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Space of Kähler metrics III – On the lower bound of the Calabi energy and geodesic distance

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1 Introduction

Inspired by the beautiful work of Donaldson [15], the author initiated a series of works [8,9] aiming to understand the geometric structure of the space of Kähler potentials and its application to interesting problems in Kähler geometry. The present work should be viewed as part III of this series. It consists of three inter-related parts:

1. First, we prove a folklore conjecture on the greatest lower bound of the Calabi energy in any Kähler class. This was known in the 1990s for Kähler metrics that are invariant under a maximal compact subgroup [23] of the automorphism group. See the acknowledgments for further remarks on this result.
2. Secondly, we give upper and lower bounds on the K-energy in terms of the geodesic distance and the Calabi energy. This is used to prove a theorem on convergence of Kähler metrics in holomorphic coordinates, with uniform bound on the Ricci curvature and the diameter. This kind of problem is difficult because Kähler geometry is extrinsic while the well-known Cheeger–Gromov convergence theorem (with bound on curvature, diameters) is intrinsic.

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3. Thirdly, we set up a framework for the existence of geodesic rays when an asymptotic direction is given. In particular, if the initial geodesic ray is tamed by a bounded ambient geometry (c.f. Definitions 3.2 and 3.3), then one can derive some relative $C^{1,1}$ estimates for other geodesic rays in the same direction. More in depth discussions on geodesic rays will be delayed to the beginning of Sect. 3. In a sequel to this paper, we will give more regularity estimates on geodesic rays.

1.1 Donaldson's conjectures. According to Calabi [6], extremal Kähler metrics are characterized as the critical points of the L^2 norm of the scalar curvature function in any given Kähler class. The class of extremal Kähler metrics includes the more famous Kähler Einstein metrics as a special case. In [15], Donaldson set out an ambitious program to attack core problems in Kähler geometry. He formally outlined a connection between geometric problems in an infinite dimensional space and the current interesting problems in Kähler geometry. In particular, he proposed three inter-related conjectures:

1. Any pair of Kähler potentials are uniquely connected by C^∞ geodesic segments.
2. The space of Kähler potentials is a metric space.
3. The non-existence of constant scalar curvature metrics is equivalent to a geodesic ray where the K-energy functional (cf. (2.7) for definition) decays at ∞.

In [9], following Donaldson's program, the author established the existence of $C^{1,1}$ geodesics by solving a Dirichlet boundary value problem for a homogenous complex Monge–Ampere equation. Consequently, the second conjecture of Donaldson is completely verified. Moreover, one important application is to show that Calabi's extremal Kähler metric (CextrK) is unique if the first Chern class is non-positive [9]. The uniqueness problem (with no assumption on c_1) is completely settled now. In algebraic manifolds with discrete automorphism groups, it was proved by S.K. Donaldson [16]. T. Mabuchi [26] removed the assumption on the automorphism group while Chen–Tian [12] completed the proof for general Kähler manifolds.

Chen–Tian [12] showed that the solution to the disc version of the geodesic problem is smooth except at most on a co-dimension 2 set with respect to generic boundary data. This is the best regularity result on $C^{1,1}$ solutions (to the disc version of the geodesic equation) established in [9]. For the convenience of the readers, we will briefly describe main results of [12] in Sect. 2; in particular, the partial regularity theory established in [12] for solutions of the disc version of the geodesic equation plays a crucial role in this paper (for obtaining a lower bound on the Calabi energy).

1.2 The Yau–Tian–Donaldson conjecture. The Calabi conjecture on the existence of Kähler Einstein (KE) metrics has driven the subject for the second half of the last century. In the late 1990s, S.T. Yau conjectured that the existence of Kähler Einstein metrics on Fano manifolds is equivalent to some form of stability with respect to the anti-canonical polarization. According to G. Tian [33] and Donaldson [15], this equivalence relation should be extended to include the case of a constant scalar curvature (cscK) metric in a general Kähler class. In a fundamental paper, G. Tian [33] introduced the notion of K-Stability and proved that the existence of KE metric implies weak K-stability. More recently, in a fundamental paper [16], Donaldson proved that, on an algebraic manifold with discrete automorphism group, the existence of a cscK metric implies that the underlying polarization is Chow stable. In this paper, Donaldson actually formulated a new version (but equivalent) of K-stability in terms of weights of Hilbert points. In Kähler toric varieties, the existence of a csck metric implies that the underlying polarization is semi-K-stable [18]. In [12], Chen–Tian proved that the existence of a cscK metric implies that the K-energy is bounded below on this Kähler class. As a corollary, this implies that the semi-K-stability of the underlying polarization (cf. [27]). After we announced our work [12], S.K. Donaldson [17] proved a similar lower bound in the algebraic setting.

1.3 On the existence of geodesic rays. The stability result of Donaldson's was extended by T. Mabuchi to the case of extremal Kähler metrics with a modified notion of stability. For general Kähler classes, the usual notion of stability does not apply because the manifold cannot be embedded in $\mathbb{C}P^N$ for any $N \gg 1$. In [15], Donaldson envisioned that a geodesic segment or geodesic ray in the space of Kähler potentials should play a similar role that a one parameter subgroup of $SL(N+1)$ plays in deforming a projective Kähler manifold. In the third conjecture of Donaldson's program, he defined a set of equivalence relationships:

1. There exists no cscK metric on M in the Kähler class $[\omega_0]$.
2. There exists a geodesic ray $\phi(t)$ in the space of Kähler potentials $\mathcal{H}$ initiating from some ϕ_0 such that the K-energy is strictly decreasing as $t \to \infty$.
3. For any $\phi \in \mathcal{H}$, there exists a geodesic ray initiating from ϕ such that the K-energy function is strictly decreasing as $t \to \infty$.

The first step toward proving this conjecture is establishing the existence of a geodesic ray parallel to some given geodesic ray. According to Calabi–Chen [8], the infinite dimensional space $\mathcal{H}$ is a non-positively curved space. By the triangle comparison theorem, we can show that there always exists a geodesic ray initiating from a given Kähler potential in the direction of any given geodesic ray. However, the geodesic rays obtained this way possess very little regularity; and it is very hard to use these "geodesics" in practice.[1] As a first step in this direction, we prove

[1] Note that in Donaldson's conjecture above, the geodesic ray should be geodesic ray with sufficient regularities.

Theorem 1.1 (cf. Theorem 3.9). *If there exists a geodesic ray (cf. Definition 3.2) $\rho : [0, \infty) \to \mathcal{H}$ which is tamed by a bounded ambient geometry, then for any Kähler potential $\varphi_0 \in \mathcal{H}$, there exists a relative $C^{1,1}$ geodesic ray $\varphi : [0, \infty) \to \mathcal{H}$ initiating from φ_0. Moreover, this geodesic ray is parallel to the original geodesic ray.*

Heuristically, a geodesic ray tamed by a bounded ambient geometry corresponds to the notion of a special degeneration of complex structure in the algebraic case. In Sect. 3, we will discuss at length, various issues related to stability (in terms of geodesic rays). It is expected that these notions are more-or-less equivalent to the corresponding notions in the algebraic setting.

1.4 On the lower bound of geodesic distance and the collapsing of Kähler manifolds. The famous work of Cheeger–Gromov states that the set of Riemannian metrics satisfying the following three conditions:

1. uniform curvature bound,
2. diameter is bounded from above,
3. volume is bounded from below,

is compact in the $C^{1,\alpha}$ topology for all $\alpha \in (0, 1)$ up to diffeomorphisms. For any sequence of Kähler metrics in a fixed Kähler class, the volume is *a priori* fixed. With uniform control of the curvature and diameter from above, a subsequence of Kähler metrics may converge to a limiting Kähler manifold with perhaps a different complex structure. In general, we don't know what additional geometrical condition is needed to ensure that the limiting complex structure is the same with the original complex structure. In fact, such a sequence might collapse in some Zariski open subset of the original Kähler manifold (i.e., the volume form vanishes in this subset) while the subsequence of Kähler metrics converges as Riemmanian metrics up to diffeomorphism (cf. [30]). In the discussion below, we will refer this phenomenon as "Kähler collapsing".

One intriguing and challenging question is: when this "Kähler collapsing" occurs, does the geodesic distance (in the space of Kähler metrics) necessarily diverge to ∞?[2] This in turn leads to another question: given a sequence of Kähler potentials, how do we estimate from below the geodesic distance from a fixed reference point? For instance, if the diameter of the sequence diverges to ∞ does the geodesic distance also diverge to ∞?

We first prove a theorem which links the K-energy, the Calabi energy and the geodesic distance together. The author believes that this theorem is very interesting in its own right.

[2] For a sequence of metrics mentioned above which does not converge in the original complex structure, one is expected to prove, via implicit function theory, that the geodesic distance (to some fixed Kähler metric) must diverge to ∞.

Theorem 1.2. *Let φ_0, φ_1 be two arbitrary smooth Kähler potentials in the same Kähler class. The following inequality holds*

$$\mathbf{E}(\varphi_1) - d(\varphi_0, \varphi_1) \cdot \sqrt{Ca(\varphi_1)} \leq \mathbf{E}(\varphi_0). \tag{1.1}$$

Here $d(\varphi_0, \varphi_1)$ is the geodesic distance in the space of Kähler potentials.

In other words, fixing φ_0 and letting φ_1 change, if the geodesic distance and the Calabi energy of φ_1 are bounded above, so is the K-energy. From an analytic point of view, this is quite surprising since we do not know how to control the K-energy, even assuming a uniform bound on the Riemmanian curvature. On the other hand, fixing φ_1 and letting φ_0 change, this formula gives a lower bound for the K-energy in terms of geodesic distance. Clearly, this inequality is a natural generalization of Theorem 1.1.2 of [12] that the K-energy has a lower bound if there is a cscK metric. In fact, we conjecture that, in a fixed Kähler class, if the infimum of the Calabi energy is 0, then the K-energy must be bounded below.

An immediate corollary is:

Corollary 1.3. *For any constant C, there exists a constant C' such that whenever φ is a Kähler potential with $Ca(\varphi) < C$ and $|\varphi|_\infty < C$, then its K-energy satisifies $E(\varphi) < C'$.*

We say that the K-energy functional is “proper” if it is bounded from below by a certain norm function which will be introduced in Sect. 2. We say that the K-energy functional is “quasi-proper” if the K-energy functional is bounded below by its highest order term (cf. Sect. 2). On a Kähler Einstein manifold, the K-energy functional is always proper [33]. On a general Kähler manifold, Tian conjectured that a cscK metric exists if and only if the K-energy functional is proper. When the first Chern class is non-positive, there is a sufficient condition for the K-energy functional in that Kähler class to be either proper or quasi proper[3].

Now we are ready to answer the question about Kähler collapsing:

Theorem 1.4 (No Kähler collapsing). *Let $(M, [\omega])$ be a Kähler manifold for which the K-energy is either proper or quasi-proper. Let S be a set of Kähler metrics in the class $[\omega]$ with Ricci curvature uniformly bounded and diameter bounded above[4]. If S lies in a bounded geodesic ball then all metrics in S are uniformly equivalent to each other in the $C^{1,\alpha}$ topology for any $\alpha \in (0, 1)$ in holomorphic coordinates. In particular, Kähler collapsing does not occur.*

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[3] For instance, in complex dimension 2, if

$$\frac{[\omega] \cdot [-C_1(M)]}{[\omega]^{[2]}} [-C_1(M)] - [\omega] > 0$$

then the K-energy is quasi-proper [10]. For higher dimensional Kähler manifolds, readers are referred to Song–Weinkove [3].

[4] The diameter bound can be replaced by a bound on a certain Sobolev constant.

Note that the geodesic distance appears to be a very weak notion. The bound on Ricci curvature is much weaker than the conditions stated in Cheeger–Gromov's theorem. However, the combination of the geodesic distance and Ricci curvature bounds seems to be very powerful.

In a subsequent paper, we will drop the assumption that the Ricci curvature is bounded from above. The assumption that the K-energy functional is either proper or quasi-proper in $(M, [\omega])$ is just technical. We hope that this will be removed in a subsequent work.

1.5 On the lower bound of the Calabi energy. It is well-known that the Calabi energy is locally convex near an extremal Kähler metric. According to Calabi [6], an extremal Kähler metric is invariant under a maximal compact subgroup of the automorphism groups of the underlying complex structure. In the 1990s, A. Hwang [23] proved that the Calabi energy of an invariant Kähler metric is bounded below by the absolute value of the Calabi–Futaki invariant (evaluated at the canonical extremal vector field). If there is an extremal Kähler metric in this class, the absolute value of the Calabi–Futaki invariant is precisely the Calabi energy of the extremal Kähler metric. In the 1980s, it was conjectured that the same lower bound holds for all metrics in the same Kähler class. There have been many attempts to generalize this to all Kähler metrics; however this problem proved to be a very difficult one.

Aside from this folklore conjecture on the Calabi energy, there are other important motivations for studying the greatest lower bound of the Calabi energy, such as stability and degeneration of Kähler manifolds. For our strategy to work, the main technical obstacle has been the insufficient regularity of the $C^{1,1}$ geodesic. However, the partial regularity theory established earlier (Theorem 1.3.4 in [12]) for solutions of the disc version of the geodesic equation is very helpful. By Theorem 1.3.5 of [12], we know that the K-energy functional, when restricted to the disc version of the geodesic equation, is sub-harmonic in the disc. This is used crucially to establish the sharp lower bound for the Calabi energy in terms of any effective destabilizing smooth geodesic ray (cf. Definition 3.13). In particular, we prove the following folklore conjecture about the greatest lower bound of the Calabi energy in each Kähler class.

Theorem 1.5. *For any Kähler metric ω_φ in $[\omega]$, we have*

$$Ca(\omega_\varphi) \geq \mathcal{F}_{\mathcal{X}_c}([\omega]),$$

where $\mathcal{X}_c$ is the extremal vector field of $(M, [\omega])$ and $\mathcal{F}$ is the Calabi–Futaki invariant. Equality holds when ω_φ is an extremal Kähler metric.

More generally, we have

Theorem 1.6. *Let $(M, [\omega])$ be a Kähler manifold and let ω_φ be any Kähler metric in the class $[\omega]$. Then*

$$Ca(\omega_\varphi) \geq \sup_\rho \yen(\rho)^2,$$

where the supremum runs over all possible effective destabilizing smooth geodesic rays $\rho : [0, \infty) \to \mathcal{H}$.

Definitions of an effective destabilizing smooth geodesic ray and the ¥ invariant are given in Definition 3.13 and Definition 3.10 respectively. To extend above theorem to a more general setting, we need to introduce two more notions.

Definition 1.7. *Let* $(M, [\omega_0], J_0)$ *be a Kähler manifold. Another Kähler manifold* $(M', [\omega'], J')$ *is said to lie in the closure of the diffeomorphism orbit of* $(M, [\omega_0], J_0)$ *if there exists a sequence of Kähler forms* $\{\omega_{\varphi_m}, m \in \mathbb{N}\} \subset [\omega]$ *and a sequence of diffeomorphisms* $\{f_m \in \mathrm{Diff}(M), m \in \mathbb{N}\}$ *such that* $(M, f_m^*\omega_{\varphi_m}, f_m^* J_0)$ *converges to* (M', ω', J') *in the* $C^{1,\alpha}$ *topology for some* $\alpha \in (0, 1)$.

Definition 1.8. *Let* $(M, [\omega_0], J_0)$ *be a Kähler manifold. Suppose that* $(M', [\omega'], J')$ *is another Kähler manifold which lies in the closure of the diffeomorphism orbit of* $(M, [\omega_0], J_0)$. $(M', [\omega'], J')$ *is said to destabilize the original Kähler manifold if there exists an effective destabilizing smooth geodesic ray* $\rho(t)$ *in* $(M, [\omega_0], J_0)$ *such that there is a subsequence of* $(M, \omega_{\rho(t)})(t \to \infty)$ *which converges to a metric on* $(M', [\omega'], J')$ *up to diffeomorphism.*

Now we are ready to state a general theorem.

Theorem 1.9. *Let* $(M, [\omega_0], J_0)$ *be a Kähler manifold. The following inequality holds*

$$\inf_{g\in\mathcal{H}} Ca(\omega_g) \geq \sup_{(M',[\omega'],J')} \inf_{(X,X)=1, X\in\mathcal{K}(J')} \frac{(X, \mathcal{X}_c)^2}{(X, X)}.$$

Here the inner product is the Futaki–Mabuchi inner product on the Lie algebra $\mathcal{K}(J')$ *of the maximum compact subgroup* $K(J')$ *of* $Aut(M', J')$. *The supremum runs over all possible Kähler manifolds* $(M', [\omega'], J')$ *in the closure of the diffeomorphism orbit of* $(M, [\omega_0], J_0)$ *that destabilize* $(M, [\omega_0], J_0)$.

One should be able to define a weak notion of destabilizing Kähler manifold, while retaining the validity of the inequality in Theorem 1.9.

Definition 1.10. *Suppose the Kähler manifold* $(M, [\omega], J)$ *satisfies the following inequality*

$$\mathcal{F}_{\mathcal{X}_c}([\omega_0]) \geq \sup_{(M',[\omega'],J')} \mathcal{F}_{\mathcal{X}_c}([\omega']), \tag{1.2}$$

where $(M', [\omega'], J')$ *runs over all Kähler manifolds in the closure of the diffeomorphism orbit of* $(M, [\omega], J)$. *Then we call* $(M, [\omega], J)$ *stable in the sense of differential geometry.*

What is the relation of stability in the sense of differential geometry with other algebro-geometric notions of stability such as K-stability? For algebraic manifolds, the notion of K-stability is expected to be stronger than this one.

In the light of these theorems, one expects that there is a deep interrelation between geodesic rays, test configurations and their respective roles in defining stability. We propose notions of stability in terms of smooth geodesic rays which might be viewed as a natural extension of what is given in [15]. Moreover, the relation between geodesic stability and K-stability may also be an interesting topic to be explored in near future. More extensive discussions on this topic will be deferred to Sect. 3.

Organization. In Sect. 2, we give a brief outline of known results in the space of Kähler potentials. In Sect. 3, we prove the existence of geodesic rays parallel to some "nice" geodesic rays. In Sect. 4, we give the greatest lower bound estimate for the Calabi energy. In Sect. 5, we give a lower bound estimate of the geodesic distance and rule out the possibility of "Kähler collapsing" in bounded geodesic balls in $\mathcal{H}$. In Sect. 6, we propose some conjectures in Kähler geometry.

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2 Brief outline of geometry in the space of Kähler potentials

2.1 Quick introduction to Kähler geometry. Let ω be a fixed Kähler metric on M. In a local holomorphic coordinate, ω can be expressed as

$$\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dw^{\alpha} \wedge dw^{\bar{\beta}} > 0.$$

The Ricci curvature can be conveniently expressed as

$$R_{\alpha\bar{\beta}} = -\frac{\partial^2 \log \det(g_{i\bar{j}})}{\partial w^{\alpha} \partial w^{\bar{\beta}}}.$$

The scalar curvature can be defined as

$$R = -g^{\alpha\bar{\beta}} \frac{\partial^2 \log \det(g_{i\bar{j}})}{\partial w^{\alpha} \partial w^{\bar{\beta}}}.$$

The Calabi energy is defined to be

$$Ca(\omega) = \int_M (R(\omega) - \underline{R})^2 \omega^n. \tag{2.1}$$

Here $\underline{R}$ is the average scalar curvature, whose value is the same for all metrics in a fixed Kähler class. Following Calabi [6, 7], a Kähler metric is called extremal if the complex gradient vector field

$$\mathcal{X}_c = g^{\alpha\bar{\beta}} \frac{\partial R}{\partial w^{\bar{\beta}}} \frac{\partial}{\partial w^{\alpha}} \tag{2.2}$$

is a holomorphic vector field. According to [21], the extremal vector field $\mathcal{X}_c$ is *a priori* determined in each Kähler class, up to holomorphic conjugation.

If X is a holomorphic vector field, then for any Kähler potential φ we can define θ_X up to an additive constant by

$$L_X \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\theta_X(\varphi). \tag{2.3}$$

Then, the well-known Calabi–Futaki invariant [20, 7] is

$$\mathcal{F}_X([\omega]) = \int_M \theta_X(\varphi) \cdot (\underline{R} - R(\varphi))\omega_\varphi^n. \tag{2.4}$$

Note that this is a Lie algebra character which depends on the Kähler class only.

2.2 Weil–Petersson type metric in the space of Kähler potentials. It follows from Hodge theory that the space of Kähler metrics with Kähler class $[\omega]$ can be identified with the space of smooth Kähler potentials

$$\mathcal{H} = \{\varphi \mid \omega_\varphi = \omega + \bar{\partial}\partial\varphi > 0, \text{ on } M\}/\sim,$$

where $\varphi_1 \sim \varphi_2$ if and only if $\varphi_1 = \varphi_2 + c$ for some constant c. A tangent vector in $T_\varphi\mathcal{H}$ is just a function ψ such that

$$\int_M \psi\omega_\varphi^n = 0.$$

Its norm in the L^2-metric on $\mathcal{H}$ is given by (cf. [25])

$$\|\psi\|_\varphi^2 = \int_M \psi^2\omega_\varphi^n.$$

This metric was subsequently re-defined in [31] and [15]. In all three papers, [25, 31] and [15], the authors defined this Weil–Petersson type metric from various points of view and proved formally that this infinite dimensional space has non-positive curvature. Using this definition, we can define a distance function in $\mathcal{H}$: For any two Kähler potentials $\varphi_0, \varphi_1 \in \mathcal{H}$, let $d(\varphi_0, \varphi_1)$ be the infimum of the length of all possible curves in $\mathcal{H}$ that connect φ_0 with φ_1.

A straightforward computation shows that a geodesic path $\varphi : [0,1] \to \mathcal{H}$ of this L^2 metric must satisfy the following equation

$$\varphi''(t) - g_{\varphi}{}^{\alpha\bar{\beta}} \frac{\partial^2 \varphi}{\partial t \partial w^{\alpha}} \frac{\partial^2 \varphi}{\partial t \partial w^{\bar{\beta}}} = 0,$$

where

$$g_{\varphi,\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial w^{\alpha} \partial w^{\bar{\beta}}} > 0.$$

According to S. Semmes [31], a smooth path $\{\phi(t), t \in [0,1]\} \subset \mathcal{H}$ satisfies the geodesic equation if and only if the function ϕ on $[0,1] \times S^1 \times M$ satisfies the homogeneous complex Monge–Ampere equation

$$\left(\pi_2^* \omega + \partial\bar{\partial}\phi\right)^{n+1} = 0, \quad \text{on } \Sigma \times M, \tag{2.5}$$

where $\Sigma = [0,1] \times S^1$ and $\pi_2 : \Sigma \times M \mapsto M$ is the projection. In fact, one can consider (2.5) over any Riemann surface Σ with boundary condition $\phi = \phi_0$ along $\partial\Sigma$, where ϕ_0 is a smooth function on $\partial\Sigma \times M$ such that $\phi_0(z, \cdot) \in \mathcal{H}$ for each $z \in \partial\Sigma$.[5] The equation (2.5) can be regarded as the infinite dimensional version of the WZW equation for maps from Σ into $\mathcal{H}$ (cf. [15]).[6]

Next we introduce three well-known functionals on $\mathcal{H}$. Note that our definition of the first two functionals may differ from some appearing in the literatures. First, the so-called I functional is defined as

$$\frac{dI(\varphi(t))}{dt} = \int_M \frac{\partial \varphi}{\partial t} \omega_{\varphi(t)}^n, \quad \varphi(t) \in \mathcal{H}.$$

The advantage of this functional is that it is constant along any geodesic segment (or ray). An explicit formula for the I functional is

$$I(\varphi) = \int_M \varphi \omega^n - \sum_{k=0}^{n-1} \frac{k+1}{n+1} \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_{\varphi}^{n-k-1}. \tag{2.6}$$

Secondly, the so-called J functional is defined as

$$J(\varphi) = \int_M \varphi\left(\omega^n - \omega_{\varphi}^n\right) = \int_M \partial\varphi \wedge \bar{\partial}\varphi \wedge \left(\sum_{k=0}^{n-1} \omega^k \wedge \omega_{\varphi}^{n-k-1}\right) > 0.$$

Finally, the K-energy functional (introduced by T. Mabuchi) is defined as a closed form $d\mathbf{E}$. Namely, for any $\psi \in T_{\varphi}\mathcal{H}$, we have

$$(d\mathbf{E}, \psi)_{\varphi} = \int_M \psi \cdot (\underline{R} - R(\varphi)) \omega_{\varphi}^n. \tag{2.7}$$

According to [10] (cf. Tian [33]), the K-energy functional can be expressed explicitly through the second derivatives of the Kähler potentials. For any

[5] We often regard ϕ_0 as a smooth map from $\partial\Sigma$ into $\mathcal{H}$.

[6] The original WZW equation is for maps from a Riemann surface into a Lie group.

$\phi \in \mathcal{H}_\omega$, we have

$$\mathbf{E}(\phi) = \int_M \ln \frac{\omega_\phi^n}{\omega^n}\omega^n + \underline{R} \cdot I(\varphi) \\ - \sum_{p=0}^{n-1} \frac{1}{p+1} \int_M \phi\, Ric(\omega) \wedge \omega^{n-p-1} \wedge (\sqrt{-1}\partial\bar\partial\phi)^p.$$

From this formulation, it is clear that the K-energy is well defined for any $C^{1,1}$ Kähler potential. We will exploit this important property later in this paper.

The K-energy functional is called proper (cf. [33]) in $(M, [\omega])$ if there exists a small constant $\delta > 0$ and two constants C_1, C_2 such that

$$\mathbf{E}(\varphi) \geq C_1 \cdot J(\varphi)^\delta - C_2.$$

The K-energy of $(M, [\omega])$ is called quasi-proper if there exists a small constant $\delta > 0$ and a constant C such that

$$\mathbf{E}(\varphi) \geq \delta \int_M \log \frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n - C. \tag{2.8}$$

Finally, we point out that the first derivative of the K-energy along a holomorphic family is precisely the Calabi–Futaki invariant. In other words, for any holomorphic vector field X we have

$$\mathcal{F}_X([\omega]) = (d\mathbf{E}, \theta_X)_\varphi.$$

2.3 Brief exposition of Theorem 1.3.4 in Chen–Tian [12]. We wish to point out that complex Monge–Ampere equations have been studied extensively (cf. [5,24,2] etc. and references therein). However, regularity for solutions of homogeneous complex Monge–Ampere (HCMA) equations beyond $C^{1,1}$ has been conspicuously missing. Indeed, there is a known example where solutions for a certain HCMA equation are only $C^{1,1}$, while the boundary value is smooth.

The best regularity result about the solution to the disc version of the geodesic equation is due to Chen–Tian [12] where they showed that the solution is smooth except at most on a codimension 2 set for generic boundary data. The $C^{1,1}$ bound derived in [9] plays a crucial role in [12].

Suppose that ϕ is a $C^{1,1}$ solution of (2.5), we denote by $\mathcal{R}_\phi$ the set of all $(z, x) \in \Sigma \times M$ near which ϕ is smooth and $\omega_{\phi(z',\cdot)} = \omega + \sqrt{-1}\partial\bar\partial\phi(z', \cdot)$ is a Kähler metric when restricted to the M factor. We may regard $\mathcal{R}_\phi$ as the regular set of ϕ. It is open, but *a priori*, it might be empty. We have a distribution $\mathcal{D}_\phi \subset T(\Sigma \times M)$ over $\mathcal{R}_\phi$:

$$\mathcal{D}_\phi|_{(z,x)} = \left\{v \in T_z\Sigma \times T_xM \mid i_v\left(\pi_2^*\omega + \sqrt{-1}\partial\bar\partial\phi\right) = 0\right\}, \quad \forall (z, x) \in \mathcal{R}_\phi. \tag{2.9}$$

Here i_v denotes the interior product. Since the form is closed, $\mathcal{D}_\phi$ is integrable when ϕ is smooth. We say that $\mathcal{R}_\phi$ is saturated in $\mathcal{V} \subset \Sigma \times M$ if every maximal integral sub-manifold of $\mathcal{D}_\phi$ in $\mathcal{R}_\phi \cap \mathcal{V}$ is a closed disk in $\mathcal{V}$.

As on any product manifold, we may write any vector in $\mathcal{D}_\phi$ as

$$\frac{\partial}{\partial z} + X \in \mathcal{D}_\phi|_{(z,x)}, \quad \text{where } X \in T_x^{1,0}M \tag{2.10}$$

for some choice of local coordinate z on Σ.

Definition 2.1. *A solution ϕ of* (2.5) *is called partially smooth if it is $C^{1,1}$-bounded on $\Sigma \times M$ and $\mathcal{R}_\phi$ is open and saturated in $\Sigma \times M$, but dense in $\partial\Sigma \times M$, such that the varying volume form $\omega^n_{\phi(z,\cdot)}$ extends to a continuous (n,n) form on $\Sigma^0 \times M$, where $\Sigma^0 = (\Sigma \backslash \partial\Sigma)$.*

Clearly, if ϕ is a partially smooth solution, then its regular set $\mathcal{R}_\phi$ consists of all points where the vertical volume form $\omega^n_{\phi(z,\cdot)}$ is positive in $\Sigma \times M$.

Definition 2.2. *We say that a solution ϕ of* (2.5) *is almost smooth if*

1. *It is partially smooth.*
2. *The distribution $\mathcal{D}_\phi$ extends to a continuous distribution in a saturated set $\tilde{\mathcal{V}} \subset \Sigma \times M$, such that the complement $\tilde{\mathcal{S}}_\phi$ of $\tilde{\mathcal{V}}$ has codimension at least 2 and ϕ is C^1 continuous on $\tilde{\mathcal{V}}$. The set $\tilde{\mathcal{S}}_\phi$ is referred to as the singular set of ϕ.*
3. *The leaf vector field X is uniformly bounded in $\mathcal{D}_\phi$.*

A smooth solution is certainly an almost smooth solution of (2.5). If the boundary values of a sequence of almost smooth solutions converge in some $C^{k,\beta}$ topology, then the sequence converges to a partially smooth solution in the $C^{1,\beta}$-topology ($k > 2, 0 < \beta < 1$).

Theorem 2.3 (Theorem 1.3.4 in [12]). *Suppose that Σ is a unit disc. For any $C^{k,\alpha}$ map $\phi_0 : \partial\Sigma \to \mathcal{H}$ ($k \geq 2$, $0 < \alpha < 1$) and for any $\epsilon > 0$, there exists a $\phi_\epsilon : \partial\Sigma \to \mathcal{H}$ in the ϵ-neighborhood of ϕ_0 in $C^{k,\alpha}(\Sigma \times M)$-norm, such that* (2.5) *has an almost smooth solution with boundary value ϕ_ϵ.*

An almost smooth solution of (2.5) has uniform $C^{1,1}$ bounds and is smooth almost everywhere. A detailed explanation (including definitions) can be found in [12]. However, the importance of this theorem lies in the following

Theorem 2.4 (Theorem 1.3.5 in [12]). *Suppose that ϕ is an almost smooth solution to* (2.5). *For every point $z \in \Sigma$, let $\mathbf{E}(z)$ be the K-energy (or modified K-energy) evaluated on $\phi(z,\cdot) \in \overline{\mathcal{H}}$. Then the first derivative of $\mathbf{E}$ is a uniformly continuous function in Σ up to the boundary. Moreover, it is a bounded sub-harmonic function on Σ in the sense of distributions, such that*

$$\int_{\mathcal{R}_\phi} \left| \mathcal{D}\frac{\partial \phi}{\partial \bar z}\right|^2_{\omega_{\phi(z,\cdot)}} \omega^n_{\phi(z,\cdot)} dz d\bar z \leq \int_{\partial\Sigma} \left.\frac{\partial \mathbf{E}}{\partial \mathbf{n}}\right|_{\partial\Sigma} ds,$$

where ds is the length element of $\partial\Sigma$. For any smooth function θ, $\mathcal{D}\theta$ denotes the $(2,0)$-part of the Hessian of θ with respect to the metric $\omega_{\phi(z,\cdot)}$.

3 On the existence of geodesic rays

3.1 Definitions and main results. As suggested in [15], one may view geodesic rays as substituting for the degenerations of projective Kähler manifolds arising from one parameter subgroups of $SL(N, C)$. It is natural to compare geodesic rays to the degenerations described by test configurations $\pi : \mathcal{X} \to \Delta$ where all fibers $\pi^{-1}(t)$ are biholomorphic except when $t = 0$. The central fiber usually carries a different complex structure with singularities. By blowing up a few points in the central fiber if necessary, it might be possible to make the total space smooth (or at least having some kind of bounded geometry). For any test configuration, it might be possible to prove that there is always a relatively $C^{1,1}$ geodesic ray which is asymptotically close to the test configuration near the central fiber. If the central fiber is smooth or smooth except along a sub-variety of codimension 4, one may speculate that the geodesic ray must be smooth generically except perhaps at a singular locus of codimension two or higher.

Motivated by the study of test configurations in the algebraic setting, in this section we restrict our attentions to the case of "nice" geodesic rays $\omega_{\rho(t)}$ $(t \in [0, \infty))$ which satisfy the following conditions:

1. The non-compact family $(M, \omega_{\rho(t)})$ can be compactified in some sense.
2. The limit of $(M, \omega_{\rho(t)})$ as $t \to \infty$ in a suitable topology is smooth in the "compactiftication" or has mild singularities (codimension 4 and higher).

The most special case of geodesic rays are those arising from a fixed gradient complex holomorphic vector field. In this case, the curvature of $(M, \omega_{\rho(t)})$ is uniformly bounded and the injectivity radius is uniformly bounded from below.

Let us introduce various concepts on geodesic ray.

Definition 3.1. *A path $\rho : [0, \infty) \to \mathcal{H}$ is called strictly convex if $\pi_2^*\omega_0 + i\partial\bar{\partial}\rho$ defines a Kähler metric in $([0, \infty) \times S^1) \times M$.*

Here $\pi_2 : ([0, \infty) \times S^1) \times M \to M$ is the natural projection map.

Definition 3.2. *A smooth geodesic ray $\rho : [0, \infty) \to \mathcal{H}$ is called special if it is one of the following types:*

1. Effective *if the Calabi energy of $\omega_{\rho(t)}$ in M is dominated by $\epsilon \cdot t^2$ for any $\epsilon > 0$ as $t \to \infty$.*
2. Normal *if the curvature of $\omega_{\rho(t)}$ in M is uniformly bounded for $t \in [0, \infty)$.*
3. Bounded geometry *if $(M, \omega_{\rho(t)})$ $(t \in [0, \infty))$ has uniform bounds on curvature and uniform positive lower bounds on the injective radius.*

Definition 3.3 (Bounded ambient geometry). *A Kähler metric $h = \pi_2^*\omega_0 + i\partial\bar{\partial}\bar{\rho}$ in $([0, \infty) \times S^1) \times M$ is said to have bounded ambient geometry if*

1. *It has a uniform bound on its curvature.*
2. $([0, T] \times S^1 \times M, h)$ *has a uniform lower bound on injectivity radius and the bound is independent of* $T \to \infty$.
3. *The vector length* $\left|\frac{\partial}{\partial t}\right|_h$ *has a uniform upper bound.*

Definition 3.4 (Tamed by a bounded ambient geometry). *A smooth geodesic ray* $(M, [\omega_{\rho(t)}])$ *is said to be tamed by a bounded ambient geometry h, if there is a uniform bound for the relative potential* $\rho - \bar{\rho}$.

For most purposes, "weakly tamed by a bounded ambient geometry weakly" is sufficient.

Definition 3.4a (Weakly tamed by a bounded ambient geometry). *A smooth geodesic ray* $(M, \omega_{\rho(t)})$ *is said to be weakly tamed by a bounded ambient geometry h, if there is a constant C such that*[7]

1. $\sup_t |n + 1 + \Delta_h(\rho - \bar{\rho})| \leq C$;
2. $\sup_t \left|\frac{\partial(\rho - \bar{\rho})}{\partial t}\right|_h \leq C$.

Remark 3.5. A smooth geodesic ray, tamed by a bounded ambient geometry, might be compared to a test configuration where the total space is smooth. In the future, we should broaden our definition of bounded ambient geometry to include the following situations:

1. The upper bound of the curvature of the ambient Kähler metric might not be uniform.
2. The injectivity radius may have a lower bound which depends on the distance to some singular sub-variety of higher codimension as well as on t.
3. The restriction of the ambient Kähler metric h to $\{t_i\} \times S^1 \times M$ may be some finite geodesic distance from $(M, \omega_{\rho(t_i)})$ while the later has certain geometric bounds (on, for example, the Calabi energy or Sobolev constant, cf. Theorem 1.4).

Of course, the regularity of the geodesic might have to be weakened a bit as well.

Let h be a Kähler metric in $([0, \infty) \times S^1) \times M$ with bounded ambient geometry. Suppose that its Kähler form $\tilde{\omega}$ is given by

$$\pi_2^* \omega_0 + \sum_{i,j=1}^{n} \frac{\partial^2 \bar{\rho}}{\partial w^i \partial w^{\bar{j}}} dw^i dw^{\bar{j}} + 2\,Re\left(\sum_{i=1}^{n} \frac{\partial^2 \bar{\rho}}{\partial w^i \partial \bar{z}} dw^i d\bar{z}\right) + \frac{\partial^2 \bar{\rho}}{\partial z \partial \bar{z}} dz d\bar{z}. \tag{3.1}$$

Here $z = t + \sqrt{-1}\theta$. In other words

$$\tilde{\omega} = \pi_2^* \omega_0 + \sqrt{-1}\partial\bar{\partial}\bar{\rho}. \tag{3.2}$$

In the remainder of this section, we will use h to denote any Kähler metric in $([0, \infty) \times S^1) \times M$ with bounded ambient geometry.

[7] It is sufficient to only assume these two inequalities hold for some sequence of $t_i \to \infty$ for technical purposes. This also applies to Definition 3.4.

Remark 3.6. Using Cauchy–Kowalevska's classical theorem, Arezzo–Tian [1] proved that a special degeneration of a complex structure when the central fiber is analytic is asymptotically equivalent to a smooth geodesic ray near the central fiber.

Example 3.7. Suppose that X is a gradient holomorphic vector field and let ω_0 be a Kähler form invariant under $Im(X)$. Let $\sigma(t), t \in [0, \infty)$ be the automorphism group generated by X. Set

$$\omega_{\rho(t)} = \sigma_t^* \omega_0.$$

A straightforward calculation shows that $\rho(t)$ $(t \in (-\infty, \infty))$ is a geodesic line. Let $\sigma = \sigma_1$ and let $g_1 = \sigma^* g_0$ and g_0 be the two Kähler metrics corresponding to ω_0 and $\sigma^*\omega_0$. Note that

$$z\frac{\partial}{\partial z} + X$$

induces a $\mathbb{C}^*$ action $\bar{\sigma}$ on $\Delta \times M$ which coincides with σ in the M directions and the multiplicity action in the Δ directions. Let $z_0 = 1$ and $z_k = \bar{\sigma}^k z_0 \to 0$. Set

$$M_{l,k} = \left\{ \frac{1}{2^l} \leq |z| \leq \frac{1}{2^k} \right\} \times M, \quad \forall l, k \in \mathbb{N},$$

and

$$M_{\infty,0} = (\Delta \setminus \{0\}) \times M.$$

It is easy to see that there is a smooth S^1 invariant Kähler metric $\bar{h}$ on $M_{1,0}$ such that

1. $\bar{h}|_{|z|=0} = g_0$ and $\bar{h}|_{|z|=1} = g_1$.
2. h and $\bar{\sigma}^* \bar{h}$ give rise a smooth metric in $M_{2,0}$.

Using $\bar{h}$, we can define a Kähler h on $(\Delta \setminus \{0\}) \times M$ simply by

$$h(z, \cdot) = \bar{\sigma}^{k*} \bar{h}, \quad \forall (z, \cdot) \in M_{k+1,k}.$$

By definition, h is a smooth metric in $(\Delta \setminus \{0\}) \times M$ which has bounded curvature and uniform positive lower bound on injectivity radius.

In fact, any normal geodesic ray is expected to be tamed by some bounded ambient geometry, at least when it has bounded geometry.

Definition 3.8. *Any two smooth geodesic rays $\rho_1, \rho_2 : [0, \infty) \to \mathcal{H}$, are called parallel if there exists a constant C such that*

$$\sup_{t \in [0,\infty)} |\rho_1(t) - \rho_2(t)|_{C^0} \leq C.$$

This notion of "parallelism" here is stronger than the usual one for finite dimensional hyperbolic manifolds (where "parallelism" is defined in terms of the geodesic distance between two geodesic rays). Since this is our first attempt in this direction, we are content with this stronger version. However, my sense is that when one of the geodesic ray is "nice" in some sense, then the two notions of "parallelism" are equivalent. This type of question is closely related to our discussions in Sect. 5 below and the reader is encouraged to read that part if interested.

In principle, given a smooth geodesic ray, there is some geodesic ray which initiates from any Kähler potential and which is parallel to the given geodesic ray as in Definition 3.8. However, these new "geodesic rays" usually only have very weak regularity. Now, let us re-state our main Theorem 1.1 here.

Theorem 3.9. *If there exists a smooth geodesic ray $\rho : [0,\infty) \to \mathcal{H}$ which is tamed by a bounded ambient geometry, then for any Kähler potential $\varphi_0 \in \mathcal{H}$, there exists a relative $C^{1,1}$ geodesic ray $\varphi : [0,\infty) \to \mathcal{H}$ initiating from φ_0 and parallel to ρ. More specifically, there exist two uniform constants λ and C such that*

$$0 \leq n + 1 + \tilde{\Delta}(\varphi(t,x) - \rho(t,x)) \leq C \exp \lambda(\rho(t,x) - \bar{\rho}(t,x)).$$

Here $\tilde{\Delta}$ is taken with respect to the ambient Kähler metric h. In particular, when

$$\rho(t,x) - \bar{\rho}(t,x)$$

is uniformly bounded, the resulting geodesic ray has a uniform $C^{1,1}$ bound in terms of the ambient metric h. The constants λ, C depend on h.

In [29], Phong–Sturm approximated $C^{1,1}$ geodesic segments (established in [9]) in algebraic manifolds via finite dimensional approximations. In light of the preceding theorem, it would be nice to find finite dimensional approximations of these relative $C^{1,1}$ geodesic rays also.

Definition 3.10. *For every smooth geodesic ray $\rho(t)(t \in [0,\infty))$, we can define an invariant as*

$$\yen(\rho) = \lim_{t\to\infty} \int_M \frac{\partial \rho(t)}{\partial t}(\underline{R} - R(\rho(t)))\omega_{\rho(t)}^n. \tag{3.3}$$

Remark 3.11. For a smooth geodesic ray, the K-energy is convex and the above invariant is well defined. The tricky part is how to extend this notion to the case of relative $C^{1,1}$ geodesic rays.

If a geodesic ray arises from a one parameter holomorphic transformation, the integral in (3.3) is just the usual Calabi–Futaki invariant.

A natural question is: If two geodesic rays are parallel to each other, are their ¥ invariants the same? The answer is a partial "yes":

Proposition 3.12. *For any smooth geodesic ray with bounded ambient geometry, any other smooth geodesic ray which is parallel to it must have the same* ¥ *invariant.*

Definition 3.13. *A smooth geodesic ray* $\rho : [0, \infty) \to \mathcal{H}$ *is called stable (resp. semi-stable) if* $¥(\rho) > 0$ *(resp.* ≥ 0*). It is called destabilizing if* $¥(\rho) < 0$ *and it is called effectively destabilizing if in addition,*

$$\limsup_{t\to\infty} \frac{1}{t^2} \cdot \int_M (R(\rho(t)) - \underline{R})^2 \omega^n_{\rho(t)} = 0.$$

Following the approach used in the algebraic case, we define (cf. [15]):

Definition 3.14. *A Kähler manifold* $(M, [\omega])$ *is called (effectively) geodesically stable if there is no (effective) destabilizing smooth geodesic ray. It is called weakly geodesically stable if the invariant* ¥ *is always non-negative for every smooth geodesic ray.*

We now re-state Theorem 1.6 more precisely as:

Theorem 3.15. *Suppose* $\rho : [0, \infty) \to \mathcal{H}$ *is an effective destabilizing smooth geodesic ray in* $\mathcal{H}$*, then*

$$\int_M (R(\varphi) - \underline{R})^2 \omega^n_\varphi \geq ¥(\rho)^2, \quad \forall \varphi \in \mathcal{H}.$$

In fact, we have

$$\inf_{\varphi\in\mathcal{H}} \int_M (R(\varphi) - \underline{R})^2 \omega^n_\varphi \geq \sup_\rho ¥(\rho)^2, \tag{3.4}$$

where the sup *in the right hand side of* (3.4) *runs over all possible effective destabilizing smooth geodesic rays.*

As an immediate corollary, we have

Corollary 3.16. *If there is a Kähler metric of cscK metric in* $[\omega]$*, then* $(M, [\omega])$ *is weakly effectively geodesic stable.*

One can generalize these results to the case of extremal Kähler metrics with non-constant scalar curvature. Recall that $\mathcal{X}_c$ is the canonical extremal vector field in $(M, [\omega])$ (cf. (2.2)) while $\theta(\mathcal{X}_c)$ is defined as (2.3).

Definition 3.10a. *For every smooth geodesic ray* $\rho(t \in [0, \infty))$*, we can define an invariant as*

$$\tilde{¥}(\rho) = \lim_{t\to\infty} \int_M \frac{\partial \rho(t)}{\partial t} (\underline{R} - R(\rho(t)) - \theta(\mathcal{X}_c))\omega^n_{\rho(t)}. \tag{3.5}$$

A smooth geodesic ray $\rho : [0, \infty) \to \mathcal{H}$ is called stable (resp. semi-stable) if $\tilde{¥}(\rho) > 0$ (resp. ≥ 0). It is called a destabilizing for $\mathcal{H}$ if $\tilde{¥}(\rho) < 0$ and effective destabilizing for $\mathcal{H}$ if in addition

$$\limsup_{t\to\infty} \frac{1}{t^2} \cdot \int_M (R(\rho(t)) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_{\rho(t)} = 0.$$

With essentially the same proof, we have

Theorem 3.15a. *Suppose $\rho : [0, \infty) \to \mathcal{H}$ is a smooth effectively destabilizing geodesic ray in $\mathcal{H}$, then*

$$\int_M (R(\varphi) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_\varphi \geq \tilde{¥}(\rho)^2, \quad \forall \varphi \in \mathcal{H}.$$

In fact, we have

$$\inf_{\varphi\in\mathcal{H}} \int_M (R(\varphi) - \underline{R} - \theta(\mathcal{X}_c))^2 \omega^n_\varphi \geq \sup_\rho \tilde{¥}(\rho)^2, \tag{3.6}$$

where the sup in the right hand side runs over all possible effectively destabilizing smooth geodesic rays. In particular, $(M, [\omega])$ is weakly effectively geodesically stable if there exists an extremal Kähler metric in the Kähler class $[\omega]$.

3.2 Proof of Theorem 3.9. In this subsection, we will give a proof of the existence of a geodesic ray parallel to some initial geodesic ray which is tamed by some bounded ambient geometry. One of the main challenges here has been to find the right condition to ensure the existence of a parallel geodesic ray with regularity beyond the L^2 topology on the evolved Kähler potentials. We follow the main steps in [9] under the current circumstances: for any given Kähler potential φ_0, we can pick a sequence of Kähler potentials $\{\rho(t_i)\}(i \in \mathbb{N})$ along the given geodesic ray $\rho : [0, \infty) \to \mathcal{H}$. Connecting φ_0 to $\rho(t_i)(i \in \mathbb{N})$ via the unique $C^{1,1}$ geodesic segment established in [9], we obtain a sequence of $C^{1,1}$ geodesic segments and hope to take a limit as $t_i \to \infty$. The main difficulty is to obtain some uniform $C^{1,1}$ bound which allows us to take a limit. However, without any additional assumptions, such an approach runs into a serious problem as we shall explain now. First, there is no absolute C^0 estimate, which is crucial to Yau's calculation of the second derivatives. Secondly, when we consider the blowing-up estimate, the non-compactness of the underlying Kähler manifold presents a tough challenge. This is because the quantities we want to control are measured against a background metric. In the case of a geodesic segment, these background metrics at each time slice are identical to each other. In the case of a geodesic ray, the background metric in fact varies with time. It is then important that this non-compact family of background Kähler metrics has some sort of compactness before sense can be made of any blowing-up arguments. Thirdly, in deriving boundary estimates as in [9], we need the assumption that the restriction of Kähler metric in $\partial\Sigma \times M$ has a uniform positive lower bound with respect to some fixed metric. This is clearly not available since the sequence of metrics along a geodesic ray are expected to either diverge wildly or converge to a Kähler metric in a different complex structure. In a typical scenario, this sequence of metrics (along a geodesic ray) will degenerate along generic points in the Kähler manifold and will blow up along some divisor. To overcome this difficulty, we assume the existence of a bounded ambient metric which tames the initial geodesic ray (cf. Definitions 3.2–3.4) to find a relative C^0 bound on the modified potential. In order to find a C^2 estimate in terms of this relative C^0 estimate, we need to exploit the structure of the HCMA equation more carefully. In particular, if the modified potential does not have a uniform C^0 bound, we need to re-design the blowing up procedure in [9] to obtain a growth control of the $C^{1,1}$ bound on the modified potential. We believe that such a technique may be applicable to some other interesting cases in the future.

3.2.1 Relative C^0 estimates. Let us first set up some notation. Let $T \gg 1$ be a large positive number. Let $\Sigma_T = [0, T] \times S^1$. In the $(n+1)$ dimensional Kähler manifold $\Sigma_T \times M$, we want to solve the Dirichlet problem for HCMA equation (2.5) where the boundary data is invariant in the S^1 direction. As in [9], for any T and for any smooth boundary data, we can obtain a unique $C^{1,1}$ solution $\phi(t)$ that solves the HCMA equation (2.5). In other words, we have

$$\left(\pi_2^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi\right)^{n+1} = 0, \tag{3.7}$$

where

$$\phi(0) = \varphi_0, \quad \text{and} \quad \phi(T) = \rho(T). \tag{3.8}$$

Here, $\rho : [0, \infty) \to \infty$ is the smooth geodesic ray we wish to construct a geodesic ray parallel to. Obviously, the $C^{1,1}$ estimate on ϕ depends on T and may blow up as $T \to \infty$. In fact, this $C^{1,1}$ estimate must blow up if it represents a geodesic ray.

Lemma 3.17. *For any smooth geodesic ray $\rho(t)(t \in [0, \infty))$ and for any initial metric $\varphi_0 \in \mathcal{H}$, there exists a uniform constant C such that for any $T \in (0, \infty)$, there exists a unique $C^{1,1}$ geodesic $\phi_T(t)(t \in [0, T])$ which connects φ_0 to $\rho(T)$ such that*

$$-C \le \phi_T(t, x) - \rho(t, x) \le C. \tag{3.9}$$

The Dirichlet boundary value problem (3.7) and (3.8) can be re-written as a Dirichlet problem on $\Sigma_T \times M$ such that

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2(\phi - \bar{\rho})}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)_{(n+1)\times(n+1)} = 0, \tag{3.10}$$

with boundary condition

$$\phi|_{\{0\}\times S^1\times M} = \varphi_0, \quad \text{and} \quad \phi|_{\{T\}\times S^1\times M} = \rho(T). \tag{3.11}$$

Here h is the bounded ambient metric defined in (3.1) and (3.2). Now we are ready to give a proof of Lemma 3.17.

Proof. Following [9], let ϕ_T^ϵ be the ϵ-approximated geodesic between φ_0 and $\rho(T)$ and let ρ_T^ϵ be the ϵ-approximated geodesic between $\rho(0)$ and $\rho(T)$. Then, we have (in $\Sigma_T \times M$)

$$\begin{aligned}\det\left(h_{\alpha\bar\beta} + \frac{\partial^2\left(\phi_T^\epsilon - \bar\rho\right)}{\partial w^\alpha \partial w^{\bar\beta}}\right)_{(n+1)\times(n+1)} &= \epsilon \cdot \det h \\ &= \det\left(h_{\alpha\bar\beta} + \frac{\partial^2\left(\rho_T^\epsilon - \bar\rho\right)}{\partial w^\alpha \partial w^{\bar\beta}}\right)_{(n+1)\times(n+1)}\end{aligned}$$

with

$$\phi_T^\epsilon(0) = \varphi_0, \quad \rho_T^\epsilon(0) = \rho(0), \quad \phi_T^\epsilon(T) = \rho_T^\epsilon(T) = \rho(T).$$

Set

$$\begin{aligned} C &= \max_{\partial\Sigma_T\times M} \left|\left(\phi_T^\epsilon - \bar\rho\right) - \left(\rho_T^\epsilon - \bar\rho\right)\right| \\ &= \max_{\partial\Sigma_T\times M} \left|\phi_T^\epsilon - \rho_T^\epsilon\right| \\ &= \max_M |\varphi_0 - \rho(0)|. \end{aligned}$$

Note that this constant C is independent of T. By the maximum principle for the Monge Ampere equation, we have

$$\max_{\Sigma_T\times M} \left|\left(\phi_T^\epsilon - \bar\rho\right) - \left(\rho_T^\epsilon - \bar\rho\right)\right| \le \max_{\partial\Sigma_T\times M} \left|\left(\phi_T^\epsilon - \bar\rho\right) - \left(\rho_T^\epsilon - \bar\rho\right)\right|.$$

In other words,

$$\max_{\Sigma_T\times M} \left|\phi_T^\epsilon - \rho_T^\epsilon\right| < C.$$

Let $\epsilon \to 0$, we obtain the desired estimate.

$$\max_{\Sigma_T\times M} |\phi_T - \rho| < C.$$

□

From the proof, it is clear that we do not need to make any assumption on h in this lemma except that the initial geodesic ray be tamed by h. Now the main problem is to decide if any higher derivative of the sequence $\{\phi_T\}$ can be controlled uniformly in some sense as $T \to \infty$?

3.2.2 Relative $C^{1,1}$ estimates for the HCMA equation in unbounded domains. In this subsection, we want to solve (3.10) and (3.11) for any large $T > 0$. In this subsection, we do need to assume that the initial geodesic ray ρ is tamed by the ambient metric h. Set

$$\tilde{\psi}_T(t,x) = \phi_T(t,x) - \bar{\rho}(t,x). \tag{3.12}$$

For a sequence of numbers $T_i \to \infty$ we have

$$\tilde{\psi}_{T_i}(T_i,x) = \phi_{T_i}(t_i,x) - \bar{\rho}(T_i,x) = \rho(T_i,x) - \bar{\rho}(T_i,x).$$

According to Lemma 3.17, the modified potential $\tilde{\psi}_T(t,x)$ has uniform C^0 bound. For simplicity, we drop the explicit dependency on T in this subsection.

As in [9], we want to use the method of continuity. So we set up the problem as

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2\tilde{\psi}}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)_{(n+1)\times(n+1)} = \epsilon \det(h_{i\bar{j}})_{n\times n}, \tag{3.13}$$

with boundary condition

$$\tilde{\psi}|_{\{0\}\times S^1\times M} = \varphi_0 - \bar{\rho}(0), \quad \text{and} \quad \tilde{\psi}|_{\{T\}\times S^1\times M} = \rho(T) - \bar{\rho}(T). \tag{3.14}$$

For any T fixed, this Dirichlet boundary value has a unique $C^{1,1}$ solution as in [9]. The challenge at hand is how to obtain a $C^{1,1}$ estimate when $T \to \infty$ and $\epsilon \to 0$.

Put $\omega_{\bar{\rho}(t)} = \frac{\sqrt{-1}}{2} h_{\alpha\bar{\beta}} dw^\alpha \wedge w^{\bar{\beta}}$ and $\omega_{\phi(t)} = \frac{\sqrt{-1}}{2} g'_{\alpha\bar{\beta}} dw^\alpha \wedge dw^{\bar{\beta}}$ where

$$g'_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} + \frac{\partial^2\tilde{\psi}}{\partial w^\alpha \partial w^{\bar{\beta}}}.$$

Denote two Laplacian operators by

$$\Delta' = \sum_{\alpha,\beta=1}^{n} g'^{\alpha\bar{\beta}} \frac{\partial^2}{\partial w^\alpha \partial w^{\bar{\beta}}}, \quad \tilde{\Delta} = \sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}} \frac{\partial^2}{\partial w^\alpha \partial w^{\bar{\beta}}}.$$

Following [9], we can prove

Lemma 3.18. *There exist two constants λ, C which depend only on the ambient metric h (independent of T) and initial Kähler potentials $\varphi_0, \bar{\rho}(0)$ such that*

$$e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t)) \leq C \cdot \max_{t=0,t=T} e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t)).$$

Proof. Following Yau's proof of the Calabi conjecture in [34], we want to calculate $\Delta'\left(n+1+\tilde{\Delta}\tilde{\psi}\right)$ first.

For any point in $\Sigma_T \times M$, let us fix a coordinate system so that at this point, both $\omega_{\bar{\rho}(t)} = \sqrt{-1} h_{\alpha\bar{\beta}} dw^\alpha \otimes dw^{\bar{\beta}}$ and the complex Hessian of $\tilde{\psi}$ are in diagonal forms. Moreover, the connection of $\omega_{\bar{\rho}(t)}$ vanishes at the same point. In other words, we assume that $h_{i\bar{j}} = \delta_{i\bar{j}}$ and $\tilde{\psi}_{i\bar{j}} = \delta_{i\bar{j}}\tilde{\psi}_{i\bar{i}}$. Thus

$$g'^{i\bar{s}} = \frac{\delta_{i\bar{s}}}{1+\tilde{\psi}_{i\bar{i}}}.$$

For convenience, put

$$F = \ln \epsilon + \log \det(h_{i\bar{j}}).$$

Then our equation reduces to

$$\log \det \left(h_{i\bar{j}} + \frac{\partial^2 \tilde{\psi}}{\partial w_i \partial w_{\bar{j}}} \right) = F + \log \det(h_{i\bar{j}}).$$

We first follow the standard calculation of C^2 estimates in [34]. Differentiate both sides with respect to $\frac{\partial}{\partial w_k}$

$$(g')^{i\bar{j}} \left(\frac{\partial h_{i\bar{j}}}{\partial w_k} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} \right) - h^{i\bar{j}} \frac{\partial h_{i\bar{j}}}{\partial w_k} = \frac{\partial F}{\partial w_k},$$

and differentiating again with respect to $\frac{\partial}{\partial \bar{w}_l}$ yields

$$\begin{aligned}
&(g')^{i\bar{j}} \left(\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l} \right) + h^{t\bar{j}} h^{i\bar{s}} \frac{\partial h_{t\bar{s}}}{\partial \bar{w}_l} \frac{\partial h_{i\bar{j}}}{\partial w_k} - h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} \\
&\quad - (g')^{t\bar{j}} (g')^{i\bar{s}} \left(\frac{\partial h_{t\bar{s}}}{\partial \bar{w}_l} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l} \right) \left(\frac{\partial h_{i\bar{j}}}{\partial w_k} + \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} \right) = \frac{\partial^2 F}{\partial w_k \partial \bar{w}_l}.
\end{aligned}$$

Assume that we have normal coordinates at the given point, i.e., $h_{i\bar{j}} = \delta_{ij}$ and the first order derivatives of g vanish. Now taking the trace of both sides, we have

$$\begin{aligned}
\tilde{\Delta} F &= h^{k\bar{l}} (g')^{i\bar{j}} \left(\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l} \right) \\
&\quad - h^{k\bar{l}} (g')^{t\bar{j}} (g')^{i\bar{s}} \frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l} \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} - h^{k\bar{l}} h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l}.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\Delta'(\tilde{\Delta} \tilde{\psi}(t)) &= (g')^{k\bar{l}} \frac{\partial^2}{\partial w_k \partial \bar{w}_l} \left(h^{i\bar{j}} \frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j} \right) \\
&= (g')^{k\bar{l}} h^{i\bar{j}} \frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l} + (g')^{k\bar{l}} \frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l} \frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j},
\end{aligned}$$

and we will substitute $\frac{\partial^4 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k \partial \bar{w}_l}$ in $\Delta'(\tilde{\Delta} \tilde{\psi}(t))$ so that the above reads

$$\begin{aligned}
\Delta'(\tilde{\Delta} \tilde{\psi}(t)) &= -h^{k\bar{l}} (g')^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + h^{k\bar{l}} (g')^{t\bar{j}} (g')^{i\bar{s}} \frac{\partial^3 \tilde{\psi}(t)}{\partial w_t \partial \bar{w}_s \partial \bar{w}_l} \frac{\partial^3 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j \partial w_k} \\
&\quad + h^{k\bar{l}} h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} + \tilde{\Delta} F + (g')^{k\bar{l}} \frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l} \frac{\partial^2 \tilde{\psi}(t)}{\partial w_i \partial \bar{w}_j},
\end{aligned}$$

which we can rewrite after substituting $\frac{\partial^2 h_{i\bar{j}}}{\partial w_k \partial \bar{w}_l} = -R_{i\bar{j}k\bar{l}}$ and $\frac{\partial^2 h^{i\bar{j}}}{\partial w_k \partial \bar{w}_l} = R_{j\bar{i}k\bar{l}}$ as

$$\begin{aligned}\Delta'(\tilde{\Delta}\tilde{\psi}(t)) = \tilde{\Delta}F + h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}}\tilde{\psi}(t)_{t\bar{s}l}\tilde{\psi}(t)_{i\bar{j}k}& \\ + (g')^{i\bar{j}}h^{k\bar{l}}R_{i\bar{j}k\bar{l}} - h^{i\bar{j}}h^{k\bar{l}}R_{i\bar{j}k\bar{l}} + (g')^{k\bar{l}}R_{j\bar{i}k\bar{l}}\tilde{\psi}(t)_{i\bar{j}}.&\end{aligned}$$

Restrict to the coordinates we chose in the beginning so that both g and $\tilde{\psi}(t)$ are in diagonal form. The above transforms to

$$\begin{aligned}\Delta'(\tilde{\Delta}\tilde{\psi}(t)) &= \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{j\bar{j}}}\tilde{\psi}(t)_{i\bar{j}k}\tilde{\psi}(t)_{\bar{i}j\bar{k}} + \tilde{\Delta}F \\ &\quad + R_{i\bar{i}k\bar{k}}\left(-1 + \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} + \frac{\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}}\right).\end{aligned}$$

Set now $C = \inf_{i\neq k} R_{i\bar{i}k\bar{k}}$ and observe that

$$\begin{aligned}R_{i\bar{i}k\bar{k}}\left(-1 + \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} + \frac{\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}}\right) &= \frac{1}{2}R_{i\bar{i}k\bar{k}}\frac{(\tilde{\psi}(t)_{k\bar{k}} - \tilde{\psi}(t)_{i\bar{i}})^2}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\psi(t)_{k\bar{k}})} \\ &\geq \frac{C}{2}\frac{(1+\tilde{\psi}(t)_{k\bar{k}} - 1 - \psi(t)_{i\bar{i}})^2}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\tilde{\psi}(t)_{k\bar{k}})} \\ &= C\left(\frac{1+\tilde{\psi}(t)_{i\bar{i}}}{1+\tilde{\psi}(t)_{k\bar{k}}} - 1\right),\end{aligned}$$

which yields

$$\begin{aligned}\Delta'(\tilde{\Delta}\tilde{\psi}(t)) &\geq \frac{1}{(1+\tilde{\psi}(t)_{i\bar{i}})(1+\tilde{\psi}(t)_{j\bar{j}})}\tilde{\psi}(t)_{i\bar{j}k}\tilde{\psi}(t)_{\bar{i}j\bar{k}} + \tilde{\Delta}F \\ &\quad + C\left((n+1+\tilde{\Delta}\tilde{\psi}(t))\sum_i \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} - 1\right).\end{aligned}$$

We need to apply one more trick to obtain the requested estimates. Namely,

$$\begin{aligned}&\Delta'(e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t))) \\ &\quad = e^{-\lambda\tilde{\psi}(t)}\Delta'(\tilde{\Delta}\psi(t)) + 2\nabla' e^{-\lambda\tilde{\psi}(t)}\nabla'(n+\tilde{\Delta}\psi(t)) \\ &\qquad + \Delta'(e^{-\lambda\tilde{\psi}(t)})(n+1+\tilde{\Delta}\tilde{\psi}(t)) \\ &\quad = e^{-\lambda\tilde{\psi}(t)}\Delta'(\tilde{\Delta}\tilde{\psi}(t)) - \lambda e^{-\lambda\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_i(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}} \\ &\qquad - \lambda e^{-\lambda\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_{\bar{i}}(\tilde{\Delta}\tilde{\psi}(t))_i \\ &\qquad - \lambda e^{-\lambda\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\tilde{\Delta}\tilde{\psi}(t)) \\ &\qquad + \lambda^2 e^{-\lambda\tilde{\psi}(t)}(g')^{i\bar{i}}\tilde{\psi}(t)_i\tilde{\psi}(t)_{\bar{i}}(n+1+\tilde{\Delta}\tilde{\psi}(t))\end{aligned}$$

$$
\begin{aligned}
&\ge e^{-\lambda\tilde{\psi}(t)}\Delta'(\tilde{\Delta}\tilde{\psi}(t)) \\
&\quad - e^{-\lambda\tilde{\psi}(t)}(g')^{i\bar{i}}(n+\tilde{\Delta}\tilde{\psi}(t))^{-1}(\tilde{\Delta}\tilde{\psi}(t))_i(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}} \\
&\quad - \lambda e^{-\lambda\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\tilde{\Delta}\tilde{\psi}(t)),
\end{aligned}
$$

which follows from the Schwarz lemma applied to the middle two terms. We will write out one term here; the other goes in an analogous way.

$$
\begin{aligned}
&\big(\lambda e^{-\frac{\lambda}{2}\tilde{\psi}(t)}\tilde{\psi}(t)_i(n+\tilde{\Delta}\tilde{\psi}(t))^{\frac{1}{2}}\big)\big(e^{-\frac{\lambda}{2}\tilde{\psi}(t)}(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}}(n+1+\tilde{\Delta}\tilde{\psi}(t))^{-\frac{1}{2}}\big) \\
&\qquad \le \frac{1}{2}\big(\lambda^2 e^{-\lambda\tilde{\psi}(t)}\tilde{\psi}(t)_i\tilde{\psi}(t)_{\bar{i}}(n+1+\tilde{\Delta}\tilde{\psi}(t)) \\
&\qquad\qquad + e^{-\lambda\tilde{\psi}(t)}(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}}(\tilde{\Delta}\tilde{\psi}(t))_i(n+\tilde{\Delta}\tilde{\psi}(t))^{-1}\big).
\end{aligned}
$$

Consider now the following

$$
\begin{aligned}
&-(n+1+\tilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}(\tilde{\Delta}\tilde{\psi}(t))_i(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}} + \Delta'\tilde{\Delta}\tilde{\psi}(t) \\
&\ge -(n+1+\tilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}|\tilde{\psi}(t)_{k\bar{k}i}|^2 + \tilde{\Delta}F \\
&\quad + \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{i}\bar{j}}\tilde{\psi}(t)_{i\bar{k}j} + C(n+1+\tilde{\Delta}\tilde{\psi}(t))\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}.
\end{aligned}
$$

On the other hand, using the Schwarz inequality, we have

$$
\begin{aligned}
&(n+\tilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}|\tilde{\psi}(t)_{k\bar{k}i}|^2 \\
&\quad = (n+1+\tilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\left|\frac{\tilde{\psi}(t)_{k\bar{k}i}}{(1+\tilde{\psi}(t)_{k\bar{k}})^{\frac{1}{2}}}(1+\tilde{\psi}(t)_{k\bar{k}})^{\frac{1}{2}}\right|^2 \\
&\quad \le (n+1+\tilde{\Delta}\psi(t))^{-1}\left(\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{k}i}\tilde{\psi}(t)_{\bar{k}k\bar{i}}\right)(1+\tilde{\psi}(t)_{l\bar{l}}) \\
&\quad = \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{k\bar{k}i}\tilde{\psi}(t)_{\bar{k}k\bar{i}} \\
&\quad = \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\psi(t)_{i\bar{k}k}\tilde{\psi}(t)_{k\bar{i}\bar{k}} \\
&\quad \le \frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\frac{1}{1+\tilde{\psi}(t)_{k\bar{k}}}\tilde{\psi}(t)_{i\bar{k}j}\tilde{\psi}(t)_{k\bar{i}\bar{j}},
\end{aligned}
$$

so that we get

$$
\begin{aligned}
&-(n+\tilde{\Delta}\tilde{\psi}(t))^{-1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}(\tilde{\Delta}\tilde{\psi}(t))_i(\tilde{\Delta}\tilde{\psi}(t))_{\bar{i}} + \Delta'\tilde{\Delta}\tilde{\psi}(t) \\
&\qquad\qquad \ge \tilde{\Delta}F + C(n+1+\tilde{\Delta}\tilde{\psi}(t))\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}.
\end{aligned}
$$

Putting all these together, we obtain

$$\begin{aligned}
&\Delta'(e^{-\lambda\tilde{\psi}(t)}(n+\tilde{\Delta}\tilde{\psi}(t)))\\
&\qquad \geq e^{-\lambda\tilde{\psi}(t)}\left(\tilde{\Delta}F + C(n+1+\tilde{\Delta}\tilde{\psi}(t))\sum_{i=1}^{n}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\right)\\
&\qquad\quad - \lambda e^{-\lambda\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\tilde{\Delta}\tilde{\psi}).
\end{aligned} \tag{3.15}$$

Consider

$$\tilde{\Delta}F = h^{\alpha\bar{\beta}}\frac{\partial^2 \log\det(h_{i\bar{j}})}{\partial w^\alpha \partial w^{\bar{\beta}}} = -R(\rho(t)).$$

Plugging this into the inequality (3.15), we obtain

$$\begin{aligned}
&\Delta'(e^{-\lambda\tilde{\psi}(t)}(n+\tilde{\Delta}\psi(t)))\\
&\qquad \geq e^{-\lambda\tilde{\psi}(t)}\left(C(n+1+\tilde{\Delta}\psi(t))\sum_{i=1}^{n}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\right)\\
&\qquad\quad - \lambda e^{-\lambda\tilde{\psi}(t)}\Delta'\tilde{\psi}(t)(n+1+\tilde{\Delta}\tilde{\psi}(t)) - R(\rho(t))e^{-\lambda\tilde{\psi}(t)}.
\end{aligned}$$

Now

$$\begin{aligned}
\Delta'\tilde{\psi}(t) &= \Delta'\tilde{\psi}(t) = tr_{g'}(\tilde{\omega} + i\partial\bar{\partial}\tilde{\psi} - \tilde{\omega})\\
&= n+1 - tr_{g'}h.
\end{aligned}$$

Plugging this into the above inequality, we obtain

$$\begin{aligned}
&\Delta'(e^{-\lambda\tilde{\psi}(t)}(n+\tilde{\Delta}\tilde{\psi}(t)))\\
&\qquad \geq e^{-\lambda\tilde{\psi}(t)}\left((C+\lambda\delta)(n+1+\tilde{\Delta}\tilde{\psi}(t))\sum_{i=1}^{n+1}\frac{1}{1+\psi(t)_{i\bar{i}}}\right)\\
&\qquad\quad - |\lambda| c_3 e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t)) - R(\rho(t))e^{-\lambda\tilde{\psi}(t)}.
\end{aligned}$$

Let $\lambda\delta = -C + 1$, we then have

$$\begin{aligned}
&\Delta'(e^{-\lambda\tilde{\psi}(t)}(n+\tilde{\Delta}\tilde{\psi}(t)))\\
&\qquad \geq e^{-\lambda\tilde{\psi}(t)}\left((n+\tilde{\Delta}\tilde{\psi}(t))\sum_{i=1}^{n+1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\right)\\
&\qquad\quad - c_5 e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t)) - c_2 e^{-\lambda\tilde{\psi}}.
\end{aligned}$$

Here c_5 is a uniform constant.

Claim. The maximum value of $e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t))$ must occur in $\partial\Sigma_T \times M$.

Otherwise, if the maximum occurs in the interior, we have

$$e^{-\lambda\tilde{\psi}(t)}\left((n+1+\tilde{\Delta}\tilde{\psi}(t))\sum_{i=1}^{n+1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}}\right) \\ - c_5 e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t)) - c_2 e^{-\lambda\tilde{\psi}} \le 0.$$

However,

$$\sum_{i=1}^{n+1}\frac{1}{1+\tilde{\psi}(t)_{i\bar{i}}} \to \infty$$

as $\epsilon \to 0$. This leads to a contradiction when T is finite. Thus,

$$e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t)) \le \max_{t=0,t=T} e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t)).$$

In other words,

$$e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t)) \le C\cdot \max_{t=0,t=T} e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t))$$

for some uniform constant C since

$$\tilde{\psi} = \rho - \bar{\rho} + \phi_T - \rho$$

and

$$\phi_T - \rho$$

is uniformly bounded (independent of T). □

Before stating the next lemma, we need to explain a bit about the relationship between a geodesic ray and its bounded ambient geometry. By Definitions 3.2–3.4, there exists a constant C such that either

$$\sup_{[0,\infty)\times M} |\rho(t,\cdot) - \bar{\rho}(t,\cdot)| \le C$$

or the following three inequalities hold

1. $|n+1+\Delta_h(\rho-\bar{\rho})| \le C.$
2. $\left|\frac{\partial(\rho-\bar{\rho})}{\partial t}\right|_h \le C.$
3. The vector $\left|\frac{\partial}{\partial t}\right|_h$ has uniform upper bound.

Note that the assumption of the uniform bound of $\rho - \bar{\rho}$ is stronger. On a first reading, one could perhaps just assume a uniform bound on $\rho - \bar{\rho}$ throughout this subsection, although the second set of assumptions is more general.

As in [9], we have

Lemma 3.19. *If ψ is a solution of* (3.13) *at* $0 < \epsilon < 1$, *then there exists a constant C which depends only on $(\Sigma_T \times M, h)$ such that if $e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t))$ attains the maximal value at $t = T$, then for any $\{t\}\times S^1 \times M$*

which has h-distance to $\{T\} \times S^1 \times M$ *less than* 1*, we have*

$$\max_{\{t\}\times S^1\times M}(n+1+\tilde{\Delta}\tilde{\psi}) \leq C \max_{[T-\mu,T]\times S^1\times M}\left(|\nabla\tilde{\psi}|_h^2+1\right), \tag{3.16}$$

for any $\mu > 0$ *where the h distance from* $\{T-\mu\}\times S^1\times M$ *to* $\{T\}\times M$ *is small* ($\ll 1$)*. On the other hand, if* $e^{-\lambda(\rho-\bar{\rho})}(n+1+\tilde{\Delta}\tilde{\psi}(t))$ *attains the maximal value at* $t=0$*, then*

$$\max_{[0,1]\times S^1\times M}(n+1+\tilde{\Delta}\tilde{\psi}) \leq C \max_{[0,\mu]\times S^1\times M}\left(|\nabla\tilde{\psi}|_h^2+1\right). \tag{3.17}$$

Now, we are ready to give a proof of Theorem 3.9.

Proof. For simplicity, denote the h-distance between two hyper-surfaces $\{t_1\}\times S^1\times M$ and $\{t_2\}\times S^1\times M$ as $d_h(t_1,t_2)$. Following a blowing up argument in [9], we can prove that there is a uniform $C^{1,1}$ estimate for $t\in[T-1,T]\cup[0,1]$, depending on the t for which

$$e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t))$$

realizes its maximum. For simplicity, let us assume that

$$e^{-\lambda\tilde{\psi}(t)}(n+1+\tilde{\Delta}\tilde{\psi}(t))$$

obtains its maximum at $\{T\}\times S^1\times M$. Thus, we have

$$\max_{\{t\}\times S^1\times M}(n+1+\tilde{\Delta}\tilde{\psi}) \leq C \max_{[T-\mu,T]\times S^1\times M}\left(|\nabla\tilde{\psi}|_h^2+1\right), \quad \forall\mu>0$$

for any t where $d_h(t,T)\leq 1$ and $d_h(T-\mu,T)\ll 1$. Unlike in [9], here we need to blowup at the maximum point of

$$|\nabla\tilde{\psi}|_h\cdot\left(\frac{1}{2}-d_h(t,T)\right), \quad \forall t\in\left[\frac{T}{2},T\right].$$

Note that we don't assume that $\tilde{\psi}$ has a uniform C^0 bound. To bypass this difficulty, we note that

$$\tilde{\psi}=\varphi-\rho+\rho-\bar{\rho}.$$

By assumption, the first and second derivatives of the two functions $\tilde{\psi}$ and $\varphi-\rho$ are equivalent. Therefore, we really blowup at the maximum of

$$|\nabla(\varphi-\rho)|_h\cdot\left(\frac{1}{2}-d_h(t,T)\right), \quad \forall t\in\left[\frac{T}{2},T\right].$$

As in [9], we can prove that $|\nabla(\varphi-\rho)|_h$ is uniformly bounded for $t\in[T-\frac{1}{2},T]$. Consequently, this implies uniform control on $|\nabla\tilde{\psi}|_h$. The crucial observation here is that the distance function $d_h(t,T)$ is a positive function which vanishes in the boundary, and the hessian of $d_h(t, T)$ with respect to the metric h is positive and bounded.

Using Lemma 3.19, we have

$$n + 1 + \tilde{\Delta}\tilde{\psi} \le C$$

where C is a constant independent of T. Using the estimates of Lemma 3.18, we obtain a uniform growth control on geodesic ray. Theorem 3.9 then follows. □

4 On the lower bound of the Calabi energy

In this section, we will give a lower bound estimate for the modified Calabi energy in the absence of a cscK metric or an extremal Kähler metric. Note that for an algebraic manifold, the corresponding theorem is given in [17].

4.1 The classical theory of Futaki–Mabuchi and A. Hwang. Let $K = K(J)$ be a maximal compact subgroup of the automorphism group of the Kähler manifold and let $\mathcal{K} = \mathcal{K}(J)$ be its Lie algebra of gradient holomorphic vector fields in M. According to E. Calabi, if there is a cscK metric or CextrK metric, the cscK metric or CextrK metric must be symmetric with respect to one of these maximal compact subgroups (up to holomorphic conjugation). Therefore, it makes perfect sense to consider a restricted class $\mathcal{H}_K \subset \mathcal{H}$ where all Kähler metrics are invariant under K. For simplicity, suppose ω is invariant with respect to action of K. Recall that the Lichnerowicz operator is defined as:

$$\mathcal{L}_g(f) = f_{,\alpha\beta} dz^\alpha \otimes dz^\beta,$$

where the right hand side is the (2,0) component of the Hessian form of f with respect to the Kähler metric g. For any metric $g \in \mathcal{H}_K$, define $Ker(\mathcal{L}_g)$ to be the real part of the kernel[8] of the operator $\mathcal{L}_g$ in $C^\infty(M)$. It is easy to see the correspondence between $Ker(\mathcal{L}_g)$ and $\mathcal{K}$ in the following formula

$$X = g^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_X}{\partial w^{\bar{\beta}}}$$

where

$$X \in \mathcal{K}, \quad \theta_X \in Ker(\mathcal{L}_g) \quad \text{and} \quad \int_M \theta_X \omega_g^n = 0.$$

It is also easy to see that this correspondence is 1–1 as long as $g \in \mathcal{K}$. Futaki–Mabuchi define a bilinear form in $\mathcal{K}$ by

$$(X, Y) = \int_M \theta_X \theta_Y \omega_g^n.$$

[8] Usually, the Kernel space cannot be split as real part and imaginary part. However, in the case when the metric is invariant under $K(J)$, its Lie algebra $\mathcal{K}(J)$ always corresponds to this real part of the Kernel space.

Here θ_Y is the holomorphic potential of Y. Futaki–Mabuchi proved that such a bilinear form is positive definite and well defined on $\mathcal{H}_K$. From the definition of θ_X, it is easy to see that if $g \in \mathcal{H}_K$, then θ_X is real since

$$L_{Im(X)}\omega_g = 0 \quad \forall X \in \mathcal{K} \text{ and } g \in \mathcal{H}_K.$$

Thus, the Futaki–Mabuchi bilinear form is positive definite. To show it is well defined, we need to show that it is invariant as the metric varies inside $\mathcal{H}_K$. Let $\omega_{g(t)} = \omega_g + t\sqrt{-1}\partial\bar{\partial}\varphi \in \mathcal{H}_K$. Let

$$X = g(t)^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_X(t)}{\partial w^{\bar{\beta}}}, \quad \text{and}$$
$$Y = g(t)^{\alpha\bar{\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial \theta_Y(t)}{\partial w^{\bar{\beta}}}.$$

Then,

$$\theta_X(t) = \theta_X + tX(\varphi), \qquad \theta_Y(t) = \theta_Y + tY(\varphi),$$

where

$$L_{Im(X)}\varphi = L_{Im(Y)}\varphi = 0.$$

Set

$$(X, Y)_t = \int_M \theta_X(t)\theta_Y(t)\omega_{g(t)}^n.$$

It is straightforward to compute

$$\begin{aligned}
\frac{d}{dt}(X, Y)_t &= \int_M \left(\theta_Y(t)\frac{d}{dt}\theta_X(t) + \theta_X(t)\frac{d}{dt}\theta_Y(t) + \theta_X(t)\theta_Y(t)\Delta_{g(t)}\varphi\right)\omega_{g(t)}^n \\
&= \int_M \Bigg(\theta_Y(t)X(\varphi) + \theta_X(t)Y(\varphi) \\
&\qquad - g(t)^{\alpha\bar{\beta}}\frac{\partial\theta_X(t)}{\partial w^{\bar{\beta}}}Y(\varphi)\frac{\partial\varphi}{\partial w^\alpha} - g(t)^{\alpha\bar{\beta}}\frac{\partial\theta_Y(t)}{\partial w^{\bar{\beta}}}X(\varphi)\frac{\partial\varphi}{\partial w^\alpha}\Bigg)\omega_{g(t)}^n \\
&= \int_M (\theta_Y(t)X(\varphi) + \theta_X(t)Y(\varphi) - \theta_Y(t)X(\varphi) - \theta_X(t)Y(\varphi))\,\omega_{g(t)}^n \\
&= 0.
\end{aligned}$$

Thus, the Futaki–Mabuchi bilinear form is well defined. Now, the Calabi–Futaki character defines a linear map from $\mathcal{K}$ to $\mathbb{R}$, so by the Riesz representation theorem, there is a unique vector field $\mathcal{X}_c \in \mathcal{K}$ such that

$$\mathcal{F}_X([\omega]) = (X, \mathcal{X}_c), \quad \forall X \in \mathcal{K}.$$

Since both $\mathcal{F}_X$ and the Futaki–Mabuchi form are independent of the metric, $\mathcal{X}_c$ is *a priori* determined. When there is an extremal Kähler metric, then $\mathcal{X}_c$ coincides with the complex gradient vector field of the scalar curvature function.

Theorem 4.1 (Hwang). *The following inequality holds*

$$\inf_{g\in\mathcal{H}_K} Ca(\omega_g) \geq \mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Equality holds if there is an extremal Kähler metric in $[\omega]$.

Proof. Suppose $g \in \mathcal{K}$ and using the L^2 norm with respect to ω_g^n to decompose $R(g) - \underline{R}$ as

$$R(g) - \underline{R} = -\rho - \rho^{\perp} = -\Delta_g F, \quad \text{where } \rho \in Ker(\mathcal{L}_g),$$

it is easy to see that

$$\mathcal{X}_c = g^{\alpha\bar{\beta}} \frac{\partial}{\partial w^{\alpha}} \frac{\partial \rho}{\partial w^{\bar{\beta}}}.$$

Thus,

$$\begin{aligned} Ca(\omega_g) &= \int_M (R(g) - \underline{R})^2 \omega_g^n \\ &= \int_M \rho^2 \omega_g^n + \int_M (\rho^{\perp})^2 \omega_g^n \\ &\geq -\int_M \rho(R(g) - \underline{R})\omega_g^n = \int_M \rho \Delta_g F \omega_g^n = -\int_M \nabla\rho \cdot \nabla F \omega_g^n \\ &= \int_M \mathcal{X}_c(F)\omega_g^n = \mathcal{F}_{\mathcal{X}_c}([\omega]). \end{aligned}$$

□

At the time, A. Hwang thought the same proof could be extended to cover the non-invariant case. Unfortunately, the Futaki–Mabuchi form is no longer positive definite and the whole argument collapses. Much effort has been made by other mathematicians to bridge the gap, though none has been successful. Nonetheless, its generalization to more general settings is both very interesting and important.

4.2 Approximating $C^{1,1}$ geodesic segments via long oval discs. For any two Kähler potentials $\phi_0, \phi_1 \in \mathcal{H}$, we want to use the almost smooth solution to approximate the $C^{1,1}$ geodesic between ϕ_0 and ϕ_1. This approach was first taken in [12].

Let us setup some notation first. Let $\Sigma^{(\infty)} = (-\infty, \infty) \times [0, 1] \subset \mathbf{R}^2$ denote the infinitely long strip. For any integer l, let $\Sigma^{(l)}$ be a long "oval shape" disc such that $\Sigma^{(l)}$ is the union of $[-l, l] \times [0, 1]$ with a half circle centered at $(-l, \frac{1}{2})$ with radius $\frac{1}{2}$ at the left, and a half circle centered at $(l, \frac{1}{2})$ with radius $\frac{1}{2}$ at the right. Note that we want to smooth out the corner at the four corner points $\{\pm l\} \times \{0, 1\}$ so that $\Sigma^{(l)}$ is a smooth domain[9]. By

[9] The author wishes to stress that we do this "smoothing" once for all: Namely, we smooth $\Sigma^{(1)}$ first. For any $l > 1$, we may construct $\Sigma^{(l)}$ $(l \geq 1)$ by replacing the central line segment $\{0\} \times [0, 1]$ in $\Sigma^{(1)}$ by a cylinder $[-l + 1, l - 1] \times [0, 1]$.

construction, $\{\Sigma^{(l)}\}$ is a sequence of long ovals in this infinite strip $\Sigma^{(\infty)}$ where $\Sigma^{(0)}$ is a disc of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$. Obviously this is different from the Σ_T defined in Sect. 3.

Let ψ be a convex family of Kähler potentials in $\Sigma^{(\infty)}$ given by

$$\psi(s, t, \cdot) = \bar{\phi}(s, t, \cdot), \quad \forall (s, t) \in \Sigma^{(\infty)}. \tag{4.1}$$

Here $\bar{\phi}(s, t, \cdot)$ can be any convex path connecting ϕ_0, ϕ_1. For instance, we may set

$$\bar{\phi}(s, t, \cdot) = (1 - t)\phi_0 + t\phi_1 - Kt(1 - t)$$

where K is a large enough constant. Here K must depend on ϕ_0, ϕ_1 to ensure this family of potentials is convex. In this subsection, we assume that our boundary map ψ is independent of s variable.

Consider Dirichlet problem for the HCMA equation (2.5) on the long oval shape domain $\Sigma^{(l)}$ with boundary value

$$\phi|_{\partial\Sigma^{(l)}\times M} = \psi|_{\partial\Sigma^{(l)}\times M}.$$

As in [9], we want to solve this via approximation method. For any $\epsilon > 0$, consider the Dirichlet problem:

$$\left(\pi_2^*\omega + \partial\bar{\partial}\phi\right)^{n+1} = \epsilon \cdot \left(\pi_2^*\omega + \partial\bar{\partial}\psi\right)^{n+1}, \quad \forall (s, t, \cdot) \in \Sigma^{(l)} \times M \tag{4.2}$$

with fixed boundary data

$$\phi|_{\partial\Sigma^{(l)}\times M} = \psi|_{\partial\Sigma^{(l)}\times M}. \tag{4.3}$$

Denote the solution to this Dirichlet problem as $\phi^{(l,\epsilon)}$ for any $l \geq 1$ and $\epsilon \in (0, 1)$. For each fixed l, one can prove as in [9] that the solutions $\phi^{(l,\epsilon)}$ to (4.2) and (4.3) have $C^{1,1}$ upper bounds independent of ϵ. The main problem in this subsection is to determine if this upper bound is also independent of l.

Theorem 4.2. *For every l fixed, there is a $C^{1,1}$ solution $\phi^{(l)}$ to* (4.2) *and* (4.3) *with $\epsilon = 0$. More importantly, this upper bound on $|\partial\bar{\partial}\phi^{(l,\epsilon)}|$ is independent of $l > 1$ and $\epsilon \in (0, 1]$.*

Remark 4.3. Here we only want to control this second mixed derivatives of $\{\phi^{(l,\epsilon)}\}$ in its appropriate domain $\Sigma^{(l)} \times M$. We do not attempt here to control any higher derivatives at the present work.

We defer the proof of this crucial theorem until later in the section.

As in [9], let $\phi^\epsilon(t, \cdot)$ denote the ϵ-approximated S^1 invariant solution for the geodesic equation between ϕ_0 and ϕ_1. In other words, we have

$$\left(\pi_2^*\omega + \partial\bar{\partial}\phi^\epsilon\right)^{n+1} = \epsilon \cdot \left(\pi_2^*\omega + \partial\bar{\partial}\psi\right)^{n+1}, \quad \forall (t, \cdot) \in [0, 1] \times S^1 \times M.$$

Because ϕ^ϵ is independent of s, we can view it as a solution in $\Sigma^{(\infty)} \times M$. In other words, we have

$$\left(\pi_2^*\omega + \partial\bar{\partial}\phi\right)^{n+1} = \epsilon \cdot \left(\pi_2^*\omega + \partial\bar{\partial}\psi\right)^{n+1}, \quad \forall (t, \cdot) \in \Sigma^{(\infty)} \times M \tag{4.4}$$

with Dirichlet boundary data

$$\phi^\epsilon(s, 0, \dots) = \phi_0, \quad \phi^\epsilon(s, 1, \dots) = \phi_1.$$

Here we abuse notations by letting

$$\phi^\epsilon(s, t, \dots) = \phi^\epsilon(t, \dots).$$

Let ϕ^0 denote the $C^{1,1}$ geodesic between ϕ_0, ϕ_1. By the maximum principle, we know that ϕ^ϵ monotonically increases as ϵ decreases to 0. In particular, we have

$$\psi(s, t, \cdot) \le \phi^\epsilon(s, t, \cdot) \le \phi^0(t, \cdot), \quad \forall (s, t) \in \Sigma^{(\infty)}. \tag{4.5}$$

In particular, there is an error term $o(\epsilon)$ such that $\lim_{\epsilon \to 0} o(\epsilon) = 0$ in any $C^{1,\alpha}$ norm and

$$\phi^\epsilon(s, t, \cdot) = \phi^0(t) + o(\epsilon). \tag{4.6}$$

Lemma 4.4. *In any fixed compact subset of $\Sigma^{(\infty)} \times M$, $\{\phi^{(l,\epsilon)}\}$ converges uniformly continuously to ϕ^0 as $l \to \infty$ and $\epsilon \to 0$. In particular, the same convergence result holds for $\{\phi^{(l)}\}$ in any fixed compact sub-domain of $\Sigma^{(\infty)} \times M$.*

To prove this lemma, we need to introduce a sequence of harmonic functions $h^{(l)}$ in $\Sigma^{(l)}$ such that the boundary value of $h^{(l)}$ in $\partial\Sigma^{(l)}$ is

$$h^{(l)}(s, t) = \begin{cases} 0 & |s| \le l - 1, \\ K & |s| \ge l, \\ \in [0, K] & \text{otherwise} \end{cases}$$

where K is some large enough positive constant:

$$K > 2 \max |\phi^0 - \psi| + 1.$$

The following lemma is critical

Lemma 4.5. *In any compact sub-domain $\mathcal{O} \subset \Sigma^{(\infty)}$, we have*

$$\lim_{l \to \infty} \max_{\mathcal{O}} h^{(l)} = 0.$$

The proof, which we omit, is elementary. Now we are ready to prove Lemma 4.4.

Proof. Note that for any $1 \geq \epsilon > 0$ and l fixed, we have

$$\begin{aligned}\left(\pi_2^*\omega + \partial\bar{\partial}\phi^{(l,\epsilon)}\right)^{n+1} &= \epsilon \cdot \left(\pi_2^*\omega + \partial\bar{\partial}\psi\right)^{n+1} \\ &= \left(\pi_2^*\omega + \partial\bar{\partial}\phi^{\epsilon}\right)^{n+1}\end{aligned}$$

with boundary data

$$\begin{aligned}\phi^{(l,\epsilon)}|_{\partial\Sigma^{(l)}\times M} &= \psi|_{\partial\Sigma^{(l)}\times M} \\ &\leq \phi^{\epsilon}|_{\partial\Sigma^{(l)}\times M}.\end{aligned}$$

By the maximum principle for Monge–Ampere type equations, we have

$$\phi^{(l,\epsilon)} \leq \phi^{\epsilon}, \quad \forall (s,t,\cdot) \in \Sigma^{(l)} \times M.$$

Combining this with inequality (4.5), we have

$$\phi^{(l,\epsilon)} \leq \phi^{0}, \quad \forall (s,t,\cdot) \in \Sigma^{(l)} \times M.$$

It is easy to see that for any (l,ϵ) we have

$$\psi(s,t,\cdot) \leq \phi^{(l,\epsilon)}, \quad \forall (s,t,\cdot) \in \Sigma^{(l)} \times M.$$

Note that $h^{(l)}$ is a harmonic function on $\Sigma^{(l)}$ and can be viewed as a pluri-harmonic function in $\Sigma^{(l)} \times M$. Thus,

$$\pi_2^*\omega + \partial\bar{\partial}(\phi^{\epsilon} - h^{(l)}) = \pi_2^*\omega + \partial\bar{\partial}\phi^{\epsilon}, \quad \forall (s,t,\cdot) \in \Sigma^{(l)} \times M.$$

Thus, we have

$$\left(\pi_2^*\omega + \partial\bar{\partial}\phi^{(l,\epsilon)}\right)^{n+1} = \left(\pi_2^*\omega + \partial\bar{\partial}(\phi^{\epsilon} - h^{(l)})\right)^{n+1}$$

with boundary data

$$\begin{aligned}\phi^{(l,\epsilon)}|_{\partial\Sigma^{(l)}\times M} &= \psi|_{\partial\Sigma^{(l)}\times M} \\ &\geq \phi^{\epsilon} - h^{(l)}|_{\partial\Sigma^{(l)}\times M}.\end{aligned}$$

The last inequality holds because $\psi = \phi^{\epsilon}$ when $t = 0, 1$. In $\partial\Sigma^{(l)}$ with $t \neq 0, 1$, we have

$$h^{(l)} = K > \phi^0 - \psi \geq \phi^{\epsilon} - \psi.$$

By the maximum principle for Monge–Ampere equation, we have

$$\phi^{\epsilon} - h^{(l)} \leq \phi^{(l,\epsilon)}.$$

In particular, we have

$$\phi^0 + o(\epsilon) - h^{(l)} \leq \phi^{(l,\epsilon)} \leq \phi^0.$$

Thus, the sequence of Kähler potentials $\phi^{(l,\epsilon)}$ converges to ϕ^0 in any fixed compact subset of $\Sigma \times M$ in the C^0 norm. □

In fact, more is true.

Proposition 4.6. *For any $\alpha \in (0, 1)$, the sequence of Kähler potentials $\phi^{(l)}$ converges to ϕ^0 in any fixed compact subset of $\Sigma^{(\infty)} \times M$ in the $C^{1,\alpha}$ norm.*

This is just a corollary of Theorem 4.2 and Lemma 4.4. We are now ready to prove Theorem 4.2.

To obtain uniform $C^{1,1}$ bound of $\{\phi^{(l)}\}$ independent of $l \to \infty$, we need to choose some appropriate background Kähler metric first. Let h be the Kähler metric corresponding to the Kähler form $\tilde{\omega}$ given by

$$\pi_2^* \omega_0 + \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial w^i \partial w^{\bar{j}}} dw^i dw^{\bar{j}} + 2\,Re\left(\sum_{i=1}^{n} \frac{\partial^2 \psi}{\partial w^i \partial \bar{z}} dw^i d\bar{z}\right) + \frac{\partial^2 \psi}{\partial z \partial \bar{z}} dz d\bar{z}. \tag{4.7}$$

Here $z = w^0 = s + \sqrt{-1}t$. In other words

$$\tilde{\omega} = \pi_2^* \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi.$$

Note that h is a metric defined on $\Sigma^{(\infty)} \times M$ with uniform bounds on curvature and injectivity radius.

The Dirichlet boundary value problem (2.5) can be re-written as a Dirichlet problem on $\Sigma^{(l)} \times M$ such that

$$\det\left(h_{\alpha\bar{\beta}} + \frac{\partial^2(\phi - \psi)}{\partial w^\alpha \partial w^{\bar{\beta}}}\right)_{(n+1)\times(n+1)} = \epsilon \cdot \det(h_{\alpha\bar{\beta}})_{(n+1)\times(n+1)}, \tag{4.8}$$

with boundary condition

$$\phi|_{\partial\Sigma^{(l)} \times M} = \psi|_{\partial\Sigma^{(l)} \times M}. \tag{4.9}$$

Lemma 4.4 implies a uniform C^0 bound on the solution $\phi^{(l,\epsilon)} - \phi^0$ where ψ is a sub-solution of $\phi^{(l,\epsilon)}$. To obtain a super-solution, we define a sequence of (essentially) harmonic functions $\bar{\psi}^{(l)}$ by solving

$$\tilde{\Delta}(\bar{\psi}^{(l)} - \psi) + (n+1) = 0, \quad \forall (s, t, \cdot) \in \Sigma^{(l)} \times M$$

with boundary data

$$\bar{\psi}^{(l)}|_{\partial\Sigma^{(l)} \times M} = \psi|_{\partial\Sigma^{(l)} \times M}.$$

Here $\tilde{\Delta}$ is the Laplacian operator of the Kähler metric h. Since h is a smooth metric with uniform bound on curvature and injectivity radius in $\Sigma^{(l)} \times M$ and ψ is a smooth function with uniform bounds (independent of l), the standard elliptic PDE theory (interior and boundary estimates for harmonic functions) implies that there is a uniform bound on $\bar{\psi}^{(l)}$ (up to two derivatives, for instance). Note that

$$\psi \le \phi^{(l,\epsilon)} \le \bar{\psi}^{(l)}$$

with equality holding on $\partial\Sigma^{(l)} \times M$. Consequently, we have the following

Lemma 4.7. *The first derivatives of $\phi^{(l,\epsilon)}$ in $\partial\Sigma^{(l)} \times M$ are uniformly bounded (independent of l).*

We want to solve (4.8) for any large $l > 1$. We follow Yau's estimate in [34] and we want to set up some notation first. Put $\omega_{\phi(t)} = \sqrt{-1} g'_{\alpha\bar\beta} dw^\alpha \otimes dw^{\bar\beta}$ where

$$g'_{\alpha\bar\beta} = h_{\alpha\bar\beta} + \frac{\partial^2(\phi(t) - \psi(t))}{\partial w^\alpha \partial w^{\bar\beta}}.$$

Denote the Laplacian operator of this metric as:

$$\Delta' = \sum_{\alpha,\beta=0}^{n} g'^{\alpha\bar\beta} \frac{\partial^2}{\partial w^\alpha \partial w^{\bar\beta}}.$$

Following [9], we have

Lemma 4.8. *There exists a constant C which depends only on the Kähler metric h (independent of l) such that*

$$e^{-\lambda(\phi-\psi)}(n + 1 + \tilde\Delta(\phi - \psi)) \le \max_{\partial\Sigma^{(l)}\times M} e^{-\lambda(\phi-\psi)}(n + 1 + \tilde\Delta(\phi - \psi)).$$

Lemma 4.7 implies uniform first derivative bounds of the solution along the boundary. Using the same boundary estimate as in [9], we have

Lemma 4.9 [9]. *There exists a uniform constant C which depends only on the injectivity radius and curvature of h such that*

$$\max_{\partial\Sigma^{(l)}\times M} (n + 1 + \tilde\Delta(\phi - \psi)) \le C \max_{\Sigma^{(l)}\times M} \left(|\nabla(\phi - \psi)|_h^2 + 1\right). \tag{4.10}$$

Combining Lemma 4.8 and Lemma 4.9, we have

$$\max_{\Sigma^{(l)}\times M} (n + 1 + \tilde\Delta(\phi - \psi)) \le C \max_{\Sigma^{(l)}\times M} \left(|\nabla(\phi - \psi)|_h^2 + 1\right). \tag{4.11}$$

Following a blowing up argument in [9], we can prove that there is a uniform $C^{1,1}$ estimate for $\Sigma^{(l)} \times M$. Theorem 4.2 is then proved.

4.3 The first derivative of the K-energy. It is well-known that the first derivative of the K-energy functional is monotonically increasing along any smooth geodesic segment or ray. This is not clear when the geodesic segment is only $C^{1,1}$. Using Theorems 1.3.4 and 1.3.5 of Chen–Tian [12], we can show that the difference of the first derivatives of the K-energy at the two ends of any $C^{1,1}$ geodesic segment always has a preferred sign. This property turns out to be sufficient for our purposes later.

Lemma 4.10.[10] *For any two smooth Kähler metrics $\varphi_0, \varphi_1 \in \mathcal{H}$, let $\varphi(t, \cdot)$ be the unique $C^{1,1}$ geodesic connecting these two metrics with $\varphi(0, x) = \varphi_0$*

[10] This lemma is a natural application of Theorems 1.3.4, 1.3.5 in [12].

and $\varphi(1,x)=\varphi_1$. *Then,*

$$\left(d\mathbf{E}|_{\varphi_0}, \left.\frac{\partial\varphi}{\partial t}\right|_{t=0}\right) \leq \left(d\mathbf{E}|_{\varphi_1}, \left.\frac{\partial\varphi}{\partial t}\right|_{t=1}\right).$$

Remark 4.11. Even though the K-energy is well defined along any $C^{1,1}$ geodesic path, its derivative is in general not well defined. However, the evaluation of the K-energy form on $\frac{\partial\varphi}{\partial t}$ at the two end points is well defined. This lemma might be interpreted as a weak convexity property for the K-energy along a $C^{1,1}$ geodesic.

We follow the notation of Sect. 4.2. According to Theorem 4.2, there is a $C^{1,1}$ solution $\phi^{(l)}$ to the geodesic equation with boundary data $\psi : \partial\Sigma^{(l)} \to \mathcal{H}$ where the $C^{1,1}$ bound is independent of l as $l \to \infty$. According to Theorem 2.3, for any $\delta > 0$ there exists a modification $\psi^{l,\delta}$ such that there is an almost smooth solution $\phi_\delta^{(l)} : \Sigma^{(l)} \to \mathcal{H}$ corresponding to the boundary map

$$\phi_\delta^{(l)}\big|_{\partial\Sigma^{(l)}\times M} = \psi^{l,\delta} : \partial\Sigma^{(l)} \to \mathcal{H}.$$

Following Theorem 2.3 again, we can modify the boundary map such that the $C^{2,\alpha}(0<\alpha<1)$ norm satisfies

$$\max_{\partial\Sigma^{(l)}\times M} |\psi^{l,\delta} - \psi|_{C^{2,\alpha}} = o(\delta)$$

where $\lim_{\delta\to 0} o(\delta) = 0$. According to the maximum principle for the Monge–Ampere equation, we have the following

$$\max_{\Sigma^{(l)}\times M} \left|\phi^{(l)} - \phi_\delta^{(l)}\right| = o(\delta).$$

According to Theorem 4.2, we know that $\phi^{(l,\delta)}$ has a uniform $C^{1,1}$ upper bound which is independent of l, δ. By a standard application, we have

$$\max_{\Sigma^{(l)}\times M} \left|\phi^{(l)} - \phi_\delta^{(l)}\right|_{C^{1,\alpha}(\Sigma^{(l)}\times M)} = o(\delta).$$

Set

$$\mathbf{E}_\delta^{(l)}(s,t) = \mathbf{E}\big(\phi_\delta^{(l)}(s,t)\big), \quad \mathbf{E}^{(l)}(s,t) = \mathbf{E}(\phi^{(l)}(s,t)), \quad \forall (s,t) \in \Sigma^{(l)}, \tag{4.12}$$

and

$$\mathbf{E}(s,t) = \mathbf{E}(\varphi(t)), \quad \forall (s,t) \in \Sigma^{(l)}. \tag{4.13}$$

By definition, we have

$$\mathbf{E}_\delta^{(l)}(s,1) = \mathbf{E}^{(l)}(s,1) = \mathbf{E}(\varphi_1), \quad \mathbf{E}_\delta^{(l)}(s,0) = \mathbf{E}^{(l)}(s,0) = \mathbf{E}(\varphi_0).$$

Following Theorem 2.4 (cf. Theorem 1.3.5 of [12]), $\mathbf{E}_\delta^{(l)}$ is a sub-harmonic function with respect to its variables (s, t):

$$\frac{\partial^2 \mathbf{E}_\delta^{(l)}}{\partial t^2} + \frac{\partial^2 \mathbf{E}_\delta^{(l)}}{\partial s^2} \geq 0. \tag{4.14}$$

More importantly, $\mathbf{E}_\delta^{(l)}$ has uniformly continuous first derivative. Now we are ready to give the proof of Lemma 4.10.

Proof. Let $\kappa : (-\infty, \infty) \to \mathbf{R}$ be a smooth non-negative function such that $\kappa \equiv 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and vanishes outside of $y[-\frac{3}{4}, \frac{3}{4}]$. Set

$$\kappa^{(l)}(s) = \frac{1}{v}\kappa\left(\frac{s}{l}\right), \quad \text{where } v = \int_{-\infty}^{\infty} \kappa(s)ds.$$

For any integer $m < l$, we can also define

$$f^{(ml)}(t) = \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{d\mathbf{E}_\delta^{(l)}}{dt}(s, t)ds.$$

This is clearly well defined since $\mathbf{E}_\delta^{(l)}$ has continuous first derivative on $(s, t) \in \Sigma^{(l)}$. In particular, we have

$$f^{(ml)}(0) = \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{d\mathbf{E}_\delta^{(l)}}{dt}\bigg|_{(s,0)} ds,$$

$$f^{(ml)}(1) = \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{d\mathbf{E}_\delta^{(l)}}{dt}\bigg|_{(s,1)} ds.$$

Fixing (l, δ), let $\mathbf{F}_\epsilon$ be any smooth sub-harmonic function in $\Sigma^{(l)}$ such that

$$\lim_{\epsilon \to 0} \mathbf{F}_\epsilon = \mathbf{E}_\delta^{(l)}$$

uniformly in $C^1(\Sigma^{(l)})$. Note that $\mathbf{F}_\epsilon$ is just an approximation of $\mathbf{E}_\delta^{(l)}$ in $\Sigma^{(l)}$ and it has nothing to do with the K-energy functional. Set

$$f_\epsilon^{(ml)}(t) = \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{d\mathbf{F}_\epsilon}{dt}(s, t)ds.$$

Now

$$\begin{aligned} f_\epsilon^{(ml)}(1) - f_\epsilon^{(ml)}(0) &= \int_0^1 \frac{df_\epsilon^{(ml)}}{dt}dt \\ &= \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{\partial^2 \mathbf{F}_\epsilon}{\partial t^2}dsdt \\ &= \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\Delta_{s,t}\mathbf{F}_\epsilon dsdt \\ &\quad - \int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s)\frac{\partial^2 \mathbf{F}_\epsilon}{\partial s^2}dsdt \end{aligned}$$

$$\geq -\int_0^1 \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{\partial^2 \mathbf{F}_\epsilon}{\partial s^2} ds dt$$
$$= -\int_0^1 \int_{-\infty}^{\infty} \frac{d^2 \kappa^{(m)}(s)}{ds^2} \mathbf{F}_\epsilon(s,t) ds dt$$
$$= -\frac{1}{m^2}\frac{1}{v}\int_0^1 \int_{-\infty}^{\infty} \frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \mathbf{F}_\epsilon(s,t) ds dt.$$

Here we have used the fact that $\mathbf{F}_\epsilon$ is a sub-harmonic function in its variables (s,t). In other words, we have

$$f_\epsilon^{(ml)}(1) - f_\epsilon^{(ml)}(0) \geq -\frac{1}{m^2}\frac{1}{v}\int_0^1 \int_{-\infty}^{\infty} \frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \mathbf{F}_\epsilon(s,t) ds dt.$$

Taking the limit as $\epsilon \to 0$, we have

$$f^{(ml)}(1) - f^{(ml)}(0) \geq -\frac{1}{m^2}\frac{1}{v}\int_0^1 \int_{-\infty}^{\infty} \frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \mathbf{E}_\delta^{(l)}(s,t) ds dt.$$

Note that $|\mathbf{E}_\delta^{(l)}(s,t)|$ has a uniform bound C. Thus,

$$\begin{aligned}\frac{1}{m^2}\frac{1}{v}\left|\int_{-\infty}^{\infty} \frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \mathbf{E}_\delta^{(l)}(s,t) ds\right| &\leq \frac{1}{m^2}\frac{1}{v}\int_{-\infty}^{\infty} \left|\frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \mathbf{E}_\delta^{(l)}(s,t)\right| ds \\ &\leq \frac{C}{vm^2}\int_{-\infty}^{\infty}\left|\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{m}} ds \\ &= \frac{C}{vm}\int_{-\infty}^{\infty}\left|\frac{d^2\kappa}{ds^2}\right|_{s} ds = \frac{C}{vm}\int_{-1}^{1}\left|\frac{d^2\kappa}{ds^2}\right|_{s} ds \\ &\leq \frac{C}{m}\end{aligned}$$

for some uniform constant C. Therefore, we have

$$\begin{aligned}f^{(ml)}(1) - f^{(ml)}(0) &\geq -\frac{1}{m^2}\frac{1}{v}\int_0^1 \int_{-\infty}^{\infty} \frac{d^2\kappa}{ds^2}\bigg|_{\frac{s}{m}} \left|\mathbf{E}_\delta^{(l)}(s,t)\right| ds dt \\ &\geq -\frac{C}{2m}. \end{aligned} \tag{4.15}$$

Recall that

$$\begin{aligned}f^{(ml)}(1) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \frac{d\mathbf{E}_\delta^{(l)}}{dt}(s,1) ds \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \phi_\delta^{(l)}}{\partial t} \omega_{\varphi_1}^n ds.\end{aligned}$$

For any fixed m, by Proposition 4.5, $\phi_\delta^{(l)}$ converges uniformly to φ in $\Sigma^{(m)} \times M$. In particular, $\frac{\partial \phi^{(l)}}{\partial t}$ converges strongly in the C^α-norm to $\frac{\partial \varphi}{\partial t}$ in

$\Sigma^{(m)} \times M$. Thus, fixing m and letting $l \to \infty$, we have

$$\begin{aligned}\lim_{l\to\infty} f^{(ml)}(1) &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t}\Big|_{t=1} \omega_{\varphi_1}^n ds \\ &= \lim_{l\to\infty} \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \phi_\delta^{(l)}}{\partial t}\Big|_{t=1} \omega_{\varphi_1}^n ds \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s) \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t}\Big|_{t=1} \omega_{\varphi_1}^n ds \\ &= \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t}\Big|_{t=1} \omega_{\varphi_1}^n.\end{aligned}$$

Similarly, we can prove

$$\lim_{l\to\infty} f^{(ml)}(0) = \int_M (\underline{R} - R(\varphi_1)) \frac{\partial \varphi}{\partial t}\Big|_{t=0} \omega_{\varphi_0}^n.$$

Plugging this into inequality (4.15), we have

$$\left(d\mathbf{E}, \frac{\partial \varphi}{\partial t}\Big|_{t=1}\right)_{\varphi_1} - \left(d\mathbf{E}, \frac{\partial \varphi}{\partial t}\Big|_{t=0}\right)_{\varphi_0} \geq -\frac{C}{m}.$$

As $m \to \infty$, we have

$$\left(d\mathbf{E}, \frac{\partial \varphi}{\partial t}\Big|_{t=1}\right)_{\varphi_1} \geq \left(d\mathbf{E}, \frac{\partial \varphi}{\partial t}\Big|_{t=0}\right)_{\varphi_0}.$$

The lemma is then proved. □

4.4 The greatest lower bound of the Calabi energy. Note that the first derivative of the K-energy functional is always non-decreasing along a smooth geodesic ray. Thus, the ¥ invariant is always well defined along any smooth geodesic ray. We are ready to prove Theorem 1.6 (cf. Theorem 3.15).

Proof. Suppose $\rho : [0, \infty) \to \infty$ is a geodesic ray parametrized by arc length such that

$$\lim_{t\to\infty} \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\right)_{\rho(t)} < 0$$

with

$$\left\|\frac{\partial \rho}{\partial t}\right\|_{\omega_{\rho(t)}} = 1, \quad \forall t \in (-\infty, \infty).$$

For any Kähler potential $\varphi_0 \in \mathcal{H}$, consider the unique $C^{1,1}$ geodesic segment connecting φ_0 to $\rho(l)$. Suppose the length of this geodesic segment is $\tau(l)(\forall l > 0)$ and denote this geodesic segment as $\Psi_l : s \in [0, \tau(l)] \to \mathcal{H}$.

Applying Lemma 4.9 of the preceding subsection, we have

$$\begin{aligned}
&\left(d\mathbf{E}, \frac{\partial \Psi_l}{\partial s}\bigg|_{s=0}\right)_{\varphi_0} \leq \left(d\mathbf{E}, \frac{\partial \Psi_l}{\partial s}\bigg|_{s=\tau(l)}\right)_{\rho(l)} \\
&\quad = \left(d\mathbf{E}, \frac{\partial \Psi_l}{\partial s}\bigg|_{s=\tau(l)} - \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)} + \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)} \\
&\quad \leq \left(\int_M (R(\rho(l)) - \underline{R})^2 \omega_\rho^n\right)^{\frac{1}{2}} \cdot \left(\int_M \left(\frac{\partial \rho}{\partial t}\bigg|_{t=l} - \frac{\partial \Psi_l}{\partial t}\bigg|_{s=\tau(l)}\right)^2 \omega_{\rho(l)}^n\right)^{\frac{1}{2}} \\
&\qquad + \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)} \\
&\quad \leq Ca(\omega_{\rho(l)})^{\frac{1}{2}} \cdot \left(2 - 2\left(\frac{\partial \rho}{\partial t}\bigg|_{t=l}, \frac{\partial \Psi_l}{\partial s}_{s=\tau(l)}\right)_{\rho(l)}\right)^{\frac{1}{2}} + \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)}.
\end{aligned}$$

Denote the three points $\varphi_0, \rho(0), \rho(l)$ as C, B, A in $\mathcal{H}$. For l large enough, the geodesic triangle $\overline{ABC}$ is a thin triangle with two long sides $CA = \overline{\varphi_0 \rho(l)}$, $BA = \overline{\rho(0)\rho(l)}$ with lengths $\tau(l), l$, respectively, while the length of side $CB = \overline{\varphi_0 \rho(0)}$ is fixed. By triangle comparison, we have

$$l - |BC| < \tau(l) < l + |BC|.$$

With l large enough, we can essentially treat $\tau(l) = l$. According to Calabi–Chen [8], the infinite dimensional space $\mathcal{H}$ is a non-positively curved manifold in the sense of Alexandrov. Thus, the small angle at $A = \rho(l)$ on this long, thin geodesic hinge approaches 0 as $l \to \infty$. Moreover, it is smaller than the small angle of the corresponding long, thin hinge in the Euclidean plane. Thus, we have

$$0 \leq l^2 \cdot \left(1 - \left(\frac{\partial \rho}{\partial t}\bigg|_l, \frac{\partial \Psi_l}{\partial s}_{s=\tau(l)}\right)_{\rho(l)}\right) \leq C$$

for some uniform constant C. On the other hand, if the geodesic ray is effectively destabilizing, we have

$$\limsup_{t\to\infty} Ca(\omega_{\rho(t)}) \cdot \frac{1}{t^2} = 0.$$

Now plugging this into a previous inequality, we obtain

$$\begin{aligned}
&\left(d\mathbf{E}, \frac{\partial \Psi_l}{\partial s}\bigg|_{s=0}\right)_{\varphi_0} \\
&\leq Ca(\omega_{\rho(l)})^{\frac{1}{2}} \cdot \left(2 - 2\left(\frac{\partial \rho}{\partial t}\bigg|_{t=l}, \frac{\partial \Psi_l}{\partial s}_{s=\tau(l)}\right)_{\rho(l)}\right)^{\frac{1}{2}} + \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)} \\
&= o\left(\frac{1}{l}\right) + \left(d\mathbf{E}, \frac{\partial \rho}{\partial t}\bigg|_{t=l}\right)_{\rho(l)} < 0.
\end{aligned}$$

The last inequality holds for large enough l since we assume that ρ is a destabilizing geodesic ray. By the definition of the ¥ invariant, we have

$$\begin{aligned}
-¥(\rho) &= -\liminf_{l\to\infty}\left(d\mathbf{E}, \frac{\partial\rho}{\partial t}\Big|_l\right)_{\rho(l)} \\
&\leq \liminf_{l\to\infty}\left(o\left(\frac{1}{l}\right) - \left(d\mathbf{E}, \frac{\partial\Psi_l}{\partial s}\Big|_{s=0}\right)_{\varphi_0}\right) \\
&\leq \liminf_{l\to\infty}\left(\int_M (R(\omega_{\varphi_0}) - \underline{R})^2\omega_{\varphi_0}^n\right)^{\frac{1}{2}}\left(\int_M \left(\frac{\partial\Psi_l}{\partial s}\Big|_{s=0}\right)^2 \omega_{\varphi_0}^n\right)^{\frac{1}{2}} \\
&= (Ca(\omega_{\varphi_0}))^{\frac{1}{2}}.
\end{aligned}$$

In other words, we have

$$Ca(\omega_{\varphi_0}) \geq ¥(\rho)^2.$$

Our theorem follows from here directly. □

Theorem 3.15a can be proved similarly and we omit the proof here. Now we are ready to prove Theorem 1.5.

Proof. Let $\mathcal{X}_c$ be the *a priori* extremal vector field. Suppose $g \in \mathcal{H}_K$. Suppose that $\omega_{\rho(t)}(t \in (-\infty, \infty))$ is the one parameter family of Kähler metrics generated by pulling the Kähler metrics ω_g in the direction of $Re(X)$. It is straightforward to check that $\rho(t)$ satisfies the geodesic equation and

$$\frac{d\mathbf{E}(\omega_{\rho(t)})}{dt} = \pm\mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Select one direction so that

$$\frac{d\mathbf{E}(\omega_{\rho(t)})}{dt} = -\mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Note that the length element of this geodesic line is

$$\begin{aligned}
\int_M \left(\frac{\partial\rho}{\partial t}\right)^2 \omega_{\rho(t)}^n &= (\theta_{\mathcal{X}_c}, \theta_{\mathcal{X}_c}) = -(\theta_{\mathcal{X}_c}, R(\rho) - \underline{R}) \\
&= -\int_M \frac{\partial\rho}{\partial t}(R(\rho(t)) - \underline{R})\omega_{\rho(t)}^n = \mathcal{F}_{\mathcal{X}_c}.
\end{aligned}$$

Now, if we re-parametrize the geodesic by arc length, then the ¥ invariant along this geodesic line satisfies

$$¥(\rho)^2 = \mathcal{F}_{\mathcal{X}_c}([\omega]).$$

Our theorem then follows from Theorem 1.6. □

5 On the lower bound of the geodesic distance

Let us prove Theorem 1.2 first. All of calculations here are carried in the barrier sense as in Sect. 4.3 (for simplicity, we just proceed here as if everything is smooth).

Proof. We follow the notations of Sect. 4.3. Set

$$\mathbf{E}^{(ml)}(t) = \int_{-\infty}^{\infty} k^{(m)}(s)\mathbf{E}^{(l)}(s,t)ds, \quad \forall m \le l \in \mathbb{N}.$$

Then,

$$\mathbf{E}^{(ml)}(0) = \mathbf{E}(\varphi_0), \quad \mathbf{E}^{(ml)}(1) = \mathbf{E}(\varphi_1)$$

and

$$\frac{d\mathbf{E}^{(ml)}}{dt}(t) = f^{(ml)}(t), \quad \forall t \in [0,1].$$

Following the same calculation as in Sect. 4.3, for any $0 \le t_1 < t_2 \le 1$

$$\begin{aligned} f^{(ml)}(t_2) - f^{(ml)}(t_1) &\ge -\frac{1}{m^2}\frac{1}{v}\int_{t_1}^{t_2}\int_{-\infty}^{\infty} \left.\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{m}} \mathbf{E}^{(l)}(s,t)dsdt \\ &\ge -\int_{t_1}^{t_2} \frac{1}{m^2}\frac{1}{v}\left|\int_{-\infty}^{\infty} \left.\frac{d^2\kappa}{ds^2}\right|_{\frac{s}{m}} \mathbf{E}^{(l)}(s,t)ds\right|dt \\ &\ge -\int_{t_1}^{t_2}\frac{C}{m}dt = -\frac{C}{2m}. \end{aligned} \tag{5.1}$$

Let $t_1 = 0$ and replace t_2 by t, we have

$$f^{(ml)}(0) - \frac{C}{2m} \le f^{(ml)}(t).$$

Let $t_2 = 1$ and replace t_1 by t, we have

$$f^{(ml)}(t) \le f^{(ml)}(1) + \frac{C}{2m}.$$

Therefore,

$$\begin{aligned} \mathbf{E}(\varphi_1) - \mathbf{E}(\varphi_0) &= \mathbf{E}^{(ml)}(1) - \mathbf{E}^{(ml)}(0) \\ &= \int_0^1 \frac{d\mathbf{E}^{(ml)}}{dt}(t)dt = \int_0^1 f^{(ml)}(t)dt \\ &\le \int_0^1 \left(f^{(ml)}(1) + \frac{C}{2m}\right)dt \\ &= \int_{-\infty}^{\infty} \kappa^{(m)}(s)\int_M (\underline{R} - R(\varphi_1))\frac{\partial\phi^{(l)}}{\partial t}\omega_{\varphi_1}^n ds + \frac{C}{2m}. \end{aligned}$$

As before, let $l \to \infty$, so $\phi^{(l)}(s,t)$ converges to the geodesic $\varphi(t)$ strongly in $C^{1,\alpha}$ norm. Then, letting $m \to \infty$, we have

$$\begin{aligned}\mathbf{E}(\varphi_1) - \mathbf{E}(\varphi_0) &\leq \int_M (\underline{R} - R(\varphi_1))\frac{\partial\varphi}{\partial t}\omega_{\varphi_1}^n \\ &\leq \sqrt{Ca(\varphi_1)}\sqrt{\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \omega_{\varphi_1}^n} \\ &= \sqrt{Ca(\varphi_1)} \cdot d(\varphi_0, \varphi_1).\end{aligned}$$

□

Corollary 1.3 follows from this theorem since the $|\varphi|_\infty$ bound will imply the geodesic distance of φ to a fixed Kähler potential is bounded.

Before we prove Theorem 1.4, we need to prove a proposition first.

Proposition 5.1 [11]. *Let $Ric(\omega_\varphi) \geq -C_1$ then there is a uniform constant C such that :*

$$\inf_M \log\frac{\omega_\varphi^n}{\omega^n} \geq -4C\exp\left(2 + 2\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n\right).$$

If $C_1 = 0$, then C is a dimensional constant. Otherwise, C depends on C_1 and $|\varphi|_{L^\infty}$ or $\sup\varphi_- + \int_M \varphi_+\omega^n$.

Proof. Set

$$F = \log\frac{\omega_\varphi^n}{\omega^n}.$$

The lower Ricci curvature bound implies that

$$Ric(\omega) - i\partial\bar{\partial}F \geq -C_1\omega_\varphi.$$

Taking the trace of both sides, we have

$$\Delta(F - C_1\varphi) \leq C_2$$

for some constant C_2.

Choose a constant c such that

$$\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \leq c.$$

In a fixed Kähler class, we have

$$\int_M \omega^n = Vol(M) = 1.$$

Put ϵ to be $\exp(-2-2c)$. Observe that

$$\log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \geq -e^{-1}\omega^n,$$

we have

$$\begin{aligned} c &\geq \Big(\int_{\epsilon\omega_\varphi^n > \omega^n} + \int_{\epsilon\omega_\varphi^n \leq \omega^n} \Big)\Big(\log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \Big) \\ &\geq \int_{\epsilon\omega_\varphi^n > \omega^n} \left(\log \frac{1}{\epsilon} \right) \omega_\varphi^n + \int_{\epsilon\omega_\varphi^n \leq \omega^n} (-e^{-1}\omega^n) \\ &> 2(1+c) \int_{\epsilon\omega_\varphi^n > \omega^n} \omega_\varphi^n - 1. \end{aligned}$$

It follows that

$$\int_{\epsilon\omega_\varphi^n > \omega^n} \omega^n < \frac{1}{2},$$

and consequently,

$$\int_{\omega^n \leq 4\omega_\varphi^n} \omega^n \geq \epsilon \int_{\frac{\epsilon}{4}\omega^n \leq \epsilon\omega_\varphi^n \leq \omega^n} \omega_\varphi^n \geq \frac{1}{4},$$

because we know

$$\int_{\omega^n \leq 4\omega_\varphi^n} \omega_\varphi^n > \frac{3}{4}$$

and

$$\int_{\omega^n \leq \epsilon\omega_\varphi^n} > \frac{1}{2}.$$

Now by Green's formula, we have

$$(F - C_1\varphi)(p) = -\int_M G(p,q)\Delta(F - C_1\varphi)\omega^n(q) + \int_M (F - C_1\varphi)\omega^n,$$

where $G(p,q) \geq 0$ is a Green function of ω. If either $|\varphi|_{L^\infty}$ is bounded, or

$$\sup \varphi_- \leq C, \quad \text{and} \quad \int_M \varphi_+ \omega^n \leq C,$$

then

$$\begin{aligned} \inf_M F &\geq \inf_M F \int_{\omega^n \geq 4\omega_\varphi^n} \omega^n + \int_{\omega^n < 4\omega_\varphi^n} F\omega^n - C_1 \sup \varphi + C_1 \int_M (-\varphi_-)\omega^n \\ &\geq \int_M F\omega^n - C \\ &\geq \inf_M F \int_{\omega^n \geq 4\omega_\varphi^n} \omega^n + \int_{\omega^n < 4\omega_\varphi^n} F\omega^n - C \end{aligned}$$

$$\geq \inf_M F \int_{\omega^n \geq 4\omega_\varphi^n} \omega^n - \log 4 \int_{\omega^n < 4\omega_\varphi^n} \omega^n - C$$
$$\geq \left(1 - \frac{\epsilon}{4}\right) \inf_M F - C,$$

where we can assume $\inf_M F < 0$. Therefore, we have

$$\inf_M F \geq -4C \exp(2 + 2c).$$

By the way we choose the constant c in the beginning of the proof, the proposition is proved. □

One more lemma is needed.

Lemma 5.2. *Pick a constant C, then there is a second constant C' such that whenever $(M, [\omega])$ is a Kähler manifold with Sobolev constant $C_{sob} < C$, all the Kähler potentials ψ on $(M, [\omega])$ which satisfy $\|\psi_+\| L^p < C$ also satisfy $\|\psi_+\|_\infty < C'$.*

The proof is based on a version of Moser iteration and it is well-known to the experts. We will include it here for the convenience of readers.

Proof. Without loss of generality, may assume $\psi \geq 1$ for simplicity. We start from

$$n + \Delta' \psi > 0,$$

here Δ' is the Laplacian operator of ω'. For any $p \geq 1$, we have

$$\begin{aligned} \int_M n \cdot \psi^p \omega'^n &\geq p \int_M |\nabla \psi|^2 \psi^{p-1} \omega'^n \\ &= p \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 \psi^{p-1} \omega'^n \\ &= \frac{4p}{(p+1)^2} \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 \omega'^n. \end{aligned}$$

Since ω' has a uniform Sobolev constant, we have

$$\begin{aligned} C_{sob}\Big(\int_M \psi^{\frac{p+1}{2} \cdot \frac{2m}{m-2}}\Big)^{\frac{m-2}{m}} &\leq \int_M |\nabla \psi^{\frac{p+1}{2}}|^2 + (\psi^{\frac{p+1}{2}})^2 \\ &\leq \frac{(p+1)^2}{4p} n \int_M \psi^p \omega'^n + \int_M \psi^{p+1} \omega'^n. \end{aligned}$$

Thus, there is a uniform constant C which is independent of p such that

$$\Big(\int_M \psi^{(p+1) \cdot \frac{m}{m-2}}\Big)^{\frac{m-2}{m}} \leq C(p+1) \int_M \psi^{p+1} \omega'^n.$$

Set

$$p_1 = 1 > 0, \quad p_2 = p_1 \frac{m}{m-2}, \ldots, \quad p_{j+1} = p_j \cdot \frac{m}{m-2}, \ldots$$

Then,

$$\|\psi\|_{p_{j+1}} \leq C^{\frac{1}{p_j}} p_j^{\frac{1}{p_j}} \|\psi\|_{p_j}, \quad \forall j = 1, 2, \ldots$$

In other words,

$$\|\psi\|_{p_{j+1}} \leq \|\psi\|_{p_1} \cdot C^{\sum_{k=1}^{j} \left(\frac{1}{p_j} + 1 p_j \log p_j\right)} \|\psi\|_{p_1}.$$

Let $j \to \infty$, then

$$\|\psi\|_{L^\infty} \leq C\|\psi\|_{L^2}.$$

□

Now we are ready to prove Theorem 1.4.

Proof. We consider a family of Kähler potentials φ such that:

1. $Ric(\omega_\varphi)$ is uniformly bounded from below.
2. The diameter of (M, ω_φ) is uniformly bounded from above.
3. The geodesic distance $d(0, \varphi)$ is uniformly bounded from above.

The first two conditions imply that there are uniform Sobolev and Poincare constants for (M, ω_φ). In the proof here, C represents a constant which may change from line to line, but can always be chosen to be independent of φ.

Normalize φ by adding a constant so that

$$I(\varphi) = 0.$$

According to a theorem in [9], we have

$$\max\left(\int_M \varphi_- \omega^n, \int_M \varphi_+ \omega_\varphi^n\right) \leq d(0, \varphi) \leq C. \tag{5.2}$$

Here φ_+, φ_- are positive and negative parts of φ respectively.

Normalize the K-energy to be 0 at ω_0. Theorem 1.2 implies that

$$\mathbf{E}(\varphi) \leq \mathbf{E}(0) + \sqrt{Ca(\varphi)}d(0, \varphi) \leq C.$$

If the K-energy functional is quasi-proper (cf., the detailed expression of the K-energy functional in (2.7)), we obtain

$$\int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n \leq C.$$

According to G. Tian [32], there is a positive constant $\alpha > 0$ which depends only on the Kähler class $[\omega]$ such that for any $\varphi \in \mathcal{H}$, we have

$$\int_M e^{-\alpha(\varphi - \sup \varphi)} \omega^n \leq C.$$

Or

$$\int_M e^{-\alpha(\varphi - \sup\varphi) - \log\frac{\omega_\varphi^n}{\omega^n}} \omega_\varphi^n \le C.$$

Consequently, we have

$$\int_M -\alpha(\varphi - \sup\varphi) - \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \le C.$$

Therefore,

$$\begin{aligned}
\alpha \sup\varphi &\le \alpha \int_M \varphi\omega_\varphi^n + \int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \\
&\le \alpha \int_M \varphi\omega_\varphi^n + \int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n \\
&\le C.
\end{aligned}$$

It follows that

$$\int_M |\varphi|\omega^n \le C. \tag{5.3}$$

By the detailed expression for $I(\varphi)$ (cf. (2.6)), we have

$$J(\varphi) \le C. \tag{5.4}$$

Alternatively, when the K-energy is proper, we can obtain estimates (5.3) and (5.4) as well. Note that if the K-energy functional $\mathbf{E}$ is proper in $\mathcal{H}$, we have

$$0 \le J(\varphi) \le C. \tag{5.5}$$

Again, from the detailed expression of $I(\varphi)$ (cf. (2.6)), we have

$$\left| \int_M \varphi\omega^n \right| \le C.$$

Comparing with estimate (5.2), we have

$$\int_M \varphi_+\omega^n \le C \tag{5.6}$$

and

$$\int_M \varphi_-\omega_\varphi^n \le C, \tag{5.7}$$

or

$$\int_M |\varphi|\omega^n \le C. \tag{5.8}$$

Since $Ric(\omega_\varphi) \geq -C$ and the diameter is bounded from below, we have a uniform Poincare constant for (M, ω_φ). Using the Poincare inequality, we have

$$\int_M \varphi^2 \omega^n + \int_M \varphi^2 \omega_\varphi^n \leq C\Big(J(\varphi) + \Big(\int_M |\varphi| \omega_\varphi^n\Big)^2\Big) \leq C.$$

Recall that

$$n + \Delta\varphi \geq 0.$$

Using Moser iteration, and the J functional bound (5.4), we obtain

$$0 \leq \varphi_+ \leq C.$$

Recall that

$$n + \Delta_\varphi(-\varphi) \geq 0.$$

By the assumption that the Sobolev constant of (M, ω_φ) is bounded and the L^2 norm is bounded above, the inequality (5.7) implies

$$0 \leq \varphi_- \leq C.$$

In other words, we have

$$|\varphi|_{L^\infty} \leq C.$$

To derive an upper bound on the volume form, first note that $Ric(\omega_\varphi)$ is bounded from above, thus

$$\Delta\left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right) \geq -C \tag{5.9}$$

for some constants C_2, C. Thus

$$\begin{aligned}
&\left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)(x) \\
&\qquad = -\int_M G(x, y)\Delta\left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)\omega^n + \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + C_2\varphi\right)\omega^n \\
&\qquad \leq +C + \log \int_M \frac{\omega_\varphi^n}{\omega^n}\omega^n + \int_M \varphi\omega^n \\
&\qquad \leq C + \int_M \varphi\omega^n.
\end{aligned}$$

Using the fact that $|\varphi|_{L^\infty}$ is bounded, we have

$$\log \frac{\omega_\varphi^n}{\omega^n} \leq C$$

for some constant C.

To prove the metric is equivalent, we follow Yau's proof of the Calabi conjecture (cf. [13]). Following the notation of Sect. 3.2.2, set

$$u = \exp(-\lambda\varphi)(n + \Delta\varphi), \quad \text{and} \quad F = \log \frac{\omega_\varphi^n}{\omega^n}.$$

At the maximal point p of the function u, similar to inequality (3.15), we have

$$\Delta_\varphi\{\exp(-\lambda\varphi)(n + \Delta\varphi)\}(p) \leq 0.$$

At the point p, we have

$$0 \geq \Delta F - n^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}} - \lambda n(n + \Delta\varphi) + \left(\lambda + \inf_{i \neq l} R_{i\bar{i}l\bar{l}}\right) \exp\left\{\frac{-F}{n-1}\right\}(n + \Delta\varphi)^{\frac{n}{n-1}}.$$

Applying inequality (5.9), we have

$$0 \geq -C\Delta\varphi - C_2 - n^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}} - \lambda n(n + \Delta\varphi) + \left(\lambda + \inf_{i \neq l} R_{i\bar{i}l\bar{l}}\right) \exp\left\{\frac{-F}{n-1}\right\}(n + \Delta\varphi)^{\frac{n}{n-1}}.$$

Since we already have control of F from both above and below here, we can choose λ large enough, to imply that $(n + \Delta\varphi)(p)$ are uniformly bounded from above. Therefore,

$$u = \exp(-\lambda\varphi)(n + \Delta\varphi), \quad \text{and} \quad F = \log \frac{\omega_\varphi^n}{\omega^n}$$

are uniformly bounded from above (since $|\varphi|_{L^\infty}|$ is uniformly bounded). Thus

$$0 < n + \Delta\varphi \leq C.$$

Thus, ω_φ is uniformly equivalent to ω. Consequently, $|\Delta F|$ is uniformly bounded. Thus, the metric is uniformly $C^{1,\alpha}$ bounded for any $\alpha \in (0, 1)$. □

6 Further remarks: some open problems on geodesic rays and geodesic stability

Conjecture/Question 6.1. For every smooth test configuration, there is a corresponding relative $C^{1,1}$ geodesic ray.

In [1], Arezzo–Tian constructed a smooth geodesic ray out of special degenerations where the central fiber is analytic. One may be able to prove this directly with the method of continuity. A more challenging question is the following

Conjecture/Question 6.2. For a smooth test configuration where the central fiber is smooth except on some codimension 2 sub-variety, there is a geodesic ray associated to this test configuration. Moreover, the geodesic ray is smooth everywhere except along a sub-variety of codimension at least 2.

One interesting question is whether the ¥ invariant defined here and the generalized Futaki invariant defined elsewhere are actually the same. The author believes this is the case. This in particular means that the notion of stability introduced here more-or-less corresponds to the notion of stability in the algebraic setting, as is discussed so eloquently by Yau, Tian [33], Donaldson [15] and others.

Conjecture/Question 6.3. If the initial geodesic ray has bounded geometry, then there is a geodesic ray which is parallel to this initial ray from any generic Kähler potential which is smooth everywhere except on a sub-variety of codimension 2 or higher.

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