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On the uniqueness of smooth, stationary black holes in vacuum

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Abstract. A fundamental conjecture in general relativity asserts that the domain of outer communication of a regular, stationary, four dimensional, vacuum black hole solution is isometrically diffeomorphic to the domain of outer communication of a Kerr black hole. So far the conjecture has been resolved, by combining results of Hawking [17], Carter [4] and Robinson [28], under the additional hypothesis of non-degenerate horizons and *real analyticity* of the space-time. We develop a new strategy to bypass analyticity based on a tensorial characterization of the Kerr solutions, due to Mars [24], and new geometric Carleman estimates. We prove, under a technical assumption (an identity relating the Ernst potential and the Killing scalar) on the bifurcate sphere of the event horizon, that the domain of outer communication of a smooth, regular, stationary Einstein vacuum spacetime of dimension 4 is locally isometric to the domain of outer communication of a Kerr spacetime.

Contents

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1. Introduction

A fundamental conjecture in general relativity[1] asserts that the domains of outer communication of regular[2], stationary, four dimensional, vacuum black hole solutions are isometrically diffeomorphic to those of Kerr black holes. One expects, due to gravitational radiation, that general, asymptotically flat, dynamic, solutions of the Einstein-vacuum equations settle down, asymptotically, into a stationary regime. A similar scenario is supposed to hold true in the presence of matter. Thus the conjecture, if true, would characterize all possible asymptotic states of the general evolution.

So far the conjecture has been resolved, by combining results of Hawking [17], Carter [4], and Robinson [28], under the additional hypothesis of non-degenerate horizons and *real analyticity* of the space-time. The assumption of real analyticity is both hard to justify and difficult to dispense of. One can show, using standard elliptic theory, that stationary solutions are real analytic in regions where the corresponding Killing vector-field $\mathbf{T}$ is time-like, but there is no reason to expect the same result to hold true in the ergo-region (in Kerr, the Killing vector-field $\mathbf{T}$, which is time-like in the asymptotic region, becomes space-like in the ergo-region). In view of the main application of the conjectured result to the general problem of evolution, mentioned above, there is also no reason to expect that, by losing gravitational radiation, general solutions become, somehow, analytic. Thus the assumption of analyticity is a serious limitation of the present uniqueness results. Unfortunately one of the main step in the current proof, due to Hawking [17], depends heavily on analyticity. As we argue below, to extend Hawking's argument to a smooth setting requires solving an *ill posed problem*. Roughly speaking Hawking's argument is based on the observation that, though a general stationary space may seem quite complicated, its behavior along the event horizon is remarkably simple. Thus Hawking has shown that in addition to the original, stationary, Killing field, which has to be tangent to the event horizon, there must exist, infinitesimally along the horizon, an additional Killing vector-field. To extend this information, from the event horizon to the domain of outer communication, requires one to solve a boundary value problem, with data on the horizon, for a linear differential equation. Such problems are typically *ill posed* (i.e. solutions may fail to exist in the smooth category.) In the analytic category, however, the problem can be solved by a straightforward Cauchy–Kowalewsky type argument. Thus, by assuming analyticity for the stationary metric, Hawking bypasses this fundamental difficulty, and thus is able to extend this additional Killing field to the entire domain of outer communication. As a consequence, the space-time under consideration is not just stationary but also axi-symmetric, situation for which Carter–Robinson's uniqueness

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[1] See reviews by B. Carter [5] and P. Chrusciel [9,10] for a history and review of the current status of the conjecture.

[2] The notion of regularity needed here requires a careful discussion concerning the geometric hypothesis on the space-time.

theorem [4,28] applies. It is interesting to remark that this final step does not require analyticity.

Though ill posed problems do not, in general, admit solutions, one can, when a solution is known to exist, often prove uniqueness (we refer the reader to the introduction in [19] for a more thorough discussion of this issue). This fact has led us to develop a different strategy for proving uniqueness based on a characterization of the Kerr solution, due to Mars [24], and geometric Carleman estimates applied to covariant wave equations on a general, stationary, black hole background. We discuss this strategy in more details in the following subsection, after we recall a few basic definitions and results concerning stationary black holes. Our main result, stated in Subsect. 1.2 below, proves uniqueness of the Kerr family among all, smooth, appropriately regular, stationary solutions, with a regular, bifurcate, event horizon, under an additional assumption which has to be satisfied along the bifurcate sphere S_0 of the event horizon. More precisely we assume a pointwise complex scalar identity relating the Ernst potential σ and the Killing scalar $\mathcal{F}^2$ on S_0.

1.1. Stationary, regular, black holes. In this subsection we review some of the main definitions and results concerning stationary black holes (see also the discussion in the introduction to Sect. 3). We will also give a more detailed discussion of our new approach to the problem of uniqueness. Precise assumptions concerning our result will be made only in the next subsection.

The main objects in the theory of stationary, vacuum, black holes are $3+1$ dimensional space-times $(\mathbf{M}, \mathbf{g})$ which are smooth, strongly causal, time oriented, solutions of the Einstein vacuum equations, see [17] for precise definitions, and which are also *stationary*, *asymptotically flat*. More precisely one considers, see for example p. 2 in [15], space-times $(\mathbf{M}, \mathbf{g})$ endowed with a 1-parameter group of isometries Φ_t, generated by a Killing vector-field $\mathbf{T}$, and which possess a smooth space-like slice Σ_0 with an asymptotically flat end $\Sigma_0^{(end)} \subset \Sigma_0$ on which $g(\mathbf{T}, \mathbf{T}) < 0$. To ensure strong causality we assume that $\mathbf{M}$ is the maximal globally hyperbolic extension of Σ_0. This implies, in particular, that all orbits of $\mathbf{T}$ are complete, see [8], and must intersect Σ_0, see [14]. Define $\mathbf{M}^{(end)} = \bigcup_{t\in\mathbb{R}} \Phi_t(\Sigma_0^{(end)})$. Take $\mathbf{B}$ to be the complement of $\mathcal{I}^-(\mathbf{M}^{(end)})$, $\mathbf{W}$ the complement of $\mathcal{I}^+(\mathbf{M}^{(end)})$, where $\mathcal{I}^{\pm}(S)$ denote the causal future and past sets of a set $S \subset \mathbf{M}$. In other words $\mathbf{B}$ (called the *black hole* region), respectively $\mathbf{W}$ (called the *white hole* region), is the set of points in $\mathbf{M}$ for which no future directed, respectively past directed, causal curve meets $\mathbf{M}^{(end)}$. Also we take $\mathbf{E}$ (called domain of outer communication) the complement of $\mathbf{W} \cup \mathbf{B}$, i.e. $\mathbf{E} = \mathcal{I}^-(\mathbf{M}^{(end)}) \cap \mathcal{I}^+(\mathbf{M}^{(end)})$. We further define the future event horizon $\mathcal{H}^+$ to be the boundary of $\mathcal{I}^-(\mathbf{M}^{(end)})$ and the past event horizon $\mathcal{H}^-$ to be the boundary of $\mathcal{I}^+(\mathbf{M}^{(end)})$,

$$\mathcal{H}^+ = \delta\mathbf{B}, \quad \mathcal{H}^- = \delta\mathbf{W}.$$

By definition both $\mathcal{H}^+$ and $\mathcal{H}^-$ are achronal (i.e. no two points on $\mathcal{H}^+$, or $\mathcal{H}^-$ can be connected by time-like curves) boundaries generated by null geodesic segments. According to the topological censorship theorem, see [13] or [16], the domain of outer communication $\mathbf{E}$ is simply connected. This implies that all connected components of event horizons must have the topology of $\mathbb{S}^2 \times \mathbb{R}$. In our work we shall assume that the event horizon has only one component.

It follows immediately from the definitions above that the flow Φ_t must keep $\mathcal{H}^+$ and $\mathcal{H}^-$ invariant, therefore the generating vector-field $\mathbf{T}$ must be tangent to $\mathcal{H}$. One further assumes that Φ_t has no fixed points on $\mathcal{H}$ with the possible exception of $S_0 = \mathcal{H}^+ \cap \mathcal{H}^-$. Then either $\mathbf{T}$ is space-like or null at all points of $\mathcal{H}$. If $\mathbf{T}$ is null on $\mathcal{H}$, in which case $\mathcal{H}$ is said to be a Killing horizon for $\mathbf{T}$, Sudarski–Wald [30] have proved that the space-time must be static, i.e. $\mathbf{T}$ is hypersurface orthogonal. Static solutions, on the other hand, are known to be isomorphic to Schwarzschild metrics, see [3, 11, 21]. In this paper we are interested only in the case when $\mathbf{T}$ is space-like at some points on the horizon.

The existence of partial Cauchy hypersurface Σ_0 implies, in particular, the existence of a foliation Σ_t on $\mathbf{E}$, which induces a foliation S_t on the horizon $\mathcal{H}$ with a well defined area. A key result of Hawking [17] (see also [12] where the area theorem is proved under very general differentiability assumptions), shows that the area of S_t is a monotonous function of t. Using this fact, together with the tangency of the Killing field $\mathbf{T}$, one can show that the null second fundamental forms of both $\mathcal{H}^+$ and $\mathcal{H}^-$ must vanish identically, see [17]. Specializing to the future event horizon $\mathcal{H}^+$, Hawking [17] (see also [20]) has proved the existence of a non-vanishing vector-field K, tangent to the null generators of $\mathcal{H}^+$ which is Killing to any order along $\mathcal{H}^+$. Moreover $\mathbf{D}_K K = \kappa K$ with κ, constant along $\mathcal{H}^+$, called the surface gravity of $\mathcal{H}^+$. If $\kappa \neq 0$ we say that $\mathcal{H}^+$ is *non-degenerate*. In the non-degenerate case the work of Racz and Wald [27] supports the hypothesis, which we make in our work (see next subsection), that $\mathcal{H}^+$ and $\mathcal{H}^-$ are smooth null hypersurfaces intersecting smoothly on a 2 surface S_0 with the topology of the standard sphere. We say, in this case, that the horizon $\mathcal{H}$ is a smooth *bifurcate horizon*.

Under the restrictive assumption of real analyticity of the metric g one can show, see [10, 17], that the Hawking vector-field K can be extended to a neighborhood of the entire domain of communication.[3] One can then show that the spacetime $(\mathbf{M}, \mathbf{g})$ is not just stationary but also axi-symmetric. One can then appeal to the results of Carter [4] and Robinson [28] which show that the family of Kerr solutions with $0 \leq a < m$ exhaust the class of non-degenerate, stationary axi-symmetric, connected, four dimensional, vacuum black holes. This concludes the present proof of uniqueness, based on analyticity.

[3] In [15] it is shown that K can be extended in the complement of the domain of outer communication $\mathbf{E}$ without the restrictive analyticity assumption. However their argument does not apply to the domain of outer communication $\mathbf{E}$.

Without analyticity any hope of extending K outside $\mathcal{H}$, in $\mathbf{E}$, by a direct argument encounters a fundamental difficulty. Indeed one needs to extend K such that it satisfies the Killing equation,

$$\mathbf{D}_\mu K_\nu + \mathbf{D}_\mu K_\nu = 0. \tag{1.1}$$

Differentiating the Killing equation and using the Ricci flat condition $\mathrm{Ric}(\mathbf{g}) = 0$ one derives the covariant wave equation $\square_{\mathbf{g}} K = 0$. The obstacle we encounter is that the boundary value problem $\square_{\mathbf{g}} K = 0$ with K prescribed on $\mathcal{H}$ is *ill posed*, which means that it is impossible to extend K by solving $\square_{\mathbf{g}} K = 0$, if the metric is smooth but fails to be real analytic. To understand the ill posed character of the situation it helps to consider the following simpler model problem in the domain $\mathbf{E} = \{(t, x) \in \mathbb{R}^{1+3}/|x| > 1 + |t|\}$ of Minkowski space $\mathbb{R}^{1+3}$

$$\square\phi = F(\phi, \partial\phi), \quad \phi|_{\delta\mathbf{E}} = \phi_0. \tag{1.2}$$

Here $\square$ is the usual D'Alembertian of $\mathbb{R}^{1+3}$ and F a smooth function of ϕ and its partial derivatives $\partial_\alpha\phi$, vanishing for $\phi = \partial\phi = 0$. One can regard $\mathbf{E}$ as a model of the domain of outer communication and its boundary $\mathcal{H} = \delta\mathbf{E}$ as analogous to the bifurcate event horizon considered above. The problem is still ill posed; even in the case $F \equiv 0$ we cannot, in general, find solutions for arbitrary smooth boundary data ϕ_0. Yet, as typical to many ill posed problems, even if existence fails we can still prove uniqueness. In other words if (1.2) has two solutions ϕ_1, ϕ_2 which agree on $\mathcal{H} = \delta\mathbf{E}$ then they must coincide everywhere in $\mathbf{E}$, see [19]. The result is based on Carleman estimates, i.e. on space-time L^2 a-priori estimates with carefully chosen weights. A more realistic model problem is to consider smooth space-time metrics $\mathbf{g}$ in $\mathbb{R}^{1+3}$ which verify the Einstein vacuum equations and agree, up to curvature, with the standard Minkowski metric on the boundary $\mathcal{H} = \delta\mathbf{E}$. Can we prove that $\mathbf{g}$ must be flat also in $\mathbf{E}$? It is easy to see, using the Einstein equations, that the Riemann curvature tensor R of such metrics must verify a covariant wave equation of the form $\square_{\mathbf{g}} R = R * R$, with $R * R$ denoting an appropriate quadratic product of components of R. We are thus led to a question similar to the one above; knowing that R vanishes on the boundary of $\mathbf{E}$ can we deduce that it also vanishes on $\mathbf{E}$? Using methods similar to those of [19] we can prove that R must vanish in a neighborhood of $\mathcal{H}$. We also expect that, under additional global assumptions on the metric $\mathbf{g}$, one can show that R vanishes everywhere on $\mathbf{E}$ and therefore $\mathbf{g}$ is locally Minkowskian.

These considerations lead us to look for a tensor-field $\mathcal{S}$, associated to our stationary metric $\mathbf{g}$, which satisfies the following properties.

(1) If $\mathcal{S}$ vanishes in $\mathbf{E}$ then the metric $\mathbf{g}$ is locally isometric to a Kerr solution.
(2) $\mathcal{S}$ verifies a covariant wave equation of the form,

$$\square_{\mathbf{g}} \mathcal{S} = \mathcal{A} * \mathcal{S} + \mathcal{B} * \mathbf{D}\mathcal{S}, \tag{1.3}$$

with $\mathcal{A}$ and $\mathcal{B}$ two arbitrary smooth tensor-fields.
(3) $\mathcal{S}$ vanishes identically on the bifurcate event horizon $\mathcal{H}$.

An appropriate space-time tensor verifying condition (1) has been proposed by M. Mars in [24], based on some previous work of W. Simon [29]; we refer to it as the Mars–Simon tensor. In this paper we shall show that $\mathcal{S}$ verifies the desired wave equation in (2) and give a sufficient, simple condition on the bifurcate sphere S_0, which insures that $\mathcal{S}$ vanishes on the event horizon $\mathcal{H}$. We then prove, based on a global unique continuation argument, that $\mathcal{S}$ must vanish everywhere in the domain of outer communication $\mathbf{E}$. In view of Mars's result [24] we deduce that $\mathbf{E}$ is locally isometric with a Kerr solution.

The unique continuation strategy is based on two Carleman estimates. The first one establishes the vanishing of solutions to covariant wave equations, with zero boundary conditions on a neighborhood of S_0 on the event horizon, to a full space-time neighborhood of S_0. The proof of this result can be extended to the exterior of a regular, bifurcate null hypersurface (i.e. with a regular bifurcate sphere), in a general, smooth, Lorentz manifold. Our second, conditional, Carleman estimate is significantly deeper as it depends heavily on the specific properties of stationary solutions of the Einstein vacuum equations. We use it, together with an appropriate bootstrap argument, to extend the region of vanishing of the Mars–Simon tensor from a neighborhood of S_0 to the entire domain of outer communication $\mathbf{E}$. The proof of both Carleman estimates (see also discussion in the first subsection of Sect. 3), but especially the second, rely on calculations based on null frames and complex null tetrads. We develop our own formalism, which is, we hope, a useful compromise between that of Newmann–Penrose [26] and that used in [7,22]. Strictly speaking the formalism used in [7] does not apply in the situation studied here as it presupposes that the horizontal distribution generated by the null pair is integrable. The horizontal distribution generated by the principal null directions in Kerr do not verify this property.

1.2. Precise assumptions and the main theorem. We state now our precise assumptions. We assume that $(\mathbf{M}, \mathbf{g})$ is a smooth,[4] time oriented, vacuum Einstein spacetime of dimension $3 + 1$ and $\mathbf{T} \in \mathbf{T}(\mathbf{M})$ is a smooth Killing vector-field on $\mathbf{M}$. In addition, we make the following assumptions and definitions.

AF (Asymptotic flatness)**.** We assume that there is an open subset $\mathbf{M}^{(end)}$ of $\mathbf{M}$ which is diffeomorphic to $\mathbb{R} \times (\{x \in \mathbb{R}^3 : |x| > R\})$ for some R sufficiently large. In local coordinates $\{t, x^i\}$ defined by this diffeomorphism, we assume that, with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$,

$$\mathbf{g}_{00} = -1 + \frac{2M}{r} + O(r^{-2}), \quad \mathbf{g}_{ij} = \delta_{ij} + O(r^{-1}), \quad \mathbf{g}_{0i} = O(r^{-2}), \tag{1.4}$$

[4] $\mathbf{M}$ is assumed to be a connected, orientable, paracompact C^∞ manifold without boundary.

for some $M > 0$, and

$$\mathbf{T} = \partial_t \quad \text{therefore} \quad \partial_t \mathbf{g}_{\mu\nu} = 0.$$

We define the domain of outer communication (exterior region)

$$\mathbf{E} = \mathcal{I}^-(\mathbf{M}^{(end)}) \cap \mathcal{I}^+(\mathbf{M}^{(end)}).$$

We assume that there is an imbedded space-like hypersurface $\Sigma_0 \subseteq \mathbf{M}$ which is diffeomorphic to $\{x \in \mathbb{R}^3 : |x| > 1/2\}$ and, in $\mathbf{M}^{(end)}$, Σ_0 agrees with the hypersurface corresponding to $t = 0$. Let T_0 denote the future directed unit vector orthogonal to Σ_0. We assume that every orbit of $\mathbf{T}$ in $\mathbf{E}$ is complete and intersects the hypersurface Σ_0, and

$$|\mathbf{g}(\mathbf{T}, T_0)| > 0 \quad \text{on } \Sigma_0 \cap \mathbf{E}. \tag{1.5}$$

SBS (Smooth bifurcate sphere)**.** Let

$$S_0 = \delta(\mathcal{I}^-(\mathbf{M}^{(end)})) \cap \delta(\mathcal{I}^+(\mathbf{M}^{(end)})).$$

We assume that $S_0 \subseteq \Sigma_0$ and S_0 is an imbedded 2-sphere which agrees with the sphere of radius 1 in $\mathbb{R}^3$ under the identification of Σ_0 with $\{x \in \mathbb{R}^3 : |x| > 1/2\}$. Furthermore, we assume that there is a neighborhood $\mathbf{O}$ of S_0 in $\mathbf{M}$ such that the sets

$$\mathcal{H}^+ = \mathbf{O} \cap \delta(\mathcal{I}^-(\mathbf{M}^{(end)})) \quad \text{and} \quad \mathcal{H}^- = \mathbf{O} \cap \delta(\mathcal{I}^+(\mathbf{M}^{(end)}))$$

are smooth imbedded hypersurfaces diffeomorphic to $S_0 \times (-1, 1)$, We assume that these hypersurfaces are null, non-expanding[5], and intersect transversally in S_0. Finally, we assume that the vector-field $\mathbf{T}$ is tangent to both hypersurfaces $\mathcal{H}^+ = \mathbf{O} \cap \delta(\mathcal{I}^-(\mathbf{M}^{(end)}))$ and $\mathcal{H}^- = \mathbf{O} \cap \delta(\mathcal{I}^+(\mathbf{M}^{(end)}))$, and does not vanish identically on S_0.[6]

T (Technical assumptions)**.** Let $F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta$ denote the Killing form on $\mathbf{M}$, and $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i\,{}^*F_{\alpha\beta}$, where ${}^*F_{\alpha\beta} = \frac{1}{2}\in_{\alpha\beta\gamma\delta} F^{\gamma\delta}$. Let $\mathcal{F}^2 = \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$. The Ernst 1-form associated to $\mathbf{T}$ is defined as $\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu}$. It is easy to check, see (4.18), that σ_μ is exact and, therefore, there exists a complex scalar σ defined in an open neighborhood of Σ_0, called the Ernst potential, such that $\mathbf{D}_\mu \sigma = \sigma_\mu$. In view of the asymptotic flatness assumption **AF**, we can choose σ such that $\sigma \to 1$ at infinity along Σ_0. Our main technical assumptions are

$$-4M^2\mathcal{F}^2 = (1 - \sigma)^4 \quad \text{on } S_0, \tag{1.6}$$

[5] A null hypersurface is said to be non-expanding if the trace of its null second fundamental form vanishes identically.

[6] In view of a well known result, see [23], any non-vanishing Killing field on S_0 can only vanish at a finite number of isolated points.

and

$$\Re((1-\sigma)^{-1}) > 1/2 \quad \text{at some point on } S_0. \tag{1.7}$$

Remark 1.1. As we have discussed in the previous subsection some of the assumptions made above have been deduced from more primitive assumptions. For example, the completeness of orbits of **E** can be deduced by assuming that **M** is the maximal global hyperbolic extension of Σ_0, see [8]. Our precise space-time asymptotic flatness conditions can be deduced by making asymptotic flatness assumptions only on Σ_0, see [2,1]. The assumption (1.5) can be replaced, at the expense of some additional work in Sect. 8, by a suitable regularity assumption on the space of orbits of **T**. The non-expanding condition in **SBS** can be derived using the area theorem, see [17,12]. The regular bifurcate structure of the horizon, assumed in **SBS**, is connected to the more primitive assumption of non-degeneracy of the horizon, see [27].

Remark 1.2. Assumption (1.7) is consistent with the natural condition $0 \le a < M$ satisfied by the two parameters of the Kerr family. The key technical assumption in this paper is the identity (1.6), which is assumed to hold on the bifurcate sphere S_0. This assumption is made in order to insure that the corresponding Mars–Simon tensor vanishes on $\mathcal{H}^- \cup \mathcal{H}^+$. We emphasize, however, that we do not make any technical assumptions in the open set **E** itself; the identity (1.6) is only assumed to hold on the bifurcate sphere S_0, which is a codimension 2 set, while the inequality (1.7) is only assumed at one point of S_0. We hope to further relax these technical conditions and interpret them as part of the "regularity" assumptions on the black hole in future work.

Remark 1.3. In Boyer–Lindquist coordinates the Kerr metric takes the form,

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{\rho^2}\left(d\phi - \frac{2aMr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2, \tag{1.8}$$

where,

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr,$$
$$\Sigma^2 = (r^2+a^2)\rho^2 + 2Mra^2(\sin\theta)^2.$$

On the horizon we have $r = r_+ := M + \sqrt{M^2 - a^2}$ and $\Delta = 0$. The domain of outer communication **E** is given by $r > r_+$. One can show that the complex Ernst potential σ and the complex scalar $\mathcal{F}^2$ are given by

$$\sigma = 1 - \frac{2M}{r + ia\cos\theta}, \qquad \mathcal{F}^2 = -\frac{4M^2}{(r+ia\cos\theta)^4}. \tag{1.9}$$

Thus,

$$-4M^2\mathcal{F}^2 = (1-\sigma)^4 \tag{1.10}$$

everywhere in the exterior region. Writing $y + iz := (1-\sigma)^{-1}$ we observe that,

$$y = \frac{r}{2M} \geq \frac{r_+}{2M} > \frac{1}{2}$$

everywhere in the exterior region.

Main theorem. *Under the assumptions* **AF**, **SBS**, *and* **T** *the domain of outer communication* **E** *of* **M** *is locally isometric to the domain of outer communication of a Kerr space-time with mass M and* $0 < a < M$.

As mentioned earlier, the basic idea of the proof is to show that the Mars–Simon tensor is well-defined and vanishes in the entire domain of outer communication, by relying on Carleman type estimates. We provide below a more detailed outline of the proof.

In Sect. 3, we prove a sufficiently general geometric Carleman inequality, Proposition 3.3, with weights that satisfy suitable conditional pseudo-convexity assumptions. This Carleman inequality is applied in Sect. 6 to prove Proposition 6.1 and Sect. 8 to prove Proposition 8.5.

In Sect. 4 we define, in a simply connected neighborhood $\widetilde{\mathbf{M}}$ of $\Sigma_0 \cap \overline{\mathbf{E}}$, the Killing form $\mathcal{F}_{\alpha\beta}$ and the Ernst potential σ. We then introduce the Mars–Simon tensor, see [24],

$$\mathcal{S}_{\alpha\beta\mu\nu} = \mathcal{R}_{\alpha\beta\mu\nu} + 6(1-\sigma)^{-1}\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - (1/3)\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}\big)$$

as a self-dual Weyl tensor, which is well defined and smooth in the open set

$$\mathbf{N}_0 = \{x \in \widetilde{\mathbf{M}} : 1 - \sigma(x) \neq 0\}.$$

It is important to observe that $\mathbf{N}_0$ contains a neighborhood of the bifurcate sphere S_0, since $\Re\sigma = -\mathbf{T}^\alpha\mathbf{T}_\alpha$, which is nonpositive on S_0. In particular, the Mars–Simon tensor is well defined in a neighborhood of S_0. The main result of the section, stated in Theorem 4.5, is the identity

$$\mathbf{D}^\sigma\mathcal{S}_{\sigma\alpha\mu\nu} = \mathcal{J}(\mathcal{S})_{\alpha\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\lambda\mathcal{S}_{\lambda\rho\gamma\delta}\big(\mathcal{F}_\alpha{}^\rho\delta_\mu^\gamma\delta_\nu^\delta - (2/3)\mathcal{F}^{\gamma\delta}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}\big), \tag{1.11}$$

which shows that $\mathcal{S}$ verifies a divergence equation with a source term $\mathcal{J}(\mathcal{S})$ proportional to $\mathcal{S}$. It is then straightforward to deduce, see Theorem 4.7, that $\mathcal{S}$ verifies a covariant wave equation with a source proportional to $\mathcal{S}$ and first derivatives of $\mathcal{S}$.

In Sect. 5 we show that $\mathcal{S}$ vanishes on the horizon $\delta(\mathcal{I}^-(\mathbf{M}^{(end)})) \cup \delta(\mathcal{I}^+(\mathbf{M}^{(end)}))$, in a neighborhood of the bifurcate sphere S_0. The proof depends on special properties of the horizon, such as the vanishing of the null second fundamental forms and certain null curvature components, and the divergence equation (1.11). The proof also depends on the main technical assumption (1.6) to show that the component $\rho(\mathcal{S})$ vanishes on S_0 (this is the only place where this technical assumption is used).

In Sect. 6 we show that $\mathcal{S}$ vanishes in a full space-time neighborhood $\mathbf{O}_{r_1} \cap \mathbf{E}$ of S_0 in $\mathbf{E}$, see Proposition 6.1. For this we derive the Carleman inequality of Lemma 6.2, as a consequence of the more general Proposition 3.3. The weight function used in this Carleman inequality is constructed with the help of two optical functions u_+ and u_-, defined in a space-time neighborhood of S_0. We then apply this Carleman inequality to the covariant wave equation verified by $\mathcal{S}$, to prove Proposition 6.1.

Once we have regions of space-time in which $\mathcal{S}$ vanishes we can rely on some of the remarkable computations of Mars [24]. In Sect. 7 we work in an open set $\mathbf{N} \subseteq \mathbf{N}_0$ (thus $1 - \sigma \neq 0$ in $\mathbf{N}$), $S_0 \subseteq \mathbf{N}$, with the property that $\mathcal{S} = 0$ in $\mathbf{N} \cap \mathbf{E}$ and $\mathbf{N} \cap \mathbf{E}$ is connected. Such sets exist, in view of the main result of Sect. 6. Following Mars [24], we define the real functions y and z in $\mathbf{N}$ by

$$y + iz = (1 - \sigma)^{-1},$$

see Remark 1.3 for explicit formulas in the Kerr spaces. The function y satisfies the important identity (7.19), found by Mars,

$$\mathbf{D}_\alpha y \mathbf{D}^\alpha y = \frac{y^2 - y + B}{4M^2(y^2 + z^2)} \tag{1.12}$$

in $\mathbf{N} \cap \mathbf{E}$, where $B \in [0, \infty)$ is a constant which has the additional property that $z^2 \leq B$ in $\mathbf{N} \cap \mathbf{E}$ (in the Kerr space $B = a^2/(4M^2)$). We then use this identity and the fact that $\Re(1 - \sigma) = 1 + \mathbf{g}(\mathbf{T}, \mathbf{T})$ to prove the key bound on the coordinate norm of the gradient

$$|D^1 y| \leq \widetilde{C} \quad \text{in } \mathbf{N} \cap \mathbf{E}, \tag{1.13}$$

with a uniform constant $\widetilde{C}$ (see Proposition 7.2). This bound, together with $z^2 \leq B$, shows that the function $1 - \sigma = (y + iz)^{-1}$ cannot vanish in a neighborhood of the closure of $\mathbf{N} \cap \mathbf{E}$, as long as $\mathcal{S} = 0$ in $\mathbf{N} \cap \mathbf{E}$ and $\mathbf{N} \cap \mathbf{E}$ is connected. This observation is important in Sect. 8, as part of the bootstrap argument, to show that $1 - \sigma \neq 0$ in $\Sigma_0 \cap \mathbf{E}$. Finally, in Lemma 7.3 we work in a canonical complex null tetrad and compute the Hessian $\mathbf{D}^2 y$ in terms of the functions y and z, and the connection coefficient ζ.

In Sect. 8 we use a bootstrap argument to complete the proof of the main theorem. Our main goal is to show that $1 - \sigma \neq 0$ and $\mathcal{S} = 0$ in $\Sigma_0 \cap \mathbf{E}$. We start by showing that $y = y_{S_0}$ is constant on the bifurcate sphere S_0, and use (1.12) to show that $y_{S_0}^2 - y_{S_0} + B = 0$; using (1.7) it follows that $B \in [0, 1/4)$ and $y_{S_0} \in (1/2, 1]$. We use then the wave equation

$$\mathbf{D}^\alpha \mathbf{D}_\alpha y = \frac{2y - 1}{4M^2(y^2 + z^2)},$$

which is a consequence of $\mathcal{S} = 0$, and the fact that $y_{S_0} > 1/2$, to show that y must increase in a small neighborhood $\mathbf{O}_\epsilon \cap \mathbf{E}$. We can then start our bootstrap argument: for $R > y_{S_0}$ let $\mathcal{U}_R$ denote the unique connected component of the set $\{x \in \Sigma_0 \cap \mathbf{E} : \sigma(x) \neq 1 \text{ and } y(x) < R\}$ whose closure in Σ_0 contains S_0. We need to show, by induction over R, that $\mathcal{S} = 0$ in $\mathcal{U}_R$ for any $R > y_{S_0}$; assuming this, it would follow from (1.13) that $\sigma \neq 1$ in $\Sigma_0 \cap \mathbf{E}$ and $\bigcup_{R>y_{S_0}} \mathcal{U}_R = \Sigma_0 \cap \mathbf{E}$, which would complete the proof of the main theorem. The key inductive step in proving that $\mathcal{S} = 0$ in $\mathcal{U}_R$ is to show that if x_0 is a point on the boundary of $\mathcal{U}_R$ in $\Sigma_0 \cap \mathbf{E}$, and if $\mathcal{S} = 0$ in $\mathcal{U}_R$, then $\mathcal{S} = 0$ in a neighborhood of x_0 (see Proposition 8.5). For this we use a second Carleman inequality, Lemma 8.6, with a weight that depends on the function y. To prove this second Carleman estimate we use the general Carleman estimate Proposition 3.3 and the remarkable pseudo-convexity properties of the Hessian of the function y computed in Lemma 7.3.

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2. Geometric preliminaries

2.1. Optical functions. We define two optical functions u_-, u_+ in a neighborhood of the bifurcate sphere S_0, included in the neighborhood $\mathbf{O}$ of hypothesis **SBS**. Choose a smooth future-past directed null pair (L_+, L_-) along S_0 (i.e. L_+ is future oriented while L_- is past oriented),

$$\mathbf{g}(L_-, L_-) = \mathbf{g}(L_+, L_+) = 0, \quad \mathbf{g}(L_+, T_0) = -1, \quad \mathbf{g}(L_+, L_-) = 1. \tag{2.1}$$

We extend L_+ (resp. L_-) along the null geodesic generators of $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) by parallel transport, i.e. $\mathbf{D}_{L_+} L_+ = 0$ (resp. $\mathbf{D}_{L_-} L_- = 0$). We define the function u_- (resp. u_+) along $\mathcal{H}^+$ (resp. $\mathcal{H}^-$) by setting $u_- = u_+ = 0$ on the bifurcate sphere S_0 and solving $L_+(u_-) = 1$ (resp. $L_-(u_+) = 1$). Let S_{u_-} (resp. S_{u_+}) be the level surfaces of u_- (resp. u_+) along $\mathcal{H}^+$ (resp. $\mathcal{H}^-$). We define L_- at every point of $\mathcal{H}^+$ (resp. L_+ at every point of $\mathcal{H}^-$) as the unique, past directed (resp. future directed), null vector-field orthogonal to the surface S_{u_-} (resp. S_{u_+}) passing through that point and such that $\mathbf{g}(L_+, L_-) = 1$. We now define the null hypersurface $\mathscr{H}_{u_-}$ to be the congruence of null geodesics initiating on $S_{u_-} \subset \mathcal{H}^+$ in the direction of L_-. Similarly we define $\mathscr{H}_{u_+}$ to be the congruence of null geodesics initiating on $S_{u_+} \subset \mathcal{H}^-$ in the direction of L_+. Both congruences are well defined in a sufficiently small neighborhood $\mathbf{O}$ of S_0 in $\mathbf{M}$. The null hypersurfaces $\mathscr{H}_{u_-}$ (resp. $\mathscr{H}_{u_+}$) are the level sets of a function u_- (resp. u_+) vanishing on $\mathcal{H}^-$ (resp. $\mathcal{H}^+$). Moreover we can arrange that both u_-, u_+ are positive in the domain of outer communication $\mathbf{E}$. By construction they are both null optical functions, i.e.

$$\mathbf{g}^{\mu\nu}\partial_\mu u_+ \partial_\nu u_+ = \mathbf{g}^{\mu\nu}\partial_\mu u_- \partial_\nu u_- = 0. \tag{2.2}$$

We define

$$\Omega = \mathbf{g}^{\mu\nu}\partial_\mu u_+ \partial_\nu u_-. \tag{2.3}$$

In view of our construction we have,

$$u_+|_{\mathcal{H}^+} = u_-|_{\mathcal{H}^-} = 0, \quad \Omega|_{\mathcal{H}^+\cup\mathcal{H}^-} = 1. \tag{2.4}$$

Let

$$L_+ = \mathbf{g}^{\mu\nu}\partial_\mu u_+ \partial_\nu, \quad L_- = \mathbf{g}^{\mu\nu}\partial_\mu u_- \partial_\nu. \tag{2.5}$$

We have,

$$\mathbf{g}(L_+, L_+) = \mathbf{g}(L_-, L_-) = 0, \quad \mathbf{g}(L_+, L_-) = \Omega.$$

Define the sets,

$$\mathbf{O}_\epsilon = \{x \in \mathbf{O} : |u_-| < \epsilon, |u_+| < \epsilon\}.$$

For sufficiently small $\epsilon_0 > 0$ we have,

$$\Omega > \frac{1}{2} \quad \text{in } \mathbf{O}_{\epsilon_0}, \quad \overline{\mathbf{O}_{\epsilon_0}} \subset \mathbf{O}. \tag{2.6}$$

We also have, for $\epsilon \leq \epsilon_0$, $\mathbf{O}_\epsilon \cap \overline{\mathbf{E}} = \{0 \leq u_- < \epsilon, 0 \leq u_+ < \epsilon\}$. If ϕ is a smooth function in $\mathbf{O}_\epsilon$, vanishing on $\mathcal{H}^+ \cap \mathbf{O}_\epsilon$, one can show that there exists a smooth function ϕ' defined on $\mathbf{O}_\epsilon$ such that,

$$\phi = u_+ \cdot \phi' \quad \text{on } \mathbf{O}_\epsilon. \tag{2.7}$$

Similarly, if ϕ is a smooth function in $\mathbf{O}_\epsilon$, vanishing on $\mathcal{H}^- \cap \mathbf{O}_\epsilon$, then there exists another smooth function ϕ' defined on $\mathbf{O}_\epsilon$ such that,

$$\phi = u_- \cdot \phi' \quad \text{on } \mathbf{O}_\epsilon. \tag{2.8}$$

2.2. Quantitative bounds. Using the hypothesis (1.5) we may assume that for every $0 < \epsilon < \epsilon_0$ there is a sufficiently large constant $\widetilde{A}_\epsilon$ such that,

$$|\mathbf{g}(\mathbf{T}, T_0)| > \widetilde{A}_\epsilon^{-1}, \quad \forall x \in (\Sigma_0 \cap \mathbf{E}) \setminus \mathbf{O}_\epsilon. \tag{2.9}$$

In view of the normalization (2.1) we may assume (after possibly decreasing the value of ϵ_0) that, for some constant A_0,

$$u_+/u_- + u_-/u_+ \leq A_0 \quad \text{on } \mathbf{O}_{\epsilon_0} \cap \mathbf{E} \cap \Sigma_0. \tag{2.10}$$

We construct a system of coordinates which cover a neighborhood of the space-like hypersurface Σ_0. For any $R \in (0, 1]$ let $B_R = \{x \in \mathbb{R}^4 : |x| < R\}$ denote the open ball of radius R in $\mathbb{R}^4$. In view of the asymptotic flatness assumption **AF**, there is a constant $A_0 \in [\epsilon_0^{-1}, \infty)$ such that (2.10) holds and, in addition, for any $x_0 \in \Sigma_0 \cap \overline{\mathbf{E}}$ there is an open set $B_1(x_0) \subseteq \mathbf{M}$ containing x_0 and a smooth coordinate chart $\Phi^{x_0} : B_1 \to B_1(x_0)$, $\Phi^{x_0}(0) = x_0$, with the property that

$$\sup_{x_0\in\Sigma_0\cap\overline{\mathbf{E}}} \sup_{x\in B_1(x_0)} \sum_{j=0}^{6} \sum_{\alpha_1,\ldots,\alpha_j,\beta,\gamma=1}^{4} \left(|\partial_{\alpha_1}\ldots\partial_{\alpha_j}\mathbf{g}_{\beta\gamma}(x)| + \left|\partial_{\alpha_1}\ldots\partial_{\alpha_j}\mathbf{g}^{\beta\gamma}(x)\right|\right)$$
$$\leq A_0; \tag{2.11}$$
$$\sup_{x_0\in\Sigma_0\cap\overline{\mathbf{E}}} \sup_{x\in B_1(x_0)} \sum_{j=0}^{6} \sum_{\alpha_1,\ldots,\alpha_j,\beta=1}^{4} \left|\partial_{\alpha_1}\ldots\partial_{\alpha_j}\mathbf{T}^{\beta}(x)\right| \leq A_0.$$

We may assume that $B_1(x_0) \subseteq \mathbf{O}_{\epsilon_0}$ if $x_0 \in S_0$. We define $\widetilde{\mathbf{M}}$ to be the union of the balls $B_1(x_0)$ over all points $x_0 \in \Sigma_0 \cap \overline{\mathbf{E}}$. We can arrange such that $\widetilde{\mathbf{M}}$ is simply connected.

Since S_0 is compact, we may assume (after possibly increasing the value of A_0) that

$$\sup_{x_0\in S_0} \sup_{x\in B_1(x_0)} \left[\sum_{j=0}^{6} \sum_{\alpha_1,\ldots,\alpha_j=1}^{4} |\partial_{\alpha_1}\ldots\partial_{\alpha_j}u_{\pm}(x)| + \left(\sum_{\alpha=1}^{4}|\partial_\alpha u_{\pm}(x)|\right)^{-1}\right] \leq A_0. \tag{2.12}$$

Finally, we may also assume, in view of (1.7), that there is a point $x_0 \in S_0$ such that,

$$\Re\left((1-\sigma(x_0))^{-1}\right) > \frac{1}{2} + A_0^{-1}. \tag{2.13}$$

To summarize, we fixed constants ϵ_0 and $A_0 \geq \epsilon_0^{-1}$ such that (2.10)–(2.13) hold.

3. Unique continuation and Carleman inequalities

3.1. General considerations. As explained in Sect. 1 our proof of the main theorem is based on a global, unique continuation strategy applied to (1.3). We say that a linear differential operator L, in a domain $\Omega \subset \mathbb{R}^d$, satisfies the unique continuation property with respect to a smooth, oriented, hypersurface $\Sigma \subset \Omega$, if any smooth solution of $L\phi = 0$ which vanishes on one side of Σ must in fact vanish in a small neighborhood of Σ. Such a property depends, of course, on the interplay between the properties of the operator L and the hypersurface Σ. A classical result of Hörmander, see for example Chapt. 28 in [18], provides sufficient conditions for a scalar linear equation which guarantee that the unique continuation property holds. In the particular case of the scalar wave equation, $\square_{\mathbf{g}}\phi = 0$, and a smooth surface Σ, defined by the equation $h = 0$, $\nabla h \neq 0$, Hörmander's pseudo-convexity condition takes the simple form,

$$\mathbf{D}^2 h(X, X) < 0 \quad \text{if } \mathbf{g}(X, X) = \mathbf{g}(X, \mathbf{D}h) = 0 \tag{3.1}$$

at all points on the surface Σ, where we assume that ϕ is known to vanish on the side of Σ corresponding to $h < 0$.

In our situation, we plan to apply the general philosophy of unique continuation to the covariant wave equation (see Theorem 4.7),

$$\square_{\mathbf{g}}\mathcal{S} = \mathcal{A} * \mathcal{S} + \mathcal{B} * \mathbf{D}\mathcal{S}, \tag{3.2}$$

verified by the Mars–Simon tensor $\mathcal{S}$, see Definition 4.3. We prove in Sect. 5, using the main technical assumption (1.6), that $\mathcal{S}$ vanishes on the horizon $\mathcal{H}^+ \cup \mathcal{H}^-$ and we would like to prove, by unique continuation, that $\mathcal{S}$ vanishes in the entire domain of outer communication. In implementing such a strategy one encounters the following difficulties:

(1) Equation (3.2) is tensorial, rather than scalar.
(2) The horizon $\mathcal{H}^+ \cup \mathcal{H}^-$ is characteristic and non smooth in a neighborhood of the bifurcate sphere.
(3) Though one can show that an appropriate variant of Hörmander's pseudo-convexity condition holds true along the horizon, in a neighborhood of the bifurcate sphere, we have no guarantee that such condition continues to be true slightly away from the horizon, within the ergosphere region of the stationary space-time where $\mathbf{T}$ is space-like.

Problem (1) is not very serious; we can effectively reduce (3.2) to a system of scalar equations, diagonal with respect to the principal symbol. Problem (2) can be dealt with by an adaptation of Hörmander's pseudo-convexity condition. We note however that such an adaptation is necessary since, given our simple vanishing condition of $\mathcal{S}$ along the horizon, we cannot directly apply Hörmander's result in [18]. Problem (3) is by far the most serious. Indeed, even in the case when $\mathbf{g}$ is a Kerr metric (1.8), one can show that there exist null geodesics trapped within the ergosphere region $m + \sqrt{m^2 - a^2} \leq r \leq m + \sqrt{m^2 - a^2 \cos^2\theta}$. Indeed surfaces of the form $r\Delta = m(r^2 - a^2)^{1/2}$, which intersect the ergosphere for a sufficiently close to m, are known to contain such null geodesics, see [6]. One can show that the presence of trapped null geodesics invalidates Hörmander's pseudo-convexity condition. Thus, even in the case of the scalar wave equation $\square_{\mathbf{g}}\phi = 0$ in such a Kerr metric, one cannot guarantee, by a classical unique continuation argument (in the absence of additional conditions) that ϕ vanishes beyond a small neighborhood of the horizon.

In order to overcome this difficulty we exploit the geometric nature of our problem and make use of the invariance of $\mathcal{S}$ with respect to $\mathbf{T}$. Thus the tensor $\mathcal{S}$ satisfies, in addition to (3.2), the identity

$$\mathcal{L}_{\mathbf{T}}\mathcal{S} = 0. \tag{3.3}$$

Observe that (3.3) can, in principle, transform (3.2) into a much simpler elliptic problem, in any domain which lies strictly outside the ergosphere (where $\mathbf{T}$ is strictly time-like). Unfortunately this possible strategy is not available to us since, as we have remarked above, we cannot hope to extend the vanishing of $\mathcal{S}$, by a simple analogue of Hörmander's pseudo-convexity condition, beyond the first trapped null geodesics.

Our solution is to extend Hörmander's classical pseudo-convexity condition (3.1) to one which takes into account both (3.2) and (3.3). These considerations lead to the following qualitative, $\mathbf{T}$-conditional, pseudo-convexity condition,

$$\begin{aligned}&\mathbf{T}(h) = 0;\\ &\mathbf{D}^2 h(X, X) < 0 \quad \text{if } \mathbf{g}(X, X) = \mathbf{g}(X, \mathbf{D}h) = \mathbf{g}(\mathbf{T}, X) = 0.\end{aligned} \tag{3.4}$$

In a first approximation one can show that this condition can be verified in all Kerr spaces $a \in [0, m)$, for the simple function $h = r$ (see [19]), where r is one of the Boyer–Lindquist coordinates. Thus (3.4) is a good substitute for the more general condition (3.1). The fact that the two geometric identities (3.2) and (3.3) cooperate exactly in the right way, via (3.4), thus allowing us to compensate for both the failure of condition (3.1) as well as the failure of the vector field $\mathbf{T}$ to be time-like in the ergoregion, seems to us to be a very remarkable property of the Kerr spaces. In the next subsection we give a quantitative version of the condition and derive a Carleman estimate of sufficient generality to cover all our needs.

3.2. A Carleman estimate of sufficient generality. Unique continuation properties are often proved using Carleman inequalities. In this subsection we prove a sufficiently general Carleman inequality, Proposition 3.3, under a quantitative conditional pseudo-convexity assumption. This general Carleman inequality is used in Sect. 6 to show that $\mathcal{S}$ vanishes in a small neighborhood of the bifurcate sphere S_0 in $\overline{\mathbf{E}}$, and then in Sect. 8 to prove that $\mathcal{S}$ vanishes in the entire exterior domain. The two applications are genuinely different, since, in particular, the horizon is a bifurcate surface which is not smooth and the weights needed in this case have to be "singular" in an appropriate sense. In order to be able to cover both applications and prove unique continuation in a quantitative sense, which is important especially in Sect. 8, we work with a more technical notion of conditional pseudo-convexity than (3.4), see Definition 3.1 below.

Assume, as in the previous section, that $x_0 \in \Sigma_0 \cap \overline{\mathbf{E}}$ and $\Phi^{x_0} : B_1 \to B_1(x_0)$ is the corresponding coordinate chart. For simplicity of notation, let $B_r = B_r(x_0)$. For any smooth function $\phi : B \to \mathbb{C}$, where $B \subseteq B_1$ is an open set, and $j = 0, 1, \dots$ let

$$|D^j\phi(x)| = \sum_{\alpha_1,\dots,\alpha_j=1}^{4} |\partial_{\alpha_1} \dots \partial_{\alpha_j}\phi(x)|.$$

Assume that $V = V^\alpha \partial_\alpha$ is a vector-field on B_1 with the property that

$$\sup_{x\in B_1} \sum_{j=0}^{4}\sum_{\beta=1}^{4} |D^j V^\beta| \le A_0. \tag{3.5}$$

In our applications, $V = 0$ or $V = \mathbf{T}$.

Definition 3.1. *A family of weights $h_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}_+$, $\epsilon \in (0, \epsilon_1)$, $\epsilon_1 \le A_0^{-1}$, will be called V-conditional pseudo-convex if for any $\epsilon \in (0, \epsilon_1)$*

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x\in B_{\epsilon^{10}}} \sum_{j=1}^{4} \epsilon^j \left|D^j h_\epsilon(x)\right| \le \epsilon/\epsilon_1, \quad |V(h_\epsilon)(x_0)| \le \epsilon^{10}, \tag{3.6}$$

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) \ge \epsilon_1^2, \tag{3.7}$$

and there is $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ such that for all vectors $X = X^\alpha \partial_\alpha \in \mathbf{T}_{x_0}(\mathbf{M})$

$$\begin{aligned} &\epsilon_1^2[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2] \\ &\le X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}\left(\left|X^\alpha V_\alpha(x_0)\right|^2 + \left|X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)\right|^2\right). \end{aligned} \tag{3.8}$$

A function $e_\epsilon : B_{\epsilon^{10}} \to \mathbb{R}$ will be called a negligible perturbation if

$$\sup_{x\in B_{\epsilon^{10}}} \left|D^j e_\epsilon(x)\right| \le \epsilon^{10} \quad \textit{for } j = 0, \dots, 4. \tag{3.9}$$

Remark 3.2. One can see that the technical conditions (3.6)–(3.8) are related to the qualitative condition (3.4), at least when $h_\epsilon = h + \epsilon$ for some smooth function h. The assumption $|V(h_\epsilon)(x_0)| \le \epsilon^{10}$ is a quantitative version of $V(h) = 0$. The assumption (3.8) is a quantitative version of the inequality in the second line of (3.4), in view of the large factor ϵ^{-2} on the terms $|X^\alpha V_\alpha(x_0)|^2$ and $|X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2$, and the freedom to choose μ in a large range. The assumption (3.7) is a quantitative version of the condition $\nabla h \ne 0$ (assuming that (3.8) already holds).

It is important that the Carleman estimates we prove are stable under small perturbations of the weight, in order to be able to use them to prove unique continuation. We quantify this stability in (3.9).

We observe that if $\{h_\epsilon\}_{\epsilon\in(0,\epsilon_1)}$ is a V-conditional pseudo-convex family, and e_ϵ is a negligible perturbation for any $\epsilon \in (0, \epsilon_1]$, then

$$h_\epsilon + e_\epsilon \in [\epsilon/2, 2\epsilon] \quad \text{in } B_{\epsilon^{10}}.$$

The pseudo-convexity conditions of Definition 3.1 are probably not as general as possible, but are suitable for our applications both in Sect. 6, with "singular" weights h_ϵ, and Sect. 8, with "smooth" weights h_ϵ. We also note that it is important to our goal to prove a global result (see Sect. 8), to be able to track quantitatively the size of the support of the functions for which Carleman estimates can be applied; in our notation, this size depends only on the parameter ϵ_1 in Definition 3.1.

Proposition 3.3. *Assume x_0, V are as above, $\epsilon_1 \le A_0^{-1}$, $\{h_\epsilon\}_{\epsilon\in(0,\epsilon_1)}$ is a V-conditional pseudo-convex family, and e_ϵ is a negligible perturbation for any $\epsilon \in (0, \epsilon_1]$. Then there is $\epsilon \in (0, \epsilon_1)$ sufficiently small and $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \ge \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}})$*

$$\begin{aligned}&\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|D^1\phi|\|_{L^2}\\ &\le \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}\phi\big\|_{L^2} + \epsilon^{-6}\|e^{-\lambda f_\epsilon}V(\phi)\|_{L^2},\end{aligned} \tag{3.10}$$

where $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$.

Proof. As mentioned earlier, many Carleman estimates such as (3.10) are known, for the particular case when $V = 0$, in more general settings. The optimal proof, see Chap. 28 of [18], is based on the Fefferman–Phong inequality. Here we provide a self-contained, elementary, proof which, though not optimal, it is perfectly adequate to our needs.

We will use the notation $\widetilde{C}$ to denote various constants in $[1,\infty)$ that may depend only on the constant ϵ_1. We will use the notation $\widetilde{C}_\epsilon$ to denote various constants in $[1,\infty)$ that may depend only on ϵ. We emphasize that these constants do not depend on the (very large) parameter λ or the function ϕ in (3.10). The value of ϵ will be fixed at the end of the proof and depends only on ϵ_1. We divide the proof into several steps.

Step 1. Clearly, we may assume that ϕ is real-valued. Let $\psi = e^{-\lambda f_\epsilon}\phi \in C_0^\infty(B_{\epsilon^{10}})$. In terms of ψ, inequality (3.10) takes the form,

$$\begin{aligned}&\lambda\|\psi\|_{L^2} + \|e^{-\lambda f_\epsilon}|D^1(e^{\lambda f_\epsilon}\psi)|\|_{L^2}\\ &\le \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + \epsilon^{-6}\|e^{-\lambda f_\epsilon}V(e^{\lambda f_\epsilon}\psi)\|_{L^2}.\end{aligned} \tag{3.11}$$

We reduce the proof of (3.11) by a sequence of steps. We claim first that for (3.11) to hold true, it suffices to prove that there exist $\epsilon \ll 1$ and $\widetilde{C}_\epsilon \gg 1$ such that

$$\lambda\|\psi\|_{L^2} + \||D^1\psi|\|_{L^2} \le \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + 8\epsilon^{-4}\|V(\psi)\|_{L^2}, \tag{3.12}$$

for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\psi \in C_0^\infty(B_{\epsilon^{10}})$. Indeed, using (3.6) and (3.9) (thus $|V(h_\epsilon + e_\epsilon)(x)| \leq \widetilde{C}\epsilon^8$ for $x \in B_{\epsilon^{10}}$), the observation $h_\epsilon + e_\epsilon \in [\epsilon/2, 2\epsilon]$ in $B_{\epsilon^{10}}$, and the definition $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$, we have

$$
\begin{aligned}
&e^{-\lambda f_\epsilon}|D^1(e^{\lambda f_\epsilon}\psi)| \leq |D^1\psi| + \widetilde{C}\epsilon^{-1}\lambda|\psi|;\\
&|e^{-\lambda f_\epsilon}V(e^{\lambda f_\epsilon}\psi) - V(\psi)| \leq \widetilde{C}\epsilon^7\lambda|\psi|.
\end{aligned}
$$

Thus, assuming (3.12), we deduce,

$$
\begin{aligned}
&\lambda\|\psi\|_{L^2} + \|e^{-\lambda f_\epsilon}|D^1(e^{\lambda f_\epsilon}\psi)|\|_{L^2} \leq \lambda\|\psi\|_{L^2} + \||D^1\psi|\|_{L^2} + \widetilde{C}\epsilon^{-1}\lambda\|\psi\|_{L^2}\\
&\leq (1+\widetilde{C}\epsilon^{-1})\big(\widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + 8\epsilon^{-4}\|V(\psi)\|_{L^2}\big)\\
&\leq (1+\widetilde{C}\epsilon^{-1})\big[\widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + 8\epsilon^{-4}\|e^{-\lambda f_\epsilon}V(e^{\lambda f_\epsilon}\psi)\|_{L^2}\\
&\qquad\qquad + 8\widetilde{C}\epsilon^3\lambda\|\psi\|_{L^2}\big]\\
&\leq \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + \widetilde{C}\epsilon^{-5}\|e^{-\lambda f_\epsilon}V(e^{\lambda f_\epsilon}\psi)\|_{L^2} + \widetilde{C}\epsilon^2\lambda\|\psi\|_{L^2},
\end{aligned}
$$

and the inequality (3.11) follows for $\epsilon \ll \widetilde{C}^{-1}$.

Step 2. We write

$$
\begin{aligned}
e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi) &= \square_{\mathbf{g}}\psi + 2\lambda\mathbf{D}^\alpha(f_\epsilon)\mathbf{D}_\alpha\psi\\
&\qquad + \lambda^2\mathbf{D}_\alpha(f_\epsilon)\mathbf{D}^\alpha(f_\epsilon)\cdot\psi + \lambda\square_{\mathbf{g}}(f_\epsilon)\cdot\psi,\\
&= L_\epsilon\psi + \lambda\square_{\mathbf{g}}(f_\epsilon)\cdot\psi,
\end{aligned}
\tag{3.13}
$$

with $L_\epsilon := \square_{\mathbf{g}} + 2\lambda\mathbf{D}^\alpha(f_\epsilon)\mathbf{D}_\alpha + \lambda^2\mathbf{D}_\alpha(f_\epsilon)\mathbf{D}^\alpha(f_\epsilon)$, and show that (3.12) follows from,

$$
\lambda\|\psi\|_{L^2} + \||D^1\psi|\|_{L^2} \leq \widetilde{C}_\epsilon\lambda^{-1/2}\|L_\epsilon\psi\|_{L^2} + 4\epsilon^{-4}\|V(\psi)\|_{L^2} \tag{3.14}
$$

for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\psi \in C_0^\infty(B_{\epsilon^{10}})$. Indeed,

$$
\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} \geq \|L_\epsilon\psi\|_{L^2} - \lambda\|\square_{\mathbf{g}}(f_\epsilon)\psi\|_{L^2}.
$$

Observe that, according to (3.6), we have $|\square_{\mathbf{g}}(f_\epsilon)| \leq \widetilde{C}_\epsilon$ on $B_{\epsilon^{10}}$. Thus, if (3.14) holds,

$$
\begin{aligned}
&\lambda\|\psi\|_{L^2} + \||D^1\psi|\|_{L^2}\\
&\leq \widetilde{C}_\epsilon\lambda^{-1/2}\big(\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + \lambda\|\square_{\mathbf{g}}(f_\epsilon)\psi\|_{L^2}\big) + 4\epsilon^{-4}\|V(\psi)\|_{L^2}\\
&\leq \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + \widetilde{C}_\epsilon^2\lambda^{1/2}\|\psi\|_{L^2} + 4\epsilon^{-4}\|V(\psi)\|_{L^2}
\end{aligned}
$$

or,

$$
\begin{aligned}
&\big(\lambda - \widetilde{C}_\epsilon^2\lambda^{1/2}\big)\|\psi\|_{L^2} + \||D^1\psi|\|_{L^2}\\
&\leq \widetilde{C}_\epsilon\lambda^{-1/2}\big\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}(e^{\lambda f_\epsilon}\psi)\big\|_{L^2} + 4\epsilon^{-4}\|V(\psi)\|_{L^2}
\end{aligned}
$$

from which we easily derive (3.12), by redefining the constant $\widetilde{C}_\epsilon$ and taking λ sufficiently large relative to $\widetilde{C}_\epsilon$.

Step 3. We write L_ϵ in the form,

$$\begin{aligned} L_\epsilon &= \square_{\mathbf{g}} + 2\lambda W + \lambda^2 G \\ W &= \mathbf{D}^\alpha(f_\epsilon)\mathbf{D}_\alpha, \quad G = \mathbf{D}_\alpha(f_\epsilon)\mathbf{D}^\alpha(f_\epsilon). \end{aligned} \tag{3.15}$$

We observe that inequality (3.14) follows as a consequence of the following statement: there exist $\epsilon \ll 1$, $\mu_1 \in [-\epsilon^{-3/2}, \epsilon^{-3/2}]$, and $\widetilde{C}_\epsilon \gg 1$ such that

$$\begin{aligned} &2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2 + \int_{B_{\epsilon^{10}}} L_\epsilon\psi \cdot (2\lambda W(\psi) - 2\lambda w\psi)d\mu \\ &\geq \widetilde{C}_\epsilon^{-1}\|\lambda W(\psi) - \lambda w\psi\|_{L^2}^2 + \lambda^3\|\psi\|_{L^2}^2 + \lambda\||D^1\psi|\|_{L^2}^2, \end{aligned} \tag{3.16}$$

for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\psi \in C_0^\infty(B_{\epsilon^{10}})$, where

$$w = \mu_1 - (1/2)\square_{\mathbf{g}} f_\epsilon. \tag{3.17}$$

The reason for choosing w of this form will become clear in Step 6. Assuming that (3.16) holds true and denoting by RHS the right-hand side of that inequality, we have

$$\begin{aligned} \text{RHS} &\leq \int_{B_{\epsilon^{10}}} \widetilde{C}_\epsilon^{1/2} L_\epsilon\psi \cdot \widetilde{C}_\epsilon^{-1/2}(2\lambda W(\psi) - 2\lambda w\psi)d\mu + 2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2 \\ &\leq \widetilde{C}_\epsilon^{-1}\|\lambda W(\psi) - \lambda w\psi\|_{L^2}^2 + \widetilde{C}_\epsilon\|L_\epsilon\psi\|_{L^2}^2 + 2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2. \end{aligned}$$

Hence

$$\lambda^3\|\psi\|_{L^2}^2 + \lambda\||D^1\psi|\|_{L^2}^2 \leq \widetilde{C}_\epsilon\|L_\epsilon\psi\|_{L^2}^2 + 2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2$$

from which (3.14) follows easily.

Step 4. We claim now that inequality (3.16) is a consequence of the inequality

$$\begin{aligned} &2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2 + \int_{B_{\epsilon^{10}}} \left(\square_{\mathbf{g}}\psi + \lambda^2 G\psi\right) \cdot (2\lambda W(\psi) - 2\lambda w\psi)d\mu \\ &\quad + 2\lambda^2\|W(\psi)\|_{L^2}^2 \\ &\geq 2\lambda^3\|\psi\|_{L^2}^2 + 2\lambda\||D^1\psi|\|_{L^2}^2. \end{aligned} \tag{3.18}$$

To prove that (3.18) implies (3.16) we write

$$L_\epsilon\psi = \square_{\mathbf{g}}\psi + \lambda^2 G \cdot \psi + (\lambda W(\psi) - \lambda w\psi) + (\lambda W(\psi) + \lambda w\psi).$$

Thus, assuming (3.18),

$$\begin{aligned}
&2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2 + \int_{B_{\epsilon 10}} L_\epsilon\psi \cdot (2\lambda W(\psi) - 2\lambda w\psi)d\mu \\
&= 2\lambda\epsilon^{-8}\|V(\psi)\|_{L^2}^2 + \int_{B_{\epsilon 10}} \big(\Box_{\mathbf{g}}\psi + \lambda^2 G\psi\big)\cdot(2\lambda W(\psi) - 2\lambda w\psi)d\mu \\
&\quad + 2\|\lambda W(\psi) - \lambda w\psi\|_{L^2}^2 + 2\lambda^2\big(\|W(\psi)\|_{L^2}^2 - \|w\psi\|_{L^2}^2\big) \\
&\geq 2\lambda^3\|\psi\|_{L^2}^2 + 2\lambda\||D^1\psi|\|_{L^2}^2 + 2\|\lambda W(\psi) - \lambda w\psi\|_{L^2}^2 - 2\lambda^2\|w\psi\|_{L^2}^2 \\
&\geq 2\|\lambda W(\psi) - \lambda w\psi\|_{L^2}^2 + \lambda^3\|\psi\|_{L^2}^2 + 2\lambda\||D^1\psi|\|_{L^2}^2,
\end{aligned}$$

if $\widetilde{C}_\epsilon$ is sufficiently large and $\lambda \geq \widetilde{C}_\epsilon$, which gives (3.16). In the last inequality we use the bound $|w| \leq \widetilde{C}\epsilon^{-2}$ (see (3.17)) thus $2\lambda^3\|\psi\|_{L^2}^2 - 2\lambda^2\|w\psi\|_{L^2}^2 \geq \lambda^3\|\psi\|_{L^2}^2$ for λ sufficiently large.

Step 5. Let $Q_{\alpha\beta}$ denote the enery-momentum tensor of $\Box_{\mathbf{g}}$, i.e.

$$Q_{\alpha\beta} = \mathbf{D}_\alpha\psi\mathbf{D}_\beta\psi - \frac{1}{2}g_{\alpha\beta}\big(\mathbf{D}^\mu\psi\mathbf{D}_\mu\psi\big).$$

Direct computations show that

$$\begin{aligned}
\Box_{\mathbf{g}}\psi\cdot(2W(\psi) - 2w\psi) &= \mathbf{D}^\alpha\big(2W^\beta Q_{\alpha\beta} - 2w\psi\cdot\mathbf{D}_\alpha\psi + \mathbf{D}_\alpha w\cdot\psi^2\big) \\
&\quad - 2\mathbf{D}^\alpha W^\beta\cdot Q_{\alpha\beta} + 2w\mathbf{D}^\alpha\psi\cdot\mathbf{D}_\alpha\psi - \Box_{\mathbf{g}}w\cdot\psi^2,
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
G\psi\cdot(2W(\psi) - 2w\psi) &= \mathbf{D}^\alpha\big(\psi^2 G\cdot W_\alpha\big) \\
&\quad - \psi^2\big(2wG + W(G) + G\cdot\mathbf{D}^\alpha W_\alpha\big).
\end{aligned} \tag{3.20}$$

Since $\psi \in C_0^\infty(B_{\epsilon 10})$ we integrate by parts to conclude that

$$\begin{aligned}
&\int_{B_{\epsilon 10}} \big(\Box_{\mathbf{g}}\psi + \lambda^2 G\cdot\psi\big)\cdot(2W(\psi) - 2w\psi)d\mu \\
&= \int_{B_{\epsilon 10}} 2w\mathbf{D}^\alpha\psi\cdot\mathbf{D}_\alpha\psi - 2\mathbf{D}^\alpha W^\beta\cdot Q_{\alpha\beta}d\mu \\
&\quad + \lambda^2\int_{B_{\epsilon 10}} \psi^2\big(-2wG - W(G) - G\cdot\mathbf{D}^\alpha W_\alpha - \lambda^{-2}\Box_{\mathbf{g}}w\big)d\mu.
\end{aligned}$$

Thus, after dividing by λ, for (3.18) it suffices to prove that the pointwise bounds

$$|D^1\psi|^2 \leq \epsilon^{-8}|V(\psi)|^2 + \lambda|W(\psi)|^2 + \big(w\mathbf{D}^\alpha\psi\cdot\mathbf{D}_\alpha\psi - \mathbf{D}^\alpha W^\beta\cdot Q_{\alpha\beta}\big), \tag{3.21}$$

and

$$2 \leq -2wG - W(G) - G \cdot \mathbf{D}^\alpha W_\alpha - \lambda^{-2}\square_{\mathbf{g}} w, \tag{3.22}$$

hold on $B_{\epsilon^{10}}$.

Step 6. Recall that $w = \mu_1 - (1/2)\square_{\mathbf{g}} f_\epsilon$, $W^\alpha = \mathbf{D}^\alpha(f_\epsilon)$ and $G = \mathbf{D}_\alpha(f_\epsilon)\mathbf{D}^\alpha(f_\epsilon)$. Observe that

$$\begin{aligned} &w\mathbf{D}^\alpha\psi \cdot \mathbf{D}_\alpha\psi - \mathbf{D}^\alpha W^\beta \cdot Q_{\alpha\beta} \\ &= (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)[(w + (1/2)\square_{\mathbf{g}} f_\epsilon)\mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha\mathbf{D}_\beta f_\epsilon] \end{aligned}$$

and

$$-2wG - W(G) - G \cdot \mathbf{D}^\alpha W_\alpha = -G(2w + \square_{\mathbf{g}} f_\epsilon) - 2\mathbf{D}^\alpha f_\epsilon \mathbf{D}^\beta f_\epsilon \cdot \mathbf{D}_\alpha\mathbf{D}_\beta f_\epsilon.$$

Thus (3.21) and (3.22) are equivalent to the pointwise inequalities

$$\begin{aligned} |D^1\psi|^2 \leq \epsilon^{-8}|V(\psi)|^2 &+ \lambda\big|\mathbf{D}_\alpha f_\epsilon \cdot \mathbf{D}^\alpha\psi\big|^2 \\ &+ (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)(\mu_1\mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha\mathbf{D}_\beta f_\epsilon), \end{aligned} \tag{3.23}$$

and

$$1 \leq -\mu_1 G - \mathbf{D}^\alpha f_\epsilon \mathbf{D}^\beta f_\epsilon \cdot \mathbf{D}_\alpha\mathbf{D}_\beta f_\epsilon + (1/4)\lambda^{-2}\square_{\mathbf{g}}^2(f_\epsilon) \tag{3.24}$$

on $B_{\epsilon^{10}}$, for some $\epsilon \ll 1$ and λ sufficiently large.

Let $\widetilde{h}_\epsilon = h_\epsilon + e_\epsilon$ and $\widetilde{H}_\epsilon = \mathbf{D}^\alpha\widetilde{h}_\epsilon\mathbf{D}_\alpha\widetilde{h}_\epsilon$. We use now the definition $f_\epsilon = \ln\widetilde{h}_\epsilon$. Since $\widetilde{h}_\epsilon \in [\epsilon/2, 2\epsilon]$, for (3.23) and (3.24) it suffices to prove that there are constants $\epsilon \ll 1$ and $\mu_1 \in [-\epsilon^{-3/2}, \epsilon^{-3/2}]$ such that the pointwise bounds

$$\begin{aligned} |D^1\psi|^2 \leq \epsilon^{-8}|V(\psi)|^2 &+ \epsilon^{-8}\big|\mathbf{D}_\alpha\widetilde{h}_\epsilon \cdot \mathbf{D}^\alpha\psi\big|^2 \\ &+ (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)\big(\mu_1\mathbf{g}_{\alpha\beta} - \widetilde{h}_\epsilon^{-1}\mathbf{D}_\alpha\mathbf{D}_\beta\widetilde{h}_\epsilon\big), \end{aligned} \tag{3.25}$$

and

$$2 \leq \widetilde{h}_\epsilon^{-4}\widetilde{H}_\epsilon^2 - \widetilde{h}_\epsilon^{-3}\mathbf{D}^\alpha\widetilde{h}_\epsilon\mathbf{D}^\beta\widetilde{h}_\epsilon\mathbf{D}_\alpha\mathbf{D}_\beta\widetilde{h}_\epsilon - \widetilde{h}_\epsilon^{-2}\mu_1\widetilde{H}_\epsilon \tag{3.26}$$

hold on $B_{\epsilon^{10}}$ for any $\psi \in C_0^\infty(B_{\epsilon^{10}})$. Indeed, the bound (3.23) follows from (3.25) if $\lambda \geq 2\epsilon^{-7}$. The bound (3.24) follows from (3.26) if $|\lambda^{-2}\square_{\mathbf{g}}^2(f_\epsilon)| \leq 1$, which holds true if $\lambda \geq \widetilde{C}\epsilon^{-2}$.

Step 7. We prove now that the bound (3.26) holds for any $\mu_1 \in [-\epsilon^{-3/2}, \epsilon^{-3/2}]$. We start from the assumption (3.7)

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon\mathbf{D}_\beta h_\epsilon - \epsilon\mathbf{D}_\alpha\mathbf{D}_\beta h_\epsilon)(x_0) \geq \epsilon_1^2.$$

For $x \in B_{\epsilon^{10}}$ let

$$K(x) = \mathbf{D}^\alpha h_\epsilon(x)\mathbf{D}^\beta h_\epsilon(x)(\mathbf{D}_\alpha h_\epsilon\mathbf{D}_\beta h_\epsilon - h_\epsilon \cdot \mathbf{D}_\alpha\mathbf{D}_\beta h_\epsilon)(x).$$

It follows from the second bound in (3.6) that $|D^1K(x)| \le \widetilde{C}\epsilon^{-1}$, thus, since $\epsilon = h_\epsilon(x_0)$, $K(x) \ge \epsilon_1^2/2$ for any $x \in B_{\epsilon^{10}}$ if ϵ is sufficiently small.

Let

$$\widetilde{K}(x) = \mathbf{D}^\alpha \widetilde{h}_\epsilon(x)\mathbf{D}^\beta \widetilde{h}_\epsilon(x)(\mathbf{D}_\alpha \widetilde{h}_\epsilon \mathbf{D}_\beta \widetilde{h}_\epsilon - \widetilde{h}_\epsilon \cdot \mathbf{D}_\alpha \mathbf{D}_\beta \widetilde{h}_\epsilon)(x).$$

It follows from the assumption (3.9) on e_ϵ and the assumption (3.6) that $|\widetilde{K}(x) - K(x)| \le \widetilde{C}\epsilon$, thus $\widetilde{K}(x) \ge \epsilon_1^2/4$ on $B_{\epsilon^{10}}$, provided that ϵ is sufficiently small. By multiplying with $\widetilde{h}_\epsilon^{-4}$ we have

$$\widetilde{h}_\epsilon^{-4}\epsilon_1^2/4 \le \widetilde{h}_\epsilon^{-4}\widetilde{K}(x) = \widetilde{h}_\epsilon^{-4}\widetilde{H}_\epsilon^2 - \widetilde{h}_\epsilon^{-3}\mathbf{D}^\alpha \widetilde{h}_\epsilon \mathbf{D}^\beta \widetilde{h}_\epsilon \cdot \mathbf{D}_\alpha \mathbf{D}_\beta \widetilde{h}_\epsilon$$

on $B_{\epsilon^{10}}$. The bound (3.26) follows for ϵ small enough since $\widetilde{h}_\epsilon(x) \in [\epsilon/2, 2\epsilon]$ on $B_{\epsilon^{10}}$ and $|\widetilde{h}_\epsilon^{-2}\mu_1 \widetilde{H}_\epsilon| \le \widetilde{C}|\mu_1|\epsilon^{-2} \le \widetilde{C}\epsilon^{-7/2}$.

Step 8. We prove now the bound (3.25). We start from the assumption (3.8)

$$\begin{aligned}&\epsilon_1^2[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2]\\ &\le X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}\big(\big|X^\alpha V_\alpha(x_0)\big|^2 + \big|X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)\big|^2\big),\end{aligned} \tag{3.27}$$

for some $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$ and all vectors $X = X^\alpha \partial_\alpha \in \mathbf{T}_{x_0}(\mathbf{M})$. Let

$$K_{\alpha\beta} = \mu\epsilon^{-1}h_\epsilon \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon + \epsilon^{-2}V_\alpha V_\beta + \epsilon^{-2}\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon.$$

We work in the local frame $\partial_1, \partial_2, \partial_3, \partial_4$. In view of (3.6),

$$\big|D^1 K_{\alpha\beta}(x)\big| \le \widetilde{C}\epsilon^{-3}$$

for any $\alpha, \beta = 1, 2, 3, 4$ and $x \in B_{\epsilon^{10}}$. It follows from (3.27) and $\epsilon^{-1}h_\epsilon(x_0) = 1$ that

$$\sum_{\alpha,\beta=1}^{4} X^\alpha X^\beta K_{\alpha\beta}(x) \ge \big(\epsilon_1^2/2\big)[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2] \tag{3.28}$$

for any $x \in B_{\epsilon^{10}}$ and $(X^1, X^2, X^3, X^4) \in \mathbb{R}^4$, provided that ϵ is sufficiently small. Let

$$\widetilde{K}_{\alpha\beta} = \mu\epsilon^{-1}\widetilde{h}_\epsilon \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta \widetilde{h}_\epsilon + \epsilon^{-2}V_\alpha V_\beta + \epsilon^{-2}\mathbf{D}_\alpha \widetilde{h}_\epsilon \mathbf{D}_\beta \widetilde{h}_\epsilon,$$

and observe that, in view of (3.9) and (3.6), $|\widetilde{K}_{\alpha\beta}(x) - K_{\alpha\beta}(x)| \le \widetilde{C}\epsilon^5$ for any $\alpha, \beta = 1, 2, 3, 4$ and $x \in B_{\epsilon^{10}}$. Thus, using (3.28), if ϵ is sufficiently small then

$$\sum_{\alpha,\beta=1}^{4} X^\alpha X^\beta \widetilde{K}_{\alpha\beta}(x) \ge \big(\epsilon_1^2/4\big)[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2]$$

for any $x \in B_{\epsilon^{10}}$ and $(X^1, X^2, X^3, X^4) \in \mathbb{R}^4$. We multiply this by $\widetilde{h}_\epsilon^{-1} \in [\epsilon^{-1}/2, 2\epsilon^{-1}]$ and use the definition of $\widetilde{K}_{\alpha\beta}$ to conclude that

$$\sum_{\alpha,\beta=1}^{4} X^\alpha X^\beta \big(\mu\epsilon^{-1}\mathbf{g}_{\alpha\beta} - \widetilde{h}_\epsilon^{-1}\mathbf{D}_\alpha \mathbf{D}_\beta \widetilde{h}_\epsilon\big) + 2\epsilon^{-3}\Big|\sum_{\alpha=1}^{4} X^\alpha V_\alpha\Big|^2 + 2\epsilon^{-3}\Big|\sum_{\alpha=1}^{4} X^\alpha \mathbf{D}_\alpha h_\epsilon\Big|^2 \geq \widetilde{h}_\epsilon^{-1}\big(\epsilon_1^2/4\big)[(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2].$$

The bound (3.25) follows for ϵ sufficiently small, with $\mu_1 = \mu\epsilon^{-1} \in [-(\epsilon\epsilon_1)^{-1}, (\epsilon\epsilon_1)^{-1}]$. This completes the proof of the proposition. □

4. The Mars–Simon tensor $\mathcal{S}$

4.1. Preliminaries. Assume $(\mathbf{N}, \mathbf{g})$ is a smooth vacuum Einstein spacetime of dimension 4. Given an antisymmetric 2-form, real or complex valued, $G_{\alpha\beta} = -G_{\beta\alpha}$ we define its Hodge dual,

$${}^*G_{\alpha\beta} = \frac{1}{2}\in_{\alpha\beta}{}^{\mu\nu} G_{\mu\nu}.$$

Observe that ${}^*({}^*G) = -G$. This follows easily from the identity,

$$\in_{\alpha\beta\rho\sigma}\in^{\mu\nu\rho\sigma} = -2\delta_\alpha^\mu \wedge \delta_\beta^\nu = -2\big(\delta_\alpha^\mu\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\mu\big).$$

Given 2 such forms F, G we have the identity

$$F_{\mu\sigma}G_\nu{}^\sigma - ({}^*F)_{\nu\sigma}({}^*G)_\mu{}^\sigma = \frac{1}{2}\mathbf{g}_{\mu\nu}F_{\alpha\beta}G^{\alpha\beta} \tag{4.1}$$

which follows easily from the identity

$$\begin{aligned}\in^{\alpha_2\alpha_3\alpha_4\alpha_1}\in_{\beta_2\beta_3\beta_4\alpha_1} &= -\delta_{\beta_2}^{\alpha_2}\wedge\delta_{\beta_3}^{\alpha_3}\wedge\delta_{\beta_4}^{\alpha_4}\\ &= \delta_{\beta_4}^{\alpha_2}\delta_{\beta_3}^{\alpha_3}\delta_{\beta_2}^{\alpha_4} + \delta_{\beta_2}^{\alpha_2}\delta_{\beta_4}^{\alpha_3}\delta_{\beta_3}^{\alpha_4} + \delta_{\beta_3}^{\alpha_2}\delta_{\beta_2}^{\alpha_3}\delta_{\beta_4}^{\alpha_4}\\ &\quad - \delta_{\beta_2}^{\alpha_2}\delta_{\beta_3}^{\alpha_3}\delta_{\beta_4}^{\alpha_4} - \delta_{\beta_3}^{\alpha_2}\delta_{\beta_4}^{\alpha_3}\delta_{\beta_2}^{\alpha_4} - \delta_{\beta_4}^{\alpha_2}\delta_{\beta_2}^{\alpha_3}\delta_{\beta_3}^{\alpha_4}.\end{aligned}$$

An antisymmetric 2-form $\mathcal{F}$ is called self-dual if,

$${}^*\mathcal{F} = -i\mathcal{F}.$$

It follows easily form (4.1) that if $\mathcal{F}, \mathcal{G}$ are two self-dual 2-forms then

$$\mathcal{F}_{\mu\sigma}\mathcal{G}_\nu{}^\sigma + \mathcal{F}_{\nu\sigma}\mathcal{G}_\mu{}^\sigma = \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{F}_{\alpha\beta}\mathcal{G}^{\alpha\beta}. \tag{4.2}$$

We also have, for any self-dual $\mathcal{F}$,

$$\mathcal{F}_{\mu\sigma}(\Re\mathcal{F})_{\nu}{}^{\sigma} = \mathcal{F}_{\nu\sigma}(\Re\mathcal{F})_{\mu}{}^{\sigma} \tag{4.3}$$

where $\Re\mathcal{F}$ denotes the real part of $\mathcal{F}$.

A tensor $W \in \mathbf{T}_4^0(\mathbf{N})$ will be called partially antisymmetric if

$$W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu}. \tag{4.4}$$

Given such a tensor-field we define its Hodge dual

$${}^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\in_{\gamma\delta}{}^{\rho\sigma} W_{\alpha\beta\rho\sigma}.$$

As before, ${}^*({}^*W) = -W$ for any partially antisymmetric tensor W. A complex partially antisymmetric tensor $\mathcal{U}$ of rank 4 is called self-dual if ${}^*\mathcal{U} = (-i)\mathcal{U}$. The following extension of identity (4.2) holds for such tensors,

$$\mathcal{F}_{\mu}{}^{\sigma}\mathcal{U}_{\alpha\beta\nu\sigma} + \mathcal{F}_{\nu}{}^{\sigma}\mathcal{U}_{\alpha\beta\mu\sigma} = \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{F}^{\gamma\delta}\mathcal{U}_{\alpha\beta\gamma\delta}. \tag{4.5}$$

A partially antisymmetric tensor of rank 4 is called a Weyl field if

$$\begin{cases} W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta}; \\ W_{\alpha[\beta\mu\nu]} = W_{\alpha\beta\mu\nu} + W_{\alpha\mu\nu\beta} + W_{\alpha\nu\beta\mu} = 0; \\ \mathbf{g}^{\beta\nu} W_{\alpha\beta\mu\nu} = 0. \end{cases} \tag{4.6}$$

It is well-known that if W is a Weyl field then ${}^*W$ is also a Weyl field. In particular

$${}^*W_{\alpha\beta\mu\nu} = {}^*W_{\mu\nu\alpha\beta} = \frac{1}{2}\in_{\alpha\beta}{}^{\rho\sigma} W_{\mu\nu\rho\sigma}. \tag{4.7}$$

The Riemann curvature tensor R of an Einstein vacuum spacetime provides an example of a Weyl field. Moreover R verifies the Bianchi identities,

$$\mathbf{D}_{[\sigma} R_{\gamma\delta]\alpha\beta} = 0.$$

In this paper we will have to consider Weyl fields W which verify equations of the form

$$\mathbf{D}^{\alpha} W_{\alpha\beta\gamma\delta} = J_{\beta\gamma\delta} \tag{4.8}$$

for some Weyl current $J \in \mathbf{T}_3^0(\mathbf{N})$. It follows from (4.8) that

$$\mathbf{D}^{\alpha}\, {}^*W_{\alpha\beta\gamma\delta} = {}^*J_{\beta\gamma\delta} = \frac{1}{2}\in_{\gamma\delta}{}^{\rho\sigma} J_{\beta\rho\sigma}. \tag{4.9}$$

The following proposition follows immediately from definitions and (4.7).

Proposition 4.1. *If W is a Weyl field and* (4.8) *is satisfied then*

$$\mathbf{D}_{[\sigma} W_{\gamma\delta]\alpha\beta} = \in_{\mu\sigma\gamma\delta}{}^*J^{\mu}{}_{\alpha\beta}. \tag{4.10}$$

4.2. Killing vector-fields and the Ernst potential. We assume now that $\mathbf{T}$ is a Killing vector-field on $\mathbf{N}$, i.e.

$$\mathbf{D}_\alpha \mathbf{T}_\beta + \mathbf{D}_\beta \mathbf{T}_\alpha = 0. \tag{4.11}$$

We define the 2-form,

$$F_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{T}_\beta$$

and recall that F verifies the Ricci identity

$$\mathbf{D}_\mu F_{\alpha\beta} = \mathbf{T}^\nu R_{\nu\mu\alpha\beta}, \tag{4.12}$$

with R the curvature tensor of the spacetime. In view of the first Bianchi identity for R we infer that,

$$\mathbf{D}_{[\mu} F_{\alpha\beta]} = \mathbf{D}_\mu F_{\alpha\beta} + \mathbf{D}_\alpha F_{\beta\mu} + \mathbf{D}_\beta F_{\mu\alpha} = 0. \tag{4.13}$$

Also, since we are in an Einstein vacuum spacetime,

$$\mathbf{D}^\beta F_{\alpha\beta} = 0. \tag{4.14}$$

We now define the complex valued 2-form,

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i\,{}^*F_{\alpha\beta}. \tag{4.15}$$

Clearly, $\mathcal{F}$ is self-dual solution of the Maxwell equations, i.e. $\mathcal{F}^* = (-i)\mathcal{F}$ and

$$\mathbf{D}_{[\mu} \mathcal{F}_{\alpha\beta]} = 0, \quad \mathbf{D}^\beta \mathcal{F}_{\alpha\beta} = 0. \tag{4.16}$$

We define also the Ernst 1-form associated to the Killing vector-field $\mathbf{T}$,

$$\sigma_\mu = 2\mathbf{T}^\alpha \mathcal{F}_{\alpha\mu} = \mathbf{D}_\mu\big(-\mathbf{T}^\alpha \mathbf{T}_\alpha\big) - i\in_{\mu\beta\gamma\delta} \mathbf{T}^\beta \mathbf{D}^\gamma \mathbf{T}^\delta. \tag{4.17}$$

It is easy to check (see, for example, [25, Sect. 3]) that

$$\begin{cases} \mathbf{D}_\mu \sigma_\nu - \mathbf{D}_\nu \sigma_\mu = 0; \\ \mathbf{D}^\mu \sigma_\mu = -\mathcal{F}^2; \\ \sigma_\mu \sigma^\mu = \mathbf{g}(\mathbf{T}, \mathbf{T})\mathcal{F}^2. \end{cases} \tag{4.18}$$

Since $d(\sigma_\mu dx^\mu) = 0$ and the set $\widetilde{\mathbf{M}}$ is simply connected we infer that there exists a function $\sigma : \widetilde{\mathbf{M}} \to \mathbb{C}$, called the Ernst potential, such that $\sigma_\mu = \mathbf{D}_\mu \sigma$, $\sigma \to 1$ at infinity along Σ_0, and $\Re\sigma = -\mathbf{T}^\alpha \mathbf{T}_\alpha$.

4.3. The Mars–Simon tensor. In the rest of this section we assume that $\mathbf{N} \subseteq \widetilde{\mathbf{M}}$ is an open set with the property that

$$1 - \sigma \neq 0 \quad \text{in } \mathbf{N}. \tag{4.19}$$

We define the complex-valued self-dual Weyl tensor

$$\mathcal{R}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} + \frac{i}{2}\in_{\mu\nu}{}^{\rho\sigma} R_{\alpha\beta\rho\sigma} = R_{\alpha\beta\mu\nu} + i\,{}^*R_{\alpha\beta\mu\nu}. \tag{4.20}$$

We define the tensor $\mathcal{I} \in \mathbf{T}^0_4(\mathbf{N})$,

$$\mathcal{I}_{\alpha\beta\mu\nu} = (\mathbf{g}_{\alpha\mu}\mathbf{g}_{\beta\nu} - \mathbf{g}_{\alpha\nu}\mathbf{g}_{\beta\mu} + i\in_{\alpha\beta\mu\nu})/4. \tag{4.21}$$

Clearly,

$$\mathcal{I}_{\alpha\beta\mu\nu} = -\mathcal{I}_{\beta\alpha\mu\nu} = -\mathcal{I}_{\alpha\beta\nu\mu} = \mathcal{I}_{\mu\nu\alpha\beta}. \tag{4.22}$$

On the other hand,

$$\mathcal{I}_{\alpha[\beta\gamma\delta]} = \mathcal{I}_{\alpha\beta\gamma\delta} + \mathcal{I}_{\alpha\gamma\delta\beta} + \mathcal{I}_{\alpha\delta\beta\gamma} = \frac{3i}{4}\in_{\alpha\beta\gamma\delta}. \tag{4.23}$$

Using the definition (4.21) we derive

$${}^*\mathcal{I}_{\alpha\beta\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\sigma}\mathcal{I}_{\alpha\beta\rho\sigma} = (-i)\mathcal{I}_{\alpha\beta\mu\nu}. \tag{4.24}$$

Thus $\mathcal{I}$ is a self-dual partially antisymmetric tensor. We can therefore apply (4.5) and (4.22) to derive

$$\mathcal{F}_\mu{}^\sigma \mathcal{I}_{\nu\sigma\alpha\beta} + \mathcal{F}_\nu{}^\sigma \mathcal{I}_{\mu\sigma\alpha\beta} = \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{F}^{\gamma\delta}\mathcal{I}_{\gamma\delta\alpha\beta}. \tag{4.25}$$

We observe also that

$$\mathcal{F}^{\mu\nu}\mathcal{I}_{\alpha\beta\mu\nu} = \mathcal{F}_{\alpha\beta}. \tag{4.26}$$

Following [24], we define the tensor-field $\mathcal{Q} \in \mathbf{T}^0_4(\mathbf{N})$,

$$\mathcal{Q}_{\alpha\beta\mu\nu} = (1-\sigma)^{-1}\Big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}\Big). \tag{4.27}$$

We show now that $\mathcal{Q}$ is a self-dual Weyl field on $\mathbf{N}$.

Proposition 4.2. *The tensor-field $\mathcal{Q}$ is a self-dual Weyl field, i.e.*

$$\begin{cases} \mathcal{Q}_{\alpha\beta\mu\nu} = -\mathcal{Q}_{\beta\alpha\mu\nu} = -\mathcal{Q}_{\alpha\beta\nu\mu} = \mathcal{Q}_{\mu\nu\alpha\beta}; \\ \mathcal{Q}_{\alpha\beta\mu\nu} + \mathcal{Q}_{\alpha\mu\nu\beta} + \mathcal{Q}_{\alpha\nu\beta\mu} = 0; \\ \mathbf{g}^{\beta\nu}\mathcal{Q}_{\alpha\beta\mu\nu} = 0, \end{cases}$$

and

$$\frac{1}{2}\in_{\mu\nu\rho\sigma}\mathcal{Q}_{\alpha\beta}{}^{\rho\sigma} = (-i)\mathcal{Q}_{\alpha\beta\mu\nu}.$$

Proof. The identities

$$\mathcal{Q}_{\alpha\beta\mu\nu} = -\mathcal{Q}_{\beta\alpha\mu\nu} = -\mathcal{Q}_{\alpha\beta\nu\mu} = \mathcal{Q}_{\mu\nu\alpha\beta}$$

follow immediately from the definition. To prove

$$\mathcal{Q}_{\alpha[\beta\mu\nu]} = \mathcal{Q}_{\alpha\beta\mu\nu} + \mathcal{Q}_{\alpha\mu\nu\beta} + \mathcal{Q}_{\alpha\nu\beta\mu} = 0$$

it suffices to check, in view of the identity (4.23),

$$\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} + \mathcal{F}_{\alpha\mu}\mathcal{F}_{\nu\beta} + \mathcal{F}_{\alpha\nu}\mathcal{F}_{\beta\mu} = \frac{i}{4}\in_{\alpha\beta\mu\nu} \cdot \mathcal{F}^2. \tag{4.28}$$

Since $\mathcal{F}$ is a 2-form, the left-hand side of (4.28) is a 4-form on $\mathbf{N}$ (which has dimension 4). Thus, for (4.28) it suffices to check

$$\in^{\alpha\beta\mu\nu}(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} + \mathcal{F}_{\alpha\mu}\mathcal{F}_{\nu\beta} + \mathcal{F}_{\alpha\nu}\mathcal{F}_{\beta\mu}) = -6i\mathcal{F}^2.$$

This follows since the left-hand side of the above equation is equal to $6\mathcal{F}_{\alpha\beta}{}^*\mathcal{F}^{\alpha\beta} = -6i\mathcal{F}^2$.

We compute

$$\mathbf{g}^{\beta\nu}\mathcal{Q}_{\alpha\beta\mu\nu} = (1-\sigma)^{-1}\left(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu}{}^{\beta} - \frac{1}{3}\mathcal{F}^2 \cdot \mathbf{g}^{\beta\nu}\mathcal{I}_{\alpha\beta\mu\nu}\right) = 0.$$

Also

$$\frac{1}{2}\in_{\mu\nu\rho\sigma}\mathcal{Q}_{\alpha\beta}{}^{\rho\sigma} = (1-\sigma)^{-1}\left(\mathcal{F}_{\alpha\beta}{}^*\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^{2\,*}\mathcal{I}_{\alpha\beta\mu\nu}\right) = (-i)\mathcal{Q}_{\alpha\beta\mu\nu}.$$

This completes the proof of the proposition. □

We define now the Mars–Simon tensor.

Definition 4.3. *We define the self-dual Weyl field* $\mathcal{S}$,

$$\mathcal{S} = \mathcal{R} + 6\mathcal{Q}. \tag{4.29}$$

Remark 4.4. Since $\Re\sigma = -\mathbf{T}^\alpha\mathbf{T}_\alpha \leq 0$ on S_0, it follows from the definition of the constant A_0 in Sect. 2 that $\Re(1-\sigma) \geq 1/2$ in a neighborhood $\mathbf{O}_{\epsilon_2} \subseteq \widetilde{\mathbf{M}}$ of S_0, for some $\epsilon_2 \leq \epsilon_0$ that depends only on A_0. In particular, the tensor $\mathcal{S}$ is well defined in $\mathbf{O}_{\epsilon_2}$.

4.4. A covariant wave equation for $\mathcal{S}$. Our main goal now is to show that $\mathcal{S}$ verifies a covariant wave equation. We first calculate its spacetime divergence $\mathbf{D}^\alpha\mathcal{S}_{\alpha\beta\mu\nu}$. Clearly, it suffices to calculate $\mathbf{D}^\alpha\mathcal{Q}_{\alpha\beta\mu\nu}$. Recalling the definition of the 1-form $\sigma_\alpha = 2\mathbf{T}^\nu \mathcal{F}_{\nu\alpha}$ and using the definition (4.27) we compute

$$\begin{aligned}\mathbf{D}_\rho \mathcal{Q}_{\alpha\beta\mu\nu} &= (1-\sigma)^{-1}\mathbf{D}_\rho \mathcal{F}_{\alpha\beta} \cdot \mathcal{F}_{\mu\nu} + (1-\sigma)^{-1}\mathcal{F}_{\alpha\beta} \cdot \mathbf{D}_\rho \mathcal{F}_{\mu\nu} \\ &\quad - \frac{1}{3}(1-\sigma)^{-1}\mathbf{D}_\rho \mathcal{F}^2 \cdot \mathcal{I}_{\alpha\beta\mu\nu} \\ &\quad + (1-\sigma)^{-2}\sigma_\rho \left(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2 \mathcal{I}_{\alpha\beta\mu\nu} \right).\end{aligned} \tag{4.30}$$

Using (4.12), (4.26), and $\mathcal{R} = \mathcal{S} - 6\mathcal{Q}$, we have

$$\begin{aligned}\mathbf{D}_\rho \mathcal{F}_{\gamma\delta} &= \mathbf{T}^\nu \mathcal{R}_{\nu\rho\gamma\delta} = \mathbf{T}^\nu \mathcal{S}_{\nu\rho\gamma\delta} - 6 \cdot \mathbf{T}^\nu \mathcal{Q}_{\nu\rho\gamma\delta} \\ &= -3(1-\sigma)^{-1}\sigma_\rho \mathcal{F}_{\gamma\delta} + 2(1-\sigma)^{-1}\mathcal{F}^2 \cdot \mathbf{T}^\nu \mathcal{I}_{\nu\rho\gamma\delta} + \mathbf{T}^\nu \mathcal{S}_{\nu\rho\gamma\delta}.\end{aligned} \tag{4.31}$$

Thus,

$$\begin{aligned}(1-\sigma)^{-1}\mathcal{F}_{\alpha\beta} \cdot \mathbf{D}_\rho \mathcal{F}_{\mu\nu} &= -3(1-\sigma)^{-2} \cdot \sigma_\rho \mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} \\ &\quad + 2(1-\sigma)^{-2}\mathcal{F}^2 \mathcal{F}_{\alpha\beta}\mathbf{T}^\lambda \mathcal{I}_{\lambda\rho\mu\nu} + \mathcal{J}_1(\mathcal{S})_{\rho\alpha\beta\mu\nu},\end{aligned} \tag{4.32}$$

where

$$\mathcal{J}_1(\mathcal{S})_{\rho\alpha\beta\mu\nu} = (1-\sigma)^{-1} \cdot \mathcal{F}_{\alpha\beta}\mathbf{T}^\lambda S_{\lambda\rho\mu\nu}.$$

Observe that, in view of (4.31) and (4.26)

$$\mathbf{D}_\rho \mathcal{F}^2 = 2\mathbf{D}_\rho \mathcal{F}_{\gamma\delta} \cdot \mathcal{F}^{\gamma\delta} = -4(1-\sigma)^{-1}\mathcal{F}^2 \sigma_\rho + 2\mathbf{T}^\nu \mathcal{S}_{\nu\rho\gamma\delta}\mathcal{F}^{\gamma\delta}. \tag{4.33}$$

Thus

$$-\frac{1}{3}(1-\sigma)^{-1}\mathbf{D}_\rho \mathcal{F}^2 \cdot \mathcal{I}_{\alpha\beta\mu\nu} = \frac{4}{3}(1-\sigma)^{-2}\mathcal{F}^2 \cdot \sigma_\rho \mathcal{I}_{\alpha\beta\mu\nu} + \mathcal{J}_2(\mathcal{S})_{\rho\alpha\beta\mu\nu}, \tag{4.34}$$

where,

$$\mathcal{J}_2(\mathcal{S})_{\rho\alpha\beta\mu\nu} = -\frac{2}{3}(1-\sigma)^{-1} \cdot \mathbf{T}^\lambda \mathcal{S}_{\lambda\rho\gamma\delta}\mathcal{F}^{\gamma\delta}\mathcal{I}_{\alpha\beta\mu\nu}.$$

We combine (4.30), (4.32), and (4.34) to write

$$\begin{aligned}\mathbf{D}_\rho \mathcal{Q}_{\alpha\beta\mu\nu} &= (1-\sigma)^{-1}\mathbf{D}_\rho \mathcal{F}_{\alpha\beta} \cdot \mathcal{F}_{\mu\nu} - 2(1-\sigma)^{-2}\sigma_\rho \mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} \\ &\quad + 2(1-\sigma)^{-2}\mathcal{F}^2 \mathcal{F}_{\alpha\beta}\mathbf{T}^\lambda \cdot \mathcal{I}_{\lambda\rho\mu\nu} + (1-\sigma)^{-2}\mathcal{F}^2 \sigma_\rho \mathcal{I}_{\alpha\beta\mu\nu} \\ &\quad + \mathcal{J}_1(\mathcal{S})_{\rho\alpha\beta\mu\nu} + \mathcal{J}_2(\mathcal{S})_{\rho\alpha\beta\mu\nu}.\end{aligned} \tag{4.35}$$

We are now ready to compute the divergence $\mathbf{D}^\sigma \mathcal{Q}_{\beta\sigma\mu\nu}$. Using (4.35) and the Maxwell equations (4.16) we derive

$$\begin{aligned}\mathbf{D}^\beta \mathcal{Q}_{\alpha\beta\mu\nu} &= \mathcal{J}''(\mathcal{S})_{\alpha\mu\nu} - 2(1-\sigma)^{-2}\sigma_\rho \mathcal{F}_\alpha{}^\rho \mathcal{F}_{\mu\nu} \\ &\quad + 2(1-\sigma)^{-2}\mathcal{F}^2 \mathcal{F}_\alpha{}^\rho \mathbf{T}^\lambda \mathcal{I}_{\lambda\rho\mu\nu} + (1-\sigma)^{-2}\mathcal{F}^2\sigma^\beta \mathcal{I}_{\alpha\beta\mu\nu};\\ \mathcal{J}''(\mathcal{S})_{\alpha\mu\nu} &= \mathbf{g}^{\rho\beta}(\mathcal{J}_1(\mathcal{S})_{\rho\alpha\beta\mu\nu} + \mathcal{J}_2(\mathcal{S})_{\rho\alpha\beta\mu\nu}).\end{aligned}$$

Using (4.2) and the definition of σ_ρ we derive,

$$\begin{aligned}-2(1-\sigma)^{-2}\cdot\sigma_\rho \mathcal{F}_\alpha{}^\rho \mathcal{F}_{\mu\nu} &= -2(1-\sigma)^{-2}\cdot 2\mathbf{T}^\lambda \mathcal{F}_{\lambda\rho}\mathcal{F}_\alpha{}^\rho \mathcal{F}_{\mu\nu}\\ &= -2(1-\sigma)^{-2}\cdot 2\mathbf{T}^\lambda\cdot\frac{1}{4}\mathbf{g}_{\lambda\alpha}\mathcal{F}^2\cdot\mathcal{F}_{\mu\nu} \\ &= -(1-\sigma)^{-2}\mathcal{F}^2\mathbf{T}_\alpha\mathcal{F}_{\mu\nu}.\end{aligned} \tag{4.36}$$

Using (4.25), (4.26), and the definitions,

$$\begin{aligned}&2(1-\sigma)^{-2}\mathcal{F}^2\cdot\mathcal{F}_\alpha{}^\rho\mathbf{T}^\lambda\mathcal{I}_{\lambda\rho\mu\nu} + (1-\sigma)^{-2}\mathcal{F}^2\cdot\sigma^\beta\mathcal{I}_{\alpha\beta\mu\nu}\\ &= 2(1-\sigma)^{-2}\mathcal{F}^2\big(\mathcal{F}_\alpha{}^\rho\mathbf{T}^\lambda\mathcal{I}_{\lambda\rho\mu\nu} + \mathbf{T}^\lambda\mathcal{F}_\lambda{}^\rho\mathcal{I}_{\alpha\rho\mu\nu}\big)\\ &= 2(1-\sigma)^{-2}\mathcal{F}^2\mathbf{T}^\lambda\big(\mathcal{F}_\alpha{}^\rho\mathcal{I}_{\lambda\rho\mu\nu} + \mathcal{F}_\lambda{}^\rho\mathcal{I}_{\alpha\rho\mu\nu}\big)\\ &= 2(1-\sigma)^{-2}\mathcal{F}^2\mathbf{T}^\lambda\cdot\frac{1}{2}\mathbf{g}_{\lambda\alpha}\mathcal{F}^{\rho\sigma}\mathcal{I}_{\rho\sigma\mu\nu}\\ &= (1-\sigma)^{-2}\mathcal{F}^2\mathbf{T}_\alpha\mathcal{F}_{\mu\nu}.\end{aligned} \tag{4.37}$$

Thus, using (4.36) and (4.37), we derive,

$$\mathbf{D}^\sigma\mathcal{Q}_{\alpha\sigma\mu\nu} = \mathcal{J}''(\mathcal{S})_{\alpha\mu\nu},$$

with

$$\mathcal{J}''(\mathcal{S})_{\alpha\mu\nu} = (1-\sigma)^{-1}\mathbf{T}^\lambda\mathcal{S}_{\lambda\rho\gamma\delta}\big(\mathcal{F}_\alpha{}^\rho\delta_\mu^\gamma\delta_\nu^\delta - (2/3)\mathcal{F}^{\gamma\delta}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}\big).$$

Since, according to the Bianchi identities, and the Einstein equations, we have $\mathbf{D}^\sigma\mathcal{R}_{\beta\sigma\mu\nu} = 0$ we deduce the following.

Theorem 4.5. *The Mars–Simon tensor $\mathcal{S}$ verifies,*

$$\mathbf{D}^\sigma\mathcal{S}_{\sigma\alpha\mu\nu} = \mathcal{J}(\mathcal{S})_{\alpha\mu\nu}. \tag{4.38}$$

where,

$$\mathcal{J}(\mathcal{S})_{\alpha\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\lambda\mathcal{S}_{\lambda\rho\gamma\delta}\big(\mathcal{F}_\alpha{}^\rho\delta_\mu^\gamma\delta_\nu^\delta - (2/3)\mathcal{F}^{\gamma\delta}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}\big).$$

As a consequence of the theorem we deduce from Proposition 4.1 and the self-duality of $\mathcal{S}$ and $\mathcal{J}$,

$$\mathbf{D}_{[\sigma}\mathcal{S}_{\mu\nu]\alpha\beta} = -i\in_{\rho\sigma\mu\nu}\mathcal{J}^\rho{}_{\alpha\beta}(\mathcal{S}). \tag{4.39}$$

In the following calculations the precise form of $\mathcal{J}(\mathcal{S})$ is not important, we only need to keep track of the fact that it is a multiple of $\mathcal{S}$.

Definition 4.6. *We denote by $\mathcal{M}(\mathcal{S})$ any k-tensor with the property that there is a smooth tensor-field $\mathcal{A}$ such that*

$$\mathcal{M}(\mathcal{S})_{\alpha_1\ldots\alpha_k} = \mathcal{S}_{\beta_1\ldots\beta_4}\mathcal{A}^{\beta_1\ldots\beta_4}{}_{\alpha_1\ldots\alpha_k}. \tag{4.40}$$

Similarly we denote by $\mathcal{M}(\mathcal{S},\mathbf{D}\mathcal{S})$ any k-tensor with the property that there exist smooth tensor-fields $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{M}(\mathcal{S},\mathbf{D}\mathcal{S})_{\alpha_1\ldots\alpha_k} = \mathcal{S}_{\beta_1\ldots\beta_4}\mathcal{A}^{\beta_1\ldots\beta_4}{}_{\alpha_1\ldots\alpha_k} + \mathbf{D}_{\beta_5}\mathcal{S}_{\beta_1\ldots\beta_4}\mathcal{B}^{\beta_1\ldots\beta_5}{}_{\alpha_1\ldots\alpha_k}. \tag{4.41}$$

We state the main result of this section.

Theorem 4.7. *We have*

$$\square_{\mathbf{g}}\mathcal{S} = \mathcal{M}(\mathcal{S},\mathbf{D}\mathcal{S}). \tag{4.42}$$

Proof. The result follows easily from (4.38) and (4.39)

$$\mathbf{D}^\alpha\mathcal{S}_{\alpha\beta\gamma\delta} = \mathcal{J}(\mathcal{S})_{\beta\gamma\delta}$$
$$\mathbf{D}_{[\sigma}\mathcal{S}_{\alpha\beta]\gamma\delta} = -i\in_{\rho\sigma\alpha\beta}\mathcal{J}^\rho{}_{\gamma\delta}(\mathcal{S}).$$

Indeed, differentiating once more the second equation we derive,

$$\mathbf{D}^\sigma(\mathbf{D}_\sigma\mathcal{S}_{\alpha\beta\gamma\delta} + \mathbf{D}_\alpha\mathcal{S}_{\beta\sigma\gamma\delta} + \mathbf{D}_\beta\mathcal{S}_{\sigma\alpha\gamma\delta}) = -i\in_{\rho\sigma\alpha\beta}\mathbf{D}^\sigma\mathcal{J}^\rho{}_{\gamma\delta}(\mathcal{S}).$$

Thus, after commuting covariant derivatives and using the first equation we derive,

$$\square_{\mathbf{g}}\mathcal{S}_{\alpha\beta\gamma\delta} = \mathcal{M}(\mathcal{S},\mathbf{D}\mathcal{S})_{\alpha\beta\gamma\delta}$$

as desired. □

5. Vanishing of $\mathcal{S}$ on the horizon

In this section we prove that the Mars–Simon tensor $\mathcal{S}$ vanishes on $\mathcal{H}^+\cup\mathcal{H}^-$.

Proposition 5.1. *The Mars–Simon tensor $\mathcal{S}$ vanishes along the horizon $\mathcal{H}^+\cup\mathcal{H}^-$.*

The rest of the section is concerned with the proof of Proposition 5.1. Recall, see Remark 4.4, that the tensor $\mathcal{S}$ is well defined on S_0. We will use the notation in the appendix. Assume $\mathcal{N}$ is a null hypersurface (in our case $\mathcal{N}=\mathcal{H}^+$ or $\mathcal{N}=\mathcal{H}^-$) and let $l\in\mathbf{T}(\mathcal{N})$ denote a null vector-field orthogonal to $\mathcal{N}$. The Lie bracket $[X,Y]$ of any two vector-fields X,Y tangent to $\mathcal{N}$ is again tangent to $\mathcal{N}$ and therefore

$$\mathbf{g}(\mathbf{D}_Xl,Y) - \mathbf{g}(\mathbf{D}_Yl,X) = -\mathbf{g}(l,[X,Y]) = 0 \quad \text{and}$$
$$\mathbf{g}(\mathbf{D}_ll,X) = -\mathbf{g}(l,\mathbf{D}_lX) = 0.$$

In particular we infer that along $\mathcal{N}$ ${}^{(h)}\xi$ vanishes identically and ${}^{(h)}\chi$ is symmetric.

Definition 5.2. *Given a null hypersurface $\mathcal{N}$ and l a fixed non-vanishing null vector-field on it we define $\chi(X,Y) = \mathbf{g}(\mathbf{D}_X l, Y)$, $X, Y \in \mathbf{T}(\mathcal{N})$, the null second fundamental form of $\mathcal{N}$. We denote by $\operatorname{tr}\chi$ the trace[7] of χ with respect to the induced metric and by $\hat{\chi}$ the traceless part of χ, i.e. $\hat{\chi} = \chi - \frac{1}{2}\gamma \operatorname{tr}\chi$, with γ the degenerate metric on $\mathcal{N}$ induced by g.*

In view of the definitions (A.13), writing $m = (e_1 + ie_2)/\sqrt{2}$, with e_1, e_2 an arbitrary horizontal orthonormal frame, we deduce that,

$$\theta = (\chi_{11} + \chi_{22})/2 = \operatorname{tr}\chi/2$$
$$\vartheta = (\chi_{11} - \chi_{22})/2 + i\chi_{12}.$$

We now restrict our considerations to that of a non-expanding null hypersurface. In other words we assume that $\theta = \operatorname{tr}\chi/2$ vanishes identically along $\mathcal{N}$. In view of the null structure equation (A.21) and the vanishing of $\xi = {}^{(h)}\xi(m)$, we deduce that $|\vartheta|^2 = 0$ along $\mathcal{N}$ therefore $\vartheta \equiv 0$. Therefore the full null second fundamental form of $\mathcal{N}$ vanishes identically. We now consider the null structure equation (A.19). Since ξ, θ, ϑ vanish we deduce that $\Psi_{(2)}(R)$ must vanish along $\mathcal{N}$. Similarly we deduce that $\Psi_{(1)}(R)$ vanishes along $\mathcal{N}$ from (A.29). Finally, we consider the Bianchi equations with zero source J. From (A.41) we deduce that $D\Psi_{(0)}$ vanishes identically along $\mathcal{N}$. Observe also that $\Psi_{(0)}(R)$ is invariant under general changes of the null pair $(l, \underline{l})$ which keep l orthogonal to $\mathcal{N}$. Indeed $\Psi_{(0)}(R)$ is always invariant under the scale transformations $l' = fl, \underline{l}' = f^{-1}\underline{l}$. On the other hand if we keep l fixed and perform the general transformations $\underline{l}' = \underline{l} + Al + Bm + \overline{Bm}$ we easily find that $\Psi'_{(0)}(R)$ differs from $\Psi_{(0)}(R)$ by a linear combination of $\Psi_{(2)}(R)$ and $\Psi_{(1)}(R)$.

We have thus proved the following.

Proposition 5.3. *Let $(l, \underline{l})$ be a null pair in an open set $\mathbf{N}$ with l orthogonal to a non-expanding null hypersurface in $\mathcal{N} \subset \mathbf{N}$. Then ${}^{(h)}\xi$ and ${}^{(h)}\chi$ vanish identically on $\mathcal{N}$. Moreover the curvature components $\Psi_{(2)}(R)$ and $\Psi_{(1)}(R)$ (or equivalently, $\alpha(R)$, $\beta(R)$) vanish along $\mathcal{N}$ and the invariant $\Psi_{(0)}(R)$ (or equivalently $\rho(R + i\,{}^*R)$) is constant along the null generators.*

We apply this proposition to the surfaces $\mathcal{H}^+$ and $\mathcal{H}^-$ to establish the following facts. Recall that $\mathcal{R} = R + i\,{}^*R$.

(1) The null second fundamental form χ, respectively $\underline{\chi}$, vanishes identically along $\mathcal{H}^+$, respectively $\mathcal{H}^-$.
(2) The null curvature components $\alpha = \alpha(\mathcal{R})$ and $\beta = \beta(\mathcal{R})$ (respectively $\underline{\alpha}(\mathcal{R})$, $\underline{\beta}(\mathcal{R})$), vanish identically along $\mathcal{H}^+$ (respectively $\mathcal{H}^-$).

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[7] The trace is well defined since $\chi(X,l) = \gamma(X,l) = 0$ for all $X \in \mathbf{T}(\mathcal{N})$.

(3) The null curvature component $\rho(\mathcal{R})$ is invariant and constant along the null generators of both $\mathcal{H}^+$ and $\mathcal{H}^-$.
(4) All null curvature components, except $\rho(\mathcal{R})$, vanish along the bifurcate sphere S_0. We also have $\chi = \underline{\chi} = 0$ on S_0.

Consider an adapted null frame $e_1, e_2, e_3 = \underline{l}, e_4 = l$ in $\mathbf{O}$ with l tangent to the null generators of $\mathcal{H}^+$ and $\underline{l}$ tangent to the null generators of $\mathcal{H}^-$. Thus,

$$\mathbf{g}(l,l) = \mathbf{g}(\underline{l},\underline{l}) = 0, \quad \mathbf{g}(l,\underline{l}) = -1, \quad \mathbf{g}(l,e_a) = \mathbf{g}(\underline{l},e_a) = 0,$$
$$\mathbf{g}(e_a,e_b) = \delta_{ab}, \quad a,b = 1,2, \quad \in_{12} = \in(e_1,e_2,e_3,e_4) = 1.$$

We introduce the notation,

$$\alpha_a(\mathcal{F}) = \mathcal{F}(e_a,l), \quad \underline{\alpha}_a(\mathcal{F}) = \mathcal{F}(e_a,\underline{l}), \quad \rho(\mathcal{F}) = \mathcal{F}(\underline{l},l). \tag{5.1}$$

Observe that the null components $\alpha_a(\mathcal{F}), \underline{\alpha}_a(\mathcal{F}), \rho(\mathcal{F})$ completely determine the antisymmetric, self-dual tensor $\mathcal{F}$. Indeed, $-i\mathcal{F}_{34} = ({}^*\mathcal{F})_{34} = \frac{1}{2}\in_{34ab}\mathcal{F}^{ab} = \frac{1}{2}\in_{ab}\mathcal{F}^{ab}$. Hence,

$$\mathcal{F}_{ab} = -i\in_{ab}\rho(\mathcal{F}). \tag{5.2}$$

We claim that $\alpha(\mathcal{F})$ vanishes on $\mathcal{H}^+$ while $\underline{\alpha}(\mathcal{F})$ vanishes on $\mathcal{H}^-$,

$$\alpha(\mathcal{F}) = 0 \quad \text{on } \mathcal{H}^+, \qquad \underline{\alpha}(\mathcal{F}) = 0 \quad \text{on } \mathcal{H}^-. \tag{5.3}$$

Indeed since $g(\mathbf{T},l) = 0$ on $\mathcal{H}^+$ (see the assumption **SBS**) and the null second fundamental form χ vanishes identically on $\mathcal{H}^+$,

$$F(e_a,l) = -g(\mathbf{T},\mathbf{D}_{e_a}l) = -\chi(\mathbf{T},e_a) = 0 \quad \text{on } \mathcal{H}^+.$$

On the other hand,

$${}^*F_{a4} = \frac{1}{2}\in_{a4\mu\nu}F^{\mu\nu} = \in_{a4b3}F^{b3} = -\in_{a4b3}F_{b4} = 0.$$

Hence $\alpha_a(\mathcal{F}) = \mathcal{F}_{a4} = F_{a4} + i\,{}^*F_{a4} = 0$ on $\mathcal{H}^+$. The proof of vanishing of $\underline{\alpha}(\mathcal{F})$ on $\mathcal{H}^-$ is similar. We infer that both $\alpha(\mathcal{F})$ and $\underline{\alpha}(\mathcal{F})$ have to vanish along the bifurcate sphere S_0. We also observe,

$$\mathcal{F}^2 = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = 2\mathcal{F}^{34}\mathcal{F}_{34} + \mathcal{F}^{ab}\mathcal{F}_{ab} = -4\mathcal{F}_{34}^2 = -4\rho(\mathcal{F})^2 \quad \text{on } S_0.$$

Since $\mathcal{F}^2$ does not vanish on S_0 we infer that $\rho(\mathcal{F})$ cannot vanish on S_0.

Consider now the Mars–Simon tensor (4.29). To show that the Weyl tensor $\mathcal{S}$ vanishes along the $\mathcal{H}^+ \cup \mathcal{H}^-$ it suffices to show that all its null components (see appendix) $\alpha(\mathcal{S}), \beta(\mathcal{S}), \rho(\mathcal{S}), \underline{\alpha}(\mathcal{S}), \underline{\beta}(\mathcal{S})$, relative to an arbitrary, adapted, null frame $(e_1, e_2, \underline{l}, l)$, vanish along $\mathcal{H}^+ \cup \mathcal{H}^-$. We first show that

$$\alpha(\mathcal{S}) = \beta(\mathcal{S}) = 0 \quad \text{on } \mathcal{H}^+, \qquad \underline{\alpha}(\mathcal{S}) = \underline{\beta}(\mathcal{S}) = 0 \quad \text{on } \mathcal{H}^-. \tag{5.4}$$

Indeed,

$$\mathcal{I}(l, e_a, l, e_b) = 0, \quad \mathcal{I}(e_a, l, \underline{l}, l) = 0, \quad \mathcal{I}(l, \underline{l}, l, \underline{l}) = -1/4.$$

Therefore along $\mathcal{H}^+$, where $\alpha(\mathcal{F}), \alpha(\mathcal{R}), \beta(\mathcal{R})$ vanish,

$$\alpha(\mathcal{S})_{ab} = \beta(\mathcal{S})_a = 0,$$

using the formula $\mathcal{S} = \mathcal{R} + 6\mathcal{Q}$. Similarly we infer that $\underline{\alpha}(\mathcal{S}) = \underline{\beta}(\mathcal{S}) = 0$ along $\mathcal{H}^-$.

We show now that $\rho(\mathcal{S})$ vanishes on S_0. This is where we need the main technical assumption (1.6) along S_0,

$$(1-\sigma)^4 = -4M^2\mathcal{F}^2.$$

Differentiating it along S_0 we find,

$$0 = \mathbf{D}_a(\mathcal{F}^2(1-\sigma)^{-4}) = (1-\sigma)^{-4}\big(\mathbf{D}_a\mathcal{F}^2 + 4(1-\sigma)^{-1}\sigma_a\big).$$

On the other hand, recalling formula (4.33)

$$D_\alpha\mathcal{F}^2 + 4(1-\sigma)^{-1}\mathcal{F}^2\sigma_\alpha = 2\mathbf{T}^\lambda\mathcal{S}_{\lambda\alpha\gamma\delta}\mathcal{F}^{\gamma\delta}.$$

We deduce that

$$\mathbf{T}^\lambda\mathcal{S}_{\lambda a\gamma\delta}\mathcal{F}^{\gamma\delta} = 0 \quad \text{on } S_0. \tag{5.5}$$

Recall $\mathbf{T}$ is tangent on S_0 and can only vanish at a discrete set of points (see assumption **SBS** in Subsect. 1.2). Therefore, at a point where $\mathbf{g}(\mathbf{T},\mathbf{T})^{1/2}$, does not vanish we can introduce an orthonormal frame e_1, e_2 with $\mathbf{T} = \mathbf{g}(\mathbf{T},\mathbf{T})^{1/2}e_1$.

We now expand the left-hand side of (5.5) using (5.2) while setting the index $a = 2$,

$$\begin{aligned}
0 = \mathbf{T}^\lambda\mathcal{S}_{\lambda 2\gamma\delta}\mathcal{F}^{\gamma\delta} &= 2\mathbf{T}^\lambda\mathcal{S}_{\lambda 234}\mathcal{F}^{34} + \mathbf{T}^\lambda\mathcal{S}_{\lambda 2cd}\mathcal{F}^{cd}\\
&= -2\mathbf{T}^\lambda\mathcal{S}_{\lambda 234}\mathcal{F}_{34} - i\mathbf{T}^\lambda\mathcal{S}_{\lambda 2cd}\in^{cd}\rho(\mathcal{F})\\
&= -\mathbf{g}(\mathbf{T},\mathbf{T})^{1/2}\rho(\mathcal{F})\big(2\mathcal{S}_{1234} + i\mathcal{S}_{12cd}\in^{cd}\big) = 4i\mathbf{g}(\mathbf{T},\mathbf{T})^{1/2}\rho(\mathcal{F})\rho(\mathcal{S}).
\end{aligned}$$

The last equality follows from (see (A.12))

$$\mathcal{S}_{1234} = -i\rho(\mathcal{S}), \quad \mathcal{S}_{12cd} = -\in_{cd}\rho(\mathcal{S}).$$

Therefore, at all points of S_0 where $\mathbf{T}$ does not vanish we infer that $\rho(\mathcal{S}) = 0$ (since $\rho(\mathcal{F})$ cannot vanish on S_0, due to (1.6) and (1.7)). Since the set of such points is dense in S_0 we conclude that $\rho(\mathcal{S})$ vanishes identically on the bifurcate sphere S_0. We have thus proved the following.

Proposition 5.4. *The components $\alpha(\mathcal{S})$, $\beta(\mathcal{S})$ vanish along $\mathcal{H}^+$ while $\underline{\alpha}(\mathcal{S})$, $\underline{\beta}(\mathcal{S})$ vanish along $\mathcal{H}^-$. In addition, if* (1.6) *holds then $\rho(\mathcal{S})$ also vanishes on S_0.*

To show that $\rho(\mathcal{S})$, $\underline{\beta}(\mathcal{S})$, $\underline{\alpha}(\mathcal{S})$ vanish on $\mathcal{H}^+$ we need to use the Bianchi equations (see Theorem 4.5),

$$\mathbf{D}^\sigma \mathcal{S}_{\sigma\alpha\mu\nu} = \mathcal{J}(\mathcal{S})_{\alpha\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda\rho\gamma\delta}\left(\mathcal{F}_\alpha{}^\rho \delta_\mu^\gamma \delta_\nu^\delta - \frac{2}{3}\mathcal{F}^{\gamma\delta}\mathcal{I}_\alpha{}^\rho{}_{\mu\nu}\right). \tag{5.6}$$

Assume, without loss of generality, that the null generating vector-field l is geodesic along $\mathcal{H}^+$, i.e. $\mathbf{D}_l l = 0$. Since both $\beta(\mathcal{S}) = \alpha(\mathcal{S}) = 0$ along $\mathcal{H}$ we deduce[8] directly that $\rho(\mathcal{S})$ must verify the equation,

$$\nabla_l \rho(\mathcal{S}) = -\mathcal{J}(\mathcal{S})_{434}. \tag{5.7}$$

To deduce that $\rho(\mathcal{S})$ vanishes identically on $\mathcal{H}^+$ it only remains to verify that $\mathcal{J}(\mathcal{S})_{434}$ vanishes on $\mathcal{H}^+$. Clearly

$$\mathcal{J}(\mathcal{S})_{434} = -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda\rho\gamma\delta}\left(\mathcal{F}_4{}^\rho \delta_3^\gamma \delta_4^\delta - \frac{2}{3}\mathcal{F}^{\gamma\delta}\mathcal{I}_4{}^\rho{}_{34}\right).$$

Observe that the only choice of the index ρ for which the expression inside brackets does not vanish is $\rho = 4$. Thus

$$\begin{aligned}\mathcal{J}(\mathcal{S})_{434} &= -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda 4\gamma\delta}\left(\mathcal{F}_{34}\delta_3^\gamma\delta_4^\delta + \frac{1}{6}\mathcal{F}^{\gamma\delta}\right)\\ &= -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda 434}\mathcal{F}_{34} - (1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda 4\gamma\delta}\mathcal{F}^{\gamma\delta}.\end{aligned}$$

Since $\alpha(\mathcal{S})$, $\beta(\mathcal{S})$ vanish the only pair of indices $\gamma\delta$ for which $\mathbf{T}^\lambda \mathcal{S}_{\lambda 4\gamma\delta}$ does not vanish is when either of the two indices is a 3 and the other is $a \in \{1,2\}$. Since $\alpha(\mathcal{F}) = 0$, it follows that $\mathcal{J}(\mathcal{S})_{434}$ vanishes identically as stated. Thus $\rho(\mathcal{S})$ is constant along generators and vanishes on S_0. We conclude that $\rho(\mathcal{S})$ vanishes identically on $\mathcal{H}^+$.

To show that $\underline{\beta}(\mathcal{S})$ also vanishes we derive a transport equation for it along the generators of $\mathcal{H}^+$. In view of the vanishing of $\alpha(\mathcal{S})$, $\beta(\mathcal{S})$, $\rho(\mathcal{S})$ we can directly deduce[9] (see also appendix) it from (5.6),

$$\nabla_l \underline{\beta}(\mathcal{S})_a = \mathcal{J}(\mathcal{S})_{4a3}.$$

Thus, since $\underline{\beta}(\mathcal{S})$ vanishes on S_0, to deduce that it vanishes everywhere on $\mathcal{H}^+$ we only need to verify that $\mathcal{J}(\mathcal{S})_{4a3}$ vanishes identically on $\mathcal{H}^+$.

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[8] Alternatively we can use the null Bianchi identities of the appendix.

[9] We also refer the reader to the appendix for the definition of the horizontal covariant derivative ∇_l.

Now,

$$\begin{aligned}\mathcal{J}(\mathcal{S})_{4a3} &= -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda\rho\gamma\delta}\left(\mathcal{F}_4{}^\rho \delta_a^\gamma \delta_3^\delta - \frac{2}{3}\mathcal{F}^{\gamma\delta}\mathcal{I}_4{}^\rho{}_{a3}\right)\\ &= -6(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda 4a3}\mathcal{F}_{43} + 8(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda b34}\mathcal{F}_{43}\mathcal{I}_4{}^b{}_{a3}\\ &\quad + 4(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda bcd}\mathcal{F}^{cd}\mathcal{I}_4{}^b{}_{a3} + 8(1-\sigma)^{-1}\mathbf{T}^\lambda \mathcal{S}_{\lambda b4c}\mathcal{F}^{4c}\mathcal{I}_4{}^b{}_{a3}.\end{aligned}$$

Since $\alpha(\mathcal{S})$, $\beta(\mathcal{S})$ and $\rho(\mathcal{S})$ vanish, it follows that $\mathcal{S}_{b4a3} = \mathcal{S}_{ab34} = \mathcal{S}_{abcd} = \mathcal{S}_{4bcd} = \mathcal{S}_{4b4c} = 0$, which gives $\mathcal{J}(\mathcal{S})_{4a3} = 0$.

To show that $\underline{\alpha}(\mathcal{S})$ also vanishes on $\mathcal{H}^+$ we derive another transport equation for it. Since all other components of $\mathcal{S}$ have already been shown to vanish we easily derive, from (5.6),

$$\nabla_l \underline{\alpha}(\mathcal{S})_{ab} = -\mathcal{J}(\mathcal{S})_{a3b}. \tag{5.8}$$

Since $\underline{\alpha}(\mathcal{S})$ vanishes on S_0 it only remains to check that $\mathcal{J}(\mathcal{S})_{a3b}$ vanishes identically. This can be checked as before taking advantage of the cancellations of all the other null components of $\mathcal{S}$. Therefore $\mathcal{S}$ vanishes along the entire event horizon.

6. Vanishing of $\mathcal{S}$ in a neighborhood of the bifurcate sphere

Let $\mathbf{O}_\epsilon = \{x \in \mathbf{O} : |u_-| < \epsilon, |u_+| < \epsilon\}$ as in Sect. 2. In this section we show that the tensor $\mathcal{S}$ vanishes in a neighborhood of the bifurcate sphere S_0 in $\mathbf{E}$.

Proposition 6.1. *There is $r_1 = r_1(A_0) > 0$ such that*

$$\mathcal{S} \equiv 0 \quad \textit{in } \mathbf{O}_{r_1} \cap \mathbf{E}.$$

The rest of this section is concerned with the proof of Proposition 6.1. Recall, see Remark 4.4, that the tensor $\mathcal{S}$ is well defined and smooth on $\mathbf{O}_{\epsilon_2}$ for some $\epsilon_2 = \epsilon_2(A_0) \in (0, \epsilon_0)$. Recall that we have

$$\mathbf{g}(L_\pm, L_\pm) = 0, \quad \mathbf{g}(L_+, L_-) = \Omega > \frac{1}{2} \quad \text{in } \mathbf{O}_{\epsilon_0}.$$

Moreover both L_+, L_- are orthogonal to the 2-surfaces $S_{u_-,u_+} = \mathcal{H}_{u_-} \cap \mathcal{H}_{u_+}$. We choose, locally at any point $p \in S_{u_-,u_+}$, an orthonormal frame $(L_a)_{a=1,2}$ tangent to S_{u_-,u_+}. Thus, relative to the null frame $L_1, L_2, L_3 = L_-, L_4 = L_+$ the metric $\mathbf{g}$ takes the form,

$$\begin{cases}\mathbf{g}_{ab} = \delta_{ab}, \quad \mathbf{g}_{a3} = \mathbf{g}_{a4} = 0, \quad a, b = 1, 2\\ \mathbf{g}_{33} = \mathbf{g}_{44} = 0, \quad \mathbf{g}_{34} = \Omega.\end{cases} \tag{6.1}$$

Also, for the inverse metric,

$$\begin{cases} \mathbf{g}^{ab} = \delta^{ab}, \quad \mathbf{g}^{a3} = \mathbf{g}^{a4} = 0, \quad a, b = 1, 2 \\ \mathbf{g}^{33} = g^{44} = 0, \quad \mathbf{g}^{34} = \Omega^{-1}. \end{cases} \tag{6.2}$$

We denote by $O(1)$ any quantity with absolute value uniformly bounded by a positive constant which depends only on A_0 (in particular $L_\alpha(\Omega) = O(1)$, $\alpha = 1, 2, 3, 4$). In view of the definitions of $u_\pm$ and $L_\pm$ we have,

$$L_1(u_\pm) = L_2(u_\pm) = L_-(u_-) = L_+(u_+) = 0, \quad L_-(u_+) = L_+(u_-) = \Omega. \tag{6.3}$$

For $\epsilon \in (0, \epsilon_0]$ we define the weight function in $\mathbf{O}_{\epsilon^2}$,

$$h_\epsilon = \epsilon^{-1}(u_+ + \epsilon)(u_- + \epsilon). \tag{6.4}$$

Observe that,

$$\begin{aligned} &L_4(h_\epsilon) = \epsilon^{-1}(u_+ + \epsilon)\Omega, \quad L_3(h_\epsilon) = \epsilon^{-1}(u_- + \epsilon)\Omega, \\ &L_a(h_\epsilon) = 0, \quad a = 1, 2. \end{aligned} \tag{6.5}$$

Also, using (6.3) and (6.5)

$$\begin{cases} (\mathbf{D}^2 h_\epsilon)_{33} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{44} = O(1), \\ (\mathbf{D}^2 h_\epsilon)_{34} = \epsilon^{-1}\Omega^2 + O(1), \quad (\mathbf{D}^2 h_\epsilon)_{ab} = O(1), \quad a, b = 1, 2, \\ (\mathbf{D}^2 h_\epsilon)_{3a} = O(1), \quad (\mathbf{D}^2 h_\epsilon)_{4a} = O(1), \quad a = 1, 2. \end{cases} \tag{6.6}$$

Assume $x_0 \in S_0$ is a fixed point and define, using the coordinate chart $\Phi^{x_0} : B_1 \to B_1(x_0)$, $N^{x_0} : B_1(x_0) \to [0, \infty)$,

$$N^{x_0}(x) = |(\Phi^{x_0})^{-1}(x)|^2. \tag{6.7}$$

We state now the main Carleman estimate needed in the proof of Proposition 6.1.

Lemma 6.2. *There is $\epsilon \in (0, \epsilon_2)$ sufficiently small and $\widetilde{C}_\epsilon$ sufficiently large such that for any $x_0 \in S_0$, any $\lambda \geq \widetilde{C}_\epsilon$, and any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$*

$$\lambda \| e^{-\lambda f_\epsilon} \phi \|_{L^2} + \| e^{-\lambda f_\epsilon} |D^1 \phi| \|_{L^2} \leq \widetilde{C}_\epsilon \lambda^{-1/2} \left\| e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi \right\|_{L^2}, \tag{6.8}$$

where $f_\epsilon = \ln(h_\epsilon + \epsilon^{12} N^{x_0})$, see definitions (6.4) *and* (6.7).

Proof. It is clear that $B_{\epsilon^{10}}(x_0) \subseteq \mathbf{O}_{\epsilon^2}$ for ϵ sufficiently small (depending only on the constant A_0), thus the weight f_ϵ is well defined in $B_{\epsilon^{10}}(x_0)$. We apply Proposition 3.3 with $V = 0$. It is clear that $\epsilon^{12} N^{x_0}$ is a negligible perturbation, in the sense of (3.9), for ϵ sufficiently small. It remains to prove that there is $\epsilon_1 = \epsilon_1(A_0) > 0$ such that the family of weights $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ satisfies conditions (3.6)–(3.8).

Let $\widetilde{C}$ denote constants that may depend only on A_0. The definition (6.4) easily gives $h_\epsilon(x_0) = \epsilon$, $|D^1 h_\epsilon| \leq \widetilde{C}$ on $B_{\epsilon^{10}}(x_0)$, and $|D^j h_\epsilon| \leq \widetilde{C}\epsilon^{-1}$ on $B_{\epsilon^{10}}(x_0)$ for $j = 2, 3, 4$. Thus condition (3.6) is satisfied provided $\epsilon_1 \leq \widetilde{C}^{-1}$.

Using (6.2), (6.5), (6.6), and $\Omega(x_0) = 1$ we compute in the frame L_1, L_2, L_3, L_4

$$\mathbf{D}^\alpha h_\epsilon(x_0)\mathbf{D}^\beta h_\epsilon(x_0)(\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) = 2 + \epsilon O(1) \geq 1$$

if ϵ_1 is sufficiently small. Thus condition (3.7) is satisfied provided $\epsilon_1 \leq \widetilde{C}^{-1}$.

Assume now $Y = Y^\alpha L_\alpha$ is a vector in $\mathbf{T}_{x_0}(\mathbf{M})$. We fix $\mu = \epsilon_1^{-1/2}$ and compute, using (6.5), (6.6), and $\Omega(x_0) = 1$,

$$\begin{aligned}
&Y^\alpha Y^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}\left|Y^\alpha \mathbf{D}_\alpha h_\epsilon\right|^2 \\
&= \mu((Y^1)^2 + (Y^2)^2 + 2Y^3Y^4) - 2\epsilon^{-1}Y^3Y^4 \\
&\quad + \epsilon^{-2}(Y^3 + Y^4)^2 + O(1)\sum_{\alpha=1}^{4}(Y^\alpha)^2 \\
&\geq (\mu/2)[(Y^1)^2 + (Y^2)^2] + (\epsilon^{-1}/2)[(Y^3)^2 + (Y^4)^2] \\
&\geq (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2
\end{aligned}$$

if ϵ_1 is sufficiently small. We notice now that we can write $Y = X^\alpha \partial_\alpha$ in the coordinate frame $\partial_1, \partial_2, \partial_3, \partial_4$, and $|X^\alpha| \leq \widetilde{C}(|Y^1| + |Y^2| + |Y^3| + |Y^4|)$ for $\alpha = 1, 2, 3, 4$. Thus condition (3.8) is satisfied provided $\epsilon_1 \leq \widetilde{C}^{-1}$, which completes the proof of the lemma. □

We prove now Proposition 6.1.

Proof of Proposition 6.1. In view of Lemma 6.2, there are constants $\epsilon = \epsilon(A_0) \in (0, \epsilon_0)$ and $\widetilde{C}_\epsilon \geq 1$ such that, for any $x_0 \in S_0$, $\lambda \geq \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$

$$\lambda\|e^{-\lambda f_\epsilon}\phi\|_{L^2} + \|e^{-\lambda f_\epsilon}|D^1\phi|\|_{L^2} \leq \widetilde{C}_\epsilon \lambda^{-1/2}\left\|e^{-\lambda f_\epsilon}\square_{\mathbf{g}}\phi\right\|_{L^2}, \tag{6.9}$$

where

$$f_\epsilon = \ln\left(\epsilon^{-1}(u_+ + \epsilon)(u_- + \epsilon) + \epsilon^{12}N^{x_0}\right). \tag{6.10}$$

The constant ϵ will remain fixed in this proof. For simplicity of notation, we replace the constants $\widetilde{C}_\epsilon$ in (6.9) with $\widetilde{C}$; since ϵ is fixed, these constants may depend only on the constant A_0. We will show that $\mathcal{S} \equiv 0$ in $B_{\epsilon^{40}}(x_0) \cap \mathbf{E}$ for any $x_0 \in S_0$. This suffices to prove the proposition.

We fix $x_0 \in S_0$ and, for $(j_1, \ldots, j_4) \in \{1, 2, 3, 4\}^4$, we define using the vector-fields ∂_α induced by the coordinate chart Φ^{x_0}

$$\phi_{(j_1 \ldots j_4)} = \mathcal{S}(\partial_{j_1}, \ldots, \partial_{j_4}). \tag{6.11}$$

The functions $\phi_{(j_1 \ldots j_4)} : B_{\epsilon^{10}}(x_0) \to \mathbb{C}$ are smooth. Let $\eta : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[1/2, \infty)$ and equal to 1 in $[3/4, \infty)$.

For $\delta \in (0, 1]$ we define,

$$\begin{aligned}\phi^{\delta,\epsilon}_{(j_1\dots j_4)} &= \phi_{(j_1\dots j_4)} \cdot \mathbf{1_E} \cdot \eta(u_+u_-/\delta) \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) \\ &= \phi_{(j_1\dots j_4)} \cdot \widetilde{\eta}_{\delta,\epsilon}.\end{aligned} \tag{6.12}$$

Clearly, $\phi^{\delta,\epsilon}_{(j_1\dots j_4)} \in C^\infty_0(B_{\epsilon^{10}}(x_0) \cap \mathbf{E})$. We would like to apply the inequality (6.9) to the functions $\phi^{\delta,\epsilon}_{(j_1\dots j_4)}$, and then let $\delta \to 0$ and $\lambda \to \infty$ (in this order).

Using the definition (6.12), we have

$$\square_{\mathbf{g}}\phi^{\delta,\epsilon}_{(j_1\dots j_4)} = \widetilde{\eta}_{\delta,\epsilon} \cdot \square_{\mathbf{g}}\phi_{(j_1\dots j_4)} + 2\mathbf{D}_\alpha\phi_{(j_1\dots j_4)} \cdot \mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon} + \phi_{(j_1\dots j_4)} \cdot \square_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}.$$

Using the Carleman inequality (6.9), for any $(j_1, \dots, j_4) \in \{1, 2, 3, 4\}^4$ we have

$$\begin{aligned}&\lambda \cdot \left\| e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon}\phi_{(j_1\dots j_4)}\right\|_{L^2} + \left\| e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon}\left|D^1\phi_{(j_1\dots j_4)}\right|\right\|_{L^2} \\ &\leq \widetilde{C}\lambda^{-1/2} \cdot \left\| e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon}\square_{\mathbf{g}}\phi_{(j_1\dots j_4)}\right\|_{L^2} \\ &\quad + \widetilde{C}\Big[\left\| e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha\phi_{(j_1\dots j_4)}\mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon}\right\|_{L^2} \\ &\qquad + \left\| e^{-\lambda f_\epsilon} \cdot \phi_{(j_1\dots j_4)}\big(|\square_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}| + \left|D^1\widetilde{\eta}_{\delta,\epsilon}\right|\big)\right\|_{L^2}\Big],\end{aligned} \tag{6.13}$$

for any $\lambda \geq \widetilde{C}$. We estimate now $|\square_{\mathbf{g}}\phi_{(j_1\dots j_4)}|$. Using Theorem 4.7 and the definition (4.41), in $B_{\epsilon^{10}}(x_0)$ we estimate pointwise

$$|\square_{\mathbf{g}}\phi_{(j_1\dots j_4)}| \leq M \sum_{l_1,\dots,l_4} \big(\left|D^1\phi_{(l_1\dots l_4)}\right| + |\phi_{(l_1\dots l_4)}|\big), \tag{6.14}$$

for some large constant M. We add inequalities (6.13) over $(j_1, \dots, j_4) \in \{1, 2, 3, 4\}^4$. The key observation is that, in view of (6.14), the first term in the right-hand side of (6.13) can be absorbed into the left-hand side for λ sufficiently large. Thus, for any λ sufficiently large and $\delta \in (0, 1]$,

$$\begin{aligned}&\lambda \sum_{j_1,\dots,j_4} \left\| e^{-\lambda f_\epsilon} \cdot \widetilde{\eta}_{\delta,\epsilon}\phi_{(j_1\dots j_4)}\right\|_{L^2} \\ &\leq \widetilde{C} \sum_{j_1,\dots,j_4} \Big[\left\| e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha\phi_{(j_1\dots j_4)}\mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon}\right\|_{L^2} \\ &\qquad + \left\| e^{-\lambda f_\epsilon} \cdot \phi_{(j_1\dots j_4)}\big(|\square_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}| + \left|D^1\widetilde{\eta}_{\delta,\epsilon}\right|\big)\right\|_{L^2}\Big].\end{aligned} \tag{6.15}$$

We would like to let $\delta \to 0$ in (6.15). For this, we observe first that the functions $\mathbf{D}_\alpha\phi_{(j_1\dots j_4)}\mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon}$ and $(|\square_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}| + |D^1\widetilde{\eta}_{\delta,\epsilon}|)$ vanish outside the set $\mathbf{A}_\delta \cup \widetilde{\mathbf{B}}_\epsilon$, where

$$\begin{aligned}\mathbf{A}_\delta &= \{x \in B_{\epsilon^{10}}(x_0) \cap \mathbf{E} : u_+(x)u_-(x) \in (\delta/2, \delta)\}; \\ \mathbf{B}_\epsilon &= \big\{x \in B_{\epsilon^{10}}(x_0) \cap \mathbf{E} : N^{x_0} \in (\epsilon^{20}/2, \epsilon^{20})\big\}.\end{aligned}$$

In addition, since $\phi_{(j_1\dots j_4)} = 0$ on $\mathbf{O}_{\epsilon_2} \cap [\delta(\mathcal{I}^-(\mathbf{M}^{(end)})) \cup \delta(\mathcal{I}^+(\mathbf{M}^{(end)}))]$ (see Sect. 5), it follows from (2.7) and (2.8) that there are smooth functions $\phi'_{(j_1\dots j_4)} : \mathbf{O}_{\epsilon_2} \to \mathbb{C}$ such that

$$\phi_{(j_1\dots j_4)} = u_+u_- \cdot \phi'_{(j_1\dots j_4)} \quad \text{in } \mathbf{O}_{\epsilon_2}. \tag{6.16}$$

We show now that

$$|\Box_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}| + \left|D^1\widetilde{\eta}_{\delta,\epsilon}\right| \leq \widetilde{C}(\mathbf{1}_{\mathbf{B}_\epsilon} + (1/\delta)\mathbf{1}_{\mathbf{A}_\delta}). \tag{6.17}$$

The inequality for $|D^1\widetilde{\eta}_{\delta,\epsilon}|$ follows directly from the definition (6.12). Also, using again the definition,

$$\begin{aligned}\left|\mathbf{D}^\alpha\mathbf{D}_\alpha\widetilde{\eta}_{\delta,\epsilon}\right| \leq{}& \left|\mathbf{D}^\alpha\mathbf{D}_\alpha(\mathbf{1}_{\mathbf{E}} \cdot \eta(u_+u_-/\delta))\right| \cdot (1 - \eta(N^{x_0}/\epsilon^{20}))\\ &+ \widetilde{C}(\mathbf{1}_{\mathbf{B}_\epsilon} + (1/\delta)\mathbf{1}_{\mathbf{A}_\delta}).\end{aligned}$$

Thus, for (6.17), it suffices to prove that

$$\mathbf{1}_{\mathbf{E}\cap B_{\epsilon^{10}}(x_0)} \cdot \left|\mathbf{D}^\alpha\mathbf{D}_\alpha(\eta(u_+u_-/\delta))\right| \leq \widetilde{C}/\delta \cdot \mathbf{1}_{\mathbf{A}_\delta}. \tag{6.18}$$

Since u_+, u_-, η are smooth functions, for (6.18) it suffices to prove that

$$\delta^{-2}\left|\mathbf{D}^\alpha(u_+u_-)\mathbf{D}_\alpha(u_+u_-)\right| \leq \widetilde{C}/\delta \quad \text{in } \mathbf{A}_\delta, \tag{6.19}$$

which follows from (6.3).

We show now that

$$\left|\mathbf{D}_\alpha\phi_{(j_1\dots j_4)}\mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon}\right| \leq \widetilde{C}_{\phi'}(\mathbf{1}_{\mathbf{B}_\epsilon} + \mathbf{1}_{\mathbf{A}_\delta}), \tag{6.20}$$

where the constant $\widetilde{C}_{\phi'}$ depends on the smooth functions $\phi'_{(j_1\dots j_4)}$ defined in (6.16). Using the formula (6.16), this follows easily from (6.19).

It follows from (6.16), (6.17), and (6.20) that

$$\left|\mathbf{D}_\alpha\phi_{(j_1\dots j_4)}\mathbf{D}^\alpha\widetilde{\eta}_{\delta,\epsilon}\right| + |\phi_{j_1\dots j_4}|\left(|\Box_{\mathbf{g}}\widetilde{\eta}_{\delta,\epsilon}| + \left|D^1\widetilde{\eta}_{\delta,\epsilon}\right|\right) \leq \widetilde{C}_{\phi'}(\mathbf{1}_{\mathbf{B}_\epsilon} + \mathbf{1}_{\mathbf{A}_\delta}).$$

Since $\lim_{\delta\to 0} \|\mathbf{1}_{\mathbf{A}_\delta}\|_{L^2} = 0$, we can let $\delta \to 0$ in (6.15) to conclude that

$$\lambda \sum_{j_1,\dots,j_4} \left\|e^{-\lambda f_\epsilon} \cdot \mathbf{1}_{B_{\epsilon^{10}/2}(x_0)\cap\mathbf{E}} \cdot \phi_{(j_1\dots j_4)}\right\|_{L^2} \leq \widetilde{C}_{\phi'}\left\|e^{-\lambda f_\epsilon} \cdot \mathbf{1}_{\mathbf{B}_\epsilon}\right\|_{L^2} \tag{6.21}$$

for any λ sufficiently large. Finally, using the definition (6.10), we observe that

$$\inf_{B_{\epsilon^{40}}(x_0)\cap\mathbf{E}} e^{-\lambda f_\epsilon} \geq e^{-\lambda \ln[\epsilon+\epsilon^{32}/2]} \geq \sup_{\mathbf{B}_\epsilon} e^{-\lambda f_\epsilon}.$$

It follows from (6.21) that

$$\lambda \sum_{j_1,\dots,j_4} \|\mathbf{1}_{B_{\epsilon^{40}}(x_0)\cap\mathbf{E}} \cdot \phi_{(j_1\dots j_4)}\|_{L^2} \leq \widetilde{C}_{\phi'}\|\mathbf{1}_{\mathbf{B}_\epsilon}\|_{L^2}$$

for any λ sufficiently large. We let $\lambda \to \infty$ to conclude that $\phi_{(j_1 \dots j_4)} = 0$ in $B_{\epsilon^{40}}(x_0) \cap \mathbf{E}$, which completes the proof of the proposition. □

7. Consequences of the vanishing of $\mathcal{S}$

We assume in this section that $\mathbf{N} \subseteq \widetilde{\mathbf{M}}$ is a open set, $S_0 \subseteq \mathbf{N}$, $\mathbf{N} \cap \mathbf{E}$ is connected, and

$$\begin{cases} 1 - \sigma \neq 0 \quad \text{in } \mathbf{N}; \\ \mathcal{S}_{\alpha\beta\mu\nu} = \mathcal{R}_{\alpha\beta\mu\nu} + 6(1-\sigma)^{-1}\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}\big) = 0 \quad \text{in } \mathbf{N} \cap \mathbf{E}. \end{cases} \tag{7.1}$$

It follows from the assumption (1.6), and the identities (4.33) and (7.1) (which give $\mathbf{D}_\rho(\mathcal{F}^2(1-\sigma)^{-4}) = 0$ in $\mathbf{N} \cap \mathbf{E}$) that

$$-4M^2\mathcal{F}^2 = (1-\sigma)^4 \quad \text{in } \mathbf{N} \cap \mathbf{E}. \tag{7.2}$$

We define the smooth function $P = y + iz : \mathbf{N} \to \mathbb{C}$,

$$P = y + iz = (1-\sigma)^{-1}. \tag{7.3}$$

Since $-\mathcal{F}^2/4 = (4MP^2)^{-2} \neq 0$ (see (7.2)), there are null vector-fields $l, \underline{l}$, locally around every point in $\mathbf{N}$, such that

$$\begin{aligned} &\mathcal{F}_{\alpha\beta} l^\beta = (4MP^2)^{-1} l_\alpha, \quad \mathcal{F}_{\alpha\beta}\underline{l}^\beta = -(-4MP^2)^{-1}\underline{l}_\alpha, \quad \text{and} \\ &l^\alpha \underline{l}_\alpha = -1 \quad \text{in } \mathbf{N} \cap \mathbf{E}. \end{aligned} \tag{7.4}$$

We fix a complex-valued null vector-field m on $\mathbf{N}$ such that $(m, \overline{m}, \underline{l}, l) = (e_1, e_2, e_3, e_4)$ is a complex null tetrad, see the definitions in Subsect. A.2. We may also assume that $(m, \overline{m}, \underline{l}, l)$ has positive orientation, i.e.

$$\in_{\alpha\beta\mu\nu} m^\alpha \overline{m}^\beta \underline{l}^\mu l^\nu = i.$$

We prove now some identities. Most of these identities, with the exception of Proposition 7.2 and the computation of the Hessian of y in Lemma 7.3, were derived by Mars [24]; for the sake of completeness we rederive them in our notation. It follows from (7.4) and (7.2) that, in $\mathbf{N} \cap \mathbf{E}$,

$$\mathcal{F}_{\alpha\beta} = \frac{1}{4MP^2}\big(-l_\alpha \underline{l}_\beta + l_\beta \underline{l}_\alpha - i\in_{\alpha\beta\mu\nu} l^\mu \underline{l}^\nu\big). \tag{7.5}$$

Using (7.5), we compute easily

$$\mathcal{F}_{41} = \mathcal{F}_{42} = \mathcal{F}_{31} = \mathcal{F}_{32} = 0 \quad \text{and} \quad \mathcal{F}_{43} = \mathcal{F}_{21} = 1/(4MP^2). \tag{7.6}$$

Using (7.1) and (7.6) we compute

$$\begin{aligned}
&\mathcal{R}_{4141} = \mathcal{R}_{4242} = 0 \quad \text{thus} \quad \Psi_{(2)}(R) = 0;\\
&\mathcal{R}_{3131} = \mathcal{R}_{3232} = 0 \quad \text{thus} \quad \underline{\Psi}_{(2)}(R) = 0;\\
&\mathcal{R}_{1434} = \mathcal{R}_{2434} = 0 \quad \text{thus} \quad \Psi_{(1)}(R) = 0;\\
&\mathcal{R}_{1343} = \mathcal{R}_{2343} = 0 \quad \text{thus} \quad \underline{\Psi}_{(1)}(R) = 0;\\
&\mathcal{R}_{2314} = \frac{1}{4M^2P^3}, \ \mathcal{R}_{1324} = 0 \quad \text{thus} \quad \Psi_{(0)}(R) = \frac{1}{8M^2P^3}.
\end{aligned} \tag{7.7}$$

We use now the first 4 Bianchi identities (A.37)–(A.40) to conclude that

$$\xi = \underline{\xi} = \vartheta = \underline{\vartheta} = 0 \quad \text{in } \mathbf{N} \cap \mathbf{E}. \tag{7.8}$$

The remaining 4 Bianchi identities, (A.41)–(A.44) give

$$DP = \theta P, \quad \underline{D}P = \underline{\overline{\theta}}P, \quad \delta P = \eta P, \quad \overline{\delta}P = \underline{\overline{\eta}}P. \tag{7.9}$$

We analyze now the functions y and z. By contracting (7.5) with $2\mathbf{T}^\alpha$ and using $2\mathbf{T}^\alpha \mathcal{F}_{\alpha\beta} = \sigma_\beta = \mathbf{D}_\beta \sigma$ we derive

$$\mathbf{D}_\beta y = \frac{1}{2M}\big[-(\mathbf{T}^\alpha l_\alpha)\underline{l}_\beta + (\mathbf{T}^\alpha \underline{l}_\alpha) l_\beta\big] \quad \text{and} \quad \mathbf{D}_\beta z = \frac{-1}{2M}\in_{\alpha\beta\mu\nu}\mathbf{T}^\alpha l^\mu \underline{l}^\nu. \tag{7.10}$$

In particular,

$$\delta y = \overline{\delta} y = Dz = \underline{D}z = 0. \tag{7.11}$$

Using (7.9) it follows that

$$Dy = \theta P, \quad \underline{D}y = \underline{\overline{\theta}}P, \quad \delta z = -i\eta P, \quad \overline{\delta}z = -i\underline{\overline{\eta}}P. \tag{7.12}$$

In particular $\theta P = \overline{\theta P}$, $\underline{\overline{\theta}}P = \underline{\theta}\,\overline{P}$, $-\eta P = \underline{\eta}\,\overline{P}$, and, using again (7.10),

$$\mathbf{T}^\alpha l_\alpha = 2M\theta P, \quad \mathbf{T}^\alpha \underline{l}_\alpha = -2M\underline{\overline{\theta}}P. \tag{7.13}$$

Using (7.11) and (7.12) we rewrite (7.10) in the form

$$\mathbf{D}_\beta y = -\underline{\overline{\theta}}Pl_\beta - \theta P\underline{l}_\beta, \quad \mathbf{D}_\beta z = -i\underline{\overline{\eta}}Pm_\beta - i\eta P\overline{m}_\beta. \tag{7.14}$$

A direct computation using the definition of P shows that

$$\mathbf{D}_\alpha P\, \mathbf{D}^\alpha P = \frac{\mathbf{D}_\alpha \sigma \mathbf{D}^\alpha \sigma}{(1-\sigma)^4} = -\frac{\mathbf{T}^\alpha \mathbf{T}_\alpha}{4M^2}.$$

The real part of this identity and $-\mathbf{T}^\alpha \mathbf{T}_\alpha = \Re\sigma$ give

$$\mathbf{D}_\alpha y \mathbf{D}^\alpha y - \mathbf{D}_\alpha z \mathbf{D}^\alpha z = \frac{-\mathbf{T}^\alpha \mathbf{T}_\alpha}{4M^2} = \frac{1}{4M^2}\left(1 - \frac{y}{y^2+z^2}\right). \tag{7.15}$$

Using (7.14) this gives

$$8M^2(\eta\underline{\overline{\eta}} - \theta\underline{\overline{\theta}})P^3\overline{P} = y^2 - y + z^2. \tag{7.16}$$

Lemma 7.1. *There is a constant* $B \in [0, \infty)$ *such that*

$$\mathbf{D}_\alpha z \mathbf{D}^\alpha z = \frac{B - z^2}{4M^2(y^2 + z^2)} \quad \text{in } \mathbf{N} \cap \mathbf{E}. \tag{7.17}$$

In addition $z^2 \leq B$ *in* $\mathbf{N} \cap \mathbf{E}$.

Proof. For (7.17) it suffices to prove that

$$4M^2P\overline{P} \cdot \mathbf{D}_\alpha z \mathbf{D}^\alpha z + z^2 = B. \tag{7.18}$$

Let $Z = 4M^2P\overline{P} \cdot \mathbf{D}_\alpha z \mathbf{D}^\alpha z$. To show $\underline{D}(Z + z^2) = 0$ we use the formula $Z = 8M^2P^2\overline{P}^2\underline{\eta}\underline{\overline{\eta}}$ (which follows from (7.14) and $-\eta P = \underline{\eta}\overline{P}$), the identities (7.9), $\underline{\overline{\theta}}P = \underline{\theta}\overline{P}$, and the Ricci equation (see (A.24), (7.7), and (7.8))

$$\underline{D}\underline{\eta} = \underline{\theta}(\eta - \underline{\eta}) - \Gamma_{123}\underline{\eta}.$$

Indeed,

$$\begin{aligned}
\underline{D}(Z + z^2) &= 8M^2P^2\overline{P}^2\underline{\eta}\underline{\overline{\eta}}\left(\frac{2\underline{D}P}{P} + \frac{2\underline{D}\overline{P}}{\overline{P}} + \frac{\underline{D}\underline{\eta}}{\underline{\eta}} + \frac{\underline{D}\underline{\overline{\eta}}}{\underline{\overline{\eta}}}\right) \\
&= 8M^2P^2\overline{P}^2\underline{\eta}\underline{\overline{\eta}}\,[2\underline{\overline{\theta}} + 2\underline{\theta} - \underline{\theta}(\overline{P}/P + 1) - \Gamma_{123} \\
&\qquad - \underline{\overline{\theta}}(P/\overline{P} + 1) - \Gamma_{213}] \\
&= 0.
\end{aligned}$$

To show $D(Z + z^2) = 0$ we use the formula $Z = 8M^2P^2\overline{P}^2\eta\overline{\eta}$ (which follows from (7.14) and $-\eta P = \underline{\eta}\overline{P}$), the identities (7.9), $\theta P = \overline{\theta}\overline{P}$, and the Ricci equation (see (A.23), (7.7), and (7.8))

$$D\eta = \theta(\underline{\eta} - \eta) - \Gamma_{124}\eta.$$

Indeed,

$$\begin{aligned}
D(Z + z^2) &= 8M^2P^2\overline{P}^2\eta\overline{\eta}\left(\frac{2DP}{P} + \frac{2D\overline{P}}{\overline{P}} + \frac{D\eta}{\eta} + \frac{D\overline{\eta}}{\overline{\eta}}\right) \\
&= 8M^2P^2\overline{P}^2\eta\overline{\eta}[2\theta + 2\overline{\theta} - \theta(P/\overline{P} + 1) - \Gamma_{124} \\
&\qquad - \overline{\theta}(\overline{P}/P + 1) - \Gamma_{214}] \\
&= 0.
\end{aligned}$$

Finally, to show that $\delta(Z + z^2) = 0$ we use the formula

$$Z + z^2 = -8M^2P^2\overline{P}^2\theta\underline{\theta} - y^2 + y,$$

which follows from (7.16) and $\bar{\underline{\theta}} P = \underline{\theta}\overline{P}$, the identities (7.9), $\theta P = \bar{\theta}\overline{P}$, and the Ricci equations (see (A.29), (A.30), (7.7), and (7.8))

$$\begin{cases} \delta\theta = -\zeta\theta - \eta(\theta - \bar{\theta}); \\ \delta\underline{\theta} = \zeta\underline{\theta} - \underline{\eta}(\underline{\theta} - \bar{\underline{\theta}}). \end{cases}$$

Indeed,

$$\begin{aligned} \delta(Z + z^2) &= -8M^2P^2\overline{P}^2\theta\underline{\theta}\left(\frac{\delta\theta}{\theta} + \frac{\delta\underline{\theta}}{\underline{\theta}} + \frac{2\delta P}{P} + \frac{2\delta\overline{P}}{\overline{P}}\right) \\ &= -8M^2P^2\overline{P}^2\theta\underline{\theta}[-\zeta - \eta(1 - P/\overline{P}) + \zeta \\ &\qquad\qquad - \underline{\eta}(1 - \overline{P}/P) + 2\eta + 2\underline{\eta}] \\ &= 0. \end{aligned}$$

This completes the proof of (7.18). □

It follows from (7.17) and (7.15) that

$$\mathbf{D}_\alpha y \mathbf{D}^\alpha y = \frac{y^2 - y + B}{4M^2(y^2 + z^2)}. \tag{7.19}$$

Using (7.13) and (7.14), it follows that

$$-\theta\bar{\underline{\theta}}P^2 = \frac{y^2 - y + B}{8M^2(y^2 + z^2)} = \frac{(\mathbf{T}^\alpha l_\alpha) \cdot (\mathbf{T}^\alpha \underline{l}_\alpha)}{4M^2}. \tag{7.20}$$

We express also the vector $\mathbf{T}$ in the complex null tetrad $(m, \overline{m}, \underline{l}, l)$. Using (7.5) and (7.10),

$$\mathbf{T}_\alpha = (\mathcal{F}^2/4)^{-1}\mathcal{F}_\alpha{}^\mu \mathbf{T}^\beta \mathcal{F}_{\beta\mu} = -(\mathbf{T}^\beta \underline{l}_\beta) l_\alpha - (\mathbf{T}^\beta l_\beta)\underline{l}_\alpha - 2M \in_{\alpha\beta\mu\nu} \mathbf{D}^\beta z l^\mu \underline{l}^\nu. \tag{7.21}$$

We prove now a uniform bound on the gradient of the function y.

Proposition 7.2. *There is a constant $\widetilde{C} = \widetilde{C}(A_0)$ that depends only on A_0 such that*

$$|D^1 y| \leq \widetilde{C} \quad \textit{in } \mathbf{N}. \tag{7.22}$$

Proof. For $p \in \Phi^{x_0}(B_1)$, $x_0 \in \Sigma_0$, the gradient $|D^1 y|$ is defined using the coordinate chart Φ^{x_0}, i.e.

$$|D^1 y|(p) = \sum_{j=1}^{4} |\partial_j(y)(p)|.$$

In view of the definition $y = \Re[(1 - \sigma)^{-1}]$ and the smoothness of σ, the bound (7.22) is clear if $|1 - \sigma(p)| \geq 1/4$. Assume that $|1 - \sigma(p)| < 1/4$.

Since

$$\Re(1-\sigma) = 1 + \mathbf{g}(\mathbf{T},\mathbf{T}),$$

it follows that $\mathbf{g}(\mathbf{T},\mathbf{T})(p) < -3/4$. In particular, $p \in \mathbf{N} \cap \mathbf{E}$. We define the vector-field,

$$Y = \mathbf{g}^{\alpha\beta}\partial_\alpha y \partial_\beta. \tag{7.23}$$

In view of (7.19) and $T(y) = 0$, we have

$$|\mathbf{g}(Y,Y)| = \left|\frac{y^2 - y + B}{4M^2(y^2+z^2)}\right| \le \widetilde{C} \quad \text{and} \quad \mathbf{g}(\mathbf{T},Y) = \mathbf{T}(y) = 0 \quad \text{at } p.$$

Since $\mathbf{g}(\mathbf{T},\mathbf{T}) < -3/4$ it follows that Y_p is a space-like vector with norm (as induced by the coordinate chart Φ^{x_0}) dominated by $\widetilde{C}$. The bound (7.22) follows since $\partial_j y = \mathbf{g}(Y, \partial_j)$. □

7.1. The connection coefficients and the Hessian of y**.** Assume now that $\mathbf{N}'$ is a subset of $\mathbf{N} \cap \mathbf{E}$ with the property that

$$y^2 - y + B > 0 \quad \text{in } \mathbf{N}'.$$

Using (7.20), we can normalize the vector l such that

$$\mathbf{T}^\alpha l_\alpha = 2M \quad \text{in } \mathbf{N}'. \tag{7.24}$$

Thus, using (7.13) and (7.20), we compute

$$\theta = 1/P \quad \text{and} \quad \underline{\theta} = -W/\overline{P} = -\frac{1}{\overline{P}} \cdot \frac{y^2 - y + B}{8M^2(y^2+z^2)} \quad \text{in } \mathbf{N}'. \tag{7.25}$$

Using the null structure equation (A.21) (see also (7.8))

$$D\theta = -\theta^2 - \omega\theta,$$

together with (7.9) and (7.25), we compute

$$\omega = 0 \quad \text{in } \mathbf{N}'. \tag{7.26}$$

Using the null structure equation (A.22) (see also (7.8))

$$\underline{D}\underline{\theta} = -\underline{\theta}^2 - \underline{\omega}\,\underline{\theta},$$

together with (7.11), (7.12), and (7.25), we compute

$$\underline{\omega} = \frac{y^2 - z^2 - 2y(B - z^2)}{8M^2(y^2+z^2)^2} \quad \text{in } \mathbf{N}'. \tag{7.27}$$

We can express $\underline{\omega}$ in the form,

$$\underline{\omega} = HW, \quad H = \frac{y^2 - z^2 - 2y(B - z^2)}{(y^2 - y + B)(y^2 + z^2)}. \tag{7.28}$$

Using the null structure equation (A.29) (see also (7.8) and (7.7))

$$\delta\theta = -\zeta\theta - \eta(\theta - \overline{\theta}),$$

together with (7.9) and (7.25), we compute

$$\zeta = \frac{\eta P}{\overline{P}} = -\underline{\eta} \quad \text{in } \mathbf{N}'. \tag{7.29}$$

Using (7.16) and (7.25),

$$|\zeta|^2 = \underline{\eta}\underline{\overline{\eta}} = \frac{B - z^2}{8M^2(y^2 + z^2)^2} \quad \text{and} \quad \eta = -\frac{\underline{\eta}\overline{P}}{P} \quad \text{in } \mathbf{N}'. \tag{7.30}$$

Finally, using (7.14), we rewrite (7.21) in the form

$$\mathbf{T} = -2M(Wl + \underline{l} - \overline{\zeta}Pm - \zeta\overline{P}\overline{m}) \quad \text{in } \mathbf{N}'. \tag{7.31}$$

We summarize these computations in the first part of the following lemma.

Lemma 7.3. *Let* $\mathbf{N}$ *be the set defined by* (7.1) *and* $\mathbf{N}'$ *the subset of* $\mathbf{N} \cap \mathbf{E}$ *for which* $y^2 - y + B > 0$*, with* B *the constant of Lemma 7.1. In* $\mathbf{N}'$ *we have, with* $P = y + iz = (1 - \sigma)^{-1}$,

$$\xi = \underline{\xi} = \vartheta = \underline{\vartheta} = \omega = 0,$$

$$\underline{\omega} = HW = \frac{y^2 - z^2 - 2y(B - z^2)}{8M^2(y^2 + z^2)^2}, \quad H = \frac{y^2 - z^2 - 2y(B - z^2)}{(y^2 - y + B)(y^2 + z^2)},$$

$$W = \frac{y^2 - y + B}{8M^2(y^2 + z^2)} > 0, \quad |z|^2 \le B,$$

$$\theta = 1/P, \quad \underline{\theta} = -W/\overline{P}, \quad |\zeta|^2 = \frac{B - z^2}{8M^2(y^2 + z^2)^2}, \quad \eta = \zeta\frac{\overline{P}}{P}, \quad \underline{\eta} = -\zeta,$$

$$\delta y = \overline{\delta} y = Dz = \underline{D}z = 0, \quad Dy = 1, \quad \underline{D}y = -W, \quad \mathbf{D}_\alpha y \mathbf{D}^\alpha y = 2W.$$

We also have, for the Hessian of y,

$$\begin{cases} (\mathbf{D}^2 y)_{44} = (\mathbf{D}^2 y)_{33} = 0, & (\mathbf{D}^2 y)_{43} = (\mathbf{D}^2 y)_{34} = -WH \\ (\mathbf{D}^2 y)_{41} = (\mathbf{D}^2 y)_{14} = \zeta, & (\mathbf{D}^2 y)_{42} = (\mathbf{D}^2 y)_{24} = \overline{\zeta} \\ (\mathbf{D}^2 y)_{31} = (\mathbf{D}^2 y)_{13} = \eta W, & (\mathbf{D}^2 y)_{32} = (\mathbf{D}^2 y)_{23} = \overline{\eta} W \\ (\mathbf{D}^2 y)_{12} = (\mathbf{D}^2 y)_{21} = W\frac{2y}{y^2+z^2}, & (\mathbf{D}^2 y)_{11} = (\mathbf{D}^2 y)_{22} = 0. \end{cases} \tag{7.32}$$

Proof. It only remains to prove formulas in (7.32). These formulas follow easily using $(\mathbf{D}^2 y)_{\alpha\beta} = e_\alpha(e_\beta y) - \Gamma^\mu{}_{\beta\alpha} e_\mu(y)$, the first part of the lemma, and the table (A.16). □

8. The main bootstrap argument

In this section we show that

$$1 - \sigma \neq 0 \quad \text{and} \quad \mathcal{S} = 0 \quad \text{on } \Sigma_0 \cap \mathbf{E}. \tag{8.1}$$

In view of our assumption **AF**, this suffices to show that $\mathcal{S} = 0$ in $\mathbf{E}$. Our main theorem is then consequence of the main result of Mars in [24].

We show first that the function y is constant on $\mathcal{H}^+ \cup \mathcal{H}^-$ and increases in $\mathbf{E}$.

Lemma 8.1. *There is a constant* $y_{S_0} \in (1/2, 1]$ *such that*

$$y = y_{S_0} \quad \textit{on } \mathcal{H}^+ \cup \mathcal{H}^-. \tag{8.2}$$

In addition $B \in [0, 1/4)$, *where* B *is the constant in Lemma 7.1. Finally, for sufficiently small* $\epsilon = \epsilon(A_0) > 0$,

$$y > y_{S_0} + \widetilde{C}^{-1} u_+ u_- \quad \textit{on } \mathbf{O}_\epsilon \cap \mathbf{E}, \tag{8.3}$$

where $\mathbf{O}_\epsilon$ *are the open sets defined in Sect. 2, and* $\widetilde{C} = \widetilde{C}(A_0) > 0$.

Proof. Let $\mathbf{N} = \mathbf{O}_{r_1}$ denote the set constructed in Proposition 6.1. Since $\mathcal{S} = 0$ in $\mathbf{N}$, we can apply the computations of the previous section. It follows from (5.3) that if $\widetilde{l}$ is tangent to the null generators of $\mathcal{H}^+$ then $\mathcal{F}_{\alpha\beta}\widetilde{l}^\beta = C\widetilde{l}_\alpha$ for some scalar C. Thus $\widetilde{l}$ is parallel to either l or $\underline{l}$ on $\mathcal{H}^+$. Similarly, the null generator of $\mathcal{H}^-$ is also parallel to either l or $\underline{l}$ on $\mathcal{H}^-$. Thus the vector m is tangent to the bifurcate sphere S_0. Using $\delta y = 0$, see (7.11), it follows that y is constant on S_0. Using (7.12) and Proposition 5.3 it follows that y is constant on $\mathcal{H}^+ \cup \mathcal{H}^-$, which gives (8.2). Also, using (7.20) on S_0 and the fact that $\mathbf{T}$ is tangent to S_0, it follows that

$$y_{S_0}^2 - y_{S_0} + B = 0.$$

Since $B \in [0, \infty)$ and $y_{S_0} > 1/2$ (using assumption (1.7)), it follows that $B \in [0, 1/4)$ and

$$y_{S_0} = \frac{1 + \sqrt{1 - 4B}}{2} \in (1/2, 1].$$

To prove (8.3) we consider the open sets $\mathbf{O}_\epsilon$ and the functions $u_\pm : \mathbf{O}_\epsilon \to \mathbb{R}$ defined in Sect. 2. It follows from (8.2) combined with (2.7) and (2.8) that

$$y = y_{S_0} + u_+ u_- \cdot y', \tag{8.4}$$

for some smooth function $y' : \mathbf{O}_\epsilon \to \mathbb{R}$, with $|D^1 y'| \leq \widetilde{C}$. The identities $P = (1-\sigma)^{-1}$, $\mathbf{D}^\mu \mathbf{D}_\mu \sigma = -\mathcal{F}^2$, $\mathbf{D}_\mu \sigma \mathbf{D}^\mu \sigma = \mathbf{T}^\alpha \mathbf{T}_\alpha \cdot \mathcal{F}^2 = -\Re\sigma \cdot \mathcal{F}^2$ (see (4.18)), and $\mathcal{F}^2 = -(1-\sigma)^4/(4M^2)$ (see (7.2)) show that,

$$\mathbf{D}^\mu \mathbf{D}_\mu P = (1-\sigma)^{-2} \mathbf{D}^\mu \mathbf{D}_\mu \sigma + 2(1-\sigma)^{-3} \mathbf{D}^\mu \sigma \mathbf{D}_\mu \sigma$$
$$= \frac{1}{4M^2}(1-\sigma)(1+\bar{\sigma}) = \frac{2\overline{P}-1}{4M^2 P\overline{P}}.$$

Therefore,

$$\mathbf{D}^\mu \mathbf{D}_\mu y = \frac{2y-1}{4M^2(y^2+z^2)}. \tag{8.5}$$

We substitute $y = y_{S_0} + u_+ u_- \cdot y'$ (see (8.4)) and evaluate on S_0

$$\frac{2y_{S_0}-1}{4M^2\big(y_{S_0}^2+z^2\big)} = \mathbf{D}^\mu \mathbf{D}_\mu (y_{S_0} + u_+ u_- \cdot y') = 2\mathbf{D}^\mu (u_+) \mathbf{D}_\mu (u_-) \cdot y' = 4y'.$$

Since $y_{S_0} > 1/2 + \widetilde{C}^{-1}$ it follows that $y' > \widetilde{C}^{-1}$ on S_0. Thus, for $\epsilon \in (0, r_1)$ sufficiently small,

$$y > y_{S_0} + \widetilde{C}^{-1} u_+ u_- \quad \text{in } \mathbf{O}_\epsilon \cap \mathbf{E},$$

as desired. □

We define the set

$$\Sigma_0' = \{x \in \Sigma_0 \cap \mathbf{E} : \sigma(x) \neq 1\}.$$

Clearly, Σ_0' is an open subset of $\Sigma_0 \cap \mathbf{E}$ which contains a neighborhood of S_0 in $\Sigma_0 \cap \mathbf{E}$. We define the function (which agrees with the function y defined earlier on open sets)

$$y : \Sigma_0' \to \mathbb{R}, \quad y(x) = \Re[(1-\sigma)^{-1}].$$

For any $R > y_{S_0}$ let $\mathcal{V}_R = \{x \in \Sigma_0' : y(x) < R\}$ and $\mathcal{U}_R$ the unique connected component of $\mathcal{V}_R$ whose closure in Σ_0 contains S_0 (this unique connected component exists since $y(x) = y_{S_0} < R$ on S_0). We prove now the first step in our bootstrap argument.

Proposition 8.2. *There is a real number* $R_1 \geq y_{S_0} + \widetilde{C}^{-1}$*, for some constant* $\widetilde{C} = \widetilde{C}(A_0) > 0$*, such that* $\mathcal{S} = 0$ *in* $\mathcal{U}_{R_1}$.

Proof. With ϵ as in Lemma 8.1, it follows from Proposition 6.1 that $\mathcal{S} = 0$ in $\mathbf{O}_\epsilon \cap \mathbf{E}$. Also, since $u_+/u_- + u_-/u_+ \leq A_0$ in $\Sigma_0 \cap \mathbf{E} \cap \mathbf{O}_\epsilon$, it follows from (8.3) that

$$y - y_{S_0} \in \big[\widetilde{C}^{-1}\big(u_+^2 + u_-^2\big), \widetilde{C}\big(u_+^2 + u_-^2\big)\big] \quad \text{in } \Sigma_0 \cap \mathbf{E} \cap \mathbf{O}_\epsilon.$$

Thus, for R_1 sufficiently close to y_{S_0}, the set $\mathcal{U}_{R_1}$ is included in $\mathbf{O}_\epsilon$, and the proposition follows. □

With R_1 as in Proposition 8.2, the main result in this section is the following:

Proposition 8.3. *For any $R_2 \geq R_1$ we have $\mathcal{S} = 0$ in $\mathcal{U}_{R_2}$.*

The proof of Proposition 8.3, which will be completed in Subsect. 8.2, is done by induction. In view of Proposition 8.2, we may assume that the claims in Proposition 8.3 hold for some value $R_2 \geq R_1$. We therefore make the following induction hypothesis:

Induction hypothesis. For a fixed $R_2 \geq R_1$ the tensor $\mathcal{S}$ vanishes on the set $\mathcal{U}_{R_2}$, which is the unique connected component of the set $\mathcal{V}_{R_2} = \{x \in \Sigma_0 \cap \mathbf{E} : y(x) < R_2, \sigma(x) \neq 1\}$ whose closure in Σ_0 contains the bifurcate sphere S_0.

To complete the proof of the proposition we have to advance these claims for $R_2' = R_2 + r'$, where $r' > 0$ depends only on the constants A_0, $\widetilde{A}_{\widetilde{C}^{-1}}$ (here $\widetilde{A}_{\widetilde{C}^{-1}} = \widetilde{A}_\epsilon$ with $\epsilon = \widetilde{C}^{-1}$, see (2.9) for the definition of $\widetilde{A}_\epsilon$), and R_2 (as before, the constants $\widetilde{C}$ may depend only on A_0). In the rest of this section we let $\widetilde{C}_{R_2}$ denote various constants in $[1, \infty)$ that may depend only on A_0, $\widetilde{A}_{\widetilde{C}^{-1}}$, and R_2. It is important that such constants do not depend on other parameters, such as the point $x_0 \in \delta_{\Sigma_0 \cap \mathbf{E}}(\mathcal{U}_{R_2})$ chosen below.

Assume $x_0 \in \delta_{\Sigma_0 \cap \mathbf{E}}(\mathcal{U}_{R_2})$ is a point on the boundary of $\mathcal{U}_{R_2}$ in $\Sigma_0 \cap \mathbf{E}$. Clearly,

$$y(x_0) = R_2.$$

Thus $|1 - \sigma(x_0)| = (R_2^2 + z(x_0)^2)^{-1/2}$. Since $1 - \sigma$ is a smooth function on $\widetilde{\mathbf{M}}$ and $z(x_0)^2 \leq B < 1/4$ (see Lemma 7.3), there is $r_2' = r_2'(A_0, R_2) > 0$ such that $|1 - \sigma(x)| \in (1/(2R_2), 2/R_2)$ in $B_{r_2'}(x_0)$. Thus the function

$$y : B_{r_2'}(x_0) \to \mathbb{R}, \quad y(x) = \Re[(1 - \sigma(x))^{-1}],$$

is well defined; observe that, with ∂_j defined according to the coordinate charts defined in Subsect. 2.2,

$$\sup_{x \in B_{r_2'(x_0)}} (|y(x)| + |D^1 y(x)| + \ldots + |D^4 y(x)|) \leq \widetilde{C}_{R_2}. \tag{8.6}$$

By choosing r_2' sufficiently small it follows from $y(x_0) = R_2$ and (8.6) that

$$y(x) \in ((y_{S_0} + R_1)/2, 2R_2) \quad \text{for any } x \in B_{r_2'}(x_0). \tag{8.7}$$

In view of (2.9) there is $\delta_2 > \widetilde{C}_{R_2}^{-1}$ small[10] such that the set $(-\delta_2, \delta_2) \times (B_{r_2'}(x_0) \cap \Sigma_0)$ is diffeomorphic to the set $\bigcup_{|t|<\delta_2} \Phi_t(B_{r_2'}(x_0) \cap \Sigma_0)$. We let

[10] The constants r_2' and δ_2 are fixed in this paragraph such that $r_2', \delta_2 \ll 1$. We later fix the constants $r_2 \ll \min(r_2', \delta_2)$ (Lemma 8.4), $r_3 \ll r_2$ (Proposition 8.5), and $r' \ll r_3$ (proof of Proposition 8.3). All of these constants are bounded from below by some constant $\widetilde{C}_{R_2}^{-1}$.

$Q : \bigcup_{|t|<\delta_2} \Phi_t(B_{r_2'}(x_0) \cap \Sigma_0) \to B_{r_2'}(x_0) \cap \Sigma_0$ denote the induced smooth projection which takes every point $\Phi_t(x)$ into x.

We now define the connected open set of $\widetilde{\mathbf{M}}$, which we denote by $\mathbf{N}_{R_2}$,

$$\begin{aligned} \mathbf{N}_{R_2} = &\text{ connected component of } \Big[\Big(\bigcup_{t\in\mathbb{R}} \Phi_t(\mathcal{U}_{R_2})\Big) \cup \mathbf{O}_{r_1}\Big] \cap \widetilde{\mathbf{M}} \\ &\text{ containing } \mathcal{U}_{R_2}, \end{aligned} \tag{8.8}$$

where r_1 is as in Proposition 6.1. Since $\mathbf{T}$ is a Killing vector-field, $\mathcal{L}_{\mathbf{T}}\mathcal{S} = 0$ in $\widetilde{\mathbf{M}}$ and $\mathbf{T}(1-\sigma) = 0$. In view of our induction hypothesis $\mathcal{S} = 0$ in $\mathcal{U}_{R_2}$ and $\mathbf{T}$ does not vanish in $\mathbf{E}$; it follows that

$$1 - \sigma \neq 0 \text{ in } \mathbf{N}_{R_2} \quad \text{and} \quad \mathcal{S} = 0 \text{ in } \mathbf{N}_{R_2} \cap \mathbf{E}.$$

Thus the computations in Sect. 7 can be applied in the open set $\mathbf{N}_{R_2}$.

Lemma 8.4. *With $x_0 \in \delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})$ as before, there is $r_2 \in (0, r_2']$ such that*

$$\{x \in B_{r_2}(x_0) : y(x) < R_2\} \subseteq \bigcup_{|t|<\delta_2} \Phi_t(\mathcal{U}_{R_2}). \tag{8.9}$$

Proof. In view of (7.19),

$$\mathbf{D}^\alpha y \mathbf{D}_\alpha y = \frac{y^2 - y + B}{4M^2(y^2+z^2)} \quad \text{in } \mathbf{N}_{R_2}.$$

Thus, if $r_2'' \leq \widetilde{C}_{R_2}^{-1}$ is sufficiently small then $\mathbf{D}^\alpha y \mathbf{D}_\alpha y \geq \widetilde{C}_{R_2}^{-1}$ in $B_{r_2''}(x_0)$. It follows that there exists $r_2 = r_2(A_0, \widetilde{A}_{\widetilde{C}^{-1}}, R_2) > 0$ and an open set B', $B_{r_2}(x_0) \subseteq B' \subseteq B_{r_2''}(x_0)$, such that the set $\{x \in B' : y(x) < R_2\}$ is connected. Let $Q : B' \to B_{r_2'}(x_0) \cap \Sigma_0$ denote the projection defined above. The set $Q(\{x \in B' : y(x) < R_2\}) \subseteq B_{r_2'}(x_0) \cap \Sigma_0$ is connected and contains the set $\{x \in B' \cap \Sigma_0 : y(x) < R_2\}$. Since $y(Q(x)) = y(x)$, it follows from the definition of $\mathcal{U}_{R_2}$ (as a connected component of the set $\mathcal{V}_{R_2}$) that

$$Q(\{x \in B' : y(x) < R_2\}) \subseteq \mathcal{U}_{R_2}.$$

The claim (8.9) follows. □

We define now $\mathbf{N}' = \mathbf{N}_{R_2} \cap B_{r_2}(x_0)$. Since $y^2 - y + B > \widetilde{C}_{R_2}^{-1}$ in $\mathbf{N}'$, the calculations following (7.24) in the previous sections are also applicable in $\mathbf{N}'$. Recall the function H defined in (7.28),

$$H = \frac{y^2 - z^2 - 2y(B - z^2)}{(y^2 - y + B)(y^2 + z^2)}.$$

Since $B \in [0, 1/4)$ (see Lemma 8.1) and $y \geq y_{S_0} + \widetilde{C}_{R_2}^{-1} \geq 1/2 + \widetilde{C}_{R_2}^{-1}$, it follows that $H \geq \widetilde{C}_{R_2}^{-1}$ in $\mathbf{N}'$.

8.1. Vanishing of $\mathcal{S}$ is a neighborhood of x_0. Assume $x_0 \in \delta_{\Sigma_0 \cap \mathbf{E}}(\mathcal{U}_{R_2})$ is as before, and $r_2 > 0$ is constructed as in Lemma 8.4. We show now that the tensor $\mathcal{S}$ vanishes in a neighborhood of x_0.

Proposition 8.5. *There is $r_3 = r_3(A_0, \widetilde{A}_{\widetilde{C}^{-1}}, R_2) \in (0, r_2)$ such that $\mathcal{S} = 0$ in $B_{r_3}(x_0)$.*

As in Sect. 6, the main ingredient needed to prove Proposition 8.5 is a Carleman inequality. We define the smooth function $N^{x_0} : \Phi^{x_0}(B_1) = B_1(x_0) \to [0, \infty)$

$$N^{x_0}(x) = |(\Phi^{x_0})^{-1}(x)|^2.$$

Lemma 8.6. *There is $\epsilon \in (0, r_2]$ sufficiently small and $\widetilde{C}_\epsilon$ sufficiently large such that for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$*

$$\begin{aligned} &\lambda \|e^{-\lambda \widetilde{f}_\epsilon} \phi\|_{L^2} + \|e^{-\lambda \widetilde{f}_\epsilon} |D^1 \phi|\|_{L^2} \\ &\leq \widetilde{C}_\epsilon \lambda^{-1/2} \left\| e^{-\lambda \widetilde{f}_\epsilon} \square_{\mathbf{g}} \phi \right\|_{L^2} + \epsilon^{-6} \|e^{-\lambda \widetilde{f}_\epsilon} \mathbf{T}(\phi)\|_{L^2}, \end{aligned} \tag{8.10}$$

where, with $R_2 = y(x_0)$,

$$\widetilde{f}_\epsilon = \ln\left[y - R_2 + \epsilon + \epsilon^{12} N^{x_0}\right]. \tag{8.11}$$

Proof. We will use the notation $\widetilde{C}_{R_2}$ to denote various constants in $[1, \infty)$ that may depend only on the constants A_0, $\widetilde{A}_{\widetilde{C}^{-1}}$, and R_2. We would like to apply Proposition 3.3 with $V = \mathbf{T}$, $h_\epsilon = y - R_2 + \epsilon$ and $e_\epsilon = \epsilon^{12} N^{x_0}$. The condition (3.9) for the negligible perturbation e_ϵ is clearly satisfied if ϵ is sufficiently small. It remains to show that there is ϵ_1 sufficiently small such that the family of weights $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ satisfies the pseudo-convexity conditions (3.6)–(3.8).

Clearly, $h_\epsilon(x_0) = \epsilon$ and $\mathbf{T}(h_\epsilon)(x_0) = 0$ since $\mathbf{T}(\sigma) = 0$. Also $|D^j y| \leq \widetilde{C}_{R_2}$ for $j = 1, 2, 3, 4$ in $B_{r_2}(x_0)$, see (8.6), thus condition (3.6) is satisfied if ϵ_1 is sufficiently small.

To prove (3.7) and (3.8) we use the complex null tetrad $l = e_4$, $\underline{l} = e_3$, $m = e_1$, $\overline{m} = e_2$, normalized as in (7.24). With $\mathbf{D}_{(\alpha)} = \mathbf{D}_{e_\alpha}$, using Lemma 7.3 and the definition $h_\epsilon = y - R_2 + \epsilon$ we have

$$\mathbf{D}_{(1)} h_\epsilon = \mathbf{D}_{(2)} h_\epsilon = 0, \quad \mathbf{D}_{(3)} h_\epsilon = -W, \quad \mathbf{D}_{(4)} h_\epsilon = 1, \tag{8.12}$$

and, using also $\eta = \zeta \frac{\overline{P}}{P}$,

$$\begin{cases} \mathbf{D}_{(4)}\mathbf{D}_{(4)} h_\epsilon = \mathbf{D}_{(3)}\mathbf{D}_{(3)} h_\epsilon = 0, & \mathbf{D}_{(4)}\mathbf{D}_{(3)} h_\epsilon = \mathbf{D}_{(3)}\mathbf{D}_{(4)} h_\epsilon = -WH \\ \mathbf{D}_{(4)}\mathbf{D}_{(1)} h_\epsilon = \mathbf{D}_{(1)}\mathbf{D}_{(4)} h_\epsilon = \zeta, & \mathbf{D}_{(4)}\mathbf{D}_{(2)} h_\epsilon = \mathbf{D}_{(2)}\mathbf{D}_{(4)} h_\epsilon = \overline{\zeta} \\ \mathbf{D}_{(3)}\mathbf{D}_{(1)} h_\epsilon = \mathbf{D}_{(1)}\mathbf{D}_{(3)} h_\epsilon = W\zeta \frac{\overline{P}}{P}, & \mathbf{D}_{(3)}\mathbf{D}_{(2)} h_\epsilon = \mathbf{D}_{(2)}\mathbf{D}_{(3)} h_\epsilon = W\overline{\zeta} \frac{P}{\overline{P}} \\ \mathbf{D}_{(1)}\mathbf{D}_{(2)} h_\epsilon = \mathbf{D}_{(2)}\mathbf{D}_{(1)} h_\epsilon = W \frac{2R_2}{R_2^2 + z^2}, & \mathbf{D}_{(1)}\mathbf{D}_{(1)} h_\epsilon = \mathbf{D}_{(2)}\mathbf{D}_{(2)} h_\epsilon = 0, \end{cases} \tag{8.13}$$

where all the functions are evaluated at x_0. Thus

$$\mathbf{D}^\alpha h_\epsilon \mathbf{D}^\beta h_\epsilon (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon) = 4W^2 - 2\epsilon W^2 H,$$

which is bounded from below by ϵ_1^2 if ϵ_1 is sufficiently small, since $W(x_0) \geq \widetilde{C}_{R_2}^{-1}$ and $|H(x_0)| \leq \widetilde{C}_{R_2}$. The condition (3.7) is therefore satisfied.

We prove now condition (3.8) for a vector $X = X^{(1)}e_1 + \overline{X^{(1)}}e_2 + Ye_3 + Ze_4$, $Y, Z \in \mathbb{R}$, $X^{(1)} \in \mathbb{C}$. Recall, see (7.31),

$$\mathbf{T}/(2M) = \overline{\zeta} P e_1 + \zeta \overline{P} e_2 - e_3 - W e_4.$$

Thus, using also (8.12)

$$\begin{aligned}
&\epsilon^{-2}\big(\big|X^\alpha \mathbf{T}_\alpha\big|^2 + \big|X^\alpha \mathbf{D}_\alpha h_\epsilon\big|^2\big) \\
&= \epsilon^{-2}(Z - WY)^2 + \epsilon^{-2} 4M^2 (\zeta \overline{P} X^{(1)} + \overline{\zeta} P \overline{X^{(1)}} + YW + Z)^2 \\
&\geq (\epsilon^{-2}/2)(Z - WY)^2 + (\epsilon^{-1}/2)(\zeta \overline{P} X^{(1)} + \overline{\zeta} P \overline{X^{(1)}} + 2YW)^2
\end{aligned} \tag{8.14}$$

for ϵ sufficiently small. Using (8.13)

$$\begin{aligned}
&X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon) \\
&= 2X^{(1)}\overline{X^{(1)}}\left(\mu - \frac{2R_2 W}{R_2^2 + z^2}\right) + 2YZ(-\mu + WH) \\
&\quad - 2\zeta X^{(1)}[Z + WY(\overline{P}/P)] - 2\overline{\zeta}\,\overline{X^{(1)}}[Z + WY(P/\overline{P})].
\end{aligned}$$

Let $L = \zeta \overline{P} X^{(1)} + \overline{\zeta} P \overline{X^{(1)}}$. We write $Z = WY + Z - WY$, and then $L = -2WY + L + 2WY$, and use

$$1 + (\overline{P}/P) = \overline{P}\frac{2R_2}{R_2^2 + z^2}, \quad 1 + (P/\overline{P}) = P\frac{2R_2}{R_2^2 + z^2},$$

to rewrite

$$\begin{aligned}
&X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon) \\
&= 2X^{(1)}\overline{X^{(1)}}\left(\mu - \frac{2R_2 W}{R_2^2 + z^2}\right) + 2Y^2(-W\mu + W^2 H) \\
&\quad - \frac{4R_2}{R_2^2 + z^2} WY \cdot L + (Z - WY)[2Y(-\mu + WH) - 2\zeta X^{(1)} - 2\overline{\zeta}\,\overline{X^{(1)}}] \\
&= 2X^{(1)}\overline{X^{(1)}}\left(\mu - \frac{2R_2 W}{R_2^2 + z^2}\right) + 2Y^2\left(-W\mu + W^2 H + \frac{4R_2 W^2}{R_2^2 + z^2}\right) \\
&\quad - \frac{4R_2}{R_2^2 + z^2} WY \cdot (L + 2WY) \\
&\quad + (Z - WY)[2Y(-\mu + WH) - 2\zeta X^{(1)} - 2\overline{\zeta}\,\overline{X^{(1)}}].
\end{aligned} \tag{8.15}$$

We set now $\mu = 3R_2W/(R_2^2 + z^2)$ and combine (8.14) and (8.15). Since $H(x_0) \geq 0$ it follows that

$$\begin{aligned} &X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon) + \epsilon^{-2}\big(\big|X^\alpha \mathbf{T}_\alpha\big|^2 + \big|X^\alpha \mathbf{D}_\alpha h_\epsilon\big|^2\big) \\ &\geq (\epsilon^{-2}/2)(Z - WY)^2 + (\epsilon^{-1}/2)(L + 2YW)^2 + 2|X^{(1)}|^2 \frac{R_2 W}{R_2^2 + z^2} \\ &\quad + 2Y^2 \frac{R_2 W^2}{R_2^2 + z^2} - \widetilde{C}_{R_2}(|Z - WY| + |L + WY|)(|Y| + |X^{(1)}|) \\ &\geq (\epsilon^{-2}/4)(Z - WY)^2 + (\epsilon^{-1}/4)(L + 2YW)^2 \\ &\quad + |X^{(1)}|^2 \frac{R_2 W}{R_2^2 + z^2} + Y^2 \frac{R_2 W^2}{R_2^2 + z^2} \end{aligned}$$

if ϵ is sufficiently small, since $W \geq \widetilde{C}_{R_2}^{-1}$. It follows that

$$\begin{aligned} &X^\alpha X^\beta(\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon) + \epsilon^{-2}\big(\big|X^\alpha \mathbf{T}_\alpha\big|^2 + \big|X^\alpha \mathbf{D}_\alpha h_\epsilon\big|^2\big) \\ &\geq \widetilde{C}_{R_2}^{-1}(Z^2 + |X^{(1)}|^2 + Y^2), \end{aligned}$$

thus the condition (3.8) is satisfied for ϵ_1 sufficiently small. This completes the proof of the lemma. □

We prove now Proposition 8.1.

Proof of Proposition 8.1. We use the Carleman estimate in Lemma 8.6 and Lemma 8.4. In view of Lemma 8.6, there are constants $\epsilon \in (0, r_2]$ and $\widetilde{C}_\epsilon \geq 1$ such that for any $\lambda \geq \widetilde{C}_\epsilon$ and any $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$,

$$\begin{aligned} &\lambda \|e^{-\lambda \widetilde{f}_\epsilon}\phi\|_{L^2} + \|e^{-\lambda \widetilde{f}_\epsilon}|D^1\phi|\|_{L^2} \\ &\leq \widetilde{C}_\epsilon \lambda^{-1/2}\big\|e^{-\lambda \widetilde{f}_\epsilon}\square_{\mathbf{g}}\phi\big\|_{L^2} + \epsilon^{-6}\|e^{-\lambda \widetilde{f}_\epsilon}\mathbf{T}(\phi)\|_{L^2}, \end{aligned} \tag{8.16}$$

where

$$\widetilde{f}_\epsilon = \ln\big[y - R_2 + \epsilon + \epsilon^{12}N^{x_0}\big]. \tag{8.17}$$

The constant ϵ will remain fixed in this proof. For simplicity of notation, we replace the constants $\widetilde{C}_\epsilon$ with $\widetilde{C}_{R_2}$; since ϵ is fixed, these constants may depend only on the constants A_0, $\widetilde{A}_{\widetilde{C}^{-1}}$, and R_2. We will show that $\mathcal{S} \equiv 0$ in the set $B_{\epsilon^{100}} = B_{\epsilon^{100}}(x_0)$.

In view of Theorem 4.7 and the fact that $\mathbf{T}$ is a Killing vector-field

$$\begin{cases} \square_{\mathbf{g}}\mathcal{S}_{\alpha_1\ldots\alpha_4} = \mathcal{S}_{\beta_1\ldots\beta_4}\mathcal{A}^{\beta_1\ldots\beta_4}{}_{\alpha_1\ldots\alpha_4} + \mathbf{D}_{\beta_5}\mathcal{S}_{\beta_1\ldots\beta_4}\mathcal{B}^{\beta_1\ldots\beta_5}{}_{\alpha_1\ldots\alpha_4}; \\ \mathcal{L}_{\mathbf{T}}\mathcal{S} = 0, \end{cases} \tag{8.18}$$

in $B_{\epsilon^{10}}(x_0)$, for some smooth tensor-fields $\mathcal{A}$ and $\mathcal{B}$. Also, using Lemma 8.4 and the fact that $\mathcal{S}$ vanishes in $\mathcal{U}_{R_2}$ (the bootstrap assumption),

$$\mathcal{S} = 0 \quad \text{in } \{x \in B_{\epsilon^{10}}(x_0) : y(x) < R_2\}. \tag{8.19}$$

As in the proof of Proposition 6.1, for $(j_1, \ldots, j_4) \in \{1, 2, 3, 4\}^4$ we define, using the coordinate chart Φ,

$$\phi_{(j_1 \ldots j_4)} = \mathcal{S}(\partial_{j_1}, \ldots, \partial_{j_k}).$$

The functions $\phi_{(j_1 \ldots j_4)} : B_{\epsilon^{10}}(x_0) \to \mathbb{C}$ are smooth. Let $\eta : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[1/2, \infty)$ and equal to 1 in $[3/4, \infty)$. We define

$$\phi^{\epsilon}_{(j_1 \ldots j_4)} = \phi_{(j_1 \ldots j_4)} \cdot (1 - \eta(N(x)/\epsilon^{40})) = \phi_{(j_1 \ldots j_4)} \cdot \widetilde{\eta}_\epsilon.$$

Clearly, $\phi^{\epsilon}_{(j_1 \ldots j_4)} \in C^\infty_0(B_{\epsilon^{10}}(x_0))$ and

$$\begin{cases} \Box_{\mathbf{g}} \phi^{\epsilon}_{(j_1 \ldots j_4)} = \widetilde{\eta}_\epsilon \cdot \Box_{\mathbf{g}} \phi_{(j_1 \ldots j_4)} + 2 \mathbf{D}_\alpha \phi_{(j_1 \ldots j_4)} \cdot \mathbf{D}^\alpha \widetilde{\eta}_\epsilon + \phi_{(j_1 \ldots j_4)} \cdot \Box_{\mathbf{g}} \widetilde{\eta}_\epsilon \\ \mathbf{T}\big(\phi^{\epsilon}_{(j_1 \ldots j_4)}\big) = \widetilde{\eta}_\epsilon \cdot \mathbf{T}(\phi_{(j_1 \ldots j_4)}) + \phi_{(j_1 \ldots j_4)} \cdot \mathbf{T}(\widetilde{\eta}_\epsilon). \end{cases}$$

Using the Carleman inequality (8.16), for any $(j_1, \ldots, j_4) \in \{1, 2, 3, 4\}^4$ we have

$$\begin{aligned} &\lambda \cdot \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \widetilde{\eta}_\epsilon \phi_{(j_1 \ldots j_4)} \big\|_{L^2} + \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \widetilde{\eta}_\epsilon \big| D^1 \phi_{(j_1 \ldots j_4)} \big| \big\|_{L^2} \\ &\leq \widetilde{C}_{R_2} \lambda^{-1/2} \cdot \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \widetilde{\eta}_\epsilon \Box_{\mathbf{g}} \phi_{(j_1 \ldots j_4)} \big\|_{L^2} + \widetilde{C}_{R_2} \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \widetilde{\eta}_\epsilon \mathbf{T}(\phi_{(j_1 \ldots j_4)}) \big\|_{L^2} \\ &\quad + \widetilde{C}_{R_2} \big[\big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \ldots j_4)} \mathbf{D}^\alpha \widetilde{\eta}_\epsilon \big\|_{L^2} \\ &\qquad + \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \phi_{(j_1 \ldots j_4)} \big(|\Box_{\mathbf{g}} \widetilde{\eta}_\epsilon| + \big| D^1 \widetilde{\eta}_\epsilon \big| \big) \big\|_{L^2} \big], \end{aligned} \tag{8.20}$$

for any $\lambda \geq \widetilde{C}_{R_2}$. Using the identities in (8.18), in $B_{\epsilon^{10}}(x_0)$ we estimate pointwise

$$\begin{cases} |\Box_{\mathbf{g}} \phi_{(j_1 \ldots j_4)}| \leq \widetilde{C}_{R_2} \sum_{l_1, \ldots, l_4} \big(\big| D^1 \phi_{(l_1 \ldots l_4)} \big| + |\phi_{(l_1 \ldots l_4)}| \big); \\ |\mathbf{T}(\phi_{(j_1 \ldots j_4)})| \leq \widetilde{C}_{R_2} \sum_{l_1, \ldots, l_4} |\phi_{(l_1 \ldots l_4)}|. \end{cases} \tag{8.21}$$

We add up the inequalities (8.20) over $(j_1, \ldots, j_4) \in \{1, 2, 3, 4\}^4$. The key observation is that, in view of (8.21), the first two terms in the right-hand side can be absorbed into the left-hand side for λ sufficiently large. Thus, for any $\lambda \geq \widetilde{C}_{R_2}$

$$\begin{aligned} &\lambda \sum_{j_1, \ldots, j_4} \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \widetilde{\eta}_\epsilon \phi_{(j_1 \ldots j_4)} \big\|_{L^2} \\ &\leq \widetilde{C}_{R_2} \sum_{j_1, \ldots, j_4} \big[\big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \ldots j_4)} \mathbf{D}^\alpha \widetilde{\eta}_\epsilon \big\|_{L^2} \\ &\qquad + \big\| e^{-\lambda \widetilde{f}_\epsilon} \cdot \phi_{(j_1 \ldots j_4)} \big(|\Box_{\mathbf{g}} \widetilde{\eta}_\epsilon| + \big| D^1 \widetilde{\eta}_\epsilon \big| \big) \big\|_{L^2} \big]. \end{aligned} \tag{8.22}$$

Using the hypothesis (8.19) and the definition of the function $\widetilde{\eta}_\epsilon$, we have

$$\begin{aligned} &\big| \mathbf{D}_\alpha \phi_{(j_1 \ldots j_4)} \mathbf{D}^\alpha \widetilde{\eta}_\epsilon \big| + \phi_{(j_1 \ldots j_4)} \big(|\Box_{\mathbf{g}} \widetilde{\eta}_\epsilon| + \big| D^1 \widetilde{\eta}_\epsilon \big| \big) \\ &\leq \widetilde{C}_{R_2} \cdot \mathbf{1}_{\{x \in B_{\epsilon^{10}}(x_0) : y(x) \geq R_2 \text{ and } N(x) \geq \epsilon^{50}\}}. \end{aligned}$$

Using the definition (8.17), we observe also that

$$\inf_{B_{\epsilon^{100}}} e^{-\lambda \widetilde{f}_\epsilon} \geq e^{-\lambda \ln(\epsilon+\epsilon^{70})} \geq \sup_{\{x\in B_{\epsilon^{10}}(x_0):y(x)\geq R_2 \text{ and } N(x)\geq \epsilon^{50}\}} e^{-\lambda \widetilde{f}_\epsilon}.$$

It follows from these last two inequalities and (8.22) that

$$\lambda \sum_{j_1,\dots,j_4} \|\mathbf{1}_{B_{\epsilon^{100}}} \cdot \phi_{(j_1\dots j_4)}\|_{L^2} \leq \widetilde{C}_{R_2} \sum_{j_1,\dots,j_4} \|\mathbf{1}_{\{x\in B_{\epsilon^{10}}(x_0):y(x)\geq R_2 \text{ and } N(x)\geq\epsilon^{50}\}}\|_{L^2},$$

for any $\lambda \geq \widetilde{C}_{R_2}$. The proposition follows by letting $\lambda \to \infty$. □

8.2. Proof of Proposition 8.3 and the main theorem. In this subsection we complete the proof of the main theorem.

Proof of Proposition 8.3. In view of Proposition 8.5, the tensor $\mathcal{S}$ vanishes in the connected open set $\mathbf{N}' = \mathbf{N}_{R_2} \cup (\bigcup_{x_0\in\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})} B_{r_3}(x_0))$. It remains to show that for some $r' \ll r_3$ we have

$$\begin{aligned} &\mathcal{U}_{R_2+r'} \subseteq \mathcal{U}_{R_2} \cup \Big(\bigcup_{x_0\in\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})} G_{r_3/\tilde{C}}(x_0)\Big), \\ &G_r(x_0) = \{x \in B_r(x_0) \cap \Sigma_0 : y(x) < R_2 + r'\}, \end{aligned} \tag{8.23}$$

where $\widetilde{C}$ is sufficiently large so that,

$$\overline{\Big(\bigcup_{x_0\in\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})} G_{r_3/\tilde{C}}(x_0)\Big)} \subseteq \Big(\bigcup_{x_0\in\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})} \overline{G_{r_3/4}(x_0)}\Big), \tag{8.24}$$

with the bars denoting the closures in Σ_0. We observe that such a constant exists in view of the fact that $\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})$ is compact and the function y tends to infinity in the asymptotic region of Σ_0 (in view of our assumption **AF**).

Assume, by contradiction, that (8.23) does not hold, thus there exists $p \in \mathcal{U}_{R_2+r'}$ which does not belong in the open set (in Σ_0) in the right-hand side of (8.23). Let $\gamma : [0, 1] \to \mathcal{U}_{R_2+r'} \cup S_0$ denote a smooth curve such that $\gamma(0) \in S_0$ and $\gamma(1) = p$. Let $p' = \gamma(t')$ denote the first point on this curve which is not in the open set in the right-hand side of (8.23). Clearly, p' does not belong to the closure of $\mathcal{U}_{R_2}$, thus

$$p' \in \overline{\bigcup_{x_0\in\delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})} G_{r_3/\tilde{C}}(x_0)}.$$

In view of (8.24) we infer that, for some $x_0 \in \delta_{\Sigma_0\cap\mathbf{E}}(\mathcal{U}_{R_2})$,

$$p' \in \{x \in B_{r_3/2}(x_0) \cap \Sigma_0 : y(x) < R_2 + r'\}. \tag{8.25}$$

Recall our smooth vector-field $Y = \mathbf{g}^{\alpha\beta}\partial_\alpha y \partial_\beta$, see (7.23) and discussion following it, with the property that $\mathbf{g}(Y, Y) \geq \widetilde{C}_{R_2}^{-1}$ in $B_{r_3}(x_0)$. We consider the integral curve starting from the point p' and flowing (backwards) a short distance $\widetilde{C}_{R_2}^{-1}$ (much smaller than r_3) along Y, and project this integral curve to Σ_0 using the smooth projection $Q : \bigcup_{|t|<\delta_2} \Phi_t(B_{r_3}(x_0) \cap \Sigma_0) \to B_{r_3}(x_0) \cap \Sigma_0$. The resulting curve is a smooth curve in $B_{r_3}(x_0) \cap \Sigma_0$; if r' sufficiently small then this curve contains a point p'' such that $y(p'') < R_2$. In view of Lemma 8.4, $p'' \in \mathcal{U}_{R_2}$, thus there is a point $p''' \in \delta_{\Sigma_0 \cap \mathbf{E}}(\mathcal{U}_{R_2})$ on the curve joining p' and p''. Then $p' \in B_{r_3/\widetilde{C}}(p''')$, which gives a contradiction. □

To complete the proof of the main theorem we use Propositions 8.3 and 7.2. Using Proposition 8.3, it follows that the tensor $\mathcal{S}$ vanishes in the connected component of the set Σ_0' whose closure in Σ_0 contains S_0. Assume $(\Sigma_0 \cap \mathbf{E}) \setminus \Sigma_0' \neq \emptyset$ and let $p \in (\Sigma_0 \cap \mathbf{E}) \setminus \Sigma_0'$. Assume $\gamma : [0, 1] \to \Sigma_0 \cap \overline{\mathbf{E}}$ is a smooth curve such that $\gamma(0) \in S_0$ and $\gamma(1) = p$. Let $p' = \gamma(t')$ denote the first point on this curve which is not in $\Sigma_0' \cup S_0$. Thus $\gamma(t'')$ belongs to the connected component of the set Σ_0' whose closure in Σ_0 contains S_0 for any $t'' < t'$. Since $\mathcal{S}$ vanishes in this connected component, it follows from Lemma 7.2 that the function y is bounded by a constant at all points $\gamma(t'')$, $t'' < t'$. Thus $p' \in \Sigma_0'$, contradiction.

It follows that $\Sigma_0' = \Sigma \cap \mathbf{E}$ and $\mathcal{S} = 0$ in $\Sigma \cap \mathbf{E}$, which establishes the claim (8.1).

Appendix A. The main formalism

A.1. Horizontal structures. Assume $(\mathbf{N}, \mathbf{g})$ is a smooth[11] vacuum Einstein space-time of dimension 4. Assume $(l, \underline{l})$ is a null pair on $\mathbf{N}$, i.e.

$$\mathbf{g}(l, l) = \mathbf{g}(\underline{l}, \underline{l}) = 0 \quad \text{and} \quad \mathbf{g}(l, \underline{l}) = -1.$$

We say that a vector-field X is *horizontal* if

$$\mathbf{g}(l, X) = \mathbf{g}(\underline{l}, X) = 0.$$

Let $\mathbf{O}(\mathbf{N})$ denote the vector space of horizontal vector-fields on $\mathbf{N}$. We define the induced metric, and induced volume form,

$$\begin{cases} \gamma(X, Y) = \mathbf{g}(X, Y) & \forall X, Y \in \mathbf{O}(\mathbf{N}), \\ \in(X, Y) = \in(X, Y, \underline{l}, l) & \forall X, Y \in \mathbf{O}(\mathbf{N}). \end{cases} \tag{A.1}$$

where $\in$ denotes the standard volume form on $\mathbf{N}$. If $(e_a)_{a=1,2}$ is an orthonormal basis of horizontal vector-fields, i.e. $\gamma(e_a, e_b) = \delta_{ab}$, we write $\in_{ab} = \in(e_a, e_b)$ and without loss of generality we assume that $\in_{12} = 1$.

In general the commutator $[X, Y]$ of two horizontal vector-fields may fail to be horizontal. We say that the pair $(l, \underline{l})$ is *integrable* if the set of

[11] As before, $\mathbf{N}$ is assumed to be a connected, orientable, paracompact C^∞ manifold without boundary.

horizontal vector-fields forms an integrable distribution, i.e. $X, Y \in \mathbf{O}(\mathbf{N})$ implies that $[X, Y] \in \mathbf{O}(\mathbf{N})$. For any vector-field $X \in \mathbf{T}(\mathbf{N})$ we define its horizontal projection

$$^{(h)}X = X + \mathbf{g}(X, \underline{l})l + \mathbf{g}(X, l)\underline{l}.$$

Using this projection we define the horizontal covariant derivative $\nabla_X Y$, $X \in \mathbf{T}(\mathbf{N})$, $Y \in \mathbf{O}(\mathbf{N})$,

$$\nabla_X Y = {}^{(h)}(\mathbf{D}_X Y) = \mathbf{D}_X Y - g(\mathbf{D}_X \underline{l}, Y)l - g(\mathbf{D}_X l, Y)\underline{l}.$$

The definition shows easily that,

$$\begin{cases} \nabla_{fX+f'X'} Y = f\nabla_X Y + f'\nabla_{X'} Y; \\ \nabla_X (fY + f'Y') = f\nabla_X Y + X(f)Y + f'\nabla_X Y' + X(f')Y'; \\ X\gamma(Y, Y') = \gamma(\nabla_X Y, Y') + \gamma(Y, \nabla_X Y'), \end{cases} \tag{A.2}$$

for any $X, X' \in \mathbf{T}(\mathbf{N})$, $Y, Y' \in \mathbf{O}(\mathbf{N})$, $f, f' \in C^\infty(\mathbf{N})$. In particular we see that ∇ is compatible with the horizontal metric γ.

In what follows we identify covariant and contravariant horizontal tensor-fields using the induced metric $^{(h)}\gamma$. For any $k \in \mathbb{Z}_+$ let $\mathbf{O}_k(\mathbf{N})$ denote the vector space of k *horizontal tensor-fields*

$$U : \mathbf{O}(\mathbf{N}) \times \ldots \times \mathbf{O}(\mathbf{N}) \to \mathbb{C}.$$

Given a horizontal tensor-field $U \in \mathbf{O}_k(\mathbf{N})$ and $X \in \mathbf{T}(\mathbf{N})$ we define the covariant derivative $\nabla_X U \in \mathbf{O}_k(\mathbf{N})$ by the formula

$$\begin{aligned} \nabla_X U(Y_1, \ldots, Y_k) = X(U(Y_1, \ldots, Y_k)) - U(\nabla_X Y_1, \ldots, Y_k) \\ - \ldots - U(Y_1, \ldots, \nabla_X Y_k). \end{aligned} \tag{A.3}$$

According to the definition the mapping $(X, Y_1, \ldots, Y_k) \to \nabla_X U(Y_1, \ldots, Y_k)$ is a multilinear mapping on $\mathbf{T}(\mathbf{N}) \times \mathbf{O}(\mathbf{N}) \times \ldots \times \mathbf{O}(\mathbf{N})$.

We define the null second fundamental forms $^{(h)}\chi$, $^{(h)}\underline{\chi} \in \mathbf{O}_2(\mathbf{N})$ by

$$\begin{cases} ^{(h)}\chi(X, Y) = g(\mathbf{D}_X l, Y), \\ ^{(h)}\underline{\chi}(X, Y) = g(\mathbf{D}_X \underline{l}, Y). \end{cases} \tag{A.4}$$

Observe that $^{(h)}\chi$ and $^{(h)}\underline{\chi}$ are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas,

$$\begin{aligned} ^{(h)}\chi(X, Y) - {}^{(h)}\chi(Y, X) &= \mathbf{g}(\mathbf{D}_X l, Y) - \mathbf{g}(\mathbf{D}_Y l, X) = -\mathbf{g}(l, [X, Y]) \\ ^{(h)}\underline{\chi}(X, Y) - {}^{(h)}\underline{\chi}(Y, X) &= \mathbf{g}(\mathbf{D}_X \underline{l}, Y) - \mathbf{g}(\mathbf{D}_Y \underline{l}, X) = -\mathbf{g}(\underline{l}, [X, Y]). \end{aligned}$$

The trace of an horizontal 2-tensor U is defined according to

$$\mathrm{tr}(U) := \delta^{ab} U_{ab}$$

where $(e_a)_{a=1,2}$ is an arbitrary orthonormal frame of horizontal vector-fields. Observe that the definition does not depend on the particular frame.

We denote by $\mathrm{tr}\,\chi$ and $\mathrm{tr}\,\underline{\chi}$ the traces of ${}^{(h)}\chi$ and ${}^{(h)}\underline{\chi}$. If $U \in \mathbf{O}_k(\mathbf{N})$ with $k = 1, 2$ we define its dual, expressed relative to an arbitrary orthonormal frame $(e_a)_{a=1,2} \in \mathbf{O}(\mathbf{N})$,

$$ {}^*U_a = \in_{ab} U_b, \quad {}^*U_{ab} = \in_{ac} U_{cb}. $$

Clearly ${}^*({}^*\omega) = -\omega$. If $\omega \in \mathbf{O}(\mathbf{N})_2$ is symmetric traceless then so is its dual ${}^*\omega$.

We define also the horizontal 1-forms ${}^{(h)}\xi, {}^{(h)}\underline{\xi}, {}^{(h)}\eta, {}^{(h)}\underline{\eta}, {}^{(h)}\zeta \in \mathbf{O}_1(\mathbf{N})$ by

$$ \begin{cases} {}^{(h)}\xi(X) = \mathbf{g}(\mathbf{D}_l l, X), & {}^{(h)}\underline{\xi}(X) = \mathbf{g}(\mathbf{D}_{\underline{l}} \underline{l}, X), \\ {}^{(h)}\eta(X) = \mathbf{g}(\mathbf{D}_{\underline{l}} l, X), & {}^{(h)}\underline{\eta}(X) = \mathbf{g}(\mathbf{D}_l \underline{l}, X), \\ {}^{(h)}\zeta(X) = \mathbf{g}(\mathbf{D}_X l, \underline{l}), \end{cases} \tag{A.5} $$

and the real scalars

$$ \omega = \mathbf{g}(\mathbf{D}_l l, \underline{l}), \quad \underline{\omega} = \mathbf{g}(\mathbf{D}_{\underline{l}} \underline{l}, l). \tag{A.6} $$

Assume that $W \in \mathbf{T}_4^0(\mathbf{N})$ is a Weyl field, i.e.

$$ \begin{cases} W_{\alpha\beta\mu\nu} = -W_{\beta\alpha\mu\nu} = -W_{\alpha\beta\nu\mu} = W_{\mu\nu\alpha\beta}; \\ W_{\alpha\beta\mu\nu} + W_{\alpha\mu\nu\beta} + W_{\alpha\nu\beta\mu} = 0; \\ \mathbf{g}^{\beta\nu} W_{\alpha\beta\mu\nu} = 0. \end{cases} \tag{A.7} $$

We define the null components of the Weyl field W, $\alpha(W), \underline{\alpha}(W), \varrho(W) \in \mathbf{O}_2(\mathbf{N})$ and $\beta(W), \underline{\beta}(W) \in \mathbf{O}_1(\mathbf{N})$ by the formulas

$$ \begin{cases} \alpha(W)(X, Y) = W(l, X, l, Y), \\ \underline{\alpha}(W)(X, Y) = W(\underline{l}, X, \underline{l}, Y), \\ \beta(W)(X) = W(X, l, \underline{l}, l), \\ \underline{\beta}(W)(X) = W(X, \underline{l}, l, \underline{l}), \\ \varrho(W)(X, Y) = W(X, \underline{l}, Y, l). \end{cases} \tag{A.8} $$

Recall that if W is a Weyl field its Hodge dual ${}^*W$, defined by ${}^*W_{\alpha\beta\mu\nu} = \frac{1}{2} \in_{\mu\nu}{}^{\rho\sigma} W_{\alpha\beta\rho\sigma}$, is also a Weyl field. We easily check the formulas,

$$ \begin{cases} \underline{\alpha}({}^*W) = {}^*\underline{\alpha}(W), & \alpha({}^*W) = -{}^*\alpha(W) \\ \underline{\beta}({}^*W) = {}^*\underline{\beta}(W), & \beta({}^*W) = -{}^*\beta(W) \\ \varrho({}^*W) = {}^*\varrho(W). \end{cases} \tag{A.9} $$

It is easy to check that $\alpha, \underline{\alpha}$ are symmetric traceless horizontal tensor-fields in $\mathbf{O}_2(\mathbf{N})$. On the other hand $\varrho \in \mathbf{O}_2(\mathbf{N})$ is however neither symmetric nor traceless. It is convenient to express it in terms of the following two scalar quantities,

$$ \rho(W) = W(l, \underline{l}, l, \underline{l}), \quad {}^*\rho(W) = {}^*W(l, \underline{l}, l, \underline{l}). \tag{A.10} $$

Observe also that,

$$\rho({}^*W) = {}^*\rho(W), \quad {}^*\rho({}^*W) = -\rho.$$

Thus,

$$\varrho(X,Y) = \frac{1}{2}(-\rho\gamma(X,Y) + {}^*\rho\in(X,Y)), \quad \forall X, Y \in \mathbf{O}(\mathbf{N}). \tag{A.11}$$

We have,

$$W(X,Y,\underline{l},l) = \varrho(W)(X,Y) - \varrho(W)(Y,X) = {}^*\rho(W)\in(X,Y).$$

Also, since ${}^*({}^*W) = -W$, we deduce that

$$W(X,Y,X',Y') = \in(X,Y){}^*W(X',Y',\underline{l},l) = \in(X,Y)\in(X',Y'){}^*\rho({}^*W).$$

Therefore,

$$\begin{cases} W(X,Y,\underline{l},l) = \in(X,Y){}^*\rho(W) \\ W(X,Y,X',Y') = -\in(X,Y)\in(X',Y')\rho(W) \\ W(X,Y,Z,\underline{l}) = \in(X,Y)\underline{\beta}(W)(Z). \end{cases}$$

We also consider the case of a self-dual Weyl field $\mathcal{W} = W + i{}^*W$, i.e. ${}^*\mathcal{W} = -i\mathcal{W}$. Defining the null decomposition $\underline{\alpha}(\mathcal{W})$, $\underline{\beta}(\mathcal{W})$, $\rho(\mathcal{W})$, ${}^*\rho(\mathcal{W})$, $\beta(\mathcal{W})$, $\alpha(\mathcal{W})$ as in (A.8) and (A.10) and setting ${}^*\rho(\mathcal{W}) := \rho({}^*\mathcal{W})$ as in (A.10), we find,

$${}^*\rho(\mathcal{W}) = -i\rho(\mathcal{W}).$$

Relative to a null frame $e_1, e_2, e_3 = \underline{l}, l = e_4$ we have,

$$\mathcal{W}_{ab34} = -i\in_{ab}\rho(\mathcal{W}), \quad \mathcal{W}_{abcd} = -\in_{ab}\in_{cd}\rho(\mathcal{W}), \quad \mathcal{W}_{abc3} = \in_{ab}\underline{\beta}_c(\mathcal{W}). \tag{A.12}$$

A.2. Complex null tetrads. We extend by linearity the definition of horizontal vector-fields to complex ones. We say that a complex vector-field m on $\mathbf{N}$ is *compatible* with the null pair $(\underline{l}, l)$ if,

$$\mathbf{g}(l,m) = \mathbf{g}(\underline{l},m) = \mathbf{g}(m,m) = 0, \quad \mathbf{g}(m,\overline{m}) = 1.$$

In that case we say that $(m, \overline{m}, \underline{l}, l)$ forms *a complex null tetrad*. Clearly m is compatible if and only if $m = \frac{1}{\sqrt{2}}(X + iY)$ for some real vectors $X, Y \in \mathbf{O}(\mathbf{N})$ with $g(X,Y) = 0$, $g(X,X) = g(Y,Y) = 1$. Given a compatible vector-field m and ${}^{(h)}U \in \mathbf{O}_1(\mathbf{N})$ we can define the complex scalar $U_1 : \mathbf{N} \to \mathbb{C}$,

$$U_1 = {}^{(h)}U(m).$$

Similarly, given ${}^{(h)}V \in \mathbf{O}_2(\mathbf{N})$ we can define the complex scalars V_{21}, V_{11} : $\mathbf{N} \to \mathbb{C}$,

$$V_{21} = {}^{(h)}V(\overline{m}, m), \quad V_{11} = {}^{(h)}V(m, m).$$

The complex scalars U_1, respectively V_{21} and V_{11}, determine uniquely the real horizontal tensors fields ${}^{(h)}U$ and ${}^{(h)}V$ respectively.

Given a compatible vector-field m we define (compare with (A.4)–(A.6))

$$\begin{aligned}
&\theta = {}^{(h)}\chi(\overline{m}, m) = \mathbf{g}(\mathbf{D}_{\overline{m}} l, m), && \underline{\theta} = {}^{(h)}\underline{\chi}(\overline{m}, m) = \mathbf{g}(\mathbf{D}_{\overline{m}} \underline{l}, m),\\
&\vartheta = {}^{(h)}\chi(m, m) = \mathbf{g}(\mathbf{D}_m l, m), && \underline{\vartheta} = {}^{(h)}\underline{\chi}(m, m) = \mathbf{g}(\mathbf{D}_m \underline{l}, m),\\
&\xi = {}^{(h)}\xi(m) = \mathbf{g}(\mathbf{D}_l l, m), && \underline{\xi} = {}^{(h)}\underline{\xi}(m) = \mathbf{g}(\mathbf{D}_{\underline{l}} \underline{l}, m),\\
&\eta = {}^{(h)}\eta(m) = \mathbf{g}(\mathbf{D}_{\underline{l}} l, m), && \underline{\eta} = {}^{(h)}\underline{\eta}(m) = \mathbf{g}(\mathbf{D}_l \underline{l}, m),\\
&\omega = \mathbf{g}(\mathbf{D}_l l, \underline{l}), && \underline{\omega} = \mathbf{g}(\mathbf{D}_{\underline{l}} \underline{l}, l),\\
&\zeta = {}^{(h)}\zeta(m) = \mathbf{g}(\mathbf{D}_m l, \underline{l}).
\end{aligned} \tag{A.13}$$

The complex scalars $\theta, \underline{\theta}, \vartheta, \underline{\vartheta}, \xi, \underline{\xi}, \eta, \underline{\eta}, \zeta$ and the real scalars $\omega, \underline{\omega}$ are the main connection coefficients of the null tetrad.

Similarly, given a real-valued Weyl field W we define (compare with (A.8))

$$\begin{cases}
\Psi_{(2)} = \Psi_{(2)}(W) = \alpha(W)(m, m) = W(l, m, l, m),\\
\underline{\Psi}_{(2)} = \underline{\Psi}_{(2)}(W) = \underline{\alpha}(W)(m, m) = W(\underline{l}, m, \underline{l}, m),\\
\Psi_{(1)} = \Psi_{(1)}(W) = \beta(W)(m) = W(m, l, \underline{l}, l),\\
\underline{\Psi}_{(1)} = \underline{\Psi}_{(1)}(W) = \underline{\beta}(W)(m) = W(m, \underline{l}, l, \underline{l}),\\
\Psi_{(0)} = \Psi_{(0)}(W) = \varrho(W)(\overline{m}, m) = W(\overline{m}, \underline{l}, m, l).
\end{cases} \tag{A.14}$$

Notice that, in view of (A.7), $\alpha(W)(\overline{m}, m) = \underline{\alpha}(W)(\overline{m}, m) = \varrho(W)(m, m) = 0$, so the scalars $\Psi_{(2)}, \underline{\Psi}_{(2)}, \Psi_{(1)}, \underline{\Psi}_{(1)}, \Psi_{(0)}$ uniquely determine the real-valued Weyl field W. In addition, if

$$^*W_{\alpha\beta\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\sigma} W_{\alpha\beta\rho\sigma}$$

is the dual dual of W, and the null tetrad $(m, \overline{m}, \underline{l}, l)$ has positive orientation (i.e. $\in_{\alpha\beta\mu\nu} m^\alpha \overline{m}^\beta \underline{l}^\mu l^\nu = i$) then

$$\begin{aligned}
&\Psi_2(^*W) = (-i)\Psi_2(W), \quad \Psi_1(^*W) = (-i)\Psi_1(W),\\
&\Psi_0(^*W) = (-i)\Psi_0(W),\\
&\underline{\Psi}_{(2)}(^*W) = i\underline{\Psi}_2(W), \quad \underline{\Psi}_{(1)}(^*W) = i\underline{\Psi}_1(W).
\end{aligned} \tag{A.15}$$

In what follows we denote,

$$e_1 = m, \quad e_2 = \overline{m}, \quad e_3 = \underline{l}, \quad e_4 = l.$$

We define the connection coefficients ${\Gamma^\mu}_{\alpha\beta}$, $\Gamma_{\mu\alpha\beta}$ by the formulas

$$\mathbf{D}_{e_\beta} e_\alpha = {\Gamma^\mu}_{\alpha\beta} e_\mu$$

and

$$\Gamma_{\mu\alpha\beta} = \mathbf{g}_{\mu\nu} {\Gamma^\nu}_{\alpha\beta} = \mathbf{g}(e_\mu, \mathbf{D}_{e_\beta} e_\alpha).$$

Clearly

$$\Gamma_{\mu\alpha\beta} + \Gamma_{\alpha\mu\beta} = 0.$$

We easily check the formulas,

$$\begin{aligned}
&\Gamma_{144} = \xi, &&\Gamma_{244} = \overline{\xi}, &&\Gamma_{133} = \underline{\xi}, &&\Gamma_{233} = \overline{\underline{\xi}},\\
&\Gamma_{143} = \eta, &&\Gamma_{243} = \overline{\eta}, &&\Gamma_{134} = \underline{\eta}, &&\Gamma_{234} = \overline{\underline{\eta}},\\
&\Gamma_{142} = \theta, &&\Gamma_{241} = \overline{\theta}, &&\Gamma_{132} = \underline{\theta}, &&\Gamma_{231} = \overline{\underline{\theta}},\\
&\Gamma_{141} = \vartheta, &&\Gamma_{242} = \overline{\vartheta}, &&\Gamma_{131} = \underline{\vartheta}, &&\Gamma_{232} = \overline{\underline{\vartheta}},\\
&\Gamma_{344} = \omega, &&\Gamma_{433} = \underline{\omega}, &&\Gamma_{341} = \zeta, &&\Gamma_{342} = \overline{\zeta}.
\end{aligned} \tag{A.16}$$

Using the definition (A.3) we see easily that if ${}^{(h)}U \in \mathbf{O}_1(\mathbf{N})$, ${}^{(h)}V \in \mathbf{O}_2(\mathbf{N})$, and $\alpha \in \{1, 2, 3, 4\}$ then

$$\nabla_\alpha {}^{(h)}U_1 = (e_\alpha + \Gamma_{12\alpha})\big({}^{(h)}U_1\big), \tag{A.17}$$

and

$$\nabla_\alpha {}^{(h)}V_{11} = (e_\alpha + 2\Gamma_{12\alpha})\big({}^{(h)}V_{11}\big), \quad \nabla_\alpha {}^{(h)}V_{21} = e_\alpha\big({}^{(h)}V_{21}\big). \tag{A.18}$$

A.3. The null structure equations and the Bianchi identities. We define

$$D = l = e_4, \quad \underline{D} = \underline{l} = e_3, \quad \delta = m = e_1, \quad \overline{\delta} = \overline{m} = e_2.$$

Let R denote the Riemann curvature tensor on $\mathbf{M}$. We compute

$$\begin{aligned}
R_{\alpha\beta\mu\nu} &= \mathbf{g}(e_\alpha, [\mathbf{D}_{e_\mu}(\mathbf{D}_{e_\nu} e_\beta) - \mathbf{D}_{e_\nu}(\mathbf{D}_{e_\mu} e_\beta) - \mathbf{D}_{[e_\mu, e_\nu]} e_\beta])\\
&= \mathbf{g}\big(e_\alpha, \big[\mathbf{D}_{e_\mu}\big({\Gamma^\rho}_{\beta\nu} e_\rho\big) - \mathbf{D}_{e_\nu}\big({\Gamma^\rho}_{\beta\mu} e_\rho\big) - \big({\Gamma^\rho}_{\nu\mu} - {\Gamma^\rho}_{\mu\nu}\big)\mathbf{D}_{e_\rho} e_\beta\big]\big)\\
&= e_\mu(\Gamma_{\alpha\beta\nu}) - e_\nu(\Gamma_{\alpha\beta\mu}) + {\Gamma^\rho}_{\beta\nu}\Gamma_{\alpha\rho\mu} - {\Gamma^\rho}_{\beta\mu}\Gamma_{\alpha\rho\nu}\\
&\quad + \big({\Gamma^\rho}_{\mu\nu} - {\Gamma^\rho}_{\nu\mu}\big)\Gamma_{\alpha\beta\rho}.
\end{aligned}$$

Using this formula and the table (A.16) we derive the null structure equations. Using $R_{1441} = -\Psi_{(2)}(R)$ we derive

$$(D + 2\Gamma_{124})\vartheta - (\delta + \Gamma_{121})\xi = \xi(2\zeta + \eta + \underline{\eta}) - \vartheta(\omega + \theta + \overline{\theta}) - \Psi_{(2)}(R). \tag{A.19}$$

Using $R_{1331} = -\underline{\Psi}_2(R)$ we derive

$$\begin{aligned}&(\underline{D}+2\Gamma_{123})\underline{\vartheta}-(\delta+\Gamma_{121})\underline{\xi}\\ &=\underline{\xi}(-2\zeta+\underline{\eta}+\eta)-\underline{\vartheta}(\underline{\omega}+\underline{\theta}+\underline{\bar{\theta}})-\underline{\Psi}_{(2)}(R).\end{aligned} \tag{A.20}$$

Using $R_{1442} = 0$ we derive

$$D\theta-(\bar{\delta}+\Gamma_{122})\xi=-\theta^2-\omega\theta-\vartheta\bar{\vartheta}+\bar{\xi}\eta+\xi(2\bar{\zeta}+\underline{\bar{\eta}}). \tag{A.21}$$

Using $R_{1332} = 0$ we derive

$$\underline{D}\underline{\theta}-(\bar{\delta}+\Gamma_{122})\underline{\xi}=-\underline{\theta}^2-\underline{\omega}\,\underline{\theta}-\underline{\vartheta}\,\underline{\bar{\vartheta}}+\underline{\bar{\xi}}\,\underline{\eta}+\underline{\xi}(-2\bar{\zeta}+\bar{\eta}). \tag{A.22}$$

Using $R_{1443} = -\Psi_{(1)}(R)$ we derive

$$(D+\Gamma_{124})\eta-(\underline{D}+\Gamma_{123})\xi=-2\underline{\omega}\xi+\theta(\underline{\eta}-\eta)+\vartheta(\underline{\bar{\eta}}-\bar{\eta})-\Psi_{(1)}(R). \tag{A.23}$$

Using $R_{1334} = -\underline{\Psi}_{(1)}(R)$ we derive

$$(\underline{D}+\Gamma_{123})\underline{\eta}-(D+\Gamma_{124})\underline{\xi}=-2\omega\underline{\xi}+\underline{\theta}(\eta-\underline{\eta})+\underline{\vartheta}(\bar{\eta}-\underline{\bar{\eta}})-\underline{\Psi}_{(1)}(R). \tag{A.24}$$

Using $R_{1431} = 0$ we derive

$$(\underline{D}+2\Gamma_{123})\vartheta-(\delta+\Gamma_{121})\eta=\eta^2+\underline{\xi}\xi-\underline{\vartheta}\theta+\vartheta(\underline{\omega}-\underline{\bar{\theta}}). \tag{A.25}$$

Using $R_{1341} = 0$ we derive

$$(D+2\Gamma_{124})\underline{\vartheta}-(\delta+\Gamma_{121})\underline{\eta}=\underline{\eta}^2+\xi\underline{\xi}-\vartheta\underline{\theta}+\underline{\vartheta}(\omega-\bar{\theta}). \tag{A.26}$$

Using $R_{1432} = -\Psi_{(0)}(R)$ we derive

$$\underline{D}\theta-(\bar{\delta}+\Gamma_{122})\eta=\xi\underline{\bar{\xi}}+\eta\bar{\eta}-\vartheta\underline{\bar{\vartheta}}+\theta(\underline{\omega}-\underline{\theta})-\Psi_{(0)}(R). \tag{A.27}$$

Using $R_{1342} = -\overline{\Psi_{(0)}(R)}$ we derive

$$D\underline{\theta}-(\bar{\delta}+\Gamma_{122})\underline{\eta}=\underline{\xi}\bar{\xi}+\underline{\eta}\,\underline{\bar{\eta}}-\underline{\vartheta}\bar{\vartheta}+\underline{\theta}(\omega-\theta)-\overline{\Psi_{(0)}(R)}. \tag{A.28}$$

Using $R_{1421} = -\Psi_{(1)}(R)$ we derive

$$(\bar{\delta}+2\Gamma_{122})\vartheta-\delta\theta=\zeta\theta-\bar{\zeta}\vartheta+\eta(\theta-\bar{\theta})+\xi(\underline{\theta}-\underline{\bar{\theta}})-\Psi_{(1)}(R). \tag{A.29}$$

Using $R_{1321} = -\underline{\Psi}_{(1)}(R)$ we derive

$$(\bar{\delta}+2\Gamma_{122})\underline{\vartheta}-\delta\underline{\theta}=-\zeta\underline{\theta}+\bar{\zeta}\underline{\vartheta}+\underline{\eta}(\underline{\theta}-\underline{\bar{\theta}})+\underline{\xi}(\theta-\bar{\theta})-\underline{\Psi}_{(1)}(R). \tag{A.30}$$

Using $R_{3441} = -\Psi_{(1)}(R)$ we derive

$$\begin{aligned}(D+\Gamma_{124})\zeta - \delta\omega = \omega(\zeta + \underline{\eta}) + \overline{\theta}(\underline{\eta} - \zeta) + \vartheta(\overline{\underline{\eta}} - \overline{\zeta}) \\ - \xi(\overline{\underline{\theta}} + \underline{\omega}) - \overline{\xi}\underline{\vartheta} - \Psi_{(1)}(R).\end{aligned} \tag{A.31}$$

Using $R_{4331} = -\underline{\Psi}_{(1)}(R)$ we derive

$$\begin{aligned}(\underline{D}+\Gamma_{123})(-\zeta) - \delta\underline{\omega} = \underline{\omega}(-\zeta + \eta) + \overline{\underline{\theta}}(\eta + \zeta) + \underline{\vartheta}(\overline{\eta} + \overline{\zeta}) \\ - \underline{\xi}(\overline{\theta} + \omega) - \overline{\underline{\xi}}\vartheta - \underline{\Psi}_{(1)}(R).\end{aligned} \tag{A.32}$$

Using $R_{3443} = \Psi_{(0)}(R) + \overline{\Psi_{(0)}(R)}$ we derive

$$\begin{aligned}D\underline{\omega} + \underline{D}\omega = \overline{\xi}\underline{\xi} + \xi\overline{\underline{\xi}} - \overline{\eta}\underline{\eta} - \eta\overline{\underline{\eta}} + \zeta(\overline{\eta} - \overline{\underline{\eta}}) + \overline{\zeta}(\eta - \underline{\eta}) \\ - (\Psi_{(0)}(R) + \overline{\Psi_{(0)}(R)}).\end{aligned} \tag{A.33}$$

Using $R_{3421} = \Psi_{(0)}(R) - \overline{\Psi_{(0)}(R)}$ we derive

$$\begin{aligned}(\delta - \Gamma_{121})\overline{\zeta} - (\overline{\delta} + \Gamma_{122})\zeta = (\overline{\vartheta}\underline{\vartheta} - \vartheta\overline{\underline{\vartheta}}) + (\theta\overline{\underline{\theta}} - \overline{\theta}\underline{\theta}) + \underline{\omega}(\theta - \overline{\theta}) \\ - \omega(\underline{\theta} - \overline{\underline{\theta}}) - (\Psi_{(0)}(R) - \overline{\Psi_{(0)}(R)}).\end{aligned} \tag{A.34}$$

We derive now the Bianchi identities. Assume W is a real-valued Weyl field, see (A.7), and

$$\mathbf{D}^{\alpha} W_{\alpha\beta\mu\nu} = J_{\beta\mu\nu},$$

for some Weyl current $J \in \mathbf{T}_3^0(\mathbf{M})$. Then, using Proposition 4.1,

$$\mathbf{D}_{[\rho} W_{\alpha\beta]\mu\nu} = \mathbf{D}_{\rho} W_{\alpha\beta\mu\nu} + \mathbf{D}_{\alpha} W_{\beta\rho\mu\nu} + \mathbf{D}_{\beta} W_{\rho\alpha\mu\nu} = \in_{\sigma\rho\alpha\beta} {}^{*}J^{\sigma}{}_{\mu\nu}, \tag{A.35}$$

where

$${}^{*}J^{\sigma}{}_{\mu\nu} = \frac{1}{2} \in_{\mu\nu}{}^{\gamma\delta} J^{\sigma}{}_{\gamma\delta}.$$

Using (A.7), we derive the following

$$\begin{aligned}
&W_{3141} = W_{3242} = W_{4241} = W_{3231} = 0,\\
&W_{4141} = \Psi_{(2)}, \quad W_{4242} = \overline{\Psi_{(2)}}, \quad W_{3131} = \underline{\Psi}_{(2)}, \quad W_{3232} = \overline{\underline{\Psi}_{(2)}},\\
&W_{2314} = \Psi_{(0)}, \quad W_{1324} = \overline{\Psi_{(0)}}, \quad W_{4343} = W_{1212} = -\Psi_{(0)} - \overline{\Psi_{(0)}},\\
&W_{1234} = \overline{\Psi_{(0)}} - \Psi_{(0)},\\
&W_{1434} = W_{2141} = \Psi_{(1)}, \quad W_{2434} = W_{1242} = \overline{\Psi_{(1)}},\\
&W_{1343} = W_{2131} = \underline{\Psi}_{(1)}, \quad W_{2343} = W_{1232} = \overline{\underline{\Psi}_{(1)}}.
\end{aligned} \tag{A.36}$$

We use the table (A.36) and the formula (A.35) to derive the Bianchi identities. Using $\mathbf{D}_{[2} W_{41]41} = -J_{414}$ we derive

$$\begin{aligned}&(\overline{\delta} + 2\Gamma_{122})\Psi_{(2)} - (D + \Gamma_{124})\Psi_{(1)} \\ &= -(2\overline{\zeta} + \underline{\eta})\Psi_{(2)} + (4\theta + \omega)\Psi_{(1)} + 3\xi\Psi_{(0)} - J_{414}.\end{aligned} \tag{A.37}$$

Using $\mathbf{D}_{[2}W_{31]31} = -J_{313}$ we derive

$$\begin{aligned}&(\overline{\delta} + 2\Gamma_{122})\underline{\Psi}_{(2)} - (\underline{D} + \Gamma_{123})\underline{\Psi}_{(1)}\\ &= -(-2\overline{\zeta} + \overline{\eta})\underline{\Psi}_{(2)} + (4\underline{\theta} + \underline{\omega})\underline{\Psi}_{(1)} + 3\underline{\xi}\overline{\Psi_{(0)}} - J_{313}.\end{aligned} \tag{A.38}$$

Using $\mathbf{D}_{[3}W_{41]41} = J_{114}$ we derive

$$\begin{aligned}&(\underline{D} + 2\Gamma_{123})\Psi_{(2)} - (\delta + \Gamma_{121})\Psi_{(1)}\\ &= (2\underline{\omega} - \overline{\underline{\theta}})\Psi_{(2)} + (\zeta + 4\eta)\Psi_{(1)} + 3\vartheta\Psi_{(0)} + J_{114}.\end{aligned} \tag{A.39}$$

Using $\mathbf{D}_{[4}W_{31]31} = J_{113}$ we derive

$$\begin{aligned}&(D + 2\Gamma_{124})\underline{\Psi}_{(2)} - (\delta + \Gamma_{121})\underline{\Psi}_{(1)}\\ &= (2\omega - \overline{\theta})\underline{\Psi}_{(2)} + (-\zeta + 4\underline{\eta})\underline{\Psi}_{(1)} + 3\underline{\vartheta}\overline{\Psi_{(0)}} + J_{113}.\end{aligned} \tag{A.40}$$

Using $\mathbf{D}_{[2}W_{34]41} = -J_{214}$ we derive

$$\begin{aligned}&- D\Psi_{(0)} - (\overline{\delta} + \Gamma_{122})\Psi_{(1)}\\ &= -\overline{\underline{\vartheta}}\Psi_{(2)} + (2\overline{\underline{\eta}} + \overline{\zeta})\Psi_{(1)} + 3\theta\Psi_{(0)} + 2\xi\overline{\underline{\Psi}_{(1)}} - J_{214}.\end{aligned} \tag{A.41}$$

Using $\mathbf{D}_{[2}W_{43]31} = -J_{213}$ we derive

$$\begin{aligned}&- \underline{D}\overline{\Psi_{(0)}} - (\overline{\delta} + \Gamma_{122})\underline{\Psi}_{(1)}\\ &= -\overline{\vartheta}\underline{\Psi}_{(2)} + (2\overline{\eta} - \overline{\zeta})\underline{\Psi}_{(1)} + 3\underline{\theta}\overline{\Psi_{(0)}} + 2\underline{\xi}\overline{\Psi_{(1)}} - J_{213}.\end{aligned} \tag{A.42}$$

Using $\mathbf{D}_{[1}W_{42]31} = J_{413}$ we derive

$$\begin{aligned}&\delta\overline{\Psi_{(0)}} + (D + \Gamma_{124})\underline{\Psi}_{(1)}\\ &= -2\underline{\vartheta}\overline{\Psi_{(1)}} - 3\underline{\eta}\overline{\Psi_{(0)}} + (\omega - 2\overline{\theta})\underline{\Psi}_{(1)} + \overline{\xi}\underline{\Psi}_{(2)} + J_{413}.\end{aligned} \tag{A.43}$$

Using $\mathbf{D}_{[1}W_{32]41} = J_{314}$ we derive

$$\begin{aligned}&\delta\Psi_{(0)} + (\underline{D} + \Gamma_{123})\Psi_{(1)}\\ &= -2\vartheta\overline{\underline{\Psi}}_1 - 3\eta\Psi_{(0)} + (\underline{\omega} - 2\overline{\underline{\theta}})\Psi_{(1)} + \overline{\underline{\xi}}\Psi_{(2)} + J_{314}.\end{aligned} \tag{A.44}$$

A.4. Symmetries of the formalism. We discuss now the main symmetries of the formalism introduced in this section.

1. Interchange of the vectors l and $\underline{l}$. We define the complex tetrad $(m', \overline{m'}, \underline{l}', l')$,

$$e_1' = m' = m, \quad e_2' = \overline{m'} = \overline{m}, \quad e_3' = \underline{l}' = l, \quad e_4' = l' = \underline{l}. \tag{A.45}$$

Using this new complex tetrad we define the scalars θ', $\underline{\theta}'$, ϑ', $\underline{\vartheta}'$, ξ', $\underline{\xi}'$, η', $\underline{\eta}'$, ω', $\underline{\omega}'$, ζ' as in (A.13). Given a real-valued Weyl field W, we define the scalars $\Psi'_{(2)}, \underline{\Psi}'_{(2)}, \Psi'_{(1)}, \underline{\Psi}'_{(1)}, \Psi'_{(0)}$ as in (A.14). We define the connection coefficients $\Gamma'_{\mu\alpha\beta} = g(e'_\mu, \mathbf{D}_{e'_\beta} e'_\alpha)$. The definitions show easily that

$$\begin{aligned}
&\theta' = \underline{\theta}, \quad \underline{\theta}' = \theta, \quad \vartheta' = \underline{\vartheta}, \quad \underline{\vartheta}' = \vartheta, \quad \xi' = \underline{\xi}, \quad \underline{\xi}' = \xi,\\
&\eta' = \underline{\eta}, \quad \underline{\eta}' = \eta, \quad \omega' = \underline{\omega}, \quad \underline{\omega}' = \omega, \quad \zeta' = -\zeta,\\
&\Psi'_{(2)} = \underline{\Psi}_{(2)}, \quad \underline{\Psi}'_{(2)} = \Psi_{(2)}, \quad \Psi'_{(1)} = \underline{\Psi}_{(1)}, \quad \underline{\Psi}'_{(1)} = \Psi_{(1)}, \quad \Psi'_{(0)} = \overline{\Psi_{(0)}},\\
&\delta' = \delta, \quad \overline{\delta'} = \overline{\delta}, \quad \underline{D}' = D, \quad D' = \underline{D},\\
&\Gamma'_{121} = \Gamma_{121}, \quad \Gamma'_{122} = \Gamma_{122}, \quad \Gamma'_{123} = \Gamma_{124}, \quad \Gamma'_{124} = \Gamma_{123}.
\end{aligned} \tag{A.46}$$

The Ricci equations (A.19)–(A.34) and the Bianchi identities (A.37)–(A.44) are invariant with respect to the transformation (A.45). For example, the equation corresponding to (A.19) in the complex tetrad $(m', \overline{m'}, \underline{l}', l')$ reads

$$\begin{aligned}
&(D' + 2\Gamma'_{124})\vartheta' - (\delta' + \Gamma'_{121})\xi'\\
&= \xi'(2\zeta' + \eta' + \underline{\eta}') - \vartheta'(\omega' + \theta' + \overline{\theta'}) - \Psi'_{(2)}(R).
\end{aligned}$$

After using the table (A.46), this is equivalent to

$$\begin{aligned}
&(\underline{D} + 2\Gamma_{123})\underline{\vartheta} - (\delta + \Gamma_{121})\underline{\xi}\\
&= \underline{\xi}(-2\zeta + \underline{\eta} + \eta) - \underline{\vartheta}(\underline{\omega} + \underline{\theta} + \overline{\underline{\theta}}) - \underline{\Psi}_{(2)}(R),
\end{aligned}$$

which is (A.20).

2. Interchange of the vectors m and $\overline{m}$. We define the complex tetrad $(m', \overline{m'}, \underline{l}', l')$,

$$e'_1 = m' = \overline{m}, \quad e'_2 = \overline{m'} = m, \quad e'_3 = \underline{l}' = \underline{l}, \quad e'_4 = l' = l. \tag{A.47}$$

Using this new complex tetrad we define the scalars θ', $\underline{\theta}'$, ϑ', $\underline{\vartheta}'$, ξ', $\underline{\xi}'$, η', $\underline{\eta}'$, ω', $\underline{\omega}'$, ζ' as in (A.13). Given a real-valued Weyl field W, we define the scalars $\Psi'_{(2)}, \underline{\Psi}'_{(2)}, \Psi'_{(1)}, \underline{\Psi}'_{(1)}, \Psi'_{(0)}$ as in (A.14). We define the connection coefficients $\Gamma'_{\mu\alpha\beta} = g(e'_\mu, \mathbf{D}_{e'_\beta} e'_\alpha)$. The definitions show easily that

$$\begin{aligned}
&\theta' = \overline{\theta}, \quad \underline{\theta}' = \overline{\underline{\theta}}, \quad \vartheta' = \overline{\vartheta}, \quad \underline{\vartheta}' = \overline{\underline{\vartheta}}, \quad \xi' = \overline{\xi}, \quad \underline{\xi}' = \overline{\underline{\xi}},\\
&\eta' = \overline{\eta}, \quad \underline{\eta}' = \overline{\underline{\eta}}, \quad \omega' = \omega, \quad \underline{\omega}' = \underline{\omega}, \quad \zeta' = \overline{\zeta},\\
&\Psi'_{(2)} = \overline{\Psi_{(2)}}, \quad \underline{\Psi}'_{(2)} = \overline{\underline{\Psi}_{(2)}}, \quad \Psi'_{(1)} = \overline{\Psi_{(1)}}, \quad \underline{\Psi}'_{(1)} = \overline{\underline{\Psi}_{(1)}}, \quad \Psi'_{(0)} = \overline{\Psi_{(0)}},\\
&\delta' = \overline{\delta}, \quad \overline{\delta'} = \delta, \quad \underline{D}' = \underline{D}, \quad D' = D,\\
&\Gamma'_{121} = \overline{\Gamma_{121}}, \quad \Gamma'_{122} = \overline{\Gamma_{122}}, \quad \Gamma'_{123} = \overline{\Gamma_{123}}, \quad \Gamma'_{124} = \overline{\Gamma_{124}}.
\end{aligned} \tag{A.48}$$

The Ricci equations (A.19)–(A.34) and the Bianchi identities (A.37)–(A.44) are invariant with respect to the transformation (A.47). For example, the equation corresponding to (A.19) in the complex tetrad $(m', \overline{m'}, \underline{l}', l')$ reads

$$\begin{aligned}&\big(D' + 2\Gamma'_{124}\big)\vartheta' - \big(\delta' + \Gamma'_{121}\big)\xi'\\ &= \xi'(2\zeta' + \eta' + \underline{\eta}') - \vartheta'(\omega' + \theta' + \overline{\theta'}) - \Psi'_{(2)}(R).\end{aligned}$$

After using the table (A.48), this is equivalent to

$$(D + 2\overline{\Gamma_{124}})\overline{\vartheta} - (\overline{\delta} + \overline{\Gamma_{121}})\overline{\xi} = \overline{\xi}(2\overline{\zeta} + \overline{\eta} + \underline{\overline{\eta}}) - \overline{\vartheta}(\omega + \overline{\theta} + \theta) - \overline{\Psi_{(2)}(R)},$$

which is equivalent to (A.19) after complex conjugation.

3. Rescaling of the null pair $l, \underline{l}$. We define the complex tetrad $(m', \overline{m'}, \underline{l}', l')$,

$$e'_1 = m' = m, \quad e'_2 = \overline{m'} = \overline{m}, \quad e'_3 = \underline{l}' = A^{-1}\underline{l}, \quad e'_4 = l' = A \cdot l, \tag{A.49}$$

for some smooth function $A : \mathbf{N} \to \mathbb{R} \setminus \{0\}$. Using this new complex tetrad we define the scalars $\theta', \underline{\theta}', \vartheta', \underline{\vartheta}', \xi', \underline{\xi}', \eta', \underline{\eta}', \omega', \underline{\omega}', \zeta'$ as in (A.13). Given a real-valued Weyl field W, we define the scalars $\Psi'_{(2)}, \underline{\Psi}'_{(2)}, \Psi'_{(1)}, \underline{\Psi}'_{(1)}, \Psi'_{(0)}$ as in (A.14). We define the connection coefficients $\Gamma'_{\mu\alpha\beta} = g(e'_\mu, \mathbf{D}_{e'_\beta} e'_\alpha)$. The definitions show easily that

$$\begin{aligned}&\theta' = A\theta, \quad \underline{\theta}' = A^{-1}\underline{\theta}, \quad \vartheta' = A\vartheta, \quad \underline{\vartheta}' = A^{-1}\underline{\vartheta},\\ &\xi' = A^2\xi, \quad \underline{\xi}' = A^{-2}\underline{\xi}, \quad \eta' = \eta, \quad \underline{\eta}' = \underline{\eta},\\ &\Psi'_{(2)} = A^2\Psi_{(2)}, \quad \underline{\Psi}'_{(2)} = A^{-2}\underline{\Psi}_{(2)},\\ &\Psi'_{(1)} = A\Psi_{(1)}, \quad \underline{\Psi}'_{(1)} = A^{-1}\underline{\Psi}_{(1)}, \quad \Psi'_{(0)} = \Psi_{(0)},\\ &\delta' = \delta, \quad \overline{\delta'} = \overline{\delta}, \quad \underline{D}' = A^{-1}\underline{D}, \quad D' = AD,\\ &\Gamma'_{121} = \Gamma_{121}, \quad \Gamma'_{122} = \Gamma_{122},\\ &\omega' = A\omega - D(A), \quad \underline{\omega}' = A^{-1}\underline{\omega} - \underline{D}(A^{-1}), \quad \zeta' = \zeta - \delta(A)/A,\\ &\Gamma'_{123} = A^{-1}\Gamma_{123}, \quad \Gamma'_{124} = A\Gamma_{124}.\end{aligned} \tag{A.50}$$

The Ricci equations (A.19)–(A.34) and the Bianchi identities (A.37)–(A.44) are invariant with respect to the transformation (A.49). For example, the equation corresponding to (A.19) in the complex tetrad $(m', \overline{m'}, \underline{l}', l')$ reads

$$\begin{aligned}&\big(D' + 2\Gamma'_{124}\big)\vartheta' - \big(\delta' + \Gamma'_{121}\big)\xi'\\ &= \xi'(2\zeta' + \eta' + \underline{\eta}') - \vartheta'(\omega' + \theta' + \overline{\theta'}) - \Psi'_{(2)}(R).\end{aligned}$$

After using the table (A.50), this is equivalent to

$$\begin{aligned}&(AD + 2A\Gamma_{124})(A\vartheta) - (\delta + \Gamma_{121})(A^2\xi)\\&= A^2\xi(2\zeta - 2\delta(A)/A + \eta + \underline{\eta})\\&\quad - A\vartheta(A\omega - D(A) + A\theta + A\overline{\theta}) - A^2\Psi_{(2)}(R).\end{aligned}$$

This is equivalent to (A.19), after simplifying the term $AD(A)\vartheta - 2A\delta(A)\xi$ and multiplying by A^{-2}.

4. Rotation of the vector m**.** We define the complex tetrad $(m', \overline{m'}, \underline{l}', l')$,

$$e_1' = m' = Bm, \quad e_2' = \overline{m'} = B^{-1}\overline{m}, \quad e_3' = \underline{l}' = \underline{l}, \quad e_4' = l' = l, \tag{A.51}$$

for some smooth function $B : \mathbf{N} \to \mathbb{C}$, $|B| \equiv 1$. Using this new complex tetrad we define the scalars $\theta', \underline{\theta}', \vartheta', \underline{\vartheta}', \xi', \underline{\xi}', \eta', \underline{\eta}', \omega', \underline{\omega}', \zeta'$ as in (A.13). Given a real-valued Weyl field W, we define the scalars $\Psi'_{(2)}$, $\underline{\Psi}'_{(2)}$, $\Psi'_{(1)}$, $\underline{\Psi}'_{(1)}$, $\Psi'_{(0)}$ as in (A.14). We define the connection coefficients $\Gamma'_{\mu\alpha\beta} = g(e'_\mu, \mathbf{D}_{e'_\beta} e'_\alpha)$. The definitions show easily that

$$\begin{aligned}&\theta' = \theta, \quad \underline{\theta}' = \underline{\theta}, \quad \vartheta' = B^2\vartheta, \quad \underline{\vartheta}' = B^2\underline{\vartheta}, \quad \xi' = B\xi, \quad \underline{\xi}' = B\underline{\xi},\\&\eta' = B\eta, \quad \underline{\eta}' = B\underline{\eta}, \quad \omega' = \omega, \quad \underline{\omega}' = \underline{\omega}, \quad \zeta' = B\zeta,\\&\Psi'_{(2)} = B^2\Psi_{(2)}, \quad \underline{\Psi}'_{(2)} = B^2\underline{\Psi}_{(2)},\\&\Psi'_{(1)} = B\Psi_{(1)}, \quad \underline{\Psi}'_{(1)} = B\underline{\Psi}_{(1)}, \quad \Psi'_{(0)} = \Psi_{(0)},\\&\delta' = B\delta, \quad \overline{\delta'} = B^{-1}\overline{\delta},\\&\Gamma'_{121} = B\Gamma_{121} - \delta(B), \quad \Gamma'_{122} = B^{-1}\Gamma_{122} + \overline{\delta}(B^{-1}),\\&\underline{D}' = \underline{D}, \quad D' = D,\\&\Gamma'_{123} = \Gamma_{123} - \underline{D}(B)/B, \quad \Gamma'_{124} = \Gamma_{124} - D(B)/B.\end{aligned} \tag{A.52}$$

The Ricci equations (A.19)–(A.34) and the Bianchi identities (A.37)–(A.44) are invariant with respect to the transformation (A.51). For example, the equation corresponding to (A.19) in the complex tetrad $(m', \overline{m'}, \underline{l}', l')$ reads

$$\begin{aligned}&\big(D' + 2\Gamma'_{124}\big)\vartheta' - \big(\delta' + \Gamma'_{121}\big)\xi'\\&= \xi'(2\zeta' + \eta' + \underline{\eta}') - \vartheta'(\omega' + \theta' + \overline{\theta'}) - \Psi'_{(2)}(R).\end{aligned}$$

After using the table (A.52), this is equivalent to

$$\begin{aligned}&(D + 2\Gamma_{124} - 2D(B)/B)(B^2\vartheta) - (B\delta + B\Gamma_{121} - \delta(B))(B\xi)\\&= B\xi(2B\zeta + B\eta + B\underline{\eta}) - B^2(\omega + \theta + \overline{\theta}) - B^2\Psi_{(2)}(R).\end{aligned}$$

This is equivalent to (A.19), after simplifying the left-hand side and multiplying by B^{-2}.

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