta"] & X^{\mathrm{ss}}(\mathcal{L})/\!/G. \end{tikzcd}$$

Then, θ is finite and surjective.

Proof The inclusion of $C^{\mathrm{ss}}(\mathcal{L}, G^\lambda)$ in $X^{\mathrm{ss}}(\mathcal{L})$ is a direct consequence of Proposition 8. Since $C^{\mathrm{ss}}(\mathcal{L}, G^\lambda)/\!/G^\lambda$ is projective, to prove the proposition, it is sufficient to prove that θ is dominant and its fibers are finite.

Since (C, λ) is dominant, C^+ must intersect $X^{\mathrm{ss}}(\mathcal{L})$. Let $x \in C^+ \cap X^{\mathrm{ss}}(\mathcal{L})$. Set $z = \lim_{t\to 0} \lambda(t)x$. By Assertion (ii) of Lemma 2, z is semistable for $\mathcal{L}$. This implies that $\pi(C^+ \cap X^{\mathrm{ss}}(\mathcal{L})) = \pi(C^{\mathrm{ss}}(\mathcal{L}, G^\lambda))$. It follows that θ is dominant.

Let $\xi \in X^{\mathrm{ss}}(\mathcal{L})/\!/G$. Let $\mathcal{O}_\xi$ be the unique closed G-orbit in $\pi^{-1}(\xi)$. The points in the fiber $\theta^{-1}(\xi)$ correspond bijectively to the closed G^λ-orbits in $\pi^{-1}(\xi) \cap C$. But, [28, Corollary 2 and Remark 1] implies that these orbits are contained in $\mathcal{O}_\xi$. We conclude that $\theta^{-1}(\xi)$ is finite, by using [40, Theorem A] which implies that $\mathcal{O}_\xi^\lambda$ contains only finitely many G^λ-orbits. □

6.2 If $X = \hat{G}/\hat{B} \times Y \ldots$

In this section, we will explain how Proposition 9 can be improved if $X = \hat{G}/\hat{B} \times Y$ with notation of Sect. 3.3.2.

Theorem 8 *We use notation of Proposition 9, assuming in addition that $X = \hat{G}/\hat{B} \times Y$ with a normal projective G-variety Y.*

Then, θ is an isomorphism and (C, λ) is a well covering pair.

Proof Since X is normal, so is $X^{\mathrm{ss}}(\mathcal{L})/\!/G$. But, Proposition 9 shows that θ is finite; it is sufficient to prove that it is birational. Since the base field has characteristic zero, it remains to prove that θ is bijective to obtain the first assertion.

Let $\xi \in X^{\mathrm{ss}}(\mathcal{L})/\!/G$ and $\mathcal{O}_\xi$ denote the only closed G-orbit in $\pi^{-1}(\xi)$. As noticed in the proof of Proposition 9, the points in $\theta^{-1}(\xi)$ correspond bijectively to the G^λ-orbits in $\mathcal{O}_\xi \cap C$. So, by [40] it is sufficient to prove that $\mathcal{O}_\xi \cap C$ is irreducible.

Consider the first projection $p_1 : X \longrightarrow \hat{G}/\hat{B}$ and fix $x \in \mathcal{O}_\xi \cap C$. Let $\hat{B}$ denote the stabilizer in $\hat{G}$ of $p_1(x)$. Since $\mathcal{O}_\xi$ is closed in $\pi^{-1}(\xi)$, G_x

is diagonalizable. Let $\hat{T}$ be a maximal torus of $\hat{G}$ such that $G_x \subset \hat{T} \subset \hat{B}$. Consider the morphisms $q : \mathcal{O}_\xi \longrightarrow \hat{G}/\hat{T}$, $g.x \mapsto g\hat{T}/\hat{T}$ and $\hat{q} : \hat{G}/\hat{T} \longrightarrow \hat{G}/\hat{B}$, $\hat{g}\hat{T}/\hat{T} \mapsto \hat{g}\hat{B}/\hat{B}$ induced by these inclusions; we have, $p_1 = \hat{q} \circ q$.

We claim that $\hat{q}^{-1}(\hat{G}^\lambda \hat{B}/\hat{B})^\lambda = \hat{G}^\lambda \hat{T}/\hat{T}$. Since $\hat{G}^\lambda$ is connected, each irreducible component of $\hat{q}^{-1}(\hat{G}^\lambda \hat{B}/\hat{B})^\lambda$ is $\hat{G}^\lambda$-stable, and so, it maps onto $\hat{G}^\lambda \hat{T}/\hat{T}$. In particular, it intersects $\hat{q}^{-1}(\hat{B}/\hat{B}) = \hat{B}/\hat{T}$. But, $\hat{B}/\hat{T}$ is isomorphic to the Lie algebra of the unipotent radical of $\hat{B}$ as a $\hat{T}$-variety; in particular, $(\hat{B}/\hat{T})^\lambda$ is irreducible and so is $\hat{q}^{-1}(\hat{G}^\lambda \hat{B}/\hat{B})^\lambda$. The claim now follows from [40].

Note that C is the product of one irreducible component of $(\hat{G}/\hat{B})^\lambda$ and one of Y^λ; this implies that $p_1(C) = \hat{G}^\lambda \hat{B}/\hat{B}$. So, $\mathcal{O}_\xi \cap C$ is contained in $p_1^{-1}(\hat{G}^\lambda \hat{B}/\hat{B})^\lambda$ and so in $\mathcal{O}_\xi \cap q^{-1}(\hat{G}^\lambda \hat{T}/\hat{T})$. But, since $\hat{T}$ is contained in $\hat{G}^\lambda$, $\hat{G}^\lambda \hat{T}/\hat{T} \cap G\hat{T}/\hat{T} = G^\lambda \hat{T}/\hat{T}$. It follows that $\mathcal{O}_\xi \cap C = G^\lambda.x$ is irreducible. This ends the proof of the first assertion.

Consider now $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$. We claim that for any $x \in C^+ \cap X^{\mathrm{ss}}(\mathcal{L})$, $\eta^{-1}(x)$ is only one point. Since η is dominant and the ground field has characteristic zero, the claim implies that η is birational. Since $X^{\mathrm{ss}}(\mathcal{L}) \cap C$ is nonempty, the claim implies that η is bijective over on open subset of X intersecting C. Since X is normal, Zariski's main theorem (see [12, §8.12]) implies that (C, λ) is well covering.

Let us prove the claim. Let $g \in G$ such that $g^{-1}x \in C^+$. We have to prove that $g \in P(\lambda)$. Set $x' = g^{-1}x$, $z = \lim_{t\to 0} \lambda(t)x$ and $z' = \lim_{t\to 0} \lambda(t)x'$. Obviously $x' \in X^{\mathrm{ss}}(\mathcal{L})$, and by Assertion (ii) of Lemma 2, z, $z' \in X^{\mathrm{ss}}(\mathcal{L})$ as well. It is also clear that $\pi(x) = \pi(x') = \pi(z) = \pi(z') =: \xi$.

Let $x_0 \in C \cap \pi^{-1}(\xi)$ whose the orbit is closed in $X^{\mathrm{ss}}(\mathcal{L})$. Set $H = G_{x_0}$. Consider the set Σ (resp. Σ_C) of y in X (resp. C) such that x_0 is contained in the closure of $H.y$ (resp. of $H^\lambda.y$). By [30] (see also [37]), $\pi^{-1}(\xi) \simeq G \times_H \Sigma$ and $\pi_C^{-1}(\theta^{-1}(\xi)) \simeq G^\lambda \times_{H^\lambda} \Sigma_C$. Consider the natural G-equivariant morphism $\gamma : G \times_H \Sigma \longrightarrow G/H$. Since $\pi^{-1}(\xi) \cap C = \pi_C^{-1}(\theta^{-1}(\xi))$, it equals $G^\lambda.\Sigma_C$. So, $\gamma(\pi^{-1}(\xi) \cap C) = G^\lambda H/H$. Since γ is "continuous" and G-equivariant, we deduce that $\lim_{t\to 0} \lambda(t)\gamma(x)$ and $\lim_{t\to 0} \lambda(t)\gamma(x')$ belong to $G^\lambda H/H$. Lemma 12 below proves that there exist p and p' in $P(\lambda)$ such that $\gamma(x) = pH/H$ and $\gamma(x') = p'H/H$. Since $gx' = x$, we have $g \in pHp'^{-1}$. By Proposition 5, H is diagonalizable. But, H contains the image of λ; so, $H \subset G^\lambda$. Finally, $g \in P(\lambda)$. □

It remains to prove the following lemma. In fact, it is an adaptation of the main result of [40]:

Lemma 12 *Let $\mathcal{O}$ be any G-homogeneous space. Let λ be a one parameter subgroup of G. Let C be an irreducible component of $\mathcal{O}^\lambda$. Set $C^+ := \{x \in \mathcal{O} :$ $\lim_{t\to 0} \lambda(t)x$ exists and belongs to $C\}$.*

Then, C^+ is a $P(\lambda)$-orbit.

Proof Let $x \in C$. The differential of the map $g \mapsto g.x$ induces a surjective linear map $\phi : \mathfrak{g} \longrightarrow T_x\mathcal{O}$. Since x is fixed by λ, λ acts on $T_x\mathcal{O}$; it also acts by the adjoint action on $\mathfrak{g}$, and ϕ is equivariant. In particular, the restriction $\bar{\phi} : \mathfrak{g}_{\geq 0} \longrightarrow (T_x\mathcal{O})_{\geq 0}$ of ϕ is also surjective. But, on one hand $(T_x\mathcal{O})_{\geq 0} = T_xC^+$ by Theorem 5 and on the other hand $\mathfrak{g}_{\geq 0}$ is the Lie algebra of $P(\lambda)$. One can conclude that $T_xC^+ = T_xP(\lambda)x$. Since $P(\lambda)x$ is smooth, this implies that $P(\lambda)x$ is open in C^+. Since C^+ is irreducible, it contains a unique open $P(\lambda)$-orbit $\mathcal{O}_0$ which contains C.

Note that $C^+ - \mathcal{O}_0$ is $P(\lambda)$-stable and closed in C^+. Since for any $y \in C^+$ $\lim_{t\to 0}\lambda(t)y \in C \subset \mathcal{O}_0$, this implies that $C^+ - \mathcal{O}_0$ is empty. □

Remark Here, is an example which proves that the assumption on X in Theorem 8 is useful. Let V be a vector space of dimension 2. Make the group $G = \mathbb{K}^* \times \mathrm{SL}(V)$ acting on $X = \mathbb{P}(\mathbb{K} \oplus V \oplus V)$ by:

$$(t, g).[\tau : v_1 : v_2] = [\tau, gv_1, t^2gv_2]. \tag{4}$$

Consider (with obvious notation), the following one parameter subgroup

$$\lambda(t) = \left(t^{-1}, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right).$$

Note that $C = \{[1 : 0 : 0]\}$ is an irreducible component of X^λ. Formula (4) gives also a linearization $\mathcal{L}$ of the line bundle $\mathcal{O}(1)$. Then, $\mu^{\mathcal{L}}(C, \lambda) = 0$, and $\mathcal{L} \in \mathcal{AC}^G(X)$ (more precisely, $X^{\mathrm{ss}}(\mathcal{L})$ is the set of $[\tau : v_1 : v_2]$ such that $\tau \neq 0$). The map η associated to (C, λ) is birational, but the fiber over the only point of C is $\mathbb{P}^1$: so (C, λ) is covering but not well covering.

7 If $X = G/Q \times \hat{G}/\hat{Q}$...

7.1 Interpretations of the GIT-cones

7.1.1

From now on, we assume that G is embedded in a connected reductive group $\hat{G}$. Let us fix maximal tori T (resp. $\hat{T}$) and Borel subgroups B (resp. $\hat{B}$) of G (resp. $\hat{G}$) such that $T \subset B \subset \hat{B} \supset \hat{T} \supset T$. Let Q (resp. $\hat{Q}$) be a parabolic subgroup of G (resp. $\hat{G}$) containing B (resp. $\hat{B}$); let L (resp. $\hat{L}$) denote the Levi subgroup of Q (resp. $\hat{Q}$) containing T (resp. $\hat{T}$).

In this section, X denotes the variety $G/Q \times \hat{G}/\hat{Q}$ endowed with the diagonal action of G. We will apply the results of the preceding sections to X with $\Lambda = \mathrm{Pic}^G(X)$.

7.1.2

Let us describe $\mathrm{Pic}^G(X)_{\mathbb{Q}}$. Consider the natural action of $G \times \hat{G}$ on X. Using notation of Sect. 3.3.1 we have the following isomorphism $X(Q) \times X(\hat{Q}) \longrightarrow \mathrm{Pic}^{G\times\hat{G}}(X)$, $(\nu, \hat{\nu}) \longmapsto \mathcal{L}_{(\nu,\hat{\nu})}$.

Lemma 13 *The following complex is exact*

$$0 \longrightarrow X(\hat{G})_{\mathbb{Q}} \xrightarrow{\hat{\nu} \mapsto \mathcal{L}_{(-\hat{\nu}_{|Q},\, \hat{\nu}_{|\hat{Q}})}} \mathrm{Pic}^{G\times\hat{G}}(X)_{\mathbb{Q}} \xrightarrow{r^{\Delta G}_{\mathbb{Q}}} \mathrm{Pic}^G(X)_{\mathbb{Q}} \longrightarrow 0,$$

where the second linear map is induced by the restriction $r^{\Delta G}$ of the action of $G \times \hat{G}$ to G diagonally embedded in $G \times \hat{G}$.

Proof Let χ and $\hat{\chi}$ be characters of G and $\hat{G}$ respectively. The trivial bundle on X linearized by $(\chi, \hat{\chi})$ belongs to $\mathrm{Pic}^{G\times\hat{G}}(X)$. The image of this line bundle in $\mathrm{Pic}^G(X)$ is the trivial line bundle linearized by the character $\chi + \hat{\chi}_{|G}$ of G. In particular, any G-linearization of the trivial bundle belongs to the image of $r^{\Delta G}$.

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. Let $\mathcal{L}' \in \mathrm{Pic}(X)$ be obtained from $\mathcal{L}$ by forgetting the action of G. By [24], there exists a $G \times \hat{G}$-linearization $\mathcal{M}$ of $\mathcal{L}'^{\otimes n}$ for some positive integer n. Then $\mathcal{M}^* \otimes \mathcal{L}^{\otimes n}$ is the trivial line bundle over X; so, it belongs to the image of $r^{\Delta G}$. Finally, $\mathcal{L}^{\otimes n}$ belongs to the image of $r^{\Delta G}$. This proves that $r^{\Delta G}_{\mathbb{Q}}$ is surjective.

Let $\mathcal{L}$ be in the kernel of $r^{\Delta G}$. Since $\mathcal{L}$ is trivial as a line bundle, there exist characters χ and $\hat{\chi}$ of G and $\hat{G}$ such that $\mathcal{L} = \mathcal{L}_{(\chi_{|Q},\, \hat{\chi}_{|\hat{Q}})}$. The G-linearization of this line bundle is trivial if and only if $\chi + \hat{\chi}_{|G}$ is trivial. This ends the proof of the lemma. $\square$

7.1.3

The relations between the three GIT-cones of X and the branching rule problem are as follow:

Proposition 10

(i) *$\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$ is a closed convex polyhedral cone in* $\mathrm{Pic}^G(X)_{\mathbb{Q}}$.
(ii) *Let $(\nu, \hat{\nu}) \in X(Q)_{\mathbb{Q}} \times X(\hat{Q})_{\mathbb{Q}}$. Then, $r^{\Delta G}_{\mathbb{Q}}(\mathcal{L}_{(\nu,\hat{\nu})}) \in \mathcal{TC}^G(X)$ if and only if ν and $\hat{\nu}$ are dominant and for some positive integer n $V_{n\nu} \otimes V_{n\hat{\nu}}$ contains nonzero G-invariant vectors.*
(iii) *If $\mathcal{AC}^G(X)$ is nonempty, its closure in* $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ *is $\mathcal{SAC}^G(X)$.*
(iv) *If Q and $\hat{Q}$ are Borel subgroups of G and $\hat{G}$ then $\mathcal{AC}^G(X)$ is nonempty.*

Proof Since X is homogeneous under the action of $G \times \hat{G}$ and since any line bundle has a linearizable power, every bundle with a nonzero section is semi-ample. Hence, $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$. Note that $\Lambda^+_{\mathbb{Q}}$ is closed in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$. Proposition 4 implies that $\mathcal{SAC}^G(X)$ is a closed convex cone.

Let now ν and $\hat{\nu}$ be characters of Q and $\hat{Q}$. If ν or $\hat{\nu}$ is not dominant then $\mathcal{L}_{(\nu,\hat{\nu})}$ has no regular section. Otherwise, the Borel-Weil theorem shows that $\mathrm{H}^0(X, \mathcal{L}_{(\nu,\hat{\nu})})$ is isomorphic as a $G \times \hat{G}$-module to $V^*_\nu \otimes V^*_{\hat{\nu}}$. The second assertion of the proposition follows.

The third assertion is satisfied since $\Lambda^{++}_{\mathbb{Q}}$ is the interior of $\Lambda^+_{\mathbb{Q}}$.

Let w_0 be the longest element of the Weyl group of G. Assume that $Q = B$ and $\hat{Q} = \hat{B}$. Let $\hat{\nu}_0$ be any character of $\hat{B}$ such that $\mathcal{L}_{\hat{\nu}_0}$ is ample over $\hat{G}/\hat{B}$. Let ν be any dominant weight of the G-module $V^*_{\hat{\nu}_0}$. Then, $r^{\Delta G}(\mathcal{L}_{(-w_0\nu,\hat{\nu}_0)})$ belongs to $\mathcal{SAC}^G(X)$.

Let ν_0 be any character of B such that $\mathcal{L}_{\nu_0}$ is ample over G/B. Since the restriction $\mathbb{K}[\hat{G}] \longrightarrow \mathbb{K}[G]$ is surjective, the Frobenius theorem implies that $V^*_{\nu_0}$ is contained in an irreducible $\hat{G}$-module $V^*_{\hat{\nu}}$. Then, $r^{\Delta G}(\mathcal{L}_{(-w_0\nu_0,\hat{\nu})})$ belongs to $\mathcal{SAC}^G(X)$.

Since $\mathcal{SAC}^G(X)$ is convex, it contains $r^{\Delta G}(\mathcal{L}_{(-w_0(\nu_0+\nu),\hat{\nu}+\hat{\nu}_0)})$. But the line bundle $\mathcal{L}_{(-w_0(\nu_0+\nu),\hat{\nu}+\hat{\nu}_0)}$ is ample. The last assertion is proved.

Since $\mathcal{AC}^G(X)$ is polyhedral in $\Lambda^{++}_{\mathbb{Q}}$ and $\Lambda^+_{\mathbb{Q}}$ is polyhedral, the two last assertions imply that $\mathcal{SAC}^G(G/B \times \hat{G}/\hat{B})$ is polyhedral. But, $\mathcal{SAC}^G(G/Q \times \hat{G}/\hat{Q})$ identifies with a linear section of $\mathcal{SAC}^G(G/B \times \hat{G}/\hat{B})$; so, it is polyhedral. $\square$

7.2 Dominant and well covering pairs

Notation Let W and $\hat{W}$ denote the Weyl groups of G and $\hat{G}$. If P is a parabolic subgroup of G containing T, W_P denotes the Weyl group of the Levi subgroup of P containing T. This group W_P is canonically a subgroup of W.

We are going to describe the well covering pairs in the case when $X = G/Q \times \hat{G}/\hat{Q}$.

7.2.1

Let λ be a one parameter subgroup of T and so of $\hat{T}$. It is well known that the fixed point set of λ in X is

$$X^\lambda = \bigcup_{\substack{w \in W_{P(\lambda)} \backslash W / W_Q \\ \hat{w} \in \hat{W}_{\hat{P}(\lambda)} \backslash \hat{W} / \hat{W}_{\hat{Q}}}} G^\lambda w Q/Q \times \hat{G}^\lambda \hat{w} \hat{Q}/\hat{Q}.$$

For $(w, \hat{w}) \in W_Q\backslash W/W_{P(\lambda)} \times \hat{W}_{\hat{Q}}\backslash\hat{W}/\hat{W}_{\hat{P}(\lambda)}$, we set

$$C(w, \hat{w}) = G^\lambda w^{-1} Q/Q \times \hat{G}^\lambda \hat{w}^{-1} \hat{Q}/\hat{Q}.$$

Note that, for later use, we have introduced a $^{-1}$ in this definition. Note also that (see for example Lemma 12)

$$C^+(w, \hat{w}) = P(\lambda) w^{-1} Q/Q \times \hat{P}(\lambda) \hat{w}^{-1} \hat{Q}/\hat{Q}.$$

7.2.2

Let P be a standard (that is, containing B) parabolic subgroup of G. We consider the cohomology ring $\mathrm{H}^*(G/P, \mathbb{Z})$ of G/P. Here, we use singular cohomology with integer coefficients.

Notation The elements of W/W_P will be denoted w as elements of W: in other word, the notation does not distinguish a coset and a representative. Each times we use this abuse the reader has to check that the considered quantities does not depend on the representative.

If Y is an irreducible closed subvariety of G/P, we denote by $[Y] \in \mathrm{H}^*(G/P, \mathbb{Z})$ its cycle class in cohomology. If $w \in W/W_P$, we denote by σ_w^P the corresponding Schubert class; we have: $\sigma_w^P = [\overline{BwP/P}]$. Note that, $[\mathrm{pt}] = \sigma_e^P$. Let us recall that

$$\mathrm{H}^*(G/P, \mathbb{Z}) = \bigoplus_{w \in W/W_P} \mathbb{Z}\sigma_w^P.$$

We use similar notation for $\hat{G}/\hat{P}$.

We now consider the case when $P = P(\lambda)$ and $\hat{P} = \hat{P}(\lambda)$. Since $P = G \cap \hat{P}$, G/P identifies with the G-orbit of $\hat{P}/\hat{P}$ in $\hat{G}/\hat{P}$; let $\iota : G/P \longrightarrow \hat{G}/\hat{P}$ denote this closed immersion. The map ι induces a map ι^* in cohomology:

$$\iota^* : \mathrm{H}^*(\hat{G}/\hat{P}, \mathbb{Z}) \longrightarrow \mathrm{H}^*(G/P, \mathbb{Z}).$$

Lemma 14 *Let $(w, \hat{w}) \in W_Q\backslash W/W_P \times \hat{W}_{\hat{Q}}\backslash\hat{W}/\hat{W}_{\hat{P}}$. Then, we have:*

(i) *the pair $(C(w, \hat{w}), \lambda)$ is dominant if and only if $[\overline{QwP/P}] \cdot \iota^*([\overline{\hat{Q}\hat{w}\hat{P}/\hat{P}}]) \neq 0$;*
(ii) *the pair $(C(w, \hat{w}), \lambda)$ is covering if and only if $[\overline{QwP/P}] \cdot \iota^*([\overline{\hat{Q}\hat{w}\hat{P}/\hat{P}}]) = [\mathrm{pt}]$.*

Proof Consider the map:

$$\eta : G \times_P C^+(w, \hat{w}) \longrightarrow X.$$

Since the characteristic of $\mathbb{K}$ is zero, η is birational (resp. dominant) if and only if for x in an open subset of X, $\eta^{-1}(x)$ is reduced to one point (resp. nonempty). Consider the projection $p : G \times_P C^+(w, \hat{w}) \longrightarrow G/P$. For any x in X, p induces an isomorphism from $\eta^{-1}(x)$ onto the following locally closed subvariety of G/P:

$$F_x := \{hP \in G/P : h^{-1}x \in C^+(w, \hat{w})\}.$$

Let $(g, \hat{g}) \in G \times \hat{G}$ and set $x = (gQ/Q, \hat{g}\hat{Q}/\hat{Q}) \in X$. We have:

$$\begin{aligned} F_x &= \{hP/P \in G/P : h^{-1}gQ/Q \in Pw^{-1}Q/Q \text{ and} \\ &\qquad h^{-1}\hat{g}\hat{Q}/\hat{Q} \in \hat{P}\hat{w}^{-1}\hat{Q}/\hat{Q}\} \\ &= \{hP/P \in G/P : h^{-1} \in (Pw^{-1}Qg^{-1}) \cap (\hat{P}\hat{w}^{-1}\hat{Q}\hat{g}^{-1})\} \\ &= \iota(gQwP/P) \cap (\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}). \end{aligned}$$

Let us fix g arbitrarily. By Kleiman's transversality theorem (see [22]), there exists an open subset of $\hat{g}$'s in $\hat{G}$ such that the intersection $\overline{gQwP/P} \cap \overline{\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}}$ is transverse. Moreover (see for example [3]), one may assume that $(gQwP/P) \cap (\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P})$ is dense in $\overline{gQwP/P} \cap \overline{\hat{g}\hat{Q}\hat{w}\hat{P}/\hat{P}}$. We deduce that the following are equivalent:

(i) for general $\hat{g}$, F_x is reduced to a point (resp. $F_x \neq \emptyset$),
(ii) $[\overline{QwP/P}] \cdot \iota^*([\overline{\hat{Q}\hat{w}\hat{P}/\hat{P}}]) = [\mathrm{pt}]$ (resp. $\neq 0$).

Since η is G-equivariant, the above Condition (i) is clearly equivalent to the fact that η is birational respectively dominant. □

7.2.3

Notation From now on, $\mathfrak{g}$ and $\mathfrak{b}$ will denote the Lie algebras of G and B, R (resp. R^+) will denote the set of roots (resp. positive roots) of $\mathfrak{g}$. We denote by ρ the half sum of positive roots of $\mathfrak{g}$. We will also use the following similar notation for $\hat{G}$: $\hat{\mathfrak{g}}$, $\hat{\mathfrak{b}}$, $\hat{R}$, $\hat{R}^+$, $\hat{\rho}$.

Let $w \in W/W_P$ and consider the associated B-orbit in G/P. We define γ_w^P to be the sum of weights of T in the normal space at wP/P of BwP/P in G/P. Similarly, we define $\hat{\gamma}_{\hat{w}}^{\hat{P}}$.

Lemma 15 *The character γ_w^P is the sum of weights of T in $\mathfrak{g}/(\mathfrak{b}+w.\mathfrak{p})$. We have*:

$$\gamma_w^P = -(\rho + \tilde{w}\rho),$$

where $\tilde{w}$ denotes the longest element in the coset $w \in W/W_P$.

Proof Consider the map $G \longrightarrow G/P$, $g \mapsto gwP/P$ and its tangent map $\varphi : \mathfrak{g} \longrightarrow T_{wP/P}G/P$. Since φ is T-equivariant, surjective and $\mathfrak{b} + w.\mathfrak{p} = \varphi^{-1}(T_{wP/P}BwP/P)$, γ_w^P is the sum of weights of T in $\mathfrak{g}/(\mathfrak{b}+w.\mathfrak{p})$.

Consider the G-equivariant surjective map $\pi : G/B \longrightarrow G/P$. The definition of $\tilde{w}$ implies that $\pi(\tilde{w}B/B) = wP/P$ and $B\tilde{w}B/B$ is open in $\pi^{-1}(\overline{BwP/P})$. It follows that γ_w^P is the sum of weights of T in the normal space at $\tilde{w}B/B$ of $B\tilde{w}B/B$ in G/B; that is, the sum of weights of T in $\mathfrak{g}/(\mathfrak{b}+\tilde{w}.\mathfrak{b})$. Since the sum of all the roots is zero, we obtain that

$$\begin{aligned}
-\gamma_w^P &= \sum_{\alpha \in R^+ \cup \tilde{w}R^+} \alpha \\
&= \frac{1}{2}\left(\sum_{\alpha \in R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+} \alpha + \sum_{\alpha \in R^+ \backslash \tilde{w}R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+ \backslash R^+} \alpha\right) \\
&= \frac{1}{2}\left(2\rho + 2\tilde{w}\rho + \sum_{\alpha \in R^+ \backslash \tilde{w}R^+} \alpha + \sum_{\alpha \in \tilde{w}R^+ \backslash R^+} \alpha\right) \\
&= \rho + \tilde{w}\rho
\end{aligned}$$

where the last equality holds since $R^+ \backslash \tilde{w}R^+ = -(\tilde{w}R^+ \backslash R^+)$. $\square$

Remark In [3], Belkale-Kumar defined characters $\chi_{w^{-1}}$ for w of minimal length in its coset in $W_P \backslash W$. We have $\gamma_{w^{-1}}^P = -\tilde{w}^{-1}w^P(\chi_{w^{-1}})$, where w^P denote the longest element of W_P.

7.2.4

Notation If Y is a smooth variety of dimension n, $\mathcal{T}Y$ denotes its tangent bundle. The line bundle $\bigwedge^n \mathcal{T}Y$ over Y will be called the *determinant bundle* and denoted by $\mathcal{D}et Y$. If $\varphi : Y \longrightarrow Y'$ is a morphism between smooth varieties, we denote by $T\varphi : \mathcal{T}Y \longrightarrow \mathcal{T}Y'$ its tangent map, and by $\mathcal{D}et\varphi : \mathcal{D}et Y \longrightarrow \mathcal{D}et Y'$ its determinant.

We denote by $r_T : X(\hat{T}) \longrightarrow X(T)$ the restriction morphism.

Let us fix again a dominant one parameter subgroup λ of T, $w \in W$ and $\hat{w} \in \hat{W}$. To simplify notation, we set $P = P(\lambda)$, $C = C(w, \hat{w})$ and $C^+ =$

$C^+(w, \hat{w})$. Consider

$$\eta : G \times_P C^+ \longrightarrow X = G/Q \times \hat{G}/\hat{Q}.$$

Consider the restriction of $T\eta$ to C^+:

$$T\eta_{|C^+} : \mathcal{T}(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{T}(X)_{|C^+},$$

and the restriction of $\mathcal{D}et\eta$ to C^+:

$$\mathcal{D}et\eta_{|C^+} : \mathcal{D}et(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{D}et(X)_{|C^+}.$$

Since η is G-equivariant, the morphism $\mathcal{D}et\eta_{|C^+}$ is P-equivariant; it can be thought as a P-invariant section of the line bundle $\mathcal{D}et(G \times_P C^+)^*_{|C^+} \otimes \mathcal{D}et(X)_{|C^+}$ over C^+. We denote by $\mathcal{L}_{P,w,\hat{w}}$ this last P-linearized line bundle on C^+.

Lemma 16 *Let $(w, \hat{w}) \in W_Q\backslash W/W_P \times \hat{W}_{\hat{Q}}\backslash \hat{W}/\hat{W}_{\hat{P}}$. Let $\tilde{w} \in W$ (resp. $\tilde{\hat{w}} \in \hat{W}$) be the longest element in the class w (resp. $\hat{w}$).*

Then, the torus T acts on the fiber over the point $(\tilde{w}^{-1}Q/Q, \tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q})$ in $\mathcal{L}_{P,w,\hat{w}}$ by the character

$$\tilde{w}^{-1}\gamma_w^P + r_T(\tilde{\hat{w}}^{-1}\gamma_{\hat{w}}^{\hat{P}}) - \gamma_e^P.$$

Proof If Z is a locally closed subvariety of a variety Y and z is a point of Z, we denote by $N_z^Y(Z)$ the quotient T_zY/T_zZ of the tangent spaces at z of Y and Z. If V is a T-module $\mathrm{Wt_T(V)}$ denotes the multiset of weights of T in V. Let χ denote the character of the action of T on the fiber over the point $x = (\tilde{w}^{-1}Q/Q, \tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q})$ in $\mathcal{L}_{P,w,\hat{w}}$. Let $\mathfrak{p}$ denote the Lie algebra of P.

Since η induces the identity on C^+, we have:

$$\chi = -\sum_{\alpha \in \mathrm{Wt_T(N_x^{G\times_P C^+}(C^+))}} \alpha + \sum_{\alpha \in \mathrm{Wt_T(N_x^X(C^+))}} \alpha.$$

Moreover, we have the following T-equivariant isomorphisms:

$$\begin{aligned}
N_x^{G\times_P C^+}(C^+) &\simeq N_e^G(P) \simeq \mathfrak{g}/\mathfrak{p},\\
N_x^X(C^+) &\simeq N^{G/Q}_{\tilde{w}^{-1}Q/Q}(Pw^{-1}Q/Q) \oplus N^{\hat{G}/\hat{Q}}_{\tilde{\hat{w}}^{-1}\hat{Q}/\hat{Q}}(\hat{P}\hat{w}^{-1}\hat{Q}/\hat{Q})\\
&\simeq N^{G/B}_{\tilde{w}^{-1}B/B}(P\tilde{w}^{-1}B/B) \oplus N^{\hat{G}/\hat{B}}_{\tilde{\hat{w}}^{-1}\hat{B}/\hat{B}}(\hat{P}\tilde{\hat{w}}^{-1}\hat{B}/\hat{B})\\
&\simeq \mathfrak{g}/(\mathfrak{p} + \tilde{w}^{-1}\mathfrak{b}) \oplus \hat{\mathfrak{g}}/(\hat{\mathfrak{p}} + \tilde{\hat{w}}^{-1}\hat{\mathfrak{b}})
\end{aligned}$$

Now, the lemma is direct consequence of Lemma 15. □

One can now describe the well covering pairs of X:

Proposition 11 *Let λ be a dominant one parameter subgroup of T. Let $(w, \hat{w}) \in W/W_P \times \hat{W}/\hat{W}_{\hat{P}}$ be such that BwP/P and $\hat{B}\hat{w}\hat{P}/\hat{P}$ are open in QwP/P and $\hat{Q}\hat{w}\hat{P}/\hat{P}$ respectively.*

The following are equivalent:

(i) *The pair $(C(w, \hat{w}), \lambda)$ is well covering.*
(ii) $\sigma_w^P.\iota^*(\sigma_{\hat{w}}^{\hat{P}}) - \sigma_e^P = 0$, *and* $\langle w\lambda, \gamma_w^P\rangle + \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}}\rangle - \langle \lambda, \gamma_e^P\rangle = 0$.

Proof By Lemma 14, we may (and shall) assume that (C, λ) is covering. Note that $w\lambda$ is well defined, since W_P is precisely the stabilizer of λ in W. We choose $\tilde{w}$ and $\tilde{\hat{w}}$ as in Lemma 16. Note that

$$\langle \lambda, \tilde{w}^{-1}\gamma_w^P + r_T(\tilde{\hat{w}}^{-1}\gamma_{\hat{w}}^{\hat{P}}) - \gamma_e^P\rangle = \langle w\lambda, \gamma_w^P\rangle + \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}}\rangle - \langle \lambda, \gamma_e^P\rangle.$$

In particular, by Lemma 16, $\langle w\lambda, \gamma_w^P\rangle + \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}}\rangle - \langle \lambda, \gamma_e^P\rangle = 0$ if and only if λ acts trivially on the restriction of $\mathcal{L}_{P,w,\hat{w}}$ to C.

Assume that (C, λ) is well covering. Then $\mathcal{D}et\eta_{|C}$ is not identically zero. Since $\mathcal{D}et\eta_{|C}$ is a G^λ-invariant section of $\mathcal{L}_{P,w,\hat{w}|C}$, λ which pointwise fixes C has to act trivially on $\mathcal{L}_{P,w,\hat{w}|C}$ (see for example Lemma 1).

Conversely, assume Condition (ii) is satisfied. Since η is birational, $\mathcal{D}et\eta$ is G-invariant and nonzero; hence, $\mathcal{D}et\eta_{|C^+}$ is P-invariant and nonzero. Since $\mu^{\mathcal{L}_{P,w,\hat{w}}}(C, \lambda) = 0$, Lemma 5 shows that the restriction of $\mathcal{D}et\eta_{|C}$ is not identically zero. Since η is birational, this implies that η is an isomorphism over an open subset intersecting C. □

Remark As pointed out in the introduction, the notion of well covering pair is an adaptation of Belkale-Kumar's Levi-movability. Now, Proposition 11 is an adaptation of [3, Theorem 15].

7.3 The case $X = G/B \times \hat{G}/\hat{B}$

7.3.1

Let us recall that $\mathcal{LR}(G, \hat{G})$ denotes the cone of pairs $(\nu, \hat{\nu}) \in X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$ such that for some positive integer n, $n\nu$ and $n\hat{\nu}$ are dominant weights such that $V_{n\nu} \otimes V_{n\hat{\nu}}$ contains nonzero G-invariant vectors.

From now on, $X = G/B \times \hat{G}/\hat{B}$. By Proposition 10, a point $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if $r^{\Delta G}(\mathcal{L}_{(\nu,\hat{\nu})})$ belongs to $\mathcal{TC}^G(X) = \mathcal{SAC}^G(X)$.

Proposition 12 *The following are equivalent*:

(i) *no nonzero ideal of* $\mathfrak{g}$ *is an ideal of* $\hat{\mathfrak{g}}$;
(ii) *the interior of* $\mathcal{LR}(G, \hat{G})$ *in* $X(T)_{\mathbb{Q}} \times X(\hat{T})_{\mathbb{Q}}$ *is not empty*;
(iii) *the interior of* $\mathcal{AC}^G(X)$ *in* $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ *is not empty.*

Proof The equivalence between the two last assertions follows immediately from Proposition 10. By [32, Corollaire 1], the codimension of $\mathcal{LR}(G, \hat{G})$ in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ is the dimension of the general isotropy of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$. This general isotropy is also the kernel of the action of T on $\hat{G}/G$. So, it is contained in $\bigcap_{\hat{g}\in\hat{G}} \hat{g}G\hat{g}^{-1}$. Since this group is normal in $\hat{G}$ and G, it is finite if and only if no nonzero ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. □

7.3.2 Admisible subtore

Consider the G-module $\hat{\mathfrak{g}}/\mathfrak{g}$. Let $\mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$ be the set of nontrivial weights of T in $\hat{\mathfrak{g}}/\mathfrak{g}$. For $I \subset \mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$, we will denote by T_I the identity component of the intersection of the kernels of the $\chi \in I$. A subtorus of the form T_I is said to be *admissible*. The subtorus T_I is said to be *dominant* if $Y(T_I)_{\mathbb{Q}}$ is spanned by its intersection with the dominant chamber of $Y(T)_{\mathbb{Q}}$. Notice that $\mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$ being stable by the action of W, any admissible subtorus is conjugated by an element of W to a dominant admissible subtorus. A one parameter subgroup of T is said to be *admissible* if its image is.

To each $\chi \in \mathrm{Wt_T}(\hat{\mathfrak{g}}/\mathfrak{g})$, we associate the hyperplane $\mathcal{H}_\chi$ in $Y(T)_{\mathbb{Q}}$ spanned by the $\lambda \in Y(T)$ such that $\chi \circ \lambda$ is trivial. The $\mathcal{H}_\chi$'s form a W-invariant arrangement of hyperplanes in $Y(T)_{\mathbb{Q}}$. Moreover, $Y(T_I)_{\mathbb{Q}}$ is the intersection of the $\mathcal{H}_\chi$'s with $\chi \in I$.

The role of admissible subtori of T is explained by Corollary 3 and the following

Lemma 17 *Let S be a subtorus of T. We consider the action of G^S on the complete flag variety X' of the group $G^S \times \hat{G}^S$. If there exists $x \in X'$ such that the identity component of G^S_x is S then S is admissible*

Proof Since S acts trivially on X', the condition of the lemma on the isotropy of x is open in x; in particular, for $x \in X'$ general, we have $(G^S_x)^\circ = S$.

Hence, the general isotropy of B^S/S acting on $\hat{G}^S/\hat{B}^S$ is finite; that is, by the Bruhat theorem, the general isotropy of B^S/S acting on $\hat{B}^S/\hat{T}$ is finite. Since U^S acts freely on $\hat{B}^S/\hat{T}$, for $x \in \hat{U}^S/U^S$ general, $T_x^\circ = S$. But, $\hat{U}^S/U^S$ is isomorphic to $\hat{\mathfrak{u}}^S/\mathfrak{u}^S$ as a T-variety. So, S is the identity component of the kernel of the action of T on $\hat{\mathfrak{u}}^S/\mathfrak{u}^S = (\hat{\mathfrak{u}}/\mathfrak{u})^S$. It follows that S is admissible. □

7.3.3 Inequalities for $\mathcal{LR}(G, \hat{G})$

The following theorem is a generalization of [3, Theorem 28] for general branching problem:

Theorem 9 *We assume that no nonzero ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$. Any dominant weight $(\nu, \hat{\nu})$ belongs to $\mathcal{LR}(G, \hat{G})$ if and only if*

$$\langle w\lambda, \nu\rangle + \langle \hat{w}\lambda, \hat{\nu}\rangle \geq 0, \tag{5}$$

for any indivisible dominant admissible one parameter subgroup λ of T and for any $(w, \hat{w}) \in W/W_{P(\lambda)} \times \hat{W}/\hat{W}_{\hat{P}(\lambda)}$ such that

(i) $\sigma_w^{P(\lambda)} \cdot \iota^*(\sigma_{\hat{w}}^{\hat{P}(\lambda)}) = \sigma_e^{P(\lambda)} \in \mathrm{H}^*(G/P(\lambda), \mathbb{Z})$, *and*
(ii) $\langle w\lambda, \gamma_w^{P(\lambda)}\rangle + \langle \hat{w}\lambda, \gamma_{\hat{w}}^{\hat{P}(\lambda)}\rangle = \langle \lambda, \gamma_e^{P(\lambda)}\rangle$.

Proof Since $\mathcal{LR}(G, \hat{G})$ is the pullback of $\mathcal{SAC}^G(X)$ which is the closure of $\mathcal{AC}^G(X)$, it is sufficient to prove the theorem for $\mathcal{AC}^G(X)$. Let (C, λ) be a well covering pair as in Theorem 3. Since C is isomorphic to the complete flag variety of $G^\lambda \times \hat{G}^\lambda$, Lemma 17 shows that λ is admissible. Now, the theorem follows immediately from Theorem 3 and Proposition 11. □

7.3.4 Irredundancy

The following is our irredundancy result:

Theorem 10 *In Theorem 9, Inequalities (5) are pairwise distinct and no one can be omitted.*

Proof Note that the stabilizer in W (resp. $\hat{W}$) of $\lambda \in Y(T)$ (resp. $\lambda \in Y(\hat{T})$) is precisely $W_{P(\lambda)}$ (resp. $\hat{W}_{\hat{P}(\lambda)}$). It follows that the inequalities are pairwise distinct.

Let (C, λ) be a well covering pair of X corresponding to one inequality. Consider the associated face $\mathcal{TF}(C)$ of $\mathcal{TC}^G(X)$. We have to prove that $\mathcal{TF}(C)$ has codimension one. Consider the restriction morphism ρ : $\mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}^{G^\lambda}(C)$. Since $\mathrm{Pic}^{G\times\hat{G}}(X) \simeq \mathrm{Pic}^{G^\lambda\times\hat{G}^\lambda}(C)$, Lemma 13 implies that ρ is surjective. By Theorem 4, we have to prove that $\mathcal{AC}^{G^\lambda}(C)$ has codimension one in $\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$; or, equivalently that, $\mathcal{LR}(G^\lambda, \hat{G}^\lambda)$ has codimension one. Since λ is admissible, [32, Corollaire 1] gives this codimension.

One ends the proof of the theorem noting that $\mathcal{TF}(C)$ is not a face of the dominant chamber (for the group $G \times \hat{G}$). □

Remark In Theorems 9 and 10, the weight $(\nu,\hat\nu)$ is assumed to be dominant a priori. This imposes linear inequalities which can be redundant as shown by the following example. Set $\hat G = \mathrm{SL}_3$ and $G \simeq \mathrm{SL}_2$ diagonally embedded in $\hat G$. With usual notation, the irreducible representations of $\hat G$ correspond to the nondecreasing sequences of two nonnegative integers: $\hat\nu = (\hat\nu_1 \geq \hat\nu_2 \geq 0)$. Those of G correspond to nonnegative integers $\nu \geq 0$. Applying Theorem 9, one can recover the well known following equality:

$$\mathcal{LR}(G,\hat G) = \{(\nu,\hat\nu) : \hat\nu_1 \geq \nu \geq \hat\nu_2\}.$$

Clearly, the inequality $\hat\nu_1 \geq \hat\nu_2$ is a consequence of $\hat\nu_1 \geq \nu$ and $\nu \geq \hat\nu_2$ and so is redundant.

7.3.5 Smaller faces

We now state our result about the smaller faces of $\mathcal{LR}(G,\hat G)$.

Notation With notation of Lemma 16, we set $\theta_w^P = \tilde w^{-1}\gamma_w^P$ and $\theta_{\hat w}^{\hat P} = \tilde{\hat w}^{-1}\gamma_{\hat w}^{\hat P}$.

Theorem 11 *We assume that no nonzero ideal of $\mathfrak{g}$ is an ideal of $\hat{\mathfrak{g}}$.*

(i) *Let $\mathcal{F}$ be a face of $\mathcal{LR}(G,\hat G)$ of codimension $r \geq 1$ which intersects the interior of the dominant chamber. Then there exist a dominant admissible subtorus T_I (with $I \subset \mathrm{Wt}_{\mathrm{T}}(\hat{\mathfrak{g}}/\mathfrak{g})$) of T of dimension r, a dominant indivisible one parameter subgroup λ of T_I, and an irreducible component $C(w,\hat w)$ of X^λ (and X^{T_I}) such that:*
   (a) $\sigma_w^{P(\lambda)} \cdot \iota^*(\sigma_{\hat w}^{\hat P(\lambda)}) = \sigma_e^{P(\lambda)} \in \mathrm{H}^*(G/P(\lambda),\mathbb{Z})$,
   (b) $\theta_w^{P(\lambda)} + r_T(\hat\theta_{\hat w}^{\hat P(\lambda)}) - \theta_e^{P(\lambda)}$ *is trivial on T_I, and*
   (c) *$\mathcal{F}$ is the set of $(\nu,\hat\nu) \in \mathcal{LR}(G,\hat G)$ such that $\langle w\lambda,\nu\rangle + \langle \hat w\lambda,\hat\nu\rangle = 0$.*

(ii) *Conversely, let λ be a dominant one parameter subgroup of T and $C = C(w,\hat w)$ be an irreducible component X^λ. Set $I = \{\chi \in \mathrm{Wt}_{\mathrm{T}}(\hat{\mathfrak{g}}/\mathfrak{g}) \mid \chi\circ\lambda \text{ is trivial}\}$ and denote by r the dimension of T_I. If*
   (a) $\sigma_w^{P(\lambda)} \cdot \iota^*(\sigma_{\hat w}^{\hat P(\lambda)}) = \sigma_e^{P(\lambda)} \in \mathrm{H}^*(G/P(\lambda),\mathbb{Z})$ *and*
   (b) $\theta_w^{P(\lambda)} + r_T(\hat\theta_{\hat w}^{\hat P(\lambda)}) - \theta_e^{P(\lambda)}$ *is trivial on T_I,*
   then the set of $(\nu,\hat\nu) \in \mathcal{LR}(G,\hat G)$ such that $\langle w\lambda,\nu\rangle + \langle \hat w\lambda,\hat\nu\rangle = 0$ is a face of $\mathcal{LR}(G,\hat G)$ of codimension r.

Proof A face as in the first assertion corresponds to one $\mathcal{AF}$ of $\mathcal{AC}^G(X)$ of codimension r. Let S be a torus in G, λ be an indivisible one parameter subgroup of S and C be an irreducible component of X^S satisfying Corollary 3.

Up to conjugacy, we may assume that S is contained in T and that S and λ are dominant.

Lemma 17 implies that S is admissible. Now, we obtain Assertion (i) using Proposition 11.

Conversely, let w, $\hat{w}$, λ and I be as in Assertion (ii) of the theorem. By Proposition 11, $(C(w, \hat{w}), \lambda)$ is a well covering pair. Consider the associated face $\mathcal{TF}(C(w, \hat{w}))$ of $\mathcal{TC}^G(X)$. It remains to prove that the codimension of $\mathcal{TF}(C(w, \hat{w}))$ equals the dimension r of T_I.

By Lemma 13, the restriction $\rho : \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}^{G^\lambda}(C(w, \hat{w}))$ is surjective. By Theorem 4, it remains to prove that $\mathcal{TC}^{G^\lambda}(C(w, \hat{w}))$ has codimension r in $\mathrm{Pic}^{G^\lambda}(C(w, \hat{w}))_{\mathbb{Q}}$. By [32, Corollaire 1], this codimension is the dimension of the kernel of the action of T on $\hat{\mathfrak{g}}^\lambda/\mathfrak{g}^\lambda = (\hat{\mathfrak{g}}/\mathfrak{g})^\lambda$. By definition, T_I is the identity component of this kernel. □

7.3.6 *Redundant inequalities*

Theorem 9 gives a minimal list of inequalities which determine $\mathcal{LR}(G, \hat{G})$ as a part of the dominant chamber. Other classical inequalities like those given by the Horn conjecture or the Berenstein-Sjamaar theorem (see [4]) come from dominant pairs. We now want to understand the face corresponding to such a redundant inequality.

Let us fix a dominant one parameter subgroup λ. Let $(\sigma_w^{P(\lambda)}, \sigma_{\hat{w}}^{\hat{P}(\lambda)})$ be a pair of Schubert classes in $G/P(\lambda)$ and $\hat{G}/\hat{P}(\lambda)$ such that $\sigma_w^{P(\lambda)} \cdot \iota^*(\sigma_{\hat{w}}^{\hat{P}(\lambda)}) \neq 0$ in $\mathrm{H}^*(G/P(\lambda), \mathbb{Z})$. By Lemma 3, for any $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$, we have $\langle w\lambda, \nu\rangle + \langle \hat{w}\lambda, \hat{\nu}\rangle \geq 0$. In particular, the set $\mathcal{F}(w, \hat{w}, \lambda)$ of $(\nu, \hat{\nu}) \in \mathcal{LR}(G, \hat{G})$ such that $\langle w\lambda, \nu\rangle + \langle \hat{w}\lambda, \hat{\nu}\rangle = 0$ is a face of $\mathcal{LR}(G, \hat{G})$. By Theorem 9, if the pair $(C(w, \hat{w}), \lambda)$ is not well covering then $\mathcal{F}(w, \hat{w}, \lambda)$ has not codimension one. The following theorem improves this remark proving that $\mathcal{F}(w, \hat{w}, \lambda)$ is contained in a codimension two boundary of the dominant chamber.

Theorem 12 *We assume that* $(C(w, \hat{w}), \lambda)$ *is dominant but not well covering. Then,* $\mathcal{F}(w, \hat{w}, \lambda)$ *contains no point* $(\nu, \hat{\nu})$ *with* ν *OR* $\hat{\nu}$ *strictly dominant.*

Proof Set $x = (w^{-1}B/B, \hat{w}^{-1}\hat{B}/\hat{B})$ and $C = C(w, \hat{w})$. We assume that $\mathcal{F}(w, \hat{w}, \lambda)$ contains a weight $(\nu, \hat{\nu})$ with $\hat{\nu}$ strictly dominant. We are going to prove that (C, λ) is well covering.

Consider the parabolic subgroup Q of G containing B such that $\mathcal{L}_{(\nu,\hat{\nu})}$ is an ample line bundle on $G/Q \times \hat{G}/\hat{B}$ (denoted by $\overline{X}$ from now on). Consider the natural $G \times \hat{G}$-equivariant morphism $p : X \longrightarrow \overline{X}$. Set $\overline{x} = p(x)$. Let $\overline{C}$ denote the irreducible component of $\overline{X}^\lambda$ containing $\overline{x}$ and $\overline{C}^+$ the corresponding Białynicki-Birula cell.

Since $\overline{C}$ (resp. $\overline{C}^+$) is an orbit of $G^\lambda \times \hat{G}^\lambda$ (resp. of $P(\lambda) \times \hat{P}(\lambda)$), we have $p(C) = \overline{C}$ and $p(C^+) = \overline{C}^+$. Since (C, λ) is dominant, we deduce that $(\overline{C}, \lambda)$ is dominant. Now, Theorem 8 implies that $(\overline{C}, \lambda)$ is well covering.

Let $y \in C$ be general and $g \in G$ such that $g^{-1}y \in C^+$. Since $p(C) = \overline{C}$, $p(y)$ is general in $\overline{C}$. But, $(\overline{C}, \lambda)$ is well covering and $g^{-1}p(y) \in \overline{C}^+$; so, $g \in P(\lambda)$. This proves that (C, λ) is well covering.

The case when $\mathcal{F}(w, \hat{w}, \lambda)$ contains a weight $(\nu, \hat{\nu})$ with ν strictly dominant works similarly. Note that Theorem 8 can be applied with "$\hat{G} = G$". □

Example As an example, we consider the first redundant inequality given by the Horn conjecture. The computations was made using [7] and [15]: I want to thank their authors. Consider $G = \mathrm{SL}_6$ and the cone $\mathcal{LR}(G, G \times G)$. As usually, we identify the dominant weights of SL_6 with the sequences $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_5 \geq 0)$. We denote by (α, β, γ) the elements of $\mathcal{LR}(G, G \times G)$. Let $\lambda(t)$ be the diagonal matrix with diagonal entries $(t, t, t, t^{-1}, t^{-1}, t^{-1})$. The variety $G/P(\lambda)$ is the Grassmann variety $\mathrm{Gr}(3, 6)$. Let $\sigma \in \mathrm{H}^3(\mathrm{Gr}(3, 6), \mathbb{Z})$ be the Schubert class associated to the part $\{2, 4, 6\}$ (in usual way). We have $\sigma \cdot \sigma \cdot \sigma = 2.\sigma_e$. In particular, (σ, σ, σ) corresponds to a (conjugacy class of) dominant pair (C, λ) of G/B^3. The corresponding inequality is

$$\alpha_2 + \alpha_4 + \beta_2 + \beta_4 + \gamma_2 + \gamma_4 \geq 0.$$

The extremal rays of $\mathcal{TF}(C)$ are $(\omega_2, \omega_2, \omega_2)$, $(\omega_4, \omega_4, \omega_4)$ and the 6 elements obtained by permuting the three entries of $(0, \omega_2, \omega_4)$.

Let $\mathcal{F}l(2, 4, 6)$ denote the partial flag manifold of flags $V_2 \subset V_4 \subset \mathbb{K}^6$ where V_2 and V_4 have dimension 2 and 4. With obvious identification $\mathcal{TF}(C)$ has nonempty interior in $\mathrm{Pic}^G(\mathcal{F}l(2, 4, 6)^3)$; so, $\mathcal{TF}(C) = \mathcal{TC}^G(\mathcal{F}l(2, 4, 6)^3)$.

7.4 Application to the tensor product

In this section, G is assumed to be semisimple. We also fix an integer $s \geq 2$ and set $\hat{G} = G^s$, $\hat{T} = T^s$ and $\hat{B} = B^s$. We embed G diagonally in $\hat{G}$. Then $\mathcal{SAC}^G(X) \cap X(T)^{s+1}$ identifies with the set of $(s+1)$-uples $(\nu_0, \ldots, \nu_s)$ of dominant weights such that for n big enough $V_{n\nu_0} \otimes \cdots \otimes V_{n\nu_s}$ contains a nonzero G-invariant vector.

The set of weights of T in $\hat{\mathfrak{g}}/\mathfrak{g}$ is simply the root system R of G. Let Δ be the set of simple roots of G for $T \subset B$. Let I be a part of Δ. Let $L(I)$ denote the Levi subgroup of G containing T and having $\Delta - I$ as simple roots. Let T_I denote the identity component of the center of $L(I)$; T_I is dominant. Note that the dimension of T_I is the cardinality of I. Moreover, any dominant admissible subtorus of T is obtained in such a way. We will also denote by

$P(I)$ the standard parabolic subgroup with Levi subgroup $L(I)$. We denote by W_I the Weyl group of $L(I)$.

Let λ be a dominant one parameter subgroup of T. For $(w, \hat{w}) = (w_0, \dots, w_s) \in W \times \hat{W} = W^{s+1}$, and $(\nu, \hat{\nu}) = (\nu_0, \dots, \nu_s) \in \mathrm{Pic}^G(X)_{\mathbb{Q}} = X(T)_{\mathbb{Q}}^{s+1}$ we have:

- $r_T(\hat{w}\hat{\nu}) = \sum_{i=1}^s w_i \nu_i$, and $r_T(\hat{\theta}_{\hat{w}}^{\hat{P}(\lambda)}) = \sum_{i=1}^s \theta_{w_i}^{P(\lambda)}$,
- $\iota^*(\sigma_{\hat{w}}^{\hat{P}(\lambda)}) = \sigma_{w_1}^{P(\lambda)} \cdot \dots \cdot \sigma_{w_s}^{P(\lambda)}$,

In [3], Belkale-Kumar defined a new product denoted $\odot_0$ on the cohomology groups $\mathrm{H}^*(G/P, \mathbb{Z})$ for any parabolic subgroup P of G. By [3, Proposition 17], this product $\odot_0$ has the following very interesting property.

For $w_i \in W/W_I$, the following are equivalent:

(i) $\sigma_{w_0}^{P(I)} \cdot \dots \cdot \sigma_{w_s}^{P(I)} = \sigma_e^{P(I)}$ and, the restriction of $\theta_{w_0}^{P(I)} + \dots + \theta_{w_s}^{P(I)} - \theta_e^{P(I)}$ to T_I is trivial;

(ii) $\sigma_{w_0}^{P(I)} \odot_0 \dots \odot_0 \sigma_{w_s}^{P(I)} = \sigma_e^{P(I)}$.

Using this Belkale-Kumar result our Theorem 11 gives the following corollary. If α is a root of G, $\alpha^\vee$ denotes the corresponding coroot. If α is a simple root, $\omega_{\alpha^\vee}$ denotes the corresponding fundamental one parameter subgroup.

Corollary 4

(i) *A point $(\nu_0, \dots, \nu_s) \in X(T)_{\mathbb{Q}}^{s+1}$ belongs to the cone $\mathcal{LR}(G, G^s)$ if and only if*
  (a) *each ν_i is dominant; that is $\langle \alpha^\vee, \nu_i \rangle \geq 0$ for any simple root α.*
  (b) *for any simple root α; for any $(w_0, \dots, w_s) \in (W/W_\alpha)^{s+1}$ such that $\sigma_{w_0}^{P(\alpha)} \odot_0 \dots \odot_0 \sigma_{w_s}^{P(\alpha)} = \sigma_e^{P(\alpha)} \in \mathrm{H}^*(G/P(\alpha), \mathbb{Z})$, we have:*

$$\sum_{i=0}^{s} \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle \geq 0.$$

(ii) *We assume either that $\mathfrak{g}$ does not contain any factor of rank one or $s \geq 3$. In the above description of $\mathcal{LR}(G, G^s)$, the inequalities are pairwise distinct and no one can be omitted (neither in (a) nor (b)).*

(iii) *Let $\mathcal{F}$ be a face of $\mathcal{LR}(G, G^s)$ of codimension $r \geq 1$ which intersects the interior of the dominant chamber. There exist a subset I of r simple roots and $(w_0, \dots, w_s) \in (W/W_I)^{s+1}$ such that:*
  (a) $\sigma_{w_0}^{P(I)} \odot_0 \dots \odot_0 \sigma_{w_s}^{P(I)} = \sigma_e^{P(I)} \in \mathrm{H}^*(G/P(I), \mathbb{Z})$,
  (b) *the subspace spanned by $\mathcal{F}$ is the set of $(\nu_0, \dots, \nu_s) \in X(T)_{\mathbb{Q}}^{s+1}$ such that:*

$$\forall \alpha \in I \quad \sum_i \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle = 0.$$

(iv) *Conversely, let* I *be a subset of* r *simple roots and* $(w_0, \dots, w_s) \in (W/W_I)^{s+1}$ *such that* $\sigma_{w_0}^{P(I)} \odot_0 \cdots \odot_0 \sigma_{w_s}^{P(I)} = \sigma_e^{P(I)} \in \mathrm{H}^*(G/P(I), \mathbb{Z})$. *Then, the set of* $(\nu_0, \dots, \nu_s) \in \mathcal{LR}(G, G^s)$ *such that*

$$\forall \alpha \in I \quad \sum_i \langle w_i \omega_{\alpha^\vee}, \nu_i \rangle = 0,$$

is a face of codimension r *of* $\mathcal{LR}(G, G^s)$.

Proof Inequalities (a) are pairwise distinct and are not repeated in Inequalities (b). Moreover, by [32, Proposition 7] they define codimension one faces of $\mathcal{LR}(G, G^s)$.

The rest of the corollary is a simple rephrasing of Theorems 9, 10 and 11. □

Remark Assertion (i) of Corollary 4 is [3, Theorem 22]. The proof made here is very similar to that of Belkale-Kumar. The main difference is the use of abundance in the proof of Theorem 3 instead of [3, Theorem 26].

The description of the smaller faces of $\mathcal{C}^G((G/B)^{s+1})$ gives an application of the Belkale-Kumar product $\odot_0$ for any complete homogeneous space.

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