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Rates of convergence for the homogenization of fully nonlinear uniformly elliptic pde in random media

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Abstract We establish a logarithmic-type rate of convergence for the homogenization of fully nonlinear uniformly elliptic second-order pde in strongly mixing media with similar, i.e., logarithmic, decorrelation rate. The proof consists of two major steps. The first, which is actually the only place in the paper where probability plays a role, establishes the rate for special (quadratic) data using the methodology developed by the authors and Wang to study the homogenization of nonlinear uniformly elliptic pde in general stationary ergodic random media. The second is a general argument, based on the new notion of δ-viscosity solutions which is introduced in this paper, that shows that rates known for quadratic can be extended to general data. As an application of this we also obtain here rates of convergence for the homogenization in periodic and almost periodic environments. The former is algebraic while the latter depends on the particular equation.

1 Introduction

We establish rates of convergence for the homogenization of general uniformly elliptic fully nonlinear second-order pde in periodic, almost periodic and strongly mixing stationary random environments.

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Most of the paper is devoted to the latter setting where we establish a logarithmic-type rate of convergence for the homogenization of fully nonlinear uniformly elliptic second-order pde in strongly mixing stationary media with similar, i.e., logarithmic, decorrelation rate. The proof consists of two major steps. The first, which is actually the only place in the paper where probability plays a role, yields the rate for special (quadratic) data using the methodology developed by the authors and Wang [5] to study the homogenization of nonlinear uniformly elliptic pde in general stationary ergodic random media. The second is a general argument, based on the new notion of δ-viscosity solutions which is introduced in this paper, that shows that rates known for quadratic can be extended to general data. As an application of this we also obtain here rates of convergence for the homogenization in periodic and almost periodic environments. The former is algebraic while the latter depends on the particular equation.

Finding rates for quadratic data, which are uniform for quadratics of the same size, in the random setting is rather technical. It requires an understanding of the methodology of [5] and a good knowledge of some basic facts from the theory of fully nonlinear uniformly elliptic equations, both of which are explained in detail in the paper. Rates for quadratic data in periodic and almost periodic media are easier to obtain in view of the fact that the corresponding cell problems have respectively correctors and approximate correctors.

Extending qualitative results known for quadratic data/test functions to general data/test functions is, of course, the folklore of viscosity solutions of second-order elliptic equations. It was not known, however, how to extend this general principle to quantitative statements like, for example, rates. This was done for the first time for fully nonlinear equations by the authors in [4] to obtain error estimates for monotone approximations to uniformly elliptic second order pde. The approach of [4] can be reformulated and extended using δ-viscosity solutions, which in the sequel we will call simply δ-solutions. It turns out that δ-solutions are within distance δ^α from the actual solution for some uniform α. To extend hence a rate from quadratic to general data, it suffices to show that the rate for the former yields that the solution of the general problem is actually a δ-solution for an appropriate choice of δ.

The role of the δ-solutions can be described briefly as follows: If the limiting equations have smooth ($C^{2,\alpha}$ solutions), a fact which is known in general only for convex/concave nonlinearities, then we can first replace the solutions by their second-order Taylor's expansion with a uniform error and then use the rate we know for the quadratic expansion to find the rate. In general, however, solutions are not even $C^{1,1}$. It then becomes necessary to approximate them from above and below by appropriate $C^{2,\alpha}$-expansions. This is where δ-solutions come in.

Next we describe the results. To keep the Introduction simple, in many places below we state assumptions without much rigor. The precise statements are given in the main body of the paper.

To this end, we begin with the random setting and we consider the boundary value problem

$$\begin{cases} F(D^2u_\varepsilon, \frac{x}{\varepsilon}, \omega) = 0 & \text{in } U, \\ u_\varepsilon = g & \text{on } \partial U. \end{cases} \tag{1.1}$$

Here $(\Omega, \mathcal{F}, \mu)$ is the underlying probability space endowed with a measure preserving transformation $(\tau_y)_{y\in\mathbb{R}^N}$, U is an open subset of $\mathbb{R}^N$ with regular boundary, and, for each $\omega \in \Omega$, $F(\cdot,\cdot,\omega) \in C(S^N \times \mathbb{R}^N)$ is uniformly elliptic with ellipticity constants independent of ω, where S^N is the space of $N \times N$ symmetric matrices. The nonlinearity F must, of course, satisfy the standard assumptions guaranteeing the well-posedness of viscosity solutions of (1.1) for each ε. Since these conditions play no role in the analysis here, we have chosen to omit them. As far as the randomness goes, i.e., the way F depends on ω, the key assumptions are (i) the stationarity of F with respect to (y, ω) (throughout the paper y denotes the fast variable x/ε), and (ii) the strongly mixing property of the measure preserving transformation with a prescribed rate of decorrelation.

It turns out (see Papanicolaou and Varadhan [26] and Kozlov [19] in the linear case and the authors and Wang [5] in the fully-nonlinear case) that (1.1) homogenizes in stationary ergodic environments. This means that there exists a unique uniformly elliptic $\overline{F} \in C(S^N)$, which is linear when F is linear and nonlinear when F is nonlinear, such that, if $\bar{u} \in C(\overline{U})$ is the solution of the homogeneous boundary value problem

$$\begin{cases} \overline{F}(D^2\bar{u}) = 0 & \text{in } U, \\ \bar{u} = g & \text{on } \partial U, \end{cases} \tag{1.2}$$

then, as $\varepsilon \to 0$,

$$u_\varepsilon \to \bar{u} \quad \text{in } C(\overline{U}) \text{ and a.s. in } \omega.$$

Here we obtain a rate for this convergence. To do so, it is necessary to quantify the assumption of ergodicity. We assume that F is strongly mixing, which, loosely speaking, means that $F(\cdot, y, \omega)$ and $F(\cdot, x, \omega)$ decorrelate as $|x - y| \to \infty$, with a logarithmic-type rate.

The first result is:

Theorem 1.1 *Let $u_\varepsilon \in C(\bar{U})$ and $\bar{u} \in C^{0,1}(\bar{U})$ be the solutions of* (1.1) *and* (1.2) *respectively. Assume that F is uniformly elliptic* (2.7), *bounded* (2.8), *stationary in* (y,ω) (2.5) *and strongly mixing with a logarithmic rate given by* (2.27). *There exist positive constants* $\bar{C}$, $\hat{C}$, $\bar{c}$, $\hat{c}$ *and* ε_0, *depending only on the constants in* (2.7) *and* (2.8) *and the mixing condition*, N *and* $\|\bar{u}\|_{C^{0,1}(\bar{U})}$ *but not on* ε, *such that, for all* $\varepsilon \in (0,\varepsilon_0)$, *there exists* $A_\varepsilon \subset \Omega$ *such that*

$$\mu(A_\varepsilon) \leqq \bar{C}\varepsilon^{\bar{c}|\ln\varepsilon|^{-1/2}} \quad \textit{and} \quad \|u_\varepsilon(\cdot,\omega)-\bar{u}\|_{C(\bar{U})} \leqq \hat{C}\varepsilon^{\hat{c}|\ln\varepsilon|^{-1/2}} \quad \textit{in } \Omega\setminus A_\varepsilon^c.$$

The proof of Theorem 1.1 is rather long and consists of several steps, the most important being Theorem 3.1 which yields a decay rate for the "mass" of the obstacle problems (with quadratic obstacles) that control the homogenization of (1.1) (the precise statements are presented later in the paper). This is the only part of the paper where the strongly mixing assumption and some probabilistic arguments play a role. The rate obtained in Theorem 3.1 is then used to prove, after some technical approximations, Theorem 1.1 for special quadratic data. Theorem 1.1 follows after we show, using Theorem 3.1, that solutions of (1.1) are $\delta=\delta(\varepsilon)$-viscosity solutions of the homogenized equation for appropriately chosen δ.

As far as rates are concerned nothing was known for the homogenization of nonlinear elliptic equations in random environments and in the generality we consider here. Yurinskii [30, 31] assumed an algebraic mixing rate and obtained a Hölder rate (ε^α) for linear uniformly elliptic equations. As we will see later in the paper the obstruction to prove a similar rate in the nonlinear case is the different homogeneity of the ways the equation controls the solution (Alexander-Bakelman-Pucci estimate) and the solutions control the equation (Fabes-Stroock estimate).

Several results were known about rates of convergence for the homogenization in the periodic setting for the boundary value problem

$$\begin{cases} F(D^2u_\varepsilon,\frac{x}{\varepsilon})=0 & \text{in } U,\\ u_\varepsilon = g & \text{on } \partial U, \end{cases} \tag{1.3}$$

under special structure conditions on F (convexity with respect to the Hessian). For example, using probabilistic/stochastic control related arguments Ichihara [13] obtained an algebraic (Hölder) rate for nonlinear degenerate elliptic equations with convex with respect to the Hessian nonlinearities of the type arising in the theory of backward stochastic differential equations. More recently, Camilli and Marchi [6] obtained a rate of convergence in the periodic setting under the assumption that F is uniformly elliptic and convex. In this case there exist regular (i.e., $C^{2,\alpha}$) correctors and the rate follows relatively easily.

Using δ-viscosity solutions and the fact that in the periodic setting there exist correctors (see Section 6 for the meaning of the corrector) we consider (1.3) without any convexity assumptions and obtain

Theorem 1.2 *Assume that* $F \in C(S^N \times \mathbb{R}^N)$ *is periodic in* y, *uniformly elliptic* (2.7) *and bounded* (2.8) *and let* $\bar{F} \in C(S^N)$ *be the corresponding homogenized nonlinearity. Let* $u_\varepsilon \in C(\bar{U})$ *and* $\bar{u} \in C^{0,1}(\bar{U})$ *be the solutions of* (1.3) *and* (1.2). *There exists* $\alpha \in (0, 1)$, $C > 0$, *and* $\varepsilon_0 > 0$, *depending only on the constants in* (2.7) *and* (2.8), *the dimension, the domain,* $\|\bar{u}\|_{C^{0,1}(\bar{U})}$ *but not* ε *so that, for all* $\varepsilon \in (0, \varepsilon_0)$

$$|u_\varepsilon - \bar{u}| \leqq C\varepsilon^\alpha \quad \text{in } \bar{U}.$$

The rate for the homogenization in the almost periodic setting does not follow from either of Theorem 1.1 (almost periodic media are not strongly mixing) and Theorem 1.2 (the corrector equation does not have a solution in general). Nevertheless using again δ-viscosity solutions together with the existence of approximate correctors (see Sect. 7 for the precise meaning) it is possible to obtain a rate.

We have:

Theorem 1.3 *Assume that* $F \in C(S^N \times \mathbb{R}^N)$ *is almost periodic in* y *(in the sense of* (8.2)), *uniformly elliptic* (2.7) *and bounded* (2.8) *and let* $\bar{F} \in C(S^N)$ *be the corresponding homogenized nonlinearity. Let* $u_\varepsilon \in C(\bar{U})$ *and* $\bar{u} \in C^{0,1}(\bar{U})$ *be the solutions of* (1.3) *and* (1.2). *There exist a modulus* $\rho : [0, \infty) \to [0, \infty)$ *such that* $\rho(0^+) = 0$ *and* $\varepsilon_0 > 0$ *both depending on* F, N, $\|\bar{u}\|_{C^{0,1}}$ *and the constants in* (2.7) *and* (2.8), *but not* ε, *such that, for all* $\varepsilon \in (0, \varepsilon_0)$,

$$|u_\varepsilon - \bar{u}| \leqq \rho(\varepsilon) \quad \text{in } \bar{U}.$$

We conclude the Introduction with a brief discussion of the homogenization of nonlinear first- and second-order pde. The homogenization of (1.3) in the periodic/almost periodic setting is by far simpler than the random environments. The first result for homogenization for periodic first-order nonlinear (Hamilton-Jacobi) equations was proved by Lions, Papanicolaou and Varadhan [23]. The problem was revisited by Evans [9, 10] who introduced the notion of perturbed test function and considered also second-order equations like (1.3). Caffarelli [2] put forward a different approach for the homogenization of fully nonlinear uniformly elliptic equations. Time-dependent problems were studied by Majda and Souganidis [24] and Evans and Gomes [11]. Ishii [14] considered the homogenization of almost periodic Hamilton-Jacobi equations, while Lions and Souganidis [22] analyzed second-order problems, even for degenerate elliptic equations.

The homogenization of linear, uniformly elliptic equations in both divergence and nondivergence form in random environments was established a while ago by Papanicolaou and Varadhan [25, 26] and Kozlov [19] (see also

Jikov, Kozlov and Oleinik [16]). The first nonlinear stochastic homogenization result in the variational setting was obtained by Dal Maso and Modica [8]. The first nonlinear nonvariational result for Hamilton-Jacobi equations was proved by one of the authors in [29] (see also Rezankhanlou and Tarver [27]). Lions and Souganidis investigated in [20] the issue of the existence of correctors for the random Hamilton-Jacobi case. In [21] they also established the stochastic homogenization of viscous Hamilton-Jacobi equations. A similar result was obtained independently by Kosygina, Rezankhanlou and Varadhan [18]. In spatio-temporal environments, the homogenization of (uniformly elliptic) viscous Hamilton-Jacobi was studied by Kosygina and Varadhan [17], while Schwab [28] considered first-order Hamilton-Jacobi equations. The only known result for fully nonlinear uniformly elliptic operators was obtained in [5].

The paper is organized as follows: The first three sections are devoted to the random setting. In Sect. 1 we state the main assumptions, recall the homogenization method put forward in [5], and introduce some of the quantities that lead to the convergence rate. In Sect. 2 we explain and prove the main decay estimate for the obstacle problem. In Sect. 3 we establish the rate of convergence for any quadratic data in the unit ball. Moreover, we show that this rate is uniform for bounded families of quadratics. In Sect. 4 we introduce the notion of δ-viscosity solution and prove a general algebraic estimate for the difference between δ and "classical" viscosity solutions. In Sect. 5 we use the result of Sect. 3 to show that solutions of (1.1) are $\delta(\varepsilon)$-viscosity solutions of (1.2) a fact which, in view of the results in Sect. 4, completes the proof of the error estimate in the strongly mixing media. In Sects. 6 and 7 we obtain rates of convergence for the periodic and almost periodic settings respectively. In Appendix A we review some basic facts from the theory of fully nonlinear elliptic equations which are used throughout the paper.

2 The general setting, assumptions and review of random homogenization

Since the section is rather long, for the convenience of the reader, we divide it into several parts. And begin with

(i) The general setting

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space on which $\mathbb{R}^N$ acts as a group $(\tau_y)_{y\in\mathbb{R}^N}$ of measure-preserving transformations and assume that μ is ergodic under this action, i.e., all translation invariant subsets of Ω have probability either 0 or 1.

We consider processes $\tilde{F} : S^N \times \Omega \to \mathbb{R}$ such that, for each $\omega \in \Omega$, $\tilde{F}(\cdot, \omega) \in C(S^N)$ is uniformly elliptic with, independent of ω, ellipticity constants $\lambda, \Lambda > 0$, i.e., for all $X, Y \in S^N$ with $Y \geqq 0$, and $\omega \in \Omega$,

$$\lambda|Y| \leqq \tilde{F}(X+Y,\omega) - \tilde{F}(X,Y,\omega) \leqq \Lambda|Y|, \tag{2.1}$$

and bounded uniformly in ω, i.e., there exists $\Lambda > 0$ independent of ω such that

$$|\tilde{D}F(0,\omega)| \leqq \Lambda. \tag{2.2}$$

To keep the notation simple we write $|\cdot|$ for both the Euclidean distance and the Lebesgue measure in $\mathbb{R}^N$ as well as the usual norm in S^N. For the same reason we take the constants in the upper bounds in (2.1) and (2.2) to be the same.

It follows that there exists some, independent of ω, $\tilde{c} > 0$ such that, for all $X, Y \in S^N$ and a.s. in ω,

$$|\tilde{F}(X,\omega) - \tilde{F}(Y,\omega)| \leqq \tilde{c}|X-Y| \quad \text{and} \quad |\tilde{F}(X,\omega)| \leqq \tilde{c}(1+|X|). \tag{2.3}$$

We further assume that

$$\begin{cases} y \longmapsto \tilde{F}(X,\tau_y\omega) \text{ is uniformly continuous in } y \text{ for bounded } X \\ \text{and uniformly in } \omega, \text{ i.e., for any } r>0, \\ \lim_{\delta\to 0}\sup_{|y|\leqq\delta}\sup_{|X|\leqq r}\sup_\omega |\tilde{F}(X,\tau_y\omega) - \tilde{F}(X,\omega)| = 0. \end{cases} \tag{2.4}$$

Next, given $\tilde{F}$ satisfying (2.1), (2.2) and (2.4), we define, for $(X, y) \in S^N \times \mathbb{R}^N$ and $\omega \in \Omega$,

$$F(X,y,\omega) = \tilde{F}(X,\tau_y\omega).$$

It follows that, for all $X \in S^N$, $y, y' \in \mathbb{R}^N$ and $\omega \in \Omega$,

$$F(X,y+y',\omega) = F(X,y',\tau_y\omega) \quad \text{and} \quad F(X,0,\omega) = \tilde{F}(X,\omega), \tag{2.5}$$

$$\begin{cases} F(\cdot,\cdot,\omega) \in C(S^N \times \mathbb{R}^N) \quad \text{and} \quad y \longmapsto F(X,y,\omega) \\ \text{is uniformly continuous in } y \text{ uniformly for } X \text{ bounded and all } \omega, \end{cases} \tag{2.6}$$

$$\lambda|Y| \leq F(X+Y,y,\omega) - F(X,y,\omega) \leqq \Lambda|Y| \quad \text{if } Y \geqq 0, \tag{2.7}$$

$$|F(0,y,\omega)| \leqq \Lambda, \tag{2.8}$$

and

$$|F(X,y,\omega) - F(Y,y,\omega)| \leqq \tilde{c}|X-Y| \quad \text{and} \quad |F(X,y,\omega)| \leqq \tilde{c}(1+|X|). \tag{2.9}$$

As discussed in [25], Ω can be taken to be the set of all continuous functions on S^N with the value of $\omega \in \Omega$ at $y \in \mathbb{R}^N$ defined by $\omega(\cdot, y)$. The σ-algebra $\mathcal{F}$ is then generated by cylinder sets with base points having rational coordinates in $\mathbb{R}^N$ and range sets spheres in $C(S^N)$ with rational centers and radii. The probability measure μ on $(\Omega, \mathcal{F})$ is invariant with respect to the translation group $(\tau_y)_{y\in\mathbb{R}^N}$ defined, for $y, y' \in \mathbb{R}^N$, by

$$(\tau_y\omega)(y') = \omega(y' - y),$$

which is assumed to be ergodic, and its support are all the functions in $C(S^N)$ satisfying (2.1), (2.2) and (2.4).

Loosely speaking Ω can be thought of as the set of all $\tilde{F}$'s satisfying (2.7) and (2.6) and the probability measure as the "frequency" by which particular $\tilde{F}$'s appear. Of course, the probability of selecting some (many) equations may be zero. Stationarity, which is defined by (2.5), can be described as follows: Spatial translations of F appear with "the same frequency", i.e., given $y \in \mathbb{R}^N$, the equation (we use the term equation for F) $F(\cdot, \cdot + y, \omega)$ appears with the same frequency as $F(\cdot, \cdot, \omega)$. Recall that a stochastic process $f : \mathbb{R}^N \times \Omega \to \mathbb{R}$ is called stationary if, for any positive integer k and all $y_1, \ldots, y_k \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$, the joint distribution of the random vector $(f(y_1 + h, \cdot), \ldots, f(y_k + h, \cdot))$ is independent of h.

To shorten statements in the sequel we will say that a process $F(\cdot, \cdot, \omega) \in C(S^N \times \mathbb{R}^N)$ is stationary ergodic, if $(\tau_y)_{y\in\mathbb{R}^N}$ is ergodic and (2.5) holds. Moreover, we will formulate everything using F and not $\tilde{F}$.

(ii) The homogenization process for fully nonlinear pde

The result of [5] is:

Theorem 2.1 *Assume that $F : S^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ is stationary ergodic, and, a.s. in ω, $F(\cdot, \cdot, \omega) \in C(S^N \times \mathbb{R}^N)$ satisfies (2.7), (2.2) and (2.6). There exists a uniformly elliptic $\bar{F} \in C(S^N)$ such that, if $u_\varepsilon(\cdot, \omega) \in C(\bar{U})$ and $\bar{u} \in C(\bar{U})$ are the solutions of (1.1) and (1.2) respectively, then, as $\varepsilon \to 0$, a.s. in ω and uniformly in U, $u_\varepsilon(\cdot, \omega) \to u$.*

The homogenization result of [5] relies on the Crandall-Lions notion of viscosity solutions. The effective nonlinearity $\bar{F}$ is identified either implicitly by finding all $P \in S^N$ belonging to each level set of $\bar{F}$ or, equivalently, explicitly by finding, for each $P \in S^N$, the level set $\bar{F}(P)$ containing P. Below we use the latter characterization.

Throughout the paper and depending on the context, we denote by P either the matrix $P \in S^N$ or the quadratic polynomial $P(x) = \frac{1}{2}(Px, x)$. Moreover, $B_r(x)$ (resp. B_r) stands for the ball in $\mathbb{R}^N$ centered at x (resp. the origin) with radius r and $Q_r(x)$ (resp. Q_r) stands for the cube in $\mathbb{R}^N$ centered at x

(resp. the origin) with side length r. We say that a constant is universal if it depends only on λ, Λ, Λ' and $\tilde{c}$ in (2.7), (2.8) and (2.9), the dimension N, and, in general, the domain. We will often write c and C for uniform constants that may change from line to line. Finally, we will consider boundary value problems in either balls or cubes depending on the type of argument we are using. The particular choice (ball or cube) is irrelevant for the final result.

To find, for a given $P \in S^N$, $\bar{F}(P)$, we consider, for each $\ell \in \mathbb{R}$, the a.s. in ω limiting behavior, as $\varepsilon \to 0$, of the solution $u_\varepsilon(\cdot, \omega)$ of the boundary value problem

$$\begin{cases} F(D^2 u_\varepsilon, \frac{x}{\varepsilon}, \omega) = \ell & \text{in } B_1, \\ u_\varepsilon = P & \text{on } \partial B_1, \end{cases} \tag{2.10}$$

where, for notational simplicity, the dependence of u_ε on ℓ and, whenever possible, ω is suppressed.

It is shown in [5] that there exists a unique constant $\bar{F}(P)$ such that, as $\varepsilon \to 0$, a.s. in ω and uniformly in B_1,

$$\begin{cases} \lim_{\varepsilon\to 0} u_\varepsilon(\cdot, \omega) \geqq P \quad \text{if } \ell \geqq \bar{F}(P), \\ \quad \text{i.e., } u_\varepsilon \text{ aligns with } P \text{ from above, and} \\ \lim_{\varepsilon\to 0} u_\varepsilon(\cdot, \omega) \leqq P \quad \text{if } \ell \leqq \bar{F}(P), \\ \quad \text{i.e., } u_\varepsilon \text{ aligns with } P \text{ from below,} \end{cases} \tag{2.11}$$

and, hence, for $\ell = \bar{F}(P)$, a.s. in ω, $u_\varepsilon(\cdot, \omega)$ becomes an "approximate corrector" (aligns with P), i.e., uniformly on $\bar{B}_1$ and a.s. in ω,

$$\lim_{\varepsilon \to 0} u_\varepsilon(\cdot, \omega) = P. \tag{2.12}$$

In the course of the analysis it is often more convenient to solve the boundary value problem in $B_{1/\varepsilon}$ and, instead of (2.10), to consider the a.s. limiting behavior of the solution $u^\varepsilon(\cdot, \omega)$ of

$$\begin{cases} F(D^2 u^\varepsilon, x, \omega) = \ell & \text{in } B_{1/\varepsilon}, \\ u^\varepsilon = P & \text{on } \partial B_{1/\varepsilon}. \end{cases} \tag{2.13}$$

Since $u_\varepsilon(x, \omega) = \varepsilon^2 u^\varepsilon(\frac{x}{\varepsilon}, \omega)$, the a.s. behavior of $u_\varepsilon(\cdot, \omega)$ in B_1, as $\varepsilon \to 0$, is equivalent to the asymptotic quadratic behavior of $u^\varepsilon(\cdot, \omega)$ at infinity.

The methodology described above lacks "monotonicity" with respect to ε—if $\varepsilon_1 < \varepsilon_2$ then u^{ε_1} has no obvious monotonicity relationship to u^{ε_2}—one of the basic concepts of both the ergodic and viscosity theories and the main technical tool to obtain the above described "alignment". To force such a property in the problem, we replace (2.10) and (2.13) by the corresponding obstacle problems with obstacle P from either above or below.

The properties of the obstacle problem pertinent to our work can be found in Caffarelli and Kinderlehrer [3] for the Laplacian and [5] for nonlinear equations.

(iii) The obstacle problem

We review next the general properties of the obstacle problem which play a role in our analysis. To simplify the presentation, we omit for now the dependence on ε and ω.

Let V be a bounded open subset of $\mathbb{R}^N$ with regular boundary. A function u^+ (resp. u^-) $\in C(\bar{V})$ is a solution of the obstacle problem for F from above (resp. below) with obstacle P, if it is the least super-solution (resp. largest sub-solution) of $F = 0$ in V above (resp. below) P. When necessary, we write $u_V^{\pm}$ to denote the solutions of the obstacle problems in V.

It is an immediate consequence of the definition and the properties of viscosity solutions (see Theorem 2.1 and the discussion in [5]) that

$$u^{\pm} = P \quad \text{on } \partial V, \tag{2.14}$$

and

$$\begin{cases} F(D^2u^+, x) \leqq \ell \text{ and } u^+ \geqq P \quad \text{in } V \text{ and} \\ F(D^2u^+, x) = \ell \quad \text{in } \{u^+ > P\} \\ (\text{resp. } F(D^2u^-, x) \geqq \ell \text{ and } u^+ \leqq P \quad \text{in } V \text{ and} \\ F(D^2u^-, x) = \ell \quad \text{in } \{u^- < P\}). \end{cases} \tag{2.15}$$

Moreover, the Harnack inequality (see Theorem 2.1 of [5]) implies that $u_V^{\pm}$ separates uniformly from the obstacle, i.e.,

$$\begin{cases} \text{there exists } c_1 > 0, \text{ depending only on the ellipticity constants and } N, \\ \text{such that, if } x \in \{u^+ = P\} \text{ (resp. } \{u^- = P\}\text{), then, for all } y \in V, \\ 0 \leqq (u^+ - P)(y) \leqq c_1|y - x|^2 \quad (\text{resp. } 0 \leqq (P - u^-)(y) \leqq c_1|y - x|^2). \end{cases} \tag{2.16}$$

The regularity properties of the obstacle problem and, in particular, (2.16) yield that u^+ and u^- are respectively the unique solutions of the boundary value problems

$$\begin{cases} F(D^2u^+, x) = \ell - (\ell - F(P, x))_+ \mathbf{1}_{\Lambda^+} & \text{in } V, \\ u^+ = P & \text{on } \partial V, \end{cases} \tag{2.17}$$

and

$$\begin{cases} F(D^2u^-, x) = \ell + (\ell - F(P, x))_- \mathbf{1}_{\Lambda^-} & \text{in } V, \\ u^- = P & \text{on } \partial V, \end{cases} \tag{2.18}$$

where

$$\Lambda^+ = \{x \in V : u^+(x) = P(x)\} \quad \text{and} \quad \Lambda^- = \{x \in V : u^-(x) = P(x)\}$$

are the coincidence (contact) sets of u^+ and u^-, $\mathbf{1}_A$ denotes the characteristic function of the set A and r_+ and r_- are respectively the positive and negative parts of $r \in \mathbb{R}$. In the sequel we refer to the quantities

$$(\ell - F(P,x))_+ \mathbf{1}_{\Lambda^+} \quad \text{and} \quad (\ell - F(P,x))_- \mathbf{1}_{\Lambda^-}$$

and

$$\int_{V \cap \Lambda^+} (\ell - F(P,x))_+^N \, dx \quad \text{and} \quad \int_{V \cap \Lambda^-} (\ell - F(P,x))_-^N \, dx$$

as, respectively, the mass densities and total masses of the obstacle problems.

The obstacle problem has a very important monotonicity property with respect to the domain. Indeed, if $V_1, V_2 \subset \mathbb{R}^N$ are open bounded subsets of $\mathbb{R}^N$ with regular boundaries such that $V_1 \subset V_2$, then $u^+_{V_2}$ and $u^-_{V_2}$ qualify respectively as admissible super-solution and sub-solution in V_1. It follows that, in V_1, $u^+_{V_2} \geqq u^+_{V_1}$ and $u^-_{V_2} \leqq u^-_{V_1}$, and thus the coincidence sets are nonincreasing with respect to the domain, i.e.,

$$\text{if } V_1 \subset V_2, \quad \text{then } \Lambda^\pm_{V_2} \cap V_1 \subset \Lambda^\pm_{V_1}. \tag{2.19}$$

An immediate consequence of this monotonicity as well as the additivity property of the Lebesgue measure is that both

$$\int_{V \cap \Lambda^\pm} (\ell - F(P,x))_\pm^N \, dx \quad \text{and} \quad |V \cap \Lambda^\pm|$$

are clearly subadditive with respect to the domain V.

(iv) Two measures of separation

Next we describe two quantitative measurements of the separation between the solutions of the upper and lower obstacle problems and the solution of the corresponding boundary value problem with the same quadratic data P.

To this end, take $V = Q_1$, let u be the solution of the "unrestricted" boundary value problem

$$\begin{cases} F(D^2u, x) = \ell & \text{in } Q_1, \\ u = P & \text{on } \partial Q_1, \end{cases} \tag{2.20}$$

and consider the solutions u^+ and u^- of the obstacle problems in Q_1 with obstacle P from above and below respectively.

Since u^+ and u^- are respectively super- and sub-solutions of (2.20) in Q_1, the comparison property of viscosity solution yields

$$u^- \leqq u \leqq u^+ \quad \text{on } Q_1. \tag{2.21}$$

The first measurement is the Alexandrov-Bakelman-Pucci estimate (ABP for short) (see Theorem 3.2 of [1])) which controls the separation $(u^\pm - u)_\pm$ from above in terms of the total mass of the obstacle problems. Indeed the ABP estimate yields a uniform constant c_2 such that

$$\|(u^\pm - u)_\pm\|_{C(\bar{Q}_1)} \leqq c_2 \|(\ell - F(P,\cdot))_\pm \mathbf{1}_{\Lambda^\pm}\|_{L^N(Q_1)} \quad \text{in } \bar{Q}_1. \tag{2.22}$$

Since, in view of (2.9),

$$0 \leqq (\ell - F(P,x))_\pm \leqq \tilde{\tilde{c}}(1 + |P| + |\ell|) \quad \text{in } Q_1,$$

with

$$\tilde{\tilde{c}} = \max(1, \tilde{c}),$$

it follows from (2.22) that, for a uniform constant $c_3 > 0$,

$$(u^\pm - u) \leqq c_3(1 + |P| + |\ell|)|\Lambda^\pm|^{1/N} \quad \text{in } \bar{Q}_1. \tag{2.23}$$

To go in the opposite direction, i.e., to control the separation from below, we note that the right hand sides of (2.17) and (2.18) have a fixed sign. Hence, the Fabes-Stroock estimate [12] (see also Corollary B.5 in [5] and Theorem A.2 in Appendix A) gives, for some uniform constants $c_4 > 0$ and $M > N$,

$$\begin{aligned} &c_4(1 + |P| + |\ell|)^{(1-M)} \|(\ell - F(P,x))_\pm \mathbf{1}_{\Lambda^\pm}\|^M_{L^N(Q_1)} \\ &\quad \leqq |u^\pm - u| \quad \text{in } Q_{2/3}. \end{aligned} \tag{2.24}$$

The lower and upper bounds in (2.22), (2.23) and (2.24) have different homogeneity. This is the reason we obtain a logarithmic and not an algebraic rate in the error estimate.

In the linear setting it should be possible to obtain better lower and upper estimates using the Green's function and the associated invariant measures. This will provide a different proof for the error estimates in [30, 31]. We plan to return to this issue in a future publication.

(v) The methodology for homogenization process

We return now to the homogenization problem. We introduce the relevant obstacle problems in B_1, their scaled versions in $B_{1/\varepsilon}$ and, for the convenience of the reader, we present next a brief review of the arguments introduced in [5] to identify $\bar{F}(P)$.

To this end, fix $\omega \in \Omega$, $P \in S^N$ and $\ell \in \mathbb{R}$, and let $u_\varepsilon^+(\cdot,\omega;\ell)$ (resp. $u_\varepsilon^-(\cdot,\omega;\ell)$) be the solution of the obstacle P from above (resp. below), i.e., u_ε^+ (resp. u_ε^-) is the smallest super-solution (resp. largest sub-solution) of (2.10) above (resp. below) P in B_1. In view of the previous discussion $u_\varepsilon^\pm$ solve the boundary value problems

$$\begin{cases} F(D^2u_\varepsilon^\pm, \frac{x}{\varepsilon},\omega) = \ell \mp (\ell - F(P,\frac{x}{\varepsilon},\omega))_\pm \mathbf{1}_{\Lambda_\varepsilon^\pm} & \text{in } B_1,\\ u_\varepsilon^\pm = P & \text{on } \partial B_1. \end{cases} \tag{2.25}$$

where $\Lambda_\varepsilon^\pm(\omega;\ell)$ is the coincidence set of $u_\varepsilon^\pm(\cdot,\omega;\ell)$.

Let $u^{\varepsilon,\pm}(\cdot,\omega;\ell)$ be the quadratic rescalings of $u_\varepsilon^\pm(\cdot,\omega;\ell)$ given by

$$u_\varepsilon^\pm(x,\omega;\ell) = \varepsilon^2 u^{\varepsilon,\pm}\left(\frac{x}{\varepsilon},\omega;\ell\right),$$

which solve the obstacle problems

$$\begin{cases} F(D^2u^{\varepsilon,\pm}, y,\omega) = \ell \mp (\ell - F(P,y,\omega))_\pm \mathbf{1}_{\Lambda_\varepsilon^\pm} & \text{in } B_{1/\varepsilon},\\ u^{\varepsilon,\pm} = P & \text{on } \partial B_{1/\varepsilon}, \end{cases}$$

where $\Lambda^{\varepsilon,\pm}(\omega;\ell)$ are the respective contact sets.

A simple rescaling yields

$$\Lambda_\varepsilon^\pm(\omega;\ell) = \varepsilon^{-1}\Lambda^{\varepsilon,\pm}(\omega;\ell).$$

The subadditivity property of $|\Lambda_V^\pm|$ with respect to V and the assumptions of stationarity and ergodicity allows for the use of the subadditive ergodic theorem (see [5]) to obtain that, a.s. in ω,

$$\lim_{\varepsilon\to 0}|\Lambda_\varepsilon^\pm(\omega;\ell)| = \lim_{\varepsilon\to 0}\varepsilon^{-N}|\Lambda^{\varepsilon,\pm}(\omega;\ell)| = m^\pm(\ell).$$

The monotonicity property of the obstacle problems and (2.8) yield that

$$\begin{cases} \ell \to m^+(\ell) \quad \text{is nonincreasing and}\\ m^+(\ell) = 0 \quad (\text{resp. } m^+(\ell)=1)\\ \quad \text{for } \ell \text{ uniformly small negative (resp. large positive),} \end{cases}$$

and, similarly,

$$\begin{cases} \ell \to m^-(\ell) \quad \text{is nondecreasing and}\\ m^-(\ell) = 1 \quad (\text{resp. } m^-(\ell)=0),\\ \quad \text{for } \ell \text{ uniformly small negative (resp. large positive),} \end{cases}$$

If

$$\ell^+ = \inf\{\ell \in \mathbb{R} : m^+(\ell) = 0\} \quad \text{and} \quad \ell^- = \sup\{\ell \in \mathbb{R} : m^-(\ell) = 0\},$$

the first separation measurement implies that, as $\varepsilon \to 0$, a.s. in ω and uniformly on $\bar{B}_1$,

$$\text{if } \ell > \ell^+, \quad \text{then } u_\varepsilon^+(\cdot, \omega) - u_\varepsilon(\cdot, \omega) \to 0,$$

and

$$\text{if } \ell < \ell^-, \quad \text{then } u_\varepsilon(\cdot, \omega) - u_\varepsilon^-(\cdot, \omega) \to 0.$$

Moreover, the quadratic separation property of the obstacle problem yields, again as $\varepsilon \to 0$, that a.s. in ω and uniformly on $\bar{B}_1$,

$$\text{if } \ell < \ell^+, \quad \text{then } \lim_{\varepsilon \to 0} (u_\varepsilon^+(\cdot, \omega) - P) = 0,$$

and

$$\text{if } \ell > \ell^-, \quad \text{then } \lim_{\varepsilon \to 0} (P - u_\varepsilon^-(\cdot, \omega)) = 0.$$

It then follows from a perturbation argument and the fact that

$$u_\varepsilon^- \leqq u_\varepsilon \leqq u_\varepsilon^+ \quad \text{in } B_1$$

that (2.12) holds with $\bar{F}(P) = \ell^\pm$.

We remark that the whole argument works if, instead of $|\Lambda_\varepsilon^\pm(\omega, \ell)|$, we use the L^N-norms (in B_1) of the total masses of the two obstacle problems. Since the homogenization process is controlled by these norms, it is clear that to obtain rates of convergence it is enough to find decay rates for these norms. The argument leading to the rate is based on an iterative estimate on how these measures decay with respect to ε for strongly mixing configurations.

(vi) The mixing conditions

Mixing conditions are used in probability as a "measurement" of the (roughly speaking) independence or, better, decorrelation at large distance of measurable subsets of Ω.

To state the strongly mixing condition we recall that the Ω is taken to be the set of all elliptic operators satisfying (2.1) and (2.2) and their translations. Given a subset K of $\mathbb{R}^N$ and A of Ω, consider the "cylinder set" A_K consisting of all F's whose traces in K coincide with that of an element in A. Ergodicity implies that, if for example, K is the unit cube Q_1 and $y_k \to \infty$, the intersection of the translations $\tau_{y_k} A_{Q_1}$ with each other, then, as $k \to \infty$, $\mu(\bigcap_{y_k} \tau_{y_k} A_{Q_1}) \to 0$ unless $\mu(A_{Q_1}) = 1$, i.e., each operator defined on $\mathbb{R}^N$ is composed of a "random" mix of different local pieces. The strongly mixing condition is a rate for the above limit.

For $r > 0$, let $\mathcal{S}$ and $\mathcal{S}(r)$ be the smallest σ-algebras generated by the measurable subsets $\{F(\cdot, y, \cdot) : y \in Q_1\}$ and $\{F(\cdot, y, \cdot) : \operatorname{dist}(y, Q_1) \geqq r\}$ of Ω.

The probability space $(\Omega, \mathcal{F}, \mu)$ is called strongly mixing if

$$\lim_{r\to\infty} \sup_{\substack{A\in S\\ B\in S(r)}} |\mu(A \cap B) - \mu(A)\mu(B)| = 0. \tag{2.26}$$

To obtain a rate of convergence for the homogenization, we quantify the limit in (2.26) assuming the following logarithmic-type rate:

$$\begin{cases} \text{there exists } c > 0 \text{ such that, for all } \delta > 0, \\ \sup_{A\in S, B\in S(r)} |\mu(A \cap B) - \mu(A)\mu(B)| \leqq \delta \quad \text{if } r > \delta^{c(\ln\delta)}. \end{cases} \tag{2.27}$$

Throughout the proofs in the paper we will be using $\delta = 3^{-k}$, in which case (2.27) reads

$$\sup_{A\in S, B\in S(r)} |\mu(A \cap B) - \mu(A)\mu(B)| \leqq 3^{-k} \quad \text{if } r > 3^{k^2}. \tag{2.28}$$

Next we present a simple example of strongly mixing media satisfying (2.28). To this end, begin with a regular checkerboard of cubes Q^i of length size 1 and center at $i \in \mathbb{Z}^N$, fix two bounded uniformly elliptic maps $F_1, F_2 \in C(S^N)$, consider a sequence $(\omega_n)_{n\in\mathbb{Z}^N}$ with values F_1 and F_2 and let

$$F(\cdot, y, \omega) = \omega_n \quad \text{if } y \in Q^i.$$

Then $\Omega = \{F_1, F_2\}^{\mathbb{Z}^N}$ and the probability measure is the infinite product of the trivial equidistributed probability measure on $\{F_1, F_2\}$. Since, for each $k \in \mathbb{Z}^N$, and $\tilde{\omega}_\cdot = \omega_{\cdot+k}$,

$$F(X, y + k, \omega) = F(X, y, \tilde{\omega}),$$

it is clear that, if $A \in S$ and $B \in S(r)$ with $r \geqq 1$, then A and B are independent, and, hence, (2.27) holds.

Another simple example of a random medium satisfying (2.28) is a sequence of larger and larger chessboards where, with probability $1/2$, we choose from a medium that decorrelates at length 1, with a probability $1/2^2$, from a medium that decorrelates at length 3, with probability 2^{-k}, from a medium that decorrelates at distance 3^{k^2}, etc.

The rate in (2.27), (2.28) is much slower than the power decay used in [Y1], [Y2]. Note, however, that the rate of convergence for the homogenization in Theorem 1.1 is optimal for slow decay rates like (2.26), although it may not be optimal if, instead of (2.26), we had assumed a power type decay.

The next observation, which is a direct consequence of (2.28), records the rate of decorrelation of nonnegative random variables. To this end, let $L^\infty_+(\mathcal{S})$ and $L^\infty_+(\mathcal{S}(r))$ be the spaces of nonnegative bounded random variables which are $\mathcal{S}$ and $\mathcal{S}(r)$ measurable respectively.

Finally, given a random variable $f:\Omega\to\mathbb{R}$ we write $E(f)$ and $V(f)$ for its expectation and variance respectively, i.e.,

$$E(f)=\int f\,d\mu \quad\text{and}\quad V(f)=E((f-E(f))^2).$$

We have:

Proposition 2.1 *Assume* (2.28). *Then*

$$\sup_{f\in L^\infty_+(\mathcal{S}),\ g\in L^\infty_+(\mathcal{S}(r))} |E(fg)-Ef\,Eg|\leqq 3^{-k}\|f\|_\infty\|g\|_\infty \quad \textit{if } r\geq 3^{k^2}. \tag{2.29}$$

Proof Without any loss of generality we may assume that $\|f\|_\infty=\|g\|_\infty=1$. Recalling the representation

$$f(\omega)=\int_0^1 \mathbf{1}_{\{f(\cdot)>t\}}(\omega)\,dt \quad\text{and}\quad g(\omega)=\int_0^1 \mathbf{1}_{\{g(\cdot)>t\}}(\omega)\,dt,$$

for $r>3^{k^2}$, we have

$$\begin{aligned}|E(fg)-Ef\,Eg| &= \Big|E\int_0^1\int_0^1 \mathbf{1}_{\{f(\cdot)>t\}}\mathbf{1}_{\{g(\cdot)>s\}}\,ds\,dt \\ &\quad - E\int_0^1 \mathbf{1}_{\{f(\cdot)>t\}}\,dt\ E\int_0^1 \mathbf{1}_{\{f(\cdot)>s\}}\,ds\Big| \\ &= \Big|\int_0^1\int_0^1 [\mu(\{f(\cdot)>t\}\cap\{g(\cdot)>s\}) \\ &\quad - \mu(\{f(\cdot)>t\})\mu(\{g(\cdot)>s\})]\,ds\,dt\Big| \leqq 3^{-k}. \end{aligned}$$ □

We conclude with a technical auxiliary lemma that will be used later in the paper. It concerns the decay of the second moments and the variance of averages of random variables.

We have:

Lemma 2.1 *Let $h_1, \dots, h_M$ be a family of random variables such that, for $i, j = 1, \dots, M$ and some $\sigma_{ij} > 0$, $E(h_i h_j) - Eh_i Eh_j \leqq \sigma_{ij}$. Then*

$$V\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right) \leqq \frac{1}{M^2}\sum_{i=1}^{M} V(h_i) + \frac{1}{M^2}\sum_{i,j=1}^{M} \sigma_{ij},$$
$$E\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right)^2 \leqq \left(\frac{1}{M}\sum_{i=1}^{M} E(h_i)\right)^2 + \frac{1}{M^2}\sum_{i,j=1}^{M} \sigma_{ij} + \frac{1}{M^2}\sum_{i=1}^{M} V(h_i), \tag{2.30}$$

and if, for all $i = 1, \dots, M$, $Eh_i = E$ and $V(h_i) = V$, then

$$V\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right) \leqq \frac{1}{M} V + \frac{1}{M^2}\sum_{i,j=1}^{M} \sigma_{ij}.$$

Proof The first estimate follows from the identity

$$E\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right)^2 = \frac{1}{M^2}\sum_{i,j=1}^{M} E(h_i h_j),$$

and the observation that, if $i = j$, then

$$E(h_i h_j) = Eh_i^2 = (E(h_i))^2 + V(h_i),$$

while, for $i \neq j$, in view of the assumption,

$$E(h_i h_j) \leqq E(h_i)E(h_j) + \sigma_{ij}.$$

The second inequality follows from (2.30) since

$$E\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right)^2 = \left(E\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right)\right)^2 + V\left(\frac{1}{M}\sum_{i=1}^{M} h_i\right),$$

while the last is immediate. □

3 A decay rate for the total masses of the obstacle problems

We formulate and prove here the result which is the key step of the proof of Theorem 1.1. In what follows it is more convenient to work in Q_1 instead of B_1. We consider the obstacle problem (2.25) for arbitrary P and constant $\ell \in \mathbb{R}$, which remain fixed throughout the section, and prove a decay rate for the total masses. This, in turn, will lead to a rate of convergence for the homogenization.

To obtain the rate, which is presented below in Theorem 3.1, we work at scale 1 instead of ε, i.e., we consider all problems in the cubes $Q_{1/\varepsilon}$. The reason for this choice of scale is that, in this setting, it is possible to compare directly the solutions of the obstacle problems along increasing cubes, a fact which leads to a monotonicity property for the second moments of the averaged total masses.

For the fixed $P \in S^N$ and $\ell \in \mathbb{R}$ and each in ω, let $u_m^{\pm}(\cdot, \omega)$ be the solution of (2.15) with contact sets $\Lambda_m^{\pm}(\omega)$ in the cube

$$D_m = Q_{3^m}, \tag{3.1}$$

and consider their total masses

$$h_m^{\pm}(\omega) = \frac{1}{|D_m|} \int_{D_m \cap \Lambda^{\pm}(\omega)} (\ell - F(P, y, \omega))_{\pm}^N \, dy. \tag{3.2}$$

The definition of the total mass in the previous section did not include the factor $1/|D_m|$. Indeed in Sect. 1 we introduced the obstacle problems for a fixed domain V. Here we do the same, i.e., we actually consider the cell problems in Q_ε with $\varepsilon = 3^{-m}$. The factor $1/|D_m|$ appears after rescaling to work, for the reasons explained above, at scale 1, i.e., in D_m.

Let

$$H_m^{\pm} = E(h_m^{\pm})^2$$

be the second moments of the total masses. We study the behavior, along a particular sequence $m \to \infty$, of the product

$$\mathbb{H}_m = H_m^+ H_m^-. \tag{3.3}$$

Next we explain the choice of (3.2). For notational simplicity we use again ε as parameter. We are interested in an estimate on

$$\max_{\bar{B}_1} |u_\varepsilon^{\pm} - u_\varepsilon|.$$

The Alexandrov-Bakelman-Pucci estimate yields

$$\begin{aligned}\max_{Q_1} (u_\varepsilon^{\pm} - u_\varepsilon)_{\pm}^N &\leqq c_2 \int_{Q_1 \cap \Lambda_\varepsilon^{\pm}(\omega)} \left(\ell - F\left(P, \frac{x}{\varepsilon}, \omega\right)\right)_{\pm}^N dx \\ &= c_2 \varepsilon^N \int_{Q_{1/\varepsilon} \cap \Lambda^{\varepsilon,\pm}(\omega)} (\ell - F(P, y, \omega))_{\pm}^N \, dy,\end{aligned}$$

with the right hand like (3.2) for $\varepsilon = 3^{-m}$.

For what follows it is convenient to introduce some additional notation. To this end, for each m, we write $E_m^{\pm}$ and $V_m^{\pm}$ for the expectation and variance respectively of $h_m^{\pm}$, i.e.,

$$E_m^{\pm} = E(h_m^{\pm}) \quad \text{and} \quad V_m^{\pm} = V(h_m^{\pm}) = E[(h_m^{\pm} - E_m^{\pm})^2].$$

Then

$$H_m^{\pm} = E[(h_m^{\pm})^2] = V_m^{\pm} + (E_m^{\pm})^2.$$

To further simplify the presentation, we denote by $\pm$ statements holding for both the upper and lower obstacle problems, while we write $+$ or $-$ for statements holding for either the upper or the lower obstacle problem respectively. Finally, whenever possible, we omit the explicit dependence on ω.

The monotonicity property of the obstacle problem with respect to the domain (see Proposition 3.3 of [5]) implies that the $H_m^{\pm}$'s and, hence, the $\mathbb{H}_m$'s are nonincreasing in m (we prove this fact in Lemma 3.12 below). Actually we remark that, as it follows from the results of [5] the $\mathbb{H}_m$'s always converge to zero for any choice of P and l, while this is not the case for the $H_m^{\pm}$'s. In Theorem 3.1 we actually show that, along the sequence $m_k = k^2$, there is a rate of decay.

A decay rate on $\mathbb{H}_{k^2}$, of course, does not necessarily imply a rate for $H_{k^2}^{\pm}$. Indeed, although both $H_{k^2}^{+}$ and $H_{k^2}^{-}$ are nonincreasing with respect to k, for different k one may decay strictly while the other stays the same. To overcome this difficulty, we use the subadditivity property of the obstacle problem and appropriate (small) perturbations of P and ℓ, the latter around the homogenization value $\bar{F}(P)$. This is discussed in Sect. 3.

The main result is:

Theorem 3.1 *Assume the strongly mixing condition* (2.28) *and let* $F : S^N \times \mathbb{R}^N \times \Omega \to \mathbb{R}$ *be as in Theorem* 1.1. *Fix* $P \in S^N$ *and* $\ell \in \mathbb{R}$ *and consider the solutions* $u_{k^2}^{\pm}(\cdot, \omega)$ *of the obstacle problems* (2.15) *in* D_{k^2}. *There exist universal constants* $\tau \in (0, 1)$, $C > 0$ *and a positive integer* k_0, *depending on the constants in* (2.7) *and* (2.8), *and the dimension, such that, for* $k \geqq k_0$,

$$\mathbb{H}_{k^2} \leqq C(1 + |P| + |\ell|)^{4N} 3^{(k_0 - k)\tau}. \tag{3.4}$$

The proof of (3.4) is rather long and consists of a number of crucial steps and many technicalities. To explain the main ideas, below we first introduce the general setting and some simplifications (normalization of constants), we formulate the several steps as separate lemmas, which we prove at the end of the section, we describe heuristically the crux of the argument, and, finally, we prove the Theorem.

We begin with

(i) The general setting and some preliminaries

We consider the following geometry: Each cube $D_{(k+1)^2}$ of side length $3^{(k+1)^2}$ is subdivided into $3^{(k+1)N}$ cubes $D^i_{k^2+k}$, with $i=1,\dots,3^{(k+1)N}$, of side length 3^{k^2+k}. In turn each $D^i_{k^2+k}$ is subdivided into 3^{kN} cubes $D^{ij}_{k^2}$, with $j=1,\dots,3^{kN}$, of side length 3^{k^2}. For each $\omega\in\Omega$ and i, j as above we consider the obstacle problems in the cubes $D_{(k+1)^2}$, $D^i_{k^2+k}$, $D^{ij}_{k^2}$, their respective solutions $u^{\pm}_{(k+1)^2}(\cdot,\omega)$, $u^{i,\pm}_{k^2+k}(\cdot,\omega)$, and $u^{ij,\pm}_{k^2}(\cdot,\omega)$, contact sets $\Lambda^{\pm}_{(k+1)^2}(\omega)$, $\Lambda^{i,\pm}_{k^2+k}(\omega)$, and $\Lambda^{ij,\pm}_{k^2}(\omega)$, total masses $h^{\pm}_{(k+1)^2}(\omega)$, $h^{i,\pm}_{k^2+k}(\omega)$ and $h^{ij,\pm}_{k^2}(\omega)$, and the averages

$$A^{\pm}_{(k+1)^2}(\omega)=3^{-(k+1)N}\sum_{i=1}^{3^{(k+1)N}} h^{i,\pm}_{k^2+k}(\omega) \tag{3.5}$$

and

$$A^{\pm,i}_{k^2+k}(\omega)=3^{-kN}\sum_{j=1}^{3^{kN}} h^{ij,\pm}_{k^2}(\omega). \tag{3.6}$$

The stationarity implies, for $i=1,\dots,3^{(k+1)N}$ and $j=1,\dots,3^{kN}$, that

$$E(h^{i,\pm}_{k^2+k})=E^{\pm}_{k^2+k}\quad\text{and}\quad Eh^{ij,\pm}_{k^2}=E^{\pm}_{k^2},$$

and, hence,

$$EA^{\pm}_{(k+1)^2}=E^{\pm}_{k^2+k}\quad\text{and}\quad EA^{i,\pm}_{k^2+k}=E^{\pm}_{k^2}. \tag{3.7}$$

Below we write D_{k^2+k}, $u^{\pm}_{k^2+k}$, $\Lambda^{\pm}_{k^2+k}$, and $h^{\pm}_{k^2+k}$ and D_{k^2}, $u^{\pm}_{k^2}$, $\Lambda^{\pm}_{k^2}$ and $h^{\pm}_{k^2}$ for the cubes, solutions of the obstacle problems and their contact sets and total masses corresponding the middle cube of the partition $(D^i_{k^2+k})_{i=1}^{3^{(k+1)N}}$ and the middle cube of the partition $(D^j_{k^2})_{j=1}^{3^{kN}}$ of D_{k^2+k}. Note that such middle cubes exist since we are using a configuration with side length powers of 3; actually this is one of the reasons for using 3. Finally we write $A^{\pm}_{k^2+k}$ for the average of the total masses corresponding to the partition of D_{k^2+k}.

We continue with

(ii) A simplification

We use here a scaling argument to reduce to the case that, for all ω and y,

$$(\ell-F(P,y,\omega))_{\pm}\leqq 1, \tag{3.8}$$

which implies, for all $m = 1, 2, \ldots$, that

$$0 \leqq h_m^\pm \leqq 1, \tag{3.9}$$

and, hence, for all $k \geqq 1$,

$$H_{k^2}^\pm \leqq 1 \quad \text{and} \quad H_{k^2+k}^\pm \leqq 1. \tag{3.10}$$

To this end, recalling that

$$(\ell - F(P, y, \omega))_\pm \leqq \tilde{\tilde{c}}(1 + |P| + |\ell|),$$

with

$$\tilde{\tilde{c}} = \max(1, \tilde{c}) \geqq 1,$$

rescale P and ℓ to respectively

$$\tilde{P} = (\tilde{\tilde{c}}(1 + |P| + |\ell|))^{-1} P \quad \text{and} \quad \tilde{\ell} = (\tilde{\tilde{c}}(1 + |P| + |\ell|))^{-1} \ell$$

and consider the scaled nonlinearity

$$\tilde{F}(\cdot, y, \omega) = (\tilde{\tilde{c}}(1 + |P| + |\ell|))^{-1} F(\tilde{\tilde{c}}(1 + |P| + |\ell|)\cdot, y, \omega),$$

which has the same ellipticity constants as F. It is clear that, for all $\omega \in \Omega$ and $y \in \mathbb{R}^N$,

$$\left(\frac{\ell}{\tilde{\tilde{c}}(1 + |P| + |\ell|)} - \tilde{F}\left(\frac{P}{\tilde{\tilde{c}}(1 + |P| + |\ell|)}, y, \omega\right)\right)_\pm \leqq 1,$$

and, hence, if $\tilde{h}_m^\pm$ is the total mass corresponding to $\tilde{F}$, $\tilde{P}$ and $\tilde{\ell}$, then

$$0 \leqq \tilde{h}_m^\pm \leqq 1.$$

In the sequel we drop the $\sim$'s, we assume that (3.8), (3.9) and (3.10) hold and prove that there exist uniform $\tau \in (0, 1)$ and $k_0 \geqq 1$ such that, for $k \geq k_0$,

$$\mathbb{H}_{k^2} \leqq 3^{(k_0 - k)\tau}. \tag{3.11}$$

We continue with a series of lemmas/steps leading to the proof of Theorem 3.1. All the proofs are presented at the end of the section.

(iii) The monotonicity of $H_{k^2}^\pm$ and $\mathbb{H}_{k^2}$

The monotonicity properties of the obstacle problem and the choice of the cubes and the partitions yields

Lemma 3.1 *For all $k \geqq 1$,*

$$\begin{aligned} &\text{(i)} \quad H^{\pm}_{(k+1)^2} \leqq H^{\pm}_{k^2+k} \leqq H^{\pm}_{k^2} \quad \textit{and} \\ &\text{(ii)} \quad \mathbb{H}_{(k+1)^2} \leqq \mathbb{H}_{k^2+k} \leqq \mathbb{H}_{k^2}. \end{aligned} \tag{3.12}$$

(iii) A strict decay of the variances

The strong mixing rate (2.28) yields the following strict decay for the variances of $A^{\pm}_{k^2+k}$ and $A^{\pm}_{(k+1)^2}$.

Lemma 3.2 *Assume* (3.9) *and the hypotheses of Theorem* 3.1. *Then, for all $k \geqq 1$,*

$$V(A^{\pm}_{k^2+k}) \leqq 3^{-(k-1)N} V^{\pm}_{k^2} + 3^{-k}, \tag{3.13}$$

and

$$V(A^{\pm}_{(k+1)^2}) \leqq 3^{-kN} V^{\pm}_{k^2+k} + 3^{-k}. \tag{3.14}$$

(iv) The distribution of $h^{i,\pm}_{k^2}$ and $h^{i,\pm}_{k^2+k}$

It follows from Chebyshev's inequality that, if the variances of $h^{\pm}_{k^2+k}$ and $h^{j,\pm}_{k^2}$, the latter being the total masses of the subdivision $D^j_{k^2}$, $j = 1, 2, \ldots, 3^{kN}$, of D_{k^2+k}, are smaller than a (small) multiple of $(E^{\pm}_{k^2+k})^2$ and $(E^{\pm}_{k^2})^2$ respectively, i.e., if

$$V^{\pm}_{k^2} \leqq \eta (E^{\pm}_{k})^2 \tag{3.15}$$

and

$$V^{\pm}_{k^2+k} \leqq \eta (E^{\pm}_{k^2+k})^2, \tag{3.16}$$

then, off sets of probability controlled by appropriate multiples of η, $h^{j,\pm}_{k^2}$ and $h^{\pm}_{k^2+k}$ are comparable to their averages. The exceptional sets, of course, depend on the particular j, while we would like to work with a "uniform" set with probability controlled by η. This is possible but requires justification.

To state the result, we introduce, for $j = 1, \ldots, 3^{kN}$, the random variables

$$\rho^{j,\pm}_{k^2}(\omega) = \begin{cases} 1 & \text{if } h^{j,\pm}_{k^2} \leqq \frac{1}{2} E^{\pm}_{k^2}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.17}$$

and their average

$$\bar{\rho}_k^{\pm} = 3^{-kN} \sum_{i=1}^{3^{kN}} \rho_{k^2}^{j,\pm}. \tag{3.18}$$

Let $N_k^{\pm}(\omega)$ be the number of $\rho_{k^2}^{j,\pm}(\omega)$'s that are zero, which is actually the same as the number of cubes $D_{k^2}^j$ where $h_{k^2}^{j,\pm}(\omega) > \frac{1}{2}E_{k^2}^{\pm}$, set

$$\zeta_N = \frac{1}{4}\left(\frac{2}{3}\right)^N, \tag{3.19}$$

and consider the sets

$$\tilde{B}_{k^2}^{\pm} = \{\omega \in \Omega : \bar{\rho}_k^{\pm}(\omega) > \zeta_N\}. \tag{3.20}$$

We have:

Lemma 3.3 *Assume* (3.15) *and* (3.16). *Then, for each* k, *there exist subsets* $B_{k^2+k}^{\pm}$ *of* Ω *such that*

$$h_{k^2+k}^{\pm} \in \left[\frac{1}{2}E_{k^2+k}^{\pm}, \frac{3}{2}E_{k^2+k}^{\pm}\right] \quad \text{in } \Omega \setminus B_{k^2+k}^{\pm} \quad \text{and} \quad \mu(B_{k^2+k}^{\pm}) \le 4\eta, \tag{3.21}$$

and

$$\mu(\tilde{B}_{k^2}^{\pm}) < \zeta_N^{-1}\eta \quad \text{and} \quad N_k^{\pm} > (1-\zeta_N)3^{kN} \quad \text{in } \Omega \setminus \tilde{B}_{k_0}^{\pm}. \tag{3.22}$$

If

$$\mathbf{B}_k = B_{k^2+k}^{+} \cup B_{k^2+k}^{-} \cup \tilde{B}_{k^2}^{+} \cup \tilde{B}_{k^2}^{-},$$

then both (3.21) *and* (3.22) *hold in* $\Omega \setminus \mathbf{B}_k$ *and*

$$\mu(\mathbf{B}_k) \le 8(1+\zeta_N^{-1})\eta. \tag{3.23}$$

Having completed the presentation of the technical results needed in the proof of Theorem 3.1, we continue with a heuristic description of the argument.

(v) The clearing of contact set at large scales

We present here the most important step of the proof of Theorem 3.1 as a separate lemma. The basic idea is that, if the solutions $u_{m+\ell}^{+}$ and $u_{m+\ell}^{-}$ of the obstacle problems in the cube $D_{m+\ell}$ have, for some ω, total masses $h^+_{m+\ell}(\omega)$ and $h^-_{m+\ell}(\omega)$ respectively such that $h^+_{m+\ell}(\omega)h^-_{m+\ell}(\omega) \geqq \theta$, then, for ℓ sufficiently large compared to θ, the quadratic separation of the obstacle problem and the Fabes-Stroock lemma yield that either $u^+_{m+\ell}(\omega)$ or $u^-_{m+\ell}(\omega)$ cannot have a contact point in the intersection of $\frac{2}{3}D_{m+\ell}$ with any of the cubes of side length 3^m that subdivide $D_{m+\ell}$. As a result the contact set of either $u^+_{m+\ell}(\cdot,\omega)$ or $u^-_{m+\ell}(\cdot,\omega)$ has cleared at least half the subcubes of side length 3^m that are inside $\frac{2}{3}D_{m+\ell}$.

We have:

Lemma 3.4 *Fix $\omega \in \Omega$ and assume that total masses $h^\pm_{m+\ell}(\omega)$ of the solution $u^\pm_{m+\ell}(\cdot,\omega)$ of the upper and lower obstacle problems in $D_{\ell+m}$ satisfy $h^+_{m+\ell}(\omega)h^-_{m+\ell}(\omega) \geqq \theta$ for some $\theta > 0$. If ℓ and θ are such that $3^{2\ell}\theta^{\frac{M}{2N}} > 4c_1/c_7$, where c_1, c_7 and M are the universal constants in the quadratic separation and Fabes-Stroock results, then it is not possible for both $u^+_{m+\ell}(\cdot,\omega)$ and $u^-_{m+\ell}(\cdot,\omega)$ to touch the obstacle in any of the subcubes D^i_m that subdivide $D_{m+\ell}$ and are inside $\frac{2}{3}D_{m+\ell}$.*

(vi) The basic ideas for the proof of Theorem 3.1

As already mentioned earlier the sequence $\mathbb{H}_{k^2}$ is nonincreasing in k. The goal here is to show that going from D_{k^2} to $D_{(k+1)^2}$, $\mathbb{H}_{(k+1)^2}$ is strictly less (by a fixed amount) from $\mathbb{H}_{k^2}$. To achieve this, we start observing that, if either $H^{+\text{ or }-}_{k^2}$ or $H^{+\text{ or }-}_{k^2+k}$ is already less than a critical level, say $3^{(k_0-k)\tau-1}$, then there is nothing to prove since they are both nonincreasing in k. If both $H^\pm_{k^2}$ and $H^\pm_{k^2+k}$ are above the critical level $3^{(k_0-k)\tau-1}$ and either $V^{+\text{ or }-}_{k^2}$ or $V^{+\text{ or }-}_{k^2+k}$ is bigger than a small multiple of either $(E^{+\text{ or }-}_{k^2})^2$ or $(E^{+\text{ or }-}_{k^2+k})^2$ respectively, the decay of the variance due to strongly mixing assumption (see Lemma 3.2) yields a strict decay for either $H^{+\text{ or }-}_{k^2}$ or $H^{+\text{ or }-}_{k^2+k}$ and we may, again, conclude.

Therefore the real problem is when both $H^\pm_{k^2}$ and $H^\pm_{k^2+k}$ are of order $3^{(k_0-k)\tau-1}$ and both $V^\pm_{k^2}$ and $V^\pm_{k^2+k}$ are less than small multiples of $(E^\pm_{k^2})^2$ and $(E^\pm_{k^2+k})^2$ respectively. Then the averaged masses $h^{ij,\pm}_{k^2}$ and $h^{i,\pm}_{k^2+k}$ are evenly distributed over $D^{ij}_{k^2}$ and $D^i_{k^2+k}$ respectively in the sense that, for most of the ω's, $h^{ij,\pm}_{k^2}$ and $h^{i,\pm}_{k^2+k}$ are of order $E^\pm_{k^2}$ and $E^\pm_{k^2+k}$ respectively. Of course, there are several exceptional sets whose size needs to be controlled—this is where Lemma 3.3 plays a role.

For those ω's for which $h^{+\text{ or }-}_{k^2+k}$ have actually decayed strictly with respect to $h^{i,j,+\text{ or }-}_{k^2}$ we have nothing to prove. For the ω's, for which the $h^\pm_{k^2+k}$ have not decayed and, hence, have remained comparable to $3^{(k_0-k)\tau-1}$, we use the Fabes-Stroock estimate to obtain a strict separation between $u^+_{k^2+k}$ and $u^-_{k^2+k}$ in such a way that in many of the $D^j_{k^2}$'s that subdivide D_{k^2+k}, the contact set of either $u^+_{k^2+k}(\cdot,\omega)$ or $u^-_{k^2+k}(\cdot,\omega)$ has disappeared completely forcing again a strict decay from $h^{ij,\pm}_{k^2}$ to $h^{\pm}_{(k+1)^2}$.

(iv) The main proof

We proceed now with the

Proof of Theorem 3.1 We prove (3.10) by induction. Since, in view of (3.8), (3.10) holds for $k=k_0$, we assume next that

$$\mathbb{H}_{k^2} \leqq 3^{(k_0-k)\tau},$$

and prove that, for appropriate choices of k_0 and τ,

$$\mathbb{H}_{(k+1)^2} \leqq 3^{(k_0-k-1)\tau}. \tag{3.24}$$

First we observe that, if

$$\text{either}\quad H^{+\text{ or }-}_{k^2} \leqq 3^{(k_0-k)\tau-1} \quad\text{or}\quad H^{+\text{ or }-}_{k^2+k} \leqq 3^{(k_0-k)\tau-1} \quad\text{or}\quad \mathbb{H}_{k^2+k} \leqq 3^{(k_0-k)\tau-1},$$

then, if $\tau\in(0,1)$, (3.10) yields (3.24).

Hence, in what follows, we assume that

$$H^{\pm}_{k^2} \geqq 3^{(k_0-k)\tau-1},\quad H^{\pm}_{k^2+k} \geqq 3^{(k_0-k)\tau-1} \quad\text{and}\quad \mathbb{H}_{k^2+k} \geqq 3^{(k_0-k)\tau-1}. \tag{3.25}$$

Assume next that, for some $\eta\in(0,1)$,

$$\text{either}\quad V^{+\text{ or }-}_{k^2} \geqq \eta(E^{+\text{ or }-}_{k^2})^2 \quad\text{or}\quad V^{+\text{ or }-}_{k^2+k} \geqq \eta(E^{+\text{ or }-}_{k^2+k})^2. \tag{3.26}$$

It follows from Lemmas 3.1, 3.2 and (3.26) that

$$\begin{aligned}
H^{+\text{ or }-}_{(k+1)^2} &\leqq H^{+\text{ or }-}_{k^2+k} \leq E[A^{+\text{ or }-}_{k^2+k}]^2 = V(A^{+\text{ or }-}_{k^2+k}) + (E^{+\text{ or }-}_{k^2})^2\\
&\leqq 3^{-(k-1)N}V^{+\text{ or }-}_{k^2} + 3^{-k} + (E^{+\text{ or }-}_{k^2})^2\\
&\leqq (1+\eta)^{-1}(1+3^{-(k-1)N}\eta)H^{+\text{ or }-}_{k^2} + 3^{-k-(k_0-k)\tau+1}H^{+\text{ or }-}_{k^2}\\
&\leqq 3^{-\tau}H^{+\text{ or }-}_{k^2},
\end{aligned}$$

provided η, k_0 and τ are such that, for $k \geqq k_0$,

$$(1+\eta)^{-1}(1+3^{-(k-1)N}\eta)+3^{-k-(k_0-k)\tau+1} \leqq 3^{-\tau},$$

which is possible if $\eta \in (0,1)$, $k_0 \geqq 1$ and $\tau \in (0,1)$ are chosen so that

$$(1+\eta)^{-1}(1+3^{-(k_0-1)N}\eta)+3^{1-k_0} \leqq 3^{-\tau}. \tag{3.27}$$

Next we assume the opposite of (3.26), i.e., (3.15), and show that, for some uniform k_0 and $\tau \in (0,1)$ and for all ω in a sufficiently large subset of Ω, $h^{+\text{ or }-}_{k^2+k}$ is strictly smaller than $A^{+\text{ or }-}_{k^2+k}$ by a fixed multiple of $E^{+\text{ or }-}_{k^2}$. This then suffices to yield that $H^{+\text{ or }-}_{k^2+k}$ is strictly less than $H^{+\text{ or }-}_{k^2}$ by an amount enough to imply (3.24).

Let $\mathbf{B}_k$ be the set defined in Lemma 3.3. The first step is to use Lemma 3.4 to show that

$$\text{in } \Omega \setminus \mathbf{B}_k \text{ either } u^+_{k^2+k} \text{ or } u^-_{k^2+k} \text{ cannot have a contact point in } \frac{2}{3}D_{k^2+k}. \tag{3.28}$$

It then follows that one of the $u^+_{k^2+k}(\cdot,\omega)$ or $u^-_{k^2+k}(\cdot,\omega)$, say $u^+_{k^2+k}(\cdot,\omega)$, cannot have a contact point in at least half of the $(\frac{2}{3})^N 3^{kN} = 4\zeta_N 3^{kN}$ cubes $D^i_{k^2}$ whose union is $\frac{2}{3}D_{(k^2+k)}$. Note that, for a different $\omega \in \Omega \setminus \mathbf{B}_k$, it may be that $u^-_{k^2+k}(\cdot,\omega)$ has this property. But in any case one of them, which here we take to be $u^+_{k^2+k}(\cdot,\omega)$, has the property for at least "half" of the ω's in $\Omega \setminus \mathbf{B}_k$. This means that there exists $\mathbf{C}_k \subset \Omega \setminus \mathbf{B}_k$ such that

$$\mu(\mathbf{C}_k) \geqq \frac{1}{2}\mu(\Omega \setminus \mathbf{B}_k) \geqq \frac{1}{2}(1-8(1+\zeta_N^{-1})\eta) \tag{3.29}$$

and, for $\omega \in \mathbf{C}_k$,

$$u^+_{k^2+k}(\cdot,\omega) \text{ does not touch } P \text{ for at least } 2\zeta_N 3^{kN} \text{ cubes } D^j_{k^2} \text{ inside } \frac{2}{3}D_{k^2+k}. \tag{3.30}$$

Of course for (3.28) to be meaningful we need to choose $\eta > 0$ so that

$$\eta \in (0, (8(1+\zeta_N^{-1}))^{-1}). \tag{3.31}$$

We postpone the proof of (3.28) till the end of the ongoing one and we use Lemma 3.3 to show that

$$h^+_{k^2+k} \text{ is strictly smaller than } A^+_{k^2+k} \text{ by a fixed multiple of } E^+_{k^2} \text{ in } \mathbf{C}_k. \tag{3.32}$$

Indeed fix $\omega \in \mathbf{C}_k$ as in (3.28) and recall that Lemma 3.3 yields the existence of at least $(1-\zeta_N)3^{kN}$ cubes $D^i_{k^2}$ (recall that $D_{k^2+k} = \bigcup_{i=1}^{3^{kN}} D^i_{k^2}$) where $h^{i,+}_{k^2} \geqq \frac{1}{2} E^+_{k^2}$. Since there exist $(1-4\zeta_N)3^{kN}$ cubes $D^i_{k^2}$ whose union is $D_{k^2+k} \setminus \frac{2}{3} D_{k^2+k}$, it follows that, in at least $3\zeta_N 3^{kN} = (1-\zeta_N)3^{kN} - (1-4\zeta_N)3^{kN}$ of the cubes $D^i_{k^2}$ with union $\frac{2}{3} D_{k^2+k}$, we have $h^{i,+}_{k^2} \geqq \frac{1}{2} E^+_{k^2}$. This last observation combined with (3.28) yields, for $\omega \in \mathbf{C}_k$, the existence of at least $\zeta_N 3^{kN}$ cubes $D^i_{k^2}$ inside $\frac{2}{3} D_{k^2+k}$, where $u^+_{k^2+k}(\cdot,\omega)$ does not touch and $h^{i,+}_{k^2}(\omega) \geqq \frac{1}{2} E^+_{k^2}$. We denote by i' the indices of these cubes and by i^* the indices of the rest.

We have:

$$
\begin{aligned}
h^+_{k^2+k}(\omega) &= 3^{-(k^2+k)N} \int_{D_{k^2+k}} (\ell - F(P,y,\omega))^N_+ \mathbf{1}_{\Lambda^+_{k^2+k}(\omega)}\, dy \\
&= 3^{-(k^2+k)N} \sum_{j=1}^{3^{kN}} \int_{D^j_{k^2} \cap \Lambda^+_{k^2+k}(\omega)} (\ell - F(P,y,\omega))^N_+\, dy \\
&\leqq 3^{-kN} \sum_{i^*} h^{i^*,+}_{k^2}(\omega) \\
&\leqq 3^{-kN} \left(\sum_{i^*} h^{i^*,+}_{k^2}(\omega) + \sum_{i'} \left(h^{i,+}_{k^2}(\omega) - \frac{1}{2} E^+_{k^2} \right) \right) \\
&\leqq A^+_{k^2+k}(\omega) - \frac{3^{-kN}}{2} \left(\sum_{i'} E^+_{k^2} \right) \\
&\leqq A^+_{k^2+k}(\omega) - \frac{1}{2} \zeta_N E^+_{k^2}.
\end{aligned}
$$

Finally recalling that $h^+_{k^2+k} \leqq A^\pm_{k^2+k}$ and using (3.29) and the previous estimate we obtain

$$
\begin{aligned}
E^+_{k^2+k} = E h^+_{k^2+k} &= \int_{\Omega \setminus \mathbf{C}_k} h^+_{k^2+k}(\omega)\, d\mu(\omega) + \int_{\mathbf{C}_k} h^+_{k^2+k}(\omega) d\mu(\omega) \\
&\leqq \int_{\Omega \setminus \mathbf{C}_k} A^+_{k^2+k}(\omega) d\mu(\omega) + \int_{\mathbf{C}_k} \left(A^+_{k^2+k}(\omega) - \frac{1}{2} \zeta_N E^+_{k^2} \right) d\mu(\omega) \\
&= E A^+_{k^2+k} - \frac{1}{2} \zeta_N \mu(\mathbf{C}_k) E^+_{k^2} \\
&\leqq E^+_{k^2} - \frac{1}{4} (1 - 8(1+\zeta_N^{-1})\eta) \zeta_N E^+_{k^2} \\
&= \left(1 - \frac{1}{4} (1 - 8(1+\zeta_N^{-1})\eta) \zeta_N \right) E^+_{k^2}.
\end{aligned}
$$

We summarize the above sequence of inequalities as

$$E^+_{k^2+k} \leqq L_N E^+_{k^2} \tag{3.33}$$

where

$$L_N = 1 - \frac{1}{4}\zeta_N + 2(1+\zeta_N)\eta. \tag{3.34}$$

Using that

$$\begin{aligned} &(E^\pm_{k^2+k})^2 \leqq H^\pm_{k^2+k} \leqq (1+\eta)(E^\pm_{k^2+k})^2 \quad \text{and} \\ &(E^\pm_{k^2})^2 \leqq H^\pm_{k^2} \leqq (1+\eta)(E^\pm_{k^2})^2, \end{aligned} \tag{3.35}$$

which is a consequence of (3.15), we conclude from (3.33) that

$$H^+_{k^2+k} \leqq (1+\eta) L_N^2 H^+_{k^2}. \tag{3.36}$$

It is now clear that it is possible to choose η and τ sufficiently small depending only on N so that

$$(1+\eta)\left(1 - \frac{1}{4}\zeta_N + 2(1+\zeta_N)\eta\right)^2 < 3^{-\tau}. \tag{3.37}$$

Combining (3.36), (3.37) and $H^-_{k^2+k} \leqq H^-_{k^2}$, we get

$$\mathbb{H}_{(k+1)^2} \leqq \mathbb{H}_{k^2+k} \leqq 3^{(k_0-k-1)\tau}.$$

To conclude the proof we need to prove (3.28) for which it suffices to show that the assumptions of Lemma 3.4 are satisfied.

To this end recall that, for $\omega \in \mathbf{B}_k$, we have, in view of (3.21), (3.25) and (3.35),

$$\begin{aligned} h^+_{k^2+k}(\omega) h^-_{k^2+k}(\omega) &\geqq \frac{1}{4} E^+_{k^2+k} E^-_{k^2+k} \geqq \frac{1}{4(1+\eta)} (H^+_{k^2+k} H^-_{k^2+k})^{1/2} \\ &= \frac{1}{4(1+\eta)} \mathbb{H}^{1/2}_{k^2+k} \geqq \frac{1}{4(1+\eta)}\, 3^{\frac{(k_0-k)\tau-1}{2}}. \end{aligned}$$

Next we apply Lemma 3.4 with $m = k^2$, $\ell = k$, and

$$\theta = \frac{1}{4(1+\eta)}\, 3^{\frac{(k_0-k)\tau-1}{2}}. \tag{3.38}$$

The conclusion follows if we show that it is possible to choose $\eta \in (0,1)$, $k_0 \geqq 1$ and $\tau \in (0,1)$ so that, for $k \geqq k_0$,

$$3^{2k} \theta^{\frac{M}{2N}} > 4c_1/c_7,$$

i.e.,

$$3^{(2-\frac{M}{4N}\tau)k+(k_0\tau-1)\frac{M}{4N}} \geqq 4(4(1+\eta))^{\frac{M}{2N}} c_1/c_7$$

which holds for all $k \geqq k_0$ and $\eta \in (0,1)$ provided τ and k_0 satisfy

$$0<\tau<\frac{8N}{M} \quad \text{and} \quad 3^{2k_0-\frac{M}{4N}} > 2^{\frac{3M}{4N}+2} > 4(4(1+\eta))^{\frac{M}{2N}} c_1/c_7. \tag{3.39}$$

To conclude observe that it is possible to choose $\eta \in (0,1)$, $k_0 \geqq 1$ and $\tau \in (0,1)$ so that all (3.27), (3.31), (3.37) and (3.39). □

For future reference we restate Theorem 3.1 at the ε-scale, in which case (3.4) translates to a logarithmic rate.

To this end, for $\ell \in \mathbb{R}$ and $P \in S^N$, let $u_\varepsilon(\cdot,\omega)$, $u_\varepsilon^+(\cdot,\omega)$ and $u_\varepsilon^-(\cdot,\omega)$ be respectively the solutions of

$$\begin{cases} F(D^2u_\varepsilon, \frac{x}{\varepsilon}, \omega) = \ell & \text{in } D_0, \\ u_\varepsilon = P & \text{on } \partial D_0, \end{cases} \tag{3.40}$$

and the upper and lower obstacle problems in D_0 with constant ℓ and obstacle P. Finally denote by $h_\varepsilon^+(\omega)$ and $h_\varepsilon^-(\omega)$ the total masses, i.e.,

$$h_\varepsilon^\pm(\omega) = \int_{D_0} \left(\ell - F\left(P, \frac{x}{\varepsilon}, \omega\right)\right)_\pm^N \mathbf{1}_{\{u_\varepsilon^\pm(\cdot,\omega)=P\}}(x)\,dx. \tag{3.41}$$

Let

$$\mathbb{H}_\varepsilon = E(h_\varepsilon^+)^2 E(h_\varepsilon^-)^2. \tag{3.42}$$

We have:

Theorem 3.2 *Assume the hypotheses of Theorem 3.1. Fix $P \in S^N$ and $\ell \in \mathbb{R}$. There exist universal positive constants C, c and ε_0 such that, for all $\varepsilon \leqq \varepsilon_0$,*

$$\mathbb{H}_\varepsilon \leqq C(1+|P|+|\ell|)^{4N} \varepsilon^{c|\ln\varepsilon|^{-1/2}}. \tag{3.43}$$

Proof The conclusion follows from (3.4). Indeed let k be such that $\varepsilon \cong 3^{-k^2}$. Then

$$3^{-\tau k} \cong \varepsilon^{c|\ln\varepsilon|^{-1/2}} \quad \text{and the claim holds with}$$
$$c = \tau(\ln 3)^{1/2}, \quad C = 3^{\tau k_0} \quad \text{and} \quad \varepsilon_0 \cong 3^{-k_0^2}.$$ □

We conclude the section with the proofs of the lemmas stated earlier.

Proof of Lemma 3.1 Since the monotonicity of $\mathbb{H}_{k^2}$ follows immediately from the monotonicity of $H^{\pm}_{k^2}$, here we concentrate on the latter. Moreover, since the two inequalities for $H^{\pm}_{k^2}$ are proved in a similar manner, below we present the argument for $H^{\pm}_{(k+1)^2} \leqq H^{\pm}_{k^2+k}$.

The definition of the solution of the upper and lower obstacle problems (smallest super-solution above P for the upper problem and largest sub-solution below P for the lower problem) yields that $u^{+}_{(k+1)^2}$ (resp. $u^{-}_{(k+1)^2}$) is a super-solution (resp. sub-solution) of the upper (resp. lower) obstacle problem in $D^i_{k^2+k}$, for $i = 1, \ldots, 3^{(k+1)N}$.

Hence, for all ω and $i = 1, \ldots, 3^{(k+1)N}$, we have

$$\Lambda^{\pm}_{(k+1)^2}(\omega) \cap D^i_{k^2+k} \subseteq \Lambda^{i,\pm}_{k^2+k}(\omega).$$

It follows that

$$\begin{aligned} |D_{(k+1)^2}| h^{\pm}_{(k+1)^2}(\omega) &= \int_{D_{(k+1)^2} \cap \Lambda^{\pm}_{(k+1)^2}} (\ell - F(P, y, \omega))^N_{\pm} dy \\ &= \sum_{i=1}^{3^{(k+1)N}} \int_{D^i_{k^2+k} \cap \Lambda^{\pm}_{(k+1)^2}} (\ell - F(P, y, \omega))_{\pm} dy \\ &\leqq \sum_{i=1}^{3^{(k+1)N}} \int_{D^i_{k^2+k} \cap \Lambda^{i,\pm}_{k^2+k}} (\ell - F(P, y, \omega))^N_{\pm} dy \\ &= |D_{k^2+k}| \sum_{i=1}^{3^{(k+1)N}} h^{i,\pm}_{k^2+k}(\omega), \end{aligned}$$

and, therefore,

$$h^{\pm}_{(k+1)^2}(\omega) \leqq \frac{|D_{k^2+k}|}{|D_{(k+1)^2}|} \sum_{i=1}^{3^{(k+1)N}} h^{i,\pm}_{k^2+k}(\omega) = A^{\pm}_{(k+1)^2}(\omega).$$

Thus

$$H^{\pm}_{(k+1)^2} = E^{\pm}_{(k+1)^2} \leqq E(A^{\pm}_{(k+1)^2})^2.$$

Finally, in view of the stationarity and Cauchy-Schwartz inequality,

$$E(A^{\pm}_{(k+1)^2})^2 = \frac{1}{3^{2(k+1)N}} E\left(\sum_{i=1}^{3^{(k+1)N}} h^{i,\pm}_{k^2+k} \right)^2$$

$$= \frac{1}{3^{2(k+1)N}} \sum_{i,j=1}^{3^{(k+1)N}} E(h_{k^2+k}^{i,\pm} h_{k^2+k}^{j,\pm})$$

$$\leqq \frac{1}{3^{(2(k+1)N}} \sum_{i,j=1}^{3^{(k+1)N}} (E(h_{k^2+k}^{i,\pm})^2)^{1/2} (E(h_{k^2+k}^{j,\pm})^2)^{1/2}$$

$$= E(h_{k^2+k}^{\pm})^2 = H_{k^2+k}^{\pm}. \qquad \square$$

Proof of Lemma 3.2 We only prove (3.13), since (3.14) follows similarly. The difference on the decay between (3.13) and (3.14) is due to the fact that $A^{\pm}_{(k+1)^2}$ is an average over $3^{(k+1)N}$ cubes $D^i_{k^2+k}$ while $A^{\pm}_{k^2+k}$ is the average over 3^{kN} cubes $D^i_{k^2}$.

Let z_i be the center of the cube $D^i_{k^2}$ and note that, for each z_i, there exist at most 3^N distinct cubes $D^j_{k^2}$ ($D^i_{k^2}$ included) with $|z_i - z_j| \leqq 2 \cdot 3^{k^2}$.

It follows from (2.28) and (2.29) that

$$E h_{k^2}^{i,\pm} h_{k^2}^{j,\pm} \leqq (E_{k^2}^{\pm})^2 + 3^{-k} \quad \text{if } |z_i - z_j| > 3^{k^2},$$

while, for all i, j, Hölder inequality yields

$$E(h_{k^2}^{i,\pm} h_{k^2}^{j,\pm}) - (E_k^{\pm})^2 \leqq V_{k^2}^{\pm}.$$

We have:

$$V(A_{k^2+k}^{\pm}) = 3^{-2kN} E\left[\sum_{i=1}^{3^{kN}} (h_{k^2}^{i,\pm} - E_{k^2}^{\pm})\right]^2$$

$$= 3^{-2kN} \sum_{i,j=1}^{3^{kN}} E[(h_{k^2}^{i,\pm} - E_{k^2}^{\pm})(h_{k^2}^{j,\pm} - E_{k^2}^{\pm})]$$

$$= 3^{-2kN} \sum_{i,j=1}^{3^{kN}} [E(h_{k^2}^{i,\pm} h_{k^2}^{j,\pm}) - (E_{k^2}^{\pm})^2]$$

$$= 3^{-2kN} \sum_{\substack{i,j=1 \\ |z_i - z_j| < 2\cdot 3^{k^2}}}^{3^{kN}} [E(h_{k^2}^{i,\pm} h_{k^2}^{j,\pm}) - (E_{k^2}^{\pm})^2]$$

$$+ 3^{-2kN} \sum_{\substack{i,j=1 \\ |z_i - z_j| \geq 2\cdot 3^{k^2}}}^{3^{kN}} [E h_{k^2}^{i,\pm} h_{k^2}^{j,\pm} - (E_{k^2}^{\pm})^2]$$

$$\leqq 3^{-2kN}(3^{kN}3^N V_{k^2}^{\pm} + 3^{kN}(3^{kN} - 3^N)3^{-k})$$

$$= 3^{-(k-1)N} V_{k^2}^{\pm} + 3^{-kN}(3^{kN} - 3^N)3^{-k}$$

$$\leqq 3^{-(k-1)N} V_{k^2}^{\pm} + 3^{-k}. \qquad \square$$

Proof of Lemma 3.3 The first claim (3.21) follows immediately from (3.16) with

$$B_{k^2+k}^{\pm} = \left\{\omega : |h_{k^2+k}^{\pm} - E_{k^2+k}^{\pm}| \geqq \frac{1}{2} E_{k^2+k}^{\pm}\right\}.$$

The definition of $N_k^{\pm}$ gives

$$\bar{\rho}_k^{\pm} = 3^{-kN}(3^{kN} - N_k^{\pm}). \tag{3.44}$$

Moreover, for each $i = 1, \ldots, 3^{kN}$,

$$E(\rho_{k^2}^{i,\pm}) = \mu\left(\left\{\omega \in \Omega : h_{k^2}^{i,\pm}(\omega) \leqq \frac{1}{2} E_{k^2}^{\pm}\right\}\right)$$

and

$$V_{k^2}^{\pm} = E[h_{k^2}^{i,\pm} - E(h_{k^2}^{\pm})]^2 \geqq \frac{1}{4}(E_k^{\pm})^2 \mu\left(\left\{\omega \in \Omega : h_{k^2}^{i,\pm}(\omega) \leqq \frac{1}{2} E_{k^2}^{\pm}\right\}\right).$$

Hence

$$E(\rho_{k^2}^{i,\pm}) \leqq \frac{4V_{k^2}^{\pm}}{(E_k^{\pm})^2},$$

and, in view of (3.15),

$$E(\rho_{k^2}^{i,\pm}) \leqq 4\eta.$$

The stationarity assumption then gives

$$E(\bar{\rho}_k^{\pm}(\ell)) \leqq 4\eta,$$

and, hence, again from Chebyshev's inequality,

$$\mu(\tilde{B}_k^{\pm}) \leqq 4\zeta_N^{-1}\eta,$$

while, in view of (3.44),

$$N_k^{\pm} \geqq (1-\zeta_N)3^{kN} \quad \text{in } \Omega \setminus \tilde{B}_{k^2}^{\pm}.$$

Finally,

$$\mu(\mathbf{B}_k) \leqq \mu(B_{k^2+k}^{+}) + \mu(B_{k^2+k}^{-}) + \mu(\tilde{B}_{k^2}^{+}) + \mu(\tilde{B}_{k^2}^{-}) \leqq 8\eta(1+\zeta_N^{-1}). \quad \square$$

Proof of Lemma 3.4 Since ω is a fixed parameter throughout the proof, for notational simplicity we do not display it here.

Let

$$w_{m+\ell} = u_{m+\ell}^{+} - u_{m+\ell}^{-}.$$

The linearization argument, which was developed in [1] and is described in Appendix A, implies that there exists a linear uniformly elliptic operator $\mathcal{L}$ (the "linearization" of F) with the same ellipticity constants as F such that

$$\begin{cases} \mathcal{L} w_{m+\ell} = ((\ell - F(P,\cdot))_{+}\mathbf{1}_{\Lambda_{m+\ell}^{+}} + (\ell - F(P,\cdot))_{-}\mathbf{1}_{\Lambda_{m+\ell}^{-}}) & \text{in } D_{m+\ell}, \\ w_{m+\ell} = 0 & \text{on } \partial D_{m+\ell}. \end{cases}$$

The Fabes-Stroock estimate (Theorem A.2 in Appendix A) yields

$$w_{m+\ell} \geqq c_7\, 3^{(2-M)(m+\ell)} \|f\|^{1-M} \|f\|_{L^N(D_{m+\ell})}^{M} \quad \text{in } \frac{2}{3} D_{m+\ell}, \tag{3.45}$$

with c_7 the uniform positive constant from (A.4) and

$$f(y) = (\ell - F(P,y))_{+}\mathbf{1}_{\Lambda_{m+\ell}^{+}} + (\ell - F(P,y))_{-}\mathbf{1}_{\Lambda_{m+\ell}^{-}}.$$

Since $(\ell - F(P,y))_{+}(\ell - F(P,y))_{-} = 0$, we get

$$\begin{aligned} \|f\|_{L^N(D_{m+\ell})}^{N} &= \int_{D_{m+\ell}} [(\ell - F(P,y))_{+}\mathbf{1}_{\Lambda_{m+\ell}^{+}} + (\ell - F(P,y))_{-}\mathbf{1}_{\Lambda_{m+\ell}^{-}}]^N dy \\ &= \int_{D_{m+\ell}} (\ell - F(P,y))_{+}^{N}\mathbf{1}_{\Lambda_{m+\ell}^{+}} + (\ell - F(P,y))_{-}^{N}\mathbf{1}_{\Lambda_{m+\ell}^{-}} dy \\ &= |D_{m+\ell}|(h_{m+\ell}^{+} + h_{m+\ell}^{-}) \geqq |D_{m+\ell}|\, \theta^{1/2}. \end{aligned}$$

Moreover, in view of (3.8), we have $\|f\| = 1$, and, hence, (3.45) yields

$$\begin{aligned} w_{m+\ell} &\geqq c_7 3^{(2-M)(m+\ell)} |D_{m+\ell}|^{M/N} \theta^{M/2N} \\ &= c_7 3^{2(m+\ell)} \theta^{M/2N} \quad \text{in } \frac{2}{3} D_{m+\ell}, \end{aligned} \tag{3.46}$$

and, therefore,

$$u^+_{m+\ell} - P + P - u^-_{m+\ell} = w_{m+\ell} \geqq c_7\, 3^{2(m+\ell)\frac{M}{2N}} \theta^{M/2N}. \tag{3.47}$$

Since $u^+_{m+\ell} - P \geqq 0$ and $P - u^-_{m+\ell} \geqq 0$, it follows that, for every $x \in \frac{2}{3}D_{m+\ell}$,

$$\begin{aligned} &\text{either} \quad u^+_{m+\ell} - P \geqq (1/2)c_7 3^{2(m+\ell)}\theta^{M/2N} \quad \text{or} \\ &P - u^-_{m+\ell} \geqq (1/2)c_7 3^{2(m+\ell)}\theta^{M/2N}. \end{aligned} \tag{3.48}$$

If there is a contact point for both $u^+_{m+\ell}$ and $u^-_{m+\ell}$ in the intersection of $\frac{2}{3}D_{m+\ell}$ with any of the cubes D^i_m of side length 3^m that subdivide $D_{m+\ell}$ and $\frac{2}{3}D_{m+\ell}$, the quadratic separation property of the obstacle problems yields

$$\begin{aligned} \max_{D^i_m}(u^+_{m+\ell}(\cdot,\omega) - u^-_{m+\ell}(\cdot,\omega)) &\leqq \max_{D^i_m}(u^+_{m+\ell} - P) + \max_{D^i_m}(u^-_{m+\ell} - P) \\ &\leqq 2c_1 3^{2M}. \end{aligned} \tag{3.49}$$

Combining (3.47) and (3.49) we find

$$2c_1 3^{2m} \geqq c_7 3^{2(m+\ell)}\theta^{M/2N},$$

and, hence,

$$4c_1(c_7)^{-1} \geqq 3^{2\ell}\theta^{M/2N},$$

which is impossible in view of the assumption of Lemma 2.5.

The claim now follows. □

We remark that at this point the probabilistic content of the paper is complete.

4 Uniform rate of convergence for bounded families of quadratic data in cubes

We show here that the rate of convergence obtained in the previous section for each quadratic yields a similar rate for the distance $\|u_\varepsilon - \bar{u}\|$ between the solution u_ε of the boundary value problem

$$\begin{cases} F(D^2u_\varepsilon, \frac{x}{\varepsilon}, \omega) = \bar{F}(P) & \text{in } B_1, \\ u_\varepsilon = P & \text{on } \partial B_1. \end{cases} \tag{4.1}$$

and $\bar{u} = P$, which, in view of the uniqueness of viscosity solutions, is the solution of

$$\begin{cases} \bar{F}(D^2\bar{u}) = \bar{F}(P) & \text{in } B_1, \\ \bar{u} = P & \text{on } \partial B_1, \end{cases} \tag{4.2}$$

off an "exceptional" set that depends on $|P|$ and not P.

The way that the exceptional set depends on P is a consequence of the strong stability properties of the solution of the obstacle problems with respect to the obstacle. As a matter of fact it is very important for the proof of the general rate because it implies that, for each $\varepsilon > 0$ and $R > 0$, there exists the same negligible set of bad configurations for all quadratics P such that $|P| \leq R$.

The first step is

Proposition 4.1 *Assume* (2.5), (2.7), (2.8) *and* (2.27). *There exist positive constants* $\bar{C}$, $\hat{C}$, $\bar{c}$, $\hat{c}$ *and* ε_0 *depending on the constants in* (2.7) *and* (2.8) *and the dimension such that, for each* $P \in S^N$ *and all* $\varepsilon \in (0, \varepsilon_0)$, *there exists* $A_\varepsilon \subset \Omega$ *which may depend on* P *such that, if* $u_\varepsilon(\cdot, \omega) \in C(\bar{B}_1)$, *is the solution of* (4.1), *then*

$$\begin{aligned} &\mu(A_\varepsilon) \leqq \bar{C}(1+|P|)^{2N}\varepsilon^{\bar{c}|\ln\varepsilon|^{-1/2}} \quad \textit{and} \\ &\|u_\varepsilon(\cdot,\omega) - P\| \leqq \hat{C}\varepsilon^{\hat{c}|\ln\varepsilon|^{-1/2}}\delta^2 \quad \textit{for } \omega \textit{ in } \Omega \setminus A_\varepsilon^c. \end{aligned} \tag{4.3}$$

Proof To prove (4.3) we compare $u_\varepsilon(\cdot, \omega)$ to the solutions of the upper and lower obstacle problems with obstacle P and $\ell = \bar{F}(P)$ and we use (3.43). There is, however, a slight difficulty due to the fact that the decay of $\mathbb{H}_\varepsilon$ does not yield immediately a decay for $H_\varepsilon^+ = E(h_\varepsilon^+)^2$ and $H_\varepsilon^- = E(h_\varepsilon^-)^2$. Recall that although both H_ε^+ and H_ε^- are monotone in ε, along subsequences either one can be constant for a while. To circumvent this difficulty, we work instead with slightly perturbed ℓ's, namely with $\ell = \bar{F}(P) \pm \gamma$. Adding (resp. subtracting) γ "makes" u_ε^+ (resp. u_ε^-) a "subsolution" (resp. "supersolution") of the expected homogenized equation and forces u_ε^+ (resp. u_ε^-) to "stick" (converge) to P. That is we show that the second moment $H_\varepsilon^{\gamma,+}$ (resp. $H_\varepsilon^{\gamma,-}$) corresponding to $\ell = \bar{F}(P) + \gamma$ (resp. $\ell = \bar{F}(P) - \gamma$) is bounded from below (resp. below) away from zero by a uniform $O(\gamma^{2N})$. Hence the decay of the product $\mathbb{H}_\varepsilon$ in this case yields a decay for $H_\varepsilon^{\gamma,-}$ (resp. $H_\varepsilon^{\gamma,+}$).

Here we only show the upper bound on $u_\varepsilon(\cdot, \omega) - P$, since the lower bound follows similarly.

To this end, fix $\gamma > 0$ and let $u_{\varepsilon,\gamma}(\cdot, \omega)$, $u_{\varepsilon,\gamma}^+(\cdot, \omega)$ and $u_{\varepsilon,\gamma}^-(\cdot, \omega)$ be respectively the solutions of (4.1) and the upper and lower obstacle problems in D_0 with obstacle P and all with right hand side the constant $\ell = \bar{F}(P) + \gamma$.

The stability properties of viscosity solutions imply that, for some uniform $c' > 0$,

$$\|u_{\varepsilon,\gamma}(\cdot,\omega) - u_\varepsilon(\cdot,\omega)\| \leqq c'\gamma. \tag{4.4}$$

Moreover,

$$u^-_{\varepsilon,\gamma}(\cdot,\omega) \leqq P \quad \text{in } D_0. \tag{4.5}$$

It follows that

$$u_\varepsilon(\cdot,\omega) - P \leqq u_{\varepsilon,\gamma}(\cdot,\omega) - u^-_{\varepsilon,\gamma}(\cdot,\omega) + c'\eta. \tag{4.6}$$

We need the following lemma. Its proof is presented at the end of the section.

Lemma 4.1 *Let $h^+_{\varepsilon,\gamma}(\omega)$ be the total mass of the solutions of the upper obstacle problem in D_0 with $\ell = \bar{F}(P) + \gamma$ and quadratic P. There exist a uniform constant $c_{11} > 0$ such that*

$$E(h^+_{\varepsilon,\gamma})^2 \geqq c_{11}\gamma^{2N}. \tag{4.7}$$

With (4.7) at hand, we find now an exceptional set A_ε and obtain upper bounds as in (4.3). Indeed (4.7) and (3.43) imply, for some uniform $c_{12} > 0$, that

$$E(h^-_{\varepsilon,\gamma})^2 \leqq c_{12}(1+|P|)^{4N}\gamma^{-2N}\varepsilon^{c|\ln\varepsilon|^{-1/2}}. \tag{4.8}$$

Note that in deriving (4.8) we used that $\gamma \in (0,1)$ and the fact that, for some uniform $c_{13} > 0$,

$$|\bar{F}(P)| \leqq c_{13}(1+|P|), \tag{4.9}$$

which is a consequence of (2.9).

Hence

$$Eh^-_{\varepsilon,\gamma} \leqq c_{12}^{1/2}(1+|P|)^{2N}\gamma^{-N}\varepsilon^{c'|\ln\varepsilon|^{-1/2}} \quad \text{with } c' = c/2.$$

Finally, in view of (2.22), for yet another uniform constant $c_{14} > 0$, we have

$$E(\|u_{\varepsilon,\gamma} - u^-_{\varepsilon,\gamma}\|^N_{C(\bar{D}_0)}) \leqq c_{14}(1+|P|)^{2N}\gamma^{-N}\varepsilon^{c'|\ln\varepsilon|^{-1/2}}.$$

For $\theta > 0$ let $A^\gamma_\theta \subset \Omega$ be given by

$$A^\gamma_\delta = \{\omega \in \Omega : \|u_{\varepsilon,\gamma}(\cdot,\omega) - u^-_{\varepsilon,\gamma}(\cdot,\omega)\|_{C(\bar{D}_0)} > \theta\}. \tag{4.10}$$

Chebyshev's inequality yields

$$\mu(A_\theta^\gamma) \leqq c_{14}^{1/2}(1+|P|)^{2N}\varepsilon^{c'|\ln\varepsilon|^{-1/2}}(\theta\gamma)^{-N}. \tag{4.11}$$

Combining (4.4), (4.5), (4.6) and (4.10) we find

$$\max_{\bar{D}_0}(u_\varepsilon(\cdot,\omega)-P) \leqq \theta + c'\gamma \quad \text{in} \quad \Omega\setminus A_\theta^\gamma. \tag{4.12}$$

To conclude we need to show that it is possible to choose θ and γ sufficiently small depending on ε so that (4.3) hold.

Let

$$\theta = \gamma = \varepsilon^{c''|\ln\varepsilon|^{-1/2}} \quad \text{with } c'' < c'/2N.$$

Then (4.3) follows with

$$\hat{C} = 2, \quad \hat{c} = c'', \quad \bar{C} = c_{14} \quad \text{and} \quad \bar{c} = c' - 2Nc''. \qquad \square$$

The exceptional set $A_\varepsilon = A_\theta^\gamma$ defined in the previous proof may depend, of course, on the specific P. To obtain the rate of convergence for the general problem, we must show that, for all $R > 0$ and for all $P \in S^N$ such that $|P| \leqq R$, it is possible to choose the same exceptional set which now may depend on R. As already stated earlier, this follows again from the stability properties of the obstacle problem and the uniformity already built in (3.4) and (3.43).

The first step is to show that it is possible to have the same exceptional set and rate for quadratics not far from each other. Throughout the discussion below we denote by $u_{\varepsilon,P}(\cdot,\omega)$ the solution of (4.1) with data P.

We have:

Proposition 4.2 *There exist uniform positive constants $\bar{C}, \hat{C}, \bar{c}, \hat{c}$ and ε_0 such that, for all $P_0 \in S^N$ and $\varepsilon \in (0,\varepsilon_0)$, there exists $A_\varepsilon \subset \Omega$, which may depend on P_0, such that $\mu(A_\varepsilon) \leqq \bar{C}(1+|P_0|)^{2N}\varepsilon^{\bar{c}|\ln\varepsilon|^{-1/2}}$ and*

$$\sup_{|P-P_0|\leqq \varepsilon^{\hat{c}|\ln\varepsilon|^{-1/2}}} \|u_{\varepsilon,P}(\cdot,\omega)-P\|_{C(\bar{D}_0)} \leqq \hat{C}\varepsilon^{\hat{c}|\ln\varepsilon|^{-1/2}} \quad \textit{in } \Omega\setminus A_\varepsilon.$$

Proof The stability properties of the viscosity solutions and the Lipschitz continuity of $\bar{F}$ with respect to P (a direct consequence of the uniform ellipticity) imply, for some uniform constant $c_{15} > 0$,

$$\sup_{|P-P_0|\leqq r} \|u^\pm_{\varepsilon,P}(\cdot,\omega)-P\|_{C(\bar{D}_0)} \leqq c_{15}r + \|u^\pm_{\varepsilon,P_0}(\cdot,\omega)-P_0\|_{C(\bar{D}_0)}.$$

Moreover the previous proof yield constants $\bar{C}$, $\hat{C}$, $\bar{c}$, $\hat{c}$ and ε_0 and, for each $\varepsilon \in (0, \varepsilon_0)$, $A_\varepsilon \subset \Omega$ such that

$$\|u^{\pm}_{\varepsilon, P_0}(\cdot, \omega) - P_0\|_{C(\bar{D}_0)} \leqq C\varepsilon^{\hat{c}|\ln \varepsilon|^{-1/2}} \quad \text{in } \Omega \setminus A_\varepsilon \text{ and}$$
$$\mu(A_\varepsilon) \leqq \bar{C}(1 + |P_0|)^{2N} \varepsilon^{\bar{c}|\ln \varepsilon|^{-1/2}}.$$

The claim now follows provided that r is chosen so that

$$c_{15} r \leqq \varepsilon^{\hat{c}|\ln \varepsilon|^{-1/2}}. \qquad \square$$

Next we use Proposition 4.2 and the arguments in its proof to obtain a result yielding a common exceptional set (with the desired upper bound) for all quadratic polynomials of size R. The exceptional set may, of course, depend on R.

We have:

Proposition 4.3 *Fix $R > 0$. There exist uniform positive constants $\bar{C}$, $\hat{C}$, $\bar{c}$, $\hat{c}$ and ε_0 such that, for all $P \in S^N$ such that $|P| \leqq R$ and all $\varepsilon \in (0, \varepsilon_0)$, there exists $A_\varepsilon \subset \Omega$, which may depend on R, such that*

$$\sup_{|P| \leqq R} \|u_{\varepsilon, P}(\cdot, \omega) - P\|_{C(\bar{D}_0)} \leqq \hat{C}\varepsilon^{\hat{c}|\ln \varepsilon|^{-1/2}} \quad \textit{in } \Omega \setminus A_\varepsilon$$

and

$$\mu(A_\varepsilon) \leqq \bar{C}(R^N(1 + R)^2)^N \varepsilon^{\bar{c}|\ln \varepsilon|^{-1/2}}.$$

Proof Consider a cover of $\{P \in S^N : |P| \leqq R\}$ by $M \approx (r^{-1}R)^{N^2}$ balls of radius $r > 0$ centered at $P_i \in S^N$ such that $|P_i| \leqq R$.

Proposition 4.2 gives, for each $i = 1, \ldots, M$, an exceptional set A^i_ε such that

$$\mu(A^i_\varepsilon) \leqq \bar{C}(1 + R)^{2N} \varepsilon^{\bar{c}|\ln \varepsilon|^{-1/2}} \quad \text{and}$$
$$\|u^{\pm}_{\varepsilon, P_i}(\cdot, \omega) - P_i\| \leqq \hat{C}\varepsilon^{\hat{c}|\ln \varepsilon|^{-1/2}} \quad \text{in } \Omega \setminus A^i_\varepsilon.$$

Let $A_\varepsilon = \bigcup A^i_\varepsilon$. Then

$$\sup_{|P| \leqq R} \|u^{\pm}_{\varepsilon, P}(\cdot, \omega) - P\|_{C(\bar{D}_0)} \leqq \hat{C}\varepsilon^{\hat{c}|\ln \varepsilon|^{-1/2}} \quad \text{in } \Omega \setminus A_\varepsilon,$$

and

$$\mu(A_\varepsilon) \leqq \sum_{i=1}^{M} \mu(A^i_\varepsilon) \leqq \bar{C}M(1 + R)^{2N} \varepsilon^{\bar{c}|\ln \varepsilon|^{-1/2}}.$$

Choose $r = \varepsilon^{(\tilde{c}/2N^2)|\ln\varepsilon|^{-1/2}}$. Then $M \approx R^{N^2}\varepsilon^{-(\tilde{c}/2)|\ln\varepsilon|^{-1/2}}$ and we have

$$\mu(A_\varepsilon) \leqq \bar{C}(R^N(1+R)^2)^N \varepsilon^{(\tilde{c}/2)|\ln\varepsilon|^{-1/2}}. \qquad \square$$

We conclude with the proof of Lemma 4.1. Before we get into it, we remark that, since $\bar{F}$ is uniformly elliptic, if

$$P_t(x) = P(x) + t(|x|^2 - 1),$$

there exist $0 < \bar{\lambda} \leq \bar{\Lambda}$ such that

$$\bar{\lambda} t \leqq \bar{F}(D^2 P_t) - \bar{F}(P) \leqq \bar{\Lambda} t. \tag{4.13}$$

Proof of Lemma 4.1 Let $u_{\varepsilon,\gamma}$ the solution of

$$\begin{cases} F(D^2 u_{\varepsilon,\gamma}, \frac{x}{\varepsilon}, \omega) = \bar{F}(P) + \gamma & \text{in } D_0, \\ u_{\varepsilon,\gamma} = P & \text{on } \partial D_0. \end{cases}$$

The homogenization result of [5] yields that, as $\varepsilon \to 0$ and a.s. in ω, $u_{\varepsilon,\gamma}(\cdot,\omega) \to \bar{u}_\gamma$ in $C(\bar{D}_0)$ where $\bar{u}_\gamma$ is the solution of

$$\begin{cases} \overline{F}(D^2 \bar{u}_\gamma) = \bar{F}(P) + \gamma & \text{in } D_0, \\ \bar{u}_\gamma = P & \text{on } \partial D_0. \end{cases} \tag{4.14}$$

The uniqueness of solutions to (4.14) and (4.13) yields, for some universal constant $c_{16} > 0$, that

$$\bar{u}_\gamma = P - c_{16}\gamma(1 - |x|^2). \tag{4.15}$$

Let $u^+_{\varepsilon,\gamma}$ be the solution of the upper obstacle problem with obstacle P and constant $\ell = \bar{F}(P) + \gamma$. Since $u^+_{\varepsilon,\gamma}(\cdot,\omega) \geqq P$, in the limit $\varepsilon \to 0$ we have, a.s. in ω,

$$\lim_{\varepsilon\to 0}(u^+_{\varepsilon,\gamma}(x,\omega) - \bar{u}_\gamma(x)) \geqq c_{16}\gamma(1 - |x|^2),$$

while the ABP-estimate (2.22) yields

$$\sup_{\bar{D}_0}(u^+_{\varepsilon,\gamma} - u_{\varepsilon,\gamma})^N(\cdot,\omega) \leqq c_2^N h^+_{\varepsilon,\gamma}(\omega).$$

Combining (4.15) and the last two inequalities we find

$$(c_{16}\gamma)^N \leqq (u^+_{\varepsilon,\gamma}(0,\omega) - u_{\varepsilon,\gamma}(0,\omega))^N \leqq c_2^N h^+_{\varepsilon,\gamma}(\omega).$$

Since, in view of the subadditivity (see the proof of Lemma 3.16 of [5]), $Eh^+_{\varepsilon,\gamma}$ converges decreasingly, the claim now follows with $c_{11} = (c_{16}/c_2)^N$. $\square$

5 δ-Viscosity solutions

To obtain the rate of convergence for general (not necessarily quadratic) data we introduce a class of suitable approximations to viscosity solutions, which we call δ-viscosity solutions and, for short, δ-solutions. We believe that this is of independent interest and it will have applications in other contexts involving rates of convergence.

As already mentioned in the Introduction, we show that δ-solutions are within a uniform distance from the solution of the given problem. To obtain a rate of convergence for a given approximation, it then suffices to show that the solution of the approximate problem is a δ-solution for an appropriate choice of δ.

To state the definition of the δ-solution in what follows we consider the equation

$$F(D^2u) = 0 \quad \text{in } U, \tag{5.1}$$

where U is an open subset of $\mathbb{R}^N$ and $F \in C(S^N)$ is uniformly elliptic.

We have:

Definition 5.1 Fix $\delta > 0$. $v \in C(\bar{U})$ is a δ-supersolution (resp. δ-subsolution) of (5.1) in U if, for all $x_0 \in U$ such that $B_\delta(x_0) \subset U$, a quadratic polynomial P such that $|P| \leqq C\delta^{-\sigma}$, for some universal $C, \sigma > 0$, and $P \leqq v$ (resp. $P \geqq v$) in $B_\delta(x_0)$ can touch v from below (resp. above) at x_0, i.e., $P(x_0) = v(x_0)$, only if $F(D^2P) \leqq 0$ (resp. $F(D^2P) \geqq 0$). Finally, $v \in C(U)$ is a δ-solution if it is both δ-supersolution and δ-subsolution.

The difference with the viscosity solution is the condition on the size of the polynomial and, more importantly, the requirement that for a δ-supersolution (resp. δ-subsolution) v the "test" polynomial P must be smaller (resp. larger) than v in all of $B_\delta(x_0)$. It is immediate that a viscosity supersolution (resp. subsolution) is actually a δ-supersolution (resp. δ-subsolution). On the other hand a δ-subsolution (resp. δ-supersolution) may not necessarily be a viscosity subsolution (resp. supersolution) unless it is a δ-subsolution (resp. supersolution) for all δ.

Our main result concerning δ-solutions is:

Theorem 5.1 *Let U be an open subset of $\mathbb{R}^N$ with regular boundary and consider a solution $u \in C^{0,1}(\bar{U})$ of (5.1). Assume that $v^+ \in C^{0,\eta}(\bar{U})$ (resp. $v^- \in C^{0,\eta}(\bar{U})$) is a δ-subsolution (resp. δ-supersolution) of (5.1) for some fixed $\eta \in (0,1)$. If $v^+ \leqq u + \underline{c}\delta^{\underline{\alpha}}$ (resp. $u \leqq v^- + \underline{c}\delta^{\underline{\alpha}}$) on ∂U for some positive constants $\underline{c}$ and $\underline{\alpha}$, then there exist uniform constants $c > 0$ and $\alpha \in (0, \underline{\alpha})$ such that, for δ sufficiently small,*

$$v^+ \leqq u + c\delta^\alpha \quad (\text{resp. } u \leqq v^- + c\delta^\alpha) \quad \text{in } \bar{U}. \tag{5.2}$$

The assumption that $v^{\pm} \in C^{0,\eta}(\bar{U})$ *is used only to compare* u *and* $v^{\pm}$ *in a neighborhood of the boundary. As such it can be replaced by other conditions like, for example, requiring that* $v^{+} \leqq u + \underline{c}\delta^{\underline{\alpha}}$ *(resp.* $u \leqq v^{-} + \underline{c}\delta^{\underline{\alpha}}$*) in an appropriate neighborhood of* ∂U.

The proof of Theorem 5.1 is long. It involves the choice of appropriate approximations and perturbations to u and $v^{\pm}$. Such arguments were already presented by the authors in [4] where they studied error estimates for monotone numerical approximations to solutions of uniformly elliptic pde. Again we divide the argument into several steps.

Step 1: Strong differentiability properties of solutions

We begin recalling two important elements of the regularity theory for solutions of linear, uniformly elliptic pde of the form

$$\operatorname{tr} A D^2 w = 0 \quad \text{in } B_1, \tag{5.3}$$

with A bounded measurable and uniformly elliptic. We refer to [1] for the proofs.

The first result provides some L^p-integrability for $D^2 w$. We have:

Proposition 5.1 *Let* $w \in C(B_1)$ *be a bounded solution of* (5.3). *There exists a universal constant* $p \in (0, 1)$ *such that* $D^2 w \in L^p(B_1)$.

The solution also satisfies the following maximal-type estimate.

Proposition 5.2 *Let* $w \in C(B_1)$ *be a bounded solution of* (5.3). *For* $\lambda > 0$, *let* A_λ *be the subset of* $B_{1/2}$ *at which* w *admits global (in* B_1*) tangent paraboloids from above and below with opening* λ, *i.e.*,

$$A_\lambda = \{x \in B_{1/2} : w(x) + \ell(y - x) - \lambda|y - x|^2 \leqq w(y) \leqq w(x) + \ell(y - x) + \lambda|y - x|^2 \text{ in } B_1\},$$

where ℓ *denotes a linear function. There exists* $c, \sigma > 0$ *depending on the ellipticity constants and the dimension such that, if* $D_\lambda = B_{1/2} \setminus A_\lambda$, *then* $|D_\lambda| \leqq c\lambda^{-\sigma}$.

A straightforward rescaling argument yields

Corollary 5.1 *Let* $w \in C(B_1)$ *be a bounded solution of* (5.3) *in* B_1. *If* A_λ *is the subset of* $B_{1-\delta^\theta}$ *where* w *is touched from above and below in* B_1 *by* *quadratics of opening* λ *and* $D_\lambda = B_{1-\delta^\theta} \setminus A_\lambda$, *then, for some universal constants* σ *and* $c > 0$,

$$|D_\lambda| \leqq c(\lambda\delta^{2\theta})^{-\sigma}.$$

We look again at the solution u of (5.1). The next theorem, which was proved in [4], shows that Lipschitz continuous solutions of (5.1) have in fact a controlled quadratic behavior in large subsets of B_1.

Theorem 5.2 *Let* $u \in C^{0,1}(\bar{B}_1)$ *be a solution of* (5.1) *in* B_1. *For all* $\lambda > 0$, *there exists a subset* A_λ *of* $B_{1-\delta^\theta}$ *such that*:

(i) *For each* $x_0 \in A_\lambda$, *there exists a quadratic polynomial* P_{x_0} *of opening* λ *such that* $F(D^2 P_{x_0}) = 0$, *and, for all* $x \in B_1$ *and a universal constant* $C > 0$,

$$|u(x) - [u(x_0) + P_{x_0}(x - x_0)]| \leqq C\lambda|x - x_0|^3.$$

(ii) *There exists* $\sigma > 0$ *depending only on the ellipticity constants and the dimension such that, if* $D_\lambda = B_{1-\delta^\theta} \setminus A_\lambda$, *then, for a constant* $c > 0$ *depending on* $\|u\|_{C^{0,1}}$ *and* F,

$$|D_\lambda| \leqq c(\lambda\delta^{2\theta})^{-\sigma}.$$

An immediate consequence is that, for any $x_0 \in A_\lambda$, u behaves like a "test polynomial" up to a controlled error of order $\lambda\delta^2$ in $B_\delta(x_0)$.

Step 2: One sided smoothing by sup- and inf-convolutions

Next we recall the sup- and inf-convolution regularization of $u \in C^{0,\eta}(\bar{U})$, for some $\eta \in (0, 1]$, given respectively, for $\theta > 0$, by

$$\begin{aligned} u_\theta^+(x) &= \sup_{\bar{U}}\left[u(y) - \frac{1}{2\theta}|x - y|^2\right] \quad \text{and} \\ u_\theta^-(x) &= \inf_{\bar{U}}\left[u(y) + \frac{1}{2\theta}|x - y|^2\right]. \end{aligned} \tag{5.4}$$

Sup- and inf-convolutions as well as a similar type regularizations using "parallel" surfaces are important tools in the theory of viscosity solutions. The main reason is that, in addition to regularizing a given function, they also preserve the notions of sub- and super-solution.

Let

$$U^\theta = \{x \in U : d(x, \partial U) \geqq (2[u]_{0,\eta}\theta)^{\frac{1}{2-\eta}}\},$$

where $[u]_{0,\eta}$ denotes the η-Hölder seminorm, and, for $f \in C^{0,1}(\bar{U})$ and $x \in \bar{U}^\theta$, define

$$f_\theta^-(x) = \sup_{|y-x| \leqq (2[u]_{0,\eta}\theta)^{\frac{1}{2-\eta}}} f(y) \quad \text{and} \quad f_\theta^+(x) = \inf_{|y-x| \leqq (2[u]_{0,\eta}\theta)^{\frac{1}{2-\eta}}} f(y).$$

The following technical proposition summarizing the essential properties of $u_\theta^\pm$ is classical in the theory of viscosity solutions. We refer to Jensen, Lions and Souganidis [15], Crandall, Ishii and Lions [7], Cabre and Caffarelli [1], etc., for its proof.

Proposition 5.3 *Assume $u \in C^{0,\eta}(\bar{U})$. Then:*

(i) $u_\theta^\pm \in C^{0,\eta}(\bar{U})$, $u_\theta^+ \searrow u$, $u_\theta^- \nearrow u$, *as* $\theta \to 0$, *and* $0 \leqq u_\theta^+ - u \leqq [u_{0,\eta}]^{\frac{2}{2-\eta}}(2\theta)^{\frac{\eta}{2-\eta}}$ *and* $0 \leqq u - u_\theta^- \leqq [u_{0,\eta}]^{\frac{2}{2-\eta}}(2\theta)^{\frac{\eta}{2-\eta}}$.

(ii) *For all $x \in U^\theta$, there exist concave (resp. convex) paraboloids of opening θ^{-1} that touch u_θ^+ (resp. u_θ^-) from below (resp. above), and, in the sense of distributions, $D^2u_\theta^+ \geqq -\theta^{-1}$ and $D^2u_\theta^- \leqq \theta^{-1}$. Moreover, $u_\theta^\pm$ is twice differentiable a.e. in U^θ, $u_\theta^\pm \in C^{0,1}(\bar{U}^\theta)$ and $\|Du_\theta^\pm\| \leqq [\|u\|(2\theta)^{-1}]^{1/2}$.*

(iii) *If u is a viscosity solution of $F(D^2u) = f$ in U, then u_θ^+ (resp. u_θ^-) is a subsolution (resp. supersolution) of*

$$F(D^2w) = f_\theta^+ \quad (\textit{resp. } F(D^2w) = f_\theta^-) \quad \textit{in } U^\theta.$$

Proposition 5.3 is very general and the last property holds for all degenerate second-order equations. Claims (i), (iii), and the first and last parts of claim (ii) are a direct consequence of the definition, while the a.e. twice differentiability requires some real analysis.

It turns out, however, that, when the equation is uniformly elliptic, the inf- and sup-convolutions enjoy more regularity. In particular, the regularity claimed in Theorem 5.2 carries over to $u_\theta^\pm$.

To state the result, given $x \in U^\theta$ we denote by $y_\theta^\pm(x)$ one of the points in U where the sup (inf) in (5.4) is achieved.

We have:

Proposition 5.4 *Assume that F is uniformly elliptic and let $u \in C^{0,1}(\bar{U})$ be a solution of $F(D^2u) = f$ in $B(\hat{x}_0, 2r) \subset U$. Then:*

(i) *Let P be a paraboloid touching u_θ^+ (resp. u_θ^-) form above (resp. below) at $x_0 \in B_{2r}^\theta(\hat{x}) = B(\hat{x}, 2r - \theta\|Du\|)$. Then u is touched at $y_\theta^+(x_0)$ (resp. $y_\theta^-(x_0)$) from above (resp. below) by a paraboloid $P_{x_0}^{+,\theta}$ (resp. $P_{x_0}^{-,\theta}$) and, for a uniform constant $C > 0$, in the viscosity sense,*

$$D^2u_\theta^+(x) \geqq D^2u(y_\theta^+(x)) + C\theta^2|D^2u(y_\theta^+(x))|^2$$
$$(\textit{resp. } D^2u_\theta^-(x) \leqq D^2u(y_\theta^-(x)) + C\theta^2|D^2u(y_\theta^-(x))|^2).$$

(ii) *There exists a uniform* $C > 0$ *such that, for all* $x_1, x_2 \in B^{\theta}_{2r}(\hat{x})$,

$$|x_1 - x_2| \leqq C|y^{\pm}_{\theta}(x_1) - y^{\pm}_{\theta}(x_2)|.$$

In particular, if, for some $A \subset U$, $A^{\pm}_{\theta} = \{x : y^{\pm}_{\theta}(x) \in A\}$, *then* $|A^{\pm}_{\theta}| \leqq C|A|$.

(iii) *There exist universal positive constants* t_0 *and* σ *such that, for every* $t > t_0$, *there exist open sets* $A^{\theta,\pm}_t \subset B^{\theta}_{2r}(\hat{x})$ *such that, for every* $x_0 \in A^{\theta,\pm}_t \cap B^{\theta/2}_r(\hat{x})$, *there exist paraboloids* $P^{t,\theta,\pm}_{x_0}$ *of opening* t *such that, for a uniform* $C > 0$ *and for* $x \in B^{\theta}_{2r}(\hat{x})$,

$$u^{+}_{\theta}(x) \geqq u^{+}_{\theta}(x_0) + P^{t,\theta,+}_{x_0}(x - x_0) + Cr^{-1}t|x - x_0|^3,$$
$$u^{-}_{\theta}(x) \leqq u^{-}_{\theta}(x_0) + P^{t,\theta,-}_{x_0}(x - x_0) - Cr^{-1}t|x - x_0|^3,$$

and

$$|(B^{\theta}_{2r}(\hat{x}) \setminus A^{\theta,\pm}_t) \cap B^{\theta/2}_r(\hat{x})| \leqq Cr^{N-1}(\|Du\|_{\infty} + \|Df\|_{L^N(B(\hat{x},2r))})t^{-\sigma}.$$

Proposition 5.4 is Proposition 2.1 in [4] where we refer to for the details. The key step is the observation that the (Krylov-Safonov) Harnack inequality yields that, at any point where u is touched from above (resp. below) by a paraboloid of opening θ^{-1} (resp. $-\theta^{-1}$), it is also touched from below (resp. above) by a paraboloid of opening $C\theta^{-1}$ (resp. $-C\theta^{-1}$) with $C > 0$ depending only on the ellipticity constants and the dimension. This implies that at the points $y^{\pm}_{\theta}$, u has C^1-contact from above and below with respectively the convex and concave envelops of paraboloids with opening $C\theta^{-1}$.

Step 3: Sup- and inf-convolutions of δ-solutions

Next we investigate how δ-sub- and supersolutions behave with respect to the sup- and inf-convolutions. To this end, we assume that we are given a δ-subsolution (resp. δ-supersolution) $v \in C^{0,\eta}(\bar{U})$ and we define $U^{\theta,\delta}$ by

$$U^{\theta,\delta} = \{x \in U : \mathrm{dist}(x, \partial U) \geqq (2[v]_{0,\eta}\theta)^{\frac{1}{2-\eta}} + \delta\}.$$

We have:

Proposition 5.5 *Let* $v \in C^{0,\eta}(\bar{U})$ *be a* δ*-subsolution* (*resp.* δ*-supersolution*) *of* (5.1). *The* θ*-sup-convolution* (*resp.* θ*-inf-convolution*) v^{+}_{θ} (*resp.* v^{-}_{θ}) *of* v *is a* δ*-subsolution* (*resp.* δ*-subsolution*) *of* (5.1) *in* $U^{\theta,\delta}$.

Proof We only present the argument about the δ-subsolution; the claim for δ-supersolutions is proved similarly.

Let $x_0 \in U^{\delta,\theta}$. It is an elementary calculation to check that

$$|x_0 - y_\theta^+(x_0)| \leqq (2[v]_{0,\eta}\theta)^{\frac{1}{2-\eta}}.$$

Assume that a quadratic P is such that

$$|P| \leqq C\delta^{-\beta}, \quad v_\theta^+ \leqq P \quad \text{in } B(x_0,\delta) \quad \text{and} \quad v_\theta^+(x_0) = P(x_0).$$

Then, for all $x \in B(x_0,\delta)$ and $y \in U$,

$$v(y) - \frac{|x-y|^2}{2\theta} \leqq P(x) \tag{5.5}$$

and

$$v(y_\theta^+(x_0)) - \frac{|x_0 - y_\theta^+(x_0)|^2}{2\theta} = P(x_0).$$

Let $x = y + x_0 - y_\theta^+(x_0)$ in (5.5). Then

$$v(\cdot) \leqq P(\cdot + x_0 - y_\theta^+(x_0)) + \frac{|x_0 - y_\theta^+(x_0)|^2}{2\theta} \quad \text{in } B_\delta(y_0) \subset U$$

and, since v is a δ-subsolution of (5.1), we must have

$$F(D^2P) \leq 0. \qquad \square$$

Step 4: The main comparison theorem

We proceed now with the proof of Theorem 5.1. Since it involves several steps and approximations, we present first the general plan.

We begin by moving away from the boundary and changing the right hand side of (5.1) by δ^β to have some room in our calculations to absorb small perturbations. Then we regularize u and the given δ-sub- and supersolution by sup- and inf-convolution. As already discussed these approximations are semi-convex or concave in the right direction, provide appropriate bounds for the Hessian, and have second-order expansions (with controlled error) outside small sets with measure estimated by the size of the quadratics in the expansion. The approximations are clearly δ-sub- and super-solutions around points of second-differentiability. To control what happens on the small exceptional sets, we use the Alexandrov-Bakelman-Pucci (ABP)-method by construction the convex envelope $\Gamma(w)$ of the difference w of u and the approximations. The control on the sizes of the Hessians and the exceptional sets force the contact set $\{\Gamma(w) = w\}$, where the support of $\det D^2\Gamma(w)$ is concentrated to be small. The estimate on the Hessian of the approximations then yields that, even in this exceptional case, the quantity $\det D^2\Gamma(w)|\{\Gamma(w)=w\}|$ falls within the δ^α margin of error.

Proof of Theorem 5.1 Here we only prove that, for some appropriate $\alpha \in (0,1)$,

$$\sup_U (u - v^-) \leqq C\delta^\alpha,$$

since the estimate for $v^+ - u$ follows in a similar way.

We begin by introducing the several layers of the approximations we discussed earlier for u, v^- and U.

The first step is to create some "room" in the equation for u by considering, for some $\beta \in (0,1)$ to be chosen later, the boundary value problem

$$\begin{cases} F(D^2u^\delta) = \delta^\beta & \text{in } U, \\ u^\delta = g & \text{on } \partial U. \end{cases} \tag{5.6}$$

The comparison and regularity properties of viscosity solutions (ABP-estimate and Lipschitz continuity) as well the assumptions on F and u yield, for some uniform $C > 0$, the estimates

$$\begin{aligned} &0 \leqq u - u^\delta \leqq \delta^\beta \quad \text{and} \quad |Du^\delta| \leqq C \quad \text{in } U \quad \text{and} \\ &u^\delta \leqq v^- + \underline{c}\delta^{\underline{\alpha}} \quad \text{on } \partial U. \end{aligned} \tag{5.7}$$

In the sequel we estimate the difference $u^\delta - v^-$. To this end, we consider the sup-convolution $u_\theta^{\delta,+}$ and the inf-convolution $v_\theta^{-,-}$ regularizations of u^δ and v^- respectively. At the expense of perhaps some greater generality, here we set

$$\theta = \delta^{2\zeta} \quad \text{with } \zeta \in (0, 1/2) \quad \text{and} \quad \gamma = \min\left(\underline{\alpha}, \frac{2\zeta\eta}{2-\eta}\right), \tag{5.8}$$

and we write

$$\begin{aligned} &u^{\delta,+} = u^{\delta,+}_{\delta^{2\zeta}}, \quad v_\delta^- = v^{-,-}_{\delta^{2\zeta}} \quad \text{and} \\ &\tilde{U}_\delta = U^{\delta^{2\zeta},\delta^\gamma} = \{x \in U : \operatorname{dist}(x, \partial U) \geqq (2[v^-]_{0,\eta}\delta^{2\zeta})^{\frac{1}{2-\eta}} + \delta^\gamma\}. \end{aligned}$$

The Lipschitz continuity of u, the Hölder continuity of v^-, the fact that $u^\delta \leqq v^- + \underline{c}\delta^{\underline{\alpha}}$ on ∂U, and the choice of $\tilde{U}_\delta$ yield,

$$\sup_U (u^\delta - v^-) = \max\left(\sup_{U\setminus\tilde{U}_\delta} (u^\delta - v^-), \sup_{\tilde{U}_\delta} (u^\delta - v^-)\right)$$

$$\leqq \max\Big(C'\delta^\gamma, \sup_{\tilde{U}_\delta}(u^\delta - v^-)\Big), \tag{5.9}$$

where C' is a constant depending on $\|Du\|$ and $[v]_{0,\eta}$.

Moreover,

$$\sup_{\tilde{U}_\delta}(u^\delta - v^-) \leqq \sup_{\tilde{U}_\delta}(u^{\delta,+} - v_\delta^-). \tag{5.10}$$

The Lipschitz continuity of $u^{\delta,+}$ and u^δ, the Hölder continuity of v^- and v_δ^-, the assumption about the upper bound of $u - v^-$ on ∂U, and the choice of $\tilde{U}_\delta$, also yield, for some other positive constant C, that

$$u^{\delta,+} - v_\delta^- \leqq C\delta^\gamma \quad \text{on } \partial\tilde{U}_\delta. \tag{5.11}$$

Assume next that

$$\beta < \gamma. \tag{5.12}$$

We summarize all the previous estimates in the inequality

$$\sup_U(u - v^-) \leqq C\delta^\gamma + \sup_{\tilde{U}_\delta}(u^{\delta,+} - v_\delta^-), \tag{5.13}$$

where C is a positive constant depending only on $\|Du\|$ and $[v]_{0,\eta}$.

We concentrate next on the sup in the right hand side of (5.13). To this end, recall that v_δ^- is a δ-supersolution of (5.1) while $u^{\delta,+}$ is a subsolution of (5.6).

Let

$$w = u^{\delta,+} - v_\delta^- - C\delta^\gamma$$

and consider the concave envelope Γ_w of w in B_{2R} for some $R > 0$ such that $U \subset B_R$.

On the contact set $\{w = \Gamma_w\}$ we always have

$$D^2\Gamma_w \leqq 0.$$

Since $u^{\delta,+}$ is semiconvex and v_δ^- is semiconcave (recall that $D^2u^{\delta,+} \geqq -\delta^{-2\zeta} I$ and $D^2 v_\delta^- \leqq \delta^{-2\zeta} I$), it follows that, on the contact set,

$$-2\delta^{-2\zeta} I \leqq D^2\Gamma_w \leqq 0. \tag{5.14}$$

The ABP-estimate yields, for some uniform constant C depending on the ellipticity, the dimension and the domain U, the estimate

$$\sup_{\tilde{U}_\delta} w \leqq C\left[\int_{\{\Gamma_w = w\}} \det D^2\Gamma_w\right]^{1/N} \leqq C\,\delta^{-2\zeta}\,|\{\Gamma_w = w\}|^{1/N}. \tag{5.15}$$

Assume next that, for some $c > 0$ and $\tilde{\alpha} \in (0, 1)$,

$$c\delta^{\tilde{\alpha}} \leqq \sup_{\tilde{U}_\delta} w. \tag{5.16}$$

It follows from the last three inequalities that, for yet another $C > 0$,

$$\delta^{(\tilde{\alpha}+2\zeta)N} \leqq C|\{\Gamma_w = w\}|. \tag{5.17}$$

Since $\tilde{U}_\delta$ is compact, there exist $M \approx \delta^{-N\gamma}$ balls $(B(x_i, \frac{1}{2}\delta^\gamma))_{1\leqq i\leqq M}$ covering $\tilde{U}_\delta$ and, in view of the definition of $\tilde{U}_\delta$, $B(x_i, \delta^\gamma) \subset U^{\delta^{2\zeta}}$ for all $i = 1, \dots, M$.

It then follows from (5.17) that there must exist $i \in \{1, \dots, M\}$ such that, for some other uniform $c > 0$,

$$\left| B\left(x_i, \frac{1}{2}\delta^\gamma\right) \cap \{\Gamma_w = w\}\right| \geqq CM^{-1}\delta^{(\tilde{\alpha}+2\zeta)N} \geqq C\delta^{(\tilde{\alpha}+2\zeta+\gamma)N}. \tag{5.18}$$

Next we apply Proposition 5.5 to $B(x_i, \delta^\gamma)$ with t such that

$$(\{\Gamma_w = w\} \cap B(x_i, \delta^\gamma)) \cap A_t \neq \emptyset.$$

It suffices to choose, for an appropriate $C > 0$,

$$t = C\delta^{-\frac{1}{\sigma}(\tilde{\alpha}+2\zeta+\gamma)N}.$$

Hence the contact set $\{\Gamma_w = w\}$ contains points where $u^{\delta,+}$ has a second order expansion P of opening t with error of order $t\delta^{-\gamma}$.

At any such point, if P denotes the tangent quadratic, we have

$$F(D^2P) \geqq \delta^\beta, \tag{5.19}$$

while v_δ^- satisfies, for δ sufficiently small,

$$v_\delta^- \geqq \tilde{P} \quad \text{in } B(x_i, \delta) \subset B\left(x_i, \frac{1}{2}\delta^\gamma\right) \subset \tilde{U}_\delta \quad \text{and} \quad v_\delta^-(x_i) = \tilde{P}(x_i),$$

where, for some $C > 0$,

$$\tilde{P}(x) = P(x) - \frac{Ct\delta}{2}|x|^2 \quad \text{and} \quad |D^2\tilde{P}| \leqq Ct.$$

Since v_δ^- is a δ-supersolution, we have

$$F(D^2P - Ct\delta) \leqq 0,$$

and, hence,

$$\delta^{\beta} \leqq F(D^2P) - F(D^2P - Ct\delta) \leqq C\Lambda t\delta = C\Lambda\delta^{1-\frac{1}{\sigma}(\tilde{\alpha}+2\zeta+\gamma)N}. \quad (5.20)$$

If $\tilde{\alpha}, \beta$ and γ are chosen sufficiently small so that (recall that ζ can be small in (5.8))

$$\sigma(1-\beta) > (\tilde{\alpha} + 2\zeta + \gamma)N$$

then (5.20) and, hence, (5.16) cannot hold.

Therefore, for some universal c,

$$\sup_{\tilde{U}_\delta} w \leqq c\delta^{\tilde{\alpha}}.$$

The definition of w then yields, for some universal positive constant C which also depends on the Lipschitz and Hölder constants of u and u^- respectively,

$$\sup_{\tilde{U}_\delta}(u^{\delta,+} - v_\delta^-) \leqq C(\delta^{\tilde{\alpha}} + \delta^{\gamma}). \quad (5.21)$$

The claim now follows combining (5.13) and (5.21), for $\alpha = \min(\tilde{\alpha}, \gamma)$. □

6 The general rate

If the solution $\bar{u}$ of (1.2) were known to have $C^{2,\alpha}$ estimates, something which is, for instance, true when $\bar{F}$ is convex or concave, then proving the rate of convergence would be simple. Indeed, around any point $x_0 \in U$, $\bar{u}$ would behave as a quadratic polynomial $\bar{P}$ with a uniform error, i.e.,

$$\bar{u}(x) = \bar{u}(x_0) + \bar{P}(x - x_0) + O(|x - x_0|^{2+\alpha}),$$

in which case it would be possible to use the rate we already proved for quadratic data to conclude. For general $\bar{F}$'s, however, it is known that solutions may not even be $C^{1,1}$. Then it becomes necessary to come up with special $C^{2,\alpha}$ approximations from above and below. This is where δ-solutions come in the picture.

In this section we show that solutions of (1.1) are for an appropriate choice of δ, actually δ-solutions of suitable approximations of (1.2).

We have:

Theorem 6.1 *There exist uniform positive constants $\bar{\bar{C}}, \hat{\hat{C}}, \bar{\bar{c}}$ and $\hat{\hat{c}}$ and $\alpha \in (0,1)$ such that any solution u_ε of (1.1) is a δ-subsolution of $\bar{F}(D^2w) = -\delta^{\alpha}$ and a δ-supersolution of $\bar{F}(D^2w) = \delta^{\alpha}$, for $\delta = \hat{\hat{C}}\varepsilon^{\hat{\hat{c}}|\ln\varepsilon|^{-1/2}}$, off a subset A_ε of Ω such that $\mu(A_\varepsilon) \leqq \bar{\bar{C}}\varepsilon^{\bar{\bar{c}}|\ln\varepsilon|^{-1/2}}$.*

Before we present the proof of Theorem 6.1, we give the proof of Theorem 1.1 which is, actually, immediate.

Proof of Theorem 1.1 Theorems 5.1 and 6.1 yield that, for appropriately chosen $\delta = \delta(\varepsilon)$, off an exceptional set, u_ε is within distance δ^α from the viscosity solutions of $\bar{F}(D^2\bar{u}_\delta^\pm) = \pm\delta^\alpha$, which, in turn, are δ^α away from the solution of (1.2).

Of course Theorem 5.1 requires certain regularity (uniform in ε Hölder continuity) and boundary behavior for the u_ε's. Both are consequences of the regularity theory of uniformly elliptic pde (Harnack inequality) and the existence of appropriate barriers. We leave it up to the interested reader to fill in the technical details. □

We continue with the

Proof of Theorem 6.1 We only show the proof for the δ-subsolution. The argument for the δ-supersolution follows along the same lines.

To this end, fix $x_0 \in U$ such that $B(x_0, \delta) \subset U$ and a quadratic P such that

$$|D^2 P| \leqq C\delta^{-\sigma}, \tag{6.1}$$

where C and σ are universal constants, and

$$u_\varepsilon \leqq P \quad \text{in } B(x_0, \delta) \quad \text{and} \quad u_\varepsilon(x_0) = P(x_0). \tag{6.2}$$

We argue by contradiction assuming that

$$\bar{F}(D^2 P) < -\delta^\alpha. \tag{6.3}$$

For $\eta > 0$, consider the quadratic

$$P_\delta(x) = P(x) - \eta\delta^\alpha(\delta^2 - |x - x_0|^2).$$

It is immediate from the ellipticity of F that, for an appropriate universal positive η,

$$\bar{F}(D^2 P_\delta) < 0.$$

Moreover,

$$P_\delta(x_0) - u_\varepsilon(x_0) = P(x_0) - u_\varepsilon(x_0) - \eta\delta^{2+\alpha} = -\eta\delta^{2+\alpha}.$$

Consider next the solution $u_{\varepsilon,\delta}$ of

$$\begin{cases} F(D^2 u_{\varepsilon,\delta}, \frac{x}{\varepsilon}, \omega) = \bar{F}(D^2 P_\delta) & \text{in } B_\delta(x_0), \\ u_{\varepsilon,\delta} = P_\delta & \text{on } \partial B_\delta(x_0). \end{cases} \tag{6.4}$$

A rescaling argument, Proposition 4.3, and (6.1) yield that there exist positive constants $\bar{C}', \hat{C}, \bar{c}', \hat{c}$ and ε_0 such that, for all $\varepsilon \in (0, \delta\varepsilon_0)$, there exists a set $A_{\delta,\varepsilon} \subset \Omega$ of bad configurations such that

$$\begin{cases} \|u_{\varepsilon,\delta} - P_\delta\|_{C(\overline{B_\delta(x_0)})} \leqq \hat{C}\delta^2 \varepsilon\delta^{-1^{-\hat{c}|\ln(\varepsilon\delta^{-1})|^{-1/2}}} & \text{in } \Omega \setminus A_{\delta,\varepsilon} \text{ and} \\ \mu(A_{\delta,\varepsilon}) \leqq \bar{C}'(\delta^{\sigma N}(1+\delta^{-\sigma})^2)^N (\varepsilon\delta^{-1})^{\bar{c}'|\ln(\varepsilon\delta^{-1})|^{-1/2}}. \end{cases} \tag{6.5}$$

Indeed, without loss of generality we may take $x_0 = 0$, in which case

$$u_\varepsilon^\delta(x,\omega) = \delta^{-2} u_{\delta,\varepsilon}(\delta x, \omega)$$

solves

$$\begin{cases} F(D^2 u_{\delta,\varepsilon}, \frac{x}{\varepsilon\delta^{-1}}, \omega) = \bar{F}(P) & \text{in } B_1, \\ u_{\delta,\varepsilon} = P & \text{on } \partial B_1, \end{cases}$$

and then (6.1) follows from Proposition 4.3.

Since

$$u_{\varepsilon,\delta} = P_\delta = P \geqq u_\varepsilon \quad \text{on } \partial B_\delta(x_0),$$

the comparison result of viscosity solutions yield

$$u_\varepsilon \leq u_{\varepsilon,\delta} \quad \text{in } B_\delta(x_0).$$

In particular, we have

$$0 < u_{\varepsilon,\delta}(x_0) - u_\varepsilon(x_0) \leqq \hat{C}\delta^2(\varepsilon\delta^{-1})^{\hat{c}|\ln(\varepsilon\delta^{-1})|^{-1/2}} - \eta\delta^{\alpha+2}$$

and, therefore,

$$\eta\delta^\alpha \leqq \hat{C}(\varepsilon\delta^{-1})^{\hat{c}|\ln(\varepsilon\delta^{-1})|^{-1/2}}.$$

But the last inequality gives a contradiction, for δ as in the assumption after appropriate choices of $\hat{\hat{C}}, \hat{\hat{c}}$ and $\alpha > 0$.

Finally, for δ as in the assumption, we also have, for appropriately chosen $\hat{\hat{C}}, \bar{C}', \hat{\hat{c}}$ and $\bar{c}'$,

$$\bar{C}\delta^2[\delta^{-\sigma N}(1+\delta^{-\sigma})^2]^N(\varepsilon\delta^{-1})^{\bar{c}|\ln(\varepsilon\delta^{-1})|^{-1/2}} \leqq \bar{\bar{C}}\varepsilon^{\bar{\bar{c}}|\ln\varepsilon|^{-1/2}}. \qquad \square$$

7 Rates for periodic homogenization

We obtain here error estimates for the homogenization of the general boundary value problem

$$\begin{cases} F(D^2u_\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } U, \\ u_\varepsilon = g & \text{on } \partial U, \end{cases} \tag{7.1}$$

with $F \in C(S^N \times \mathbb{R}^N)$ satisfying (2.7), (2.8) and

$$F \text{ is periodic with respect to } y \text{ in the unit cube } Q_1. \tag{7.2}$$

The periodic homogenization is easier to study since the associated cell problem has bounded (periodic solutions), i.e., it is possible to show that, if F satisfies (2.7), (2.8) and (7.2), then

$$\begin{cases} \text{for each } P \in S^N \text{ there exists a unique constant } \bar{F}(P) \text{ such that the equation} \\ F(P + D^2v, y) = \bar{F}(P) \quad \text{in } \mathbb{R}^N \\ \text{has a periodic (bounded) viscosity solution } v. \end{cases} \tag{7.3}$$

Let $\bar{u}$ be the solution of (1.2) with the $\bar{F}$ given by (7.3). It is proved in [10] and [2] that, as $\varepsilon \to 0$, $u_\varepsilon \to \bar{u}$ in $C(\bar{U})$.

Before we present the proof of Theorem 1.2, we explain how the cell problem is solved and record some key properties. To find $\bar{F}(P)$ and a solution v of (7.3), we solve the auxiliary problem

$$-\lambda v_\lambda + F(D^2v_\lambda + P, y) = 0 \quad \text{in } \mathbb{R}^N. \tag{7.4}$$

and prove that, if

$$\hat{v}_\lambda = v_\lambda - v_\lambda(0), \tag{7.5}$$

then, as $\lambda \to 0$, $\lambda v_\lambda \to \bar{F}(p)$ and $\hat{v}_\lambda \to v$.

We have:

Lemma 7.1 *Assume that F satisfies* (2.7), (2.8) *and* (7.2) *and fix $P \in S^N$. Then, for each $\lambda > 0$,* (7.4) *admits a bounded, uniformly continuous and periodic solution v_λ. Moreover, there exist a uniform constant $c > 0$ such that, for all $\lambda \in (0, 1)$,*

$$\begin{aligned} &\|\lambda v_\lambda\| \leqq c(1 + |P|), \quad \|\lambda \hat{v}_\lambda\| \leqq c(1 + |P|)\lambda \quad \textit{and} \\ &|\lambda v_\lambda(0) - \bar{F}(P)| \leqq C(1 + |P|)\lambda \end{aligned} \tag{7.6}$$

and, along subsequences $\lambda \to 0$, $\hat{v}_\lambda \to v$ in $C(\mathbb{R}^N)$ where v is a solution of (7.3) *satisfying $|v| \leqq c(1 + |P|)$.*

We refer to [10], among other places, for the proof of Lemma 7.1 and continue with the preparation for the proof of Theorem 1.2.

As in the proof of Theorem 1.1, the crucial step is to prove it with quadratic data in balls, i.e., to consider the problem

$$\begin{cases} F(D^2 u_\varepsilon, \frac{x}{\varepsilon}) = \bar{F}(P) & \text{in } B_\delta, \\ u_\varepsilon = P & \text{on } \partial B_\delta, \end{cases} \tag{7.7}$$

and to establish a rate for $\|u_\varepsilon - P\|$.

We formulate and prove this fact in

Proposition 7.1 *Assume that F satisfies the assumptions of Theorem 1.2 and consider, for fixed $\delta > 0$ and $P \in S^N$, (7.7). There exists a uniform constant $c > 0$ that depends on F and the dimension but not δ, P and ε, such that*

$$\|u_\varepsilon - P\| \leqq c(1 + |P|)\varepsilon^2 \quad \textit{in } \bar{B}_\delta. \tag{7.8}$$

Proof Let v^ε be the solution of (7.4) with $\lambda = \varepsilon^2$ and define $\hat{v}^\varepsilon$ by (7.5). Recall that Lemma 7.1 yields a uniform constant $c > 0$ such that

$$\|\hat{v}^\varepsilon\| \leqq c(1 + |P|)\varepsilon^2 \quad \text{and} \quad \|\varepsilon^2 v^\varepsilon(0) - \bar{F}(P)\| \leqq c(1 + |P|)\varepsilon^2. \tag{7.9}$$

It follows that the function

$$V^\varepsilon(x) = P(x) + \varepsilon^2 v^\varepsilon\left(\frac{x}{\varepsilon}\right) \tag{7.10}$$

solves

$$F\left(D^2 V^\varepsilon, \frac{x}{\varepsilon}\right) = \varepsilon^2 v^\varepsilon(0) \quad \text{in } \mathbb{R}^N.$$

and, hence, it is a supersolution of

$$F\left(D^2 V^\varepsilon, \frac{x}{\varepsilon}\right) = \bar{F}(P) + c(1 + |P|)\varepsilon^2 \quad \text{in } B_\delta.$$

The comparison principle for viscosity solutions yields the estimate

$$\max_{\bar{B}_\delta}(u_\varepsilon - V^\varepsilon) \leqq c(1 + |P|)\varepsilon^2 + \max_{\partial B_\delta}(u_\varepsilon - V^\varepsilon),$$

and, hence,

$$\max_{\bar{B}_\delta}(u_\varepsilon - P) \leqq c(1 + |P|)\varepsilon^2 + \max_{\bar{B}_\delta}\left|\varepsilon^2 v^\varepsilon\left(\frac{x}{\varepsilon}\right)\right| \leqq 2c(1 + |P|)\varepsilon^2.$$

The other inequality follows similarly. □

With Proposition 7.1 on hand we proceed now with the

Proof of Theorem 1.2 We follow the proof of Theorem 1.1 without, however, having to worry about ω's. As before the key step is to show that, for an appropriate choice of $\delta = \delta(\varepsilon)$, the solution u_ε of (7.1) is a δ-subsolution (resp. δ-supersolution) of

$$\bar{F}(D^2w) = -\delta^\alpha \quad (\text{resp. } \bar{F}(D^2w) = \delta^\alpha).$$

We concentrate on the first claim and observe that, as the proof of Theorem 0.1, the conclusion follows as long as we establish the analogue of Theorem 5.1. For this it suffices to find $\delta = \delta(\varepsilon)$ and $\alpha \in (0, 1)$ such that, as $\varepsilon \to 0$,

$$\delta^{-\alpha}(\delta^{-\sigma N}((1+\delta^{-\sigma})^2)^N\varepsilon^2 \to 0,$$

which is, of course, clearly possible, if

$$\delta = \varepsilon^\beta \quad \text{with } \beta \in (0, (2/\alpha + \sigma N(N+2))). \qquad \square$$

8 Rates of the homogenization in almost periodic media

We obtain here a rate for the homogenization of the boundary value problem (7.1) with $F \in C(S^N \times \mathbb{R}^N)$ satisfying (2.7), (2.8) and

$$F \text{ is almost periodic with respect to } y. \tag{8.1}$$

There are several definitions for almost periodicity. Here by (8.1) we mean that, for each $R > 0$,

$$\{F(\cdot, \cdot + \zeta) : \zeta \in \mathbb{R}^N\} \quad \text{is precompact, as } |\zeta| \to \infty, \text{ in } C(S^N \cap B_R, \mathbb{R}^N) \tag{8.2}$$

Almost periodic environments are not strongly mixing. Moreover, it is not known (see [22] for a related discussion) whether the cell problem (7.3) has a solution. Note that, in the context of almost periodic functions, to have a unique $\bar{F}(p)$ in the cell problem (7.3), the solutions v must be (see [5]) strictly subquadratic at infinity. Whether such solutions exist in general is an open problem.

The possible lack of subquadratically growing solutions to (7.3) can be circumvented in the almost periodic setting by showing that there exist approximate correctors (see, for example, [14, 22]), i.e.,

$$\begin{cases} \text{for each } P \in S^N \text{ there exists a unique constant } \bar{F}(P) \text{ such that,} \\ \text{for each } \eta > 0, \text{ there exists bounded viscosity solutions } v_\eta^\pm \text{ of} \\ F(D^2 v_\eta^+ + P, y) \leqq \bar{F}(P) + \eta \quad \text{and} \quad F(D^2 v_\eta^- + P, y) \geqq \bar{F}(P) - \eta. \end{cases} \tag{8.3}$$

It is then follows (see [22] and [14] for such arguments) that, as $\varepsilon \to 0$, $u_\varepsilon \to \bar{u}$ where $\bar{u}$ is the solution of (1.2) with $\bar{F}$ given by (8.3).

To obtain a rate of convergence it is necessary to look more carefully to the issue of the existence of approximate correctors and the analogue of Lemma 7.1.

We have:

Lemma 8.1 *Assume that F satisfies* (2.7), (2.8) *and* (8.1). *Fix $P \in S^N$ and consider v_λ and $\hat{v}_\lambda$ as in* (7.4) *and* (7.5). *There exist a uniform constant $c > 0$ and a modulus $\bar{\rho} : [0, \infty) \to [0, \infty)$ such that $\bar{\rho}(0^+) = 0$, that depend only on F and the dimension, such that, for all $\lambda \in (0, 1)$,*

$$\|\lambda v_\lambda\| \leqq c(1 + |P|), \tag{8.4}$$

and

$$\|\lambda \hat{v}_\lambda\| \leqq c(1 + |P|)\bar{\rho}(\lambda), \quad \textit{and} \quad \|\bar{F}(P) - \lambda v_\lambda(0)\| \leqq c(1 + |P|)\bar{\rho}(\lambda). \tag{8.5}$$

Proof The existence of a solution v_λ of (7.4), which is actually itself almost periodic, as well as (8.4) are standard facts in the theory of viscosity solutions.

The fact that, as $\lambda \to 0$, $\lambda \hat{v}_\lambda \to 0$ uniformly in $\mathbb{R}^N$ and, hence, $\hat{v}_\lambda$ is an approximate corrector, follows similarly to the analogous statements in [22] and [14], the only difference being that there are no, uniform in λ, Lipschitz bounds for the v_λ's. This difficulty can be overcome using the facts that, in view of (8.4), the functions $w_\lambda = \lambda v_\lambda$ are actually uniformly Hölder continuous (write the equation satisfied by w_λ and apply the Krylov-Safonov result about Hölder continuity of solutions to uniformly elliptic pde (see [1])) and, as $\lambda \to 0$, $\hat{w}_\lambda = \lambda \hat{v}_\lambda \to 0$ uniformly on compact subsets of $\mathbb{R}^N$. Indeed, in view of the Hölder continuity, (8.4) and the fact that $w_\lambda(0) = 0$, the $\hat{w}_\lambda$'s converge, along subsequences, to bounded solutions of a uniformly elliptic pde which admits only constants as bounded solutions in the whole space. We leave the rest of the details to the reader.

Next we discuss (8.5) and the existence of $\bar{\rho}$. To this end, we remark that, if v_λ^1 and v_λ^2 are the solutions of (8.2) for quadratics P_1 and P_2 respectively, then, since F satisfies (2.9), the comparison of viscosity solutions yields

$$\|\lambda v_\lambda^1 - \lambda v_\lambda^2\| \leqq \tilde{c}|P_1 - P_2|. \tag{8.6}$$

The compactness of viscosity solutions and (2.7) and (2.8) imply that the convergence, as $\lambda \to 0$, of $\lambda \hat{v}_\lambda$ and $\lambda v_\lambda(0)$ is uniform for $|P| \leqq 1$. This defines the modulus $\bar{\rho}$. Then (8.5) follows by a scaling argument similar to the one used already in Sect. 2. □

Following the reasoning of the previous section we state

Proposition 8.1 *Assume that F satisfies* (2.7), (2.8) *and* (8.1). *There exists a uniform constant $\delta > 0$ and a modulus $\bar{\rho}$ such that, if u_ε is the solution of* (7.7) *for $\delta > 0$ and $P \in S^N$, then*

$$\|u_\varepsilon - P\| \leqq c(1 + |P|)\bar{\rho}(\varepsilon^2) \quad \textit{in } \bar{B}_\delta(x_0). \tag{8.7}$$

Since (8.7) follows exactly as (7.8), provided we use Lemma 8.1 in place of Lemma 7.1, we omit the proof and proceed with the

Proof of Theorem 1.3 Looking at the proofs of Theorems 1.1 and 1.2 we see that it suffices to find $\delta = \delta(\varepsilon) \to 0$ such that, as $\varepsilon \to 0$,

$$\delta \to 0 \quad \text{and} \quad \delta^{-\alpha}(\delta^{-\sigma N}(1 + \delta^{-\sigma})^2)^N \bar{\rho}(\varepsilon^2) \to 0,$$

a fact which follows for $\delta = \bar{\rho}^\beta$ and $\beta \in (0, 2/\alpha + \sigma N(N+2))$.

The result now follows. □

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Appendix A: Brief review of some basic facts about viscosity solutions of fully nonlinear elliptic pde

We summarize here (without proofs) a few very basic results about viscosity solutions of uniformly elliptic equations

$$F(D^2 u, x) = 0 \quad \text{in } U \tag{A.1}$$

with F satisfying (2.7) and (2.8) and U a bounded subset of $\mathbb{R}^N$. For proofs as well as more information we refer to the book by Caffarelli and Cabre [1].

Let $M^\pm(P, \lambda, \Lambda)$ denote the Pucci extremal operators associated with the uniform ellipticity constants λ and Λ in (2.7). They are given by

$$M^-(P, \lambda, \Lambda) = M^-(P) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i$$

and

$$M^+(P,\lambda,\Lambda) = M^+(P) = \Lambda \sum_{i_0>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where $e_i = e_i(P)$ are the eigenvalues of $P \in S^N$.

Moreover, for $f \in C(U)$, let $\underline{S}(\lambda, \Lambda, f)$ and $\overline{S}(\lambda, \Lambda, f)$ denote the sets of continuous viscosity subsolutions and supersolutions of $M^+(D^2u, \lambda, \Lambda) \geqq f$ in U and $M^-(D^2u, \lambda, \Lambda) \leqq f$ in U respectively. Also let $S(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f)$.

It was shown (Proposition 2.13 in [1]) that, if u is a viscosity subsolution (resp. supersolution) of $F(D^2u, x) \geqq f$ in U (resp. $F(D^2u, x) \leqq f$ in U), then

$$u \in \underline{S}\left(\frac{\lambda}{N}, \Lambda, f - F(0,\cdot)\right) \quad \left(\text{resp. } \overline{S}\left(\frac{\lambda}{N}, \Lambda, f - F(0,\cdot)\right)\right).$$

It turns out (see the discussion after the proof of Proposition 2.13 in [1]) that $S(\lambda, \Lambda, f) \cap C^2(U)$ is the set of $C^2(U)$-functions u for which, for any $x \in U$, there exists a symmetric $A(x) \in S^N$, which may not be continuous in x, with eigenvalues in $[\lambda, \Lambda]$ such that $\operatorname{tr} A(x) D^2 u = f$. Therefore, roughly speaking, $S(\lambda, \Lambda, f)$ is the class of all weak solutions to all uniformly elliptic, with ellipticity constants λ and Λ, operators in the nondivergence form

$$\operatorname{tr} A D^2 u = f. \tag{A.2}$$

Moreover a viscosity solution of $F(D^2u, x) = f$ belongs to $S(\frac{\lambda}{N}, \lambda, f - F(0,\cdot))$ and, hence, any results for functions in the classes S is also valid for fully nonlinear uniformly elliptic equations.

We proceed now with the results we use in this paper. The first concerns viscosity solutions u_1 and u_2 of

$$F(D^2u_1, x) = f_1 \quad \text{in } U \quad \text{and} \quad F(D^2u_2, x) = f_2 \quad \text{in } U.$$

It is proved in Theorem 5.3 of [1] that

$$u_1 - u_2 \in S\left(\frac{\lambda}{N}, \Lambda, f_1 - f_2\right),$$

which, in view of the previous discussion, can be understood as saying that $w = u_1 - u_2$ is a weak solution of

$$\operatorname{tr} A(x) D^2 w = f_1 - f_2 \quad \text{on } U$$

for some bounded measurable $A(\cdot) \in S^N$ with ellipticity constants λ and Λ.

Next we recall the classical Alexandrov-Bakelman-Pucci estimate proved in Theorem 3.2 of [1]. Let B_R denote an open ball of radius R in $\mathbb{R}^N$. We have

Theorem A.1 *Let $u \in \overline{S}(\lambda, \Lambda, f) \cap C(\overline{B}_R)$ with $f \in C(\bar{B}_R)$, and assume that $u \geqq 0$ in ∂B_R. Then there exists a universal constant $C > 0$ such that*

$$\sup_{B_R} u_- \leqq CR\left(\int_{B_R \cap \{u=\Gamma_u\}} (f_+)^N\right)^{1/N}, \tag{A.3}$$

where Γ_u is the convex envelop in B_{2R} of u_- which is obtained by extending u to be zero outside B_R.

In view of the previous discussion, (A.3) also holds for all nonnegative continuous supersolutions of (A.1) in $U = B_R$.

The final result we want to recall here is a consequence of the so-called Fabes-Stroock estimate [12] (see also Corollary B.5 in [5] for another proof). It was proved originally for solutions of the linear nondivergence form equation (A.2) with bounded, measurable, uniformly elliptic $A \in S^N$. In view of the previous discussion, however, about linearization it clearly applies also to solutions of (A.1).

We have:

Theorem A.2 *Let $u \in C(B_R)$ solve (A.2) win B_R with $u = 0$ on ∂B_R and $f \in C(B_R)$ such that $f \in [-\|f\|, 0]$. There exists uniform constants on $C > 0$ and $M > N$ such that*

$$u \geqq CR^{2-M} \|f\|^{1-M} \left(\int_{B_R} |f|^N\right)^{\frac{M}{N}} \quad \text{in } B_{\frac{2}{3}R}. \tag{A.4}$$

We conclude remarking although stated for balls, (A.3) and (A.4) also hold for cubes.

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