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On the role of convexity in isoperimetry, spectral gap and concentration

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Abstract We show that for convex domains in Euclidean space, Cheeger's isoperimetric inequality, spectral gap of the Neumann Laplacian, exponential concentration of Lipschitz functions, and the a-priori weakest requirement that Lipschitz functions have *arbitrarily slow* uniform tail-decay, are all quantitatively equivalent (to within universal constants, independent of the dimension). This substantially extends previous results of Maz'ya, Cheeger, Gromov–Milman, Buser and Ledoux. As an application, we conclude a sharp quantitative stability result for the spectral gap of convex domains under convex perturbations which preserve volume (up to constants) and under maps which are "on-average" Lipschitz. We also provide a new characterization (up to constants) of the spectral gap of a convex domain, as one over the square of the average distance from the "worst" subset having half the measure of the domain. In addition, we easily recover and extend many previously known lower bounds on the spectral gap of convex domains, due to Payne–Weinberger, Li–Yau, Kannan–Lovász–Simonovits, Bobkov and Sodin. The proof involves estimates on the diffusion semi-group following Bakry–Ledoux and a result from Riemannian Geometry on the concavity of the isoperimetric profile. Our results extend to the more general setting of Riemannian manifolds with density which satisfy the $CD(0,\infty)$ curvature-dimension condition of Bakry-Émery.

1 Introduction

Let (Ω, d, μ) denote a metric probability space. More precisely, we assume that (Ω, d) is a separable metric space and that μ is a Borel probability measure on (Ω, d) which is not a unit mass at a point. Although it is not essential for the ensuing discussion, it will be more convenient to specialize to the case where Ω is a smooth complete oriented n-dimensional Riemannian manifold (M, g), d is the induced geodesic distance, and μ is an

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absolutely continuous measure with respect to the Riemannian volume form vol_M on M. A question which goes back at least to the 19th century (motivating the solution to the isoperimetric problem in $\mathbb{R}^n$), and arguably much before that (e.g. Dido's problem), pertains to the interplay between the metric d and the measure μ. There are various different ways to measure this relationship, which may be typically arranged according to strength, forming a hierarchy. In this work, we will be primarily concerned with three such different ways.

1.1 The hierarchy

The first way is by means of an isoperimetric inequality. Recall that Minkowski's (exterior) boundary measure of a Borel set $A \subset \Omega$, which we denote here by $\mu^+(A)$, is defined as:

$$\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon,d}) - \mu(A)}{\varepsilon},$$

where $A_{\varepsilon,d} := \{x \in \Omega; \exists y \in A \; d(x,y) < \varepsilon\}$ denotes the ε-neighborhood of A with respect to the metric d. It is clear that the boundary measure is a natural generalization of the notion of surface area to the metric probability space setting. An isoperimetric inequality measures the relation between $\mu^+(A)$ and $\mu(A)$ by means of the isoperimetric profile $I = I_{(\Omega,d,\mu)}$, defined as the pointwise maximal function $I : [0,1] \to \mathbb{R}_+$, so that $\mu^+(A) \geq I(\mu(A))$ for all Borel sets $A \subset \Omega$. A set A for which equality above is attained is called an isoperimetric minimizer. Since A and $\Omega \setminus A$ will typically (but not necessarily, consider μ with non-continuous density) have the same boundary measure, it will be convenient to also define $\tilde{I} = \tilde{I}_{(\Omega,d,\mu)}$ as the function $\tilde{I} : [0,1/2] \to \mathbb{R}_+$ given by $\tilde{I}(t) := \min(I(t), I(1-t))$.

A very useful isoperimetric inequality was considered by Cheeger [27] (and in a more general form, independently by V. G. Maz'ya [60, 61]):

Definition The space (Ω, d, μ) is said to satisfy Cheeger's isoperimetric inequality if:

$$\exists D > 0 \quad \text{such that} \quad \tilde{I}_{(\Omega,d,\mu)}(t) \geq Dt \quad \forall t \in [0,1/2].$$

The best possible constant D above is denoted by $D_{Che} = D_{Che}(\Omega, d, \mu)$.

A second way to measure the interplay between d and μ is given by functional inequalities. Let $\mathcal{F} = \mathcal{F}(\Omega, d)$ denote the space of functions which are Lipschitz on every ball in (Ω, d)—we will call such functions "Lipschitz-on-balls"—and let $f \in \mathcal{F}$. We will consider functional inequalities which measure the relation between $\|f\|_{L_p(\mu)}$ and $\||\nabla f|\|_{L_q(\mu)}$, for $0 < p, q \leq \infty$ (more general Orlicz norms will be treated in [64]). Here, the effect of the metric d is via the Riemannian metric g which is used to measure $|\nabla f| := g(\nabla f, \nabla f)^{1/2}$, although more general ways exist to define $|\nabla f|$ in the non manifold setting. Of course if f is constant there is no sense to compare against $\||\nabla f|\|_{L_q(\mu)} = 0$, so we will need to exclude these cases. To this end, we will require that either the expectation $E_\mu f$ or median $M_\mu f$ of f are 0. Here $E_\mu f = \int f d\mu$ and $M_\mu f$ is a value so that $\mu(f \geq M_\mu f) \geq 1/2$ and $\mu(f \leq M_\mu f) \geq 1/2$.

A well known example of a functional inequality was studied by Poincaré:

Definition The space (Ω, d, μ) is said to satisfy Poincaré's inequality if:

$$\exists D > 0 \quad \text{such that} \quad \forall f \in \mathcal{F} \quad D \left\| f - E_\mu f \right\|_{L_2(\mu)} \leq \||\nabla f|\|_{L_2(\mu)}.$$

The best possible constant D above is denoted by $D_{Poin} = D_{Poin}(\Omega, d, \mu)$.

It is well known (e.g. [33]) that under appropriate smoothness assumptions, Poincaré's inequality is equivalent to the existence of a spectral gap of an appropriate Laplacian operator $-\Delta_{g,\mu}$ on (M, g) associated to the measure μ with corresponding boundary conditions on its support. When μ is uniform on a domain $\Omega \subset (M, g)$, $\Delta_{g,\mu}$ coincides with the usual Laplace-Beltrami operator Δ_g with Neumann boundary conditions on Ω. The first non-trivial eigenvalue of $-\Delta_{g,\mu}$ (the "spectral gap") is then precisely $D^2_{Poin}(\Omega, d, \mu)$.

A third way to measure the relation between d and μ is given by concentration inequalities. These measure how tightly 1-Lipschitz functions are concentrated about their mean, by providing a quantitative estimate on the tail decay $\mu(|f - E_\mu f| \geq t)$. A typical situation is given by the following example:

Definition The space (Ω, d, μ) is said to have exponential concentration if:

$$\exists c, D > 0 \quad \text{such that} \quad \forall \text{ 1-Lipschitz } f \; \forall t > 0 \quad \mu(|f - E_\mu f| \geq t) \leq c \exp(-Dt).$$

Fixing $c = e$, the best possible constant D above is denoted by $D_{Exp} = D_{Exp}(\Omega, d, \mu)$. The best constant for a specific f is denoted by $D_{Exp}(f)$.

It is known that the three examples mentioned above are arranged in a hierarchy. It was shown by Cheeger [27], and in a more general form, independently by Maz'ya [60–62] (see also [37]), that Cheeger's isoperimetric inequality always implies Poincaré's inequality (or spectral gap):

Theorem 1.1 (Maz'ya, Cheeger) $D_{Poin} \geq D_{Che}/2$ (*"Cheeger's inequality"*).

The fact that Poincaré's inequality implies exponential concentration was first shown by M. Gromov and V. Milman [40] in the Riemannian setting, and subsequently by other authors in other settings as well (e.g. [3], see [55] and the references therein):

Theorem 1.2 (Gromov–Milman) *There exists a universal numeric constant* $c > 0$ *such that* $D_{Exp} \geq c D_{Poin}$.

1.2 Reversing the hierarchy

It is known and easy to show that these implications *cannot* be reversed in general. For instance, using $([-1, 1], |\cdot|, \mu_\alpha)$ where $d\mu_\alpha = \frac{1+\alpha}{2}|x|^\alpha dx$ on $[-1, 1]$, clearly $\mu_\alpha^+([0, 1]) = 0$ so $D_{Che} = 0$, whereas one can show that $D_{Poin} > 0$ for $\alpha \in (0, 1)$ using a criterion for the Poincaré inequality on $\mathbb{R}$ due to Artola, Talenti and Tomaselli (cf. Muckenhoupt [74]). In addition, if μ is supported on a set Ω with diameter bounded by a finite D, trivially one has $D_{Exp} \geq 1/D > 0$; but if we choose Ω to be disconnected, we will always have $D_{Poin} = D_{Che} = 0$. In fact, one need not impose such topological obstructions on Ω, it is also easy to construct a connected set with arbitrarily narrow "necks". We conclude that in order to have any chance of reversing the above implications, we will need to add some additional assumptions, which will prevent the existence of such narrow necks. Intuitively, it is clear that some type of convexity assumptions are a natural candidate. We start with two important examples when $(M, g) = (\mathbb{R}^n, |\cdot|)$ and $|\cdot|$ is some fixed Euclidean norm:

- Ω is an *arbitrary* bounded convex domain in $\mathbb{R}^n$ ($n \geq 2$), and μ is the uniform probability measure on Ω.

- $\Omega = \mathbb{R}^n$ ($n \geq 1$) and μ is an *arbitrary* absolutely continuous log-concave probability measure, meaning that $d\mu = \exp(-\psi)dx$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex (we refer to the paper [23] of C. Borell for more information).

In both cases, we will say that "our convexity assumptions are fulfilled". More generally, we present the following definition:

Definition We will say that our *smooth convexity assumptions* are fulfilled if:

- (M, g) denotes an n-dimensional ($n \geq 2$) smooth complete oriented connected Riemannian manifold or $(M, g) = (\mathbb{R}, |\cdot|)$, and $\Omega = M$.
- d denotes the induced geodesic distance on (M, g).
- $d\mu = \exp(-\psi)d\mathrm{vol}_M$, $\psi \in C^2(M)$, and as tensor fields on M:

$$\mathrm{Ric}_g + \mathrm{Hess}_g \psi \geq 0. \tag{1.1}$$

We will say that our *convexity assumptions* are fulfilled if μ can be approximated in total-variation by measures $\{\mu_m\}$ so that (Ω, d, μ_m) satisfy our smooth convexity assumptions.

The condition (1.1) is the well-known Curvature-Dimension condition $CD(0, \infty)$, introduced by Bakry and Émery in their influential paper [4] (in the more abstract framework of diffusion generators). Here Ric_g denotes the Ricci curvature tensor and Hess_g denotes the second covariant derivative. When the Ricci tensor satisfies a slightly relaxed condition $\mathrm{Ric}_g \geq -Kg$, $K \geq 0$, it was first shown by Buser [26] that the implication in Theorem 1.1 can be reversed. We only quote the $K = 0$ case, which in our setting reads:

Theorem 1.3 (Buser) *If μ is uniform on a closed n-dimensional manifold (M, g) and $\mathrm{Ric}_g \geq 0$ then $D_{Che} \geq c D_{Poin}$, where $c > 0$ is a universal numeric constant.*

The fact that the constant c above does not depend on the dimension n is quite remarkable. Buser's theorem was recently further generalized by M. Ledoux [56] (following the method developed by Bakry–Ledoux [5]) to the Bakry-Émery abstract setting. Again, we only quote the $CD(0, \infty)$ case:

Theorem 1.4 (Ledoux) *Under our smooth convexity assumptions $D_{Che} \geq c D_{Poin}$, where $c > 0$ is a universal numeric constant.*

1.3 Main theorem

How about reversing the implication in Theorem 1.2 under our convexity assumptions? This is one of the statements in our Main Theorem below. A second statement, which is much more surprising, concerns a very weak type of concentration inequality, which we introduce:

Definition The space (Ω, d, μ) is said to satisfy First-Moment concentration if:

$$\exists D > 0 \quad \text{such that} \quad \forall \text{ 1-Lipschitz } f \quad \left\| f - E_\mu f \right\|_{L_1(\mu)} \leq \frac{1}{D}. \tag{1.2}$$

The best possible constant D above is denoted by $D_{FM} = D_{FM}(\Omega, d, \mu)$.

Clearly, by the Markov-Chebyshev inequality, First-Moment concentration implies *linear* tail-decay:

$$\forall \text{ 1-Lipschitz } f \ \forall t > 0 \quad \mu(|f - E_\mu f| \geq t) \leq \frac{1}{D_{FM} t},$$

and decay slightly faster than linear implies (integrating by parts) First-Moment concentration. The First-Moment concentration is clearly a-priori *much weaker* than exponential concentration. Our Main Theorem, first announced in [65], asserts that under our convexity assumptions, not only is First-Moment concentration *equivalent* to exponential concentration, but in fact also to the a-priori stronger inequalities of Poincaré and Cheeger:

Theorem 1.5 *Under our convexity assumptions, the following statements are equivalent:*

1. *Cheeger's isoperimetric inequality (with D_{Che}).*
2. *Poincaré's inequality (with D_{Poin}).*
3. *Exponential concentration inequality (with D_{Exp}).*
4. *First Moment concentration inequality (with D_{FM}).*

The equivalence is in the sense that the constants above satisfy $D_{Che} \simeq D_{Poin} \simeq D_{Exp} \simeq D_{FM}$.

Here and below, $A \simeq B$ means that $C_1 B \leq A \leq C_2 B$, with $C_i > 0$ some universal numerical constants, independent of any other parameter, and in particular the dimension n. We will see in Sect. 4 that the use of the First-Moment is not essential in Statement (4); we may have required any *arbitrarily slow* uniform tail decay, instead of linear decay. In other words, if:

$$\exists \alpha : \mathbb{R}_+ \to [0,1] \quad \alpha(t) \to_{t\to\infty} 0 \quad \forall \text{1-Lipschitz } f \ \forall t > 0 \quad \mu(|f - E_\mu f| \geq t) \leq \alpha(t), \tag{1.3}$$

where α decays to 0 *arbitrarily slow*, we can deduce under our convexity assumptions that Lipschitz functions have in fact much faster *exponential* tail decay (with rate depending solely on α), and in addition the stronger inequalities of Poincaré and Cheeger, as above. In this sense, our result extends the well-known Kahane-Khinchine type inequalities in Convexity Theory (e.g. consequences of Borell's Lemma [23], see [67] for an overview) stating that *linear functionals* have comparable moments, ensuring exponential tail decay, to the same statement for the "worst" 1-Lipschitz function (see Remark 4.4).

The Main Theorem may also be interpreted as stating that under our convexity assumptions, there exists a single 1-Lipschitz function f whose level sets *on average* attain the minimum (up to constants) in Cheeger's isoperimetric inequality (see Sect. 4). In fact, one may choose this function to be of the form $f(x) = d(x, A)$, where A is some set with $\mu(A) \geq 1/2$. This is expressed in the following reformulation of the Main Theorem:

Theorem 1.6 *Under our convexity assumptions on (Ω, d, μ):*

$$D_{Che}(\Omega, d, \mu) \simeq \inf \left\{ \frac{1}{\int_\Omega d(x, A) d\mu}; A \subset \Omega, \mu(A) \geq 1/2 \right\}.$$

Equivalently, this is tantamount to saying that under our convexity assumptions, it is only necessary to use test functions of the form $f(x) = d(x, A)$ when testing (up to a universal numeric constant) for the spectral gap D^2_{Poin} in Poincaré's inequality. Clearly, without any further assumptions, all of the above statements are in general false.

1.4 Applications to spectral gap of convex domains

In Sect. 5, we deduce from our Main Theorem 1.5 several new results pertaining to the spectral gap of convex domains, and recover and extend numerous previously known results as well. We will formulate our results in Euclidean space $(\mathbb{R}^n, |\cdot|)$, even though they hold for the most part under our more general convexity assumptions.

For a bounded domain $\Omega \subset (\mathbb{R}^n, |\cdot|)$, let λ_Ω denote the uniform probability measure on Ω, and denote $D_{Poin}(\Omega) := D_{Poin}(\Omega, |\cdot|, \lambda_\Omega)$. As our main application, we deduce the following stability result for the spectral gap $D^2_{Poin}(\Omega)$ of the Neumann Laplacian on Ω under perturbations of the domain Ω. Clearly, there can be no stability result without some further assumptions, which we add in the form of convexity. We formulate the stability in terms of the Cheeger constant $D_{Che}(\Omega) := D_{Che}(\Omega, |\cdot|, \lambda_\Omega)$ (this is a-priori stronger than using $D_{Poin}(\Omega)$ by the Maz'ya–Cheeger inequality, but in fact equivalent in the class of convex domains by the Buser-Ledoux Theorems):

Theorem 1.7 *Let K, L denote two bounded convex domains in $(\mathbb{R}^n, |\cdot|)$. If*:

$$\mathrm{Vol}\,(K \cap L) \geq v_K \mathrm{Vol}\,(K)\,, \qquad \mathrm{Vol}\,(K \cap L) \geq v_L \mathrm{Vol}\,(L)\,,$$

then:

$$D_{Che}(K) \geq c \frac{v_K^2}{\log(1 + 1/v_L)} D_{Che}(L), \tag{1.4}$$

where $c > 0$ is some universal numeric constant.

Here Vol denotes the Lebesgue measure. In particular, we see that:

$$\mathrm{Vol}\,(K) \simeq \mathrm{Vol}\,(L) \simeq \mathrm{Vol}\,(K \cap L) \quad \Rightarrow \quad D_{Che}(K, |\cdot|, \lambda_K) \simeq D_{Che}(L, |\cdot|, \lambda_L).$$

Note that K, L satisfying the above condition can be very different geometrically (consider for instance a Euclidean ball of radius 1 and its intersection with a centered slab of width $10/\sqrt{n}$), and yet share essentially the same spectral gap. Also note that our stability result holds with respect to all possible Euclidean structures $|\cdot|$ simultaneously, since the assumption in the left-hand side above is independent of the Euclidean structure.

We also observe that the quantitative dependence on v_K, v_L in (1.4) is essentially best possible: the logarithmic dependence on $1/v_L$ is (up to numeric constants) optimal, and the quadratic dependence on v_K cannot be improved beyond linear (and is in fact optimal in some restricted range, see Example 5.6). In addition, Theorem 1.7 implies that when $\frac{1}{a} L \subset K \subset Lb$ with $a, b \geq 1$, $ab \leq 1 + \frac{c}{n}$, then $D_{Che}(K) \simeq D_{Che}(L)$. In fact, when $ab \leq 1 + \frac{s}{n}$ with $1 \leq s \leq n$, we obtain in Corollary 5.3 the best possible (up to numeric constants) quantitative bounds on $D_{Che}(K)/D_{Che}(L)$ as a function of s (see Example 5.7). To the best of our knowledge, no quantitative bounds on the stability of D_{Che} for convex domains under convex perturbations were previously known. Completely analogous stability results hold for log-concave probability measures as well (see Theorem 5.5). Another useful result which we deduce from our Main Theorem is that Cheeger's constant is preserved under maps which are not necessarily Lipschitz, but rather Lipschitz on average (see Theorem 5.9).

An intriguing conjecture of Kannan, Lovász and Simonovits [47] states that under a natural non-degeneracy condition on a bounded convex domain K in $(\mathbb{R}^n, |\cdot|)$, $D_{Che}(K) \simeq 1$, independently of the dimension n. The upper bound follows from standard Convexity Theory, but the lower bound is far from being resolved. There are many known lower bounds which provide dimension dependent results, and we are able to easily recover many of them, without appealing to the localization method used by Kannan–Lovász–Simonovits (which may be traced back to the work of Gromov–Milman [41]). These include results by Payne and Weinberger [76], Li and Yau [58] and Kannan–Lovász–Simonovits [47]. In fact, our estimates generalize to arbitrary Riemannian manifolds satisfying our convexity assumptions, whereas the localization method is confined to Euclidean space (and a few other special manifolds). Using our stability result, we are able to give a geometric proof of a recent lower bound on D_{Che} due to S. Bobkov [17]. We also note that a recent result of Sasha Sodin [81], implying that D_{Che} is uniformly bounded for the suitably scaled unit-balls of ℓ_p^n for $p \in [1, 2]$, is now an immediate consequence of our Main Theorem together with a result of Schechtman and Zinn [79].

1.5 Ingredients in proof of main theorem

All of the four statements in our Main Theorem 1.5 can be equivalently (up to universal constants) rewritten using a single unified framework in terms of (p, q) Poincaré inequalities:

Definition The space (Ω, d, μ) is said to satisfy a (p, q) Poincaré inequality if:

$$\exists D > 0 \quad \text{such that} \quad \forall f \in \mathcal{F} \quad D \left\| f - M_\mu f \right\|_{L_p(\mu)} \leq \left\| |\nabla f| \right\|_{L_q(\mu)} .$$

The best possible constant D above is denoted by $D_{p,q} = D_{p,q}(\Omega, d, \mu)$.

We prefer to use the median M_μ in our definition for reasons which will become apparent in Sect. 2. It is known and easy to establish that $D_{Poin} \simeq D_{2,2}$, $D_{Che} = D_{1,1}$, $D_{FM} \simeq D_{1,\infty}$, so our Main Theorem can be restated as the claim that all (p, q) Poincaré inequalities in the range $1 \leq p \leq q \leq \infty$ are equivalent under our convexity assumptions (see Theorem 2.4).

The convexity assumptions are used in an essential way in the proof of the Main Theorem in several separate places. First, we employ the $CD(0, \infty)$ condition via the semi-group gradient estimates used by Ledoux in his proof of Theorem 1.4. Contrary to previous approaches, which could only deduce isoperimetric information from functional inequalities with a $\left\| |\nabla f| \right\|_{L_q(\mu)}$ term with $q = 2$ (see [7, p. 3] and the references therein), we can handle arbitrary $q \geq 1$ (and although we do not pursue this direction here, more general Orlicz norms too). To demonstrate that our estimates are sharp, we remark that the isoperimetric inequalities we obtain are in fact equivalent (up to universal constants) to the (p, q) Poincaré inequalities used to derive them. This is summarized in Theorem 2.9, which generalizes Theorems 1.1, 1.2, 1.3 and 1.4 above into a single unified framework. Using this, we deduce from the First-Moment inequality ($p = 1, q = \infty$ above) that:

$$\tilde{I}(t) \geq c D_{FM} t^2 \quad \forall t \in [0, 1/2]. \tag{1.5}$$

To deduce Cheeger's isoperimetric inequality from (1.5), we need to use our convexity assumptions for the second time. We employ the following series of results in Riemannian Geometry, due to numerous groups of authors [11–13, 16, 34, 51, 70, 73, 82], who proved them under increasingly general conditions. A detailed survey of these results may be found in the Appendix. We learned about these results from the PhD Thesis of V. Bayle [12], which was referenced to us by Sasha Sodin, to whom we are indebted. In the formulation below, we use a slightly more general notion of smooth convexity assumptions, which is defined in Sect. 6.

Theorem 1.8 (Bavard–Pansu, Bérard–Besson–Gallot, Gallot, Morgan–Johnson, Sternberg–Zumbrun, Kuwert, Bayle–Rosales, Bayle, Morgan, Bobkov) *Under our generalized smooth convexity assumptions, the isoperimetric profile $I = I_{(\Omega,d,\mu)}$ is concave on* $(0, 1)$. *Moreover, when μ is in addition uniform on $\Omega \subset (M, g)$, then $I^{n/(n-1)}$ is concave on* $[0, 1]$, *where n is the dimension of M.*

It is not hard to show (see Sect. 6) that the isoperimetric profile I is continuous under very general assumptions. It then follows by a general argument (e.g. Corollary 6.5) that I must be symmetric about the point $1/2$. Hence, the concavity of I implies that $D_{Che} = 2I(1/2)$ under our convexity assumptions. It is then immediate to deduce Cheeger's isoperimetric inequality from (1.5). In fact, a stronger statement can be deduced when μ is uniform on Ω (see Remark 2.11).

A final ingredient in the proof is an approximation argument to handle non-smooth densities, which are typical in applications as well as essential for handling uniform measures on bounded domains (with possibly non-smooth boundaries). Contrary to many results in Convexity Theory, where approximation arguments are standard, easy and usually omitted, the isoperimetric profile and the Cheeger constant are delicate objects, which in general are *not stable* under approximation in the natural total-variation metric (see Sect. 6). We therefore employ our convexity assumptions one last time, and provide in Sect. 6 a careful argument for deducing the Main Theorem 1.5 without any smoothness assumptions, and a different approximation procedure for extending Theorem 1.8, which in particular applies to the entire class of log-concave measures in Euclidean space.

The rest of this work is organized as follows. In Sect. 2, we reformulate the Main Theorem in terms of an equivalence between (p, q) Poincaré inequalities, and using Theorem 1.8, reduce it to the statement of Theorem 2.9. The semi-group argument for proving Theorem 2.9 is described in Sect. 3. Further interpretations and an extension of the Main Theorem are described in Sect. 4. Applications for the spectral gap under our convexity assumptions are described in Sect. 5. We conclude with an approximation argument for disposing of our smoothness assumptions in Sect. 6, and an Appendix describing in more detail the results summarized in the statement of Theorem 1.8.

2 (p, q) Poincaré inequalities

We start by rewriting some of the statements of the Main Theorem 1.5.

We will use the following notation. A function $N : \mathbb{R}_+ \to \mathbb{R}_+$ will be called a Young function if $N(0) = 0$ and N is convex increasing. Besides the classical Young functions t^p $(p \geq 1)$, we will also frequently use the function $\Psi_1(t) = \exp(t) - 1$. Given a Young function N, the Orlicz norm $N(\mu)$ associated to N is defined as:

$$\|f\|_{N(\mu)} := \inf\left\{v > 0; \int_\Omega N(|f|/v)d\mu \leq 1\right\}.$$

Lemma 2.1 *Let $N(\mu)$ denote an Orlicz norm associated to the Young function N. Then*:

$$\frac{1}{2}\left\|f - E_\mu f\right\|_{N(\mu)} \leq \left\|f - M_\mu f\right\|_{N(\mu)} \leq 3\left\|f - E_\mu f\right\|_{N(\mu)}.$$

Proof Note that $\|1\|_{N(\mu)} = 1/N^{-1}(1)$. First, by Jensen's inequality (applied twice):

$$\left|E_\mu f - M_\mu f\right| \leq E_\mu(\left|f - M_\mu f\right|) \leq N^{-1}(1)\left\|f - M_\mu f\right\|_{N(\mu)},$$

hence:

$$\left\| f - E_\mu f \right\|_{N(\mu)} \leq \left\| f - M_\mu f \right\|_{N(\mu)} + \frac{|E_\mu f - M_\mu f|}{N^{-1}(1)} \leq 2 \left\| f - M_\mu f \right\|_{N(\mu)}.$$

Next, we may assume that $M_\mu f \geq E_\mu f$ (otherwise exchange f by $-f$). By the Markov-Chebyshev inequality:

$$\frac{1}{2} \leq \mu(f \geq M_\mu f) \leq \mu(|f - E_\mu f| \geq M_\mu f - E_\mu f) \leq 1/N\left(\frac{M_\mu f - E_\mu f}{\left\| f - E_\mu f \right\|_{N(\mu)}}\right),$$

hence:

$$\left| M_\mu f - E_\mu f \right| \leq N^{-1}(2) \left\| f - E_\mu f \right\|_{N(\mu)},$$

and we deduce that:

$$\left\| f - M_\mu f \right\|_{N(\mu)} \leq \left\| f - E_\mu f \right\|_{N(\mu)} + \frac{\left| E_\mu f - M_\mu f \right|}{N^{-1}(1)} \leq \left(1 + \frac{N^{-1}(2)}{N^{-1}(1)}\right) \left\| f - E_\mu f \right\|_{N(\mu)}.$$

We conclude by noting that $\frac{N^{-1}(2)}{N^{-1}(1)} \leq 2$ since N is convex. □

The last lemma implies that we can pass back and forth between using the median M_μ and the expectation E_μ when excluding constant functions in our functional inequalities, at the expense of losing a universal constant. We therefore see that Poincaré's inequality is equivalent (up to constants) to the inequality:

$$\forall f \in \mathcal{F} \quad D^M_{Poin} \left\| f - M_\mu f \right\|_{L_2(\mu)} \leq \| |\nabla f| \|_{L_2(\mu)} \tag{2.1}$$

(and in fact in this case one clearly has $D_{Poin} \geq D^M_{Poin}$). The next lemma, due to Maz'ya [63] and Federer and Fleming [32] (see also [19] for a careful derivation), rewrites Cheeger's isoperimetric inequality in functional form:

Lemma 2.2 (Maz'ya, Federer–Fleming, Bobkov–Houdré) *Cheeger's isoperimetric inequality* (*with D_{Che}*) *holds iff*:

$$\forall f \in \mathcal{F} \quad D_{Che} \left\| f - M_\mu f \right\|_{L_1(\mu)} \leq \| |\nabla f| \|_{L_1(\mu)}. \tag{2.2}$$

Sketch of Proof following Bobkov–Houdré [19] It is easy to show that Cheeger's isoperimetric inequality is recovered by applying (2.2) to Lipschitz functions which approximate χ_A, the characteristic function of a Borel set A, in an appropriate sense. Conversely, the co-area formula, which for general metric probability spaces becomes an inequality (see [19]), implies for $f \in \mathcal{F}$ with $M_\mu f = 0$:

$$\begin{aligned} \int |\nabla f| \, d\mu &\geq \int_{-\infty}^{\infty} \mu^+ \{f > t\} \, dt \\ &\geq D_{Che} \left(\int_{-\infty}^{0} (1 - \mu\{f > t\}) dt + \int_0^\infty \mu\{f > t\} \, dt \right) = D_{Che} \int |f| \, d\mu. \end{aligned}$$ □

Since for a 1-Lipschitz function f, $\||\nabla f|\|_{L_\infty(\mu)} \leq 1$, our First-Moment inequality is clearly equivalent to:

$$\forall f \in \mathcal{F} \quad D^M_{FM} \left\| f - M_\mu f \right\|_{L_1(\mu)} \leq \||\nabla f|\|_{L_\infty(\mu)}, \tag{2.3}$$

in the sense that $D_{FM} \simeq D^M_{FM}$ where D^M_{FM} is the best constant above.

Remark 2.3 The above functional reformulations remain valid for general metric probability spaces (Ω, d, μ), in which case we interpret $|\nabla f|$ for any $f \in \mathcal{F}$ as the following Borel function:

$$|\nabla f|(x) := \limsup_{d(y,x)\to 0+} \frac{|f(y) - f(x)|}{d(x, y)}$$

(and we define it as 0 if x is an isolated point—see [19, pp. 184, 189] for more details).

With the above reformulations (2.1), (2.2), (2.3) serving as motivation, the reasons behind our definition of (p, q) Poincaré inequalities in the Introduction are now clear. Note that $D_{Che} = D_{1,1}$, $D^M_{Poin} = D_{2,2}$ and $D^M_{FM} = D_{1,\infty}$. We can now restate our Main Theorem 1.5 as follows:

Theorem 2.4 *Under our convexity assumptions, all (p, q) Poincaré inequalities are equivalent in the range $1 \leq p \leq q \leq \infty$. More precisely, for any other $1 \leq p' \leq q' \leq \infty$:*

$$D_{p,q} \leq C p' D_{p',q'},$$

where $C > 0$ is a universal constant.

In fact, a more precise dependence on p and p' may be obtained in some cases. For instance, clearly $D_{p',q'} \geq D_{p,q}$ if $p' \leq p$ and $q' \geq q$ without any further convexity assumptions (by Jensen's inequality), so we see that the First-Moment inequality ($(1, \infty)$ case) is the weakest among all (p, q) Poincaré inequalities in the above range. Another immediate observation is given by:

Proposition 2.5 *Let $0 < p \leq p' \leq \infty$ and $0 < q \leq q' \leq \infty$ be such that:*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{p'} - \frac{1}{q'}.$$

Then without *any further convexity assumptions, $D_{p',q'} \geq \frac{p}{p'} D_{p,q}$.*

Proof Let $g \in \mathcal{F}$ denote a function with $M_\mu g = 0$. Define $f = \mathrm{sign}(g)|g|^{p'/p}$, and apply the (p, q) Poincaré inequality to f. Clearly $M_\mu f = 0$, so we obtain by Hölder's inequality:

$$D_{p,q} \|g\|^{p'/p}_{L_{p'}(\mu)} \leq \frac{p'}{p} \left\| |g|^{p'/p-1} |\nabla g| \right\|_{L_q(\mu)} \leq \frac{p'}{p} \|g\|^{p'/p-1}_{L_{p'}(\mu)} \||\nabla g|\|_{L_{q'}(\mu)},$$

from which the assertion follows. □

Corollary 2.6 *Maz'ya–Cheeger inequality: $D_{Poin} \geq D_{Che}/2$.*

Proof

$$D_{Poin} \geq D^M_{Poin} = D_{2,2} \geq D_{1,1}/2 = D_{Che}/2.$$ □

Corollary 2.7 *Gromov–Milman inequality*: $D_{Exp} \geq c D_{Poin}$.

Proof Since $D_{Poin} \simeq D_{2,2}$, we conclude by Proposition 2.5 that $D_{p,p} \geq c D_{Poin}/p$ for every $2 \leq p \leq \infty$. Let f be a 1-Lipschitz function. It is elementary to show (e.g. [45]) that $1/D_{Exp}(f)$ is equivalent (to within universal constants) to $\|f - E_\mu f\|_{\Psi_1(\mu)}$, and that $\|g\|_{\Psi_1(\mu)}$ is in turn equivalent to $\sup_{p\geq 1} \|g\|_{L_p(\mu)}/p$. Employing Lemma 2.1 and using the (p,p) Poincaré inequalities:

$$\frac{1}{D_{Exp}(f)} \simeq \left\| f - E_\mu f \right\|_{\Psi_1(\mu)} \simeq \left\| f - M_\mu f \right\|_{\Psi_1(\mu)} \simeq \sup_{p\geq 1} \frac{\left\| f - M_\mu f \right\|_{L_p(\mu)}}{p}$$
$$\leq \sup_{p\geq 1} \frac{\left|\left|\left|\nabla f\right|\right|\right|_{L_p(\mu)}}{\min(D_{2,2}, pD_{p,p})} \leq \frac{C}{D_{Poin}} \sup_{p\geq 1} \left|\left|\left|\nabla f\right|\right|\right|_{L_p(\mu)} = \frac{C}{D_{Poin}},$$

since f was assumed 1-Lipschitz. Taking supremum on all such functions f, we obtain the conclusion. □

Remark 2.8 The exact same proof shows that $D_{Exp} \geq c_r D_{r,r}$, for arbitrary $r \geq 1$.

We have seen that passing from (p,q) to (p',q') is manageable if $q' \geq q$ (perhaps under some additional assumptions on p, p') without any convexity assumptions. Unfortunately, we are interested in the case $q' < q$, for which an analogous statement to Proposition 2.5 is simply false without any additional assumptions (counter examples are easy to construct, as in the Introduction). Our first ingredient in the proof of Theorem 2.4 states that our convexity assumptions already suffice to extend Proposition 2.5 to the case $q' < q$, $p' < p$:

Theorem 2.9 *Let* $0 < p \leq \infty$, $1 \leq q \leq \infty$, *and set* $r = 1 + \frac{1}{p} - \frac{1}{q}$. *Assume that* $\frac{1}{2} \leq r \leq 2$. *Then under our smooth convexity assumptions*, *the following statements are equivalent*:

1.
$$\forall f \in \mathcal{F} \quad D_{p,q} \left\| f - M_\mu f \right\|_{L_p(\mu)} \leq \left|\left|\left|\nabla f\right|\right|\right|_{L_q(\mu)},$$

2.
$$\tilde{I}(t) \geq D'_r t^r \quad \forall t \in [0, 1/2],$$

where the best constants $D_{p,q}$ *and* D'_r *above satisfy*:

$$c_1 D_{p,q} \leq D'_r \leq c_2 p D_{p,q}, \tag{2.4}$$

for some universal constants $c_1, c_2 > 0$.

In fact, *the direction* (2) $\Rightarrow$ (1) *holds for* $p \geq q$ *without any convexity assumptions*.

Note that when $p = q = 2$, the direction (2) $\Rightarrow$ (1) reduces (up to constants) to Theorem 1.1 (Maz'ya–Cheeger inequality), and the direction (1) $\Rightarrow$ (2) to the Buser–Ledoux Theorems 1.3, 1.4. A generalization of Theorem 2.9 involving general Orlicz norms will be derived in [64].

There is essentially no novel content in the direction (2) $\Rightarrow$ (1), which follows from the methods of Maz'ya [63, p. 89] and Federer–Fleming [32] (see also [20]). These authors deduced the optimal constant in the Gagliardo inequality ($q = 1$, $p = \frac{n}{n-1}$), as well as the Sobolev inequalities ($1 < q < n$, $p = \frac{qn}{n-q}$), from the isoperimetric inequality in $\mathbb{R}^n$ ($r = \frac{n-1}{n}$), using the following clever generalization of Lemma 2.2:

Proposition 2.10 (Maz'ya, Federer–Fleming, Bobkov–Houdré) *Let* $0 < r \le 1$. *Without any convexity assumptions*, *the* $(1/r, 1)$ *Poincaré inequality*:

$$\forall f \in \mathcal{F} \quad D \left\| f - M_\mu f \right\|_{L_{1/r}(\mu)} \le \left\| |\nabla f| \right\|_{L_1(\mu)}$$

is equivalent to the following isoperimetric inequality:

$$\tilde{I}(t) \ge D t^r \quad \forall t \in [0, 1/2].$$

Combining Propositions 2.10 and 2.5, the direction (2) $\Rightarrow$ (1) for $p \ge q$ (equivalently $r \le 1$) immediately follows without any further assumptions. For the case $p < q$, it is almost possible to avoid using the convexity assumptions, but not completely. Instead, we employ Theorem 1.8 on the concavity of I under our (smooth) convexity assumptions, and deduce from (2) that in fact $\tilde{I}(t) \ge \frac{D'_r}{2^{r-1}} t$. The latter is equivalent by Lemma 2.2 to the statement $D_{1,1} \ge \frac{D'_r}{2^{r-1}}$, and by using Proposition 2.5 and Jensen's inequality, we deduce:

$$D_{p,q} \ge D_{p,p} \ge \frac{D_{1,1}}{p} \ge \frac{D'_r}{2^{r-1} p} \ge \frac{D'_r}{2p}.$$

The proof of (2) $\Rightarrow$ (1) is thus complete.

Before proceeding to the proof of the direction (1) $\Rightarrow$ (2) (this will be the focus of the next section), let us recall how Theorem 2.9 coupled with Theorem 1.8 conclude the proof of Theorem 2.4 and hence of our Main Theorem 1.5:

Proof of Theorem 2.4 By an approximation argument we develop in Sect. 6, it is enough to prove the theorem under our *smooth* convexity assumptions.

By Jensen's inequality, $D_{1,\infty} \ge D_{p,q}$ in the range $1 \le p \le q \le \infty$. Employing our (smooth) convexity assumptions, the direction (1) $\Rightarrow$ (2) of Theorem 2.9 implies:

$$\tilde{I}(t) \ge c D_{1,\infty} t^2 \quad \forall t \in [0, 1/2]. \tag{2.5}$$

Using our (smooth) convexity assumptions for the second time, Theorem 1.8 asserts that I is concave on $(0, 1)$. Since I is also symmetric about $1/2$ (see Corollary 6.5), we immediately deduce that:

$$\tilde{I}(t) \ge \frac{c}{2} D_{1,\infty} t \quad \forall t \in [0, 1/2],$$

which is exactly Cheeger's isoperimetric inequality, and is identical to stating $D_{1,1} \ge \frac{c}{2} D_{1,\infty}$. Using Proposition 2.5 and Jensen's inequality if necessary, we can pass from this to an arbitrary (p', q') inequality in the range $1 \le p' \le q' \le \infty$. $\square$

Remark 2.11 Note that when μ is the uniform measure on Ω, Theorem 1.8 in fact ensures that $I^{\frac{n}{n-1}}$ is concave, so we may deduce from (2.5) that in fact:

$$\tilde{I}(t) \ge \frac{c}{2^{\frac{n+1}{n}}} D_{1,\infty} t^{\frac{n-1}{n}} \quad \forall t \in [0, 1/2].$$

Proposition 2.10 implies that the latter isoperimetric inequality is equivalent to a $(\frac{n}{n-1}, 1)$ Poincaré inequality. Hence, it is clear that in this case, both our Main Theorem 1.5 and Theorem 2.4 can be strengthened.

3 The semi-group argument

In this section, we prove the direction (1) $\Rightarrow$ (2) of Theorem 2.9. Our proof closely follows Ledoux's proof [56] of Theorem 1.4.

Given a smooth complete oriented connected Riemannian manifold $\Omega = (M, g)$ equipped with a probability measure μ with density $d\mu = \exp(-\psi) d\mathrm{vol}_M$, $\psi \in C^2(M)$, we define the associated Laplacian $\Delta_{(\Omega,\mu)}$ by:

$$\Delta_{(\Omega,\mu)} := \Delta_\Omega - \nabla\psi \cdot \nabla, \tag{3.1}$$

where Δ_Ω is the usual Laplace-Beltrami operator on Ω. $\Delta_{(\Omega,\mu)}$ acts on $\mathcal{B}(\Omega)$, the space of bounded smooth real-valued functions on Ω. Let $(P_t)_{t\geq 0}$ denote the semi-group associated to the diffusion process with infinitesimal generator $\Delta_{(\Omega,\mu)}$ (cf. [30, 54]), characterized by the following system of second order differential equations:

$$\frac{d}{dt} P_t(f) = \Delta_{(\Omega,\mu)}(P_t(f)) \qquad P_0(f) = f \quad \forall f \in \mathcal{B}(\Omega).$$

For each $t \geq 0$, $P_t : \mathcal{B}(\Omega) \to \mathcal{B}(\Omega)$ is a bounded linear operator and its action naturally extends to the entire $L_p(\mu)$ spaces ($p \geq 1$). We collect several elementary properties of these operators:

- $P_t(1) = 1$.
- $f \geq 0 \Rightarrow P_t(f) \geq 0$.
- $\int P_t(f) d\mu = \int f d\mu$.
- $|P_t(f)|^p \leq P_t(|f|^p)$ for all $p \geq 1$.

The following crucial dimension-free reverse Poincaré inequality was shown by Bakry and Ledoux in [5, Lemma 4.2], extending Ledoux's approach [53] for proving Buser's Theorem (see also [5, Lemma 2.4], [56, Lemma 5.1]). It may also be interpreted as a weak, dimension-free form of the Li–Yau parabolic gradient inequality [59].

Lemma 3.1 (Bakry–Ledoux) *Assume that the following Bakry-Émery Curvature-Dimension condition holds on* Ω:

$$\mathrm{Ric}_g + \mathrm{Hess}_g \psi \geq -K g, \quad K \geq 0. \tag{3.2}$$

Then for any $t \geq 0$ *and* $f \in \mathcal{B}(\Omega)$, *we have*:

$$c(t) \, |\nabla P_t(f)|^2 \leq P_t(f^2) - (P_t(f))^2$$

pointwise, *where*:

$$c(t) = \frac{1 - \exp(-2Kt)}{K} \; (= 2t \text{ if } K = 0).$$

In fact, the proof of this lemma is very general and extends to the abstract framework of diffusion generators, as developed by Bakry and Émery [4]. We comment that in the Riemannian setting, it is known [77] (see also [44, 84]) that the gradient estimate of Lemma 3.1 is preserved when restricting to a locally convex domain (as defined in the Appendix) with smooth boundary; we refer to Sturm [83, Proposition 4.15] for a general statement about closedness of the Bakry-Émery Curvature-Dimension condition in an arbitrary metric probability space. The above lemma therefore holds under more general conditions, namely when μ is supported on a locally convex domain $\Omega \subset (M, g)$ with C^2 boundary, and $d\mu|_\Omega = \exp(-\psi) d\mathrm{vol}_M|_\Omega$, $\psi \in C^2(\overline{\Omega})$. In this case, Δ_Ω in (3.1) denotes the Neumann Laplacian on Ω, $\mathcal{B}(\Omega)$ denotes the space of bounded smooth real-valued functions on $\overline{\Omega}$ satisfying Neumann's boundary condition on $\partial\Omega$, and Lemma 3.1 remains valid.

Our convexity assumptions are that $K = 0$ in Lemma 3.1, and this is what we will henceforth assume. It is clear that our results in this section may be extended to the case of $K > 0$, but we do not pursue this direction in this work.

From Lemma 3.1, it is immediate that for any $2 \leq q \leq \infty$:

$$\forall f \in \mathcal{B}(\Omega) \quad \| |\nabla P_t(f)| \|_{L_q(\mu)} \leq \frac{1}{\sqrt{2t}} \|f\|_{L_q(\mu)}, \tag{3.3}$$

and using $q = \infty$, Ledoux easily deduces the following dual statement [56, (5.5)]:

Corollary 3.2 (Ledoux)

$$\forall f \in \mathcal{B}(\Omega) \quad \|f - P_t(f)\|_{L_1(\mu)} \leq \sqrt{2t} \, \| |\nabla f| \|_{L_1(\mu)}. \tag{3.4}$$

Proof of (1) $\Rightarrow$ (2) *of Theorem 2.9* First, our assumption on the range of r implies that by applying Proposition 2.5 if necessary, we may assume that $p \geq 1, q \geq 2$ at the expense of an additional universal constant appearing in (2.4). An additional universal constant will appear on account of Lemma 2.1, with which we pass to E_μ instead of M_μ in (1), so our assumption now reads:

$$p \geq 1, q \geq 2, \quad \forall f \in \mathcal{F} \quad D_{p,q} \left\| f - E_\mu f \right\|_{L_p(\mu)} \leq \| |\nabla f| \|_{L_q(\mu)}. \tag{3.5}$$

Let A denote an arbitrary Borel set in Ω, and let $\chi_{A,\varepsilon}(x) := (1 - \frac{1}{\varepsilon} d_g(x, A)) \vee 0$ denote a continuous approximation in Ω to the characteristic function χ_A of A. Clearly:

$$\frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon} \geq \int \left|\nabla \chi_{A,\varepsilon}\right| d\mu.$$

Applying Corollary 3.2 to functions in $\mathcal{B}(\Omega)$ which approximate $\chi_{A,\varepsilon}$ (in say $W^{1,1}(\Omega, \mu)$) and passing to the limit inferior as $\varepsilon \to 0$, it follows that:

$$\sqrt{2t} \mu^+(A) \geq \int |\chi_A - P_t(\chi_A)| \, d\mu.$$

We start by rewriting the right hand side above as:

$$\begin{aligned} &\int_A (1 - P_t(\chi_A)) d\mu + \int_{\Omega \setminus A} P_t(\chi_A) d\mu \\ &\quad = 2\left(\mu(A) - \int_A P_t(\chi_A) d\mu\right) \end{aligned}$$

$$= 2\left(\mu(A)(1-\mu(A)) - \int_\Omega (P_t(\chi_A) - \mu(A))(\chi_A - \mu(A))d\mu\right).$$

Note that by Hölder's inequality (recall that $p \geq 1$) and our assumption (3.5):

$$\int_\Omega (P_t(\chi_A) - \mu(A))(\chi_A - \mu(A))d\mu \leq \|P_t(\chi_A) - \mu(A)\|_{L_p(\mu)} \|\chi_A - \mu(A)\|_{L_{p^*}(\mu)}$$
$$\leq D_{p,q}^{-1} \||\nabla P_t(\chi_A)|\|_{L_q(\mu)} \|\chi_A - \mu(A)\|_{L_{p^*}(\mu)}.$$

Using (3.3) (recall that $q \geq 2$) to estimate $\||\nabla P_t(\chi_A)|\|_{L_q(\mu)}$, we conclude that:

$$\sqrt{2t}\mu^+(A) \geq 2\left(\mu(A)(1-\mu(A)) - \frac{1}{\sqrt{2t}D_{p,q}} \|\chi_A - \mu(A)\|_{L_q(\mu)} \|\chi_A - \mu(A)\|_{L_{p^*}(\mu)}\right). \tag{3.6}$$

We may now optimize on t. Using the rough estimate:

$$\|\chi_A - \mu(A)\|_{L_s(\mu)} \leq 2\,(\mu(A)(1-\mu(A)))^{1/s}$$

for $s \geq 1$, we evaluate (3.6) at time:

$$t = \frac{32}{D_{p,q}^2}\,(\mu(A)(1-\mu(A)))^{2(1/q-1/p)}$$

and deduce:

$$\mu^+(A) \geq \frac{D_{p,q}}{8}\,(\mu(A)(1-\mu(A)))^{2-1/q-1/p^*} \geq \frac{D_{p,q}}{8\cdot 2^r}\min(\mu(A), 1-\mu(A))^r,$$

where $r = 2 - 1/q - 1/p^* = 1 + 1/p - 1/q$. Since $r \leq 2$, this concludes the proof. □

Remark 3.3 As evident from the proof, for deducing the direction (1) ⇒ (2) of Theorem 2.9, the definition of smooth convexity assumptions given in the Introduction may be extended to encompass the more general case treated in this section. Moreover, it is possible to provide an approximation argument for deducing this direction without any smoothness assumptions. We provide the argument in [64] and omit it here, since it is not required for the results of this work.

4 Interpretations and extensions

In this section, we provide some further interpretations and extensions of our Main Theorem, which will also be needed for the applications of the next section. We assume throughout this section that our convexity assumptions on (Ω, d, μ) are satisfied.

Lemma 2.2 demonstrates that if A is a set with $\mu(A) \leq 1/2$ on which the minimal ratio $D_{Che} = \mu^+(A)/\mu(A)$ in Cheeger's isoperimetric inequality is attained (or nearly attained), then the function $f = \chi_A$ (or the sequence of Lipschitz functions which approximate it) attains the same (nearly) minimal ratio

$$\int |\nabla f|\,d\mu \Big/ \int |f|\,d\mu \tag{4.1}$$

among all functions $f \in \mathcal{F}$ with $M_\mu f = 0$. Clearly χ_A (or its approximating sequence) is far from being 1-Lipschitz. If on the other hand we define:

$$f(x) = d(x, \Omega \setminus A), \tag{4.2}$$

which *is* a 1-Lipschitz function, it is not clear that it will have a small ratio in (4.1). Our Main Theorem 1.5 (together with Lemma 2.1) states that under our convexity assumptions, any 1-Lipschitz function f_0 on (Ω, d) with $M_\mu f_0 = 0$ which is (essentially) optimal in the First-Moment inequality (say $\int |f_0| d\mu \geq 1/(3D^M_{FM})$), also essentially minimizes the ratio in (4.1). Moreover, using the co-area formula as in Lemma 2.2 and applying our Main Theorem, we have:

$$D_{Che} \leq \frac{\int_{-\infty}^{\infty} \mu^+\{f_0 > t\}\, dt}{\int_{-\infty}^{\infty} \min(\mu\{f_0 > t\}, 1 - \mu\{f_0 > t\}) dt} \leq \frac{\int |\nabla f_0|\, d\mu}{\int |f_0|\, d\mu} \leq 3D^M_{FM} \leq C D_{Che},$$

from which we also see that the ratio $\mu^+(A_t)/\min(\mu(A_t), 1 - \mu(A_t))$ for the "average" level set A_t of f_0 is essentially D_{Che}, the smallest possible.

Theorem 1.6 from the Introduction states that f_0 as above may in fact be chosen to be of the form (4.2).

Proof of Theorem 1.6 Given a Borel set $A \subset \Omega$ with $\mu(A) \geq 1/2$, we denote $g_A(x) = d(x, A)$. Clearly g_A is 1-Lipschitz and $M_\mu g_A = 0$, so one direction follows immediately by Lemma 2.2:

$$D_{Che}(\Omega, d, \mu) \leq \frac{\int |\nabla g_A|\, d\mu}{\int |g_A|\, d\mu} \leq \frac{1/2}{\int d(x, A) d\mu}.$$

For the other direction, we employ our Main Theorem (and Lemma 2.1):

$$D_{Che}(\Omega, d, \mu) \geq c D^M_{FM}(\Omega, d, \mu) = \inf \frac{c}{\int |f| d\mu},$$

where the infimum is over all 1-Lipschitz functions f on (Ω, d) with $M_\mu f = 0$. Denoting $A_1 = \{f \leq 0\}$, $A_2 = \{f \geq 0\}$, we have $\mu(A_i) \geq 1/2$, $i = 1, 2$. By continuity of f, $f|_{\partial A_1} \equiv 0$, $f|_{\partial A_2} \equiv 0$ (even though it is possible that $\partial A_1 \neq \partial A_2$), and since it is 1-Lipschitz:

$$\int |f| d\mu \leq \int_{\Omega \setminus A_2} d(x, \partial A_2) d\mu + \int_{\Omega \setminus A_1} d(x, \partial A_1) d\mu = \int d(x, A_2) d\mu + \int d(x, A_1) d\mu.$$

This concludes the proof. $\square$

The next proposition will prove to be very useful for the applications of the next section. We start with some notations. Given a Borel function f on a Borel probability space (Ω, μ) and $\delta \in [0, 1]$, let us denote by $Q_\delta(f) = Q_{\mu,\delta}(f)$ the δ-*quantile* of f:

$$Q_\delta(f) := \inf \{q \in \mathbb{R}; \mu\{f \leq q\} \geq \delta\}.$$

Let us also recall an inequality due to Paley and Zygmund [75] (see also [46, Chap. 2]), which in its simplest form reads as follows:

Lemma 4.1 (Paley–Zygmund) *Let f denote a Borel function on Ω, and assume that*:

$$\exists D > 0 \quad \text{such that} \quad \|f\|_{L_2(\mu)} \leq D \|f\|_{L_1(\mu)} < \infty.$$

Then for any $\theta \in (0,1)$, denoting $\varepsilon(\theta) = (1-\theta)^2/D^2$, one has $Q_{1-\varepsilon(\theta)}(|f|) \geq \theta \|f\|_{L_1(\mu)}$.

Proposition 4.2 *Let f_0 denote a 1-Lipschitz function with either $M_\mu f_0 = 0$ and $\|f_0\|_{L_1(\mu)} \geq 1/(2D^M_{FM})$ or $E_\mu f_0 = 0$ and $\|f_0\|_{L_1(\mu)} \geq 1/(2D_{FM})$. Then*:

$$\|f_0\|_{\Psi_1(\mu)} \leq C_0 \|f_0\|_{L_1(\mu)}, \tag{4.3}$$

and consequently:

$$Q_{1-\varepsilon_0}(|f_0|) \geq \|f_0\|_{L_1(\mu)}/2, \tag{4.4}$$

for some universal constants $C_0 > 0$ and $0 < \varepsilon_0 < 1$.

Proof Proceeding as in Corollary 2.7, and using Lemma 2.1 and the Main Theorem:

$$\|f_0\|_{\Psi_1(\mu)} \simeq \left\|f_0 - E_\mu f_0\right\|_{\Psi_1(\mu)} \simeq \frac{1}{D_{Exp(f_0)}} \leq \frac{1}{D_{Exp}} \leq \frac{C}{\max(D_{FM}, D^M_{FM})} \leq 2C \|f_0\|_{L_1(\mu)}.$$

Consequently, it is easy to check that:

$$\|f_0\|_{L_2(\mu)} \leq \sqrt{2} \|f_0\|_{\Psi_1(\mu)} \leq D_0 \|f_0\|_{L_1(\mu)},$$

for some universal constant $D_0 > 0$, and (4.4) follows by Lemma 4.1 (with $\theta = 1/2$). Note that our convexity assumptions necessarily imply that $\|f_0\|_{L_1(\mu)} < \infty$ (see Lemma 6.13), so the appeal to Lemma 4.1 is indeed legitimate. □

Corollary 4.3 *An arbitrarily slow uniform tail decay condition (1.3) implies any of the statements of the Main Theorem 1.5, with $D_{Che}, D_{Poin}, D_{Exp}, D_{FM}$ depending solely on α. Moreover, $E_\mu f$ in (1.3) may be replaced by $M_\mu f$.*

Proof Given a 1-Lipschitz function f_0 satisfying either of the assumptions of Proposition 4.2, these and (4.4) imply that:

$$\frac{1}{2\max(D_{FM}, D^M_{FM})} \leq \|f_0\|_{L_1(\mu)} \leq 2Q_{1-\varepsilon_0}(|f_0|).$$

Consequently, the tail decay condition (1.3) (whether stated with $E_\mu f$ or $M_\mu f$) ensures that $\max(D_{FM}, D^M_{FM}) \geq 1/(4\alpha^{-1}(\varepsilon_0)) > 0$, so by Lemma 2.1 the First-Moment concentration inequality is satisfied, from which the other statements of the Main Theorem follow. □

Remark 4.4 Using standard results in Convexity Theory (e.g. Borell's Lemma [23]), it is well known that when μ is a log-concave measure on $\mathbb{R}^n$ and f_0 is a *linear* (more generally, convex homogeneous) functional, then (4.3) is satisfied with some universal constant $C > 0$. In this sense, our essentially optimal 1-Lipschitz function f_0 behaves like linear functionals. A conjecture of Kannan, Lovász and Simonovits which will be described in Sect. 5, states this even more explicitly: linear functionals are essentially optimal in the (1, 1) or (2, 2) Poincaré inequalities. Using our Main Theorem, we now see that this conjecture is equivalent to stating that linear functionals are essentially optimal in the exponential concentration and First-Moment inequalities. In this sense, the Main Theorem may be thought of as a qualitative step towards resolving the conjecture: an essentially optimal function above has the form $f_0 = d(x, A)$ with $\mu(A) \geq 1/2$, and it remains to show that one can choose A to be a half-space (so that f_0 becomes linear).

5 Applications to spectral gap of convex domains

In this section, we provide several applications of our Main Theorem pertaining to the spectral gap $D^2_{Poin}(\Omega, d, \mu)$ of metric probability spaces satisfying our convexity assumptions. The results will be formulated in terms of the Cheeger constant $D_{Che}(\Omega, d, \mu)$, which by the Maz'ya–Cheeger inequality (Theorem 1.1) and the Buser-Ledoux Theorems (1.3 and 1.4) is equivalent to $D_{Poin}(\Omega, d, \mu)$ under these assumptions (see also the approximation arguments of Sect. 6 to handle non-smooth domains and densities). We will mostly restrict our attention to the case of $\mathbb{R}^n$ with some fixed Euclidean structure $|\cdot|$, although in some places we will mention our result in its full generality on Riemannian manifolds.

Given a bounded domain $\Omega \subset (M, g)$, we denote the uniform probability measure on Ω by $\lambda_\Omega := \frac{\mathrm{vol}_M|_\Omega}{\mathrm{vol}_M(\Omega)}$. We will write $D_{Che}(\Omega)$, $D_{FM}(\Omega)$, and so on, to denote $D_{Che}(\Omega, |\cdot|, \lambda_\Omega)$, $D_{FM}(\Omega, |\cdot|, \lambda_\Omega)$ for short. We will say that Ω is a convex body if Ω is a convex bounded domain in $(\mathbb{R}^n, |\cdot|)$. We will sometimes not distinguish between Ω and its closure $\overline{\Omega}$.

5.1 Stability of D_{Che} under perturbations

First, we would like to obtain a stability result for $D_{Che}(\Omega)$ (or equivalently $D_{Poin}(\Omega)$) for perturbations of Ω. Clearly, without any further assumptions, there can be no such result (as seen by adding arbitrarily small "necks" to Ω as in the Introduction), so we restrict our attention to convex domains. In this case, our Main Theorem 1.5 asserts that this is equivalent to obtaining a stability result for $D_{FM}(\Omega)$, which is much easier. To obtain the best quantitative bounds, we will also employ $D_{Exp}(\Omega)$.

Lemma 5.1 *Let $L \subset K \subset (\mathbb{R}^n, |\cdot|)$, and assume that L is a convex body. There exists a universal constant $c > 0$ such that*:

$$\mathrm{Vol}\,(L) \geq v\mathrm{Vol}\,(K) \Rightarrow D_{FM}(L) \geq \frac{c}{\log(1 + 1/v)} D_{Exp}(K).$$

Proof Let f_0 denote a 1-Lipschitz function on L with $M_{\lambda_L} f_0 = 0$ so that $\int |f_0| d\lambda_L \geq 1/(2D^M_{FM}(L))$. Since L is convex, we may clearly extend f_0 to a 1-Lipschitz function on K, say by defining $f_1 = f_0(\mathrm{Proj}_L\, x)$. Here $\mathrm{Proj}_L\, x$ denotes the unique (by convexity) y in L so that $d(x, L) = d(x, y)$. We may assume that $E_{\lambda_K} f_1 \geq 0$ (otherwise exchange f_0 with $-f_0$). Note that we can estimate $E_{\lambda_K} f_1$ as follows:

$$\frac{v}{2} \leq \lambda_K \{f_1 \leq 0\} \leq \lambda_K \left\{ \left| f_1 - E_{\lambda_K} f_1 \right| \geq E_{\lambda_K} f_1 \right\} \leq e \cdot \exp(-D_{Exp}(K) E_{\lambda_K} f_1). \tag{5.1}$$

By Proposition 4.2, there exists some universal $\varepsilon_0 > 0$ so that $\|f_0\|_{L_1(\lambda_L)} \leq Q_{\lambda_L, 1-\varepsilon_0}(|f_0|)$. Using this, the ratio between the volumes of L and K, the triangle inequality, the Markov-Chebyshev inequality and the estimate on $E_{\lambda_K} f_1$ in (5.1), we evaluate:

$$\frac{1}{2D^M_{FM}(L)} \leq \|f_0\|_{L_1(\lambda_L)} \leq Q_{\lambda_L, 1-\varepsilon_0}(|f_0|) \leq Q_{\lambda_K, 1-\varepsilon_0 v}(|f_1|)$$

$$\leq Q_{\lambda_K, 1-\varepsilon_0 v}(|f_1 - E_{\lambda_K} f_1|) + E_{\lambda_K} f_1$$
$$\leq \log\left(1 + \frac{1}{\varepsilon_0 v}\right) \| f_1 - E_{\lambda_K} f_1 \|_{\Psi_1(\lambda_K)} + \frac{\log(2e/v)}{D_{Exp}(K)} \leq C_0 \frac{\log(1+1/v)}{D_{Exp}(K)},$$

where $C_0 > 0$ is some universal constant. Using Lemma 2.1 and (2.3), the assertion follows. $\square$

Lemma 5.2 *Let $L \subset K \subset (\mathbb{R}^n, |\cdot|)$, and assume that L and K are convex bodies. Then*:

$$\mathrm{Vol}\,(L) \geq v \mathrm{Vol}\,(K) \Rightarrow D_{Che}(K) \geq v^2 D_{Che}(L).$$

Proof Note that for any $1/2 < p \leq 1$ and in fact even without assuming that L is convex:

$$\mathrm{Vol}\,(L) \geq p \mathrm{Vol}\,(K) \Rightarrow D_{Che}(K) \geq (2p-1) D_{Che}(L). \tag{5.2}$$

Indeed, since K is convex, by Theorem 1.8 (more precisely, its extension to non-smooth domains or densities given by Theorem 6.10 and Corollaries 6.11, 6.12) we know that $D_{Che}(K) = 2 I_{(K,|\cdot|,\lambda_K)}(1/2)$. Given a Borel set A with $\lambda_K(A) = 1/2$, we have:

$$\lambda_K^+(A) \geq p \lambda_L^+(A) \geq p D_{Che}(L) \min(\lambda_L(A), 1 - \lambda_L(A)).$$

By the assumption in (5.2), $1 - \frac{1}{2p} \leq \lambda_L(A) \leq \frac{1}{2p}$, and from this we easily deduce the conclusion in (5.2). Iterating this using a sequence of intermediate convex bodies (here we already need to use that L is convex) $L = L_0 \subset L_1 \subset \cdots \subset L_m = K$ so that $\mathrm{Vol}(L_i)/\mathrm{Vol}(L_{i+1}) \geq v^{1/m} > 1/2$ (for example, assuming $0 \in L$, choose $L_i = (1 + r_i) L \cap K$ for appropriate $r_i \geq 0$), we obtain that:

$$\mathrm{Vol}\,(L) \geq v \mathrm{Vol}\,(K) \Rightarrow D_{Che}(K) \geq (2 v^{1/m} - 1)^m D_{Che}(L).$$

Taking the limit as $m \to \infty$ yields the claimed power of 2 (even without any additional numerical constant!). $\square$

We can now immediately deduce Theorem 1.7 from the Introduction. Indeed, if K, L denote two convex bodies in $(\mathbb{R}^n, |\cdot|)$ such that:

$$\mathrm{Vol}\,(K \cap L) \geq v_K \mathrm{Vol}\,(K), \qquad \mathrm{Vol}\,(K \cap L) \geq v_L \mathrm{Vol}\,(L),$$

then applying Lemma 5.2, the Main Theorem 1.5 and Lemma 5.1, we obtain:

$$D_{Che}(K) \geq v_K^2 D_{Che}(K \cap L) \geq c_1 v_K^2 D_{FM}(K \cap L) \geq c_2 \frac{v_K^2}{\log(1 + 1/v_L)} D_{Exp}(L)$$
$$\geq c_3 \frac{v_K^2}{\log(1 + 1/v_L)} D_{Che}(L), \tag{5.3}$$

for some universal constants $c_i > 0$, concluding the proof of Theorem 1.7. Of course a similar upper bound on $D_{Che}(K)$ is obtained by interchanging the roles of K, L.

In Convexity Theory, many interesting ways are known to cut a convex body K so that its volume is preserved up to a constant (e.g. by slabs, parallelepipeds, balls etc.). We see that all of these preserve (up to a constant) $D_{Che}(K)$ (equivalently, the spectral gap $D^2_{Poin}(K)$).

A useful way to measure the distance between two convex bodies is given by the following variant on the usual geometric distance:

$$d_G(K, L) := \inf \left\{ ab;\ \frac{1}{a} L \subset K \subset bL, a, b \geq 1 \right\}. \tag{5.4}$$

Clearly in $(\mathbb{R}^n, |\cdot|)$:

$$\frac{\mathrm{Vol}\,(L)}{\mathrm{Vol}\,(K)} \leq d_G(K, L)^n,$$

so by passing from the outer to the inner body (in which case our estimates are logarithmic), we deduce:

Corollary 5.3 *Let K, L denote two convex bodies in $(\mathbb{R}^n, |\cdot|)$. If*:

$$d_G(K, L) \leq 1 + \frac{s}{n}$$

for some $1 \leq s \leq C_1 n$, where $C_1 > 0$ is some universal constant, then:

$$C_2 s D_{Che}(L) \geq D_{Che}(K) \geq \frac{1}{C_2 s} D_{Che}(L),$$

where $C_2 > 0$ is another universal constant.

Proof Denoting a, b the best constants in (5.4) and applying Lemma 5.1:

$$D_{Che}(K) \geq \frac{D_{Che}(bL)}{C \log(1 + d_G(K, L)^n)} \geq \frac{D_{Che}(L)}{C' bs},$$

and since $b \leq d_G(K, L) \leq C_1 + 1$, the assertion follows. □

Completely analogous results hold for absolutely continuous log-concave probability measures μ on $(\mathbb{R}^n, |\cdot|)$. We will write $D_{Che}(\mu)$ (and so on) to denote $D_{Che}(\mathbb{R}^n, |\cdot|, \mu)$ for short. Lemmas 5.1 and 5.2 were only formulated for uniform distributions λ_K, λ_L on domains K, L, since in that case, the condition:

$$L \subset K \quad \text{with } \mathrm{Vol}\,(L) \geq v \mathrm{Vol}\,(K) \tag{5.5}$$

appearing in the assumptions of both lemmas has a clear and intuitive geometric meaning.

Lemma 5.4 *Lemmas 5.1 and 5.2 remain valid for absolutely continuous log-concave probability measures μ_K, μ_L (replacing respectively K, L), if the condition (5.5) in the assumption is replaced by the condition*:

$$\frac{d\mu_K}{dx} \geq v \frac{d\mu_L}{dx},$$

and $D_{Che}(\Omega), D_{FM}(\Omega), D_{Exp}(\Omega)$ are replaced by $D_{Che}(\mu_\Omega), D_{FM}(\mu_\Omega), D_{Exp}(\mu_\Omega)$ $(\Omega = K, L)$ in the corresponding conclusion.

Proof Identical to the proof of the original lemmas; the only minor point is the construction of intermediate measures μ_{L_i} in the proof of Lemma 5.2, which may be defined e.g. by $\mu_{L_i} = \frac{\eta_{L_i}}{|\eta_{L_i}|}$, $\frac{d\eta_{L_i}}{dx}(x) = \min((1+r_i)\frac{d\mu_L}{dx}(\frac{x}{1+r_i}), \frac{d\mu_K}{dx}(x))$, for appropriate $r_i > 0$ (assuming the origin is in the interior of the support of μ_L). □

The analogue of Theorem 1.7 may then be conveniently formulated using the total-variation metric:

$$d_{TV}(\mu_1, \mu_2) := \frac{1}{2}\int \left|\frac{d\mu_1}{dx}(x) - \frac{d\mu_2}{dx}(x)\right| dx.$$

Theorem 5.5 *Let μ_1, μ_2 denote two log-concave probability measures in $(\mathbb{R}^n, |\cdot|)$. If*:

$$d_{TV}(\mu_1, \mu_2) \leq 1 - \varepsilon < 1,$$

then:

$$c(\varepsilon)^{-1} D_{Che}(\mu_2) \geq D_{Che}(\mu_1) \geq c(\varepsilon) D_{Che}(\mu_2),$$

with $c(\varepsilon) = c'\varepsilon^2/\log(1+1/\varepsilon)$ and $c' > 0$ a universal constant.

Proof Let μ_0 denote the measure whose density is $\min(\frac{d\mu_1}{dx}, \frac{d\mu_2}{dx})$, and note that $d_{TV}(\mu_1, \mu_2) = 1 - |\mu_0|$. Denoting by μ_3 the (log-concave) probability measure $\frac{\mu_0}{|\mu_0|}$, since $\frac{d\mu_i}{dx} \geq |\mu_0| \frac{d\mu_3}{dx}$, $i = 1, 2$, we may apply Lemma 5.4 and the Main Theorem to pass from μ_1 to μ_3 to μ_2 as in (5.3), concluding the proof. □

5.2 Optimality of stability

To the best of our knowledge, no quantitative results on the stability of D_{Che} or D_{Poin} for convex domains with respect to volume preserving perturbations or geometric distance were previously known. Moreover, we claim that the bounds obtained in Theorem 1.7 (or (5.3)) are optimal (up to numeric constants) with respect to v_L and close to optimal with respect to v_K (note that the dependence is logarithmic in the former yet quadratic in the latter; in other words, the deterioration in the Cheeger constant when passing from an outer convex body to an inner one is genuinely different than when passing from the inner one outward). This is witnessed by the following:

Example 5.6 Let Q^k denote a k-dimensional cube of volume 1, and let B_1^k denote the homothetic copy of the unit-ball of ℓ_1^k having volume 1. For $2 \leq k \leq n-1$, set $K_k = Q^{n-k} \times B_1^k$ and $L_k = Q^{n-k} \times [-c_1 k, c_1 k] \times c_2 B_1^{k-1}$, where $0 < c_1, c_2 < 1$ are universal constants chosen so that $L_k \subset K_k$ (it is easy to check that this is possible). Using a tensorization result of Bobkov and Houdré [19], it follows that:

$$D_{Che}(K_k) \simeq \min(D_{Che}(Q^{n-k}), D_{Che}(B_1^k)),$$
$$D_{Che}(L_k) \simeq \min(D_{Che}(Q^{n-k}), D_{Che}(B_1^{k-1}), D_{Che}([-k, k])).$$

It is known (see Sect. 5.5) that $D_{Che}(Q^m) \simeq D_{Che}(B_1^m) \simeq 1$, so by the -1-homogeneity of D_{Che}, it follows that $D_{Che}(K_k) \simeq 1$ and $D_{Che}(L_k) \simeq \frac{1}{k}$. Denoting $v_k = \frac{\mathrm{Vol}(L_k)}{\mathrm{Vol}(K_k)}$, since $\log 1/v_k \simeq k$, we conclude that:

$$D_{Che}(L_k) \simeq \frac{1}{\log(1+1/v_k)} D_{Che}(K_k),$$

uniformly for all $k = 2, \dots, n-1$. So one cannot expect better than logarithmic dependence on $1/v$ (at least when $v \geq \exp(-n)$), which coincides with the estimate given by Lemma 5.1.

On the other hand (as is well-known), if we set $L = Q^n$ and $K = Q^{n-1} \times tQ^1$ a circumscribing box with $t > 1$, since $D_{Che}(K) \simeq 1/t$ in that range, it is clear that the quadratic dependence on v in Lemma 5.2 cannot be improved beyond linear. Although we do not know whether the optimal bound is, up to a constant, closer to the linear or quadratic asymptotic, we comment that for very small perturbations (i.e. v very close to 1), it is possible to show that the exact quadratic bound in Lemma 5.2 *is* optimal (in this range of v, we of course do not allow any additional numerical constants).

The next example (which is similar yet different from the previous one) shows that the bounds in Corollary 5.3 are optimal too (up to numeric constants), as a function of s in the stated range.

Example 5.7 Continuing with the notations of Example 5.6, let us denote by r_n half the diameter of B_1^n, so that $B_1^n = r_n \mathrm{Conv}(\pm e_1, \dots, \pm e_n)$, where Conv denotes the convex-hull operation and $\{e_i\}$ is the standard orthonormal basis of $\mathbb{R}^n$. It is easy to check that $r_n/n \simeq 1$ uniformly on n. For $1 \leq s \leq c_1 n$, where $0 < c_1 < \frac{r_n}{2n}$ is some universal constant, define $K_s = B_1^n \cap \{|x_1| \leq s\}$. It is easy to check that in that range of s, $\mathrm{Vol}(K_s) \geq c_2 \mathrm{Vol}(B_1^n)$ for some universal constant $c_2 > 0$, and hence by Theorem 1.7 we deduce that $D_{Che}(K_s) \simeq 1$ uniformly on s, n. Now define:

$$L_s = \mathrm{Conv}(K_s \cap \{x_1 = s\}, K_s \cap \{x_1 = -s\}) = [-s, s] \times \left(1 - \frac{s}{r_n}\right)(B_1^n \cap \{x_1 = 0\}).$$

It follows as in Example 5.6 that:

$$\begin{aligned} D_{Che}(L_s) &\simeq \min\left(D_{Che}([-s,s]), \frac{D_{Che}(B_1^n \cap \{x_1 = 0\})}{1 - \frac{s}{r_n}}\right) \\ &\simeq \min\left(\frac{1}{s}, \frac{r_{n-1}}{r_n} D_{Che}(B_1^{n-1})\right) \simeq \frac{1}{s}. \end{aligned}$$

Since clearly $L_s \subset K_s$, it remains to note that $(1 - \frac{s}{r_n})K_s \subset L_s$, so $d_G(K_s, L_s) - 1 \simeq \frac{s}{n}$. By interchanging the roles of K_s, L_s appropriately, we observe that the estimates on $D_{Che}(K)/D_{Che}(L)$ in Corollary 5.3 are sharp both from above and from below.

Remark 5.8 It is easy to adapt the proofs of Lemma 5.1 and consequently Corollary 5.3 to obtain even sharper quantitative bounds (up to universal constants) on the stability of D_{Che} for *specific* convex bodies, such as the Euclidean ball B_2^n. For instance, in the latter case, one obtains that if $d_G(K, B_2^n) \leq 1 + \frac{s}{n}$ for $1 \leq s \leq C_1 n$, then:

$$D_{Che}(K) \geq \frac{1}{C_2\sqrt{s}} D_{Che}(B_2^n).$$

This is an improvement over Corollary 5.3 and known to be sharp for $s = n$ (folklore).

5.3 Stability of D_{Che} under Lipschitz maps

It is well known and immediate to see that isoperimetric inequalities are preserved under 1-Lipschitz mappings. Given two metric probability spaces (X, d_X, μ) and (Y, d_Y, ν), recall that a Borel map $T : (X, d_X) \to (Y, d_Y)$ is said to push forward μ onto ν, if $\nu(A) =$

$\mu(T^{-1}(A))$ for every Borel set $A \subset Y$. This is equivalent to requiring that for any Borel function g on (Y, d_Y):

$$\int_Y g(y)d\nu(y) = \int_X g(T(x))d\mu(x).$$

This will be denoted by $T_*(\mu) = \nu$. The following is then immediate from the definitions:

Fact *Assume that $T_*(\mu) = \nu$. Then*:

$$I_{(Y,d_Y,\nu)} \geq \frac{1}{\|T\|_{Lip}} I_{(X,d_X,\mu)}.$$

Here as usual:

$$\|T\|_{Lip} := \sup_{x \neq y \in X} \frac{d_Y(T(x), T(y))}{d_X(x, y)}.$$

The following result states that when our convexity assumptions hold for the target space, as far as Cheeger's isoperimetric inequality is concerned, one need not require that T be Lipschitz on the entire space, but rather just on average. We would like to thank Bo'az Klartag for a fruitful discussion regarding this point.

Theorem 5.9 *Assume that (Y, d_Y, ν) verifies our convexity assumptions and that $T_*(\mu) = \nu$ for some Lipschitz-on-balls map T. Then*:

$$D_{Che}(Y, d_Y, \nu) \geq \frac{c}{\int_X \|DT\|_{op}(x)d\mu(x)} D_{Che}(X, d_X, \mu),$$

for some universal constant $c > 0$.

Here $\|DT\|_{op}(x)$ denotes the local Lipschitz constant of T at x:

$$\|DT\|_{op}(x) := \limsup_{y \to x} \frac{d_Y(T(x), T(y))}{d_X(x, y)}.$$

When T is smooth and X, Y are linear spaces, this coincides with the operator norm of the usual derivative matrix DT at x.

Proof First, rewrite Cheeger's isoperimetric inequality on (X, d_X, μ) in functional form (Lemma 2.2):

$$\forall f \in \mathcal{F}(X, d_X) \quad D_{Che}(X, d_X, \mu) \left\| f - M_\mu f \right\|_{L_1(X,\mu)} \leq \||\nabla_X f|\|_{L_1(X,\mu)}. \tag{5.6}$$

Using this, we estimate the First-Moment constant on (Y, d_Y, ν). Given a 1-Lipschitz function g on (Y, d_Y), clearly $g \circ T$ is Lipschitz-on-balls on (X, d_X), hence in $\mathcal{F}(X, d_X)$. We then have by the definition of push-forward and our assumption (5.6):

$$\begin{aligned}\int_Y |g - M_\nu g| d\nu &= \int_X \left|g(Tx) - M_\mu(g \circ T)\right| d\mu \\ &\leq \frac{1}{D_{Che}(X, d_X, \mu)} \int_X |\nabla_X (g \circ T)|(x) d\mu(x)\end{aligned}$$

$$\leq \frac{1}{D_{Che}(X, d_X, \mu)} \int_X \|DT\|_{op}(x) \, |\nabla_Y g| \, (Tx) d\mu(x)$$
$$\leq \frac{\int_X \|DT\|_{op}(x) d\mu(x)}{D_{Che}(X, d_X, \mu)}.$$

Hence:

$$D_{FM}^{M}(Y, d_Y, \nu) \geq \frac{D_{Che}(X, d_X, \mu)}{\int_X \|DT(x)\|_{op} \, d\mu(x)}.$$

We conclude by our Main Theorem (and Lemma 2.1), which imply that $D_{Che}(Y, d_Y, \nu) \geq c D_{FM}^{M}(Y, d_Y, \nu)$ under our convexity assumptions on (Y, d_Y, ν). □

5.4 Estimating D_{Che}

In this subsection, we easily recover some previously known estimates on the Cheeger constant of convex domains in a single framework and extend some results to the Riemannian setting. We begin with the following stimulating conjecture from [47]:

Conjecture (Kannan–Lovász–Simonovits) *There exists a universal constant* $c > 0$ *such that for any convex body* K *in* $(\mathbb{R}^n, |\cdot|)$, *and more generally, for any log-concave probability measure* μ *on* $(\mathbb{R}^n, |\cdot|)$:

$$D_{Che}(\mu) \geq \frac{c}{\sigma_1(\mu)}. \tag{5.7}$$

Here $\sigma_1(\mu)^2$ denotes the largest eigenvalue of the symmetric covariance matrix $\Sigma(\mu)$ of μ:

$$\Sigma(\mu) := E_\mu(x \otimes x) - E_\mu(x) \otimes E_\mu(x).$$

We will write $\sigma_1(K)$ for $\sigma_1(\lambda_K)$.

Standard results in Convexity Theory easily imply that the opposite inequality in (5.7) holds with some universal constant $c > 0$. The reason for this is that it is easy to analyze the isoperimetric inequality for sets which are half-spaces in $\mathbb{R}^n$, and when restricting to these sets, both the upper bound and the conjectured lower bound hold with some (explicitly known) universal constants. The KLS conjecture is therefore a striking statement on the nature of isoperimetric minimizing sets for Cheeger's isoperimetric inequality in the convex setting: these sets do not minimize boundary-measure much better than just half-spaces. An explicit description of the isoperimetric minimizers is known only in a few cases, even in the Euclidean setting $(\Omega, |\cdot|, \lambda_\Omega)$ (see e.g. [78]), so it is extremely important to at least identify some *essentially* minimizing sets (up to universal constants).

Although the KLS conjecture is far from being resolved, some general lower bounds on D_{Che} are known, but these produce dimension-dependent results. We will see that our Main Theorem easily reproduces these bounds.

The following result in the Euclidean setting is due to Payne and Weinberger [76]. This was generalized to the Riemannian setting by Li and Yau [58]. We refer to the Appendix for missing definitions.

Theorem 5.10 (Payne–Weinberger, Li–Yau) *If $K \subset (M, g)$ is a locally convex bounded domain with smooth boundary and* $\mathrm{Ric}_g \geq 0$, *then*:

$$D_{Poin}(K, d_g, \lambda_K) \geq \frac{\pi}{2\mathrm{diam}(K)},$$

where diam *denotes the diameter and d_g the induced geodesic distance. In fact, when (M, g) is Euclidean space the constant 2 above may be omitted.*

Ledoux's Theorem 1.4 implies that the same lower bound (up to an additional constant) holds for $D_{Che}(K, d_g, \lambda_K)$. In the Euclidean case, this was strengthened in [47]:

Theorem 5.11 (Kannan–Lovász–Simonovits) *Let μ be a log-concave probability measure on $(\mathbb{R}^n, |\cdot|)$. Then*:

$$D_{Che}(\mu) \geq \sup_{x_0 \in \mathbb{R}^n} \frac{c}{\int |x - x_0| \, d\mu(x)},$$

for some universal constant $c > 0$.

To obtain this result, Kannan, Lovász and Simonovits developed a geometric localization technique (which in fact can be traced back to the work of M. Gromov and V. Milman [41]). As pointed out to us by Sasha Sodin, it is interesting to note that this technique uses some geometric properties of Euclidean space and does not generalize to other Riemannian manifolds (except in special cases, like that of the Euclidean Sphere, as in the work of Gromov–Milman). Our method, on the other hand, does allow us to state the following generalization of Theorem 5.11 to the Riemannian setting, which also improves over Theorem 5.10:

Theorem 5.12 *Assume that (Ω, d, μ) satisfies our convexity assumptions. Then*:

$$D_{Che}(\Omega, d, \mu) \geq \sup_{x_0 \in \Omega} \frac{c}{\int d(x, x_0) d\mu(x)},$$

for some universal constant $c > 0$.

Proof As usual, we just need to bound $D_{FM}(\Omega, d, \mu)$. Let f denote a 1-Lipschitz function on (Ω, d). Then for any $x_0 \in \Omega$, applying the triangle inequality twice:

$$\begin{aligned} \int \left| f(x) - E_\mu f \right| d\mu(x) &\leq \int |f(x) - f(x_0)| \, d\mu(x) + \left| E_\mu f - f(x_0) \right| \\ &\leq 2 \int |f(x) - f(x_0)| \, d\mu(x) \leq 2 \int d(x, x_0) d\mu(x). \end{aligned}$$

Hence:

$$D_{FM}(\Omega, d, \mu) \geq \sup_{x_0 \in \Omega} \frac{1}{2 \int d(x, x_0) d\mu(x)},$$

and the claim follows by our Main Theorem. □

Remark 5.13 An alternative approach to localization for proving isoperimetric inequalities was developed by Bobkov [18] in the Euclidean setting. Bobkov's approach was extended by Barthe [6] and subsequently by Barthe and Kolesnikov [7]. This approach is based on the Prékopa–Leindler inequality (e.g. [24]), or equivalently, on optimal transportation, which have both been recently generalized to the Riemannian-with-density-setting by Cordero-Erausquin, McCann and Schmuckenschläger [28, 29]. Using these tools we expect that it should be possible to provide an alternative proof of Theorem 5.12 following Bobkov's approach, but as pointed out to us by one of the referees, this has yet to be accomplished. We would like to thank the referee for his comments regarding our original simpleminded remark in this direction.

We would like to mention another bound on D_{Che} obtained in [47] using the localization method.

Theorem 5.14 (Kannan–Lovász–Simonovits) *Let μ be a log-concave probability measure on $(\mathbb{R}^n, |\cdot|)$ with bounded support B. Then*:

$$D_{Che}(\mu) \geq \frac{c}{\int \theta_B(x) d\mu},$$

where $\theta_B(x)$ denotes the longest symmetric interval contained in B and centered at x, and $c > 0$ is a universal constant.

We have recently managed to derive this result using our Main Theorem, but this will be described elsewhere. Instead, we would like to show how this bound may be used to recover a result of Bobkov [17]; in fact, the bound we deduce is formally stronger than Bobkov's. Bobkov employs the localization method as well, but then relies on some nice trick involving moment inequalities for polynomials in the log-concave setting. Our argument, on the other hand, is more geometric. Independently of our proof, we heard about a similar idea for bounding the boundary measure of large sets from Santosh Vempala (using localization as well).

Theorem 5.15 (Bobkov) *Let μ be a log-concave probability measure on $(\mathbb{R}^n, |\cdot|)$. Then*:

$$D_{Che}(\mu) \geq \sup_{x_0 \in \mathbb{R}^n} \frac{c}{(\mathrm{Var}_\mu(|x - x_0|^2))^{1/4}},$$

where Var_μ denotes the variance with respect to μ.

Sketch of Proof Without loss of generality, we may assume that $x_0 = 0$; for general x_0 the claimed bound follows by translating μ. Let $E := E_\mu |x|$, $S := (\mathrm{Var}_\mu |x|)^{1/2}$, and denote:

$$B := \{x \in \mathbb{R}^n ; |x| \leq E + 2S\}.$$

By Chebyshev's inequality, $\mu(B) \geq 3/4$, so if we define $\mu_0 := \mu|_B / \mu(B)$, it follows that $d_{TV}(\mu, \mu_0) \leq 1/4$. Hence $D_{Che}(\mu) \simeq D_{Che}(\mu_0)$ by Theorem 5.5. Assume that $E \geq 2S$, otherwise the support of μ_0 has diameter bounded by $8S$, and one can conclude as in Theorem 5.12. We now employ Theorem 5.14 to bound $D_{Che}(\mu_0)$:

$$D_{Che}(\mu_0) \geq \frac{c}{\int \theta_B(x) d\mu_0(x)} = \frac{c\mu(B)}{\int_B \theta_B(x) d\mu(x)}. \tag{5.8}$$

The crucial geometric observation is that for the Euclidean ball B:

$$\theta_B(x) = 2\sqrt{(E+2S)^2 - |x|^2}.$$

It remains to plug this into (5.8) and evaluate the resulting expression using integration by parts and Chebyshev's inequality. We leave it as an exercise to conclude that:

$$D_{Che}(\mu) \geq \frac{c'}{\sqrt{ES}},$$

for some universal constant $c' > 0$. This bound is in fact formally better than Bobkov's bound (by several applications of Hölder's inequality), but using some standard results in Convexity Theory, it is in fact equivalent in the interesting situations. □

5.5 D_{Che} for specific families of convex bodies

Embarrassingly, hardly any concrete examples exist of non-degenerate convex bodies K in $\mathbb{R}^n$ for which the asymptotic value of $D_{Che}(K)$ (as a function of the dimension n) is known. The KLS conjecture stating that $D_{Che}(K) \simeq 1$ for such bodies has only been confirmed in a few special cases. These include the Euclidean ball (see e.g. [25]) and the unit cube $K = [-1/2, 1/2]^n$ (Hadwiger [43], see also [8, 21]). By the tensorization results of Bobkov and Houdré [19], this is in fact true for an arbitrary log-concave product measure (appropriately normalized). When $K = \tilde{B}(\ell_p^n)$, the volume one homothetic copy of the unit-ball of ℓ_p^n, for $p \in [1, 2]$, the KLS conjecture was only recently confirmed by Sasha Sodin [81] (note that indeed $\sigma_1(\tilde{B}(\ell_p^n)) \simeq 1$). Even more recently, the case $p \geq 2$ has been confirmed by R. Latała and J. Wojtaszczyk [52] by an elegant construction of a Lipschitz map pushing forward the Gaussian measure onto the uniform measure on $\tilde{B}(\ell_p^n)$. We are not aware of any other (sufficiently different) examples.

We comment that our Main Theorem easily implies the result for $K = \tilde{B}(\ell_p^n)$, $p \in [1, 2]$, due to Sodin [81]. However, Sodin's result provides a sharp bound on the isoperimetric profile of these spaces, whereas we only deduce the bound on Cheeger's constant.

Theorem 5.16 (Sodin) *For any $n \geq 1$, $p \in [1, 2]$:*

$$D_{Che}(\tilde{B}(\ell_p^n)) \geq c > 0,$$

where $c > 0$ is a universal constant.

Proof This is immediate from the results of Schechtman and Zinn [79], who showed that D_{Exp} of these bodies is bounded from below by a universal constant. The result then follows from our Main Theorem (in fact, we only need a bound on D_{FM}). □

Another family of convex bodies for which the KLS conjecture is almost confirmed, is that of unconditional convex bodies K, i.e. convex bodies for which $(x_1, \ldots, x_n) \in K$ iff $(\pm x_1, \ldots, \pm x_n) \in K$. It was recently shown by Bo'az Klartag [48] that if K is an unconditional body with $\sigma_1(K) = 1$ then $D_{Che}(K) \geq c/\log n$, for some universal constant $c > 0$. To obtain this result, Klartag employed Theorem 1.7 to pass to an unconditional body contained inside the cube $(C \log n)[-1, 1]^n$, and then used some symmetry properties of the Laplacian's eigenfunctions to conclude his result. In fact, one can just use Theorem 1.8 on the concavity of the isoperimetric profile (in the form of Lemma 5.2) for this application.

5.6 Some dimension dependent bounds on D_{Che}

We conclude this section by stating the known dimension dependent bounds on $D_{Che}(K)$ for non-degenerate convex bodies K (in the sense that $\sigma_1(K) = 1$).

It is known in this case that $\mathrm{diam}(K) \leq cn$ (by a simple volume estimate). Theorem 5.10 (together with Theorem 1.4) then gives $D_{Che}(K) \geq c/n$. The first KLS bound (Theorem 5.11) improves this to $D_{Che}(K) \geq c/\sqrt{n}$, since:

$$\int_K \left|x - E_\mu x\right| dx \leq \left(\int_K \left|x - E_\mu x\right|^2 dx\right)^{1/2} \leq \sqrt{n}\sigma_1(K).$$

The second KLS bound (Theorem 5.14) is incomparable to the first bound, since it gives the right order for the Euclidean ball, but gives c/n for the regular simplex of volume 1 in $\mathbb{R}^n$.

Bobkov's bound (Theorem 5.15) is always at least as good as the first KLS bound (up to a constant), since (using the bound derived in the proof together with a standard application of Borell's lemma [23]):

$$\mathrm{Var}_\mu(|x - x_0|)^{1/2} \leq E_\mu(|x - x_0|^2)^{1/2} \leq C E_\mu(|x - x_0|),$$

for some universal constant $C > 0$. We see that whenever some non-trivial information on $\mathrm{Var}_\mu(|x - x_0|)$ is known, Bobkov's bound is strictly better. Such a remarkable result was proved by Bo'az Klartag [49, 50], allowing him to deduce a Central-Limit type result for the class of convex bodies (and more generally, log-concave measures). Klartag's improved estimate in [50] reads:

$$\mathrm{Var}_\mu(\left|x - E_\mu x\right|)^{1/2} \leq C_\varepsilon n^{1/2-1/10+\varepsilon} \sigma_1(\mu) \quad \forall \varepsilon > 0.$$

Combining this with Bobkov's bound, one deduces the following result, already noticed among specialists, for log-concave measures in $\mathbb{R}^n$ with $\sigma_1(\mu) = 1$:

$$D_{Che}(\mu) \geq \frac{c_\varepsilon}{n^{1/2-1/20+\varepsilon}} \quad \forall \varepsilon > 0.$$

At the moment, this is the best known bound on Cheeger's constant for general log-concave measures (or convex bodies) in $\mathbb{R}^n$.

6 Approximation argument

In this section, we develop an approximation argument for extending the following theorems to non-necessarily smooth densities (or boundaries) in our convexity assumptions:

- Theorem 1.8 on the concavity of the isoperimetric profile.
- Our Main Theorem 1.5.

We will develop different procedures for extending each of these theorems.

6.1 Stability of the isoperimetric profile

We begin by extending our definition of smooth convexity assumptions (we refer to the Appendix for the definition of *locally convex*).

Definition We will say that our *generalized smooth convexity assumptions* are fulfilled if:

- (M, g) denotes an n-dimensional ($n \geq 2$) smooth complete oriented connected Riemannian manifold or $(M, g) = (\mathbb{R}, |\cdot|)$.
- $\Omega \subset M$ is a locally convex domain with C^2 boundary.
- d denotes the induced geodesic distance on (M, g).
- $d\mu = \exp(-\psi) d\mathrm{vol}_M|_\Omega$, $\psi \in C^2(\overline{\Omega})$, and as tensor fields on Ω:

$$\mathrm{Ric}_g + \mathrm{Hess}_g \psi \geq 0.$$

This definition was already used in the statement of Theorem 1.8 on the concavity of the isoperimetric profile. The smoothness assumptions in the above definition are used in an essential way in the proof of this theorem to deduce the existence and regularity of the isoperimetric minimizers, which are otherwise false. This permits the use of variational methods from Riemannian Geometry, consequently obtaining a second-order differential inequality which the isoperimetric profile must satisfy (see the Appendix for more details). Nevertheless, the restriction to smooth densities and domains still seems like a technical artifact of the proofs. Some authors have suggested various methods to remove these smoothness assumptions (see e.g. Morgan [71] and Bayle [12, Chap. 4]), but unfortunately these are not well suited for our purposes. We therefore attempt to use a different approximation argument for extending Theorem 1.8 to a more general setting.

At first glance, it is tempting to believe that the isoperimetric profile of (Ω, d, μ) should be stable under approximating the measure μ by measures μ_m in, say, total-variation distance. However, the profile is in fact not even pointwise continuous under arbitrary approximation in total-variation. To see this, consider the measures μ_m which are uniform on the set $[0, 1] \setminus [1/2 - 1/m, 1/2 + 1/m]$, and converge to μ, the uniform measure on $[0, 1]$. Clearly $I_{\mu_m}(1/2) = 0$ for every $m \geq 3$, even though $I_\mu(1/2) = 1$. So one must take care when specifying the approximation.

Definition We say that a sequence of Borel probability measures $\{\mu_m\}$ tends to μ from above if $\{\mu_m\}$ converges to μ in total-variation and in addition there exists a sequence $\{c_m\}$ which tends to 1, so that $\mu_m(A) \geq \mu(A)/c_m$ for any Borel set A.

Lemma 6.1 *Let (Ω, d) be a metric space and let $\{\mu_m\}$ be a sequence of Borel probability measures on (Ω, d) which tends to μ from above. Then for any $t \in (0, 1)$:*

$$\liminf_{m \to \infty} I_{(\Omega,d,\mu_m)}(t) \geq \liminf_{s \to t} I_{(\Omega,d,\mu)}(s).$$

Proof Denote $I = I_{(\Omega,d,\mu)}$ and $I_m = I_{(\Omega,d,\mu_m)}$ for short. Let $\varepsilon > 0$. Then there exists m_0 such that for all $m \geq m_0$, $|\mu(B) - \mu_m(B)| < \varepsilon$ for any Borel set B. Let $\delta > 0$, then for every $m \geq m_0$ there exist a Borel set B_m such that:

$$I_m(t) + \delta \geq \mu_m^+(B_m) \geq \mu^+(B_m)/c_m \geq I(\mu(B_m))/c_m \geq \inf_{|s-t|<\varepsilon} I(s)/c_m.$$

Taking the limit as $m \to \infty$ and subsequently $\varepsilon, \delta \to 0$, we obtain the assertion. □

Definition We say that a sequence of Borel probability measures $\{\mu_m\}$ tends to μ from within if $\mu_m = \mu|_{A_m}/\mu(A_m)$ for some sequence of Borel sets A_m such that $\mu(A_m) \to 1$, and in addition $\mu^+(A_m) \to 0$.

Lemma 6.2 *Let (Ω, d) be a metric space and let $\{\mu_m\}$ be a sequence of Borel probability measures on (Ω, d) which tends to μ from within. Then for any $t \in (0, 1)$:*

$$\liminf_{m\to\infty} I_{(\Omega,d,\mu_m)}(t) \geq \liminf_{s\to t} I_{(\Omega,d,\mu)}(s).$$

Proof We continue with the same assumptions and notations as in the proof of the previous lemma and definition. In our case, we may assume that $B_m \subset A_m$. Then:

$$\begin{aligned} I_m(t) + \delta \geq \mu_m^+(B_m) &\geq \frac{\mu^+(B_m) - \mu^+(A_m)}{\mu(A_m)} \geq \frac{I(\mu(B_m)) - \mu^+(A_m)}{\mu(A_m)} \\ &\geq \inf_{|s-t|<\varepsilon} \frac{I(s) - \mu^+(A_m)}{\mu(A_m)}. \end{aligned}$$

Taking the limit as $m \to \infty$ and subsequently $\varepsilon, \delta \to 0$, we obtain the assertion. $\square$

Remark 6.3 It is quite non-trivial to come up with other conditions which ensure the conclusion of Lemmas 6.1 and 6.2. Of course convergence in the L_∞ norm of the densities with respect to the Riemannian volume form would also do, but this seems an impractical assumption since μ may have a non-continuous density. Another interesting possibility which works is to assume that μ_m are obtained by pushing μ forward using mappings T_m, so that $\|T_m\|_{Lip}$ tends to 1. Unfortunately, we do not know how to show that an arbitrary log-concave measure μ in $\mathbb{R}^n$ may be approximated by smooth log-concave measures μ_m of this type.

Next, we recall the definition of q-capacity (we will only require the case $q = 1$). Capacities were introduced in the 1960's by Maz'ya [60, 61], Federer and Fleming [32], and were used by Bobkov and Houdré in [19, 20]. We follow a variation on the definition given in [63] (for general q), which was extended by Barthe, Cattiaux and Roberto (with $q = 2$) in [10] (after being introduced in [9]). We conform to the definition implicitly used by Sodin in [81] and Sodin and the author in [66].

Definition Given a metric probability space (Ω, d, μ), $0 < q < \infty$ and $0 \leq a \leq b \leq 1$, we denote:

$$\mathrm{Cap}_q(a, b) := \inf\left\{ \| |\nabla\Phi| \|_{L_q(\mu)} \, ; \, \mu\{\Phi = 1\} \geq a, \mu\{\Phi = 0\} \geq 1 - b \right\},$$

where the infimum is on all $\Phi : \Omega \to [0, 1]$ which are Lipschitz-on-balls (recall the definition of $|\nabla\Phi|$ given in Remark 2.3).

The following proposition encapsulates the connection between 1-capacity and the isoperimetric profile $I = I_{(\Omega,d,\mu)}$. The proof is very much along the lines of the proof of Lemma 2.2, so we will omit it here; the reader is referred to Sodin [81, Proposition A] for an elementary derivation (note the slight difference in our formulation). We only remark that it suffices to use Lipschitz functions Φ in the definition of capacity above for the purpose of this proposition.

Proposition 6.4 (Maz'ya, Federer–Fleming, Bobkov–Houdré) *For all* $0 < a < b < 1$:

$$\inf_{a\leq t\leq b} I(t) \leq \mathrm{Cap}_1(a, b) \leq \inf_{a\leq t<b} I(t). \tag{6.1}$$

Since obviously $\mathrm{Cap}_1(a,b) = \mathrm{Cap}_1(1-b, 1-a)$, it follows that:

$$\inf_{a \le t \le b} I(t) \le \inf_{1-b \le t < 1-a} I(t).$$

Letting b converge to a, and replacing a, b with $1-b, 1-a$, we obtain:

Corollary 6.5 *If I is lower semi-continuous at t and $1-t$, $t \in (0,1)$, then $I(t) = I(1-t)$.*

Lemma 6.6 *Let (Ω, d) be a metric space and let $\{\mu_m\}$ be a sequence of Borel probability measures on (Ω, d) which converges in the total-variation norm to μ. Assume in addition that $I_{(\Omega,d,\mu_m)}$ are concave on $(0,1)$. Then for any $t \in (0,1)$:*

$$\liminf_{s \to t} I_{(\Omega,d,\mu)}(s) \ge \limsup_{m \to \infty} I_{(\Omega,d,\mu_m)}(t).$$

Proof As usual, denote $I = I_{(\Omega,d,\mu)}$ and $I_m = I_{(\Omega,d,\mu_m)}$ for short. Let $t \in (0,1)$ and small $\varepsilon > 0$ be given, and let $\Phi : (\Omega, d) \to [0,1]$ denote a Lipschitz function so that:

$$\mu\{\Phi = 1\} \ge t - \varepsilon, \qquad \mu\{\Phi = 0\} \ge 1 - t - \varepsilon.$$

For any small $\delta > 0$, there exists an m_0 so that for any $m \ge m_0$:

$$\mu_m\{\Phi = 1\} \ge t - \varepsilon - \delta, \qquad \mu_m\{\Phi = 0\} \ge 1 - t - \varepsilon - \delta.$$

We conclude by Proposition 6.4 and the concavity of I_m that:

$$\int |\nabla \Phi| d\mu_m \ge \inf_{t-\varepsilon-\delta \le s \le t+\varepsilon+\delta} I_m(s) \ge \min\left(\frac{t-\varepsilon-\delta}{t}, \frac{1-t-\varepsilon-\delta}{1-t}\right) I_m(t).$$

Since Φ is Lipschitz (hence $|\nabla\Phi|$ is bounded), and $\{\mu_m\}$ converge to μ in total-variation, we can pass to the limit as $m \to \infty$:

$$\int |\nabla \Phi| d\mu \ge \min\left(\frac{t-\varepsilon-\delta}{t}, \frac{1-t-\varepsilon-\delta}{1-t}\right) \limsup_{m\to\infty} I_m(t).$$

Taking infimum on all such Φ as above and using Proposition 6.4 again, we obtain:

$$\inf_{t-\varepsilon \le s < t+\varepsilon} I(s) \ge \min\left(\frac{t-\varepsilon-\delta}{t}, \frac{1-t-\varepsilon-\delta}{1-t}\right) \limsup_{m\to\infty} I_m(t).$$

Taking the limit of ε, δ to 0, we obtain the desired conclusion. □

Remark 6.7 It is clear from the proof that the concavity condition may be seriously relaxed (e.g. to equicontinuity), and the regularity condition on I_m obtained in Lemma 6.9 below may also be used.

Combining the last three lemmas we immediately obtain:

Proposition 6.8 *Let (Ω, d) be a metric space, let $\{\mu_m\}$ be a sequence of Borel probability measures on (Ω, d) which converges in the total-variation norm to μ, and assume that*

$I_{(\Omega,d,\mu_m)}$ *are all concave on* $(0,1)$. *If in addition* $\{\mu_m\}$ *tend to* μ *from above or from within, then for any* $t \in (0,1)$:

$$\liminf_{m\to\infty} I_{(\Omega,d,\mu_m)}(t) = \limsup_{m\to\infty} I_{(\Omega,d,\mu_m)}(t) = \liminf_{s\to t} I_{(\Omega,d,\mu)}(s).$$

In particular, if $I_{(\Omega,d,\mu)}$ *is in addition lower semi-continuous, we have (pointwise):*

$$\lim_{m\to\infty} I_{(\Omega,d,\mu_m)} = I_{(\Omega,d,\mu)}.$$

The following lemma, which extends the argument given by Gallot in [34, Lemma 6.2] for compact manifolds with uniform density, provides a sufficient condition for the isoperimetric profile to be continuous.

Lemma 6.9 *Let* $\Omega = (M,g)$ *denote an* n*-dimensional* $(n \geq 2)$ *smooth complete oriented connected Riemannian manifold and let* d *denote the induced geodesic distance. Let* μ *denote an absolutely continuous measure with respect to* vol_M, *such that its density is bounded from above on every ball (but not necessarily from below, nor do we assume it is continuous). Then* $I = I_{(\Omega,d,\mu)}$ *is absolutely continuous on* $[0,1]$, *and in fact is locally of Hölder exponent* $\frac{n-1}{n}$.

Proof By Lebesgue's Theorem, we know for almost every $x \in M$ (with respect to vol_M),

$$\mu(B_M(x,\varepsilon)) = \frac{d\mu}{d\mathrm{vol}_M}(x)\mathrm{Vol}_M(B_M(x,\varepsilon))(1+o(1)),$$

and clearly:

$$\mu^+(B_M(x,\varepsilon)) \leq \mu_\infty(\overline{B_M(x,2\varepsilon)})\mathrm{Vol}_M(\partial B_M(x,\varepsilon)), \tag{6.2}$$

where $B_M(x,R)$ denotes the ball in M of radius R around x, Vol_M denotes the Riemannian volume on M (and by abuse of notation the induced volume on any submanifold as well), and $\mu_\infty(\mathcal{C})$ denotes the upper bound on the density of μ on a compact set $\mathcal{C} \subset M$. By Rauch's Comparison Theorem, for any such compact set $\mathcal{C}$ (and in particular a singleton), there exists a $\varepsilon_{\mathcal{C}} < 1/2$ so that for any $x \in \mathcal{C}$ and $\varepsilon < \varepsilon_{\mathcal{C}}$:

$$\frac{3}{4}\varepsilon^n \mathrm{Vol}\,(B^n) < \mathrm{Vol}_M(B_M(x,\varepsilon)) < \frac{5}{4}\varepsilon^n \mathrm{Vol}\,(B^n)\,, \tag{6.3}$$

$$\frac{3}{4}\varepsilon^{n-1}\mathrm{Vol}\left(S^{n-1}\right) < \mathrm{Vol}_M(\partial B_M(x,\varepsilon)) < \frac{5}{4}\varepsilon^{n-1}\mathrm{Vol}\left(S^{n-1}\right), \tag{6.4}$$

where B^n and S^{n-1} denote the Euclidean unit ball and sphere, respectively, and Vol denotes Euclidean volume. Therefore as $t \to 0$:

$$I(t), I(1-t) \leq C_{n,\mu} t^{(n-1)/n}(1+o(1)),$$

where $C_{n,\mu}$ depends on n and μ only. Since clearly $I(0) = I(1) = 0$, this takes care of the continuity at 0 and 1.

Now fix $x_0 \in M$ and define $g : (0,1) \to \mathbb{R}_+$ to be the function:

$$g(\varepsilon) := \inf\{R > 0; \mu(B_M(x_0,R)) \geq 1-\varepsilon\}.$$

Given $0 < \theta < 1$, set $R_\theta = g(\theta/2)+1$, $\varepsilon_\theta = \varepsilon_{B_M(x_0,R_\theta+1)}$, and $\mu_\infty(\theta) = \mu_\infty(\overline{B_M(x_0, R_\theta + 1)})$. Let K_θ denote the (possibly negative) lower bound on the sectional curvature of K on $B_M(x_0, R_\theta)$. Rauch's Theorem also implies that:

$$\mathrm{Vol}_M(B_M(x_0, R_\theta)) \leq \mathrm{Vol}_{M_{K_\theta}}(B_{M_{K_\theta}}(R_\theta)), \tag{6.5}$$

where M_K denotes the simply connected model space with constant curvature K, Vol_{M_K} denotes the volume on M_K and $B_{M_K}(R)$ is any ball in M_K of radius R.

Given a set $A \subset M$ with $\theta = \mu(A) > 0$, note that by Fubini's Theorem, (6.3) and the definition of g, for any $\varepsilon < \varepsilon_\theta < 1/2$:

$$\begin{aligned}
&\int_{B_M(x_0,R_\theta)} \mu(A \cap B_M(x,\varepsilon)) d\mathrm{vol}_M(x) \\
&\quad = \int_A \mathrm{Vol}_M(B_M(y,\varepsilon) \cap B_M(x_0, R_\theta)) d\mu(y) \geq \int_{A\cap B_M(x_0,R_\theta-1)} \mathrm{Vol}_M(B_M(y,\varepsilon)) d\mu(y) \\
&\quad \geq \frac{3}{4}\varepsilon^n \mathrm{Vol}\,(B^n)\, \mu(A \cap B_M(x_0, g(\mu(A)/2))) \geq \frac{3}{8}\varepsilon^n \mathrm{Vol}\,(B^n)\, \mu(A).
\end{aligned} \tag{6.6}$$

We conclude from (6.6) and (6.5) that given any $A \subset M$ with $0 < \theta = \mu(A) < 1$ and $\varepsilon < \varepsilon_\theta$, there exists an $x \in B_M(x_0, R_\theta)$ such that:

$$\mu(A \cap B_M(x,\varepsilon)) \geq \frac{3}{8} \frac{\varepsilon^n \mathrm{Vol}\,(B^n)}{\mathrm{Vol}_M(B_M(x_0, R_\theta))} \mu(A) \geq \varepsilon^n \mathrm{Vol}\,(B^n)\, f(\mu(A)), \tag{6.7}$$

where f is defined as:

$$f(\theta) = \frac{3}{8} \frac{\theta}{\mathrm{Vol}_{M_{K_\theta}}(B_{M_{K_\theta}}(g(\theta/2)+1))}.$$

Now let $0 < s < t < 1$ be close enough such that there exists an $\varepsilon_1 < \varepsilon_t$ such that:

$$t - s = \varepsilon_1^n \mathrm{Vol}\,(B^n)\, f(t). \tag{6.8}$$

By definition, for any $\eta > 0$, there exists a set A such that $\mu(A) = t$ and $\mu^+(A) \leq I(t) + \eta$. By (6.7) there exists an $x \in B_M(x_0, R_t)$ such that $\mu(A \setminus B_M(x, \varepsilon_1)) \leq s$, and since μ is absolutely continuous, it follows that there exists an $\varepsilon_2 \leq \varepsilon_1$ such that $\mu(A \setminus B_M(x, \varepsilon_2)) = s$. Therefore:

$$I(s) \leq \mu^+(A \setminus B_M(x,\varepsilon_2)) \leq \mu^+(A) + \mu^+(B_M(x,\varepsilon_2)) \leq I(t) + \eta + \mu_\infty(t)\frac{5}{4}\varepsilon_1^{n-1}\mathrm{Vol}\left(S^{n-1}\right),$$

where we have used (6.2) and (6.4) in the last inequality. Sending η to 0 and plugging in (6.8), we conclude that for some constant C_n which depends on n:

$$I(s) \leq I(t) + C_n \mu_\infty(t) \left(\frac{t-s}{f(t)}\right)^{\frac{n-1}{n}}.$$

To get the inequality in the other direction, we require that $0 < s < t < 1$ are close enough so that $\varepsilon_1 < \varepsilon_{1-s}$ in addition satisfies:

$$t - s = \varepsilon_1^n \mathrm{Vol}\,(B^n)\, f(1-s).$$

Now let $A \subset M$ be such that $\mu(A) = s$ and $\mu^+(A) \leq I(s) + \eta$. Applying (6.7) for the set $M \setminus A$, we find an $x \in B_M(x_0, R_{1-s})$ and $\varepsilon_2 \leq \varepsilon_1$ such that $\mu(A \cup B_M(x, \varepsilon_2)) = t$. Repeating the above argument then gives:

$$I(t) \leq I(s) + C_n \mu_\infty (1-s) \left(\frac{t-s}{f(1-s)} \right)^{\frac{n-1}{n}} .$$

Since f is monotone, this concludes the proof. □

Our approximation argument is now clear. Given a measure μ in the setting of Lemma 6.9, we know that its isoperimetric profile I is continuous. Assume that μ can be approximated from above or from within by measures $\{\mu_m\}$ satisfying our generalized smooth convexity assumptions. By Theorem 1.8, the corresponding profiles $\{I_m\}$ (and when the densities are uniform, also the renormalized profiles $\{I_m^{n/(n-1)}\}$) are concave, and so applying Proposition 6.8, we deduce the pointwise convergence of I_m to I, which clearly preserves concavity. We therefore deduce:

Theorem 6.10 *Let $\Omega = (M, g)$ denote an n-dimensional $(n \geq 2)$ smooth complete oriented connected Riemannian manifold and let d denote the induced geodesic distance. For each $m \geq 1$, let $\{\mu_m\}$ denote a sequence of Borel probability measures on $\Omega_m \subset \Omega$ so that (Ω_m, d, μ_m) satisfies our generalized smooth convexity assumptions. Assume that $\{\mu_m\}$ tends to an absolutely continuous Borel probability measure μ from above or from within, and denote $I_m = I_{(\Omega_m, d, \mu_m)}$ and $I = I_{(\Omega, d, \mu)}$. Then $I_m \to I$ pointwise and consequently I is concave on* $[0, 1]$. *Moreover, if each μ_m is uniform over Ω_m, then $I^{n/(n-1)}$ is also concave on* $[0, 1]$.

Proof The argument has already been sketched. We only remark that it is not hard to verify the validity of the assumptions of Lemma 6.9 on μ, as the limit of $\{\mu_m\}$ as above (see e.g. [64, Remark 6.2]). □

Corollary 6.11 *Let Ω denote any (non-smooth) convex bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, let μ denote the uniform probability measure on Ω and let d denote the Euclidean metric. Then our convexity assumptions are satisfied, $I = I_{(\Omega, d, \mu)}$ is concave on* $[0, 1]$, *and so is $I^{n/(n-1)}$.*

Proof Approximate Ω from outside by smooth convex domains using standard methods (see e.g. [80]). Note that Ω_ε will only guarantee C^1 smoothness. □

Corollary 6.12 *Let $\Omega = \mathbb{R}^n$ $(n \geq 1)$, let μ denote any absolutely continuous log-concave probability measure (with possibly non-smooth density) and let $d = |\cdot|$ denote the Euclidean metric. Then our convexity assumptions are satisfied and $I = I_{(\Omega, d, \mu)}$ is concave on* $(0, 1)$ *(and if $n \geq 2$, on* $[0, 1]$*).*

Proof The case $n = 1$ follows from Theorem A.4 in the Appendix. For the case $n \geq 2$, we will need to approximate μ from above and within by a sequence of smooth log-concave probability measures. Since we did not find a standard reference for this, we outline the argument.

First, assume that the support B of μ is compact. Approximate μ by smooth log-concave probability measures $\{\nu_\varepsilon\}$ in total-variation distance using standard methods (e.g. convolution with a Gaussian mollifier). Now define $\eta_{\varepsilon,\delta}$ to be the dilatation of ν_ε given by

$\eta_{\varepsilon,\delta}(A) = \nu_\varepsilon(x_0 + (1-\delta)(A - x_0))$ for all Borel sets A, where x_0 is a point in the interior of B (another possibility would be to use sup-convolution with a small Gaussian). It is then not hard to check that for a suitable subsequence, $\eta_{\varepsilon,\delta(\varepsilon)}$ tends to μ from above, from which the assertion follows by Theorem 6.10.

In case the support of μ is not compact, we repeat the above argument for the truncated measures $\mu_r = \mu|_{rB_2^n}/\mu(rB_2^n)$, where B_2^n denotes the Euclidean unit-ball. Note that $\mu^+(rB_2^n) \to 0$ as $r \to \infty$ e.g. by the co-area formula:

$$\int_0^\infty \mu^+(rB_2^n)dr = \int_0^\infty \mu^+\{x \in \mathbb{R}^n; |x| \geq r\}\, dr = \int_{\mathbb{R}^n} |\nabla| \cdot ||d\mu = 1.$$

Hence $\{\mu_r\}$ tends to μ from within, and so by Theorem 6.10 the claim now follows for arbitrary log-concave measures. □

6.2 Stability of first-moment concentration

Up to now, we have only concluded the Main Theorem 1.5 under our *smooth* convexity assumptions. We now describe how to extend these assumptions to our general convexity assumptions.

Indeed, assume that μ can be approximated in total-variation by measures $\{\mu_m\}$ with density $\exp(-\psi_m)$ such that $\psi_m \in C^2(M)$ and $\mathrm{Ric}_g + \mathrm{Hess}_g \psi_m \geq 0$ on $\Omega = (M, g)$. We would like to show that our Main Theorem, stating that $D_{Che}(\Omega, d, \mu) \geq c D_{FM}(\Omega, d, \mu)$ for some universal constant $c > 0$, still holds. It is immediate to deduce from Lemma 6.6 that:

$$D_{Che}(\Omega, d, \mu) \geq \limsup_{m\to\infty} D_{Che}(\Omega, d, \mu_m),$$

and using our Main Theorem for the smooth measures μ_m (and Lemma 2.1), we deduce that:

$$D_{Che}(\Omega, d, \mu) \geq c \limsup_{m\to\infty} D_{FM}^M(\Omega, d, \mu_m),$$

for some universal constant $c > 0$. The First Moment constant is particularly easy to handle, since there is no $\||\nabla f|\|_{L_q}$ term which needs to be controlled. The following lemma, which is an adaptation of a classical lemma of C. Borell [23] from the Euclidean case to the Riemannian-manifold-with-density setting, enables us to reduce to the case that $\{\mu_m\}$ are all supported on some compact set:

Lemma 6.13 *Let $x_0 \in M$ and $R > 0$ be such that $\theta = \mu_m(B(x_0, R)) > 1/2$. Then*:

$$\forall t \geq 1 \quad \mu_m(M \setminus B(x_0, tR)) \leq \theta \left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}}.$$

Given this lemma, it is easy to proceed as follows. Fix $x_0 \in \Omega$ and $R > 0$ so that $\mu(B(x_0, R)) \geq 3/4$. Then for some m_0 and all $m \geq m_0$, we have $\mu_m(B(x_0, R)) \geq 2/3$, and hence by the lemma we conclude that:

$$\forall m \geq m_0 \; \forall t \geq 1 \quad \mu_m(\Omega \setminus B(x_0, tR)) \leq 2^{-\frac{t+1}{2}}.$$

Let f_m denote the 1-Lipschitz functions on Ω so that $M_{\mu_m} f_m = 0$ and $1/D_{FM}^M(\Omega, d, \mu_m) = \int |f_m| d\mu_m$ (we assume without loss of generality that the supremum is achieved). Since f_m

are continuous, $M_{\mu_m} f_m = 0$ and $\mu_m(B(x_0, R)) > 1/2$, there must exist a $x_m \in B(x_0, R)$ so that $f_m(x_m) = 0$. Since f_m are 1-Lipschitz, it follows that for any $t \geq 1$:

$$\begin{aligned}\int_{\Omega \setminus B(x_0,tR)} |f_m| d\mu_m &\leq \int_{\Omega \setminus B(x_0,tR)} d(x, x_m) d\mu_m(x) \\ &\leq d(x_m, x_0)\mu_m(\Omega \setminus B(x_0, tR)) + \int_{\Omega \setminus B(x_0,tR)} d(x, x_0) d\mu_m(x) \\ &\leq R\left(2^{-\frac{t+1}{2}} + \int_t^\infty 2^{-\frac{s+1}{2}} ds\right).\end{aligned}$$

Hence, given $\varepsilon > 0$, there exists a $t \geq 1$ so that:

$$\sup_{m \geq m_0} \left| \frac{1}{D^M_{FM}(\Omega, d, \mu_m)} - \int_{B(x_0,tR)} |f_m| d\mu_m \right| \leq \varepsilon.$$

But since our Lipschitz functions f_m are uniformly bounded on $B(x_0, tR)$ by $(t+1)R$ (by passing through x_m as before), the convergence of $\{\mu_m\}$ to μ in total-variation implies:

$$\lim_{m \to \infty} \sup_{m_1 \geq m_0} \left| \int_{B(x_0,tR)} |f_{m_1}| d\mu_m - \int_{B(x_0,tR)} |f_{m_1}| d\mu \right| = 0.$$

Finally, we note that for m large enough, by the Markov-Chebyshev inequality (we assume here without loss of generality that $M_\mu f_m \geq 0$):

$$\begin{aligned}\frac{1}{2} - \frac{1}{6} &\leq \mu_m\{f_m \leq 0\} - \frac{1}{6} \leq \mu\{f_m \leq 0\} \leq \mu\left\{\left|f_m - M_\mu f_m\right| \geq M_\mu f_m\right\} \\ &\leq \frac{1}{D^M_{FM}(\Omega, d, \mu) M_\mu f_m},\end{aligned}$$

so $|M_\mu f_m| \leq 3/D^M_{FM}(\Omega, d, \mu)$. Combining everything together, we deduce that for m large enough:

$$\begin{aligned}\frac{1}{D^M_{FM}(\Omega, d, \mu_m)} &\leq \varepsilon + \int_{B(x_0,tR)} |f_m| d\mu_m \leq 2\varepsilon + \int_{B(x_0,tR)} |f_m| d\mu \\ &\leq 2\varepsilon + \left|M_\mu f_m\right| + \int_\Omega |f_m - M_\mu f_m| d\mu \leq 2\varepsilon + \frac{4}{D^M_{FM}(\Omega, d, \mu)}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that:

$$D_{Che}(\Omega, d, \mu) \geq c \limsup_{m \to \infty} D^M_{FM}(\Omega, d, \mu_m) \geq \frac{c}{4} D^M_{FM}(\Omega, d, \mu).$$

This concludes the proof, since as usual, we may pass from D^M_{FM} to D_{FM} using Lemma 2.1.

For completeness, we provide a proof of Lemma 6.13, using the following remarkable generalization of the Prékopa-Leindler inequality (e.g. [24]) due to Cordero-Erausquin, McCann and Schmuckenschläger [29] (generalizing their own result from [28]). Given $x, y \in M$ and $s \in [0, 1]$, define:

$$Z_s(x, y) := \{z \in M; d(x, z) = sd(x, y) \text{ and } d(z, y) = (1 - s)d(x, y)\}.$$

Theorem 6.14 (Cordero-Erausquin–McCann–Schmuckenschläger) *Assume that* $d\mu = \exp(-\psi) d\mathrm{vol}_M$ *with* $\psi \in C^2(M)$ *and* $\mathrm{Ric}_g + \mathrm{Hess}_g \psi \geq 0$ *on* M. *Let* $s \in [0, 1]$ *and* $f, g, h : M \to \mathbb{R}_+$ *be such that*:

$$\forall x, y \in M \; \forall z \in Z_s(x, y) \quad h(z) \geq f^{1-s}(x) g^s(y).$$

Then:

$$\int_M h d\mu \geq \left(\int_M f d\mu\right)^{1-s} \left(\int_M g d\mu\right)^s.$$

Proof of Lemma 6.13 Let $t \geq 1$, and observe that:

$$\forall x \in B(x_0, R), \forall y \in M \setminus B(x_0, tR) \quad Z_{\frac{2}{t+1}}(x, y) \cap B(x_0, R) = \emptyset. \tag{6.9}$$

Indeed, if this is not so, there would exist a $z \in M$ so that:

$$d(x, z) = \frac{2}{t+1} d(x, y), \qquad d(z, y) = \frac{t-1}{t+1} d(x, y), \qquad d(z, x_0) < R.$$

But then:

$$d(y, x_0) \leq d(y, z) + d(z, x_0) < \frac{t-1}{t+1}(d(x, x_0) + d(x_0, y)) + R < \frac{t-1}{t+1} d(y, x_0) + \frac{2t}{t+1} R,$$

which would imply that $d(y, x_0) < tR$, a contradiction. Hence, (6.9) implies that the functions $f = \chi_{B(x_0,R)}$, $g = \chi_{M \setminus B(x_0,tR)}$ and $h = \chi_{M \setminus B(x_0,R)}$ satisfy the assumption of Theorem 6.14 with $s = \frac{2}{t+1}$. Theorem 6.14 then implies that:

$$1 - \theta \geq \theta^{\frac{t-1}{t+1}} \mu_m(M \setminus B(x_0, tR))^{\frac{2}{t+1}},$$

and the conclusion of the lemma follows. □

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Appendix

In the Appendix, we provide more details regarding the statement and ideas underlying the proof of Theorem 1.8 from the Introduction, as it plays an essential role in our argument. In the statement of this theorem, we have summarized a series of results in Riemannian Geometry concerning the concavity of the isoperimetric profile, which were proved under increasingly general convexity assumptions. An essential ingredient in the proofs of these results is provided by Geometric Measure Theory, which guarantees the existence and regularity of the isoperimetric minimizers, and permits the use of a variational argument to deduce the concavity of the profile.

A.1 Manifolds with uniform densities

First, we survey the case where the metric space (Ω, d) is given by a bounded domain (connected open set) with C^2 boundary in a smooth complete oriented connected n-dimensional $(n \geq 2)$ Riemannian manifold (M, g) along with the induced geodesic distance d in M, and the probability measure μ is given by the restriction to Ω of the Riemannian volume form vol_M on M, normalized so that $\mu(\Omega) = 1$. We summarize for completeness some remarkable results provided by Geometric Measure Theory about the existence and regularity of isoperimetric minimizers in the case we are considering, and refer to the books of Federer [31], Morgan [68], Giusti [35] and Burago and Zalgaller [25] for further information.

Theorem (Almgren [1, 2], Bombieri [22], Gonzales–Massari–Tamanini [36], Grüter [42], Morgan [69]) *For any $t \in (0, 1)$, there exists an open isoperimetric minimizer A of measure t for the isoperimetric problem on (Ω, d, μ) as above. The boundary $\Sigma = \overline{\partial A \cap \Omega}$ can be written as a disjoint union of a regular part Σ_r and a set of singularities Σ_s, with the following properties*:

- *$\Sigma_r \cap \Omega$ is a smooth, embedded hypersurface of constant mean curvature.*
- *Σ_r meets $\partial\Omega$ orthogonally.*
- *Σ_s is a closed set of Hausdorff co-dimension not smaller than* 8. *This result is sharp.*

For all the results to be described, it is essential that the Hausdorff co-dimension of the singular part of the boundary is large (although typically knowing that it is greater than 3 is sufficient). This approach was used by M. Gromov in his influential generalization of P. Lévy's isoperimetric inequality [38], [39, Appendix C]. The negligible singular part permits to consider a normal variation of the regular part, and from there on one may continue by using the readily available tools from Riemannian Geometry to calculate the first and second variations of volume and area. Before proceeding, we remark that most results we will mention deduce that the isoperimetric profile satisfies a second order differential inequality under more general convexity assumptions than stated (e.g. a negative lower bound on the Ricci curvature), and provide a characterization of the equality case as well.

The first convexity assumption which we add is that the Ricci curvature tensor Ric_g of (M, g) be non-negative. When M is a closed manifold and $\Omega = M$, and under the additional assumption that all isoperimetric minimizers are smooth submanifolds (this is always the case when $n \leq 7$), it was shown by Bavard and Pansu [11] that I is concave on $[0, 1]$. In fact, these authors attribute the same statement without the assumption on the smoothness of the isoperimetric minimizers to Bérard, Besson and Gallot. This was also formally verified by Morgan and Johnson [73, Sect. 2.1 and Proposition 3.3]. Gallot in [34, Corollary 6.6] showed that in fact the renormalized profile $I^{n/(n-1)}$ is concave in this case. This result captures the right dependence of the dimension in the exponent.

For our applications, the case where Ω is a proper subset of M is of most interest. In that case, to deduce the concavity of the isoperimetric profile, clearly one has to add some additional assumptions on Ω. When (M, g) is the Euclidean space $(\mathbb{R}^n, |\cdot|)$, it was first shown by Sternberg and Zumbrun [82] that a natural condition is that Ω be convex, in which case they showed that the profile I is indeed concave. This result was further strengthened by Kuwert [51], who showed that the renormalized profile $I^{n/(n-1)}$ is also concave. This was then generalized by Bayle and Rosales [13] to the case of a Riemannian manifold with non-negative Ricci curvature, under the assumption that Ω is *locally convex*:

Definition A domain $\Omega \subset (M, g)$ is said to be locally convex, if all geodesics in M tangent to $\partial\Omega$ are locally outside of Ω. By a result of Bishop [15], in case that Ω has C^2 boundary, this is equivalent to requiring that the second fundamental form of $\partial\Omega$ with respect to the normal pointing into Ω be positive semi-definite on all of $\partial\Omega$.

We summarize the above results in the following:

Theorem A.1 (Bavard–Pansu, Bérard–Besson–Gallot, Gallot, Morgan–Johnson, Sternberg–Zumbrun, Kuwert, Bayle–Rosales) *Let (M, g) be a smooth complete oriented connected Riemannian manifold of dimension $n \geq 2$ with non-negative Ricci curvature, and let Ω denote a locally convex bounded domain in (M, g). Let d denote the induced geodesic distance in (M, g) and μ the restriction to Ω of the canonical volume form* vol_M *on M, normalized so that $\mu(\Omega) = 1$. Assume in addition that Ω has C^2 smooth boundary. Then the isoperimetric profile $I = I_{(\Omega,d,\mu)}$ is a concave function on* $[0, 1]$. *Moreover, so is $I^{n/(n-1)}$*.

A.2 Manifolds with densities

As before, let (M, g) denote an n-dimensional ($n \geq 2$) smooth complete oriented connected Riemannian manifold with induced geodesic distance d. In addition, let $\psi \in C^2(M)$ be such that $d\mu = \exp(-\psi)d\mathrm{vol}_M$ is a probability measure on M. Since the influential work of Bakry and Émery [4] in the abstract framework of diffusion generators, it is known that a natural convexity condition on a manifold with density, which replaces the condition $\mathrm{Ric}_g \geq 0$ in the uniform density case, is to require the following $CD(0, \infty)$ Curvature-Dimension condition:

$$\mathrm{Ric}_g + \mathrm{Hess}_g \psi \geq 0 \quad \text{as 2-tensor fields.} \tag{A.1}$$

Theorem A.2 (Bayle [12], Morgan [70, 72]) *Let $\Omega = (M, g)$ and d, μ as above. Assume that (A.1) holds on Ω. Then $I = I_{(\Omega,d,\mu)}$ is a concave function on* $[0, 1]$.

This theorem was proved by Bayle in [12] under the assumption that M is a closed manifold. It was noted (without explanation) by Morgan [70, Corollary 9] that the same proof applies for a general complete manifold, as long as it has finite μ-measure. Indeed, Bayle's argument remains exactly the same; the only point one needs to check is the existence and regularity of isoperimetric minimizers in the manifold with density setting. The argument goes as follows: it was shown by Morgan in [69, Remark 3.10] that given a complete smooth Riemannian manifold with positive density $\rho \in C^k(M)$ ($k \geq 0$), if there exists an area minimizing current then its boundary is necessarily C^k regular outside a set of Hausdorff codimension at least 8. As explained e.g. in [57, 69, 70], the existence of an area minimizing current is guaranteed by the local compactness Theorem for currents (see [68]), as soon as the μ-measure of M is finite, which is always the case in our setting. Since the minimizing current is regular by the previous result, it follows that the usual notion of weighted area (i.e. Minkowski boundary measure) and the weighted area of a current coincide, and hence there exists a regular minimizer of Minkowski boundary measure.

The assumption that M has finite mass is essential for the existence of minimizers, otherwise one may construct counterexamples (see [14] or [12, p. 51]). It is also essential that the density be continuous, otherwise minimizers need not necessarily exist (consider the density $\frac{1}{4}\chi_{[0,1]\times[0,1]} + \chi_{[\frac{1}{4},1]\times[0,1]}$ on $[0,1]\times[0,1]$).

We remark that the same existence and regularity argument works for manifolds with a smooth boundary. Let $\Omega \subset (M,g)$ be a domain (connected open set) with C^2 boundary, let d be the geodesic distance induced by (M,g), and let $d\mu = \exp(-\psi) d\mathrm{vol}_M|_\Omega$ with $\psi \in C^2(\overline{\Omega})$ so that $\mu(\Omega) = 1$. One can easily check that the argument of Grüter [42] on the constant curvature of the regular part of the boundary and the orthogonality still applies, with a minor change in the conclusion. We summarize this in the following:

Theorem (Morgan [68, 69, 72], Grüter [42]) *For any $t \in (0,1)$, there exists an open isoperimetric minimizer A of measure t for the isoperimetric problem on (Ω, d, μ) as above. The boundary $\Sigma = \overline{\partial A \cap \Omega}$ can be written as a disjoint union of a regular part Σ_r and a set of singularities Σ_s, with the following properties*:

- *$\Sigma_r \cap \Omega$ is a C^2 smooth, embedded hypersurface of constant* generalized *mean curvature, defined as*:
$$H_{\Sigma_r,\psi}(x) := H_{\Sigma_r}(x) + \frac{1}{n-1} g_x(\nabla_x \psi, \nu_{\Sigma_r}(x)),$$
where $H_{\Sigma_r}(x)$ denotes the usual mean curvature of Σ_r in the direction of the unit normal $\nu_{\Sigma_r}(x)$ pointing into A (i.e. the trace of the second fundamental form divided by $(n-1)$), for $x \in \Sigma_r \cap \Omega$.
- *Σ_r meets $\partial\Omega$ orthogonally (even in the presence of a density).*
- *Σ_s is a closed set of Hausdorff co-dimension not smaller than* 8.

It is then a (tedious) exercise to follow the proof of Sternberg and Zumbrun [82] and Bayle [12] (see also [13]) and to deduce the following extension of Theorem A.2:

Theorem A.3 (after Sternberg and Zumbrun [82] and Bayle [12]) *Let $\Omega \subset (M,g)$ be a locally convex domain with C^2 boundary, and let d,μ as above. Assume that (A.1) holds on Ω. Then $I = I_{(\Omega,d,\mu)}$ is a concave function on* $[0,1]$.

In the one-dimensional case $n = 1$, it was shown by S. Bobkov [16] that all of the above theorems hold as well (here there is no point to consider a general manifold):

Theorem A.4 (Bobkov) *Let $(\Omega, d) = (\mathbb{R}, |\cdot|)$ and let μ be an arbitrary absolutely continuous log-concave measure on Ω. Then $I = I_{(\Omega,d,\mu)}$ is a concave function on* $(0,1)$.

Remark A.5 Bobkov showed that in this case, the minimizing sets are always given by half-lines, from which it is immediate that $I(t) = \min(F' \circ F^{-1}(t), F' \circ F^{-1}(1-t))$, where $F(s) = \mu(-\infty, s)$. Using that μ is log-concave, direct differentiation reveals that I is concave. Note that the case $n = 1$ is special since I may be discontinuous at 0 and 1, but this has absolutely no consequences to our applications.

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