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The Alexander-Orbach conjecture holds in high dimensions

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Abstract We examine the incipient infinite cluster (IIC) of critical percolation in regimes where mean-field behavior has been established, namely when the dimension d is large enough or when $d > 6$ and the lattice is sufficiently spread out. We find that random walk on the IIC exhibits anomalous diffusion with the spectral dimension $d_s = \frac{4}{3}$, that is, $p_t(x,x) = t^{-2/3+o(1)}$. This establishes a conjecture of Alexander and Orbach (J. Phys. Lett. (Paris) 43:625–631, 1982). En route we calculate the one-arm exponent with respect to the intrinsic distance.

1 Introduction

We study the behavior of the simple random walk on the incipient infinite cluster (IIC) of critical percolation on $\mathbb{Z}^d$. The IIC is a random infinite connected graph containing the origin which can be thought of as a critical cluster conditioned to be infinite (see formal definition in Sect. 1.2 and in particular (1.3)). The *spectral dimension* d_s of an infinite connected graph G is defined by

$$d_s = d_s(G) = -2 \lim_{n\to\infty} \frac{\log \mathbf{p}_{2n}(x,x)}{\log n} \quad \text{(if this limit exists)},$$

where $x \in G$ and $\mathbf{p}_n(x,x)$ is the return probability of the simple random walk on G after n steps (note that if the limit exists, then it is independent of the choice of x). Alexander and Orbach [4] conjectured that $d_s = 4/3$ for the IIC in all dimensions $d > 1$, but their basis for conjecturing this in low dimensions was mostly rough correspondence with numerical results and it is now believed that the conjecture is false when $d < 6$ [30, 7.4]. In this paper we establish their conjecture in high dimensions.

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Theorem 1.1 *Let $\mathbf{P}_{\mathrm{IIC}}$ be the IIC measure of critical percolation on $\mathbb{Z}^d$ with large d ($d \geq 19$ suffices) or with $d > 6$ and sufficiently spread-out lattice and consider the simple random walk on the IIC. Then $\mathbf{P}_{\mathrm{IIC}}$-a.s.*

$$\lim_{n\to\infty} \frac{\log \mathbf{p}_{2n}(0,0)}{\log n} = -\frac{2}{3}, \qquad \lim_{r\to\infty} \frac{\log \mathbb{E}\tau_r}{\log r} = 3, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{2}{3} \quad a.s.,$$

where τ_r is the hitting time of distance r from the origin (the expectation $\mathbb{E}$ is only over the randomness of the walk) and W_n is the range of the random walk after n steps.

Our main contribution is the analysis of the geometry of the IIC. The IIC admits *fractal* geometry which is dramatically different from the one of the infinite component of *supercritical* percolation. The latter behaves in many ways as $\mathbb{Z}^d$ after a "renormalization" i.e. ignoring the local structure [22] (see also [21] for a comprehensive exposition). In particular, the random walk on the supercritical infinite cluster has an invariance principle, the spectral dimension is $d_s = d$ and other $\mathbb{Z}^d$-like properties hold, see [5, 13, 14, 18, 38, 48].

Our analysis establishes that balls of radius r in the IIC typically have volume of order r^2 and that the *effective resistance* between the center of the ball and its boundary is of order r. These facts alone suffice to control the behavior of the random walk and yield Theorem 1.1, as shown by Barlow, Járai, Kumagai and Slade [11]. The key ingredient of our proofs is establishing that the critical exponents dealing with the *intrinsic* metric (i.e., the metric of the percolated graph) attain their mean-field values. It was demonstrated first in the work of Nachmias and Peres [40] that these exponents yield analogous statements to the Alexander-Orbach conjecture in the *finite* graph setting. In particular, in [40], the diameter and mixing time of critical clusters in mean-field percolation on finite graphs were analyzed.

In different settings the Alexander-Orbach conjecture was proved by various authors. When the underlying graph is an infinite regular tree, this was proved by Kesten [32] and Barlow and Kumagai [9] and in the setting of *oriented* spread-out percolation with $d > 6$, this was proved recently in the aforementioned paper [11].

1.1 Anomalous diffusion

The fact that $d_s = 4/3$ should best be contrasted against another natural definition of dimension, and that is the volume growth exponent d_f defined, for any infinite connected graph G by

$$d_f = \lim_{r\to\infty} \frac{\log |B_G(x,r)|}{\log r} \quad \text{(if the limit exists)},$$

where $B_G(x,r)$ is the ball, in the shortest-path metric with center x and radius r, and $|B_G(x,r)|$ is its volume, i.e. the number of vertices of the graph in it. The volume growth exponent is the graph analog of the Hausdorff dimension. For the IIC this limit exists and is equal to 2 for all x (Theorems 1.2 and 1.3 below). Hence we have two natural notions of dimension which give different answers. For comparison, for $\mathbb{Z}^d$ we have $d_f = d_s = d$. More generally, for any Cayley graph $d_f = d_s$ (indeed, Gromov's celebrated result [23] shows that d_f exists and is integer for any Cayley graph, and then Theorem 5.1 in [28] shows that $d_f = d_s$) and there are other rich families which satisfy this. To understand the discrepancy, we need to understand *anomalous diffusion*.

Anomalous diffusion is the phenomenon that for many natural *fractals*, or more precisely, graphical analogs of fractals, random walk on the fractal is significantly slower than in Euclidean space. In particular, while we expect a random walker on $\mathbb{Z}^d$ to be at distance

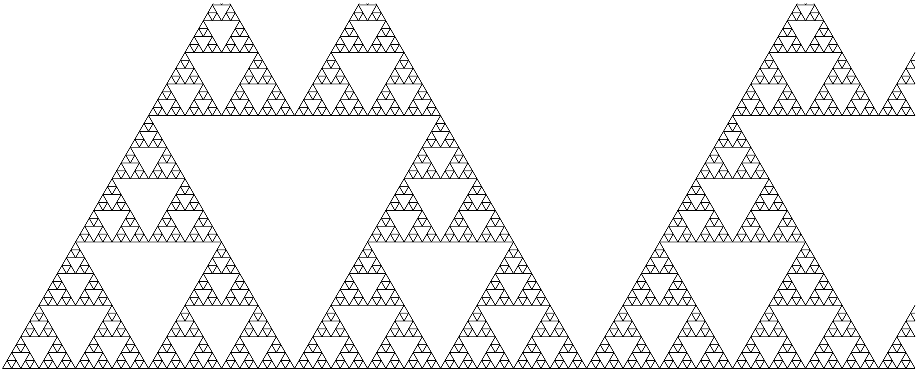

Fig. 1 A portion of the graphical Sierpinski gasket

$t^{1/2}$ at time t, on a fractal we find it in distance $t^{1/\beta}$ where $\beta \geq 2$ and often the inequality is strict.[1] In fact, we now know that any value of β between 2 and $d_f + 1$ may appear [6]. This phenomenon was first observed by physicists in the context of disordered media [4, 44] i.e. in our context. Correspondingly, the first mathematical results are Kesten's [32] who analyzed random walk on the IIC in $\mathbb{Z}^2$ and on an infinite regular tree. For the IIC on a tree Kesten's results are complete and he shows that $\beta = 3$. On the IIC in two dimensions, Kesten showed that the expected distance of the random walk from the origin after t steps is at most $t^{1/2-\epsilon}$ for some $\epsilon > 0$, hence $\beta > 2$ if it exists. Despite the great progress seen since on critical two-dimensional percolation, the exact value of β in this case is still unknown.

The attention of the mathematical community then shifted to regular fractals. The first to be analyzed were *finitely ramified* fractals, namely fractals that can be disconnected by the removal of a constant number of points in any scale. For example, arbitrarily large portions of the Sierpinski gasket (Fig. 1) can be disconnected by the removal of 3 points. In these cases β can be calculated explicitly, for example for the Sierpinski gasket $\beta = \log 5/\log 2$ [10]. A significant step forward was done by Barlow and Bass [7, 8] who showed that on any generalized Sierpinski carpet β is well defined, and the random walk exhibits many regularity properties analogous to those of random walk on $\mathbb{Z}^d$, mutatis mutandis. Unfortunately, these techniques do not allow to calculate β for many natural examples, and this remains a significant open problem.

For sufficiently "well-behaved" G we expect that $\beta = 2d_f/d_s$. Heuristically this is easy to understand since if the random walk reaches a distance of $\approx r = t^{1/\beta}$ it should see $\approx r^{d_f} = t^{d_f/\beta}$ points and assuming homogeneity we should have $\mathbf{p}_t(0,0) \approx t^{-d_f/\beta}$. This explains the connection between the various "exponents" in Theorem 1.1. Among the results of Barlow and Bass [7, 8] is the proof that indeed $\beta = 2d_f/d_s$ for any generalized Sierpinski carpet.

1.2 Percolation

Bond percolation on a graph G with parameter $p \in [0,1]$ is a probability measure $\mathbf{P}_p$ on random subgraph of the G obtained by retaining each edge independently with probability p and deleting it otherwise. Edges retained are called *open* and edges deleted are called *closed*. The graphs that interest us most are lattices in $\mathbb{R}^d$, in particular $\mathbb{Z}^d$, and regular trees.

[1] β is sometimes called the "walk dimension" and denoted by d_w.

It is well known that this model exhibits a phase transition, that is, there exists a critical probability $p_c(\mathbb{Z}^d) \in [0, 1]$ such that for all $p > p_c$ almost surely there exists an infinite connected cluster, and for any $p < p_c$ almost surely all clusters are finite. Percolation at p_c is called *critical* percolation. The subcritical and supercritical cases are understood quite well, and in neither case is it reasonable to call the resulting graphs "fractals". The subcritical case consists of finite clusters with exponential tail on their size [2, 39]. The supercritical case, as mentioned before, behaves in many ways as a perturbed version of $\mathbb{Z}^d$ and most interesting quantities behave the same as on $\mathbb{Z}^d$. For instance, $d_f = d$, $\beta = 2$, and $d_s = d$. The structure of the resulting graph in critical percolation, however, is dramatically different.

It is widely believed that critical percolation does not exhibit an infinite cluster almost surely. This has been established only for the case $d = 2$ by Kesten [31] and for sufficiently large d by Hara and Slade [25]. Proving it for all d is considered one of the most challenging problems in probability theory. Nevertheless, for all $d > 1$ it is known that in any scale there are clusters comparable to the scale [1, Theorem 1], and it is conjectured that they have fractal-like properties. Hence the natural question arises: what is the corresponding spectral dimension d_s? As already mentioned, there is a significant difference between low and high dimensions. Let us therefore spend a little effort on the difference between "low" and "high" dimensions in percolation.

Many models in mathematical physics exhibit an *upper critical dimension* and for percolation this happens at $d = 6$. The picture, as developed by physicists, is that for $d > 6$ the space is so vast that different pieces of the critical cluster no longer interact. The effect of this is that the geometry "trivializes" and for most questions the answer would be as for percolation on an infinite regular tree. This is also known as *mean-field* behavior. Aspects of this picture were confirmed rigorously but with one important caveat. The technique used, *lace expansion*, is perturbative and hence requires one of the following to hold:

- The dimension d should be large enough ($d \geq 19$ seems to be the limit of current techniques).
- The dimension should satisfy $d > 6$ but the lattice needs to be sufficiently spread out. For example, one may take some L sufficiently large and put an edge between every $x, y \in \mathbb{Z}^d$ with $|x - y| \leq L$.

Credit for these remarkable results goes to Hara and Slade [25]. For $d < 6$ it has been proved that percolation cannot attain mean-field behavior [17]. Specifically, Hara and Slade proved the that for these lattices the *triangle condition* holds. The triangle condition, suggested as an indicator of mean-field behavior by Aizenman and Newman [3] is

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}_{p_c}(0 \leftrightarrow x)\mathbf{P}_{p_c}(x \leftrightarrow y)\mathbf{P}_{p_c}(0 \leftrightarrow y) < \infty \tag{1.1}$$

where $x \leftrightarrow y$ denotes the event that x is connected to y by an open path (for simplicity we assume that the set of vertices of the lattice is always $\mathbb{Z}^d$ and denote the set of edges by $E(\mathbb{Z}^d)$). To see how to analyze the behavior of critical and near-critical percolation using the triangle condition, see [3, 12, 42]

A slightly different approach to mean-field behavior is via the *two-point function* i.e. the probability that x is connected to y by an open path. It has the estimate, for all $x, y \in \mathbb{Z}^d$,

$$\mathbf{P}_{p_c}(x \leftrightarrow y) \approx |x - y|^{2-d}, \tag{1.2}$$

where $\approx$ means that the ratio of the quantities on the left and on the right is bounded by two constants depending only on d and L. Here and below we abuse notation by considering that $0^{2-d} = 1$. A simple calculation shows that, when $d > 6$, (1.2) implies (1.1) hence the assumption on the two-point function is stronger. It was obtained using the lace expansion by Hara, van der Hofstad and Slade [27] for the spread-out model and $d > 6$, and by Hara [24] for the nearest-neighbor model with $d \geq 19$ (in fact, they obtained the right asymptotic behavior of (1.2), including the constant).

At present there is no known lattice in $\mathbb{R}^d$ for which the triangle condition is known and the two-point function is unknown (or false). Nevertheless, We believe that there is value in noting which results require the (formally) stronger two-point function estimate (1.2) and which require only the triangle condition. Reasons to keep this distinction come from the fields of long-range percolation [12, 29] and of percolation on general transitive graphs [46, 47]. In both cases the triangle condition makes more sense and was proved in many interesting examples. We will not dwell on these topics in this paper, but in general we believe that any result we prove only using the triangle condition should hold (perhaps with minor modifications) for long-range percolation and for percolation on unimodular transitive graphs.

Returning to the Alexander-Orbach conjecture, our aim is to study random walk on a typical large cluster. The term *incipient infinite cluster* was coined by Kesten, borrowing a vaguely-defined term from the physics literature. His approach in [33] for the two-dimensional case is to fix some integer n, to condition on the event $0 \leftrightarrow \partial[-n, n]^2$ and then take $n \to \infty$. In this paper we take the approach suggested by van der Hofstad and Járai [51], and that is to fix some arbitrary far point x, condition on the event $0 \leftrightarrow x$, and then take $x \to \infty$. For both approaches one still needs to show that the limit exists. This was done in [33] for the case $d = 2$, in [51] for large d as above and in [52] for the *oriented* percolation model with $d > 4$.

Formally, we endow the space of all configurations $\{0, 1\}^{E(\mathbb{Z}^d)}$ with the product topology (recall that $E(\mathbb{Z}^d)$ is the set of edges of our lattice). We consider the conditional measures given $0 \leftrightarrow x$ and finally, the IIC is the limit as $x \to \infty$ in the space of measures $\mathcal{M}(\{0, 1\}^{E(\mathbb{Z}^d)})$ with the weak topology. Put differently, for any cylinder event F (i.e., an event that can be determined by observing the status of a finite number of edges) we have

$$\mathbf{P}_{\mathrm{IIC}}(F) = \lim_{|x| \to \infty} \mathbf{P}_{p_c}(F \mid 0 \leftrightarrow x), \tag{1.3}$$

where $p_c = p_c(\mathbb{Z}^d)$ is the percolation critical probability. The convergence of the limit in the right-hand side, independently on how $x \to \infty$, is proved in [51] for d large using the lace expansion. We note in passing that the existence of the limit is not relevant for our arguments. Indeed, even if the limit would not exist, subsequence limits would exist due to compactness, and our results would hold for each one. Thus the conclusions of Theorem 1.1 hold for any lattice in $\mathbb{R}^d$ with $d > 6$ for which the two-point function estimate (1.2) holds, and for any IIC measure (i.e. any subsequence limit as above).

1.3 Intrinsic metric critical exponents

The key ingredient in our proofs is showing that the *intrinsic metric* critical exponents defined below assume their mean-field values in high dimensions.

Let G be a graph and write G_p for the result of p-bond percolation on it. Write $d_{G_p}(x, y)$ for the length of the shortest path between x and y in G_p, or ∞ if there is no such path. We call d the *intrinsic metric* on G_p—other names in the literature include the *graph metric*, the *shortest-path metric* and even the *chemical distance*. From this point on, we always perform critical percolation with $p = p_c = p_c(\mathbb{Z}^d)$. Define the random sets

$$B(x, r; G) = \{u : d_{G_{p_c}}(x, u) \le r\},$$
$$\partial B(x, r; G) = \{u : d_{G_{p_c}}(x, u) = r\}.$$

It will be occasionally important to take some $G \subset E(\mathbb{Z}^d)$ and sample $p_c(\mathbb{Z}^d)$-percolation on G. Be careful not to confuse the notation $B(x, r; G)$ which refers to a random ball in the percolation on G with $B_G(x, r)$ which is just the (deterministic) ball in G. In fact $B(x, r; G) = B_{G_{p_c(\mathbb{Z}^d)}}(x, r)$.

We usually take G to be $\mathbb{Z}^d$, and in this case it would be suppressed from the notation. Our most frequent notation is $B(x, r)$ which stands for $B(x, r; \mathbb{Z}^d)$.

Define now the event

$$H(r; G) = \{\partial B(0, r; G) \ne \emptyset\},$$

and finally define

$$\Gamma(r) = \sup_{G \subset E(\mathbb{Z}^d)} \mathbf{P}(H(r; G)).$$

Note again that we define Γ by the maximum over all subgraphs of $\mathbb{Z}^d$, but each one is "tested" with the p_c of $\mathbb{Z}^d$ rather than with its own p_c.

Theorem 1.2 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying the triangle condition* (1.1), *there exists a constant $C > 0$ such that*:

(i) $\mathbb{E}|B(0, r)| \le Cr$
(ii) $\Gamma(r) \le \frac{C}{r}$.

In particular, $\mathbf{P}(H(r)) \le C/r$.

The corresponding lower bounds to Theorem 1.2 are much easier to prove and are not needed for the proof of Theorem 1.1. We state them for the sake of completeness.

Theorem 1.3 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying the two-point function estimate* (1.2), *there exists a constant $c > 0$ such that*:

(1) $\mathbb{E}|B(0, r)| \ge cr$,
(2) $\mathbf{P}(H(r)) \ge \frac{c}{r}$.

The *extrinsic* metric corresponds to the shortest-path metric in $\mathbb{Z}^d$ while the intrinsic metric corresponds to the (random) shortest-path metric in the percolated graph $\mathbb{Z}^d_p$. The classical *one-arm* critical exponent $\rho > 0$ describes the power law decay of the probability that the origin is connected to sphere of radius r *in the extrinsic metric*, that is

$$\mathbf{P}(\exists x \text{ with } |x| = r \text{ such that } 0 \leftrightarrow x) = r^{-1/\rho + o(1)},$$

where $|x|$ denotes the usual Euclidean norm. This exponent takes the value $48/5$ in the two-dimensional triangular grid, as shown by Lawler, Schramm and Werner [37] and Smirnov [49]. In the case of an infinite regular tree we have $\rho = 1$ by a Theorem of Kolmogorov [34] (here the critical probability is $p_c = \frac{1}{\ell - 1}$, where ℓ is the vertex degree of the tree). In high dimensions it was conjectured that $\rho = 1/2$ (see [45] and the upcoming paper [35] for a proof)—a surprising belief at first, since we expect critical exponents in high dimensions to take the same value they do on a tree.

Measuring distance with respect to the *intrinsic* metric offers a simple explanation of this discrepancy. Indeed, as the extrinsic and intrinsic metrics on the tree are the same, we have that on a tree $\mathbf{P}(H(r)) \approx r^{-1}$ and by Theorem 1.2 above we learn that this is the same order in the high-dimension lattices. Similar results exist for critical Erdős-Rényi random graphs [20].

1.4 About the proof

From the point of view of analysis of fractals, the IIC is one of the simplest cases to handle because of its tree-like structure. Indeed, the main difficulty is the proof of Theorem 1.2. Once that is proved the proof proceeds roughly as follows. Write $B_{\mathrm{IIC}}(0,r)$ and $\partial B_{\mathrm{IIC}}(0,r)$ for the corresponding shortest-path metric balls in the IIC. Firstly, since $\Gamma(r) \leq r^{-1}$ and $\mathbb{E}|B(0,r)| \approx r$ we learn that $|B_{\mathrm{IIC}}(0,r)| \approx r^2$. Secondly, the intrinsic metric exponents show that there are $\geq cr$ "approximately pivotal" edges—λ-lanes in the language of [40]—between 0 and $\partial B(0,r)$ (Lemma 2.6) and therefore the *electric resistance* R_{eff} between 0 and $\partial B_{\mathrm{IIC}}(0,r)$ is $\approx r$. We conclude that $B_{\mathrm{IIC}}(0,r)$ is a graph on approximately r^2 vertices with effective resistance between 0 and $\partial B_{\mathrm{IIC}}(0,r)$ of order r—the same structure a critical branching process conditioned to survive to level r has with high probability.

Now, there are many ways to connect electric resistance and volume estimates to hitting times, and in fact we simply quote a perfectly-tailored-for-our-needs result from [11] which concludes the proof of the theorem. However, let us briefly describe a somewhat different but very natural approach. It starts with the fact [16] that, in any finite graph G, and for any two vertices x and y,

$$\mathrm{Hit}(x,y) + \mathrm{Hit}(y,x) = 2R_{\mathrm{eff}}(x,y) \cdot |E(G)|$$

where $\mathrm{Hit}(x,y)$ is the expected hitting time from x to y (or in other words, the left hand side is the expected *commute time* between x and y). Since in our case $|E(G)| \approx r^2$ we get that the commute time is $\approx r^3$. Now, in general the commute time only bounds the hitting time $\mathrm{Hit}(0, \partial B_{\mathrm{IIC}}(0,r))$ from above, but in *strongly recurrent* graphs this turns out to be sharp [36]. Thus, in time r^3 the random walk has walked only in $B_{\mathrm{IIC}}(0,r)$ and it can be shown that the end point is approximately uniformly distributed (the walk has mixed in $B_{\mathrm{IIC}}(0,r)$ in that time). Since $|B_{\mathrm{IIC}}(0,r)| \approx r^2$ we get that $\mathbf{p}_{r^3}(0,0) = r^{-2}$, as required. The details of this approach are described in the setting of finite graphs in [40] and can be adapted to this case as well.

A natural approach towards the proof of the volume growth exponent (part (i) of Theorem 1.2) is to show that $\mathbb{E}|\partial B(0,2r)| \geq c(\mathbb{E}|\partial B(0,r)|)^2$ which would show that if $\mathbb{E}|\partial B(0,r)|$ is too large for some r, it will start exploding, leading to a contradiction. We were not able to pull this approach directly—$\partial B(0,r)$ is hard to analyze—our substitute is to show that $\mathbb{E}|B(0,2r)| \geq (c/r)(\mathbb{E}|B(0,r)|)^2$. This can be proved using relatively standard "inverse BK inequalities" and the same argument then applies.

The proof of the one-arm exponent (part (ii) of Theorem 1.2) uses the precise determination of the exponent δ by Barsky and Aizenman [12], which allows us to use a regeneration argument to show, roughly, that $\Gamma(r) \leq r(\Gamma(r/4))^2 + C/r$ (the second term comes from the results of [12]), from which the estimate follows by induction. The lengths of the proofs of both pieces are equivalent, which might hide the fact that the proof of the one-arm exponent was much harder for us to obtain.

A final note is due about the use of $\Gamma(r)$. It would have been more natural to discuss only $\mathbf{P}(H(r))$ rather than $\Gamma(r)$. However, we need to use a regeneration argument. Basically we claim that, once you reached a certain level r, each vertex $v \in \partial B(0,r)$ has probability $\le \Gamma(s)$ to "reach" to $\partial B(0, r+s)$. Heuristically, one would assume that it would work even with $H(s)$, because the part of the cluster you already "explored", $B(0,r)$ only makes it more difficult to reach the level $r+s$. The problem is that $H(r)$ is not a *monotone* event. In general, if you have a graph G satisfying $\partial B_G(0,r) = \emptyset$ and you remove an edge, it could increase the distance to some vertex v, pushing it outside of $B_G(0,r)$, and restoring the event $\partial B_G(0,r) \neq \emptyset$. Hence it is not possible to use the regeneration argument with $H(r)$—there is simply no inequality in either direction relating $\mathbf{P}(H(r))$ with the conditional probability of $H(r)$ given some partial configuration of edges. The use of $\Gamma(r)$ helps us circumvent this problem. See the proofs of Lemma 2.6 and of part (ii) of Theorem 1.2.

1.5 Organization and notation

In Sect. 2 we show how the intrinsic metric critical exponents (Theorem 1.2), together with (1.2) yield our main result, Theorem 1.1. In Sect. 3 we derive the mean-field estimates of Theorems 1.2 and 1.3.

For $x, y \in \mathbb{Z}^d$ we write $x \leftrightarrow y$ for the event that x is connected to y by an open path. We write $x \overset{r}{\leftrightarrow} y$ if there is an open path of length $\le r$ connecting x and y. In order to improve readability, we denote constants which depend only on d and the lattice by C (to denote a large constant) and c (to denote a small constant) and as we do not attempt to optimize these constants we frequently use the same notation to indicate different constants. For two monotone events of percolation A and B we write $A \circ B$ for the event that A and B occurs in disjoint edges and we often use the van den Berg and Kesten inequality (BK for short) $\mathbf{P}(A \circ B) \le \mathbf{P}(A)\mathbf{P}(B)$ (see [21, 50] or [15] for more details).

2 Deriving the Alexander-Orbach conjecture from Theorem 1.2

In this entire section we assume the two-point function estimate (1.2) and Theorem 1.2. We will use results of Barlow, Járai, Kumagai and Slade [11] which are stated for *random graphs* and hence are perfectly suited for our case. It is interesting to note that log log fluctuations really do exist, and hence any result for *fixed graphs* will naturally be somewhat imprecise. To state the results of [11], we need the following definitions. Given an instance of the IIC (that is, an infinite connected graph containing the origin) write $B_{\mathrm{IIC}}(0,r)$ and $\partial B_{\mathrm{IIC}}(0,r)$ for the ball of radius r around 0 and the boundary of the ball, respectively, in the shortest path metric on the IIC. Denote by $R_{\mathrm{eff}}(0, \partial B_{\mathrm{IIC}}(0,r))$ the effective resistance between 0 and $\partial B_{\mathrm{IIC}}(0,r)$ when one considers $B_{\mathrm{IIC}}(0,r)$ as an electric network and gives each edge a resistance of 1—see [19] for a formal definition. For $\lambda > 1$ we write $J(\lambda)$ for the set of r's for which the following conditions hold:

(1) $\lambda^{-1} r^2 \le |B_{\mathrm{IIC}}(0,r)| \le \lambda r^2$,
(2) $R_{\mathrm{eff}}(0, \partial B_{\mathrm{IIC}}(0,r)) \ge \lambda^{-1} r$.

Theorems 1.5 and 1.6 of [11] relate the information of volume and effective resistance growth to the behavior of random walks. They can be stated as follows.

Theorem 2.1 [11] *If there exist some constants $K, q > 0$ such that for any large enough r we have*

$$\mathbf{P}_{\mathrm{IIC}}(r \in J(\lambda)) \ge 1 - K\lambda^{-q}, \tag{2.1}$$

then the conclusions of Theorem 1.1 hold.

We begin with some lemmas leading to the fact that condition (2.1) holds in our setting. We start with some volume estimates.

Lemma 2.2 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying (1.2), there exists a constant $C > 0$ such that for any integer $r \geq 1$ and any $x \in G$ with $|x|$ sufficiently large we have*

$$\mathbb{E}\big[|B(0,r)| \cdot \mathbf{1}_{\{0 \leftrightarrow x\}}\big] \leq Cr^2 |x|^{2-d}.$$

Here and below "$|x|$ sufficiently large" means essentially that $|x| > 4Lr$ where L is the length of the longest edge in $E(\mathbb{Z}^d)$. This point will not play any role, though.

Proof of Lemma 2.2 We have

$$\mathbb{E}\big[|B(0,r)| \cdot \mathbf{1}_{\{0 \leftrightarrow x\}}\big] = \sum_{z \in \mathbb{Z}^d} \mathbf{P}(0 \overset{r}{\leftrightarrow} z,\ 0 \leftrightarrow x).$$

If $0 \overset{r}{\leftrightarrow} z$ and $0 \leftrightarrow x$, then there must exist some y such that the events $0 \overset{r}{\leftrightarrow} y$, $y \overset{r}{\leftrightarrow} z$ and $y \leftrightarrow x$ occur disjointly. So

$$\begin{aligned}\mathbb{E}\big[|B(0,r)| \cdot \mathbf{1}_{\{0 \leftrightarrow x\}}\big] &= \sum_{z} \mathbf{P}(0 \overset{r}{\leftrightarrow} z, 0 \leftrightarrow x) \\ &\leq \sum_{z,y} \mathbf{P}(\{0 \overset{r}{\leftrightarrow} y\} \circ \{y \overset{r}{\leftrightarrow} z\} \circ \{y \leftrightarrow x\})\end{aligned}$$

and applying the BK inequality twice,

$$\leq \sum_{z,y} \mathbf{P}(0 \overset{r}{\leftrightarrow} y)\mathbf{P}(y \overset{r}{\leftrightarrow} z)\mathbf{P}(y \leftrightarrow x)$$

which by (1.2) and the fact that $|x-y| \geq |x|/2$ when x is sufficiently large

$$\leq C|x|^{2-d} \sum_{z,y} \mathbf{P}(0 \overset{r}{\leftrightarrow} y)\mathbf{P}(y \overset{r}{\leftrightarrow} z).$$

We now use part (i) of Theorem 1.2 to sum, first over z and then over y. We get

$$\leq Cr|x|^{2-d} \sum_{y} \mathbf{P}(0 \overset{r}{\leftrightarrow} y) \leq Cr^2|x|^{2-d}.$$

□

Lemma 2.3 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying (1.2), there exists a constant $C > 0$ such that for any $r \geq 1$, any $\epsilon < 1$ and any $x \in \mathbb{Z}^d$ with $|x|$ sufficiently large we have that*

$$\mathbf{P}\big(|B(0,r)| \leq \epsilon r^2,\ 0 \leftrightarrow x\big) \leq C\epsilon |x|^{2-d}.$$

Proof If $|B(0,r)| \leq \epsilon r^2$ then there must exists a (random) level $j \in [r/2, r]$ in which $|\partial B(0,j)| \leq 2\epsilon r$. Fix the smallest such j. Now, if $0 \leftrightarrow x$ then there must be some vertex $y \in \partial B(0,j)$ which is connected to x "off $B(0,j-1)$" i.e. with a path that does not use any of the vertices in $B(0, j-1)$. Let therefore A be some subgraph of $\mathbb{Z}^d$ which is "admissible" for being $B(0, j)$ i.e. $\mathbf{P}(B(0, j) = A) > 0$. We get

$$\mathbf{P}(0 \leftrightarrow x \mid B(0, j) = A) \le \sum_{y \in \partial A} \mathbf{P}(y \leftrightarrow x \text{ off } A \setminus \partial A \mid B(0, j) = A)$$

where ∂A stands for the vertices in the graph A furthest from 0. A moment's reflection shows that, for any A and any $y \in \partial A$, the event $\{y \leftrightarrow x \text{ off } A \setminus \partial A\}$ is *independent* of the event $\{B(0, j) = A\}$. Therefore we can write

$$\begin{aligned}\mathbf{P}(0 \leftrightarrow x \mid B(0, j) = A) &\le \sum_{y \in \partial A} \mathbf{P}(y \leftrightarrow x \text{ off } A \setminus \partial A)\\ &\le \sum_{y \in \partial A} \mathbf{P}(y \leftrightarrow x) \le C|\partial A||x|^{2-d}\end{aligned}$$

where the last inequality uses the two-point function estimate (1.2) and the fact that $|x - y| \ge |x|/2$. By the definition of j we have $|\partial A| \le 2\epsilon r$ and summing over all admissible A gives

$$\mathbf{P}(|B(0, r)| \le \epsilon r^2, 0 \leftrightarrow x) \le C\epsilon r|x|^{2-d} \cdot \sum_{A} \mathbf{P}(B(0, j) = A).$$

However, the events $B(0, j) = A_1$ and $B(0, j) = A_2$ are disjoint and the union of these over all A imply that $\partial B(0, \frac{1}{2}r) \ne \emptyset$. Part (ii) of Theorem 1.2 shows that the probability of this union is $\le C/r$, finishing our lemma. □

We continue with some effective resistance estimates. Recall the following definitions from [40]:

- An edge e between $\partial B(0, j-1)$ and $\partial B(0, j)$ is called a *lane* for r if there is a path with initial edge e from $\partial B(0, j-1)$ to $\partial B(0, r)$ that does not return to $\partial B(0, j-1)$.
- Say that a level j (with $0 < j < r$) has λ *lanes* for r if there are at least λ edges between $\partial B(0, j-1)$ and $\partial B(0, j)$ which are lanes for r.
- We say that 0 is λ-*lane rich* for r, if more than half of the levels $j \in [r/4, r/2]$ have λ lanes for r.

Recall also the Nash-Williams [41] inequality (see also [43, Corollary 9.2]).

Lemma 2.4 [41] *If $\{\Pi_j\}_{j=1}^J$ are disjoint cut-sets separating v from U in a graph with unit conductance for each edge, then the effective resistance from v to U satisfies*

$$R_{\text{eff}}(v \leftrightarrow U) \ge \sum_{j=1}^{J} \frac{1}{|\Pi_j|}.$$

Lemma 2.5 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying (1.2), there exists a constant $C > 0$ such that for any $r \ge 1$, for any event E measurable with respect to $B(0, r)$ and for any $x \in \mathbb{Z}^d$ with $|x|$ sufficiently large,*

$$\mathbf{P}(E \cap \{0 \leftrightarrow x\}) \le C\sqrt{r\mathbf{P}(E)}\,|x|^{2-d}.$$

Proof We first note that by Lemma 2.2 there exists some $j \in [r, 2r]$ such that

$$\mathbb{E}\big[|\partial B(0, j)| \cdot \mathbf{1}_{\{0 \leftrightarrow x\}}\big] \le Cr|x|^{2-d}. \tag{2.2}$$

Now fix some $M > 0$ (which we shall optimize in the end) and write

$$\mathbf{P}(E \cap \{0 \leftrightarrow x\}) \le \mathbf{P}(|\partial B(0, j)| > M,\ 0 \leftrightarrow x)\mathbf{P}(E \cap \{|\partial B(0, j)| \le M, 0 \leftrightarrow x\}).$$

Now, for the first term we use (2.2) and Markov's inequality and get

$$\mathbf{P}(|\partial B(0, j)| > M,\ 0 \leftrightarrow x) \le \frac{Cr|x|^{2-d}}{M}.$$

For the second term we do as in Lemma 2.3. We condition over $B(0, j)$ and note that for any A we have

$$\mathbf{P}(0 \leftrightarrow x \mid B(0, j) = A) \le C|\partial A||x|^{2-d} \le CM|x|^{2-d}.$$

Summing over all subgraphs A which satisfy E (here is where we use that E is measurable with respect to $B(0, r)$) gives that the second term is $\le \mathbf{P}(E) \cdot CM|x|^{2-d}$. Summing both terms we get

$$\mathbf{P}(E \cap \{0 \leftrightarrow x\}) \le \frac{Cr|x|^{2-d}}{M} + C\mathbf{P}(E)M|x|^{2-d}.$$

Taking $M = \sqrt{r/\mathbf{P}(E)}$ proves the lemma. □

Lemma 2.6 *For any lattice $\mathbb{Z}^d$ with $d > 6$ satisfying* (1.2), *there exists a constant $C > 0$ such for any $r \ge 1$, any $\lambda > 1$ and any $x \in \mathbb{Z}^d$ with $|x|$ sufficiently large we have that*

$$\mathbf{P}\big(R_{\mathrm{eff}}(0, \partial B(0, r)) \le \lambda^{-1} r,\ 0 \leftrightarrow x\big) \le C\lambda^{-1/2}|x|^{2-d}.$$

Proof Let $j \in [r/4, r/2]$ and denote by L_j the number of lanes between $\partial B(0, j-1)$ and $\partial B(0, j)$. Let us condition on $B(0, j)$ and take some edge between $\partial B(0, j-1)$ and $\partial B(0, j)$. Denote the end vertex of this edge in $\partial B(0, j)$ by v. The event that the edge is a lane for r implies that we have $\partial B(v, r/2; G) \ne \emptyset$ in the graph G that one gets by removing all edges which are needed to calculate $B(0, j)$, that is, all the edges with at least one vertex in $B(0, j-1)$. Thus, conditioned on $B(0, j)$, the event we are interested in is

$$\partial B(v, r/2; G) \ne \emptyset.$$

By the definition of Γ (with the G from the definition of Γ being our G) and translation invariance we get that

$$\mathbf{P}(\partial B(v, r/2; G) \ne \emptyset \mid B(0, j)) \le \Gamma(r/2)$$

which by part (ii) of Theorem 1.2 is $\le C/r$. In total we get

$$\mathbb{E}[L_j \mid B(0, j)] \le \frac{C}{r}|\partial B(0, j)|.$$

This together with part (i) of Theorem 1.2 implies that the expected number of lanes in $B(0, r/2) \setminus B(0, r/4)$ is at most a constant C. We learn that the probability that 0 is λ-lane rich is at most $C\lambda^{-1}r^{-1}$. Observe that Lemma 2.4 implies that if 0 is *not* λ-lane rich, then $R_{\mathrm{eff}}(0,\partial B(0,r)) \geq c\lambda^{-1}r$ and hence

$$\mathbf{P}\big(R_{\mathrm{eff}}(0,\partial B(0,r)) \leq \lambda^{-1}r\big) \leq C\lambda^{-1}r^{-1}$$

(where if $\partial B(0,r)=\emptyset$ then we consider the resistance to be ∞). An appeal to Lemma 2.5 and we are done. □

Proof of Theorem 1.1 By Theorem 2.1 it suffices to show that (2.1) holds. Indeed, fix $r \geq 1$ and $x \in \mathbb{Z}^d$ with $|x|$ sufficiently large. Markov's inequality with Lemma 2.2 and the lower bound for the two-point function (1.2) shows that

$$\mathbf{P}\big(|B(0,r)| \geq \lambda r^2 \mid 0 \leftrightarrow x\big) \leq C\lambda^{-1}.$$

Lemmas 2.3 and 2.6 show that

$$\mathbf{P}\big(|B(0,r)| \leq \lambda^{-1} r^2 \mid 0 \leftrightarrow x\big) \leq C\lambda^{-1},$$

and

$$\mathbf{P}\big(R_{\mathrm{eff}}(0,\partial B(0,r)) \leq \lambda^{-1}r \mid 0 \leftrightarrow x\big) \leq C\lambda^{-1/2}.$$

Taking the limit as $|x| \to \infty$ shows that (2.1) holds with constants $K=3C$ and $q=1/2$, concluding the proof. □

3 Intrinsic metric critical exponents

In this section we prove Theorem 1.2. Our assumption on the lattice is therefore that it satisfies the triangle condition (1.1). In effect we will be using a variation known as the *open* triangle condition

$$\lim_{K\to\infty} \sup_{w:|w|\geq K} \mathbf{P}(0 \leftrightarrow x)\mathbf{P}(x \leftrightarrow y)\mathbf{P}(y \leftrightarrow w) = 0. \tag{3.1}$$

For $\mathbb{Z}^d$ the triangle condition implies the open triangle condition—see [12, Lemma 2.1]. Hence, from now on we will only use (3.1).

3.1 Intrinsic metric volume exponent

Here we prove part (i) of Theorem 1.2. We use the notation

$$G(r) = \mathbb{E}|B(0,r)|.$$

The main part of the proof is the following lemma.

Lemma 3.1 *Under the setting of Theorem 1.2, there exists a constant $c_1 > 0$ such that for all r*

$$G(2r) \geq \frac{c_1 G(r)^2}{r}.$$

Let us first see how to use Lemma 3.1.

Proof of part (i) of Theorem 1.2 Assume without loss of generality that $c_1 < 1$ in Lemma 3.1 and take $C_1 > \max\{(2/c_1), 2^d\}$. Assume by contradiction that there exists r_0 such that $G(r_0) \geq C_1 r_0$. Under this assumption we prove by induction that for any integer $k \geq 0$ we have $G(2^k r_0) \geq C_1^{k+1} r_0$. The case $k = 0$ is our assumption, and Lemma 3.1 gives that

$$G(2^{k+1} r_0) \geq \frac{c_1 G(2^k r_0)^2}{2^k r_0} \geq C_1^{k+2} r_0,$$

where in the last inequality we used the induction hypothesis and the fact that $C_1 > 2/c_1$. This completes our induction.

Now, since the number of vertices of distance r from the origin is at most Cr^d for some constant C which depends on d and on the lattice, but not on r, we get that for any integer $k \geq 0$

$$C(2^k r_0)^d \geq G(2^k r_0) \geq C_1^{k+1} r_0,$$

and since $C_1 > 2^d$ we arrive at a contradiction (for some k sufficiently large) which proves the upper bound on $G(r)$. □

The next lemma is used in the proof of Lemma 3.1.

Lemma 3.2 *There exists some constant $c > 0$ such that*

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}\big(\{0 \overset{r}{\leftrightarrow} x\} \circ \{x \overset{r}{\leftrightarrow} y\}\big) \geq c G(r)^2.$$

Proof By translation invariance of $\mathbb{Z}^d$ it suffices to prove that

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}\big(\{0 \overset{r}{\leftrightarrow} x\} \circ \{0 \overset{r}{\leftrightarrow} y\}\big) \geq c G(r)^2.$$

The proof requires that we separate slightly the starting points of the two paths. Hence we shall prove that there exists some $K > 0$ such that if $u, v \in \mathbb{Z}^d$ satisfy $|u - v| \geq K$, then

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}\big(\{u \overset{r-K}{\longleftrightarrow} x\} \circ \{v \overset{r-K}{\longleftrightarrow} y\}\big) \geq \frac{1}{2} G(r-K)^2. \tag{3.2}$$

Assuming this, we will then take u, v to be antipodal vertices on the sphere of radius K (in the usual norm) of $\mathbb{Z}^d$ so that $|u - v| \geq K$. To see that the assertion of the lemma follows from the above claim, notice that if it occurs that $\{u \overset{r-K}{\longleftrightarrow} x\} \circ \{v \overset{r-K}{\longleftrightarrow} y\}$, then by changing the status of edges from closed to open only in the ball $B(0, K)$ we can make the configuration belong to the event $\{0 \overset{r}{\leftrightarrow} x\} \circ \{0 \overset{r}{\leftrightarrow} y\}$. We deduce that there exists some $c(K) > 0$ such that

$$\mathbf{P}\Big(\{0 \overset{r}{\leftrightarrow} x\} \circ \{0 \overset{r}{\leftrightarrow} y\}\Big) \geq c(K) \mathbf{P}\Big(\{u \overset{r-K}{\longleftrightarrow} x\} \circ \{v \overset{r-K}{\longleftrightarrow} y\}\Big),$$

and so by (3.2)

$$\sum_{x,y \in \mathbb{Z}^d} \mathbf{P}\Big(\{0 \overset{r}{\leftrightarrow} x\} \circ \{0 \overset{r}{\leftrightarrow} y\}\Big) \geq \frac{c(K)}{2} G(r-K)^2.$$

And it is clear that $G(r) \le CK^d G(r-K)$ and the assertion of the lemma follows, if only K can be chosen independently of r.

We proceed to prove (3.2). For any $u, v \in \mathbb{Z}^d$ and an integer $\ell > 0$ (later we put $\ell = r - K$) we have

$$\mathbf{P}\big(\{u \stackrel{\ell}{\leftrightarrow} x\} \circ \{v \stackrel{\ell}{\leftrightarrow} y\}\big) \ge \mathbf{P}\Big(u \stackrel{\ell}{\leftrightarrow} x \quad \text{and } v \stackrel{\ell}{\leftrightarrow} y \quad \text{and } \mathcal{C}(u) \ne \mathcal{C}(v)\Big),$$

where $\mathcal{C}(u)$ and $\mathcal{C}(v)$ denote the connected components containing u and v, respectively. By conditioning on $\mathcal{C}(u)$ we get that the right hand side equals

$$\sum_{\substack{A \subset E(\mathbb{Z}^d) \\ u \stackrel{\ell}{\leftrightarrow} x, u \not\leftrightarrow v \text{ in } A}} \mathbf{P}(\mathcal{C}(u) = A)\mathbf{P}\Big(v \stackrel{\ell}{\leftrightarrow} y \mid \mathcal{C}(u) = A\Big).$$

Note that for A such that $u \not\leftrightarrow v$ in A we have $\mathbf{P}(\{v \stackrel{\ell}{\leftrightarrow} y\} \mid \mathcal{C}(u) = A) = \mathbf{P}(v \stackrel{\ell}{\leftrightarrow} y \text{ off } A)$ where the event $\{v \stackrel{\ell}{\leftrightarrow} y \text{ off } A\}$ again means that there exists an open path of length at most ℓ connecting v to y which avoids the vertices of A. At this point we can remove the condition that $u \not\leftrightarrow v$ in A since in this case the event $\{v \stackrel{\ell}{\leftrightarrow} y \text{ off } A\}$ is empty. We get

$$\mathbf{P}\big(\{u \stackrel{\ell}{\leftrightarrow} x\} \circ \{v \stackrel{\ell}{\leftrightarrow} y\}\big) \ge \sum_{\substack{A \subset E(\mathbb{Z}^d) \\ u \stackrel{\ell}{\leftrightarrow} x \text{ in } A}} \mathbf{P}(\mathcal{C}(u) = A)\mathbf{P}\big(v \stackrel{\ell}{\leftrightarrow} y \quad \text{off } A\big). \tag{3.3}$$

Now, since

$$\mathbf{P}(u \stackrel{\ell}{\leftrightarrow} x)\mathbf{P}(v \stackrel{\ell}{\leftrightarrow} y) = \sum_{\substack{A \subset E(\mathbb{Z}^d) \\ u \stackrel{\ell}{\leftrightarrow} x \text{ in } A}} \mathbf{P}(\mathcal{C}(u) = A)\mathbf{P}(\{v \stackrel{\ell}{\leftrightarrow} y\}),$$

we deduce by (3.3) that

$$\begin{aligned}
&\mathbf{P}\big(\{u \stackrel{\ell}{\leftrightarrow} x\} \circ \{v \stackrel{\ell}{\leftrightarrow} y\}\big) \\
&\quad \ge \mathbf{P}(u \stackrel{\ell}{\leftrightarrow} x)\mathbf{P}(v \stackrel{\ell}{\leftrightarrow} y) - \sum_{\substack{A \subset E(\mathbb{Z}^d) \\ u \stackrel{\ell}{\leftrightarrow} x \text{ in } A}} \mathbf{P}(\mathcal{C}(u) = A)\mathbf{P}\big(v \stackrel{\ell}{\leftrightarrow} y \quad \text{only on } A\big),
\end{aligned} \tag{3.4}$$

where the event $\{v \stackrel{\ell}{\leftrightarrow} y \text{ onlyon } A\}$ means that there exists an open path between v and y of length at most ℓ and any such path must have a vertex in A. For any $A \subset \mathbb{Z}^d$ we have

$$\mathbf{P}\big(v \stackrel{\ell}{\leftrightarrow} y \quad \text{only on } A\big) \le \sum_{z \in A} \mathbf{P}\big(\{v \leftrightarrow z\} \circ \{z \stackrel{\ell}{\leftrightarrow} y\}\big).$$

Putting this into the second term of the right hand side of (3.4) and changing the order of summation gives that we can bound this term from above by

$$\sum_{z \in \mathbb{Z}^d} \mathbf{P}\big(u \stackrel{\ell}{\leftrightarrow} x,\ u \leftrightarrow z\big)\mathbf{P}\big(\{v \leftrightarrow z\} \circ \{z \stackrel{\ell}{\leftrightarrow} y\}\big).$$

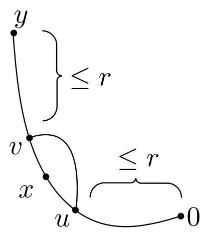

Fig. 2 The couple (x, y) is over-counted

If $u \overset{\ell}{\leftrightarrow} x$ and $u \leftrightarrow z$ then there exists z' such that the events $u \leftrightarrow z'$ and $z' \leftrightarrow z$ and $z' \overset{\ell}{\leftrightarrow} x$ occur disjointly. Using the BK inequality we bound this sum above by

$$\sum_{z,z'\in\mathbb{Z}^d} \mathbf{P}(u \leftrightarrow z')\mathbf{P}(z' \leftrightarrow z)\mathbf{P}(z' \overset{\ell}{\leftrightarrow} x)\mathbf{P}(v \leftrightarrow z)\mathbf{P}(z \overset{\ell}{\leftrightarrow} y).$$

We sum this over x and y and use (3.4) to get that

$$\sum_{x,y\in\mathbb{Z}^d} \mathbf{P}\big(\{u \overset{\ell}{\leftrightarrow} x\} \circ \{v \overset{\ell}{\leftrightarrow} y\}\big) \geq G(\ell)^2 - G(\ell)^2 \sum_{z,z'\in\mathbb{Z}^d} \mathbf{P}(u \leftrightarrow z')\mathbf{P}(z' \leftrightarrow z)\mathbf{P}(z \leftrightarrow v).$$

Since $|u - v| \geq K$, by the open triangle condition (3.1) we can take K large enough (independently of r) so that

$$\sum_{z,z'\in\mathbb{Z}^d} \mathbf{P}(v \leftrightarrow z)\mathbf{P}(z \leftrightarrow z')\mathbf{P}(z' \leftrightarrow u) \leq \frac{1}{2},$$

which immediately yields (3.2) and concludes our proof. □

Proof of Lemma 3.1 We start with a definition. For an integer $K > 0$ we say two vertices $x, y \in \mathbb{Z}^d$ are *K-over-counted* if there exists $u, v \in \mathbb{Z}^d$ with $|u - x| \geq K$ and $|v - x| \geq K$ such that

$$\{0 \overset{r}{\leftrightarrow} u\} \circ \{v \leftrightarrow x\} \circ \{x \leftrightarrow u\} \circ \{u \leftrightarrow v\} \circ \{v \overset{r}{\leftrightarrow} y\},$$

see Fig. 2. Denote by $N(K)$ the quantity

$$N(K) = \Big|\Big\{(x, y) : \{0 \overset{r}{\leftrightarrow} x\} \circ \{x \overset{r}{\leftrightarrow} y\} \quad \text{and } (x, y) \text{ are not } K\text{-over-counted}\Big\}\Big|.$$

We claim that

$$N(K) \leq CK^d \cdot 2r|B(0, 2r)|. \tag{3.5}$$

Indeed, this deterministic claim follows by observing that if $y \in B(0, 2r)$ and γ is an open simple path of length at most $2r$ connecting 0 to y, then for any $x \in \mathbb{Z}^d$ of distance at least K from γ (i.e., x is of distance at least K from *every* vertex of γ) satisfying $\{0 \overset{r}{\leftrightarrow} x\} \circ \{x \overset{r}{\leftrightarrow} y\}$ the pair (x, y) is K-over-counted. To see this, let γ_1 and γ_2 be disjoint open simple paths of length at most r connecting 0 to x and x to y respectively and take u to be the last point on $\gamma \cap \gamma_1$ and v the first point on $\gamma \cap \gamma_2$ where the ordering is induced by γ_1 and γ_2

respectively. Hence the map $(x, y) \mapsto y$ from $N(K)$ into $B(0, 2r)$ is at most $CK^d \cdot 2r$ to 1, which shows (3.5).

We now estimate $\mathbb{E}N(K)$. For any (x, y) the BK inequality and (1.2) implies that the probability that (x, y) are K-over-counted is at most

$$\sum_{\substack{u:|u-x|\geq K \\ v:|v-x|\geq K}} \mathbf{P}(0 \overset{r}{\leftrightarrow} u)\mathbf{P}(v \overset{r}{\leftrightarrow} y)\mathbf{P}(x \leftrightarrow v)\mathbf{P}(v \leftrightarrow u)\mathbf{P}(u \leftrightarrow x).$$

Writing $v' = v - x$ and $u' = u - x$ and using translation invariance we get that this sum equals

$$\sum_{\substack{u':|u'|\geq K \\ v':|v'|\geq K}} \mathbf{P}(-x \overset{r}{\leftrightarrow} u')\mathbf{P}(v' \overset{r}{\leftrightarrow} y - x)\mathbf{P}(0 \leftrightarrow v')\mathbf{P}(v' \leftrightarrow u')\mathbf{P}(u' \leftrightarrow 0).$$

We sum this over y and then over x and get that

$$\mathbb{E}\Big[\big|\big\{(x, y) \text{ are } K\text{-over-counted}\big\}\big|\Big] \leq G(r)^2 \sum_{\substack{u':|u'|\geq K \\ v':|v'|\geq K}} \mathbf{P}(0 \leftrightarrow v')\mathbf{P}(v' \leftrightarrow u')\mathbf{P}(u' \leftrightarrow 0).$$

This together with the triangle condition (1.1) and Lemma 3.2 gives that for some small $c > 0$ we can choose some large K such that $\mathbb{E}N(K) \geq cG(r)^2$. We take expectations in (3.5) and plug the estimate $\mathbb{E}N(K) \geq cG(r)^2$ in to get the assertion of the lemma. This concludes the proof of part (i) of Theorem 1.2. □

3.2 Intrinsic metric arm exponent

Here we prove part (ii) of Theorem 1.2. The proof relies on the result of Barsky and Aizenman [12] stating that *A lattice in* $\mathbb{R}^d$ *satisfying the triangle condition satisfies, as* $h \to 0$ *that*

$$\sum_{j=1}^{\infty} \mathbf{P}(|\mathcal{C}(0)| = j)(1 - e^{-jh}) \approx h^{1/2}.$$

This implies an estimate of $\mathbf{P}(|\mathcal{C}(0)|) > n$. Just fix $h = 1/n$ and get

$$\mathbf{P}\big(|\mathcal{C}(0)| > n\big) \leq \frac{C_1}{n^{1/2}}. \tag{3.6}$$

We remark that Hara and Slade achieved a significantly stronger estimate [26].

Since the event $\{|\mathcal{C}(0)| > n\}$ is monotone, we get

$$\mathbf{P}\big(|\mathcal{C}_G(0)| > n\big) \leq \frac{C_1}{n^{1/2}} \quad \text{for all } G \subset E(\mathbb{Z}^d) \tag{3.7}$$

where $\mathcal{C}_G(0)$ is the component containing 0 in percolation on G with $p = p_c(\mathbb{Z}^d)$ (and as in the definition of Γ, *not* in the critical p of G itself).

Proof of part (ii) of Theorem 1.2 Let $A \geq 1$ be a large number such that

$$3^3 A^{2/3} + C_1 A^{2/3} \leq A,$$

where C_1 is from (3.7). We will now prove that $\Gamma(r) \le 3Ar^{-1}$. This will follow by showing inductively that for any integer $k > 0$ we have

$$\Gamma(3^k) \le \frac{A}{3^k}.$$

This is trivial for $k = 0$ since $A \ge 1$. Assume the claim for all $j < k$ and we prove for k. Let $\epsilon = \epsilon(C_1) > 0$ be a small constant to be chosen later and for any $G \subset E(\mathbb{Z}^d)$ write

$$\begin{aligned} &\mathbf{P}(H(3^k; G)) \\ &\quad \le \mathbf{P}\big(\partial B(0, 3^k; G) \ne \emptyset \text{ and } |\mathcal{C}_G(0)| \le \epsilon 9^k\big) + \mathbf{P}\big(|\mathcal{C}_G(0)| > \epsilon 9^k\big) \\ &\quad \le \mathbf{P}\big(\partial B(0, 3^k; G) \ne \emptyset \text{ and } |\mathcal{C}_G(0)| \le \epsilon 9^k\big) + \frac{C_1}{\sqrt{\epsilon} 3^k}, \end{aligned} \tag{3.8}$$

where the last inequality is due to (3.7). To estimate the first term on the right-hand side we claim that

$$\mathbf{P}\big(\partial B(0, 3^k; G) \ne \emptyset \text{ and } |\mathcal{C}_G(0)| \le \epsilon 9^k\big) \le \epsilon 3^{k+1} (\Gamma(3^{k-1}))^2. \tag{3.9}$$

To see this observe that if $|\mathcal{C}_G(0)| \le \epsilon 9^k$ then there must be some level $j \in [\frac{1}{3} 3^k, \frac{2}{3} 3^k]$ such that $|\partial B(0, j; G)| \le \epsilon 3^{k+1}$. Denote by j the first such level. If, in addition, $\partial B(0, 3^k; G) \ne \emptyset$ then at least one vertex v of the $\epsilon 3^{k+1}$ vertices of level j "reaches level 3^{k-1}". Formally we do as in the proof of lemma 2.6, i.e. define G_2 to be G with all edges needed to calculate $B(0, j; G)$ removed and get that

$$\partial B(v, 3^{k-1}; G_2) \ne \emptyset$$

which, by the definition of Γ (with G_2) has probability $\le \Gamma(3^{k-1})$. Applying Markov's inequality gives

$$\mathbf{P}\big(\partial B(0, 3^k; G) \ne \emptyset \text{ and } |\mathcal{C}_G(0)| \le \epsilon 9^k \mid B(0, j; G)\big) \le \epsilon 3^{k+1} \Gamma(3^{k-1}).$$

As in the proof of Lemma 2.6, we now sum over possible values of $B(0, j; G)$ and get an extra term of $\mathbf{P}(H(0, 3^{k-1}; G))$ because we need to reach level 3^{k-1} to begin with. We can definitely bound $\mathbf{P}(H(0, 3^{k-1}; G)) \le \Gamma(3^{k-1})$ and this gives the assertion of (3.9).

We put this into (3.8) and get that

$$\mathbf{P}(H(3^k; G)) \le \epsilon 3^{k+1} (\Gamma(3^{k-1}))^2 + \frac{C_1}{\sqrt{\epsilon} 3^k} \le \frac{\epsilon 3^3 A^2 + C_1 \epsilon^{-1/2}}{3^k},$$

where in the last inequality we used the induction hypothesis. Put now $\epsilon = A^{-4/3}$. Since the last inequality holds for any $G \subset E(\mathbb{Z}^d)$ we have

$$\Gamma(3^k) \le \frac{3^3 A^{2/3} + C_1 A^{2/3}}{3^k} \le \frac{A}{3^k},$$

where the last inequality is by our choice of A. This completes our inductive proof that $\Gamma(3^k) \le A 3^{-k}$. Now, for any r choose k such that $3^{k-1} \le r < 3^k$ then we have

$$\Gamma(r) \le \Gamma(3^{k-1}) \le \frac{A}{3^{k-1}} < \frac{3A}{r}.$$

□

3.3 Corresponding lower bounds

In the following we provide the corresponding lower bounds to the estimates of Theorem 1.2.

Proof of part (i) of Theorem 1.3 Let $x \in \mathbb{Z}^d$ and write $|x|$ for the Euclidean distance of x from 0. We estimate the quantity $\mathbb{E}[d_{\mathbb{Z}^d_p}(0,x) \mid 0 \leftrightarrow x]$. If $0 \leftrightarrow x$ then we have that $d_{\mathbb{Z}^d_p}(0,x)$ is no more than the number of $y \in \mathbb{Z}^d$ such that the events $0 \leftrightarrow y$ and $y \leftrightarrow x$ occur disjointly. By the BK inequality and the two-point function estimate (1.2) we learn that

$$\mathbb{E}[d_{\mathbb{Z}^d_p}(0,x)\mathbf{1}_{\{0\leftrightarrow x\}}] \le C \sum_{y\in\mathbb{Z}^d} |y|^{2-d}|x-y|^{2-d} \le C|x|^{4-d},$$

where the last inequality is a straightforward calculation. Hence $\mathbb{E}[d_{\mathbb{Z}^d_p}(0,x) \mid 0 \leftrightarrow x] \le C|x|^2$. We learn that if x is such that $|x| \le \sqrt{r/2C}$, then Markov's inequality implies that $\mathbf{P}(d_{\mathbb{Z}^d_p}(0,x) \le r \mid 0 \leftrightarrow x) \ge 1/2$. By this and (1.2) we conclude that

$$\mathbb{E}|B(0,r;\mathbb{Z}^d)| \ge \sum_{x:|x|\le\sqrt{r/2C}} \mathbf{P}(0 \leftrightarrow x \quad \text{and} \quad d_{\mathbb{Z}^d_p}(0,x) \le r) \ge \frac{1}{2} \sum_{x:|x|\le\sqrt{r/2C}} |x|^{2-d} \ge cr,$$

where $c > 0$ is a small constant. □

Proof of part (ii) of Theorem 1.3 We use a second moment argument. Fix some $\lambda > 1$ to be chosen later. By part (i) of Theorem 1.2 we have that

$$\mathbb{E}|B(0,r)| \le C_1 r,$$

and by part (i) of Theorem 1.3 we have

$$\mathbb{E}|B(0,\lambda r)| \ge c_1 \lambda r.$$

Put $\lambda = 2C_1/c_1$ to get that

$$\mathbb{E}|B(0,\lambda r) \setminus B(0,r)| \ge c_1\lambda r - C_1 r = C_1 r.$$

We now estimate the second moment of $|B(0,\lambda r)|$. Indeed, if $0 \overset{\lambda r}{\leftrightarrow} x$ and $0 \overset{\lambda r}{\leftrightarrow} y$ then there must exist $z \in \mathbb{Z}^d$ such that the events $0 \overset{\lambda r}{\leftrightarrow} z$, $z \overset{\lambda r}{\leftrightarrow} x$ and $z \overset{\lambda r}{\leftrightarrow} y$ occur disjointly. Hence, the BK inequality gives

$$\mathbb{E}|B(0,\lambda r)|^2 \le \sum_{x,y,z} \mathbf{P}(0 \overset{\lambda r}{\leftrightarrow} z)\mathbf{P}(z \overset{\lambda r}{\leftrightarrow} x)\mathbf{P}(z \overset{\lambda r}{\leftrightarrow} y) = \left[\sum_{x\in\mathbb{Z}^d} \mathbf{P}(0 \overset{\lambda r}{\leftrightarrow} x)\right]^3 \le Cr^3,$$

where the last inequality is by part (i) of Theorem 1.2. The estimate $\mathbf{P}(Z > 0) \ge (\mathbb{E}Z)^2/\mathbb{E}Z^2$ valid for any non-negative random variable Z yields that

$$\mathbf{P}\big(|B(0,\lambda r) \setminus B(0,r)| > 0\big) \ge \frac{C_1^2 r^2}{Cr^3} \ge \frac{c}{r},$$

which concludes our proof since the event above implies $H(r)$. □

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