Inventiones mathematicae

Hamiltonian 2-forms in Kähler geometry, III Extremal metrics and stability

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Abstract. This paper concerns the existence and explicit construction of extremal Kähler metrics on total spaces of projective bundles, which have been studied in many places. We present a unified approach, motivated by the theory of Hamiltonian 2-forms (as introduced and studied in previous papers in the series) but this paper is largely independent of that theory.

We obtain a characterization, on a large family of projective bundles, of the 'admissible' Kähler classes (i.e., those compatible with the bundle structure in a way we make precise) which contain an extremal Kähler metric. In many cases every Kähler class is admissible. In particular, our results complete the classification of extremal Kähler metrics on geometrically ruled surfaces, answering several long-standing questions.

We also find that our characterization agrees with a notion of K-stability for admissible Kähler classes. Our examples and nonexistence results therefore provide a fertile testing ground for the rapidly developing theory of stability for projective varieties, and we discuss some of the ramifications. In particular we obtain examples of projective varieties which are destabilized by a non-algebraic degeneration.

Introduction

In this paper we give a systematic overview of some explicit constructions of extremal Kähler metrics on projective bundles and relate our constructions to the theory of stability for algebraic varieties.

There has been a great deal of interest recently in the relation between extremal Kähler metrics, especially constant scalar curvature (CSC) Kähler metrics, and *stability*: initiated by ideas going back to Yau [56], there are conjectures [53, 15] that the existence of an extremal or CSC Kähler metric, in an integral Kähler class Ω on a compact complex manifold M, should be equivalent to an algebraic geometric stability condition on the Kodaira embeddings of M into $P(H^0(M, L^k)^*)$ for $k \gg 1$, where $c_1(L) = \Omega/2\pi$.

Our own interest in constructions of extremal Kähler metrics has been renewed and stimulated not only by these developments, but also by a unifying principle, which we shall explain and apply here, underlying explicit examples of such metrics on projective bundles obtained in [8, 27, 30, 31, 34, 47, 54]. In short these examples have in common the presence of a *Hamiltonian 2-form of order* 1.

The general theory of Hamiltonian 2-forms has been developed in our previous papers [4, 5]. A main feature of the Kähler metrics admitting a Hamiltonian 2-form is the appearance of hidden symmetries, which, in the compact case, results in a *rigid* Hamiltonian torus action of dimension the order of the Hamiltonian 2-form.

Motivated by a non-existence result proved in Appendix A.2, in this paper we focus our attention to the case when this order is 1. Under some mild additional hypotheses, the resulting manifolds are biholomorphic to projective bundles of the form $P(E_0 \oplus E_\infty) \to S$, where E_0, E_∞ are projectively flat hermitian holomorphic vector bundles over a compact Kähler manifold S. The latter is locally a Kähler product $\prod_{a \in \mathcal{A}} S_a$ where $\mathcal{A}$ is the finite set of distinct constant eigenvalues of the hamitonian 2-form, η_a say, and each S_a has complex dimension equal to the multiplicity of η_a; furthermore, for each a in $\mathcal{A}$ there is an associated 2-form ω_a on S (with $\pm \omega_a$ being the component of the Kähler form of S on S_a and $\pm$ is chosen so that $\eta_a \omega_a$ is positive definite) such that $c_1(E_\infty)/\mathrm{rk}(E_\infty) - c_1(E_0)/\mathrm{rk}(E_0) = \sum_{a \in \mathcal{A}} [\omega_a/2\pi]$.

Complex projective bundles of this form will be referred to as *admissible bundles* or *admissible manifolds*. Likewise, the Kähler metrics, forms and classes arising on admissible manifolds according to the general theory of Hamiltonian 2-forms will be called *admissible*. By their very definition, admissible Kähler metrics admit a natural isometric Hamiltonian S^1-action whose generator is denoted by K. Section 1 is mainly devoted to make precise the above notions and conventions.

As in the first constructions of extremal Kähler metrics by E. Calabi in [7], the search for admissible extremal Kähler metrics in a given admissible class Ω on an admissible $2m$-manifold involves a polynomial of degree equal to or less than $m + 2$, here denoted by F_Ω, which we call the *extremal polynomial* of Ω. This polynomial plays a pivotal role in the whole paper. Our first main result can be stated as follows.

Theorem 1. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold, where the base S is a local Kähler product of CSC metrics $(\pm g_a, \pm \omega_a)$. Then there is an extremal Kähler metric in an admissible Kähler class Ω if and only if the extremal polynomial F_Ω is positive on $(-1, 1)$. This condition always holds if Ω is 'sufficiently small'; if it does, there is an admissible extremal Kähler metric in Ω, which is CSC if and only if the Futaki invariant $\mathfrak{F}_\Omega(K)$ vanishes (i.e., F_Ω has degree $\leq m + 1$).*

The admissible Kähler classes containing an extremal Kähler metric form a nonempty open subset of all such classes, and those containing a CSC Kähler metric form a real analytic hypersurface which is nonempty if $c_1(E_\infty)/\mathrm{rk}(E_\infty) - c_1(E_0)/\mathrm{rk}(E_0)$ is strictly indefinite (i.e., the definite forms ω_a on S_a do not all have the same sign).

Geometrically, an admissible Kähler class is *sufficiently small* if the base S is large (low curvature) compared to the fibres (high curvature). Thus the above theorem asserts the existence of extremal Kähler metrics with curvature concentrated in the fibres, which agrees with the results in [32, 45]. Note also that by the uniqueness theorem of Chen–Tian [10, 11], 'the' extremal Kähler metric in Ω (if it exists) is unique up to automorphism.

Section 2 is devoted to the proof of Theorem 1. This involves several new ingredients of independent interest and relies in a crucial way on a recent result of Chen–Tian [10, 11], quoted in this paper as Theorem 4. First of all, we consider what implications the existence of an extremal Kähler metric has for admissible bundles, Kähler classes and metrics. In Sect. 2.1, we apply the Matsushima–Lichnerowicz criterion [36, 41] to obtain information about the automorphism group of an admissible bundle when the base metric on S is CSC. In Sect. 2.2, we use this information to show that, for any admissible Kähler class Ω, the so-called *extremal vector field* of Ω is a multiple of the canonical Hamiltonian Killing vector field K and we compute the Futaki invariant $\mathfrak{F}_\Omega(K)$. In Sect. 2.3, we provide an explicit formula for the Mabuchi–Guan–Simanca (modified) K-energy functional relative to the natural S^1-action on the space of admissible Kähler metrics in Ω, and show that it is determined by the extremal polynomial F_Ω which has as a leading coefficient a nonzero multiple of $\mathfrak{F}_\Omega(K)$. By using the Chen–Tian Theorem 4, we deduce that if Ω contains an extremal Kähler metric, then F_Ω must be nonnegative on $(-1, 1)$. In the boundary case, when F_Ω is nonnegative with zeroes on $(-1, 1)$, we show non-existence of extremal Kähler metrics in Ω via a delicate limiting argument involving the LeBrun–Simanca stability theorem [35], the general theory of Hamiltonian 2-forms [5], and the uniqueness of extremal Kähler metrics modulo automorphisms established in [11].

In Sect. 2.4, we apply Proposition 1 of Sect. 1 to give a constructive proof of the existence part of Theorem 1, unifying and generalizing work of Calabi, Guan, Hwang, Hwang–Singer and the fourth author [8, 27, 30, 31, 54].

In Sect. 3 we present further existence and nonexistence results for extremal and CSC metrics by *computing* the extremal polynomial on various

examples and testing its positivity on $(-1, 1)$. In many of these examples, every Kähler class on M is admissible (see Remark 2) and, therefore, Theorem 1 describes precisely which Kähler classes do contain an extremal Kähler metric. This is the case when $M = P(\mathcal{O} \oplus \mathcal{L}) \to S$ is an admissible geometrically ruled complex surface (meaning here that the degree of $\mathcal{L}$ is non-zero); the extremal polynomial $F_\Omega(z)$ is then a quartic divisible by $1 - z^2$. We thus obtain a complete resolution of the existence question for extremal Kähler metrics on these complex surfaces. This, combined with an observation from [2], fills in the missing step in the complete classification of extremal Kähler metrics on all geometrically ruled complex surfaces, which can be specifically stated as follows.

Corollary 1. *Let $M = P(E) \to S$ be a geometrically ruled complex surface, where E is a rank 2 holomorphic vector bundle over a compact Riemann surface S of genus $\mathbf{g}$. If M admits an extremal Kähler metric, then one of the following two cases holds:*

- *E is polystable: then $M = P(E)$ admits a locally-symmetric CSC Kähler in each Kähler class;*
- *$M = P(\mathcal{O} \oplus \mathcal{L}) \to S$ is admissible, the Kähler cone is a cone on an open interval (a, b); the extremal Kähler metrics are precisely those of [7,30, 54], which are admissible and locally of cohomogeneity one, with Kähler classes parameterized by a cone on a subinterval (a, c), with $c = b$ if and only if S has genus 0 or 1.*

We refer the reader to [2] for more details. We also note that Ross–Thomas [45] and G. Székelyhidi [50,51] independently obtained similar results for rational Kähler classes on ruled surfaces, by using a different approach relying on recent works of S. Donaldson [15,17,18].

Our results provide a fertile testing ground for the conjectures relating extremal and CSC Kähler metrics to stability, and we explore this in Sect. 4. In Sect. 4.2 we relate our results to those of Ross–Thomas [45] and Hong [32]: in particular, we show that there are CSC metrics on projective bundles $P(E) \to S$ for which E is only (slope) polystable with respect to one Kähler class on S up to scale.

In Sect. 4.4 we relate Theorem 1 to the notions of K-polystability [15,52] or relative K-polystability [50] for Kähler classes, which are conjectured to be equivalent to the existence of a CSC or extremal Kähler metric in a given class. Actually, to be precise, we use a closely related notion of (relative) slope K-polystability defined by Ross–Thomas in [45], which is stronger than just specializing the notion of K-polystability in the sense [15] to certain test configurations: the notion of (relative) slope K-polystability used here might more properly be referred to as *analytic* (relative) slope K-polystability, and is the notion relevant to assigning a sign of the Futaki invariant on *analytic* test configurations (while the original definitions of [53,15] require only *algebraic* test configurations). By generalizing

a beautiful calculation of G. Székelyhidi [50] for ruled surfaces, we establish the following result.

Theorem 2. *Let Ω be an admissible integral Kähler class on the admissible projective bundle $M = P(E_0 \oplus E_\infty) \to S$, where S is CSC. If Ω is slope K-polystable, it contains a CSC Kähler metric, and if it is relatively slope K-polystable, it contains an extremal Kähler metric.*

It is natural to ask if (relative) K-polystability in the sense of [15, 50] implies the existence of a CSC (or extremal) Kähler metric in Ω. We find that this is true if dim $S \le 4$, but for dim $S \ge 6$, we are only able to show that the extremal polynomial is positive on $(-1, 1) \cap \mathbb{Q}$.

Before our work, it was believed that K-polystability implies slope K-polystability in general, but the proof in [45, 46] only shows that it implies slope K-semistability, the gap being closely related to the issue of positivity (versus nonnegativity) of the extremal polynomial at irrational points in $(-1, 1)$.

To show that this is a genuine problem, we end with some examples, on projective line bundles over a product of three Riemann surfaces, of integral admissible Kähler classes Ω such that F_Ω is positive on $(-1, 1) \cap \mathbb{Q}$ but has an irrational repeated root in $(-1, 1)$. We find these examples intriguing, since by Theorem 1, these Kähler classes do *not* contain an extremal Kähler metric, so they should be unstable. However, even though they are projective varieties, the degeneration that demonstrates this instability is not algebraic. While we cannot prove that there is no other algebraic test configuration which would detect this instability, it is difficult to imagine how such a test configuration could be constructed. Thus, presumably, our examples are algebraically (relatively) K-polystable but analytically only (relatively) K-semistable. This suggests that the non-algebraic degenerations implicit in the use of slope K-polystability may be essential to relate stability to existence of CSC and extremal Kähler metrics.[1]

We would like to thank Claude LeBrun for helpful comments, Richard Thomas for discussions concerning [45, 46], and the referee for useful suggestions.

Contents

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[1] Motivated by our examples, G. Székelyhidi introduced in [51] a stronger notion of *uniform* (relative) K-polystability for integral Kähler classes, which, in our setting, forces F_Ω to be positive on $(-1, 1)$ and, therefore, implies the existence of an admissible extremal metric.

1. Admissible bundles and Kähler metrics

This section is devoted to introducing the main ingredients of the general theory of Hamiltonian 2-forms which we will use in this paper, and giving the precise definitions of the geometric entities mentioned in the introduction.

1.1. Hamiltonian 2-forms.

A *Hamiltonian 2-form* [4] on a Kähler manifold (M, J, g, ω), of real dimension $2m > 2$ is a real $(1, 1)$-form (i.e., a J-invariant 2-form) ϕ such that

$$2\nabla_X\phi = d\operatorname{tr}\phi \wedge (JX)^\flat - (Jd\operatorname{tr}\phi) \wedge X^\flat$$

for all $X \in TM$ (where $X^\flat(Y) = g(X, Y)$ for $Y \in TM$ and $\operatorname{tr}\phi = \langle \omega, \phi \rangle_g$). The *momentum polynomial* of ϕ is then defined to be

$$p(t) := (-1)^m \operatorname{pf}(\phi - t\omega) = t^m - (\operatorname{tr}\phi)\, t^{m-1} + \cdots + (-1)^m \operatorname{pf}\phi,$$

where the *Pfaffian* is defined by $\phi \wedge \cdots \wedge \phi = (\operatorname{pf}\phi)\omega \wedge \cdots \wedge \omega$. The reason ϕ is called *Hamiltonian* is that for any $t \in \mathbb{R}$, $p(t)$ is a Hamiltonian for a Killing vector field $K(t) := J\operatorname{grad}_g p(t)$ (this is not difficult to show [4, §2]). The integer $\ell = \max_{x \in M} \dim\operatorname{span}\{K(t)_x : t \in \mathbb{R}\}$ (with $0 \le \ell \le m$) is called the *order* of ϕ.

We do not wish to impose the study of Hamiltonian 2-forms on the reader of this paper, since we only need the general theory as motivation for the classes of complex manifolds and metrics that we shall study. We therefore now state a classification result which reduces the theory of Hamiltonian 2-forms of order 1 to an ansatz for metrics on projective bundles. This result follows easily from [4,5], as we explain in Appendix A, where we also explain why we restrict attention to the order 1 case.

Theorem 3. *Let (M, g, J, ω) be a compact connected Kähler $2m$-manifold with a Hamiltonian 2-form ϕ of order 1. Then:*

- *there is an effective isometric Hamiltonian S^1 action on M generated by a vector field $K = J\operatorname{grad}_g z$ such that the stable quotient of M by the induced holomorphic $\mathbb{C}^\times$ action is a compact connected complex manifold $\hat{S}$ of real dimension $2(m - 1)$;*
- *without loss, the image of the momentum map z is $[-1, 1]$, and there are Kähler manifolds S_a of dimension $2d_a$ and real numbers x_a, indexed by $a \in \hat{\mathcal{A}} \subset \mathbb{N} \cup \{\infty\}$, such that $\hat{S}$ is covered by $\prod_{a\in\hat{\mathcal{A}}} S_a$ and $0 < |x_a| \le 1$ with equality iff $a \in \{0, \infty\}$;*
- *z is a Morse–Bott function [1] on M with critical set $z^{-1}(\{-1, 1\})$, $M^0 := z^{-1}((-1, 1))$ is a principal $\mathbb{C}^\times$-bundle over $\hat{S}$, and on each S_a there is a $(1, 1)$-form ω_a (which is the pullback of a tensor on $\hat{S}$) such that either $+\omega_a$ or $-\omega_a$ is positive definite (with the sign chosen so that ω_a/x_a is positive definite), and thus gives rise to a Kähler metric on S_a, accordingly denoted by $\pm g_a$, such that the Kähler structure on M^0 is*

(1)
$$g = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} g_a + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \quad \omega = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} \omega_a + dz \wedge \theta,$$

where θ is a connection 1-form ($\theta(K) = 1$) with $d\theta = \sum_{a \in \hat{\mathcal{A}}} \omega_a$, and Θ is a smooth function on $[-1, 1]$ satisfying

(2) $\Theta > 0 \quad$ *on* $(-1, 1)$,

(3) $\Theta(\pm 1) = 0, \quad \Theta'(\pm 1) = \mp 2;$

- *if $0 \in \hat{\mathcal{A}}$ then $x_0 = 1$, $S_0 = \mathbb{C}P^{d_0}$ and g_0 is the Fubini–Study metric of scalar curvature $2d_0(d_0 + 1)$, otherwise we set $d_0 = 0$; likewise, if $\infty \in \hat{\mathcal{A}}$ then $x_\infty = -1$, $S_\infty = \mathbb{C}P^{d_\infty}$ and $-g_\infty$ is the Fubini–Study metric of scalar curvature $2d_\infty(d_\infty + 1)$, otherwise we set $d_\infty = 0$; we also put $\mathcal{A} = \hat{\mathcal{A}} \setminus \{0, \infty\} \subset \mathbb{Z}^+$;*
- *the blow-up $\hat{M}$ of M along $z^{-1}(\{-1, 1\})$ is $\mathbb{C}^\times$-equivariantly biholomorphic to $M^0 \times_{\mathbb{C}^\times} \mathbb{C}P^1 \to \hat{S}$ and $\hat{S}$ is a fibre product of flat projective unitary $\mathbb{C}P^{d_0}$- and $\mathbb{C}P^{d_\infty}$-bundles over a Kähler manifold S covered by $\prod_{a \in \mathcal{A}} S_a$.*

If we assume that $\hat{S} = P(E_0) \times_S P(E_\infty) \to S$ for projectively-flat hermitian vector bundles $E_0, E_\infty \to S$, then these bundles can be chosen so that M is $\mathbb{C}^\times$-equivariantly biholomorphic to $P(E_0 \oplus E_\infty) \to S$, and we therefore have $\bar{c}_1(E_\infty) - \bar{c}_1(E_0) = \sum_{a \in \mathcal{A}} [\omega_a/2\pi]$, where $\bar{c}_1(E) = c_1(E)/\operatorname{rank} E$.

The final assumption of this theorem is automatic if $\pi_1(S) = 1$, when $\hat{S} = \mathbb{C}P^{d_0} \times S \times \mathbb{C}P^{d_\infty}$ and there is a line bundle $\mathcal{L} \to S$ with $c_1(\mathcal{L}) = \sum_{a \in \mathcal{A}} [\omega_a/2\pi]$ such that M is $\mathbb{C}^\times$-equivariantly biholomorphic to $P(\mathcal{O} \otimes \mathbb{C}^{d_0+1} \oplus \mathcal{L} \otimes \mathbb{C}^{d_\infty+1}) \to S$.

We shall use this theorem as an ansatz for constructing extremal Kähler metrics on projective bundles of the form $P(E_0 \oplus E_\infty) \to S$. For this we shall use the following elementary computation [27, 30, 4].

Proposition 1. *Let g be a Kähler metric of the form* (1) *on M^0 and write $F(z) = \Theta(z) p_c(z)$ with $p_c(z) := \prod_{a \in \hat{\mathcal{A}}} (1 + x_a z)^{d_a}$. Then g is extremal, with Scal_g a constant affine function of z, iff*

- *there is a polynomial P of degree $\leq \#\hat{\mathcal{A}} + 1$ such that*

(4) $$F''(z) = \Big(\prod_{a \in \hat{\mathcal{A}}} (1 + x_a z)^{d_a - 1} \Big) P(z);$$

- *for all a, $\pm g_a$ has constant scalar curvature $\mathrm{Scal}_{\pm g_a} = \pm 2 d_a s_a$ where*

(5) $$P(-1/x_a) = 2 d_a s_a x_a \prod_{b \in \hat{\mathcal{A}} \setminus \{a\}} \Big(1 - \frac{x_b}{x_a} \Big).$$

The metric g then has constant scalar curvature iff P has degree $\leq \#\hat{\mathcal{A}}$.

Compared to [4,5], we have rescaled $F(z)$ and $p_c(z)$ by $\prod_{a\in\hat{A}} x_a^{d_a}$: this is convenient as $p_c(z)$ is then positive on $(-1, 1)$. Thus Θ is positive on $(-1, 1)$ if and only if F is. Also if $\Theta(z)$ satisfies (3), then $F(z) = \Theta(z)p_c(z)$ satisfies

$$(6) \qquad F(\pm 1) = 0, \quad F'(\pm 1) = \mp 2p_c(\pm 1).$$

In the following subsections we study metrics of the form (1) and show that the conditions of Theorem 3 are sufficient for the compactification of such metrics on a projective bundle $M = P(E_0 \oplus E_\infty) \to S$.

1.2. Admissible projective bundles.
We use Theorem 3 (including the final assumption) as motivation for the class of compact complex manifolds we will study. A projective bundle of the form $M = P(E_0 \oplus E_\infty) \xrightarrow{p} S$ will be called *admissible* or an *admissible manifold* if:

- S is a covered by a product $\tilde{S} = \prod_{a\in A} S_a$ (for $A \subset \mathbb{Z}^+$) of simply-connected Kähler manifolds $(S_a, \pm g_a, \pm \omega_a)$ of real dimensions $2d_a$, with (g_a, ω_a) being pullbacks of tensors on S; here and henceforth, with slight abuse of notation, the signs $\pm$ before the Kähler structure $(\pm g_a, \pm \omega_a)$ on the complex manifold S_a mean that either $+\omega_a$ or $-\omega_a$ is a positive definite $(1, 1)$-form which defines a Kähler metric denoted by $+g_a$ or $-g_a$, respectively. We will use the notation $\pm\omega_a$ (rep. $\pm g_a$) to denote the positive definite form (resp. the Riemannian metric) on S_a.
- E_0 and E_∞ are holomorphic projectively-flat hermitian vector bundles over S of ranks $d_0 + 1$ and $d_\infty + 1$ with $\bar{c}_1(E_\infty) - \bar{c}_1(E_0) = [\omega_S/2\pi]$ and $\omega_S = \sum_{a\in A} \omega_a$.
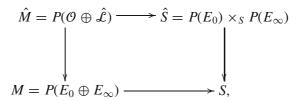
The second condition (cf. [33]) means that we can fix hermitian metrics on E_0 and E_∞ whose Chern connections have tracelike curvatures $\Omega_0 \otimes Id_{E_0}$ and $\Omega_\infty \otimes Id_{E_\infty}$ satisfying $\Omega_\infty - \Omega_0 = \sum_{a\in A} \omega_a$. We normalize the induced fibrewise Fubini–Study metrics (g_0, ω_0) and $(-g_\infty, -\omega_\infty)$ on $P(E_0)$ and $P(E_\infty)$ to have scalar curvatures $2d_0(d_0 + 1)$ and $2d_\infty(d_\infty + 1)$.

We collect a few remarks and notations that we will use. We omit pullbacks by obvious projections in these remarks.
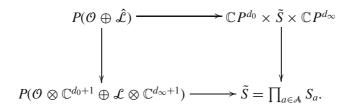
(i)   We sometimes let the index a take values in $\mathbb{N} \cup \{\infty\}$ by setting $d_a = 0$ for $a \notin A \cup \{0, \infty\}$ (so that S_a is a point and $\omega_a = 0$). This range will be assumed unless otherwise stated. We set $\hat{A} := \{a : d_a > 0\}$ so that $A = \hat{A} \cap \mathbb{Z}^+$.

(ii) The pullbacks of E_0 and E_∞ to $\tilde{S}$ are of the form $\mathcal{E}_0 \otimes \mathbb{C}^{d_0+1}$ and $\mathcal{E}_\infty \otimes \mathbb{C}^{d_\infty+1}$, where $\mathcal{L} := \mathcal{E}_0^{-1} \otimes \mathcal{E}_\infty = \bigotimes_{a\in A} \mathcal{L}_a$ for line bundles $\mathcal{L}_a \to S_a$ with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$.

(iii) $e_0 := P(E_0 \oplus 0)$ and $e_\infty := P(0 \oplus E_\infty)$ denote the 'zero' and 'infinity' subbundles of M, covered by $S_0 \times \tilde{S}$ and $\tilde{S} \times S_\infty$, where $S_0 = \mathbb{C}P^{d_0}$ and $S_\infty = \mathbb{C}P^{d_\infty}$.

(iv) The blow-up of M along $e_0 \cup e_\infty$ is $\hat{M} := P(\mathcal{O} \oplus \hat{\mathcal{L}}) \overset{\hat{p}}{\to} \hat{S}$, where $\hat{S} = P(E_0) \times_S P(E_\infty) \to S$ and $\hat{\mathcal{L}} = \mathcal{O}(1)_{E_0} \otimes \mathcal{O}(-1)_{E_\infty}$, using the (fibrewise) hyperplane and tautological line bundles; we have $c_1(\hat{\mathcal{L}}) = [\omega_{\hat{S}}/2\pi]$, where $\omega_{\hat{S}} = \sum_a \omega_a$. If $d_0 > 0$ or $d_\infty > 0$ we say *a blow-down occurs*.

(v) $\hat{e}_0$ and $\hat{e}_\infty$ denote the zero and infinity sections of $\hat{M}$. The pullback of $\hat{\mathcal{L}}$ to $S_0 \times \tilde{S} \times S_\infty$ is $\mathcal{L}_0 \otimes \mathcal{L} \otimes \mathcal{L}_\infty$, where $\mathcal{L}_0 = \mathcal{O}(1) \to S_0$ and $\mathcal{L}_\infty = \mathcal{O}(-1) \to S_\infty$.

(vi) $\hat{S}$ has a family of local Kähler product metrics $g_{\hat{S}}(z)$ with Kähler forms $z\omega_{\hat{S}} + \sum_a \omega_a/x_a$ and we set $g_{\hat{S}} = g_{\hat{S}}(0)$. (Note that $g_{\hat{S}}$ is not compatible with $\omega_{\hat{S}}$ – the latter is symplectic, but not a Kähler form in general.) We let $g_S(z)$ and $g_S = g_S(0)$ denote the induced local Kähler product metrics on S.

We summarize the set-up with the following diagram of bundles and a blow-up:

$$\hat{M} = P(\mathcal{O} \oplus \hat{\mathcal{L}}) \longrightarrow \hat{S} = P(E_0) \times_S P(E_\infty)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$M = P(E_0 \oplus E_\infty) \longrightarrow S,$$

the universal cover (omitting pullbacks) of this diagram being:

$$P(\mathcal{O} \oplus \hat{\mathcal{L}}) \longrightarrow \mathbb{C}P^{d_0} \times \tilde{S} \times \mathbb{C}P^{d_\infty}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$P(\mathcal{O} \otimes \mathbb{C}^{d_0+1} \oplus \mathcal{L} \otimes \mathbb{C}^{d_\infty+1}) \longrightarrow \tilde{S} = \prod_{a \in \mathcal{A}} S_a.$$

Remark 1. The existence of the line bundle $\hat{\mathcal{L}} \to \hat{S}$ with $c_1(\hat{\mathcal{L}}) = [\omega_{\hat{S}}/2\pi]$ implies that $\omega_{\hat{S}}$ is *integral* in the sense that $[\omega_{\hat{S}}/2\pi]$ is in the image of $H^2(\hat{S}, \mathbb{Z})$ in $H^2(\hat{S}, \mathbb{R})$. When $\hat{S}$ is a global Kähler product (i.e., we can write $M = P(\mathcal{O} \otimes \mathbb{C}^{d_0+1} \oplus \mathcal{L} \otimes \mathbb{C}^{d_\infty+1}) \to S = \prod_{a \in \mathcal{A}} S_a$), this integrality condition means that each ω_a is integral, i.e., the compact manifolds $(S_a, \pm g_a, \pm \omega_a)$ are Hodge.

We write $\omega_a = q_a \alpha_a$ for an integer $q_a \neq 0$, where α_a is a primitive integral Kähler form on S_a, so that q_a is a nonzero integer with the same sign as (g_a, ω_a), and $q_0 = 1$ and $q_\infty = -1$. We now compare $[\omega_a/2\pi]$ to the first Chern class $c_1(\mathcal{K}_a^{-1}) = [\rho_a/2\pi]$ of the anticanonical bundle of S_a, by writing $[\rho_a] = p_a[\alpha_a] + [\rho_a]_0$, for a rational number p_a, where $[\rho_a]_0 \cdot [\alpha_a]^{d_a-1} = 0$. Since any line bundle $\mathcal{P}$ with first Chern class $[\alpha_a/2\pi]$ is ample, $\mathcal{P}^{d_a+1} \otimes \mathcal{K}_a$ is nef by a re-

sult of Fujita [21] (see also [12, Theorem 8.3]), from which it follows easily that $p_a \leq d_a + 1$. If S_a is a Riemann surface of genus $\mathbf{g}_a$, then $p_a = 2(1 - \mathbf{g}_a) \leq 2$.

We set $s_a = p_a/q_a$. When $\pm g_a$ is CSC, we have $Scal_{\pm g_a} = \pm 2d_a s_a$, where the sign is that of q_a, so the scalar curvature of $\pm g_a$ has the same sign as p_a. Thus, in the case of a CSC Hodge manifold S_a, the Fujita inequality $p_a \leq d_a + 1$ is (since $|q_a| \geq 1$) equivalent to $Scal_{\pm g_a} \leq 2d_a(d_a + 1)$.

The conditions of Theorem 3 are also sufficient for the compactification of metrics of the form (1) on an admissible projective bundle $M = P(E_0 \oplus E_\infty) \rightarrow S$, where $z \colon M \rightarrow [-1, 1]$ with $e_0 = z^{-1}(1)$ and $e_\infty = z^{-1}(-1)$, and θ is a connection 1-form. Before discussing this, we introduce the Kähler classes to which they belong.

1.3. Admissible Kähler classes and canonical metrics.

Suppose that $M = P(E_0 \oplus E_\infty) \rightarrow S$ is an admissible bundle. We say that a Kähler class Ω on M is *admissible* if there are constants x_a, with $x_0 = 1$, $x_\infty = -1$, such that the pullback of Ω to $\hat{M}$ has the form

$$\sum_a [\omega_a]/x_a + \hat{\Xi}$$

up to scale, where the 2-forms ω_a are viewed as pullbacks to $\hat{M}$ of the corresponding forms on $\hat{S}$ (induced by the local product Kähler structure $\prod_a S_a$) and $\hat{\Xi}$ is Poincaré dual to $2\pi[\hat{e}_0 + \hat{e}_\infty]$. Thus $\hat{\Xi} = 2\pi c_1(V\hat{M})$, where $V\hat{M} = \mathcal{O}(2)_{\mathcal{O} \oplus \hat{\mathcal{L}}} \otimes \hat{p}^* \hat{\mathcal{L}}$ and $\mathcal{O}(-1)_{\mathcal{O} \oplus \hat{\mathcal{L}}}$ is the (fibrewise) tautological bundle of $\hat{M} = P(\mathcal{O} \oplus \hat{\mathcal{L}})$. (The first Chern class $[\omega_{\hat{s}}/2\pi]$ of $\hat{\mathcal{L}}$ itself pulls back to $\hat{M}$ to give the Poincaré dual of $[\hat{e}_0 - \hat{e}_\infty]$.)

It follows that admissible Kähler classes have the form

$$\Omega = \sum_{a \in \mathcal{A}} [\omega_a]/x_a + \Xi$$

up to scale, where the pullback of Ξ to $\hat{M}$ is $[\omega_0] - [\omega_\infty] + \hat{\Xi}$. Since pullback to a blow-up is injective on cohomology, admissible Kähler classes on M are uniquely determined by the parameters x_a.

If (g, ω) is any Kähler metric on M of the form (1) on M^0, then we claim $\Omega = [\omega]$ is admissible. For this we first note that on M^0, the Kähler form ω is a linear combination $\sum_{a \in \mathcal{A}} \omega_a/x_a + \eta$, where

$$\eta = (z + 1)\omega_0 + \sum_{a \in \mathcal{A}} z\omega_a + (z - 1)\omega_\infty + dz \wedge \theta.$$

Here ω_0, ω_∞ and θ are defined only on M^0. However, for $a \in \mathcal{A}$, ω_a extends to a closed 2-form on M (as a pullback from S), so η is globally defined and closed on M (since ω is). The pullback of η to $\hat{M}$ may be written $\omega_0 - \omega_\infty + \hat{\eta}$ with $\hat{\eta} = d(z\theta)$ on M^0, and since ω_0 and ω_∞ are well-defined

and closed on $\hat{M}$ (as pullbacks from $\hat{S}$), so is $\hat{\eta}$, and we easily see[2] that $[\hat{\eta}] = \hat{\Xi}$.

Observe that η depends implicitly on the choice of metric (g, ω) on M because the momentum map z does. However, the above shows that the cohomology class $[\eta]$ is Ξ, independent of this choice. From this realisation of Ξ, it follows easily, by pulling back to e_0 and e_∞, that for a cohomology class of the form $\sum_{a \in \mathcal{A}} [\omega_a]/x_a + \Xi$ to be a Kähler class, it is necessary that for $a \in \mathcal{A}$, $0 < |x_a| < 1$ with the sign of x_a such that ω_a/x_a is positive. Conversely, we claim that any cohomology class of this form (with $0 < |x_a| < 1$ and ω_a/x_a positive for $a \in \mathcal{A}$) is an admissible Kähler class and contains a Kähler metric of the form (1) on M^0 up to scale. To do this we construct a distinguished Kähler metric in each such class.

Let r_0 and r_∞ be the norm functions induced by the hermitian metrics on E_0 and E_∞. Then $z_0 = \frac{1}{2} r_0^2$ and $z_\infty = \frac{1}{2} r_\infty^2$ are fibrewise momentum maps for the $U(1)$ actions given by scalar multiplication in E_0 and E_∞, generated by K_0 and K_∞. We equip M with a fibrewise Fubini–Study metric $(g_{M/S}, \omega_{M/S})$: with our normalization of g_0 and g_∞, each fibre is the Kähler quotient of the corresponding fibre of $E_0 \oplus E_\infty$ by the diagonal $U(1)$ action at momentum level $z_0 + z_\infty = 2$; then on this momentum level, the function $z = z_0 - 1 = 1 - z_\infty$ descends to a fibrewise momentum map $M \to [-1, 1]$ for the quotient $U(1)$ action.

We extend $(g_{M/S}, \omega_{M/S})$ to TM by requiring that the horizontal distribution of the induced connection on M is degenerate. To obtain a nondegenerate metric, we then set

$$g_c = \sum_{a \in \mathcal{A}} \frac{1 + x_a z}{x_a} g_a + g_{M/S}, \quad \omega_c = \sum_{a \in \mathcal{A}} \frac{1 + x_a z}{x_a} \omega_a + \omega_{M/S},$$

where the (g_a, ω_a) are pulled back from S; g_c is then a positive definite Kähler metric with respect to the canonical complex structure of $M = P(E_0 \oplus E_\infty)$ by the assumptions on the parameters x_a. We refer to (g_c, ω_c) as the *canonical Kähler metric* on M in the given admissible Kähler class.

Lemma 1. *For any $0 < |x_a| < 1$ ($a \in \mathcal{A}$), the corresponding canonical Kähler metric on M is of the form* (1) *on M^0, where $\Theta = \Theta_c$ and $\Theta_c(z) = 1 - z^2$.*

Proof. The inverse image in $E_0 \oplus E_\infty$ of $M^0 = M \setminus (z^{-1}(-1) \cup z^{-1}(1))$ may be viewed as an open subset of $\mathcal{O}(-1)_{E_0} \oplus \mathcal{O}(-1)_{E_\infty}$. Then (g_c, ω_c)

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[2] On each fibre of $\hat{p}: \hat{M} \to \hat{S}$, $\hat{\eta}/4\pi$ integrates to 1 and so $[\hat{\eta}/4\pi]$ restricts to give the generator of $H^2(\hat{p}^{-1}(x), \mathbb{Z})$. Hence by the Leray–Hirsch theorem, $H^2(\hat{M}, \mathbb{R})$ is generated by $[\hat{\eta}]$ and pullbacks from S. The restriction of $[\hat{\eta}/2\pi]$ to $\hat{e}_0$ is the first Chern class $[\omega_{\hat{S}}/2\pi]$ of $\hat{\mathcal{L}}$ (and the restriction to $\hat{e}_\infty$ is the first Chern class $[-\omega_{\hat{S}}/2\pi]$ of $\hat{\mathcal{L}}^{-1}$). Thus $[\hat{\eta}/4\pi]$ is a projective version of the Thom class of a vector bundle.

is the Kähler quotient at momentum level $z_0 + z_\infty = 2$ of the metric

$$\sum_a \frac{(1 + x_a)z_0 + (1 - x_a)z_\infty}{2x_a} g_a + \frac{dz_0^2}{2z_0} + \frac{dz_\infty^2}{2z_\infty} + 2z_0\theta_0^2 + 2z_\infty\theta_\infty^2,$$

where $x_0 = 1$, $x_\infty = -1$, and θ_0, θ_∞ are connection 1-forms for the $U(1)$-line bundles $\mathcal{O}(-1)_{E_0}$, $\mathcal{O}(-1)_{E_\infty}$ with $\theta_0(K_0) = 1 = \theta_\infty(K_\infty)$, $d\theta_0 = -\omega_0 + \Omega_0$, $d\theta_\infty = \omega_\infty + \Omega_\infty$.

If we regard M^0 as an open subset of $\hat{M} = P(\mathcal{O} \oplus \hat{\mathcal{L}})$, then the diagonal action is generated by $K_0 + K_\infty$, $\theta_\infty - \theta_0$ is basic and so induces a unitary connection θ (with respect to the quotient $U(1)$-action) on $\hat{\mathcal{L}}$ with $d\theta = \omega_{\hat{S}}$. Substituting $z_0 = 1 + z$ and $z_\infty = 1 - z$ and performing the quotient yields (1) with $\Theta = \Theta_c$. (On each fibre over $\hat{S}$ this is the realization of $\mathbb{C}P^1$ as a Kähler quotient of $\mathbb{C}^2$.) $\square$

Remark 2. The existence of the canonical metric on M shows there does exist a cohomology class Ξ whose pullback to $\hat{M}$ is $\hat{\Xi}$. Ξ is then unique, and the admissible Kähler classes form a family of dimension $\#\mathcal{A} + 1$. If $b_2(S_a) = 1$ for all a and $b_1(S_a) \neq 0$ for at most one a, then every Kähler class on M is admissible.

1.4. Admissible metrics. Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible bundle and Ω an admissible Kähler class corresponding to parameters x_a. Then a Kähler metric in Ω is said to be *admissible* if it has the form (1) on M^0, up to scale, with respect to the given projective unitary bundle structure on M and local Kähler product structure on S. According to Theorem 3, in order for a (scale of a) metric of the form (1) on M^0 to define an admissible Kähler metric on M, it is necessary that Θ is a smooth function on $[-1, 1]$ satisfying (2) and (3). We now show that these conditions are also sufficient and provide a parameterization of admissible metrics.

We first note that any metric of the form (1), where Θ is a smooth function on $[-1, 1]$ satisfying (2) and (3), defines a smooth metric g on M compatible with the same *symplectic form* as the canonical Kähler metric g_c in Ω, provided that we take z to be the momentum map and θ the connection 1-form of the canonical Kähler metric; then, using (3), we find that $g - g_c$ is smooth on M, and g is positive definite on M since it is on M^0 by (2) and ω is nondegenerate on M. (See [5, §1] for details.)

With this point of view, the smooth functions Θ on $[-1, 1]$ satisfying (2) and (3) define a family of complex structures on M. However, we claim that there is an S^1-equivariant biholomorphism, in the identity component of the diffeomorphism group, between any two such complex structures, so that Θ parameterizes Kähler metrics compatible with the given (fixed) complex structure on M whose Kähler forms belong to a given admissible Kähler class Ω. This claim holds essentially because it is true for toric complex structures on $\mathbb{C}P^1$ (and for toric varieties in general), but for later use we need to make explicit the transformation of M that relates the complex and symplectic points of view, following [29, 28, 15].

A key ingredient in this transformation is the notion of a *symplectic potential* of an admissible Kähler metric defined by $\Theta(z)$, which is a function $u(z)$ on $(-1, 1)$ with $u''(z) = U(z) := 1/\Theta(z)$. Then

$$u_c(z) = \frac{1}{2}((1 - z) \log(1 - z) + (1 + z) \log(1 + z) - 2 \log 2)$$

is the unique symplectic potential for the canonical Kähler metric (g_c, J_c) given by $\Theta_c(z) = (1 - z)(1 + z)$, which satisfies $u_c(\pm 1) = 0$. We can extend this description to all admissible Kähler metrics compatible with ω, thanks to the following lemma, which is an easy application of l'Hôpital's rule and Taylor's theorem.

Lemma 2. *A smooth function $\Theta(z) = 1/U(z)$ satisfies (3) if and only if $U(z) - U_c(z)$ is smooth on $[-1, 1]$. Then $U(z)/U_c(z)$ is positive and smooth on $[-1, 1]$.*

On M^0, the symplectic potential $u(z)$ of an admissible Kähler metric is closely related to a Kähler potential of ω with respect to J by a fibrewise Legendre transform (see [4, 29]) over $\hat{S}$. Indeed, if we put

$$y = u'(z), \quad h(y) = -u(z) + yz,$$

then $d^c_J y = \theta$ and $dd^c_J h(y) = \omega - \sum_a \omega_a/x_a$ on (M^0, J).[3] Let $y_c, h_c(y_c)$ denote the corresponding quantities associated to u_c. There are local 1-forms α on $\hat{S}$ such that $\theta = dt + \alpha$, where $t : M^0 \to \mathbb{R}/2\pi\mathbb{Z}$ is locally defined up to an additive constant on each fibre. Since $\exp(y + it)$ and $\exp(y_c + it)$ give $\mathbb{C}^\times$-coordinates on the fibres, there is a $U(1)$-equivariant fibre-preserving diffeomorphism Ψ of M^0 over $\hat{S}$ with

$$\Psi^* y = y_c, \quad \Psi^* t = t, \quad \text{and hence} \quad \Psi^* J = J_c.$$

As J_c and J are integrable complex structures, Ψ extends to a $U(1)$-equivariant diffeomorphism of M leaving fixed any point on $e_0 \cup e_\infty$ (since it is fibre preserving).

Put $\tilde{\omega} := \Psi^* \omega$. Then $\tilde{\omega}$ is a Kähler form on (M, J_c) which (we claim) belongs to the same cohomology class Ω as ω. Indeed, on M^0 we have

$$\tilde{\omega} - \omega = dd^c_{J_c}(h(y_c) - h_c(y_c))$$

since $dd^c_{J_c} h(y_c) = \Psi^* dd^c_J h(y) = \tilde{\omega} - \sum_a \omega_a/x_a$, so the following implies the claim.

Lemma 3. *The function $h(y_c) - h_c(y_c)$ is smooth on M.*

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[3] It follows that if $\pm H_a$ is a local Kähler potential for $\pm \omega_a$ and $\tilde{u} = u(z) - \sum_a (1 + x_a z) H_a/x_a$, then $\tilde{y} = \partial \tilde{u}/\partial z$ is pluriharmonic and $\tilde{h} = -\tilde{u} + \tilde{y}z$ is a local Kähler potential for ω on (M^0, J) [4].

Proof. Since Ψ is a diffeomorphism with $\Psi^* y = y_c$, this holds if and only if $h(y) - h_c(y)$ is smooth on M. We already know that $h(y) - h_c(y_c) = -(u(z) - u_c(z)) + z(u'(z) - u_c'(z))$ is smooth (by Lemma 2) so it suffices to show that $h_c(y) - h_c(y_c)$ is smooth on M. However, knowing u_c explicitly, we calculate

$$h_c(y) - h_c(y_c) = -\frac{1}{2}\left(\log\left(\frac{1-\tilde{z}}{1-z}\right) + \log\left(\frac{1+\tilde{z}}{1+z}\right)\right),$$

where $\tilde{z} := \Psi^* z$ is the momentum map of $\tilde{\omega} = \Psi^* \omega$; since Ψ is S^1-equivariant and fixes $e_0 \cup e_\infty$, it follows that $\tilde{z}$, viewed as a function of z, satisfies $\tilde{z}(\pm 1) = \pm 1$; moreover, since both $\tilde{z}$ and z are momentum maps of the same $U(1)$ action on M (and are therefore Morse–Bott functions with the same critical sets), we must have $\tilde{z}'(\pm 1) \neq 0$. Thus $h_c(y) - h_c(y_c)$ is smooth on M. $\square$

Hence the moduli space $\mathcal{K}_\omega^{\mathrm{adm}}$ of admissible metrics in $\Omega = [\omega]$ is identified with the space of smooth functions Θ on $[-1, 1]$ satisfying (2)–(3) or equivalently with $\{u \in C^0([-1, 1]) : u - u_c \in C^\infty([-1, 1]), u(\pm 1) = 0$ and $u'' > 0$ on $(-1, 1)\}$.

1.5. The isometry Lie algebra. For a compact Kähler manifold (M, g), we denote by $\mathfrak{i}_0(M, g)$ the Lie algebra of all Killing vector fields with zeros. Since M is compact this is equivalently the Lie algebra of all Hamiltonian Killing vector fields.

Proposition 2. *Let g be an admissible metric on $M = P(E_0 \oplus E_\infty) \xrightarrow{p} S$ and equip S and $\hat{S} \to S$ with the metrics g_S, $g_{\hat{S}}$ induced by $\sum_a g_a/x_a$ on $\prod_a S_a$. Let $\mathfrak{z}(K, g)$ be the centralizer in $\mathfrak{i}_0(M, g)$ of the Killing vector field $K = J\operatorname{grad}_g z$.*

Then the vector space $\mathfrak{z}(K, g)$ is the direct sum of a lift of $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$ and the span of K in such a way that $p_*: \mathfrak{i}_0(M, g) \to \mathfrak{i}_0(S, g_S)$ is induced by the natural surjection $\mathfrak{i}_0(\hat{S}, g_{\hat{S}}) \to \mathfrak{i}_0(S, g_S)$.

Proof. Let X be a holomorphic vector field on $\hat{S}$ which is Hamiltonian with respect to $\omega_h := \sum_a \omega_a/x_a$; then the projection X_a of X onto the distribution $\mathcal{H}_a$ (induced by TS_a on the universal cover $\prod_a S_a$ of $\hat{S}$) is a Killing vector field with zeros, so $\iota_{X_a}\omega_h = -df_a$ for some function f_a (with integral zero). Thus $\sum_a f_a x_a$ is a Hamiltonian for X with respect to the symplectic form $\omega_{\hat{S}} = \sum_a \omega_a$: since this is the curvature $d\theta$ of the connection on M^0, X lifts to a holomorphic vector field $\tilde{X} = X_H + (\sum_a f_a x_a)K$ on M^0, which is Hamiltonian with potential $\sum_a(1 + x_a z)f_a$ and commutes with K. Here X_H is the horizontal lift to M^0 with respect to θ. $\tilde{X}$ and its potential extend to M since $M \setminus M^0$ has codimension ≥ 2 and $\tilde{X}$ has zeros.

Conversely any element of $\mathfrak{z}(K, g)$ pulls back to a holomorphic vector field V on $\hat{M}$. The projection of V to the normal bundle $\hat{p}^* T\hat{S}$ of $\hat{p}: \hat{M} \to \hat{S}$

is holomorphic hence constant on the $\mathbb{C}P^1$ fibres by Liouville's theorem (the normal bundle is trivial on each fibre), so V is projectable; since V is the pullback of a Killing vector field which commutes with K, it maps to zero iff it comes from a constant multiple of K. This gives a projection to $\mathfrak{i}_0(\hat{S}, g_{\hat{S}})$ splitting the inclusion just defined. $\qquad\square$

2. Admissible extremal Kähler metrics

2.1. Automorphisms and the Matsushima–Lichnerowicz obstruction.
On any compact Kähler manifold (M, g), the Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields lies in an exact sequence:

$$0 \to \mathfrak{h}_0(M) \to \mathfrak{h}(M) \to H^1(M, \mathbb{R})^*,$$

where $\mathfrak{h}_0(M)$ is the ideal of holomorphic vector fields with zeros, which is the Lie algebra of the *reduced automorphism group* $H_0(M) \subset \mathrm{Aut}_0(M)$, i.e., the connected component of the kernel of the Albanese map $\mathrm{Aut}_0(M) \to H^1(M, \mathbb{R})^*/H_1(M, \mathbb{Z})$. The Matsushima–Lichnerowicz theorem [36,41] says that if g is CSC, $\mathfrak{h}_0(M)$ is the complexification of the Lie algebra $\mathfrak{i}_0(M, g)$ of Hamiltonian Killing vector fields and $\mathfrak{h}(M) = \mathfrak{a}(M) \oplus \mathfrak{h}_0(M)$, $\mathfrak{a}(M)$ being the central subalgebra of parallel vector fields: thus $\mathfrak{h}(M)$ is reductive. This condition on $\mathfrak{h}(M)$ is often called the *Matsushima–Lichnerowicz obstruction* to the existence of CSC Kähler metrics on (M, J).

Let us consider the special case of geometrically ruled complex manifolds $M = P(E)$, where $p\colon P(E) \to S$ and E is a holomorphic vector bundle of rank $r+1$ over a compact Kähler $2d$-manifold (S, h, ω_h). Thus M is a kählerian $2m$-manifold with $m = r+d$: for instance, $[\mathcal{O}(1)_E] + k[p^*\omega_h]$ is a Kähler class for $k \gg 1$, where $\mathcal{O}(-1)_E$ is the (fibrewise) tautological line bundle of $P(E)$.

The projection of any holomorphic vector field $V \in \mathfrak{h}(M)$ to the normal bundle p^*TS is constant on each fibre, so V descends to a holomorphic vector field $p_*V \in \mathfrak{h}(S)$. Since $p_*[V_1, V_2] = [p_*V_1, p_*V_2]$, we have an exact sequence of Lie algebras

$$0 \to \mathfrak{h}_S(M) \to \mathfrak{h}_0(M) \to \mathfrak{h}_0(S),$$

where $\mathfrak{h}_S(M)$ is the subspace of $\mathfrak{h}(M)$ of holomorphic vector fields tangent to the fibres of p (which have zeros). Obviously $\mathfrak{h}_S(M) = H^0(S, \mathfrak{sl}(E))$ is the Lie algebra of holomorphic vector fields preserving the $\mathbb{C}P^r$-fibres of $p\colon P(E) \to S$. Since an ideal in a reductive Lie algebra is reductive, we obtain the following weaker (but often more useful) version of the Matsushima–Lichnerowicz obstruction.

Proposition 3. *Let $M = P(E) \to S$ be a geometrically ruled complex manifold which admits a CSC Kähler metric. Then $\mathfrak{h}_S(M)$ must be reductive.*

The following elementary result yields a simple application of this criterion.

Proposition 4. *Let $M = P(E_0 \oplus E_1 \oplus \cdots \oplus E_\ell)$, where E_j are holomorphic vector bundles on a compact kählerian manifold S such that $H^0(S, \mathfrak{gl}(E_j))$ is reductive and $H^0(S, \mathrm{Hom}(E_j, E_i)) = 0$ for all $i < j$. Then $\mathfrak{h}_S(M)$ is reductive iff $H^0(S, \mathrm{Hom}(E_i, E_j)) = 0$ for all $i < j$.*

Proof. Any element of the Lie algebra $\mathfrak{h}_S(M) = H^0(S, \mathfrak{sl}(E))$ may be represented as an $(\ell + 1) \times (\ell + 1)$ matrix (a_{ij}) with $a_{ij} \in H^0(S, \mathrm{Hom}(E_i, E_j))$. By assumption, this matrix is upper-triangular. The strictly upper-triangular matrices form a nilpotent ideal $\mathfrak{n}$ and if this is zero, $\mathfrak{h}_S(M)$ is clearly reductive. Conversely, taking commutators with elements of the form $Id_{E_i}/\mathrm{rank}\, E_i - Id_{E_j}/\mathrm{rank}\, E_j \in H^0(S, \mathfrak{sl}(E))$, we see that $\mathfrak{n} \subseteq [\mathfrak{h}_S(M), \mathfrak{h}_S(M)]$. Hence if $\mathfrak{n} \neq 0$, $[\mathfrak{h}_S(M), \mathfrak{h}_S(M)]$ is not semisimple, i.e., $\mathfrak{h}_S(M)$ is not reductive.   □

Corollary 2. *Let $M = P(E)$ where $E = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ is a direct sum of holomorphic line bundles over a Riemann surface Σ of genus $\mathbf{g}$. If $\mathbf{g} \geq 2$ and $\deg \mathcal{L}_i - \deg \mathcal{L}_j > \mathbf{g} - 1$ for some $0 \leq i, j \leq r$ then M admits no CSC Kähler metric. If $\mathbf{g} \leq 1$, then M admits a CSC Kähler metric if and only if $\deg \mathcal{L}_i = \deg \mathcal{L}_j$ for all i, j (i.e., $\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$ is a polystable vector bundle).*

Proof. We can assume without loss that $E = \mathcal{E}_0 \otimes \mathbb{C}^{r_0} \oplus \mathcal{E}_1 \otimes \mathbb{C}^{r_1} \oplus \cdots \oplus \mathcal{E}_\ell \otimes \mathbb{C}^{r_\ell}$ with $\deg \mathcal{E}_i \leq \deg \mathcal{E}_j$ and $\mathcal{E}_i \ncong \mathcal{E}_j$ for $i < j$. The Kodaira vanishing theorem then implies $H^0(\Sigma, \mathcal{E}_j^{-1} \otimes \mathcal{E}_i) = 0$ for any $i < j$, and we may apply Proposition 4. By assumption, $\deg \mathcal{E}_\ell - \deg \mathcal{E}_0 > \max(0, \mathbf{g} - 1)$, and so $\dim H^0(\Sigma, \mathcal{E}_0^{-1} \otimes \mathcal{E}_\ell) > 0$ by Riemann–Roch. Hence $\mathfrak{h}_\Sigma(M)$ is not reductive, and there is no CSC Kähler metric on M. The converse when $\mathbf{g} \leq 1$ follows from Narasimhan–Seshadri [43].   □

Remark 3. The assumptions of Proposition 4 hold if $E_1, \ldots, E_\ell$ are projectively-flat hermitian vector bundles such that the *slopes* $\mu(E_j) := \overline{c}_1(E_j) \cup \Omega^{d-1}$, with respect to some Kähler class Ω on S ($\dim S = 2d$), satisfy $\mu(E_i) < \mu(E_j)$ for $i < j$. Indeed in this case $\mathfrak{gl}(E_j)$ is a flat hermitian bundle and $H^0(S, \mathfrak{gl}(E_j))$ is the space of parallel sections of $\mathfrak{gl}(E_j)$[4], which is a complexification of the space of parallel sections of $\mathfrak{u}(E_j)$, hence a reductive Lie algebra. The slope condition then ensures $H^0(S, \mathrm{Hom}(E_j, E_i)) = 0$ for all $i < j$ by a theorem of Kobayashi [33].

In general, the condition that $\mathfrak{h}_S(M)$ is reductive does not imply $\mathfrak{h}_0(M)$ is. However it does if $p_* : \mathfrak{h}_0(M) \to \mathfrak{h}_0(S)$ is surjective and $\mathfrak{h}_0(S)$ is reductive. This obviously holds if $\mathfrak{h}_0(S) = 0$. It also holds if (S, g_S) is CSC and there is a metric g on M such that p_* is a surjection from $\mathfrak{i}_0(M, g)$ to $\mathfrak{i}_0(S, g_S)$. This is true for admissible bundles by Proposition 2.

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[4] This is a standard Bochner argument, as in [33]. Alternatively, note that the pullback of $\mathfrak{gl}(E_j)$ to the universal cover of S is trivialized by parallel sections, and apply the open mapping theorem: the pullback of a holomorphic section of $\mathfrak{gl}(E_j)$ has closed bounded image in this trivialization.

Proposition 5. *Let Ω be an admissible Kähler class on the admissible projective bundle $M = P(E_0 \oplus E_\infty) \to S$, where the local product metric g_S on S is CSC. Then the admissible metrics in Ω are invariant under a common maximal compact connected subgroup of $H_0(M)$, and $\mathfrak{h}_0(M)$ is reductive iff $H^0(S, \mathrm{Hom}(E_0, E_\infty)) = 0 = H^0(S, \mathrm{Hom}(E_\infty, E_0))$. This latter condition holds if $\overline{c}_1(E_\infty) - \overline{c}_1(E_0)$ is strictly indefinite.*

Proof. Let g be an admissible Kähler metric on M. We know by the Matsushima–Lichnerowicz theorem that $\mathfrak{h}(S)$ is reductive and $\mathfrak{h}_0(S)$ is the complexification of the Lie algebra $\mathfrak{i}_0(S, g_S)$ of a maximal compact subgroup of $H_0(S)$. Then by Proposition 2, both $p_*: \mathfrak{h}_0(M) \to \mathfrak{h}_0(S)$ and $p_*: \mathfrak{i}_0(M, g) \to \mathfrak{i}_0(S, g_S)$ are surjective.

To show that the Lie algebra $\mathfrak{i}_0(M, g)$ is a maximal compact subalgebra of $\mathfrak{h}_0(M)$, it therefore suffices to show that $\mathfrak{i}_0(M, g) \cap \mathfrak{h}_S(M)$ is a maximal compact subalgebra of $\mathfrak{h}_S(M)$. Since $\overline{c}_1(E_\infty) - \overline{c}_1(E_0) = \omega_S$, we can certainly arrange that $\mu(E_\infty) - \mu(E_0)$ is nonzero by the choice of a Kähler class on S. Then, by Remark 3, we are under the hypotheses of Proposition 4, and, as in its proof, we have that $\mathfrak{h}_S(M)$ is the direct sum of the reductive centralizer of K and a nilpotent ideal $\mathfrak{n}$ in $[\mathfrak{h}_S(M), \mathfrak{h}_S(M)]$. The first part now follows easily from Proposition 2 and Remark 3.

As noted above, $\mathfrak{h}_0(M)$ is reductive iff $\mathfrak{h}_S(M)$ is. By Proposition 4, the latter happens iff both $H^0(S, \mathrm{Hom}(E_0, E_\infty)) = 0 = H^0(S, \mathrm{Hom}(E_\infty, E_0))$. This indeed holds if $\overline{c}_1(E_\infty) - \overline{c}_1(E_0)$ is strictly indefinite (by the vanishing theorem of Kobayashi [33] as in Remark 3), since we can then choose Kähler classes on S such that the corresponding slopes have $\mu(E_\infty) - \mu(E_0)$ with either sign. $\square$

Since S is a local Kähler product, it is CSC iff the factors S_a ($a \in \mathcal{A}$) in the universal cover are CSC.

2.2. The Futaki invariant and extremal vector field.

On a compact Kähler $2m$-manifold (M, J, g, ω), recall that the (normalized) *Futaki invariant* of a real holomorphic vector field with zeros $V = J \,\mathrm{grad}_g f + \mathrm{grad}_g h$ is defined by

$$\mathfrak{F}_\omega(V) = \frac{\left(\int_M \mu_g \int_M (f + ih) Scal_g \mu_g - \int_M Scal_g \mu_g \int_M (f + ih) \mu_g \right)}{\mathrm{Vol}(M)^2},$$

where $\mu_g = \omega^m / m!$ is the volume form of g. Futaki [23] showed that this complex number is independent of the choice of metric in the Kähler class $\Omega = [\omega]$, and that the map $\mathfrak{F}_\Omega: \mathfrak{h}_0(M) \to \mathbb{C}$ is a character on $\mathfrak{h}_0(M)$. $\mathfrak{F}_\Omega$ is closely related to the *Futaki–Mabuchi extremal vector field* $K_\Omega := J \,\mathrm{grad}_g \,\mathrm{pr}_g Scal_g$ of (M, J, Ω, G), where G is a maximal compact connected subgroup of $H_0(M)$ and pr_g is the L_2-projection onto the space of Killing potentials with respect to any G-invariant metric g in Ω: Futaki and Mabuchi [24] showed that K_Ω is independent of this choice. Clearly $\mathfrak{F}_\Omega$ and K_Ω vanish if Ω contains a CSC metric. Calabi [8] showed that if

$\mathfrak{F}_\Omega$ vanishes then any extremal Kähler metric in Ω is a CSC metric, but the vanishing of $\mathfrak{F}_\Omega$ does not suffice in general for the existence of a CSC metric in Ω.

Let Ω be an admissible Kähler class on $M = P(E_0 \oplus E_\infty) \to S$ and suppose in addition that for $a \in \mathcal{A}$, $\pm g_a$ is a CSC Kähler metric with scalar curvature $Scal_{\pm g_a} = \pm 2d_a s_a$. Let $p_c(t) = \prod_a (1 + x_a t)^{d_a}$ and define

(7)
$$\alpha_r := \int_{-1}^1 p_c(t) t^r \, dt,$$
$$\beta_r := p_c(1) + (-1)^r p_c(-1) + \int_{-1}^1 \left(\sum_a \frac{d_a s_a x_a}{1 + x_a t} \right) p_c(t) t^r \, dt.$$

We now compute the Futaki invariant $\mathfrak{F}_\Omega(K)$ of $K = J \operatorname{grad}_g z$ and show that K_Ω is essentially $\mathfrak{F}_\Omega(K)K$, where G the maximal compact connected subgroup of $H_0(M)$ of Proposition 5 preserving admissible Kähler metrics in Ω. $\mathfrak{F}_\Omega(K)$ will reappear in the next paragraph as the leading coefficient of a polynomial associated with Ω.

Proposition 6. *Suppose M is admissible over a CSC base and Ω is an admissible Kähler class with admissible metric g. Then $\mathfrak{F}_\Omega(K) = 2(\alpha_0\beta_1 - \alpha_1\beta_0)/\alpha_0^2$. Also the L_2-projection of $Scal_g$ orthogonal to the space of Killing potentials is*

(8)
$$Scal_g + Az + B$$

where A and B are given by

(9)
$$A\alpha_1 + B\alpha_0 = -2\beta_0$$
$$A\alpha_2 + B\alpha_1 = -2\beta_1.$$

(Since $\alpha_0\alpha_2 > \alpha_1^2$, this system has a unique solution for A, B.) In particular the extremal vector field of (Ω, G) is $K_\Omega = -AK = 2(\alpha_0\beta_1 - \alpha_1\beta_0)K/(\alpha_0\alpha_2 - \alpha_1^2)$.

Proof. We may rescale Ω so that an admissible metric (g, ω) in Ω is of the form (1). We then have

$$\mu_g = \frac{\omega^m}{m!} = p_c(z) \left(\bigwedge_a \frac{(\omega_a/x_a)^{d_a}}{d_a!} \right) \wedge dz \wedge \theta,$$

where $m = 1 + \sum_a d_a$ is the complex dimension of M. Thus

$$\int_M \mu_g = 2\pi \operatorname{Vol}\left(S, \prod_a \frac{\omega_a}{x_a} \right) \alpha_0 \ (= \operatorname{Vol}(M)),$$

$$\int_M z \mu_g = 2\pi \operatorname{Vol}\left(S, \prod_a \frac{\omega_a}{x_a} \right) \alpha_1 = \operatorname{Vol}(M)\alpha_1/\alpha_0,$$

where $\operatorname{Vol}(S, \prod_a \frac{\omega_a}{x_a}) = \prod_a \operatorname{Vol}(S_a, \frac{\omega_a}{x_a})$ in the case when S is a global product.

The scalar curvature of (M, g) is given by

$$(10) \qquad Scal_g = \sum_a \frac{2d_a s_a x_a}{1 + x_a z} - \frac{F''(z)}{p_c(z)}$$

where $F(z) = \Theta(z) p_c(z)$ (see e.g. [4, (79)]). We thus calculate

$$\int_M z \, Scal_g \mu_g = 2\pi \text{Vol}\left(S, \prod_a \frac{\omega_a}{x_a}\right)$$

$$\times \int_{-1}^1 \left(\left(\sum_a \frac{2d_a s_a x_a}{1 + x_a z}\right) p_c(z) - F''(z)\right) z \, dz$$

$$= 2\pi \text{Vol}\left(S, \prod_a \frac{\omega_a}{x_a}\right)$$

$$\times \left(\int_{-1}^1 \left(\sum_a \frac{2d_a s_a x_a}{1 + x_a z}\right) p_c(z) z \, dz - [zF'(z) - F(z)]_{-1}^{+1}\right)$$

$$= 2\text{Vol}(M)\beta_1/\alpha_0,$$

where we integrate by parts, then impose the boundary conditions (3). Similarly,

$$\int_M Scal_g \mu_g = 2\text{Vol}(M)\beta_0/\alpha_0$$

and the first claim follows.

For the second claim note that the above integral formulae imply $Scal_g + Az + B$ is orthogonal to the Killing potentials $1, z$ if and only if (9) holds. By the form of $Scal_g$, the fact that the s_a are constant, and Proposition 2, the result follows. □

Note that the above expression for $\mathfrak{F}_\Omega(K)$ is manifestly independent of the choice of a smooth function $\Theta(z)$ satisfying (3), as it should be according to the general theory [23]. Indeed, as we have already discussed in Sect. 1.4, these smooth functions $\Theta(z)$ define Kähler metrics within the same Kähler class.

2.3. K-energy and the extremal polynomial.

Given a complex $2m$-manifold (M, J), a maximal compact connected subgroup G of $H_0(M)$, and a Kähler class Ω, we denote by $\mathcal{M}_\Omega$ the infinite dimensional Fréchet space of Kähler metrics in Ω and let $\mathcal{M}_\Omega^G$ be the subspace of G-invariant Kähler metrics in Ω. Following Guan [28] and Simanca [48], consider the map

$$g \mapsto \text{pr}_g^\perp Scal_g \mu_g,$$

where $\text{pr}_g^\perp$ is the L_2-projection orthogonal to the space of Killing potentials. This can be viewed (by integration) as a 1-form σ on $\mathcal{M}_\Omega^G$, which turns out

to be closed. Therefore for any $\omega_0 \in \Omega$, there exists a unique functional $E_{\omega_0}^G : \mathcal{M}_\Omega^G \to \mathbb{R}$ with

$$dE_{\omega_0}^G = -\sigma,$$

$E_{\omega_0}^G(\omega_0) = 0$. Note that changing the base point $\omega_0 \in \mathcal{M}_\Omega$ would change $E_{\omega_0}^G$ by an additive constant. We refer to $E_{\omega_0}^G$ as the (modified) *K-energy*: it agrees with the Mabuchi K-energy [38] when G is trivial.

By definition, it is clear that the critical points of $E_{\omega_0}^G$ are exactly the extremal Kähler metrics in $\mathcal{M}_\Omega^G$, since $\sigma = 0$ means that $Scal_g$ is a Killing potential. Note that by the Calabi theorem [7], any extremal Kähler metric $g \in \mathcal{M}_\Omega$ belongs to $\mathcal{M}_\Omega^G$ with $G = \mathrm{Isom}_0(M, g) \cap H_0(M)$.

Building on earlier work by Bando–Mabuchi [6], Chen [9], Donaldson [16] and others, Chen and Tian have established the following uniqueness result and necessary condition for existence of an extremal Kähler metric.

Theorem 4 [10,11]. *Extremal Kähler metrics in $\mathcal{M}_\Omega$ are unique up to automorphism and any extremal Kähler metric in $\mathcal{M}_\Omega^G$ realizes the absolute minimum of $E_{\omega_0}^G$ (for any $\omega_0 \in \mathcal{M}_\Omega^G$). In particular, if $\mathcal{M}_\Omega^G$ contains an extremal Kähler metric, then $E_{\omega_0}^G$ is bounded from below.*

Now let M be an admissible projective bundle over a CSC base as in the previous paragraph. We want to obtain a formula for the K-energy as a functional acting on $\mathcal{K}_\omega^{\mathrm{adm}}$, where ω is fixed, so we need to use the description given in Sect. 1.4 which shows how $\mathcal{K}_\omega^{\mathrm{adm}}$ is embedded into $\mathcal{M}_\Omega^G$, in which the complex structure is fixed.

This description shows that if $u_t(z)$ is a path of symplectic potentials in $\mathcal{K}_\omega^{\mathrm{adm}}$, then the smooth functions $h_t(y_c) - h_c(y_c)$ define a path $\omega + dd_{J_c}^c(h_t(y_c) - h_c(y_c))$ in $\mathcal{M}_\Omega$, where h_t are introduced by

$$h_t(y_t(z)) = -u_t(z) + y_t z, \quad y_t = u_t'(z),$$

so that, after setting $z = y_t^{-1}(y_c)$, we have

$$h_t(y_c) = -u_t\left(y_t^{-1}(y_c)\right) + y_t^{-1}(y_c)y_c.$$

Differentiating with respect to t, we get for the corresponding vector fields $\dot{u} \in T_g(\mathcal{K}_\omega^{\mathrm{adm}})$ and $\dot{h} \in T_\omega(\mathcal{M}_\Omega^G)$, the relation (cf. [28,15]):

$$\dot{h} = -\dot{u}.$$

Hence we obtain the following symplectic version of the (modified) K-energy.

Lemma 4. *The K-energy E_ω^G, restricted to the space of admissible Kähler metrics in Ω and viewed as a function on the space of symplectic potentials,*

is determined uniquely up to an additive constant by the formula

$$\left(dE_\omega^G\right)_g[\dot{u}] = \int_M \left(\mathrm{pr}_g^\perp Scal_g\right)\dot{u}\,\mu_g,$$

where $\mathrm{pr}_g^\perp$ *denotes the* L_2*-projection orthogonal to the space of Killing potentials.*

Consider an admissible metric g in Ω corresponding to the function $\Theta(z) = F(z)/p_c(z)$. Since the base S is CSC we have, by Proposition 6, $\mathrm{pr}_g^\perp Scal_g = Scal_g + Az + B$, with A and B given by (9) and $Scal_g$ by (10).

Lemma 5. *There is a unique smooth function* F_Ω *on* $[-1, 1]$ *with*

$$(11) \qquad F_\Omega''(z) = \left(Az + B + \sum_a \frac{2d_a s_a x_a}{1 + x_a z}\right) p_c(z)$$

and $F_\Omega(\pm 1) = 0$. F_Ω *satisfies (6) and is a polynomial of degree* $\leq m + 2$, *the coefficient of* z^{m+2} *being a nonzero multiple of A.*

Proof. There is clearly a unique solution to (11) with $F_\Omega(\pm 1) = 0$. One easily checks, using (9) that the solution is

$$(1 + z)p_c(-1) + (1 - z)p_c(1)$$
$$+ \int_{-1}^1 \left(\frac{1}{2}(At + B) + \sum_a \frac{d_a s_a x_a}{1 + x_a t}\right) p_c(t)|z - t|dt.$$

The derivative of this function is

$$p_c(-1) - p_c(1) + \int_{-1}^1 \left(\frac{1}{2}(At + B) + \sum_a \frac{d_a s_a x_a}{1 + x_a t}\right) p_c(t)\mathrm{sign}(z - t)dt,$$

which gives the formulae for $F_\Omega'(\pm 1)$ in (6), using the first equation of (9). $\qquad\square$

The motivation for this lemma is that now $Scal_g + Az + B = (F_\Omega''(z) - F''(z))/p_c(z)$, see (10). Furthermore, F and F_Ω satisfy the same boundary conditions (6).

Proposition 7. *Let* Ω *be an admissible Kähler class on an admissible bundle over a CSC base. Then the K-energy restricted to the space of admissible Kähler metrics* $\mathcal{K}_\omega^{\mathrm{adm}}$ *is (up to an additive constant) a positive multiple of the functional*

$$\mathcal{E}_{g_c} : u(z) \mapsto \int_{-1}^1 F_\Omega(z)\left(u''(z) - u_c''(z)\right)dz - \int_{-1}^1 p_c(z) \log\left(\frac{u''(z)}{u_c''(z)}\right)dz.$$

Proof. $\mathcal{E}_{g_c}$ is well-defined by Lemma 2 and its gradient is

$$(d\mathcal{E}_{g_c})_g[\dot{u}] = \int_{-1}^{1} F_\Omega(z)\dot{u}''(z)dz - \int_{-1}^{1} p_c(z)\frac{\dot{u}''(z)}{u''(z)}dz$$

$$= \int_{-1}^{1} (F_\Omega(z) - F(z))\dot{u}''(z)dz.$$

Integrating twice by parts, using the fact that F and F_Ω both satisfy (6), and multiplying by $2\pi \text{Vol}(S, \prod_a \omega_a/x_a)$, we obtain $\int_M (Scal_g + Az + B)\dot{u}\mu_g$. □

Remark 4. It is worth noticing that (by (3)) the functional

$$\mathcal{E} : u(z) \mapsto \int_{-1}^{1} \left(F_\Omega(z)u''(z) - p_c(z)\log u''(z) \right)dz$$

is well-defined on $\mathcal{K}_\omega^{\text{adm}}$ and is still a primitive of the restriction of the 1-form σ to $\mathcal{K}_\omega^{\text{adm}}$. It can then be regarded as a natural determination of a (modified) K-energy on $\mathcal{K}_\omega^{\text{adm}}$, with no explicit reference to any base point in $\mathcal{K}_\omega^{\text{adm}}$, and such that $\mathcal{E}_{g_c}(u) = \mathcal{E}(u) - \mathcal{E}(u_c)$.

Corollary 3. *If there is an extremal Kähler metric in* Ω*, then* $F_\Omega \geq 0$ *on* $[-1, 1]$*.*

Proof. If there is an extremal Kähler metric in Ω, then by Theorem 4 [10, 11], the K-energy is bounded from below. We now apply an argument from [15]: take any nonnegative smooth function $f(z)$ with $\text{supp}(f) \subset (-1, 1)$ and consider the sequence $u_k(z)$ with $u_k''(z) = u_c''(z) + kf(z)$ of symplectic potentials (cf. Lemma 2) for admissible Kähler metrics. We therefore get

$$\mathcal{E}_{g_c}(u_k) = -\int_{-1}^{1} p_c(z)\log\left(1 + k\frac{f(z)}{u_c''(z)}\right)dz + k\int_{-1}^{1} F_\Omega(z)f(z)dz.$$

This will tend to $-\infty$ if $\int_{-1}^{1} F_\Omega(z)f(z)dz < 0$ for some f. □

In the next paragraph, where we complete the proof of Theorem 1, we shall show that positivity of F_Ω on $(-1, 1)$ is a necessary and sufficient condition for the existence of an extremal Kähler metric in Ω.

Definition 1. Let Ω be an admissible Kähler class on M. Then the polynomial F_Ω constructed above will be called the *extremal polynomial* of Ω.

2.4. A characterization of extremal admissible Kähler classes. In this paragraph we prove Theorem 1 in three steps. First, if the extremal polynomial F_Ω of an admissible Kähler class Ω is positive on $(-1, 1)$, we construct an admissible extremal Kähler metric in Ω by adapting an argument of Guan and Hwang (cf. [27, 30, 31]): we discuss their work further in

the next section. Second, we extend the continuity argument of [54] to prove the existence of admissible extremal Kähler metrics for Ω sufficiently small. Third, we use Corollary 3, the uniqueness result of Chen–Tian [10,11] and an argument from [54] to show that an extremal Kähler metric in Ω is admissible up to automorphism. Hence we deduce that the existence of an extremal Kähler metric in Ω implies that F_Ω is positive on $(-1, 1)$.

We begin with the construction. By Proposition 1, an admissible metric (1) is extremal exactly when for each $a \in \hat{\mathcal{A}}$, $\pm g_a$ is a CSC Kähler metric with $Scal_{\pm g_a} = \pm 2d_a s_a$ and (4)–(5) hold for a polynomial P of degree $\leq N + 1$, where $N = \#\hat{\mathcal{A}}$. The metric g is CSC iff P has degree $\leq N$.

We have seen that the boundary conditions (3) imply (6) and the converse clearly holds if $p_c(\pm 1) \neq 0$ (i.e., $d_0 = 0 = d_\infty$). However, if g is extremal, then (4)–(5) imply that $F''(z) = p_c'(z)\Upsilon(z)$ with $\Upsilon(-1) = 2(d_0 + 1)$ if $d_0 > 0$ and $\Upsilon(1) = -2(d_\infty + 1)$ if $d_\infty > 0$ (because of the normalization of the Fubini–Study metrics on S_0 and S_∞). Hence, by (6), $F'(z) = p_c(z)\Psi(z)$ with $\Psi(-1) = 2(d_0 + 1)$ and $\Psi(1) = -2(d_\infty + 1)$, and $\Theta(\pm 1) = 0$. Now by l'Hôpital's rule, $\Theta'(\pm 1) = \mp 2$. Hence, for extremal Kähler metrics, the boundary conditions (3) are equivalent to (6).

In summary, to obtain a globally defined admissible extremal metric on a projective bundle $P(E_0 \oplus E_\infty) \to S$, we need, for CSC Kähler metrics $(\pm g_a, \pm \omega_a)$ satisfying $\overline{c}_1(E_\infty) - \overline{c}_1(E_0) = \sum_a [\omega_a/2\pi]$, to solve (4) and (5) for a polynomial F (of degree $\leq m + 2$) which satisfies (6) and is positive on $(-1, 1)$.

For an admissible Kähler class Ω on M, we claim that (4)–(5) and the boundary conditions (6) have a unique solution for F, given by the extremal polynomial F_Ω.

Proposition 8. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible 2m-manifold, where S is CSC. Then for any admissible Kähler class Ω on M, the extremal polynomial F_Ω is the unique polynomial F of degree $\leq m + 2$ satisfying (4)–(5) and (6).*

Proof. An admissible Kähler class on M is specified by parameters x_a such that $x_0 = 1$, $x_\infty = -1$ and otherwise $0 < |x_a| < 1$ with ω_a/x_a positive. We write $Scal_{\pm g_a} = \pm 2d_a s_a$. Equation (5) can be solved for a degree $N - 1$ polynomial P_0 by Lagrange interpolation, i.e.,

$$P_0(z) = \sum_a 2d_a s_a x_a \prod_{b \in \hat{\mathcal{A}}} (1 + x_b z)$$

and then we can write the general degree $N + 1$ solution as

$$P(z) = P_0(z) + (Az + B) \prod_{a \in \hat{\mathcal{A}}} (1 + x_a z)$$

$$= \left(Az + B + \sum_a \frac{2d_a s_a x_a}{1 + x_a z} \right) \prod_{a \in \hat{\mathcal{A}}} (1 + x_a z)$$

so that

$$F''(z) = \left(Az + B + \sum_a \frac{2d_a s_a x_a}{1 + x_a z} \right) p_c(z).$$

Integrating $F''(z)$ and $zF''(z)$ on $[-1, 1]$, (6) now implies that A, B satisfy (9). Hence $F = F_\Omega$ is the unique solution. $\qquad\square$

Remark 5. An alternative approach is to solve the initial value problem (at $z = -1$) for $F(z)$. The boundary conditions at $z = 1$ then show that A, B satisfy (9). This gives another formula for the extremal polynomial:

$$(12) \quad F_\Omega(z) = 2(1+z)p_c(-1) + \int_{-1}^z \left(At + B + \sum_a \frac{2d_a s_a x_a}{1 + x_a t} \right) p_c(t)(z-t)dt,$$

where A and B are given (as usual) by (9).

Proposition 8 shows that the existence of an *admissible* extremal Kähler metric in Ω is equivalent to the positivity of the extremal polynomial F_Ω on $(-1, 1)$. Since the leading coefficient is a nonzero multiple of A, Proposition 6 shows that such a metric will be CSC iff the Futaki invariant $\mathfrak{F}_\Omega(K)$ vanishes.

Remark 6. Since F_Ω depends continuously (in fact analytically) on the admissible Kähler class and has roots at ± 1 with fixed multiplicities $d_\infty + 1$ and $d_0 + 1$, it is positive on $(-1, 1)$ for an open subset of such classes. This observation fits in with the general stability result of LeBrun and Simanca [35].

We now show that F_Ω is positive on $(-1, 1)$ for sufficiently small Ω.

Proposition 9. *Let $M = P(E_0 \oplus E_\infty) \to S$ be admissible, where S is a local Kähler product of CSC metrics $(\pm g_a, \pm \omega_a)$. Then there is a nonempty open subset of admissible Kähler classes on M which contain an (admissible) extremal Kähler metric of positive scalar curvature. The admissible Kähler classes containing a CSC metric form a real analytic hypersurface which is nonempty if $\bar{c}_1(E_\infty) - \bar{c}_1(E_0)$ is strictly indefinite over S (i.e., the definite forms ω_a do not all have the same sign).*

Proof. As we noted in Remark 6, the extremal polynomial F_Ω is positive on $(-1, 1)$ for an open subset of admissible Kähler classes. It remains to see that this open subset is nonempty and to find the CSC metrics in the family. For this, we study the behaviour of F_Ω near $x_a = 0$ for all $a \in \mathcal{A}$.

Lemma 6. *The coefficients A and B defined by (9), as functions of x_a ($a \in \mathcal{A}$) for $|x_a|$ small are given by*

$$(13) \quad A = -2(2 + d_0 + d_\infty) \sum_{a \in \mathcal{A}} d_a x_a + O(x^2)$$

$$(14) \quad B = -(1 + d_0 + d_\infty)(2 + d_0 + d_\infty)$$
$$- 2\sum_{a \in \mathcal{A}} d_a s_a x_a + 2(d_0 - d_\infty)\sum_{a \in \mathcal{A}} d_a x_a + O(x^2)$$

where $O(x^2)$ is shorthand for $\sum_{a,b \in \mathcal{A}} O(x_a x_b)$.

The proof is given in Appendix B. It follows that in the limit $x_a \to 0$ for all $a \in \mathcal{A}$ (which does not give a Kähler class), $F''_\Omega(z)/(1+z)^{d_0}(1-z)^{d_\infty}$ is given by

$$(15) \quad -(1 + d_0 + d_\infty)(2 + d_0 + d_\infty) + \frac{2d_0(d_0 + 1)}{1 + z} + \frac{2d_\infty(d_\infty + 1)}{1 - z}.$$

If $d_0 = 0$ and $d_\infty = 0$, this is negative on $(-1, 1)$ and F_Ω is convex. By (6), for some $\varepsilon > 0$, F_Ω is positive and increasing on $(-1, -1 + \varepsilon)$ and concave if $d_0 > 0$, while it is positive and decreasing on $(1 - \varepsilon, 1)$ and concave if $d_\infty > \infty$. By (15), for sufficiently small Ω, F_Ω does not have enough inflection points to have a zero on $(-1, 1)$ and so it is positive there. Hence the set of admissible Kähler classes containing an admissible extremal Kähler metric is nonempty. Since $z \in [-1, 1]$, we see that for x_a sufficiently small, the scalar curvature $-Az - B$ of g is positive.

Now the Futaki invariant $\mathcal{F}_\Omega(K) \sim A$ is a rational function of x_a, $a \in \mathcal{A}$, so the CSC metrics form a real analytic hypersurface. It is then clear from (13) that if x_a occur with both signs, A has nonconstant sign for small x_a.   □

(Using the Matsushima–Lichnerowicz criterion, the CSC existence result in this proposition provides an alternative proof of the last part of Proposition 5.)

Proposition 10. *Let M be an admissible projective bundle over a CSC base. Then if an admissible Kähler class on M contains an extremal Kähler metric, this extremal Kähler metric is admissible up to automorphism.*

Proof. Consider the set U of admissible Kähler classes that contain an extremal Kähler metric invariant under the maximal compact subgroup G of $H_0(M)$ defined in Proposition 5. By LeBrun–Simanca [35], U is open in the set of all admissible Kähler classes. Suppose there is some admissible class Ω_0 (with parameters x_a^0) which contains an inadmissible extremal metric g_0; by the Calabi theorem, we can assume that g_0 is G-invariant, i.e., $\Omega_0 \in U$. By LeBrun–Simanca [35], this implies that in all Kähler classes sufficiently close to Ω_0, there are G-invariant extremal Kähler metrics close to g_0 (in suitable Sobolev spaces; by the Sobolev embedding theorem this also holds in the $C^\ell(M)$ topology, for any $\ell > 0$). We have two cases:

- there is an open neighbourhood of Ω_0 in U for which the extremal polynomial is not positive on $(-1, 1)$;
- there are admissible extremal Kähler metrics in classes arbitrarily close to Ω_0.

The first case contradicts the existence of admissible extremal Kähler metrics on M for sufficiently small Ω, i.e., the positivity of the extremal polynomial. Indeed, for such Ω, $F_\Omega(z)$ has at most two inflection points in $(-1, 1)$ by (15), and an easy case by case analysis (according to whether d_0, d_∞ are zero or positive) then shows that $Q_\Omega(z) := F_\Omega(z)/(1+z)^{d_0}(1-z)^{d_\infty}$, as a polynomial in z, has simple roots. However,

by assumption, for all Ω in some open neighbourhood of Ω_0, the extremal polynomial F_Ω is nonnegative (by Corollary 3) but not positive on $(-1, 1)$, so $Q_\Omega(z)$ has zero discriminant. Since it is analytic in Ω, it is identically zero, a contradiction.

In the second case we apply instead the uniqueness result [10, 11] for extremal Kähler metrics, as in [54]. Let Ω_k be a sequence of admissible Kähler classes (with parameters x_a^k) which converges to Ω_0 (i.e., x_a^k converges to x_a^0 for all a) and such that Ω_k contains an admissible extremal Kähler metric $\tilde{g}_k$ which, without loss, is not CSC. By LeBrun–Simanca [35], it follows that for $k \gg 1$ there are G-invariant extremal Kähler metrics $g_k \in \Omega_k$ which converge to g_0 in the $C^2(M)$ topology. By Chen–Tian [10,11], g_k is the pullback of $\tilde{g}_k$ by an automorphism Ψ_k of (M, J, Ω_k, G). We now claim that g_0 is the pullback by an automorphism of an admissible extremal Kähler metric in Ω_0, completing the proof.

To prove the claim, we use the theory of Hamiltonian 2-forms of order 1 from [4,5]. Since $\tilde{g}_k$ admits such a 2-form with S^1 action generated by K, so does g_k (by Proposition 6, K is a nonzero multiple of the extremal vector field of (Ω_k, G) and so is preserved by Ψ_k). Now if (g, ω) is any Kähler metric on M for which $K = J \operatorname{grad}_g z$ generates an isometric S^1 action, then it follows from [5] that this action comes from a Hamiltonian 2-form if and only if it is *rigid* (meaning that $g(K, K)$ depends only on z) and *semisimple* (meaning that for any regular value z_0 of z, the z-derivative at $z = z_0$ of the family of Kähler quotient metrics $g_{\hat{S}}(z)$ on the complex quotient $\hat{S}$ is parallel and diagonalizable with respect to $g_{\hat{S}}(z_0)$). Thus, the S^1 action generated by K is rigid and semisimple with respect to g_k, hence also with respect to g_0 by continuity, so that g_0 itself admits a Hamiltonian 2-form of order 1 with S^1 action generated by K.

We now apply Theorem 3 and Proposition 1 to g_0: it follows that g_0 is adapted to the bundle structure of $M = P(E_0 \oplus E_\infty) \to S$ and induces a CSC Kähler metric $g_{\hat{S}}{}^0$ on $\hat{S} = P(E_0) \times_S P(E_\infty)$, and a connection 1-form θ_0 on the principal $\mathbb{C}^\times$-bundle over $\hat{S}$ (whose total space is identified with M^0), such that the $(1, 1)$-form $d\theta_0 =: \omega_{\hat{S}}{}^0$ is parallel and diagonalizable with respect to $g_{\hat{S}}{}^0$. Then (by Chern–Weil theory) we have $[\omega_{\hat{S}}{}^0] = [\omega_{\hat{S}}]$, where $\omega_{\hat{S}}$ is the product CSC metric defining the admissible class Ω_0. It follows that the Kähler form of $g_{\hat{S}}{}^0$ is in the cohomology class $\sum_a [\omega_a]/x_a^0$, since, as explained in Sect. 1.3, this cohomology class is determined by the admissible class Ω_0, which can be uniquely written as the sum of the 'projective Thom class' $\hat{\Xi}$, of $\hat{M} = P(\mathcal{O} \oplus \hat{L}) \to \hat{S}$ and a pullback from $\hat{S}$.

Now by Chen–Tian [10, 11] again (see also [14, 40]), there is an automorphism ψ of $\hat{S}$ with $g_{\hat{S}} = \psi^* g_{\hat{S}}{}^0$, since these are CSC Kähler metrics in the same Kähler class. In fact, the proof of [10, 11] essentially shows that any two extremal metrics in a given Kähler class can be connected by a geodesic in the space of Kähler potentials, and therefore ψ can be chosen in the reduced (connected) automorphism group $H_0(\hat{S}) \subseteq \operatorname{Aut}_0(\hat{S})$ (see e.g. [28,26]); in particular, such a ψ acts trivially on cohomology. By Proposition 2 there is

a fibre-preserving S^1-equivariant automorphism Ψ of $\hat{M} = P(\mathcal{O} \oplus \hat{L}) \to \hat{S}$, which induces ψ on $\hat{S}$. Thus, Ψ preserves the $\mathbb{C}^\times$-bundle structure of $M^0 \to \hat{S}$ and sends the connection 1-form θ_0 to a connection 1-form $\tilde{\theta}_0$ with curvature $d\tilde{\theta}_0 = \psi^* \omega_{\hat{S}}{}^0 \in [\omega_{\hat{S}}{}^0] = [\omega_{\hat{S}}]$; now since $\psi^* \omega_{\hat{S}}{}^0$ and $\omega_{\hat{S}}$ are both parallel (and therefore harmonic) with respect to $\psi^* g_{\hat{S}}{}^0 = g_{\hat{S}}$, Hodge theory implies they are equal. We can therefore send $\tilde{\theta}_0$ to θ via a bundle isomorphism.

Thus we have constructed (on M^0 and hence, by a standard extension argument, everywhere) an automorphism sending g_0 to an admissible extremal Kähler metric in Ω_0, as required. $\square$

Theorem 1 follows from Propositions 8–10.

3. Existence and nonexistence results for extremal Kähler metrics

In this section we use Theorem 1 to construct explicit examples of extremal Kähler metrics. We also obtain some nonexistence results for CSC Kähler metrics.

3.1. Constructing admissible extremal Kähler metrics.
We begin with a root counting argument due to Hwang [30] and Guan [27] which gives a complete construction when the base S is a local Kähler product of *nonnegative* CSC Kähler metrics (in fact Hwang and Guan only considered the case that S has constant nonnegative eigenvalues of the Ricci tensor, but the proof is no different in general, and the idea to weaken this hypothesis is already explored in [31]).

Proposition 11. *Suppose that $M = P(E_0 \oplus E_\infty) \to S$ is admissible where S is a local Kähler product of nonnegative CSC metrics. Then every admissible Kähler class contains an (admissible) extremal Kähler metric.*

Proof. By the boundary conditions F_Ω is positive, and increasing or decreasing, on $(-1, -1 + \varepsilon)$ or $(1 - \varepsilon, 1)$ respectively, for some $\varepsilon > 0$. Suppose it is not positive on $(-1, 1)$. Then it has at least two maxima, one minimum and two inflection points on $(-1, 1)$. It follows that P has at least two roots in $(-1, 1)$.

Let $y_1 \leq \cdots \leq y_Q$ and $z_1 \leq \cdots \leq z_R$ ($Q, R \geq 0$) denote the roots (counted with multiplicity) of P in $[1, \infty)$ and $(-\infty, -1]$ respectively, and put $y_0 = 1$, $y_{Q+1} = \infty$, $z_0 = -\infty$, $z_{R+1} = -1$. We order $\{x_a : a \in \hat{\mathcal{A}}\}$ as

$$-1 \leq x_{a_1} < \cdots < x_{a_J} < 0 < x_{a_{J+1}} < \cdots < x_{a_N} \leq 1$$

(for some $0 \leq J \leq N$) so that g_{a_j} is negative definite (hence with s_{a_j} nonpositive) for $j \leq J$ and positive definite (hence with s_{a_j} nonnegative) for $j \geq J + 1$.

Therefore by (5), for each $0 \leq q \leq Q$, there is at most one x_{a_j} with $y_q \leq -1/x_{a_j} < y_{q+1}$, so that $Q + 1 \geq J$ with equality iff there is exactly

one x_{a_j} in each such interval. Similarly, for each $0 \leq r \leq R$, there is at most one x_{a_j} with $z_r < -1/x_{a_j} \leq z_{r+1}$, so that $R + 1 \geq N - J$ with equality iff there is exactly one x_{a_j} in each such interval. Thus P has at least $N - 2$ roots outside $(-1, 1)$.

Since P has degree $\leq N + 1$, it has at most $N - 1$ roots outside $(-1, 1)$, so we must either have $Q + 1 = J$ or $R + 1 = N - J$. If (without loss of generality) $Q + 1 = J$ then $-1/x_{a_1} < y_1$, so that $P(-1/x_{a_1}) > 0$ (by (5) again) and there must be a root of P between $-1/x_{a_1}$ $(1 \leq -1/x_{a_1} < y_1)$ and the last maximum of F_Ω in $(-1, 1)$. This now forces $R + 1 = N - J$ also, hence $P(-1/x_{a_N}) > 0$ and there must be a root of P between the first maximum of F_Ω in $(-1, 1)$ and $-1/x_{a_N}$ $(z_R < -1/x_{a_N} \leq -1)$, contradicting deg $P \leq N + 1$. $\qquad\square$

Because of this result, in the rest of this section we shall mainly be interested in the influence of negative scalar curvature factors in the base metrics. In the presence of such factors, the existence of extremal Kähler metrics is nontrivial, as was already observed in [54] for ruled surfaces. By Theorem 1, such a metric exists in a given admissible class Ω iff the extremal polynomial F_Ω is positive on $(-1, 1)$. However, the integrals α_i and β_j involved in the above construction of F_Ω are hard to compute in general (see Appendix B). For the next examples, we therefore adopt a different approach to compute F_Ω. Instead of solving (4) and (5) and integrating, we solve first the boundary conditions.

It is easy to see that (6), together with (5) for $a = 0$ and $a = \infty$ (if there are blow-downs), are solved by any F of the form

$$(16) \qquad F(z) = (1 - z^2)\big(p_c(z) + (1 + z)^{d_0+1}(1 - z)^{d_\infty+1}q(z)\big)$$

for some polynomial $q(z)$. Conversely any polynomial solution is of this form, and to obtain the extremal polynomial F_Ω, the degree of q must be $\leq (\sum_{a \in \mathcal{A}} d_a) - 1$. Now it remains to compute $F''(z)$, to solve (4) and (5) for $a \in \mathcal{A}$ (so that $F = F_\Omega$), and to check positivity. For a given projective bundle and admissible Kähler class this leads to equations on the coefficients of q. The Futaki invariant will be zero (and the metric will be CSC) iff q has degree $\leq (\sum_{a \in \mathcal{A}} d_a) - 2$.

In general, the algebraic equations on q are hopelessly complicated. However, when S has real dimension ≤ 4, they are tractable.

3.2. Extremal Kähler metrics over a Riemann surface.
We consider first extremal Kähler metrics on projective bundles over a Riemann surface, generalizing the study of projective line bundles in [54].

Let Σ be a compact Riemann surface with CSC metric $(\pm g_\Sigma, \pm \omega_\Sigma)$ and let $M = P(E_0 \oplus E_\infty) \to \Sigma$, where E_0, E_∞ are projectively-flat hermitian vector bundles with ranks $d_0 + 1 > 0$, $d_\infty + 1 > 0$, and $\bar{c}_1(E_\infty) - \bar{c}_1(E_0) = [\omega_\Sigma/2\pi]$. Let $\pm 2s$ be the scalar curvature of $\pm g_\Sigma$ and Ω be an admissible

Kähler class on M defined by $0 < |x| < 1$. The metric

$$g = (z+1)g_0 + (z+1/x)g_\Sigma + (z-1)g_\infty + \frac{p_c(z)}{F(z)}dz^2 + \frac{F(z)}{p_c(z)}\theta^2$$

is an extremal metric in the given class iff $F = F_\Omega$. From Sect. 3.1, we know that

$$F_\Omega(z) = (1+z)^{d_0+1}(1-z)^{d_\infty+1}((1+xz) + c(1-z^2)),$$

where $q(z) = c$ is a constant uniquely determined by the equation

$$F_\Omega''(-1/x) = 2sx(1-1/x)^{d_0}(1+1/x)^{d_\infty}.$$

(This holds whether or not d_0, d_∞ are zero.) We solve this to obtain

$$c(s,x) = -\frac{2x^2(2+d_0(1+x)+d_\infty(1-x)-sx)}{\left\{\begin{array}{c}(2+d_0(1+x)+d_\infty(1-x))(4+d_0(1+x)+d_\infty(1-x))\\+(4+d_0+d_\infty)(1-x^2)\end{array}\right\}}.$$

Since sx has same sign as $Scal_\Sigma$, it can be positive only when $\Sigma = \mathbb{C}P^1$, in which case $E_0 = \mathcal{L}_0 \otimes \mathbb{C}^{d_0+1}$ and $E_\infty = \mathcal{L}_\infty \otimes \mathbb{C}^{d_\infty+1}$ for some line bundles $\mathcal{L}_0$, $\mathcal{L}_\infty$. It then follows that ω_Σ is integral, and thus $s = p/q$ where $p \leq 2$ and q is an integer of same sign as $x \in (-1, 1) \setminus \{0\}$ (see Remark 1); then we have that $sx < 2$, so that $c < 0$. Therefore $\mathfrak{F}_\Omega(K)$, which is a nonzero multiple of c, doesn't vanish for any admissible Kähler class. Since $b_2(\Sigma) = 1$, every Kähler class on M is admissible, so we get an immediate nonexistence result.

Theorem 5. *Let E_0, E_∞ be projectively-flat hermitian vector bundles over a Riemann surface Σ. Then there are no CSC Kähler metrics on $M = P(E_0 \oplus E_\infty)$ unless $\bar{c}_1(E_0) = \bar{c}_1(E_\infty)$ (i.e., $E_0 \oplus E_\infty$ is polystable).*

This partially extends the converse in Corollary 2 to the case $\mathbf{g} > 1$. Compared to Theorem 1, we note that here $\bar{c}_1(E_\infty) - \bar{c}_1(E_0)$ can never be strictly indefinite. On the other hand, by Theorem 1, we have an extremal Kähler metric for sufficiently small Ω. Indeed it is easy to see that $|c|$ is small when $|x|$ is small, and hence $(1+xz)+c(1-z^2)$ is positive on $(-1, 1)$. We also know from Proposition 11 that if Σ has genus 0 or 1, then every admissible Kähler class contains an extremal Kähler metric. Let us now see what happens when $\mathbf{g} > 1$, i.e., when $sx < 0$.

Since $c < 0$, the quadratic $Q(z) = (1+xz) + c(1-z^2)$ is concave. It is clearly positive at $z = \pm 1$, so it is positive on $(-1, 1)$ unless its minimum is in $(-1, 1)$ and it is nonpositive there. The minimum value $1 + c + x^2/4c$ occurs at $z = x/2c$ and

$$c(s,1) = -\frac{2(1+d_0)-s}{2(1+d_0)(2+d_0)}, \quad c(s,-1) = -\frac{2(1+d_\infty)+s}{2(1+d_\infty)(2+d_\infty)}.$$

It follows that if $s < -d_0(d_0 + 1)$, then $c(s, 1) < -\frac{1}{2}$ and hence for $0 < x < 1$ sufficiently close to 1, we have $c(s, x) < -\frac{1}{2}$ and the minimum of $Q(z)$ is in $(-1, 0)$ and nonpositive. Similarly, if $s > d_\infty(d_\infty + 1)$ then for $-1 < x < 0$ sufficiently close to -1, we have $c(s, x) < -\frac{1}{2}$ and the minimum of $Q(z)$ is in $(0, 1)$ and nonpositive.

Hence if Σ has genus $\mathbf{g} > 1$ and $s < -d_0(d_0+1)$ or $s > d_\infty(d_\infty+1)$ then not every admissible Kähler class contains an admissible extremal metric.

Conversely if $s \geq -d_0(d_0+1)$ then it is easy to check that $c(s, x) \geq -x/2$ for all $0 < x < 1$, so the minimum of $Q(z)$ is not in $(-1, 1)$ for any such x, whereas if $s \leq d_\infty(d_\infty + 1)$, $c(s, x) \geq x/2$ for all $-1 < x < 0$ and again the minimum of $Q(z)$ is not in $(-1, 1)$ for any such x.

Theorem 6. *Let E_0, E_∞ be projectively-flat hermitian vector bundles of ranks $d_0 + 1$, $d_\infty + 1$ over a compact Riemann surface Σ of genus $\mathbf{g}$, and suppose $\bar{c}_1(E_\infty) - \bar{c}_1(E_0) = [\omega_\Sigma/2\pi]$ for a Kähler form $\pm\omega_\Sigma$ of constant curvature. Then there exist admissible extremal Kähler metrics on $P(E_0 \oplus E_\infty) \to \Sigma$. Such metrics exist in every Kähler class if $\mathbf{g} = 0$ or 1. For $\mathbf{g} > 1$, put $\rho_\Sigma = s\omega_\Sigma$. Then such metrics exist in every Kähler class if and only if $-d_0(d_0 + 1) \leq s \leq d_\infty(d_\infty + 1)$, otherwise such metrics exist for $|x|$ sufficiently small (depending on s).*

Remark 7. In absence of blow-downs, we recover the examples of [54] on (complex) pseudo-Hirzebruch surfaces $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$, where there are Kähler classes which do not contain an extremal Kähler metric (if Σ has genus $\mathbf{g} > 1$). Our result extends these examples to higher rank projective bundles. However, in the presence of blow-downs, there do exist projective bundles for which there is an extremal Kähler metric in every Kähler class, even with $\mathbf{g} > 1$.

3.3. Nonexistence of CSC Kähler metrics over a Hodge 4-manifold.
We now obtain a similar nonexistence result to Theorem 5 when dim $S = 4$.

Theorem 7. *Let $(S, \pm g_S, \pm\omega_S)$ be a CSC Hodge 4-manifold and let E_0, E_∞ be projectively-flat hermitian vector bundles of ranks $d_0 + 1$, $d_\infty + 1$ over S with $\bar{c}_1(E_\infty) - \bar{c}_1(E_0) = [\omega_S/2\pi]$. Then there are no CSC Kähler metrics in the admissible Kähler classes on $M = P(E_0 \oplus E_\infty) \to S$.*

Proof. We will prove that $\mathfrak{F}_\Omega(K)$ is nonzero for any admissible Kähler class Ω by showing that the leading coefficient of F_Ω cannot vanish. Following the discussion in Sect. 3.1 we see that for given $d_0, d_\infty \geq 0$ and $0 < |x| < 1$ we have

$$F_\Omega(z) = (1 + z)^{d_0+1}(1 - z)^{d_\infty+1}((1 + xz)^2 + (cz + e)(1 - z^2))$$

with c and e being constants uniquely determined by the conditions

$$F_\Omega''(-1/x) = 0 \quad \text{and} \quad \frac{F_\Omega''(z)}{1 + xz}\bigg|_{z=-1/x} = 4sx(1 - 1/x)^{d_0}(1 + 1/x)^{d_\infty}.$$

The leading coefficient of F_Ω vanishes iff $c = 0$. As before c (and e) are determined by s and x. In particular, $c(s, x) = n(s, x)/d(s, x)$, where

$$
\begin{aligned}
d(s, x) = {} & (1 + d_\infty)(2 + d_\infty)^2(3 + d_\infty)(1 - x)^4 \\
& + 4(2 + d_0)(1 + d_\infty)(2 + d_\infty)(3 + d_\infty)(1 - x)^3(1 + x) \\
& + 6(2 + d_0)(2 + d_\infty)(4 + d_0 + d_\infty + (1 + d_0)(1 + d_\infty)) \\
& \quad \times (1 - x)^2(1 + x)^2 \\
& + 4(1 + d_0)(2 + d_0)(3 + d_0)(2 + d_\infty)(1 - x)(1 + x)^3 \\
& + (1 + d_0)(2 + d_0)^2(3 + d_0)(1 + x)^4
\end{aligned}
$$

(which is manifestly positive for $|x| < 1$) and

$$
\begin{aligned}
-n(s, x)/2x^3 = {} & 4(6 - 3sx + sx^3) \\
& + d_0(1 + x)((5 - x)(1 + x) + (7 - x)(3 - sx)) \\
& + d_\infty(1 - x)((5 + x)(1 - x) + (7 + x)(3 - sx)) \\
& + (9 - sx)(d_0(1 + x) + d_\infty(1 - x))^2 \\
& + (d_0(1 + x) + d_\infty(1 - x))^3.
\end{aligned}
$$

Since $s = p/q$ where $p \le 3$ (see Remark 1) and q is an integer of same sign as $x \in (-1, 1) \setminus \{0\}$, we have that $sx < 3$ and a moment's thought then gives that $n(s, x)$, and therefore $c(s, x)$, is never zero. $\qquad\square$

3.4. CSC Kähler metrics over a product of two Riemann surfaces. As counterpoint to the nonexistence results of Sect. 3.2 and 3.3, we now explore explicitly the existence of CSC Kähler metrics, given by Theorem 1, in the simplest case when the base is a global product of two Riemann surfaces and there are no blow-downs.

Let Σ_a $(a = 1, 2)$ be compact Riemann surface with CSC metrics $(\pm g_a, \pm \omega_a)$ and let M be $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_1 \times \Sigma_2$ where $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{L}_a$ are pullbacks of line bundles on Σ_a with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$. Let $\pm 2s_a$ be the scalar curvature of $\pm g_a$ and $-1/x_a$ be the constant roots defining an admissible Kähler class with $x_1 \ne x_2$ (the case $x_1 = x_2$ was considered in Sect. 3.3, where we established nonexistence of CSC metrics). We thus have $p_c(z) = (1 + x_1 z)(1 + x_2 z)$ and the metric becomes

$$
g = \frac{1 + x_1 z}{x_1} g_1 + \frac{1 + x_2 z}{x_2} g_2 + \frac{p_c(z)}{F(z)} dz^2 + \frac{F(z)}{p_c(z)} \theta^2.
$$

According to Sect. 3.1, to obtain a CSC metric, $F(z)$ must be the extremal polynomial

$$
F_\Omega(z) = (1 - z^2)\big((1 + x_1 z)(1 + x_2 z) + cx_1 x_2(1 - z^2)\big),
$$

where c is a constant such that the following relations are satisfied:

$$F''_\Omega(-1/x_1) = 2s_1(x_1 - x_2) \quad \text{and} \quad F''_\Omega(-1/x_2) = 2s_2(x_2 - x_1).$$

Writing $2(1 - c) = s$ (which is $\frac{1}{6}Scal_g$ and not to be confused with the s in Sects. 3.2 and 3.3), these relations hold iff

(17) $x_1(s_1(x_1 - x_2) - 2 + (1 - s)x_1 x_2) + 3(s - 1)x_2 = 0$

(18) $x_2(s_2(x_2 - x_1) - 2 + (1 - s)x_1 x_2) + 3(s - 1)x_1 = 0,$

and these are precisely the conditions on an admissible Kähler class Ω (parameterized by (x_1, x_2) with $0 < |x_a| < 1$) coming from the vanishing of $\mathfrak{F}_\Omega(K)$ (see Proposition 8). Eliminating $s = 2(1 - c)$, we obtain (using $x_1 \neq x_2$)

(19) $x_1\big(6 + s_1 x_1(x_2^2 - 3)\big) + x_2\big(6 + s_2 x_2(x_1^2 - 3)\big) = 0.$

The normalized scalar curvatures s_a are subject to the integrality conditions $s_a = 2(1 - \mathbf{g}_a)/q_a$ for q_a a nonzero integer with the same sign as x_a, where $\mathbf{g}_a$ is the genus of Σ_a. In particular $s_a x_a < 2$. This and (19) imply that $x_1 x_2 < 0$; we thus get a nonexistence result in the case $x_1 x_2 > 0$.

Theorem 8. *Let (Σ_a, ω_a) $(a = 1, 2)$ be compact CSC Riemann surfaces and $\mathcal{L}$ be a holomorphic vector bundle over $\Sigma_1 \times \Sigma_2$ with $c_1(\mathcal{L}) = [(\omega_1 + \omega_2)/2\pi]$ (so that $c_1(\mathcal{L})$ is positive definite). Then there are no admissible Kähler classes on $M = P(\mathcal{O} \oplus \mathcal{L}) \cong P(\mathcal{O} \oplus \mathcal{L}^{-1})$ containing a CSC Kähler metric.*

Remark 8. Note that we do not need to assume that the base is a global product of compact Riemann surfaces for the nonexistence result in the above theorem. It is sufficient to have a compact base S that is a local product of Riemann surfaces with CSC and $s_a x_a < 2$, which is always satisfied, since $Scal_{\pm g_a} \leq 4$ by the integrality of the pull-back of $\pm\omega_a$ to the universal cover of S.

In contrast to this result, we have the following observation.

Lemma 7. *Let Ω be an admissible Kähler class, corresponding to a solution (x_1, x_2) of (17)–(18) with $s \geq 0$. Then Ω admits an admissible CSC Kähler metric with scalar curvature $6s$.*

Proof. If (19) holds, the extremal polynomial F_Ω of an admissible Kähler class gives rise to a globally defined CSC Kähler metric iff $F_\Omega > 0$ on $(-1, 1)$. Let $Q(z) = F_\Omega(z)/(1 - z^2)$, and observe that the coefficient of z^2 in this quadratic is $\frac{1}{2}s x_1 x_2$. Since $Q(\pm 1) > 0$, Q will be positive on $[-1, 1]$ if it is convex, i.e., if $s x_1 x_2 < 0$. If $s = 0$, $Q(z)$ is linear and positive on $[-1, 1]$. Since $x_1 x_2 < 0$, F_Ω is positive on $(-1, 1)$ whenever we have solutions of (17)–(18) with $s \geq 0$.   □

We now obtain some explicit solutions of (17)–(18). If we take $x_2(17) - x_1(18)$ and $x_1(17) - x_2(18)$, we obtain, for $x_1 \neq x_2$:

$$(s_1 + s_2)x_1x_2 = 3(s - 1)(x_1 + x_2)$$
$$2(x_1 + x_2) = s_1x_1^2 + s_2x_2^2 + (1 - s)x_1x_2(x_1 + x_2).$$

These are equivalent to (17)–(18) for $x_1^2 \neq x_2^2$. As $x_1 \neq x_2$, $x_1^2 = x_2^2$ iff $x_1 + x_2 = 0$ and then $s_1 + s_2 = 0$. The following lemma deals with this case.

Lemma 8. *If $s_1 + s_2 = 0$, then either $x_1 + x_2 = 0$ and $s = (1 - x_1^2 + 2s_1x_1)/(3 - x_1^2)$, or, without loss, $x_1 = x_2 + 1$, $s = 1$, and $s_1 = 2 = -s_2$. Conversely, these give solutions of (17)–(18).*

Proof. Clearly $s_1 + s_2 = 0$ iff $s = 1$ or $x_1 + x_2 = 0$. The formula for s in the latter case is immediate from (17). Now if $s = 1$, then without loss of generality $s_1 = -s_2$ is nonnegative and we must have either $x_1 + x_2 = 0$, or $s_1 > 0$ and $x_1 = x_2 + 2/s_1$. Since $0 < |x_a| < 1$, this forces x_1 to be positive, hence $s_1 \leq 2$, so in fact we must have $s_1 = 2$ and $x_1 = x_2 + 1$. □

In order to apply Lemma 7, we suppose in the first case above that $x_1s_1 \geq 0$: then $s > 0$ since $1 - x_1^2 > 0$ for $|x_1| < 1$. Thus in both cases $s_ax_a \geq 0$ for $a = 1, 2$ and we obtain CSC Kähler metrics on projective line bundles over $T^2 \times T^2$, $T^2 \times \mathbb{C}P^1$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. In particular any Kähler class on $P(\mathcal{O} \oplus \mathcal{O}(q, -q)) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ is admissible, so the above Lemmas and Proposition 11 yield the following conclusions.

Theorem 9. *On $P(\mathcal{O} \oplus \mathcal{O}(q, -q)) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ ($q \geq 1$), any Kähler class (parameterized, up to scale, by $0 < x_1 < 1$ and $-1 < x_2 < 0$) contains a unique admissible extremal Kähler metric. For $q > 1$ this metric is CSC if and only if $x_1 + x_2 = 0$, while for $q = 1$ it is CSC if and only if $x_1 + x_2 = 0$ or $x_1 = x_2 + 1$.*

When $q = 1$, the two 1-parameter families of CSC Kähler classes of this theorem intersect at $x_1 = 1/2$, $x_2 = -1/2$. In fact, the CSC metric in this Kähler class is the Koiso–Sakane Kähler–Einstein metric [34, 47].

We end our study of CSC Kähler metrics on $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_1 \times \Sigma_2$ by considering the case of zero scalar curvature metrics, which we do not obtain automatically from Theorem 1. If $s = 0$ then (17) defines x_2 as a function of x_1

$$x_2 = f_1(x_1) = x_1\frac{2 - s_1x_1}{x_1^2 - s_1x_1 - 3},$$

whereas (18) defines x_1 as a function of x_2

$$x_1 = f_2(x_2) = x_2\frac{2 - s_2x_2}{x_2^2 - s_2x_2 - 3}.$$

Note that $f_1(0) = f_2(0) = 0$ and the gradients dx_2/dx_1 of the two graphs at $x_1 = x_2 = 0$ are both negative. By comparing the size of the gradients, one

sees that for x_1 small and positive the graph of f_1 is above the graph of f_2. Note also that the denominator appearing in $f_a(x_a)$ is negative at $x_a = 0$.

Assume that $s_1 \leq 0$. If f_1 has no asymptotes for $0 < x_1 < 1$ then $f_1(1) \leq -1$. Otherwise, for the asymptote $x_1 = v$ closest to $x_1 = 0$ we have $\lim_{x_1 \to v^-} = -\infty$. Assume moreover that $0 < s_2$. If f_2 has no asymptotes for $-1 < x_2 < 0$ then $f_2(-1) > 1$. Otherwise, for the asymptote $x_2 = v$ closest to $x_2 = 0$, we have $\lim_{x_2 \to v^+} = +\infty$. By continuity, the graphs of f_1 and f_2 intersect in the open square $(0, 1) \times (0, -1)$ and (17)–(18) is solved for some $0 < x_1 < 1$ and $-1 < x_2 < 0$.

Theorem 10. *Let $(\Sigma_a, \pm\omega_a)$ $(a = 1, 2)$ be compact Riemann surfaces with genus $\mathbf{g}_a$ and canonical bundles $\mathcal{K}_a$, and suppose that the Kähler forms $\pm\omega_a$ are integral with constant curvature. Let $\mathcal{L}_a$ be line bundles on Σ_a with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$ and, if $\mathbf{g}_a \neq 1$, let $\mathcal{L}_a$ be $\mathcal{K}_a^{q_a/2(\mathbf{g}_a-1)}$ tensored by a flat line bundle, for an integer q_a.*

There is then an admissible scalar-flat Kähler metric on $P(\mathcal{O} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2) \to \Sigma_1 \times \Sigma_2$ in the following cases:

- *$\Sigma_1 = T^2$ and $\mathcal{L}_1$ is ample, Σ_2 has genus $\mathbf{g}_2 > 1$ and $q_2 < 0$;*
- *Σ_1 and Σ_2 both have genus $\mathbf{g}_a > 1$, $q_1 > 0$, and $q_2 < 0$.*

4. K-stability and admissible extremal Kähler metrics

4.1. Introduction to stability. It has been first suggested by S.T. Yau in [56], then formulated and worked out in several ways, in particular by G. Tian [53] and by S. Donaldson [15], that the existence of a Kähler–Einstein or, more generally, of a CSC or an extremal Kähler metric on a projective complex manifold in the Kähler class determined by some polarization L should be equivalent to some kind of *stability* for the polarized projective variety (M, L). This conjecture is drawn from a detailed formal picture which makes clear an analogy with the well-established relation between the polystability of vector bundles and the existence of Einstein–Hermitian connections.

At present the most promising candidate for the conjectured stability criterion is 'K-polystability', in the form given by Donaldson [15], following Tian [53]: a polarized projective variety (M, L) is K-polystable if any 'test configuration' for (M, L) has nonpositive Futaki invariant with equality iff the test configuration is a product. We shall explain this definition shortly. We also discuss an idea of J. Ross and R. Thomas [45, 46], who focus on test configurations arising as 'deformations to the normal cone' of subschemes of (M, L), leading to a notion of 'slope' K-polystability analogous to the slope polystability of vector bundles. We explore this analogy further in Sect. 4.2.

(Note that some authors use the term *K-stable* rather than K-polystable, but the latter term agrees better with pre-existing notions of stability.)

4.1.1. Finite dimensional motivation. Let $(X, \mathcal{L}, \Omega)$ be a polarized Kähler manifold with a hermitian metric on $\mathcal{L}$ with curvature $-i\Omega$ (thus

$c_1(\mathcal{L}) = \Omega/2\pi$). Suppose a compact connected group G acts holomorphically on X with momentum map $\mu\colon X \to \mathfrak{g}^*$ (i.e., $d\langle\mu, \xi\rangle = -\Omega(K_\xi, \cdot)$), where K_ξ is the vector field on X corresponding to $\xi \in \mathfrak{g}$, the Lie algebra of G). There is a lift of the action to $\mathcal{L}$ generated by $\tilde{K}_\xi + \langle\mu, \xi\rangle K$ for each $\xi \in \mathfrak{g}$, where $\langle\mu, \xi\rangle$ is pulled back to $\mathcal{L}$, $\tilde{K}_\xi$ is the horizontal lift, and K generates the standard $U(1)$ action on $\mathcal{L}$. The action of $\mathfrak{g}$ on X and $\mathcal{L}$ extends to an action of the complexification $\mathfrak{g}^c$ and we assume this integrates to an action of a complex Lie group G^c.

By a well-known result of Kempf–Ness and Kirwan, for any $x \in X$, there is a $g \in G^c$ such that $\mu(g \cdot x) = 0$ iff for any nonzero lift $\tilde{x}$ of x to $\mathcal{L}^*$, the orbit $G^c \cdot \tilde{x}$ is closed. Such points x are said to be *polystable*. If X^{ps} denotes the set of polystable points in X, we then have an equality between X^{ps}/G^c, the polystable quotient of X by G^c, and the symplectic quotient $X/\!/G = \mu^{-1}(0)/G$.

$G^c \cdot \tilde{x}$ is closed iff $\alpha(\mathbb{C}^\times) \cdot \tilde{x}$ is closed for any one parameter subgroup $\alpha\colon \mathbb{C}^\times \hookrightarrow G^c$. This leads to the Hilbert–Mumford criterion for polystability: x is said to be *semistable* if for any one parameter subgroup $\alpha\colon \mathbb{C}^\times \hookrightarrow G^c$, the linear action of $\mathbb{C}^\times$ on $\mathcal{L}_{x_0}^*$ has nonpositive weight $w_{x_0}(\alpha) \le 0$, where $x_0 = \lim_{\lambda \to 0} \alpha(\lambda) \cdot x$ is the limit point; x is then polystable if it is semistable and $w_{x_0}(\alpha) = 0$ only when $x_0 = x$; finally x is *stable* if it is polystable and has zero dimensional isotropy subgroup.

4.1.2. The infinite dimensional analogue.

We apply the finite dimensional picture above formally to an infinite dimensional setting in which X is the space of compatible complex structures on a compact symplectic manifold (M, ω) with $H^1(M) = 0$. The space X has a natural Kähler metric with respect to which the group G of symplectomorphisms of M acts holomorphically with a momentum map $\mu\colon X \to C_0^\infty(M, \mathbb{R})$ given by the scalar curvature of the corresponding Kähler metric on M, modified by a constant in order to lie in $\mathfrak{g}^* \cong \mathfrak{g} = C_0^\infty(M, \mathbb{R})$, the functions with total integral zero, which is the Lie algebra of the symplectomorphism group equipped with the L_2-inner product. A quick way to see this is to observe that the Mabuchi K-energy (see Sect. 2.3) of M is a Kähler potential for the metric on X: the gradient on X of the Mabuchi K-energy is the scalar curvature [26].

There is no group whose Lie algebra is the complexification $\mathfrak{g}^c$, but one can still consider the foliation of X given by the vector fields induced by $\mathfrak{g}^c$. The complex structures in a given leaf are all biholomorphic by a diffeomorphism in the connected component of the identity, and pulling back the symplectic form ω by these biholomorphisms, we may identify the leaf with the set of all Kähler metrics in a fixed Kähler class, compatible with a fixed complex structure on M. Hence there should be a CSC metric in a given Kähler class iff the momentum map μ vanishes on the corresponding leaf iff the leaf is stable in a suitable sense.

To make precise this infinite dimensional analogue, we formalize what is meant by the orbit of a 1-parameter subgroup in terms of 'test configurations'

and give a Hilbert–Mumford formulation of stability in terms of the weight of limit points.

4.1.3. Test configurations. Let (M, Ω) be a Hodge manifold, viewed as a polarized projective variety with respect to a line bundle L with $c_1(L) = \Omega/2\pi$.

Definition 2 [15]. A *test configuration* for (M, L) is a polarized scheme $(X, \mathcal{E})$ over $\mathbb{C}$ with a $\mathbb{C}^\times$ action α and a flat proper $\mathbb{C}^\times$-equivariant morphism $p: X \to \mathbb{C}$ (where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by scalar multiplication) such that the fibre $(X_t = p^{-1}(t), \mathcal{E}|_{X_t})$ is isomorphic to (M, L) for some (hence all) $t \neq 0$.

$(X_0, \mathcal{E}|_{X_0})$ is called the *central fibre*. Since $0 \in \mathbb{C}$ is fixed by the action, $(X_0, \mathcal{E}|_{X_0})$ inherits a $\mathbb{C}^\times$ action, also denoted by α.

A test configuration is said to be a *product configuration* if $X = M \times \mathbb{C}$ and α is given by a $\mathbb{C}^\times$ action on M (and scalar multiplication on $\mathbb{C}$).

Since relevant properties of test configurations are unchanged if we replace $\mathcal{E}$ by $\mathcal{E}^r$ for a positive integer r, we can let $\mathcal{E}$ be a $\mathbb{Q}$-line bundle in the definition above (i.e., $\mathcal{E}$ denotes a 'formal root' of a line bundle $\mathcal{E}^r$ for some positive integer r).

A particularly important class of test configurations are those associated to a subscheme of (M, L), as studied by J. Ross and R. Thomas [45,46]. We shall state it here for complex submanifolds of (M, L), but the same definition actually makes sense for subschemes.

Definition 3 (Deformation to the normal cone). For a polarized complex manifold (M, L) and a complex submanifold Z the normal bundle is defined by $\nu_Z = TM_{|Z}/TZ$ (in the more general situation when Z is a subscheme of M, ν_Z then denotes the *normal cone*, defined similarly). The *deformation to the normal cone* is then defined as the family $p : X \to \mathbb{C}$, where $X = \widehat{M \times \mathbb{C}}$ denotes the blow-up of $M \times \mathbb{C}$ along $Z \times \{0\}$ and p the map induced by the natural projection from $M \times \mathbb{C}$ to $\mathbb{C}$. For $t \neq 0$ in $\mathbb{C}$, $p^{-1}(t)$ is then biholomrphic to M, whereas $p^{-1}(0) = P \cup \hat{M}$, where P denotes the exceptional divisor in $\widehat{M \times \mathbb{C}}$, which is biholomorphic to $\mathbb{P}(\mathcal{O}_Z \oplus \nu_Z)$, hence is a compactification of the normal cone, and $\hat{M}$ stands for the blow-up of M along Z; notice that $P \cap \hat{M} = \mathbb{P}(\nu_Z)$, the exceptional divisor in $\hat{M}$.[5] We equip this with the polarization $\mathcal{E}_c = \pi^*L \otimes \mathcal{O}(-cP)$, where $\mathcal{O}(P)$ is the line bundle associated to the exceptional divisor P, $\pi: X \to M$ is induced by the natural projection from $M \times \mathbb{C}$ to M, and c is a positive rational number such that $\mathcal{E}_c$ is an ample $\mathbb{Q}$-line bundle. This last condition gives an upper bound ε on c, called the *Seshadri constant* of Z with respect to L.

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[5] The name *deformation to the normal cone* originates from the following fact: In the induced holomorphic embedding $Z \times \mathbb{C} \hookrightarrow \widehat{M \times \mathbb{C}}$, $Z \times \{t\} \hookrightarrow p^{-1}(t)$ is isomorphic to the natural embedding $Z \hookrightarrow M$, whereas $Z \times \{0\} \hookrightarrow p^{-1}(0) = P \cup \hat{M}$ has its image in $P = \mathbb{P}(\mathcal{O}_Z \oplus \nu_Z)$ and is isomorphic to the natural embedding $Z \hookrightarrow \nu_Z \subset \mathbb{P}(\mathcal{O}_Z \oplus \nu_Z)$ as the zero section of ν_Z, cf. [22, Chapter 5] for more details on this classical construction.

We let α be the $\mathbb{C}^\times$ action coming from the trivial action on M and multiplication on $\mathbb{C}$. This clearly defines an action on X with a lift to $\mathcal{E}_c$. Hence the deformation to the normal cone determines a family of test configurations, parameterized by $c \in (0, \varepsilon) \cap \mathbb{Q}$.

4.1.4. The Futaki invariant and K-stability. K-stability is defined using a Hilbert–Mumford criterion, i.e., in terms of a 'weight' associated to each test configuration. This weight is given by the Futaki invariant of the central fibre; however, since the latter is typically a singular projective variety, we need an algebraic geometric definition of the Futaki invariant. Such a definition has been given by Donaldson [15].

Let V be a scheme of dimension n over $\mathbb{C}$ polarized by an ample line bundle L and suppose that α is a $\mathbb{C}^\times$ action on V with a lift to L. Then α acts on the vector spaces $H_k = H^0(V, L^k)$, $k \in \mathbb{Z}^+$. If $w_k(\alpha)$ denotes the weight of the highest exterior power of H_k (that is, the trace $\operatorname{Tr} A_k$ of the infinitesimal generator A_k of the action) and d_k denotes the dimension of H_k then $w_k(\alpha)$ and d_k are given by polynomials in k for sufficiently large k, of degrees at most $n+1$ and n respectively. For sufficiently large k the quotient $w_k(\alpha)/(kd_k)$ can be expanded into a power series with no positive powers. The *Futaki invariant* $\mathfrak{F}(\alpha)$ is the residue at $k = 0$ of this quotient, that is, the coefficient of the k^{-1} term in the resulting expansion. The Futaki invariant is independent of the choice of lift of α to L. (When V is a manifold, this definition coincides with Futaki's original definition up to a normalization convention.)

Definition 4. The Futaki invariant of a test configuration is defined to be the Futaki invariant $\mathfrak{F}(\alpha)$ of the central fibre, where α denotes the induced $\mathbb{C}^\times$ action.

A Hodge manifold (M, L) is said to be K-*polystable* if the Futaki invariant of any test configuration is nonpositive, and equal to zero if and only if the test configuration is a product configuration.

For the test configurations $(X, \mathcal{E}_c)$ arising from a deformation to a normal cone, J. Ross and R. Thomas [45, 46] show that the Futaki invariants $\mathfrak{F}(\alpha_c)$ are rational in $c \in (0, \varepsilon) \cap \mathbb{Q}$, where ε is the Seshadri constant, and can be extended to $c \in (0, \varepsilon)$. With this in mind, we give the following

Definition 5. A Hodge manifold (M, L) is said to be *slope K-polystable* if for the deformation to the normal cone of any nontrivial subscheme, the Futaki invariant $\mathfrak{F}(\alpha_c)$ of the corresponding family $(X, \mathcal{E}_c)$ of test configurations and its natural extension to the whole interval $(0, \varepsilon)$ are negative.

Remark 9. The concepts of *slope K-stability*, *slope K-polystability*, *slope K-semi-stability* were introduced by J. Ross and R. Thomas [45, 46] and, in their relative versions, by G. Székelyhidi [50, 51]. The notion of slope K-polystability in [45] is stronger than the one in [46] and has been partially motivated by the limiting situations illustrated by our Example 1 below (cf. Sect. 2.1 and Remark 3.9 in [45], and Remark 10 below). Our

Definition 5 is closer to the one in [45], but is not quite the same as [45] also requires $\mathfrak{F}(\alpha_\varepsilon) < 0$ unless ε is rational and the semi-ample configuration $(X, \mathcal{E}_\varepsilon)$ is the pullback by a contraction of a product configuration. We shall not need this refinement. As observed in [45, Remark 3.9], the notion of (relative) slope K-polystability used here and in [45] might more properly be referred to as *analytic* (relative) slope K-polystability, corresponding to *analytic* test configurations (while the original definitions in [53, 15] only concern *algebraic* test configurations).

4.2. Stable bundles and CSC Kähler metrics. We now relate our results concerning CSC Kähler metrics on projective bundles to stability theory for vector bundles. Recall that if $E \to S$ is a holomorphic vector bundle over a compact kählerian $2d$-manifold $(S, [\omega_h])$, the *slope* $\mu(E)$ is the number $\overline{c}_1(E) \cdot [\omega_h]^{d-1}$; E is called (slope) *stable* or *semistable* if $\mu(F) < \mu(E)$ or $\mu(F) \leq \mu(E)$ (respectively) for any proper coherent subsheaf $F \subset E$; it is *polystable* if it is a direct sum of stable vector bundles with the same slope; then, as is well-known, 'stable' $\Rightarrow$ 'polystable' $\Rightarrow$ 'semistable', and by the Hitchin–Kobayashi correspondence (established by Donaldson [13] and Uhlenbeck–Yau [55]), E admits an Einstein–Hermitian connection iff it is polystable.

There is a close analogy between K-stability for polarized Kähler manifolds and slope stability for vector bundles. In particular, one might hope to find a direct relation between the existence problem for CSC Kähler metrics on a geometrically ruled complex manifold $P(E)$ over S and the stability of $E \to S$. Notable progress in understanding the relation between K-polystability of $P(E)$ and slope polystability of E has been made by Ross–Thomas [45, 46], using their notion of slope K-polystability: indeed if F is a coherent subsheaf of E, then $P(F)$ is a subscheme of $P(E)$ and deformation to the normal cone of $P(F)$ is a test configuration which 'destabilizes' $P(E)$ iff $\mu(F) > \mu(E)$ (see [45]).

Using the general theory of CSC Kähler metrics, the work of [45] shows that if E is not semistable with respect to an integral Kähler class $[\omega_h]$ on S, then for all $k \gg 1$ the integral classes $2\pi c_1(\mathcal{O}(1)_E) + k p^*[\omega_h]$ on $P(E) \xrightarrow{p} S$ do not contain CSC metrics. As a partial converse, Hong [32, Theorem A] shows that if E is polystable and $\mathfrak{h}_0(M) \to \mathfrak{h}_0(S)$ is surjective, then there is a CSC metric in $2\pi c_1(\mathcal{O}(1)_E) + k p^*[\omega_h]$, for each $k \gg 1$, iff the Futaki invariant $\mathfrak{F}_\Omega$ vanishes.

To put our results in this context, let $P(E) \to S$ be admissible so $E = E_0 \oplus E_\infty$ for projectively-flat (and thus polystable) hermitian vector bundles E_0 and E_∞ with $\overline{c}_1(E_\infty) - \overline{c}_1(E_0) = \sum_a [\omega_a/2\pi]$. Thus E is determined up to tensor product by a line bundle and is polystable iff it is semistable iff $\mu(E_0) = \mu(E_\infty)$. With respect to a Kähler class $[\omega_h] = [\sum_a \omega_a/f_a]$ on S (where $f_a x_a > 0$), this condition reads

$$(20) \quad 0 = \mu(E_\infty) - \mu(E_0) = \frac{(d-1)!}{2\pi} \mathrm{Vol}\Big(S, \prod_a \omega_a/f_a\Big)\Big(\sum_a d_a f_a\Big),$$

which can happen for suitable f_a iff $\bar{c}_1(E_\infty) - \bar{c}_1(E_0)$ is strictly indefinite; this is exactly the condition of Theorem 1 that ensures the existence of CSC metrics in a sufficiently small admissible Kähler class $\Omega = \Xi + p^*[\sum_{a \in \mathcal{A}} \omega_a/x_a]$, subject only to the constraint that $\mathfrak{F}_\Omega(K) = 0$. Note that Ξ is equal to $4\pi c_1(\mathcal{O}(1)_E)$ up to a basic term (depending on the choice of E) – this follows by integrating $\Xi^{d_0+d_\infty+1}$ over a fibre and using the expression for $I(d_0, d_\infty, 0)$ from Appendix B. Thus, admissible classes play a similar role to those considered by Ross–Thomas and Hong, and $k \gg 1$ corresponds to $|x_a|$ sufficiently small in our picture. (We recall that this means that the fibres are small compared to the base.) However, there is not a simple relation in general between those Ω containing a CSC metric and the $[\omega_h]$ with respect to which E is polystable: the approach of Ross–Thomas and Hong suggests taking $f_a = x_a/(1 + r_a x_a)$, for some r_a depending only on E; then (20) agrees *asymptotically* with $\mathfrak{F}_\Omega(K) = 0$ in the limit $x_a \to 0$, but the two conditions define distinct hypersurfaces in general.

Conversely, in the case E_0 and E_∞ are line bundles over a product $S = \Sigma_1 \times \Sigma_2$ of two Riemann surfaces, Theorem 8 shows that polystability of E with respect to some Kähler class on S (which is unique up to scale in this case) is also necessary for the existence of a CSC metric in an admissible Kähler class on $P(E)$.

Consider now the case that the base S is a Riemann surface Σ of genus $\mathbf{g}$; the stability of a holomorphic vector bundle is then independent of the choice of a Kähler class on Σ, and it is natural to speculate [45] that the notion of K-polystability of the projective manifold $P(E)$ should be independent of the specific Kähler class, and to conjecture that $P(E)$ admits a CSC Kähler metric iff E is polystable. At present (see [2]) this conjecture is confirmed when E is of rank 2 (i.e., on geometrically ruled surfaces), when $\mathbf{g} \leq 1$ and E is a direct sum of line bundles (cf. Corollary 2 – this always holds when $\mathbf{g} = 0$), or when E is indecomposable and $\mathbf{g} \geq 2$. Theorem 5 further confirms the conjecture in the case of decomposable bundles of the form $E = E_0 \oplus E_\infty$ with E_0 and E_∞ polystable.

4.3. Extremal Kähler metrics and relative K-polystability.

In recent work, G. Székelyhidi [50] has extended the theory of K-polystability to cover extremal Kähler metrics, not just CSC Kähler metrics. We briefly explain his ideas here.

4.3.1. Motivation.

Recall that extremal Kähler metrics are critical points for the L_2-norm of the scalar curvature for metrics in a fixed Kähler class on a complex manifold (M, J) [7]. If we identify the Kähler class with a leaf of the formal G^c orbit described in Sect. 4.1.2, we are therefore looking for critical points of $\|\mu\|^2$, where $\mu: X \to C_0^\infty(M, \mathbb{R})$ and X is the space of compatible complex structures on a compact symplectic manifold (M, ω) with $H^1(M) = 0$.

We can adapt the finite dimensional model of Sect. 4.1.1 to this problem by supposing that the Lie algebra $\mathfrak{g}$ is equipped with a G-invariant inner product $\langle\,,\,\rangle$. Now, following Székelyhidi [50], we note that the weight w_x of the linear action of the isotropy algebra $\mathfrak{g}_x$ on $\mathcal{L}_x^*$ is given by $w_x = \langle\beta_x, \cdot\rangle\colon \mathfrak{g}_x \to \mathbb{R}$ for some $\beta_x \in \mathfrak{g}_x$, which is the orthogonal projection of $\mu(x)$ onto $\mathfrak{g}_x$. We refer to β_x (or rather the induced vector field on X) as the *extremal vector field*: for in the infinite dimensional setting it agrees with the extremal vector field of Futaki and Mabuchi (see Sect. 2.2).

Clearly x is a critical point of $\|\mu\|^2$ iff $\mu(x)$ is in $\mathfrak{g}_x$, i.e., $\beta_x = \mu(x)$. Using this, Székelyhidi shows that x is in the G^c orbit of a critical point of $\|\mu\|^2$ if and only if it is polystable for the action of the subgroup of G^c whose Lie algebra is the subspace $\beta_x^\perp$ of the centralizer of β_x. The Hilbert–Mumford criterion may then be modified as follows: the *modified weight* $w_{x_0}(\alpha) - \langle\alpha, \beta_x\rangle w_{x_0}(\beta_x)/\langle\beta_x, \beta_x\rangle$ of the limit point x_0 should be nonpositive for any one parameter subgroup α of the centralizer of β_x, with equality if and only if $x_0 = x$.

4.3.2. The inner product and modified Futaki invariant.

Thus motivated, we return to the setting of Sect. 4.1.4 and define a modified Futaki invariant of a polarized scheme (V, L) (of dimension n over $\mathbb{C}$) relative to a $\mathbb{C}^\times$ action β. We first need to define an inner product between such actions.

Assume then that V has two $\mathbb{C}^\times$ actions α and β with lifts to L and infinitesimal generators A_k and B_k of the actions on H_k. Then for k sufficiently large, $\mathrm{Tr}\,(A_k B_k)$ is a polynomial of degree at most $n + 2$. The inner product $\langle\alpha, \beta\rangle$ is defined to be the coefficient of k^{n+2} of the expansion of $\mathrm{Tr}\,(A_k B_k) - w_k(\alpha)w_k(\beta)/d_k$ for large k, which is independent of the lifts of α and β to L: indeed it depends only on the trace-free parts of A_k and B_k. (When V is a manifold, this inner product coincides with Futaki–Mabuchi bilinear form [24] up to a normalization convention.)

We define the *modified Futaki invariant* [50] $\mathfrak{F}_\beta(\alpha)$ of α relative to β (assuming the action β is nontrivial) to be

$$\mathfrak{F}_\beta(\alpha) = \mathfrak{F}(\alpha) - \frac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle}\mathfrak{F}(\beta).$$

4.3.3. Relative K-stability.

Let (M, Ω, L) be as in Sect. 4.1.3 and suppose it has a nontrivial $\mathbb{C}^\times$ action β.

Definition 6 [50]. A test configuration $(X, \mathcal{E})$ for (M, L) is *compatible* with β if there is a $\mathbb{C}^\times$ action, also denoted by β, on $(X, \mathcal{E})$ preserving $p\colon X \to \mathbb{C}$ and inducing the trivial action on $\mathbb{C}$, such that β restricted to $(X_t, \mathcal{E}|_{X_t})$ coincides with the original action for $t \neq 0$ under the isomorphism with (M, L).

In this case we have an induced action on the central fibre X_0, also called β, and the modified Futaki invariant of the test configuration is defined to be $\mathfrak{F}_\beta(\alpha)$.

A polarized Hodge manifold (M, L) with nontrivial $\mathbb{C}^\times$ action β is K-*polystable relative to* β if the modified Futaki invariant of any test configuration $(X, \mathcal{E})$ compatible with β is nonpositive, and equal to zero if and only if $(X, \mathcal{E})$ is a product.

If (M, L) has a $\mathbb{C}^\times$ action β which preserves a subscheme Z, the test configurations $(X, \mathcal{E}_c)$ arising from the deformation to the normal cone are compatible with β. As in [45,46], $\mathfrak{F}_\beta(\alpha_c)$ is rational in $c \in (0, \varepsilon) \cap \mathbb{Q}$ and so extends to $c \in (0, \varepsilon)$. Thus, cf. Sect. 4.1.4, we have a notion of *slope* K-polystability relative to β.

Definition 7. A polarized Hodge manifold (M, L) with nontrivial $\mathbb{C}^\times$ action β is said to be *slope K-polystable relative to* β if the modified Futaki invariant $\mathfrak{F}_\beta(\alpha_c)$ of the family $(X, \mathcal{E}_c)$ of test configurations, corresponding to the deformation to the normal cone of any nontrivial subsheme preserved by β, is negative for $c \in (0, \varepsilon)$.

As with the definition of (absolute) slope K-polystability, strictly speaking, we should also require $\mathfrak{F}_\beta(\alpha_\varepsilon) < 0$ unless ε is rational and $(X, \mathcal{E}_\varepsilon)$ is the pullback by a contraction of a product configuration.

It follows from [25,42] that the (Futaki–Mabuchi) extremal vector field associated to a Hodge Kähler manifold (M, Ω) with a maximal compact connected subgroup G of $H_0(M)$ has closed orbits, and therefore defines an effective $\mathbb{C}^\times$ action which we will refer to as the *extremal* $\mathbb{C}^\times$ action of (M, Ω, G). By 'K-polystable relative to G', we mean relative to the extremal $\mathbb{C}^\times$ action of (Ω, G). The motivation of Sect. 4.3.1 then suggests the following conjecture [50].

Conjecture 1. Let (M, Ω, L) be a polarized Hodge manifold and G a maximal compact connected subgroup of $H_0(M)$. Then there is a G-invariant extremal Kähler metric in $\Omega = 2\pi c_1(L)$ if and only if (M, L) is K-polystable relative to G.

As a motivating example, Székelyhidi considers the deformation to the normal cone of the infinity section in a (higher genus) polarized geometrically ruled surfaces $P(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$. He finds that relative K-polystability implies existence of extremal Kähler metrics of the type constructed in [54], which are precisely the admissible metrics on these bundles. In the next section we generalize this idea to arbitrary admissible bundles. However, in doing so, we find that unless $\dim S \le 4$, we need to replace 'K-polystable' by 'slope K-polystable' in the above conjecture.

4.4. Relative K-polystability of admissible projective bundles.
We now consider the deformation to the normal cone $(X, \mathcal{E}_c, \alpha)$ of the infinity section $e_\infty = z^{-1}(-1) = P(0 \oplus E_\infty)$ for an admissible projective bundle

$M = P(E_0 \oplus E_\infty) \to S$ (with dim $S = 2d$), polarized by a line bundle L with $\Omega = 2\pi c_1(L)$ admissible.

We therefore choose the admissible Kähler class $\Omega = \Xi + \sum_a [\omega_a]/x_a$ to be integral (where $0 < |x_a| \le 1$ with equality iff $a \in \{0, \infty\}$). The Seshadri constant of this polarization is 2, so we take $c \in (0, 2) \cap \mathbb{Q}$. Since the $\mathbb{C}^\times$ action β induced by the vector field K preserves Z, X is compatible with this action. We will use the letters α, β to denote also the corresponding actions on the (polarized) central fibre (X_0, L_0) and on the vector space $H^0(X_0, L_0^k)$, where $L_0 = \mathcal{E}_c|_{X_0}$.

Let us calculate the modified Futaki invariant of this configuration. For this we first note that if $\mathcal{I}_\infty \subset \mathcal{O}_M$ is the ideal sheaf of holomorphic functions vanishing on e_∞, then for any $p \ge 0$, $\mathcal{I}_\infty^p / \mathcal{I}_\infty^{p+1}$ is supported on e_∞, and its restriction is $S^p v_\infty^*$, where v_∞ is the normal bundle to e_∞ in M.

Therefore, for k sufficiently large, we have, as in [45,50]

$$H^0(X_0, L_0^k) = \bigoplus_{i=0}^{(2-c)k} H^0(e_\infty, L|_{e_\infty}^k \otimes S^{2k-i} v_\infty^*)$$

$$\oplus \bigoplus_{j=1}^{ck} H^0(e_\infty, L|_{e_\infty}^k \otimes S^{ck-j} v_\infty^*),$$

where α acts on the first direct sum with weight 0 and on the components of the second direct sum with weight $-j$. We can choose the lift of β to L so that the weight of the induced action on $H^0(e_\infty, L|_{e_\infty}^k \otimes S^{uk+v} v_\infty^*)$ is $(u-1)k + v$.

Now $S^p v_\infty^*$ is the direct image $q_* \mathcal{O}(p)_{v_\infty}$, where $\mathcal{O}(-1)_{v_\infty}$ is the (fibrewise) tautological bundle of $q \colon \hat{e}_\infty = P(v_\infty) \to e_\infty$. Also $\hat{e}_\infty$ may be identified with $\hat{S}$ via the obvious inclusion i of $\hat{S}$ into $\hat{M} = P(\mathcal{O} \oplus \hat{\mathcal{L}})$ as the infinity section, and then $i^* \mathcal{O}(1)_{v_\infty} = \hat{\mathcal{L}}$. For convenience, we now drop the hats, so that we have

$$H^0(X_0, L_0^k) = \bigoplus_{i=0}^{(1-z)k} H^0(S, i^*L^k \otimes \mathcal{L}^{2k-i}) \oplus \bigoplus_{j=1}^{(1+z)k} H^0(S, i^*L^k \otimes \mathcal{L}^{(1+z)k-j})$$

$$= \bigoplus_{i=0}^{2k} H^0(S, i^*L^k \otimes \mathcal{L}^{2k-i}),$$

where we have abused notation by writing $c - 1 = z$; a priori this has nothing to do with the momentum map that we also denote by z, but notice that it does take (rational) values in the same interval $(-1, 1)$. To compute d_k, $\mathrm{Tr}\, A_k$, $\mathrm{Tr}\, B_k$, $\mathrm{Tr}\, A_k B_k$, $\mathrm{Tr}\, B_k^2$, and thereby $\mathfrak{F}_\beta(\alpha)$, we need only the dimensions of these vector spaces. We note that we only need to compute d_k, $\mathrm{Tr}\, A_k$ and $\mathrm{Tr}\, B_k$ to subleading order in k, whereas for $\mathrm{Tr}\, A_k B_k$ and $\mathrm{Tr}\, B_k^2$ the leading order term suffices. Consequently we will be dropping lower

order terms without further comment. We also note that since the Futaki invariant is defined in terms of ratios, we can ignore any overall multiples. Now by the Riemann–Roch formula and the ampleness of i^*L (in fact it is only semiample if $d_\infty > 0$, but we can apply a limiting argument in this case, as in [45]), for sufficiently large k we have that

$$
\begin{aligned}
h^0(S, i^*L^k \otimes \mathcal{L}^{uk+v}) &= \chi(S, i^*L^k \otimes \mathcal{L}^{uk+v}) \\
&= (\mathrm{ch}(i^*L^k \otimes \mathcal{L}^{uk+v}) \cdot \mathrm{Td}(S))[S] \\
&= \left(c_1(i^*L^k \otimes \mathcal{L}^{uk+v}) + \frac{1}{2} c_1(\mathcal{K}_S^{-1}) \right)^d [S] + O(k^{d-2}) \\
&= \left(\sum_a \frac{k + ((u-1)k + v + s_a/2)x_a}{x_a} [\omega_a/2\pi] \right)^d [S] \\
&\quad + O(k^{d-2})
\end{aligned}
$$

since the Ricci form of S_a may be written $s_a\omega_a + \rho_{a,0}$ where $\rho_{a,0} \wedge \omega_a^{d_a-1} = 0$. After an overall multiplication by $(2\pi)^d/d!$ and $\prod_a x_a^{d_a}/\mathrm{Vol}(S_a, \omega_a)$, this is

$$
k^d p_c^s(u - 1 + v/k) + O(k^{d-2}),
$$

where $p_c^s(t) = \prod_a (1 + x_a(t + s_a/2k))^{d_a}$. In order to carry out the summations over i and j we use the trapezium rule, as in [45, Lemma 4.7].

Lemma 9. *Let $f(x)$ be a polynomial and b a rational number. Then for $\varepsilon \in \{0, 1\}$ and for $k \in \mathbb{Z}^+$ such that bk is a positive integer, we have*

$$
\sum_{i=\varepsilon}^{bk} f(i/k) = k \int_0^b f(t)\, dt + \frac{1}{2}(f(b) + (-1)^\varepsilon f(0)) + O(k^{-1}).
$$

The proof is easy (see e.g. [45]): by linearity we can assume $f(x) = x^m$ and then use $\sum_{i=1}^N i^m = N^{m+1}/(m+1) + N^m/2 + O(N^{m-1})$ (which in turn is an easy induction on N). We then obtain (up to an overall multiple), that for any $r \geq 0$,

$$
\begin{aligned}
k^{-d-r} \mathrm{Tr}\, B_k^r &= k \int_0^2 (1 - t)^r p_c^s(1 - t)\, dt + \frac{1}{2}\left(p_c(1) + (-1)^r p_c(-1) \right) \\
&\quad + O(1/k) \\
&= k\alpha_r + \frac{1}{2}\beta_r + O(1/k),
\end{aligned}
$$

with $\alpha_r = \int_{-1}^1 p_c(t) t^r\, dt$ and β_r as in (7). Setting $r = 0$ gives d_k. Similarly, using the explicit formula (12) for the extremal polynomial $F_\Omega(z)$ we obtain

$$k^{-d-1}\operatorname{Tr} A_k = k \int_0^{1+z} -tp_c^s(z-t)\,dt - \frac{1}{2}(1+z)p_c(-1) + O(1/k)$$

$$= -k \int_{-1}^z p_c(t)(z-t)\,dt$$

$$- \frac{1}{2}\int_{-1}^z \left(\sum_a \frac{d_a s_a x_a}{1+x_a t}\right) p_c(t)(z-t)\,dt$$

$$- \frac{1}{2}(1+z)p_c(-1) + O(1/k)$$

$$= -k \int_{-1}^z (z-t)p_c(t)\,dt$$

$$- \frac{1}{4}F_\Omega(z) + \frac{1}{4}\int_{-1}^z (At+B)p_c(t)(z-t)\,dt + O(1/k)$$

$$k^{-d-3}\operatorname{Tr} A_k B_k = \int_0^{1+z} -t(z-t)p_c(z-t)\,dt + O(1/k)$$

$$= -\int_{-1}^z p_c(t)t(z-t)\,dt + O(1/k)$$

where A and B are the solutions of (9). Now we are ready to calculate $\langle \beta, \beta \rangle$, $\langle \alpha, \beta \rangle$, $\mathfrak{F}(\beta)$, and $\mathfrak{F}(\alpha)$. (We omit the dependence of z for convenience.)

$$\langle \beta, \beta \rangle = \frac{\alpha_2 \alpha_0 - \alpha_1^2}{\alpha_0}$$

$$\langle \alpha, \beta \rangle = -\int_{-1}^z p_c(t)t(z-t)\,dt + \frac{\alpha_1}{\alpha_0}\int_{-1}^z p_c(t)(z-t)\,dt$$

$$\mathfrak{F}(\alpha) = \operatorname{Res}_{k=0} \frac{(\operatorname{Tr} A_k)_1 + (\operatorname{Tr} A_k)_0/k}{\alpha_0(1+\beta_0/(2k\alpha_0))} = \frac{\alpha_0(\operatorname{Tr} A_k)_0 - \frac{1}{2}\beta_0(\operatorname{Tr} A_k)_1}{\alpha_0^2}$$

$$= \frac{\left\{ \begin{array}{c} -\frac{1}{4}\alpha_0 F_\Omega(z) + \frac{1}{4}\alpha_0 \int_{-1}^z (At+B)p_c(t)(z-t)\,dt \\ +\frac{1}{2}\beta_0 \int_{-1}^z p_c(t)(z-t)\,dt \end{array} \right\}}{\alpha_0^2}$$

$$\mathfrak{F}(\beta) = \operatorname{Res}_{k=0} \frac{\alpha_1 + \beta_1/2k}{\alpha_0(1+\beta_0/(2k\alpha_0))} = \frac{\beta_1\alpha_0 - \beta_0\alpha_1}{2\alpha_0^2}$$

where we have set $(\operatorname{Tr} A_k)_0 = -\frac{1}{4}F_\Omega(z) + \frac{1}{4}\int_{-1}^z (At+B)p_c(t)(z-t)\,dt$ and $(\operatorname{Tr} A_k)_1 = -k\int_{-1}^z p_c(t)(z-t)\,dt$.

Finally, we can calculate the modified Futaki invariant for our test configuration.

$$\alpha_0^2 \mathfrak{F}_\beta(\alpha) = \alpha_0^2(\mathfrak{F}(\alpha) - \langle \alpha, \beta \rangle \mathfrak{F}(\beta)/\langle \beta, \beta \rangle)$$

$$= -\frac{1}{4}\alpha_0 F_\Omega(z) + \frac{1}{4}\alpha_0 \int_{-1}^z (At + B)p_c(t)(z - t)\, dt$$

$$+ \frac{1}{2}\beta_0 \int_{-1}^z p_c(t)(z - t)\, dt$$

$$+ \frac{\alpha_0(\beta_1\alpha_0 - \beta_0\alpha_1)}{2(\alpha_2\alpha_0 - \alpha_1^2)}\left(\int_{-1}^z p_c(t)t(z - t)\, dt\right.$$

$$\left. - \frac{\alpha_1}{\alpha_0}\int_{-1}^z p_c(t)(z - t)\, dt\right)$$

$$= -\frac{1}{4}\alpha_0 F_\Omega(z) + \frac{1}{4}\alpha_0 \int_{-1}^z (At + B)p_c(t)(z - t)\, dt$$

$$- \frac{1}{4}\alpha_0 \int_{-1}^z At p_c(t)(z - t)\, dt$$

$$+ \frac{\beta_0(\alpha_2\alpha_0 - \alpha_1^2) - \alpha_1(\beta_1\alpha_0 - \beta_0\alpha_1)}{2(\alpha_2\alpha_0 - \alpha_1^2)}\int_{-1}^z p_c(t)(z - t)\, dt$$

$$= -\frac{1}{4}\alpha_0 F_\Omega(z) + \frac{1}{4}\alpha_0 \int_{-1}^z (At + B)p_c(t)(z - t)\, dt$$

$$- \frac{1}{4}\alpha_0 \int_{-1}^z At p_c(t)(z - t)\, dt - \frac{1}{4}\alpha_0 \int_{-1}^z B p_c(t)(z - t)\, dt$$

$$= -\frac{1}{4}\alpha_0 F_\Omega(z)$$

which is a negative multiple of the extremal polynomial. It follows immediately that if (M, L) is slope K-polystable relative to $K = J\,\mathrm{grad}_g\, z$, then F_Ω is positive on $(-1, 1)$ and Ω contains an admissible extremal metric by Theorem 1.

If (M, L) is slope K-polystable in the absolute sense, then $\mathfrak{F}(\alpha)$ is negative on $(-1, 1)$ and hence nonpositive at $z = 1$. Evaluating the integrals in this case (and using $F_\Omega(1) = 0$), we find that $A \geq 0$. Now if we swap the roles of the zero and infinity sections (by interchanging E_0 and E_∞) then the analogous calculation shows that $A \leq 0$ (we get the same formulae with the change of variables $z \mapsto -z$). Thus $A = 0$ and $\mathfrak{F}(\beta) = 0$. (Intuitively, the reason we get $\mathfrak{F}(\beta) = 0$ is that in these limits, deformation to the normal cone of the zero or infinity section is actually the pullback by a contraction of the product configuration associated to $\pm\beta$, cf. [45,46].)

Hence (M, L) is slope K-polystable relative to β, F_Ω is positive on $(-1, 1)$, and the admissible extremal metric is CSC.

This proves Theorem 2, providing evidence for the reverse implication in Conjecture 1 (with relative K-stability replaced by relative slope K-stability) because in our setting, the extremal vector field is a nonzero multiple of AK. This calculation also shows that the forward implication in Conjecture 1 implies Corollary 3, without referring to K-energy or the results of Chen–Tian [10,11], providing further indirect evidence. However, if we use relative K-stability instead of relative slope K-stability, we can only deduce that F_Ω is positive on $(-1, 1) \cap \mathbb{Q}$ and hence nonnegative on $(-1, 1)$. However, since Ω is integral, F_Ω has rational coefficients, and so when $\dim S(= \sum_{a \in \mathcal{A}} 2d_a) \leq 4$ it follows that F_Ω is positive on $(-1, 1)$: indeed $F_\Omega(z) = (1+z)^{d_0+1}(1-z)^{d_\infty+1} Q(z)$ where $Q(z)$ is a quadratic or cubic with rational coefficients, and the repeated roots of such a polynomial must be rational.

On the other hand, the following examples show that positivity of the extremal polynomial on $(-1, 1) \cap \mathbb{Q}$ is not sufficient for the existence of an extremal Kähler metric when $\dim S = 6$.

Example 1. Let $S = \Sigma_1 \times \Sigma_2 \times \Sigma_3$ be a product of hyperbolic Riemann surfaces Σ_a with integral Kähler classes $[\pm\omega_a]$. Then for any admissible projective line bundle M over S and any admissible integral Kähler class Ω on M with parameters $x_a \in \mathbb{Q}$, the extremal polynomial has the form

$$F_\Omega(z) = (1 - z^2)\big(p_c(z) + (1 - z^2)\big(a_0 + a_1 z + a_2 z^2\big)\big),$$

where the a_j are determined by the constant gaussian curvatures $\pm s_a$ of Σ_a (via (4)–(5)). However, since we are free to choose the genera and degrees of the line bundles defining M, the s_a can be arbitrary rational numbers subject only to the constraint that $s_a x_a < 0$ (so that the gaussian curvatures are negative). Hence we are free to choose the a_j subject to this constraint.

We claim that for any rational $r > 0$ and $x_1 > x_2 > 0 > x_3$, we can choose the a_j so that $F_\Omega(z)$ is a positive multiple of $(1 - z^2)(z^2 + rz - 1)^2$ provided that

(21) $x_1 x_2 x_3 + x_1 + x_2 + x_3 = 0.$

F_Ω then has a repeated root in $(0, 1)$ and another in $(-\infty, -1)$ and for r in an open subset of $\mathbb{Q}^+$, these roots are irrational. Obviously for any $1 > x_1 > x_2 > 0$ rational, (21) has a unique rational solution $x_3 = -(x_1 + x_2)/(1 + x_1 x_2)$ with $0 > x_3 > -1$, and it is elementary to verify our claim by equating coefficients. $F_\Omega''(z)$ is then negative for large z and has at least two roots in $(-1, 1)$ and none in $(1, \infty)$. We then check that for $r > 8/5$, $F_\Omega''(-1)$ is negative and so the other two roots (which must be real, since F_Ω must have four inflection points) are in $(-\infty, -1)$. Hence we can choose x_1, x_2 so that $s_1 x_1 < 0$ and $s_2 x_2 < 0$, with s_a defined by (4)–(5). It automatically follows that $s_3 x_3 < 0$.

These data then define a countably infinite family (parameterized by (x_1, x_2, r) in an open subset of $\mathbb{Q}^3$) of admissible projective line bundles

over products of three Riemann surfaces together with admissible rational Kähler classes (which we can scale to be integral) such that F_Ω is positive on $(-1, 1) \cap \mathbb{Q}$, but has an irrational repeated root in $(-1, 1)$. By Theorem 1 these Kähler classes then do not contain an extremal Kähler metric.

Remark 10. According to the conjectures [15,50], the above examples should define unstable projective varieties. However, the degeneration that demonstrates this instability is not algebraic. While we cannot prove that there is no other algebraic test configuration which would detect this instability, it is difficult to imagine how such a test configuration could be constructed. Thus, presumably, our examples are algebraically (relative) K-polystable but analytically only (relative) K-semistable. This suggests that the non-algebraic degenerations implicit in the use of slope K-polystability may be essential to relate stability to existence of CSC and extremal Kähler metrics.

Appendix A. Relation to previous papers

In this appendix we summarize the classification of compact Kähler $2m$-manifolds M with a Hamiltonian 2-form of order ℓ given in [5, Theorem 5], and explain how Theorem 3 follows from this classification in the case $\ell = 1$. We also give a nonexistence result for extremal Kähler metrics when $\ell = 2$.

A.1. Summary of the classification. Let (M, g, J, ω) be a compact connected Kähler $2m$-manifold with a Hamiltonian 2-form ϕ of order ℓ. Let $p(t)$ be the momentum polynomial of ϕ and $K(t) = J \operatorname{grad}_g p(t)$ be the corresponding family of Hamiltonian Killing vector fields. We summarize results from [4,5] in italics.

The vector fields $\{K(t) : t \in \mathbb{R}\}$ generate an effective isometric Hamiltonian action of an ℓ-torus $\mathbb{T}$ on M and $p(t)$ has $m - \ell$ constant roots counted with multiplicity. This action is free on a connected dense open subset M^0 of M.

We let S_Δ be the stable quotient of M by the induced action of the complexified torus $\mathbb{T}^c$ and denote by η_a, for a in a finite set with $\leq m - \ell$ elements, the *distinct* constant roots of $p(t)$ and by d_a their multiplicities.

The stable quotient S_Δ is covered by a product $\tilde{S}_\Delta = \prod_a S_a$ of Kähler $2d_a$-manifolds $(S_a, \pm g_a, \pm \omega_a)$, and the natural projection $M^0 \to S_\Delta$ is a principal $\mathbb{T}^c$-bundle.

In [4,5], we took $a \in \{1, \ldots, N\}$, but here we shall adopt (in a moment) a different notation for the index set. We let $p_c(t) = \prod_a (t - \eta_a)^{d_a}$ and write $p(t) = p_c(t) p_{nc}(t)$, where $p_{nc}(t) = \sum_{r=0}^{\ell} (-1)^r \sigma_r t^{\ell-r}$ and $\sigma_0 = 1$. The Killing vector fields $K_r := J \operatorname{grad}_g \sigma_r$, for $r = 1, \ldots, \ell$, are linearly independent on M^0.

The image Δ of the momentum map $(\sigma_1, \ldots, \sigma_\ell)$ is a simplex in $\mathfrak{t}^* \cong \mathbb{R}^\ell$, whose interior (the image of M^0) is the image under the elementary symmetric functions of a domain $D = \prod_{j=1}^\ell (\beta_{j-1}, \beta_j)$, where $\beta_0 < \beta_1 < \cdots < \beta_\ell$. The roots of $p_{\mathrm{nc}}(t)$ define smooth, functionally independent, pairwise distinct functions ξ_j $(j = 1, \ldots, \ell)$ on M^0 which extend continuously to M with image $[\beta_{j-1}, \beta_j]$. The codimension one faces of Δ may be labelled $F_0, \ldots, F_\ell$ such that on F_j, either $\xi_j = \beta_j$ or $\xi_{j+1} = \beta_j$.

The local description of the metric on M^0 is as follows (where we have set $c = 1$, according to [5, Remark 13]).

There are 1-forms $\theta_1, \ldots, \theta_\ell$ on M^0 such that $\theta_r(K_s) = \delta_{rs}$ and $d\theta_r = \sum_a (-1)^r \eta_a^{\ell-r} \omega_a$ and a function Θ of one variable satisfying

$$(22) \qquad\qquad (-1)^{\ell-j} \Theta > 0 \quad on \quad (\beta_{j-1}, \beta_j),$$

$$(23) \qquad\qquad \Theta(\beta_j) = 0, \quad \Theta'(\beta_j) = -\prod_{k \neq j}(\beta_j - \beta_k),$$

such that the Kähler structure on M^0 may be written

$$(24) \quad g = \sum_a p_{\mathrm{nc}}(\eta_a) g_a + \sum_{j=1}^\ell \frac{\Delta_j}{\Theta(\xi_j)} d\xi_j^2 + \sum_{j=1}^\ell \frac{\Theta(\xi_j)}{\Delta_j} \Big(\sum_{r=1}^\ell \sigma_{r-1}(\hat{\xi}_j)\theta_r\Big)^2,$$

$$\omega = \sum_a p_{\mathrm{nc}}(\eta_a) \omega_a + \sum_{r=1}^\ell d\sigma_r \wedge \theta_r,$$

where $\sum_a p_{\mathrm{nc}}(\eta_a) g_a$ is the pullback of a local Kähler product metric on $\hat{S}$, $\Delta_j = \prod_{k \neq j}(\xi_j - \xi_k)$, and $\sigma_r(\hat{\xi}_j)$ is the rth elementary symmetric function of $\xi_1, \ldots, \xi_\ell$ with ξ_j omitted. (σ_r itself is the rth elementary symmetric function of $\xi_1, \ldots, \xi_\ell$.)

The global description of M in [5, Theorem 5] was presented using the blow-up $\hat{M}$ of M along the inverse image of the codimension one faces $F_0, \ldots, F_\ell$ of S_Δ.

$\hat{M}$ is $\mathbb{T}^c$-equivariantly biholomorphic to the $\mathbb{C}P^\ell$-bundle $M^0 \times_{\mathbb{T}^c} \mathbb{C}P^\ell \to S_\Delta$.

The blow-up is encoded by fibrations $S_\Delta \to S_{F_j}$ for each F_j (see also [5, Proposition 6]): either $S_\Delta = S_{F_j}$, or the fibration is covered by the obvious projection $\tilde{S}_\Delta \to \prod_{b \neq a_j} S_b$ for some index a_j such that S_{a_j} is a complex projective space and $(\pm g_{a_j}, \pm \omega_{a_j})$ has constant holomorphic sectional curvature $\pm \prod_{k \neq j}(\beta_j - \beta_k)$.

We unify these cases here by introducing, if $S_\Delta = S_{F_j}$, an additional index a_j with $d_{a_j} = 0$ and $S_{a_j} = \mathbb{C}P^0$ (a point). We denote the new index set by $\hat{\mathcal{A}}$ and take $a \in \hat{\mathcal{A}}$ unless otherwise stated: the additional indices make

no difference to the previous formulae. We still have $\tilde{S}_\Delta = \prod_a S_a$, and now for all F_j, $S_\Delta \to S_{F_j}$ is a $\mathbb{C}P^{d_{a_j}}$-bundle covered by $\tilde{S}_\Delta \to \prod_{b \neq a_j} S_b$. The map $j \to a_j$ is injective [5] and so $\hat{\mathcal{A}}$ is the union of a set $\mathcal{A}$ and the injective image of $\{0, \ldots, \ell\}$ (under $j \mapsto a_j$).

For $a \in \mathcal{A}$, either $\eta_a < \beta_0$ or $\eta_a > \beta_\ell$, according to the sign of $(\pm g_a, \pm \omega_a)$, whereas for $j = \{0, \ldots, \ell\}$, $\eta_{a_j} = \beta_j$.

The formula (24) for the metric on M^0 leads to a description [5, Theorem 5] of $\hat{M}$ as a projective bundle $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S_\Delta$ together with formulae for the Chern classes of $\mathcal{L}_j$ on the covering $\tilde{S}_\Delta$. To obtain instead a description of M, we need one further ingredient, which follows easily by considering the form of the covering transformations and the fact that $\tilde{S}_\Delta \to \prod_{b \neq a_j} S_b$ covers the fibration $S_\Delta \to S_{F_j}$.

Lemma 10. *The projection $\tilde{S}_\Delta \to \tilde{S} := \prod_{a \in \mathcal{A}} S_a$ descends to realize S_Δ as a fibre product of flat projective unitary $\mathbb{C}P^{d_{a_j}}$-bundles over a quotient S of $\tilde{S}$.*

An important class of flat projective unitary $\mathbb{C}P^r$-bundles on S are those of the form $P(E)$, where E is a rank $r + 1$ projectively-flat hermitian vector bundle on S. If S is simply connected, then any flat projective unitary $\mathbb{C}P^r$-bundle is trivial, hence of the form $P(E)$ with $E \cong \mathcal{E} \otimes \mathbb{C}^{r+1}$ for a holomorphic line bundle $\mathcal{E}$. In general the obstruction to the existence of E is given by a torsion element of $H^2(S, \mathcal{O}^*)$ (cf. [19]). In particular, such an E always exists if S is a Riemann surface.

Let us suppose that $S_\Delta = P(E_0) \times_S P(E_1) \times_S \cdots \times_S P(E_\ell) \to S$, where each $E_j \to S$ is projectively-flat hermitian of rank $d_j + 1$. We are free to choose the E_j so that $\hat{M} = P(\mathcal{O}(-1)_{E_0} \oplus \mathcal{O}(-1)_{E_1} \oplus \cdots \oplus \mathcal{O}(-1)_{E_\ell})$ where $\mathcal{O}(-1)_{E_j}$ is the (fibrewise) tautological line bundle over $P(E_j)$ (trivial over the other factors of S_Δ). From the description of the blow-up in [5] (see in particular formulae (46) there), we immediately deduce the following (in which we write $\bar{c}_1(E) = c_1(E)/\operatorname{rank} E$).

M is $\mathbb{T}^c$-equivariantly biholomorphic to $P(E_0 \oplus E_1 \cdots \oplus E_\ell) \to S$ and for any $i \neq j$, $\bar{c}_1(E_j) - \bar{c}_1(E_i) = \frac{1}{2} \sum_a (\prod_{k \neq i}(\eta_a - \beta_k) - \prod_{k \neq j}(\eta_a - \beta_k))[\omega_a/2\pi]$.

Derivation of Theorem 3. In order to derive Theorem 3 from the above, it suffices to rescale g and ϕ so that we can take $\beta_0 = -1$ and $\beta_1 = 1$, see [5, Remark 13]. Then we set $\eta_a = -1/x_a$ and change the sign of ω_a for all a. We also write $\hat{S}$ for S_Δ, and replace the index set $\{0, 1\}$ by $\{0, \infty\}$ so that we can take $\hat{\mathcal{A}} = \{0, \infty\} \cup \mathcal{A}$ where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^+$, but these changes are purely cosmetic.

A.2. A nonexistence result for order 2 extremal Kähler metrics.

In this paper we study only Hamiltonian 2-forms of order 1. As a partial justification

for this restriction, we now consider the lowest interesting dimension for the order 2 case, and show that any extremal metric on a compact Kähler 6-manifold compatible with a Hamiltonian 2-form of order 2 is a Fubini–Study metric on $\mathbb{C}P^3$.

According to [4], in this situation the momentum polynomial has non-constant roots ξ_1 and ξ_2 and one constant root η so $\#\mathcal{A} \leq 1$ and $p(t) = (t - \eta)(t - \xi_1)(t - \xi_2)$. The stable quotient Σ of (M, J) by the complexified $\mathbb{T}^c$ action is a compact Riemann surface with Kähler structure $(g_\Sigma, \omega_\Sigma)$.

By [5, Remark 13], we can set $\beta_0 = -1$ and $\beta_2 = 1$ and write $\beta_1 = \beta$ (where $|\beta| < 1$). If $\mathcal{A}$ is empty, (M, J) is biholomorphic to $\mathbb{C}P^3$; otherwise $|\eta| > 1$ and (M, J) is $\mathbb{T}^c$-equivariantly biholomorphic to $M = P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \rightarrow \Sigma$, where $\mathcal{L}_j$ are holomorphic line bundles on Σ such that (without loss) $\mathcal{L}_1$ is trivial and, by [5, Theorem 5], we have

$$
\text{(25)} \qquad
\begin{aligned}
c_1(\mathcal{L}_0) &= \frac{1}{2}(\eta - 1)(\beta + 1)[\omega_\Sigma/2\pi], \\
c_1(\mathcal{L}_2) &= \frac{1}{2}(\eta + 1)(\beta - 1)[\omega_\Sigma/2\pi].
\end{aligned}
$$

The Kähler metric on M is determined by a function $\Theta(t)$ satisfying positivity and boundary conditions which imply that $\Theta(t) = F(t)/(t - \eta)$ where $F(t) = H(t)((t - \eta) + H(t)Q(t))$ for some function $Q(t)$, and $H(t) = (1 - t^2)(t - \beta)$.

If g is extremal and the extremal vector field is tangent to the fibres of $M \rightarrow \Sigma$, then by [4], $F(t)$ is a polynomial of degree at most 5 and g_Σ has scalar curvature $-F''(\eta)$. This forces $Q(t) = 0$ and so the scalar curvature of g_Σ is $2(3\eta^2 - 2\beta\eta - 1)$ which is positive since $|\eta| > 1$ and $|\beta| < 1$. Hence $\Sigma = \mathbb{C}P^1$. Since $\frac{1}{4}Scal_{g_\Sigma}[\omega_\Sigma/2\pi]$ is a primitive integral class, (25) implies that

$$
(\eta \mp 1)(\beta \pm 1) = q^\pm(3\eta^2 - 2\beta\eta - 1)
$$

for some nonzero integers $q^\pm$. We remark that these formulae show that the relation between $q^\pm$ and (η, β) is birational, in fact the restriction to $\mathbb{R}^2$ of a quadratic transformation of $\mathbb{C}P^2$. In any case, η is constant on the lines through $(q^+, q^-) = (1, 1)$, and $\beta = \pm 1$ on the lines $q^\pm = 0$ and $2q^\pm - q^\mp = 1$, the latter being the lines on which $\eta = \pm 1$. It follows straightforwardly that $|\eta| > 1$ and $|\beta| < 1$ iff $q^+ > 0$, $q^- < 0$ and $2q^+ - q^- < 1$ or vice-versa (swap plus and minus) – which is impossible as $|q_\pm| \geq 1$. We therefore have the following nonexistence result.

Theorem 11. *A compact extremal Kähler 6-manifold (M, J, g, ω) which admits a Hamiltonian 2-form of order 2 with the extremal vector field tangent to the $\mathbb{T}^c$-orbits is isometric to $\mathbb{C}P^3$ with a Fubini–Study metric.*

Appendix B. Proof of Lemma 6

In this appendix we prove Lemma 6 by computing the asymptotics as $x_a \to 0$, for $a \in \mathcal{A}$, of the solution (A, B) of the system (9), i.e., $A\alpha_1 + B\alpha_0 = -2\beta_0$, $A\alpha_2 + B\alpha_1 = -2\beta_1$, where $\alpha_r = \int_{-1}^{1} p_c(t) t^r dt$ and β_r are as in (7). In order to do this, we rewrite β_0 and β_1 as integrals using the obvious identities

$$
p_c(1) + p_c(-1) = \int_{-1}^{1} \frac{d}{dt}(t p_c(t)) dt = \int_{-1}^{1} \left(1 + \sum_a \frac{d_a x_a t}{1 + x_a t}\right) p_c(t) dt
$$

$$
p_c(1) - p_c(-1) = \int_{-1}^{1} \frac{d}{dt}(t^2 p_c(t)) dt = \int_{-1}^{1} \left(2 + \sum_a \frac{d_a x_a t}{1 + x_a t}\right) p_c(t) t \, dt
$$

to obtain

$$
\begin{aligned}
\beta_0 = \int_{-1}^{1} & (1 + t)^{d_0} (1 - t)^{d_\infty} \\
& \times \left(1 + d_0 + d_\infty + \frac{d_0^2}{1 + t} + \frac{d_\infty^2}{1 - t} + \sum_{a \in \mathcal{A}} \frac{d_a x_a (s_a + t)}{1 + x_a t}\right) \\
& \times \left(\prod_{a \in \mathcal{A}} (1 + x_a t)^{d_a}\right) dt \\
\beta_1 = \int_{-1}^{1} & (1 + t)^{d_0} (1 - t)^{d_\infty} \\
& \times \left(2 + d_0 + d_\infty + \frac{d_0^2}{1 + t} + \frac{d_\infty^2}{1 - t} + \sum_{a \in \mathcal{A}} \frac{d_a x_a (s_a + t)}{1 + x_a t}\right) \\
& \times \left(\prod_{a \in \mathcal{A}} (1 + x_a t)^{d_a}\right) t \, dt.
\end{aligned}
$$

The asymptotics of $\alpha_0, \alpha_1, \alpha_2, \beta_0$ and β_1 are given by integrals of the form

$$
I(m, n, k) = \int_{-1}^{1} (1 + t)^m (1 - t)^n t^k \, dt.
$$

Integrating by parts and using $2I(m, n, k + 1) = I(m + 1, n, k) - I(m, n + 1, k)$,

$$
I(m, n, 0) = \frac{2^{m+n+1} m! \, n!}{(m + n + 1)!}, \quad I(m, n, 1) = \frac{2^{m+n+1} (m - n) m! \, n!}{(m + n + 2)!},
$$

$$
I(m, n, 2) = \frac{2^{m+n+1} (m^2 + n^2 + m + n - 2mn + 2) m! \, n!}{(m + n + 3)!}.
$$

These are rather complicated, so we manipulate the integrals using the identities

$$I(m-1, n, 0)m^2 + I(m, n-1, 0)n^2 = \frac{1}{2}I(m, n, 0)(m+n+1)(m+n)$$

$$I(m-1, n, 1)m^2 + I(m, n-1, 1)n^2 = \frac{1}{2}I(m, n, 1)(m+n-1)$$
$$\times (m+n+2)$$

$$I(m-1, n, 2)m^2 + I(m, n-1, 2)n^2 = \frac{1}{2}I(m, n, 2)(m+n+3)(m+n)$$
$$- I(m, n, 1)(m-n)$$

and thus obtain, up to $O(x^2)$,

$$\alpha_k = I(d_0, d_\infty, k) + I(d_0, d_\infty, k+1)\sum_{a\in\mathcal{A}} d_a x_a,$$

$$\beta_0 = \frac{1}{2}I(d_0, d_\infty, 0)(1+d_0+d_\infty)(2+d_0+d_\infty) + I(d_0, d_\infty, 0)\sum_{a\in\mathcal{A}} d_a s_a x_a$$
$$+ \frac{1}{2}I(d_0, d_\infty, 1)(1+d_0+d_\infty)(2+d_0+d_\infty)\sum_{a\in\mathcal{A}} d_a x_a,$$

$$\beta_1 = \frac{1}{2}I(d_0, d_\infty, 1)(1+d_0+d_\infty)(2+d_0+d_\infty) + I(d_0, d_\infty, 1)\sum_{a\in\mathcal{A}} d_a s_a x_a$$
$$+ \left(\frac{1}{2}I(d_0, d_\infty, 2)(3+d_0+d_\infty)(2+d_0+d_\infty)\right.$$
$$\left. - I(d_0, d_\infty, 1)(d_0-d_\infty)\right)\sum_{a\in\mathcal{A}} d_a x_a.$$

A direct computation using the above formulae together with the identity $I(m, n, 1)(m+n+2) = I(m, n, 0)(m-n)$ now shows that, up to $O(x^2)$,

$$\frac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_0\alpha_2 - \alpha_1^2} = (2+d_0+d_\infty)\sum_{a\in\mathcal{A}} d_a x_a$$

$$\frac{\alpha_2\beta_0 - \alpha_1\beta_1}{\alpha_0\alpha_2 - \alpha_1^2} = \frac{1}{2}(1+d_0+d_\infty)(2+d_0+d_\infty) + \sum_{a\in\mathcal{A}} d_a s_a x_a$$
$$+ (d_\infty - d_0)\sum_{a\in\mathcal{A}} d_a x_a.$$

Multiplying by -2 completes the proof.

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