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*Inventiones mathematicae*

# Thermodynamics, dimension and the Weil–Petersson metric*

**Curtis T. McMullen**

Mathematics Department, Harvard University, 1 Oxford St,
Cambridge, MA 02138-2901, USA

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## Contents



## 1 Introduction

In this paper we show the Weil–Petersson metric on Teichmüller space can be reconstructed from the dimensions of dynamical artifacts, such as measures on the circle and limit sets on the sphere. The proof reveals a connection between Hausdorff dimension, $L^2$-norms of holomorphic forms, and the central limit theorem for geodesic flows, especially the variance of observables of mean zero.

These elements are mediated by the thermodynamic formalism, which leads to parallel results for Julia sets, polynomials and Blaschke products $f : \overline{\Delta} \to \overline{\Delta}$. Here the foliated unit tangent bundle $\mathrm{T}_1 X$ is replaced by the Riemann surface lamination $\widehat{X}$ of $f|S^1$. These parallels suggest a definition of the Weil–Petersson metric for dynamical moduli spaces, and contribute

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additional entries to the well-known dictionary between rational maps and Kleinian groups, such as those summarized in Table 1.

**Table 1**

| **Riemann surfaces** | **Dynamics** |
|---|---|
| Fuchsian group $G \subset \mathrm{Aut}(\Delta)$ | Blaschke product $f : \overline{\Delta} \to \overline{\Delta}$ |
| Quasifuchsian group $\Gamma$ | Mating $F(z)$ of two Blaschke products |
| Unit tangent bundle $\mathrm{T}_1(X)$ | Riemann surface lamination $\widehat{X}$ |
| Geodesic flow | Suspension of $f$ |
| Closed geodesic $\gamma$ | Periodic point $p$ in $S^1$ |
| Length of $\gamma$ | Log of the multiplier $\lvert (f^n)'(p) \rvert$ |
| Length of a random geodesic | Growth of $(f^n)'$ along a random orbit |
| Weil–Petersson metric on $\mathcal{T}_g$ | Metric $\int_{\widehat{X}} \lvert v'' \rvert^2$ on $\mathcal{B}_d$ |

We now turn to a detailed statement of results.

**1.1 Riemann surfaces and Kleinian groups.** Let $\mathcal{T}_g$ be the Teichmüller space of Riemann surfaces of genus $g$, and let $X_t \in \mathcal{T}_g$ be a smooth path through $X = X_0 \in \mathcal{T}_g$. The tangent vector $\dot{X}_0 = dX_t/dt|_{t=0}$ can be represented uniquely by a harmonic Beltrami differential

$$\mu = \rho^{-2}\overline{\phi},$$

where $\rho$ is the hyperbolic metric and $\phi \in Q(X)$ is a holomorphic quadratic differential. The Weil–Petersson metric on $\mathcal{T}_g$ is given by

$$\|\dot{X}_0\|^2_{\mathrm{WP}} = \|\mu\|^2_{\mathrm{WP}} = \int_{X_0} \rho^2|\mu|^2 = \int_{X_0} \rho^{-2}|\phi|^2. \tag{1.1}$$

It is naturally to scale this metric by dividing by

$$\mathrm{area}(X_0) = \int_{X_0} \rho^2 = 4\pi(g-1).$$

With this normalization, the inclusions $\mathcal{T}_g \to \mathcal{T}_h$ defined by taking finite covers of $X \in \mathcal{T}_g$ become isometries.

**Dimensions.** The family of Riemann surfaces $X_t$ can be described as a family of quotients of the disk by a smoothly varying family of Fuchsian groups,

$$X_t = \Delta/G_t.$$

There is a unique isotopy $h_t : S^1 \to S^1$ transporting the action of $G = G_0$ to that of $G_t$ and satisfying $h_0(z) = z$.

Using $h_t$ to glue $\Delta$ to $1/\Delta = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ along $S^1$, we obtain a smooth family of quasifuchsian groups

$$(\widehat{\mathbb{C}}, \Gamma_t) = (\Delta, G_0) \cup_{h_t} (1/\Delta, G_t)$$

which can be normalized so $\Gamma_0 = G_0$. This construction is the basis of Bers' embedding of Teichmüller space [Bers]. The limit set $\Lambda(\Gamma_t)$ is a Jordan curve, with $\Lambda(\Gamma_0) = S^1$, so the Hausdorff dimension $\mathrm{H.dim}(\Lambda(\Gamma_t))$ is minimized at $t = 0$.

Similarly the dimension of the pushforward $m_t$ of Lebesgue measure on the circle under $h_t$, defined by

$$\mathrm{H.dim}(m_t) = \inf\left\{\mathrm{H.dim}(E) : m_t(S^1 - E) = 0\right\},$$

achieves its maximum at $t = 0$. Our first result shows the Weil–Petersson metric can be expressed in terms of the second derivatives of these dimensions.

**Theorem 1.1** *The dimension of the quasifuchsian limit set, the dimension of the pushforward of Lebesgue measure on $S^1$, and the Weil–Petersson metric are related by*

$$\frac{d^2}{dt^2}\mathrm{H.dim}(\Lambda(\Gamma_t))\bigg|_{t=0} = -\frac{1}{4}\frac{d^2}{dt^2}\mathrm{H.dim}(m_t)\bigg|_{t=0} = \frac{1}{3}\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)}\cdot$$

See Sect. 2.

**Bending.** As an application, we have:

**Corollary 1.2** *The quasifuchsian groups obtained by bending $G_0$ with angle $\theta$ along the lifts of a simple geodesic $\gamma \subset X_0$ satisfy*

$$\frac{d^2}{d\theta^2}\mathrm{H.dim}(\Lambda(\Gamma_\theta))\bigg|_{\theta=0} = \frac{4}{3}\frac{\|d\ell_\gamma(X_0)\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)}\cdot$$

Here $\ell_\gamma : \mathcal{T}_g \to \mathbb{R}$ denotes the corresponding geodesic length function.

*Proof.* Under bending, the Riemann surfaces $X_0$ and $\overline{X}_0$ uniformized by $G_0$ are deformed by positive and negative grafting along $\gamma$. But the grafting vector field on $\mathcal{T}_g$ is simply the Weil–Petersson gradient of $\ell_\gamma$ [Mc6, Thm. 3.8], so $\|\dot{X}_0\|_{\mathrm{WP}} = \|d\ell_\gamma\|_{\mathrm{WP}}$. The factor of $1/3$ in Theorem 1.1 is replaced by a factor of $4/3$ since $\overline{X}_0$ is changing as well. □

**Vector fields.** To obtain more perspective on the Weil–Petersson metric, recall there is a unique smooth family of conformal maps

$$H_t : \Delta \to \widehat{\mathbb{C}}$$

conjugating the action of $\Gamma_0$ to $\Gamma_t$ and satisfying $H_0(z) = z$. Each $H_t$ extends to a quasiconformal map on $\widehat{\mathbb{C}}$, sending $S^1$ to $\Lambda(\Gamma_t)$. The holomorphic vector field

$$v = \left.\frac{dH_t}{dt}\right|_{t=0}$$

is canonically determined by $\dot{X}_0$ up to the addition of an infinitesimal Möbius transformation

$$(az^2 + bz + c)\frac{\partial}{\partial z} \in \mathrm{sl}_2(\mathbb{C}).$$

The derivatives $v'(z)$, $v''(z)$ and $v'''(z)$ can all be used to measure the size of $[v]$ as a deformation of $X$. In particular, the quadratic differential

$$\widetilde{\phi} = -2v'''(z)\, dz^2$$

is invariant under the action of $G$, and descends to the original quadratic differential $\phi \in Q(X)$ representing $\dot{X}_0$. Thus we can regard the Weil–Petersson metric (1.1) as a measurement of the size of $v'''(z)$, which is itself an infinitesimal form of the Schwarzian derivative.

**Power series.** Our next result (Sect. 4) is given in terms of the first derivative $v'(z)$ on the unit disk.

**Theorem 1.3** *The Hausdorff dimension of the limit set also satisfies*

$$\left.\frac{d^2}{dt^2}\,\mathrm{H.dim}(\Lambda(\Gamma_t))\right|_{t=0} = \lim_{r\to 1} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=r} |v'(z)|^2\,|dz|.$$

This formula leads to several expressions for the Weil–Petersson metric in terms of the power series for $v(z)$ or for $\widetilde{\phi}(z)$ (Sect. 9); for example, we have:

**Theorem 1.4** *The Weil–Petersson metric is given in terms of the quadratic differential $\widetilde{\phi} = \sum_0^\infty a_n z^n\, dz^2$ by*

$$\frac{1}{3}\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = \frac{1}{8}\frac{2^k}{(k-1)!}\lim_{r\to 1}(1-r)^k \sum_1^\infty n^{k-4}|a_n|^2 r^{2n},$$

*for any integer $k > 0$.*

This result can also be deduced from spectral estimates for automorphic forms (see Corollary 8.7).

**The foliated unit tangent bundle.** To study $v''$ intrinsically, we pass to the unit tangent bundle $\mathrm{T}_1 X$. Recall there is a unique smooth probability measure $d\xi$ on $\mathrm{T}_1 X$ that is invariant under the geodesic flow. The fluctuations of a smooth function $f : \mathrm{T}_1 X \to \mathbb{R}$ along geodesics obey the central limit theorem, with *variance* given by

$$\operatorname{Var}(f) = \lim_{S \to \infty} \int_{T_1 X} \frac{1}{S} \left| \int_0^S f(g_s \xi)\, ds \right|^2 d\xi \tag{1.2}$$

when $\int f\, d\xi = 0$.

The unit tangent bundle $\mathrm{T}_1 X$ carries a natural foliation $\mathcal{F}$ whose leaves are swept out by geodesics that are asymptotic in forward time. The universal cover of each leaf $L$ of $\mathcal{F}$ can be identified with the upper halfplane

$$\mathbb{H} = \{z : \operatorname{Im}(z) > 0\}$$

in such a way that the orbits of $g_s | L$ become vertical lines. These coordinates are well-defined up to $z \mapsto az + b$, and hence they determine an *affine* structure on $L$. The *nonlinearity* of $v$ along $\mathcal{F}$ is then given in affine coordinates by the holomorphic 1-form $v'' = v''(z)\, dz$, and in Sect. 9 we show:

**Theorem 1.5** *The Weil–Petersson metric also satisfies*

$$\frac{1}{3} \frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\operatorname{area}(X_0)} = 2 \int_{\mathrm{T}_1 X_0} \rho^{-2} |v''|^2\, d\xi = \operatorname{Var}(\operatorname{Re} v''/\rho).$$

**1.2 Complex dynamics.** We now turn to the formulation of parallel results in complex dynamics.

Given $d > 1$, let $\mathcal{B}_d$ denote the moduli space of degree $d$ proper holomorphic maps $f : \Delta \to \Delta$ such that $f$ has an attracting fixed point in $\Delta$. We identify maps that are conjugate by an automorphism of $\Delta$.

Any $[f] \in \mathcal{B}_d$ can be represented by a Blaschke product of the form

$$f(z) = z \prod_2^d \left( \frac{z - a_i}{1 - \overline{a}_i z} \right), \quad a_i \in \Delta,$$

and hence regarded as a rational map on the whole Riemann sphere. The assumption that $f|\Delta$ has an attracting fixed point corresponds to the assumption that $X = \Delta/G$ is compact; it insures that the Julia set $J(f)$ coincides with $S^1$, and that $f|S^1$ is expanding.

Now let $f_t(z)$ be a smooth family of Blaschke products representing a path in $\mathcal{B}_d$. In this setting, we again have a unique isotopy $h_t : S^1 \to S^1$ transporting the action of $f_0$ to that of $f_t$ and satisfying $h_0(z) = z$. If we use $h_t$ to glue $(\Delta, f_0)$ to $(1/\Delta, f_t)$, we obtain a smooth family of rational maps

$$F_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

which can be normalized so that $F_0 = f_0$. The Julia set $J(F_t)$ is a Jordan curve, with $J(F_0) = S^1$, so the Hausdorff dimension $\mathrm{H.dim}(J(F_t))$ is minimized at $t = 0$. As before, the dimension of the pushforward $m_t$ of Lebesgue measure on the circle under $h_t$ also achieves its maximum at $t = 0$. There is a unique smooth family of conformal maps

$$H_t : \Delta \to \widehat{\mathbb{C}}$$

conjugating the action of $F_0$ to $F_t$, satisfying $H_0(z) = z$, and extending continuously to a family of maps from $S^1$ to $J(F_t)$.

We will see in Sects. 2 and 4 that the results we have formulated for Fuchsian groups carry over to this setting as well, yielding:

**Theorem 1.6** *The dimension of the Julia set and the dimension of the pushforward of Lebesgue measure are related by*

$$\frac{d^2}{dt^2}\mathrm{H.dim}(J(F_t))\bigg|_{t=0} = -\frac{1}{4}\frac{d^2}{dt^2}\mathrm{H.dim}(m_t)\bigg|_{t=0}.$$

**Theorem 1.7** *In terms of the vector field $v = dH_t/dt|_{t=0}$, we also have*

$$\frac{d^2}{dt^2}\mathrm{H.dim}(J(F_t))\bigg|_{t=0} = \lim_{r\to 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|v'(z)|^2\,|dz|.$$

*Example (Polynomial Julia sets).* The preceding results readily imply:

**Theorem 1.8** *For $t$ near zero, the family of polynomials*

$$F_t(z) = z^d + t\big(b_2 z^{d-2} + b_3 z^{d-3} + \cdots + b_d\big)$$

*satisfies*

$$\mathrm{H.dim}\, J(F_t) = 1 + \frac{|t|^2}{4d^2\log d}\sum_{k=2}^{d} k^2|b_k|^2 + O(|t|^3).$$

See Sect. 5. The case $d = 2$ yields Ruelle's formula [Ru2]

$$\mathrm{H.dim}(J(z^2+c)) = 1 + |c|^2/(4\log 2) + O(|c|^3).$$

(The graph of $\mathrm{H.dim}(J(z^2+c))$ for $c \in [-1, 0.5]$ appears in [Mc7, Fig. 8].) The general formula was calculated by different means in [AMO, §8].

**Dynamical moduli spaces.** The open hyperbolic component containing $z^d$ in the moduli space of polynomials is naturally isomorphic to $\mathcal{B}_d$, giving the latter space the structure of a complex orbifold. (This isomorphism, obtained by mating $z^d$ with $f \in \mathcal{B}_d$, is analogous to Bers' embedding of Teichmüller space.) The results above suggest defining a Hermitian metric on $\mathcal{B}_d$ by

$$\left\| \frac{df_t}{dt} \right\|_{\mathrm{WP}}^2 = \frac{d^2}{dt^2} \, \mathrm{H.dim}(J(F_t)). \tag{1.3}$$

It would be interesting to investigate this metric further; for example, is it Kähler, convex and incomplete, as is the case for the Weil–Petersson metric on $\mathcal{T}_g$? (The preceding result computes this metric on the tangent space to $f(z) = z^d \in \mathcal{B}_d$.)

**The Riemann surface lamination.** To form the dynamical counterpart to the unit tangent bundle $\mathrm{T}_1 X = \mathrm{T}_1 \Delta / G$, let

$$\widehat{\Delta} = \left\{ (z_i) \in \Delta^{\mathbb{Z}} : f(z_i) = z_{i+1} \text{ and } |z_i| \to 1 \text{ as } i \to -\infty \right\},$$

and define

$$\widehat{X} = \widehat{\Delta} / \langle \widehat{f} \rangle,$$

where $\widehat{f}((z_i)) = (f(z_i)) = (z_{i+1})$. The space $\widehat{X}$ is a compact *Riemann surface lamination*, with the local structure of (a complex disk) × (a Cantor set).

There is a natural geodesic flow $g_s : \widehat{X} \to \widehat{X}$, preserving a smooth probability measure $d\xi$. Each leaf (connected component) of $\widehat{X}$ is covered by the upper halfplane, giving it a natural affine structure and hyperbolic metric $\rho$. Just as for the unit tangent bundle $\mathrm{T}_1 X$, we can define the nonlinearity $v''(z)\, dz$ and the Schwarzian $v'''(z)\, dz^2$ using affine coordinates along the leaves of $\widehat{X}$. We can also define the variance of a function on $\widehat{X}$ by (1.2), with $\mathrm{T}_1 X$ replaced by $\widehat{X}$; and in Sect. 11 we will show:

**Theorem 1.9** *The vector field $v$ satisfies*

$$\mathrm{Var}(\mathrm{Re}\, v''/\rho) = 2 \int_{\widehat{X}} \rho^{-2} |v''|^2 \, d\xi = \frac{4}{3} \int_{\widehat{X}} \rho^{-4} |v'''|^2 \, d\xi,$$

*and all three quantities coincide with* $(d^2/dt^2)\, \mathrm{H.dim}(J(F_t))|_{t=0}$.

We note that the Riemann surface lamination $\widehat{X}$ can be constructed for any $C^{1+\epsilon}$ or even *symmetric* expanding map $f : S^1 \to S^1$, and such maps are classified by the Teichmüller space of $\widehat{X}$ [Sul4]. The results above also carry over to this setting; they are formulated for Blaschke products on the unit disk since these most closely parallel Fuchsian groups.

**1.3 Random geodesics and random orbits.** Assume for convenience that the family of Blaschke products $f_t(z)$ is normalized so that $f_0(0) = 0$. Then the orbit under $f_0$ of a random point on $S^1$ is uniformly distributed with respect to the invariant measure $m_0 = |dz|/2\pi$ on $S^1$. The rate of expansion of $f_0$ along a random orbit is therefore measured by the *Lyapunov exponent*

$$L(f_0, m_0) = \int_{S^1} \log \left| f_0'(z) \right| dm_0(z).$$

Recalling that $h_t$ transports $m_0$ to $m_t$, we see that $L(f_t, m_t)$ measures the rate of expansion of $f_t$ along a random orbit for $f_0$.

Now it is well-known that the dimension, Lyapunov exponent and entropy of an ergodic measure for a rational map are related by

$$\mathrm{H.dim}(m) L(f, m) = h(f, m) \tag{1.4}$$

(see [Mn, Lemma, p. 426]). The entropy is invariant under topological conjugacy, so $h(f_t, m_t)$ is constant. Thus Theorems 1.6 and 1.9 imply:

**Theorem 1.10** *The expansion of $f_t$ along a random orbit for $f_0$ satisfies*

$$\frac{d^2}{dt^2} \log L(f_t, m_t)\bigg|_{t=0} = \frac{16}{3} \int_{\widehat{X}} \rho^{-4} |v'''|^2 \, d\xi.$$

Similarly, Theorem 1.1 gives a new proof of:

**Theorem 1.11 (Wolpert)** *The length on $X_t$ of a random geodesic on $X_0$ satisfies*

$$\frac{d^2}{dt^2} \log \ell(X_t, g_0)\bigg|_{t=0} = \frac{4}{3} \frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)}.$$

(Here the 'random geodesic' $g_0$ can be interpreted formally as the Liouville current for the hyperbolic metric on $X_0$; cf. [Bon]. The difference in the factors 4 and 16 stems from the relation $|\mu|^2 = 4\rho^{-4}|v'''|^2$.) The new proof replaces the quasiconformal methods of [Wol2] with arguments from the thermodynamic formalism.

## 1.4 Thermodynamics.

We now turn to a sketch of the proofs.

**Pressure and variance.** Let $f(z)$ be an expanding rational map with $J(f) = S^1$ and $f(\Delta) = \Delta$. Let $C^\alpha(S^1)$ denote the space of functions on $S^1$ which are Hölder continuous of exponent $\alpha$.

The thermodynamic formalism associates to each $\phi \in C^\alpha(S^1)$ a *transfer operator* $\mathcal{L}_\phi : C^\alpha(S^1) \to C^\alpha(S^1)$, defined by

$$\mathcal{L}_\phi(\psi)(y) = \sum_{f(x)=y} e^{\phi(x)} \psi(x).$$

The *pressure* $P(\phi)$ is the log of the spectral radius of $\mathcal{L}_\phi$. If $P(\phi) = 0$, then $\phi$ also determines an ergodic, $f$-invariant *equilibrium measure* $m = m(\phi)$ on $S^1$.

The *variance* of a Hölder continuous function $\psi : S^1 \to \mathbb{R}$ with $\int \psi \, dm = 0$ is given by

$$\mathrm{Var}(\psi, m) = \lim_{n\to\infty} \frac{1}{n} \int_{S^1} \Big| \sum_0^{n-1} \psi \circ f^i(z) \Big|^2 dm.$$

It is known that the variance gives the second derivative of the pressure: we have

$$P(\phi + t\psi) = P(\phi) + (t^2/2)\,\mathrm{Var}(\psi, m(\phi)) + O(t^3). \tag{1.5}$$

**Suspensions.** In Sect. 3 we show that the variance behaves well under suspensions: given a Hölder continuous roof function $\rho : S^1 \to \mathbb{R}$ with $\overline{\rho} = \int \rho \, dm > 0$, the variance of the suspension flow on $\widehat{S}^1 \times \mathbb{R}/((z,t) \sim (f(z), t + \rho(z)))$ satisfies

$$\mathrm{Var}(\psi_\rho, m_\rho) = \mathrm{Var}(\psi, m)/\overline{\rho}. \tag{1.6}$$

Here $m_\rho = (\widehat{m} \times dt)/\overline{\rho}$ and $\psi_\rho(x,t) = \psi(x)/\rho(x)$ for $0 \le t < \rho(x)$.

**Families of dynamical systems.** Now consider a smooth family of expanding rational maps $f_t(z)$ with $f_0 = f$. Since expanding maps are structurally stable, we can view $f_t(z)$ as a family of geometric structures imposed on the single topological dynamical system $(f, S^1)$. The changing geometry is recorded by the family of Hölder continuous functions

$$\phi_t(z) = -\log \left| f_t'(h_t(z)) \right|,$$

where $h_t : S^1 \to J(f_t)$ is a topological conjugacy from $f_0$ to $f_t$. Let $m$ be the unique absolutely continuous $f$-invariant probability measure on $S^1$. (If we normalize so $f(0) = 0$, then $m = |dz|/2\pi$, by a simple argument with harmonic functions (see e.g. [Mar])). Let $m_t = (h_t)_*(m)$ and $\mu_t = \mathrm{H.dim}(m_t)$.

In this framework, the Hausdorff dimension $\delta_t$ of $J(f_t)$ can be characterized as the unique solution to the equation

$$P(\delta_t \phi_t) = 0.$$

Using (1.5) and (1.4), in Sect. 2 we obtain the relation

$$\mathrm{Var}(\dot{\phi}_0, m_0)/L(f, m_0) = \ddot{\delta}_0 - \ddot{\mu}_0. \tag{1.7}$$

For a family of Blaschke products $f_t(z)$, we have $\delta_t = \mathrm{H.dim}\, J(f_t) = \mathrm{H.dim}(S^1) = 1$ and thus $\ddot{\delta}_0 = 0$. For the corresponding family of matings $F_t = (\Delta, f_0) \cup_{h_t} (1/\Delta, f_t)$, $m_t$ coincides with harmonic measure on $J(F_t)$

and thus $\mu_t = 1$ and $\ddot{\mu}_0 = 0$. In either case, we obtain a direct connection between $\mathrm{Var}(\dot{\phi}_0, m_0)$ and the second derivative of dimension. Comparing the values of $\dot{\phi}_0$ in the two cases, we obtain the equation

$$\frac{d^2}{dt^2}\,\mathrm{H.dim}(J(F_t))\bigg|_{t=0} = -\frac{1}{4}\frac{d^2}{dt^2}\,\mathrm{H.dim}(m_t)\bigg|_{t=0}$$

stated in Theorem 1.6. The factor $1/4 = (1/2)^2$ arises because $F_t$ is obtained by varying $f_0$ on only half of the Riemann sphere, and the variance is quadratic.

**Virtual coboundaries.** In the case of matings, $h_t : S^1 \to J(F_t)$ extends to a conformal map $H_t : \Delta \to \mathbb{C}$ conjugating $f$ to $F_t$. It follows that the harmonic extension of $\dot{\phi}_0$ to $\Delta$ satisfies the coboundary equation

$$\dot{\phi}_0(z) = \mathrm{Re}[v'(z) - v'(f(z))],$$

where $v = dH_t/dt$. This equation only holds on $\Delta$; in general $v'$ blows up on $S^1$. Nevertheless, in concert with the suspension relation (1.6) it leads to the formula

$$\mathrm{Var}(\dot{\phi}_0, m_0)/L(f, m_0) = (1/2)I_0(v'),$$

where

$$I_0(v') = \lim_{r\to 1} \frac{1}{2\pi|\log(1-r)|}\int_{|z|=r} |v'(z)|^2\,|dz|;$$

see Sect. 4.

**Power series and ergodicity.** In Sect. 10 we demonstrate the equation

$$I_0(v') = 2\,\mathrm{Var}(\mathrm{Re}\,v''/\rho),$$

using the fact that the growth of $|v'|^2 = |\int v''(z)\,dz|^2$ along a random ray in the disk mimics the growth of $|\int \mathrm{Re}\,v''/\rho|^2$ along a random geodesic on the Riemann surface lamination $\widehat{X}$. Using orthogonality of the functions $z^n|\Delta$, we also relate $I_0(v')$ to various $L^2$-norms of its higher derivatives $v^{(k+1)}$ on $\Delta$ (Sect. 11). Ergodicity of the geodesic flow for $\widehat{X}$ then implies these $L^2$-norms are proportional to

$$\int_{\widehat{X}} \rho^{-k}|v^{(k+1)}|^2\,d\xi,$$

completing the proof of Theorems 1.6, 1.7 and 1.9.

**Fuchsian groups.** The corresponding Theorems 1.1, 1.3 and 1.5 for families of Riemann surfaces $X_t = \Delta/G_t$ are obtained similarly, using a Markov partition introduced by Bowen and Series to replace the action of $G_0$ by a piecewise-analytic expanding map $F_0 : S^1 \to S^1$.

**The pressure metric.** Finally we observe that the pressure itself gives rise to a natural metric in the thermodynamic setting.

Let $\sigma : \Sigma \to \Sigma$ be an aperiodic subshift of finite type, and let $\mathcal{T}(\Sigma)$ denote the space of Hölder continuous functions with $P(\phi) = 0$ modulo coboundaries. By convexity, the second derivative

$$D^2 P(\psi) = \mathrm{Var}(\psi, m(\phi))$$

is non-negative on the tangent space

$$T_\phi \mathcal{T}(\Sigma) = \{\psi : \textstyle\int \psi \, dm(\phi) = 0\}/(\text{coboundaries}).$$

In fact the variance vanishes if and only if $\psi$ is cohomologous to zero [PP, Prop. 4.12], and thus the *pressure metric* on $\mathcal{T}(\Sigma)$, given by

$$\|\psi\|_P^2 = \frac{\mathrm{Var}(\psi, m(\phi))}{-\int \phi \, dm(\phi)},$$

is nondegenerate.

Now let $(\Sigma, \sigma)$ be the shift space coming from a Markov partition for a Fuchsian group of genus $g$. Then each marked Riemann surface $X = \Delta/G \in \mathcal{T}_g$ determines a Markov map $F : S^1 \to S^1$ and a symbolic encoding

$$\pi : \Sigma \to S^1,$$

satisfying $\pi(\sigma(x)) = F(\pi(x))$. Note that the Hölder continuous function

$$\phi_X(x) = -\log |F'(\pi(x))|$$

changes by a coboundary when $G$ changes by conjugacy, and that $\phi_X$ determines the lengths of closed geodesics on $X$. Thus the map $X \mapsto \phi_X$ gives a *thermodynamic embedding*

$$\mathcal{T}_g \hookrightarrow \mathcal{T}(\Sigma)$$

of Teichmüller space into the space of functions modulo coboundaries. Theorem 1.1 and (1.7) then imply:

**Theorem 1.12** *The pressure metric pulls back to a multiple of the Weil–Petersson metric under the thermodynamic embedding. More precisely, we have*

$$\|\dot{\phi}_X\|_P^2 = \frac{4}{3} \frac{\|\dot{X}\|_{\mathrm{WP}}^2}{\mathrm{area}(X)} .$$

The metric on $\mathcal{B}_d$ given by (1.3) can similarly be regarded as a pullback of the pressure metric up to scale.

*Notes and references.* The Weil–Petersson metric was introduced in [Wl]. Theorem 1.11 appears in [Wol2]; see also [Bon] and [Bu]. (Note: $3\pi$ should be $6\pi$ in [Wol2, Eq. (0.1) and Cor. 4.3], to be consistent with the definition $g_{\mathrm{WP}}(\mu, \mu) = 2 \int_X \rho^2 |\mu|^2$ used in [Wol2, p. 152 and Cor. 3.5].) E. Cawley showed that Wolpert's theorem implies a variant of Theorem 1.12.

More on the theory of Riemann surface laminations for circle maps can be found in [Sul3], [Sul4], [GS2], [GS1], and [MeSt, Ch. VI.6]. A similar theory for rational maps on $\widehat{\mathbb{C}}$ is developed in [LM]. For a survey of connections between rational maps and Kleinian groups, see [Mc4].

The mating of Blaschke products to yield the rational maps $F_t(z)$ discussed above is a special case of the gluing construction given in [Mc1, Prop. 5.5] (see also [Mc3]); the maps $F_t(z)$ can be regarded as matings of polynomials whose Julia sets are topological circles. See e.g. [Tan] and [Mil] for more on matings.

I would like to thank M. Bridgeman for a lecture on [BT1] which motivated this investigation. See also [BT2]; the results of the latter paper give the inequality

$$(d^2/dt^2)\, \mathrm{H.dim}(\Lambda(\Gamma_t))\big|_{t=0} \geq (1/3) \|\dot{X}_0\|^2_{\mathrm{WP}} / \operatorname{area}(X_0)$$

which is sharpened by Theorem 1.1, and include an inequality version of Corollary 1.2 [BT2, §11]. We remark that the pseudometric on quasifuchsian space introduced in [BT2] can also be regarded as a pullback of the pressure metric. I would also like to thank M. Zinsmeister and the referees for many useful comments.

*Notation.* The expressions $A \sim B$, $A = O(B)$ and $A \asymp B$ mean $A/B \to 1$, $|A| < CB$ and $B/C < A < CB$ for an unspecified constant $C$.

## 2 Thermodynamic formalism

In this section we recall the thermodynamic formalism for expanding conformal maps, following [PP] (see also [Ru1]). We then prove the dimension relations

$$(d^2/dt^2)\, \mathrm{H.dim}(\Lambda(\Gamma_t))\big|_{t=0} = -(1/4)(d^2/dt^2)\, \mathrm{H.dim}(m_t)\big|_{t=0}$$

and

$$(d^2/dt^2)\, \mathrm{H.dim}(J(F_t))\big|_{t=0} = -(1/4)(d^2/dt^2)\, \mathrm{H.dim}(m_t)\big|_{t=0}$$

appearing in Theorems 1.1 and 1.6 of the introduction.

**Shifts.** Let $A(i, j)$ be a $d \times d$ matrix with entries 0 or 1. Assume $A$ is *aperiodic*, meaning there is an $n > 0$ such that $A^n$ has only positive entries. The associated 1-sided *shift space* $\Sigma$ consists of all sequences $(x_0, x_1, x_2, \ldots)$ of integers $1 \le x_i \le d$ such that $A(x_i, x_{i+1}) = 1$ for all $i$. The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $\sigma((x_i)) = (x_{i+1})$. We give $\Sigma$ the metric

$$d((x_i), (y_i)) = 1/d^n,$$

where $n$ is the smallest index such that $x_n \neq y_n$. Then $(\Sigma, d)$ is a Cantor set of Hausdorff dimension one, and the map $\sigma : \Sigma \to \Sigma$ is locally expanding by a factor of $d$.

For $\alpha > 0$, let $C^\alpha(\Sigma)$ denote the Banach space of real-valued continuous functions $f$ on $\Sigma$ satisfying a Hölder estimate of the form

$$|f(x) - f(y)| \le M d(x, y)^\alpha.$$

The norm on $C^\alpha(\Sigma)$ is given by

$$\|f\|_{C^\alpha} = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}.$$

The *pullback operator* on $C^\alpha(\Sigma)$ is defined by $(\sigma^* f)(y) = f(\sigma(y))$, and we say $f_1$ and $f_2$ are *cohomologous* if

$$f_1 - f_2 = f_3 - \sigma^* f_3$$

for some $f_3 \in C^\alpha(\Sigma)$.

**Pressure.** Given $\phi \in C^\alpha(\Sigma)$, the *transfer operator* (or *Ruelle operator*) on $f \in C^\alpha(\Sigma)$ is defined by

$$\mathcal{L}_\phi(f)(y) = \sum_{\sigma(x)=y} e^{\phi(x)} f(x).$$

It is the composition of multiplication by $e^\phi$ with pushforward under $\sigma$, and satisfies:

$$\mathcal{L}_\phi(g\sigma^*(f)) = f\mathcal{L}_\phi(g). \tag{2.1}$$

The *pressure* of $\phi$ is defined in terms of the spectral radius of the transfer operator by

$$P(\phi) = \log \rho(\mathcal{L}_\phi).$$

The pressure is a convex, real-analytic function on $C^\alpha(\Sigma)$. By a generalization of the Perron–Frobenius theorem there is a positive eigenfunction $e^\psi$, unique up to scale, satisfying

$$\mathcal{L}_\phi(e^\psi) = \rho(\mathcal{L}_\phi)e^\psi;$$

and the rest of the spectrum of the transfer operator lies in a disk of radius $r < \rho(\mathcal{L}_\phi)$.

**Equilibrium measure.** Now suppose $P(\phi) = 0$, so $\mathcal{L}_\phi(e^\psi) = e^\psi$. Then there is a unique positive measure $\mu$ on $\Sigma$ satisfying

$$\int \mathcal{L}_\phi(f)\, d\mu = \int f\, d\mu$$

for all $f \in C^\alpha(\Sigma)$ and $\int e^\psi\, d\mu = 1$. We define the associated *equilibrium measure* on $\Sigma$ by

$$m(\phi) = e^\psi \mu. \tag{2.2}$$

The equilibrium measure is an ergodic, $\sigma$-invariant probability measure with positive entropy.

The pressure and the equilibrium measure depend only on the cohomology class of $\phi$. Moreover, we can modify $\phi$ by the coboundary $\psi - \sigma^*\psi$ to obtain $\mathcal{L}_\phi(1) = 1$; then $m(\phi) = \mu$, and using (2.1) we have $\mathcal{L}_\phi(\sigma^*(f)) = f$ and

$$\int \mathcal{L}_\phi(f)\, dm(\phi) = \int f\, dm(\phi)$$

for all $f \in C^\alpha(\Sigma)$.

**Decay of correlations.** Now fix an equilibrium measure $m = m(\phi)$. Consider the inner product $\langle f, g\rangle = \int_\Sigma fg\, dm$ and norm $\|f\|_2^2 = \langle f, f\rangle$ on the Banach space $C^\alpha(\Sigma)$. Adjusting by a coboundary, we can assume $\mathcal{L}_\phi(1) = 1$; we then have

$$\langle \mathcal{L}_\phi(f), g\rangle = \langle f, \sigma^*(g)\rangle. \tag{2.3}$$

Because the spectrum of $\mathcal{L}_\phi$ restricted to functions with $\int f\, dm = 0$ lies in a disk of radius $r < 1$, we have *rapid decay of correlations*; that is,

$$|\langle f, g \circ \sigma^n\rangle| = O(r^n)$$

for any $f, g \in C^\alpha(\Sigma)$ of mean zero.

**Variance.** Decay of correlations implies that the functions $f(\sigma^i x)$ behave roughly like independent random variables. The *variance* of $f \in C^\alpha(\Sigma)$ of mean zero is given by

$$\operatorname{Var}(f) = \lim\, (1/n)\|S_n(f)\|_2^2,$$

where

$$S_n(f, x) = \sum_0^{n-1} f(\sigma^i x).$$

The variance depends only on the cohomology class of $f$. It is given more explicitly by the series

$$\mathrm{Var}(f) = \langle f, f \rangle + 2 \sum_1^{\infty} \langle f, f \circ \sigma^i \rangle,$$

which is rapidly convergent by decay of correlations. We also write $\mathrm{Var}(f) = \mathrm{Var}(f, m)$ to emphasize the dependence on $m$.

The *central limit theorem* says the oscillations of $S_n(f)/\sqrt{n}$ are governed by a Gaussian distribution with variance $\mathrm{Var}(f)$.

**Theorem 2.1** *For any $a < b$, and $f \in C^\alpha(\Sigma)$ of mean zero and variance $V$, we have*

$$m \left\{ x : a < \frac{S_n(f, x)}{\sqrt{n}} < b \right\} \to \frac{1}{\sqrt{2\pi V}} \int_a^b e^{-t^2/(2V)^2}\, dt.$$

See [PP, Thm 4.13].

**Derivatives.** The following useful formulas for the derivatives of pressure appear in [PP, Props. 4.10, 4.11].

**Theorem 2.2** *Let $\phi_t$ be a smooth path in $C^\alpha(\Sigma)$, let $m_0 = m(\phi_0)$ and let $\dot{\phi}_0 = d\phi_t/dt|_{t=0}$. We then have*

$$\left. \frac{dP(\phi_t)}{dt} \right|_{t=0} = \int_\Sigma \dot{\phi}_0 \, dm_0$$

*and, if the first derivative is zero, then*

$$\left. \frac{d^2 P(\phi_t)}{dt^2} \right|_{t=0} = \mathrm{Var}(\dot{\phi}_0, m_0) + \int \ddot{\phi}_0 \, dm_0.$$

(Note: [PP] treats the second derivative in the case where $c_t = \int \phi_t \, dm_0$ is constant; to obtain the general formula above, use the fact that $P(\phi_t - c_t) = P(\phi_t) - c_t$.)

**Markov maps.** The thermodynamic formalism is well-suited to the study of expanding conformal dynamical systems.

To treat the case of both rational maps and Kleinian groups, we will consider the dynamics of an expanding, conformal *Markov map*

$$F : J(F) \to J(F)$$

relative to a *Markov partition* $J(F) = \bigcup_1^n J_i$. Here $J(F)$ is a compact subset of $\widehat{\mathbb{C}}$, and we require that:

1. As subspaces of $J(F)$, each *tile* $J_i$ is the closure of its interior;
2. The interiors of different tiles are disjoint;

3. $F|\operatorname{int} J_i$ is injective, and extends to a conformal map $F_i$ on a neighborhood of $J_i$;
4. $F_i(J_i)$ is a union of tiles, and
5. The graph of $F|J(F)$ is the union of the graphs of $F_i|J_i$.

The map $F$ is generally multivalued at points where two tiles meet.

We say $F$ is *expanding* if there is an $n > 0$ such that the spherical derivative of every branch of $F^n|J(F)$ satisfies

$$|(F^n)'(x)| > C > 1,$$

and if for every nonempty open set $U \subset J(F)$ there is an $m > 0$ such that $F^m(U) = J(F)$.

Let $A(i, j) = 1$ if $F(J_i) \supset J_j$, and 0 otherwise. By the expanding assumption, $A(i, j)$ is aperiodic, and the associated shift space $\Sigma$ admits a Hölder continuous projection

$$\pi : \Sigma \to J(F),$$

characterized by the property that $x = (x_0, x_1, x_2, \ldots)$ gives the sequences of tiles visited by the forward orbit $(z, F(z), F^2(z), \ldots)$ of $z = \pi(x)$. (Orbits of $F$ that land on the borders between tiles have multiple encodings in $\Sigma$, but these ambiguous orbits have zero mass for all equilibrium measures.) The shift map $\sigma$ gives a single-valued resolution of $F$, satisfying

$$\pi(\sigma(x)) = F_{x_0}(\pi(x)).$$

*Example.* For $F(z) = z^d$ with $J(F) = S^1$, we can take $J_1, \ldots, J_d \subset S^1$ to be the intervals bounded by consecutive $d$th roots of unity; then $\Sigma \cong (\mathbb{Z}/d)^{\mathbb{N}}$ is the 1-sided shift on $d$ symbols.

**Dimensions.** The geometry of $F$ is encoded by the Hölder continuous function

$$\phi(x) = -\log \left| F'_{x_0}(\pi(x)) \right|$$

on the symbolic dynamical system $(\Sigma, \sigma)$. For example, the Lyapunov exponent of an equilibrium measure $m$ is given by

$$L(F, \pi_* m) = -\int_\Sigma \phi \, dm. \tag{2.4}$$

This perspective is especially useful for studying families of dynamical systems; cf. [Sul2, §3].

The dimensions of interest to us can be recovered from $\phi$ by the following standard results.

**Theorem 2.3** *The Hausdorff dimension $\delta$ of $J(F)$ is the unique solution to $P(\delta\phi) = 0$; and the equilibrium measure $m_{\delta\phi}$ is equivalent to the Hausdorff measure of dimension $\delta$ on $J(F)$.*

**Theorem 2.4** *Let $m$ be an equilibrium measure on $\Sigma$. Then the Hausdorff dimension of $m$ transported to $J(F)$ satisfies*

$$\mathrm{H.dim}(\pi_* m) = h(m, \sigma)\Big( \int_\Sigma -\phi \, dm \Big)^{-1}.$$

For continuous $F$, these results appear in [Ru2, Prop. 4] and [Mn] respectively (see also [Me]); the proofs for Markov maps follow the same lines.

**Families of rational maps.** For simplicity, we now focus on the case of a rational map $F(z)$ with Julia set $J(F) \subset \mathbb{C}$. Assume $F$ is *expanding*; that is, for some $n > 0$ we have $|(F^n)'| > c > 1$ on $J(F)$ in the spherical metric. Then there exists a Markov partition $J(F) = \bigcup J_i$ making $F|J(F)$ into a Markov map as above (see e.g. [Ru1, §7.29]).

Now consider a smooth family of expanding rational maps $F_t(z)$. By the theory of holomorphic motions, there is a smooth family of homeomorphisms

$$h_t : J(F_0) \to J(F_t)$$

respecting the dynamics (see e.g. [Mc2, Ch. 4]). By *smooth* we mean there is an $\alpha > 0$ such that $h_t$ varies smoothly in the Banach space $C^\alpha(J(F_0))$ for small $t$. (Indeed, $h_t$ can be extended to a smooth family of $K_t$-quasiconformal maps on the whole sphere, with $K_t \to 1$ as $t \to 0$.)

Choosing a Markov partition for $F_0$, we obtain a continuous family of projections

$$\pi_t : \Sigma \to J(F_t)$$

satisfying $\pi_t(x) = h_t(\pi_0(x))$. Then

$$\phi_t(x) = \log \left| F_t'(\pi_t(x)) \right|$$

varies smoothly in $C^\alpha(\Sigma)$ for small $t$. By Theorem 2.3, the Hausdorff dimension $\delta_t = \mathrm{H.dim}(J(F_t))$ is characterized by

$$P(\delta_t \phi_t) = 0,$$

and the implicit function theorem implies $\delta_t$ is a smooth function of $t$.

**Theorem 2.5** *For any smooth family of expanding rational maps with $J(F_0) = S^1$, we have*

$$\mathrm{Var}(\dot\phi_0, m_0) + \ddot\delta_0 \int \phi_0 \, dm_0 + \int \ddot\phi_0 \, dm_0 = 0, \tag{2.5}$$

*where $m_0$ is the equilibrium measure for $\phi_0$.*

(Here $\dot\phi_0$, $\ddot\phi_0$ and $\ddot\delta_0$ denote derivatives with respect to $t$ evaluated at $t = 0$.)

*Proof.* Since $J(F_t)$ is homeomorphic to a circle, we have $\delta_t \geq \delta_0 = 1$ and thus $\dot{\delta}_0 = 0$. Using Theorem 2.2 to compute the quantity $d^2 P(\delta_t \phi_t)/dt^2 = 0$, we obtain the expression above. $\square$

*Remark (Lyapunov exponents).* Let $M_t = (\pi_t)_*(m_0)$. Then (2.5) can also be expressed in terms of the Lyapunov exponents $L_t = L(F_t, M_t)$ and the dimensions $\mu_t = \text{H.dim}(M_t)$, as follows:

$$\text{Var}(\dot{\phi}_0, m_0)/L_0 = \ddot{\delta}_0 + \ddot{L}_0/L_0 = \ddot{\delta}_0 - \ddot{\mu}_0.$$

**Rational maps.** Returning to the setting of Theorem 1.6, we now consider a smooth family of expanding rational maps $f_t(z)$ with $J(f_t) = S^1$ and $f_t(\Delta) = \Delta$.

Let $h_t : S^1 \to S^1$ be the unique isotopy transporting the action of $f_0$ to $f_t$ and satisfying $h_0(z) = z$. Consider the smooth, 2-parameter family of rational maps

$$F_{s,t} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

obtained by gluing $(\Delta, f_s)$ to $(1/\Delta, f_t)$ using $h_t \circ h_s^{-1}$ as in [Mc1, Prop. 5.5] (see also [Mc3]).

The family $F_{s,t}$ can be normalized so that $F_{t,t}(z) = f_t(z)$ and

$$F_{s,t}(\overline{z}) = \overline{F_{t,s}(z)}. \tag{2.6}$$

Using a Markov partition, we obtain a smoothly varying symbolic encoding

$$\pi_{s,t} : \Sigma \to J(F_{s,t}),$$

and hence smoothly varying Hölder continuous functions

$$\phi_t(x) = -\log \left| F'_{0,t}(\pi_{0,t}(x)) \right|$$

and

$$\Phi_t(x) = -\log \left| F'_{t,t}(\pi_{t,t}(x)) \right|.$$

Let $m_0 = m(\phi_0)$ and let $m_{s,t}$ denote the pushforward of $m_0$ to $J(F_{s,t})$. Then $m_{0,0}$ is equivalent to Lebesgue measure on $S^1$, and $m_{t,t} = (h_t)_*(m_{0,0})$. Thus Theorem 1.6 follows from:

**Theorem 2.6** *At $t = 0$ we have*

$$\frac{d^2}{dt^2}\text{H.dim}(J(F_{0,t})) = \frac{\text{Var}(\dot{\phi}_0, m_0)}{\int -\phi_0 \, dm_0} = -\frac{1}{4}\frac{d^2}{dt^2}\text{H.dim}(m_{t,t}). \tag{2.7}$$

*Proof.* The key point is that $F_{0,t}$ is conformally conjugate on one component $U$ of $\widehat{\mathbb{C}} - J(F_{s,t})$ to $f_0|\Delta$. Thus $m_{0,t}$ is equivalent to harmonic measure on $\partial U$, and hence $\mathrm{H.dim}(m_{0,t}) = 1$ for all $t$ [Mak]. This implies, by Theorem 2.4, that $\int \phi_t \, dm_0$ is constant, and hence $\int \ddot{\phi}_0 \, dm_0 = 0$. Applying Theorem 2.5 we then obtain

$$\mathrm{Var}(\dot{\phi}_0, m_0) + \ddot{\delta}_0 \int \phi_0 \, dm_0 = 0,$$

where $\delta_t = \mathrm{H.dim}\, J(F_{0,t})$. Rearranging terms gives the first equality in (2.7).

For the second, we observe that

$$\frac{d^2}{dt^2} \mathrm{H.dim}(m_{t,t}) = -\frac{\int \ddot{\Phi}_0 \, dm_0}{\int \Phi_0 \, dm_0}$$

by Theorem 2.4, while

$$\mathrm{Var}(\dot{\Phi}_0, m_0) + \int \ddot{\Phi}_0 \, dm_0 = 0$$

by Theorem 2.5, since $\mathrm{H.dim}(J(F_{t,t})) = 1$ for all $t$. The symmetry (2.6) of $F_{s,t}$ implies $\dot{\Phi}_0 = 2\dot{\phi}_0$, and hence

$$\mathrm{Var}(\dot{\Phi}_0, m_0) = 4 \, \mathrm{Var}(\dot{\phi}_0, m_0).$$

Since $\phi_0 = \Phi_0$, these relations give the second equality in (2.7). $\square$

**Families of Fuchsian groups.** Now consider a smooth family of cocompact Fuchsian groups $\Gamma_t$ with $\Lambda(\Gamma_t) = S^1$.

By Bowen and Series, the group $\Gamma_0$ also admits a Markov partition; that is, there is an expanding Markov map

$$f_0 : S^1 = \bigcup_1^n J_i \to S^1$$

with $f_0 | \operatorname{int} J_i = \gamma_i \in \Gamma_0$ (see [BS]).

Gluing $(\Delta, \Gamma_s)$ to $(1/\Delta, \Gamma_t)$, we obtain a two-parameter family of quasifuchsian groups $\Gamma_{s,t}$ and a corresponding family of Markov maps $F_{s,t}$. Applying the theory of holomorphic motions (as in [Sul1]), this yields a smoothly varying symbolic coding

$$\pi_{s,t} : \Sigma \to \Lambda(\Gamma_{s,t}),$$

allowing us to define

$$\phi_t(x) = -\log \left| F'_{0,t}(\pi_{0,t})(x) \right|.$$

Then $m_0 = m(\phi_0)$ corresponds to Lebesgue measure on $S^1$; setting $m_{s,t} = (\pi_{s,t})_*(m_0)$, the proof just given now yields:

**Theorem 2.7** *At $t = 0$ we have*

$$\frac{d^2}{dt^2}\,\mathrm{H.dim}(\Lambda(\Gamma_{0,t})) = \frac{\mathrm{Var}(\dot{\phi}_0, m_0)}{\int -\phi_0\, dm_0} = -\frac{1}{4}\frac{d^2}{dt^2}\,\mathrm{H.dim}(m_{t,t}).$$

In particular, we obtain the dimension relation stated in Theorem 1.1.

## 3 Variance and suspensions

In this section we relate the variance $\mathrm{Var}(f)$ for the shift map to the variance for its suspension under a roof function. This relation will allow us to express $\mathrm{Var}(f)$ for maps on the circle in terms of integrals over $\Delta$, $\mathrm{T}_1 X$ and $\widehat{X}$.

**Stopping times.** We begin by fixing an equilibrium measure $m$ on $\Sigma$, and a *roof function* $\rho \in C^\alpha(\Sigma)$, satisfying

$$\sum_0^{n-1} \rho(\sigma^i x) > 0 \tag{3.1}$$

for some $n > 0$. This implies that $\overline{\rho} = \int \rho\, dm > 0$.

Recall that the variance of $f \in C^\alpha(\Sigma)$ with $\int f\, dm = 0$ is given by

$$\mathrm{Var}(f, m) = \lim_{n\to\infty} (1/n)\|S_n(f)\|_2^2,$$

where

$$S_n(f, x) = \sum_{i=0}^{n-1} f(\sigma^i x). \tag{3.2}$$

We define the variance *relative to* $\rho$ by

$$\mathrm{Var}_\rho(f, m) = \lim\, (1/n)\|V_n(f)\|_2^2,$$

where

$$V_n(f, x) = \sum_0^{N(n,x)-1} f(\sigma^i x),$$

and the *stopping time* $N(n, x) > 0$ is the least positive integer such that

$$\sum_0^{N(n,x)-1} \rho(\sigma^i x) \geq n\,\overline{\rho}.$$

Note that $N(n, x) \asymp n$ by (3.1), and $N/n \to 1$ almost everywhere by the ergodic theorem.

The main result of this section is:

**Theorem 3.1** *For any roof function $\rho$ and equilibrium measure $m$, we have*

$$\mathrm{Var}_\rho(f, m) = \mathrm{Var}(f, m)$$

*for all $f \in C^\alpha(\Sigma)$ of mean zero.*

This result is an immediate consequence of:

**Theorem 3.2** *If $f \in C^\alpha(\Sigma)$ has mean zero, then we have*

$$\lim(1/n)\|V_n(f) - S_n(f)\|_2^2 = 0.$$

We will also deduce:

**Theorem 3.3** *For any $g \in C(\Sigma)$ and $f \in C^\alpha(\Sigma)$ with $\int f\, dm = 0$, we have*

$$\lim_{n\to\infty} \langle g, V_n(f)^2/n\rangle = \mathrm{Var}(f, m) \int g\, dm.$$

**Suspension.** The results above can also be formulated in terms of flows, as follows.

Let $(\widehat{\Sigma}, \widehat{\sigma})$ be the natural extension of $\Sigma$ to a two-sided shift. Any equilibrium measure $m$ on $\Sigma$ determines an invariant measure $\widehat{m}$ on $\widehat{\Sigma}$. The variance of $f \in C^\alpha(\widehat{\Sigma})$ is defined just as for $\Sigma$.

Let $\rho \in C^\alpha(\widehat{\Sigma})$ be a roof function satisfying (3.1). Then the *suspension* of $(\widehat{\Sigma}, \widehat{\sigma})$ under $\rho$ is the space

$$\widehat{\Sigma}_\rho = (\widehat{\Sigma} \times \mathbb{R})/((x, t) \sim (\sigma(x), t + \rho(x)))$$

equipped with the natural flow $g_s(x, t) = (x, t + s)$. The measure $m|\Sigma$ determines a natural $g_s$-invariant probability measure

$$dm_\rho = d\widehat{m}\, dt/\overline{\rho}.$$

Let $F(x, t)$ be a bounded measurable function on $\widehat{\Sigma}_\rho$ such that

$$\widehat{F}(x) = \int_0^{\rho(x)} F(x, t)\, dt$$

is Hölder continuous on $\widehat{\Sigma}$ and $\int F\, dm_\rho = \int \widehat{F} d\widehat{m} = 0$. We can then form the integrals

$$S_n(F, (x, t)) = \int_0^n F(x, t + s)\, ds$$

(even for $n \in \mathbb{R}$), and define the variation of $F$ by

$$\mathrm{Var}(F, m_\rho) = \lim_{n\to\infty} (1/n) \int_{\widehat{\Sigma}_\rho} S_n(F)^2 \, dm_\rho.$$

**Theorem 3.4** *We have* $\mathrm{Var}(F, m_\rho) = \mathrm{Var}(\widehat{F}, \widehat{m})/\overline{\rho}$.

*Proof.* By [PP, Prop. 1.2], there is a Hölder continuous function $g$ such that

$$f = \widehat{F} + g - g \circ \sigma$$

depends only on the coordinates $(x_i)$ with $i \geq 0$; in other words, $f$ is essentially a function on $\Sigma$. It follows easily that for $0 \leq t < \rho(x)$ we have

$$S_{n\overline{\rho}}(F, (x, t)) = V_n(f, x) + O(1). \tag{3.3}$$

Appealing to Theorem 3.3 and using the fact that $\int \rho(x)/\overline{\rho} = 1$, we obtain

$$\begin{aligned} \mathrm{Var}(F, m_\rho) &= \lim_{n\to\infty} \frac{1}{n\overline{\rho}} \int_{\Sigma_\rho} S_{\overline{\rho}n}(F)^2 d\widehat{m} dt/\overline{\rho} \\ &= \lim_{n\to\infty} \frac{1}{n\overline{\rho}} \int_{\widehat{\Sigma}} V_n(f, x)^2 (\rho(x)/\overline{\rho}) d\widehat{m} \\ &= \mathrm{Var}(f, m)/\overline{\rho} = \mathrm{Var}(\widehat{F}, \widehat{m})/\overline{\rho}. \end{aligned}$$

□

For use in Sect. 10 we also record:

**Theorem 3.5** *For any* $g \in C(\widehat{\Sigma})$ *with* $\int g d\widehat{m} = 1$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \int_{\widehat{\Sigma}} g(x) S_n(F, (x, 0))^2 d\widehat{m} = \mathrm{Var}(F, m_\rho).$$

*Proof.* The set of $g \in C(\widehat{\Sigma})$ for which the result holds is closed under uniform limits and composition with $\sigma^{\pm 1}$. Thus we may assume $g(x)$ only depends on $x_i$ for $i \geq 0$, since translates of functions of this type by $\sigma$ span a dense subspace of $C(\widehat{\Sigma})$. We can then apply Theorem 3.3 and (3.3) again, and continue the proof above to obtain the desired result:

$$\begin{aligned} \mathrm{Var}(F, m_\rho) = \frac{\mathrm{Var}(f, m)}{\overline{\rho}} &= \lim_{n\to\infty} \frac{1}{n\overline{\rho}} \int_{\widehat{\Sigma}} g(x) V_n(f, x)^2 d\widehat{m} \\ &= \lim_{n\to\infty} \frac{1}{n\overline{\rho}} \int_{\widehat{\Sigma}} g(x) S_{n\overline{\rho}}(F, (x, 0))^2 d\widehat{m}. \end{aligned}$$

□

**Local variation.** To prove Theorem 3.2, we begin with a bound for the variation of the function $S_n(f)$ (defined by (3.2)) on sets of small measure.

**Theorem 3.6** *Given $f \in C^\alpha(\Sigma)$ of mean zero and measurable sets $E_n$, we have*

$$\limsup_{n\to\infty} \int_{E_n} \frac{S_n(f)^2}{n}\, dm \;\leq\; \sqrt{3}\,\mathrm{Var}(f,m)(\limsup m(E_n))^{1/2}.$$

*Proof.* It suffices to treat the case where the limits superior above are actually limits.

Let $\mu_n$ be the probability measure on $\mathbb{R}$ obtained as the pushforward of $m$ under the function $t = S_n(f,x)/\sqrt{n}$, and let

$$\mu = (2\pi V)^{-1/2} \exp(-t^2/(2V)^2)\, dt$$

be the Gaussian measure on $\mathbb{R}$ of mean zero and variance $V = \mathrm{Var}(f,m)$. By the central limit theorem (Sect. 2) we have $\mu_n \to \mu$ as distributions, and hence $t^2\mu_n \to t^2\mu$ as well. By the definition of $\mu$, these limit satisfy

$$1 = \lim \int d\mu_n = \int d\mu$$

and

$$\mathrm{Var}(f,m) = \lim \int t^2\, d\mu_n = \int t^2\, d\mu;$$

that is, in both cases no mass is lost in the limit (the convergence of measures is tight).

Now the pushforward of $m|E_n$ can be expressed as $\nu_n = \phi_n\mu_n$, where $\phi_n$ is a Borel function with $0 \leq \phi_n \leq 1$. Passing to a further subsequence, we can assume $\phi_n\mu_n \to \phi\mu$ as distributions, where again $0 \leq \phi \leq 1$. Since $0 \leq \phi_n\mu_n \leq \mu_n$, these limits are also tight: we have

$$\int \phi\, d\mu = \lim \int \phi_n\, d\mu_n = \lim m(E_n)$$

and

$$\int t^2\phi\, d\mu = \lim \int t^2\phi_n\, d\mu_n = \lim \frac{1}{n} \int_{E_n} S_n(f)^2\, dm.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\left(\int t^2\phi\, d\mu\right)^2 \leq \int t^4\, d\mu \int \phi^2\, d\mu \leq 3V^2 \int \phi\, d\mu = 3V^2(\lim m(E_n)),$$

and the stated bound follows. □

**Control of the stopping time.** Applying the preceding result to $f(x) = \rho(x) - \overline{\rho}$, we obtain:

**Theorem 3.7** *For any sequence of measurable sets satisfying $m(E_n) < \epsilon$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \int_{E_n} |n - N(x,n)|^2 \, dm = O(\sqrt{\epsilon}). \tag{3.4}$$

*Proof.* Equation (3.1) implies that $|j-k| \le 1 + C(\rho) |\sum_j^k \rho(\sigma^i x)|$ for some $C(\rho) > 0$. By considering the sum from $j = n$ to $k = N(n,x)$, we obtain

$$|n - N(n,x)| \le 1 + C(\rho) \, |S_n(\rho, x) - n\overline{\rho}| \, ;$$

and since $S_n(\rho, x) - n\overline{\rho} = S_n(\rho - \overline{\rho}, x)$, Theorem 3.6 implies that

$$\limsup_{n\to\infty} \frac{1}{n} \int_{E_n} |S_n(\rho) - n\overline{\rho}|^2 \, dm = O(\sqrt{\epsilon}).$$

□

**Cancellation.** Now fix an $f \in C^\alpha(\Sigma)$ of mean zero, whose stopping-time averages $V_n(f)$ are to be studied. We will use the ergodic theorem to show there is usually abundant cancellation in these sums.

**Theorem 3.8** *For any $\epsilon > 0$, there exists an $M > 0$ and a sequence of sets with $m(E_n) > 1 - \epsilon$ such that*

$$\left| \sum_j^{k-1} f(\sigma^i x) \right| \le \epsilon |k - j| + M \tag{3.5}$$

*whenever $x \in E_j \cup E_k$.*

*Proof.* It is useful to pass to the natural extension $(\widehat{\Sigma}, \widehat{\sigma}, \widehat{m})$. Since $(\Sigma, \sigma, m)$ is ergodic, so is $(\widehat{\Sigma}, \widehat{\sigma}, \widehat{m})$. By the ergodic theorem, as $N \to \infty$ we have

$$\left| \frac{1}{N} \sum_0^N f(\widehat{\sigma}^i x) \right| + \left| \frac{1}{N} \sum_{-N}^0 f(\widehat{\sigma}^i x) \right| \to 0$$

for almost every $x \in \widehat{\Sigma}$. Thus for each $\epsilon > 0$, there exists a set with $\widehat{m}(F_0) > 1 - \epsilon$ and an $M > 0$ such that

$$\left| \sum_j^{k-1} f(\widehat{\sigma}^i x) \right| \le \epsilon |k - j| + M \tag{3.6}$$

for all $x \in F_0$ and all $j < k$, provided that $j = 0$ or $k = 0$. Setting $F_n = \widehat{\sigma}^{-n}(F_0)$, we have (3.6) whenever $x \in F_j \cup F_k$.

Let $E_n$ be the projection of $F_n$ from $\widehat{\Sigma}$ to $\Sigma$. Then we have $m(E_n) \geq \widehat{m}(F_n) = \widehat{m}(F_0) = 1 - \epsilon$, and (3.6) implies (3.5) for all $k > j \geq 0$ and all $x \in E_j \cup E_k$. □

*Proof of Theorem 3.2.* We begin by noting that if $N = N(n, x)$, then

$$|V_n(f, x) - S_n(f, x)| = \Big| \sum_{j}^{k-1} f(\sigma^i x) \Big|$$

where $(j, k) = (n, N)$ or $(N, n)$ (ordered so $j \leq k$). Thus by Theorem 3.8, for each $\epsilon > 0$ there is an $M > 0$ and a sequence of sets with $m(E_n) > 1-\epsilon$ such that

$$|V_n(f, x) - S_n(f, x)| \leq \epsilon |N(n, x) - n| + M$$

provided $x \in E_n$. This implies

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \int_{E_n} |V_n(f) - S_n(f)|^2 \, dm &\leq \epsilon^2 \limsup (1/n) \|N(n, x) - n\|_2^2 \\ &= O(\epsilon^2) \end{aligned}$$

by Theorem 3.7. Letting $\widetilde{E}_n = \Sigma - E_n$, Theorem 3.7 also implies

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \int_{\widetilde{E}_n} |V_n(f) - S_n(f)|^2 \, dm &\leq \|f\|_\infty^2 \limsup_{n\to\infty} \frac{1}{n} \int_{\widetilde{E}_n} |N(n, x) - n|^2 \, dm \\ &= O(\sqrt{\epsilon}), \end{aligned}$$

since $m\widetilde{E}_n < \epsilon$. Combining these results, we obtain

$$\limsup (1/n) \|V_n(f) - S_n(f)\|_2^2 = O(\epsilon^2 + \sqrt{\epsilon});$$

since $\epsilon > 0$ is arbitrary, the limit superior is actually zero. □

*Proof of Theorem 3.3.* By Theorem 3.2, it suffices to prove

$$\lim_{n\to\infty} \langle g, S_n(f)^2/n \rangle = \mathrm{Var}(f, m) \int g \, dm;$$

and since this equation is immediate for $g = 1$, we may assume that $g$ has mean zero. Since sequence $S_n(f)^2/n$ is bounded in $L^1(\Sigma, dm)$, the set of $g$ for which the theorem holds is closed under uniform limits, so we may also assume $g$ belongs to the dense subspace of Hölder continuous functions $C^\alpha(\Sigma) \subset C(\Sigma)$.

Since $m$ is an equilibrium measure we can write $m = m(\phi)$, where $\phi$ is normalized so that $\mathcal{L}_\phi(1) = 1$. Let

$$P(h) = h(x) - \int h \, dm$$

be the projection onto the functions of mean zero, and let $T = \mathcal{L}_\phi \circ P$. Then the spectral radius of $T$ is less than some $r < 1$, and hence

$$\|T^i\| = O(r^i)$$

as an operator on $C^\alpha(\Sigma)$. Moreover we have

$$\langle h_1, h_2 \circ \sigma \rangle = \langle T(h_1), h_2 \rangle$$

whenever $h_1$ or $h_2$ has mean zero, by (2.3).

Now observe that

$$\left\langle g, S_n(f)^2 \right\rangle = \sum_{0 \le j,k < n} \langle g, f_j f_k \rangle,$$

where $f_i(x) = f(\sigma^i x)$. To bound the terms in this sum, we note that if $j \le k$ then

$$\langle g, f_j f_k \rangle = \left\langle T^j(g), f_0 f_{k-j} \right\rangle = \left\langle T^{k-j}\left(f_0 T^j(g)\right), f_0 \right\rangle = O(r^k).$$

Consequently

$$\sum_{0 \le j,k < n} \langle g, f_j f_k \rangle = O\Big(\sum_0^\infty k r^k\Big) = O(1),$$

and hence

$$\left\langle g, S_n(f)^2/n \right\rangle = O(1/n) \to 0$$

as desired. □

*Notes.* The study of variance and the central limit theorem for geodesic flows and other dynamical systems has a long history, going back to [Si]. For recent work on the central limit theorem for suspensions, see [DP] and [MT]. (Note that the central limit theorem for $S_n(f)$ does not formally imply the existence of $\mathrm{Var}(f)$, since an interchange of limits is required.)

## 4 Virtual coboundaries

Let $f : S^1 \to S^1$ be an expanding conformal Markov map. In this section we study the variance of a function $h : S^1 \to \mathbb{R}$ in terms of a solution to the 'virtual coboundary equation'

$$h(z) = g(z) - g(f(z))$$

on the unit disk $\Delta$ (for a suitable extension of $h$). To measure the asymptotic growth of $g : \Delta \to \mathbb{C}$, we define

$$I_0(g) = \lim_{r \to 1} \frac{1}{2\pi |\log(1-r)|} \int_{|z|=r} |g(z)|^2 \, |dz|. \tag{4.1}$$

We will establish:

**Theorem 4.1** *If $h$ is the virtual coboundary of $g$, then $I_0(g)$ exists and we have:*

$$\frac{\mathrm{Var}(h \circ \pi, m)}{\int -\phi \, dm} = I_0(g). \tag{4.2}$$

Using Theorem 2.6, we will then deduce the formulas

$$(d^2/dt^2)\,\mathrm{H.dim}(\Lambda(\Gamma_t))\big|_{t=0} = (1/2) I_0(v'(z))$$

and

$$(d^2/dt^2)\,\mathrm{H.dim}(J(F_t))\big|_{t=0} = (1/2) I_0(v'(z))$$

stated as Theorems 1.3 and 1.7 in the introduction.

**Virtual coboundaries.** For simplicity of notation, we will treat the case where $f(z)$ is an expanding rational map with $J(f) = S^1$ and $f(\Delta) = \Delta$. The extension to general conformal expanding Markov maps is straightforward.

Let $\pi : \Sigma \to S^1$ express $f$ as a quotient of the shift as usual, and let $m$ be the equilibrium measure for $\phi(x) = -\log |f'(\pi(x))|$. Note that $\pi_* m$ is equivalent to Lebesgue measure on $S^1$, and that the Lyapunov exponent of $f$ is given by

$$L(f, \pi_* m) = \int -\phi \, dm.$$

Let $h : S^1 \to \mathbb{C}$ be a Hölder continuous function. We say $h$ is the *virtual coboundary* of a continuous function $g : \Delta \to \mathbb{C}$ if

$$h(z) = g(z) - g(f(z))$$

defines an extension of $h|S^1$ to a Hölder continuous function on the closed disk $\overline{\Delta}$. Intuitively this means $h|S^1$ is the coboundary of $g|S^1$, although the limiting values of $g(z)$ on $S^1$ need not exist. We remark that the expansion of $f$ implies $g(z)$ is uniformly continuous in the hyperbolic metric on $\Delta$.

**Roof function.** Now assume $h$ and $g$ are real-valued. To prove Theorem 4.1, we will show the variance of $h$ along an orbit in $S^1$ can be approximated by the variance along an orbit in $\Delta$.

By Sect. 3, it suffices to compute the variance of $h$ relative to the roof function

$$\rho(x) = -\phi(x) = \log |f'(\pi(x))|.$$

Note that $\sum_0^{n-1} \rho(\sigma^i x) > 0$ for some $n > 0$, since $f$ is expanding; and $\overline{\rho} = L(f, \pi_* m)$.

**Escape times.** Since $f|S^1$ is expanding, we can choose small annular neighborhoods $A$ and $B$ of $S^1 \subset \mathbb{C}$ such that

$$f : A \to B$$

is a proper covering map of degree $d > 1$, $\overline{A} \subset B$, and $h(z)$ is Hölder continuous on $B$. Under iteration, every point of $A$ eventually lands in $B - A$.

Let $R_n = 1 - \exp(-n\overline{\rho})$. Given $x \in \Sigma$, let $Z(n, x) = R_n \pi(x) \in \Delta$, and let the *escape time* $E(n, x) \geq 0$ be the smallest integer such that

$$f^{E(n,x)}(Z(n, x)) \notin A.$$

**Theorem 4.2** *The escape time satisfies $E(n, x) = N(n, x) + O(1)$, where $N(n, x)$ is the stopping time for $\rho$.*

*Proof.* Let $E = E(n, x)$, let $z = \pi(x)$ and let $Z = Z(n, x)$. Since $f^E(Z) \in \Delta - A$, we have $1 - |f^E(Z)| \asymp 1$. Applying the distortion theorems for univalent functions to $f^{-E}$, we also have

$$1 - |f^E(Z)| \asymp |(f^E)'(Z)|(1 - |Z|) = |(f^E)'(z)| \exp(-n\overline{\rho}).$$

Taking logarithms, we obtain

$$\log |(f^E)'(z)| = \sum_0^{E-1} \log |f'(f^i(z))| = \sum_0^{E-1} \rho(\sigma^i x) = n\overline{\rho} + O(1),$$

which implies the Theorem. □

**Theorem 4.3** *We have $V_n(h \circ \pi, x) = g(Z(n, x)) + O(1)$.*

*Proof.* Continuing with the notation above, we have

$$V_n(h \circ \pi, x) = \sum_0^{N(n,x)-1} h(f^i(z)) = \sum_0^{E-1} h(f^i(z)) + O(1)$$

since $E = E(n, x) = N(n, x) + O(1)$. Since $f$ is expanding, there is a $\lambda > 1$ such that any branch of $f^{-i}$ contracts by a factor of $O(\lambda^{-i})$, and thus

$$|f^i(z) - f^i(Z)| = O(\lambda^{i-E}) \tag{4.3}$$

because $|f^E(z) - f^E(Z)| = O(1)$.

By assumption, the extension of $h$ to $\overline{\Delta}$ defined by

$$h(z) = g(z) - g(f(z))$$

is Hölder continuous of some exponent $\alpha > 0$. Thus the bound (4.3) allows us to replace $z \in S^1$ with $Z \in \Delta$ at the cost of an exponentially small error, to obtain:

$$\begin{aligned}\sum_{i=0}^{E-1} h(f^i(z)) &= \sum_0^{E-1} h(f^i(Z)) + O(\lambda^{\alpha(i-E)}) \\ &= O(1) + \sum_0^{E-1} g(f^i(Z)) - g(f^{i+1}(Z)) \\ &= O(1) + g(Z) - g(f^E(Z)).\end{aligned}$$

Moreover $g(f^E(z)) = O(1)$ because $f^E(z)$ belongs to the compact set $\Delta - A$, and the theorem follows. □

*Proof of Theorem 4.1.* Recall from Sect. 2 that there is a unique probability measure $\mu$ on $\Sigma$ such that $\int \mathcal{L}_\phi(\psi)\, d\mu = \int \psi\, d\mu$ for all $\psi \in C^\alpha(\Sigma)$. Since $e^{-\phi} = |f'(\pi(x))|$, $\mu$ is simply Lebesgue measure on $S^1$; that is, $\pi_*(\mu) = |dz|/2\pi$. By (2.2) we can write $\mu = e^{-\psi} m$, where $\psi \in C^\alpha(\Sigma)$ and $\int e^{-\psi}\, dm = 1$.

By Theorem 3.3, the variance of $h \circ \pi$ satisfies

$$\mathrm{Var}(h \circ \pi, m) = \lim_{n\to\infty} \left\langle e^{-\psi}, V_n(h\circ\pi)^2/n \right\rangle = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma V_n(h\circ\pi)^2\, d\mu.$$

Using Theorem 4.3, we can replace $V_n(h \circ \pi)$ with $g$ and move the integral over to $S^1$, to obtain

$$\mathrm{Var}(h \circ \pi, m) = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma g(Z(n,x))^2\, d\mu(x) = \lim_{n\to\infty} \frac{1}{2\pi n} \int_{S^1} g(R_n z)^2 |dz|.$$

The last integral is over a circle of radius $r = R_n$ satisfying $|\log(1-r)| = n\overline{\rho} = n \int -\phi\, dm$, and we obtain (4.2). □

**Families of rational maps.** We now return to the setting of Theorem 1.7. Let $F_t(z)$ be a smooth family of expanding rational maps with $F_0(z) = f(z)$. Assume there is a smooth family of conformal maps $H_t : \Delta \to \widehat{\mathbb{C}}$ such that $H_0(z) = z$ and

$$F_t(H_t(z)) = H_t(F_0(z)). \tag{4.4}$$

Let $v$ denote the holomorphic vector field

$$v(z) = \dot{H}_0(z) = dH_t(z)/dt|_{t=0},$$

and let

$$h(z) = \frac{d}{dt} \log F_t'(H_t(z))\bigg|_{t=0} = \frac{\dot{F}_0'(z) + F_0''(z) v(z)}{F_0'(z)} \tag{4.5}$$

where $\dot{F}_0' = dF_t'/dt|_{t=0}$. Using the fact that $H_t$ transports the critical points of $F_0|\Delta$ to those of $F_t$, one can check that (4.5) defines $h(z)$ as a holomorphic function on $\Delta$.

**Theorem 4.4** *The function $h(z)$ has a continuous extension to $\overline{\Delta}$ which is $(1-\epsilon)$-Hölder continuous for every $\epsilon > 0$.*

*Proof.* By the theory of holomorphic motions, we can extend $v(z)$ to a quasiconformal vector field on the whole sphere. Such a vector field is $(1-\epsilon)$-Hölder continuous for every $\epsilon > 0$ (see e.g. [Mc5, Cor. A.12]). Since $F_t(z)$ is a smooth function of $(t, z)$ near $0 \times S^1$, the desired extension of $h$ is given by (4.5). □

**Theorem 4.5** *The function $h(z)$ is the virtual coboundary of $g(z) = -v'(z)$.*

*Proof.* The functional equation (4.4) yields

$$F_t'(H_t(z)) = H_t'(z)^{-1} H_t'(F_0(z)) F_0'(z)$$

upon differentiation with respect to $z$. Taking logs and differentiating with respect to $t$, we obtain the coboundary equation

$$h(z) = \frac{d}{dt} \log F_t'(H_t(z)) \bigg|_{t=0} = -\dot{H}_0'(z) + \dot{H}_0'(F_0(z)) = -v'(z) + v'(f(z)).$$

□

*Proof of Theorems 1.3 and 1.7.* Choose a Markov partition for $F_0$ and let

$$\pi_t : \Sigma \to J(F_t)$$

be the corresponding symbolic encoding for points in the Julia set of $F_t$. Then $\pi_t(x) = H_t(\pi_0(x))$, so $\phi_t(x) = -\log |F_t'(\pi_t(x)|$ satisfies

$$\dot{\phi}_0(x) = -\operatorname{Re} h(\pi(x)).$$

Consequently $\dot{\phi}_0$ is the virtual coboundary of $\operatorname{Re} v'(z)$. Combining Theorems 2.6 and 4.1, we obtain

$$\frac{d^2}{dt^2} \operatorname{H.dim}(J(F_t)) \bigg|_{t=0} = \frac{\operatorname{Var}(\dot{\phi}_0)}{\int -\phi_0 \, dm} = I_0(\operatorname{Re} v'(z)).$$

Since $v'(z)$ is holomorphic, it satisfies $I_0(\operatorname{Re} v') = (1/2) I_0(v')$, which yields Theorem 1.7.

Theorem 1.3 follows similarly, by applying Theorem 4.1 to the expanding conformal Markov map $f : S^1 \to S^1$ associated to a cocompact Fuchsian group. □

*Question.* Under what general circumstances does a smooth family of conformal maps $H_t : \Delta \to \widehat{\mathbb{C}}$ satisfy

$$\frac{d^2}{dt^2}\,\mathrm{H.dim}(H_t(S^1)) = \lim_{r\to 1} \frac{1}{4\pi|\log(1-r)|} \int_{|z|=r} \left|\dot{H}_0'(z)\right|^2 |dz|?$$

A survey of results on H.dim $H(S^1)$ for conformal maps can be found in [Pom, Ch. 10].

## 5 Dimension of Julia sets

In this section we apply Theorem 1.7 to deduce Theorem 1.8, which states that for small $t \in \mathbb{R}$ the family of monic, centered polynomials

$$F_t(z) = z^d + t\big(b_2 z^{d-2} + b_3 z^{d-3} + \cdots + b_d\big)$$

satisfies

$$\mathrm{H.dim}\, J(F_t) = 1 + \frac{|t|^2}{4d^2 \log d} \sum_{k=2}^{d} k^2 |b_k|^2 + O(|t|^3).$$

Similarly we obtain:

**Theorem 5.1** *The family of Blaschke products*

$$f_t(z) = \frac{z^d + t\big(b_2 z^{d-2} + b_3 z^{d-3} + \cdots + b_d\big)}{1 + t\big(\overline{b}_2 z^2 + \overline{b}_3 z^3 + \cdots + \overline{b}_d z^d\big)}$$

*satisfies*

$$L(f_t, m_t) = \log d + \frac{|t|^2}{d^2} \sum_{k=2}^{d} k^2 |b_k|^2 + O(|t|^3).$$

Here $L(f_t, m_t)$ is the Lyapunov exponent of the measure of maximal entropy.

*Proof of Theorem 1.8.* For $t$ small, there is a smooth family of conformal conjugacies $H_t$ between $F_0$ and $F_t$ on their basins of infinity, which is unique if we normalize so that $H_0(z) = z$. The holomorphic vector field $V = dH_t/dt$ then vanishes at infinity, and the functional equation $H_t F_0 = F_t H_t$ implies that $V = V(z)(d/dz)$ satisfies

$$V(z^d) = dz^{d-1} V(z) + \big(b_2 z^{d-2} + b_3 z^{d-3} + \cdots + b_d\big).$$

It is readily verified that this equation has a unique holomorphic solution on $|z| > 1$, namely

$$V(z) = -\frac{z}{d}\sum_{k=2}^{d}\sum_{n=0}^{\infty}\frac{b_k z^{-kd^n}}{d^n}$$
$$= -\frac{z}{d}\left(b_2 z^{-2} + \cdots + b_d z^{-d} + \frac{b_2 z^{-2d}}{d} + \frac{b_3 z^{-3d}}{d} + \cdots\right).$$

Changing variables by $z \mapsto 1/z$, the vector field $V$ on $1/\Delta$ becomes the vector field $v = v(z)(d/dz)$ on $\Delta$, where

$$v(z) = \frac{z}{d}\sum_{k=2}^{d}\sum_{n=0}^{\infty}\frac{b_k z^{kd^n}}{d^n}\cdot$$

By Theorem 1.7, we have

$$(d^2/dt^2)\,\mathrm{H.dim}\,J(F_t)\Big|_{t=0} = (1/2)I_0(v'(z)).$$

It is easy to see that

$$I_0\Big(\sum_{n=0}^{\infty} z^{kd^n}\Big) = \lim_{r\to 1}\frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty} r^{2kd^n} = \frac{1}{\log d},$$

since the terms out to $n \approx |\log(1-r)|/\log d$ are all nearly one, and the remaining terms are all nearly zero. Using orthogonality we obtain

$$I_0(v'(z)) = I_0\Big(d^{-1}\sum_k k b_k \sum_n z^{kd^n}\Big) = \frac{1}{d^2\log d}\sum_{k=2}^{d} k^2|b_k|^2,$$

which gives the desired formula. □

*Proof of Theorem 5.1.* Using the fact that $f_t(z) = F_t(z) + O(tz^{d+1})$, it is easy to see that the family of polynomials obtained by gluing $(1/\Delta, f_0)$ to $(\Delta, f_t)$ is tangent to the family $F_t(z)$ at $t = 0$. Moreover the measure of maximal entropy for $f_0(z) = z^d$ is simply Lebesgue measure on the unit circle, with entropy $h(f_0, m_0) = \log d$. Thus $\mathrm{H.dim}(m_t)L(f_t, m_t) = \log d$ as well, so Theorem 1.6 implies at $t = 0$ we have

$$\frac{d^2}{dt^2}L(f_t, m_t) = -(\log d)\frac{d^2}{dt^2}\,\mathrm{H.dim}(m_t) = 4(\log d)\frac{d^2}{dt^2}\,\mathrm{H.dim}(J(F_t)).$$

The last quantity is computed in Theorem 1.8. □

**Resultants and escape rates.** An alternative proof of Theorem 5.1 can be given using polynomial maps on $\mathbb{C}^2$ and the fact that Lebesgue measure on

$S^1$ is also the measure of maximal entropy for $f(z) = z^d$. The algebra and estimates needed are rather intricate, so we only outline the calculation.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a homogeneous polynomial map of degree $d$, covering a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$. Let $C_i \in \mathbb{C}^2$ be lifts of the $(2d-2)$ critical points of $f$, normalized so that

$$|\det(DF(z))| = d^2 \prod |z \wedge C_i|, \tag{5.1}$$

where $|z \wedge w| = |z_1 w_2 - z_2 w_1|$. Let $\mathrm{Res}(F) \in \mathbb{C}$ denote the resultant of the polynomials $F_1$ and $F_2$ giving the coordinates of $F$, and define the *escape rate* $G : \mathbb{C}^2 - \{0\} \to \mathbb{R}$ by

$$G(z) = \lim d^{-n} \log \|F^n(z)\| \tag{5.2}$$

(using any norm on $\mathbb{C}^2$). Then by a result of DeMarco [D, Cor. 1.6], we have:

**Theorem 5.2** *The Lyapunov exponent of $f : \mathbb{P}^1 \to \mathbb{P}^1$ with respect to its measure of maximal entropy is given by*

$$L(f, m) = \log d + \sum G(C_i) - \frac{2}{d} \log |\mathrm{Res}(F)|. \tag{5.3}$$

Now let $P$ be the polynomial

$$P(z) = z^d + b_2 z^{d-2} + \cdots + b_d,$$

let $P(z_1, z_2) = z_2^d P(z_1/z_2)$ be its homogenization, and let

$$Q(z) = z^d P(1/z) = 1 + b_2 z^2 + \cdots + b_d z^d$$

be its reciprocal polynomial. Assume for simplicity that the coefficients $b_i$ are real, and define

$$F(z_1, z_2) = (P(z_1, z_2), P(z_2, z_1)).$$

Then $F$ covers the Blaschke product $f(z) = P(z)/Q(z)$ corresponding to $P(z)$.

Our goal is to use (5.3) to compute $L(f, m)$ to second order in the coefficients $b_i$. Thus we will assume the coefficients $b_i$ are small, and the product of three or more coefficients is negligible.

Let $c_i \in \mathbb{C}$ denote the $(d-1)$ critical points of $f(z)$ in the unit disk $\Delta$. Since $f(z) \approx z^d$, the points $c_i$ are close to the origin. The remaining critical points of $f$ are given by $1/c_i$, so the vectors

$$\{C_i\} = \{(c_1, 1), (c_2, 1), \ldots, (c_{d-1}, 1), (1, c_1), (1, c_2), \ldots, (1, c_{d-1})\}$$

in $\mathbb{C}^2$ are lifts of the $2d - 2$ critical points of $f$ on $\mathbb{P}^1$.

Using the norm

$$\|(z_1, z_2)\| = \max(|z_1|, |z_2|),$$

(whose unit ball coincides with the region $G < 0$ for $(z^d, w^d)$), we will approximate $G(z)$ by setting $n = 1$ in (5.2). For this norm we have $\|F(c_i, 1)\| = |P(1, c_i)| = |Q(c_i)|$, and thus

$$\sum G(C_i) \approx \frac{2}{d} \log \prod |Q(c_i)|.$$

The $C_i$'s we have chosen need not satisfy (5.1), so we must also compensate by the ratio between $|\det(DF(1, 1))|$ and

$$d^2 \prod |(1, 1) \wedge C_i| = d^2 \prod |1 - c_i|^2.$$

The result is the formula

$$\begin{aligned} L(f, m) \approx \log d - \frac{2}{d} \log |\mathrm{Res}(F)| + \frac{2}{d} \log \prod |Q(c_i)| \\ + \log |\det(DF(1, 1))/d^2| - 2 \log \prod |1 - c_i|, \end{aligned}$$

which is accurate to order two.

The matrix for $\mathrm{Res}(F)$ is a perturbation of the identity matrix, allowing one to easily calculate

$$-\frac{2}{d} \log |\mathrm{Res}(F)| \approx (1/d) \sum_2^d k|b_k|^2.$$

Observing that the points $(c_i)_1^{d-1}$ are close to the zeros of $P'(z)$, we also have

$$\log \prod |Q(c_i)| \approx \sum R(c_i) = \mathrm{Res}_{z=0} \left( \frac{R(z)P''(z)}{P'(z)} \right),$$

where $R(z) = Q(z) - 1$. Applying the residue calculus and combining terms, we obtain

$$-\frac{2}{d} \log |\mathrm{Res}(F)| + \frac{2}{d} \log \prod |Q(c_i)| \approx \frac{2}{d^2} \sum_2^d k^2 |b_k|^2.$$

The calculation of $\prod |1 - c_i|$ is more delicate, because approximating $(c_i)$ by the zeros of $P'(z)$ is not accurate enough. Instead we observe that $(c_i)$ are the zeros of $S(z) = Q(z)P'(z) - P'(z)Q(z)$ inside the unit circle, and write $S(z)/(dz^{d-1}) = 1 + T(z)$ to obtain

$$\log \prod |1 - c_i| \approx \mathrm{Res}_{z=0}(\log(1 - z)T'(z)(1 - T(z))).$$

The complicated resulting expression cancels nicely with $\det(DF(1,1))$, yielding

$$\log|\det(DF(1,1))/d^2| - 2\log\prod|1-c_i| \approx -\frac{1}{d^2}\sum_2^d k^2|b_k|^2.$$

Collecting terms, we obtain

$$L(f,m) \approx \log d + \frac{1}{d^2}\sum_2^d k^2|b_k|^2$$

to order two, consistent with Theorem 5.1.

*Notes and references.* Related calculations for H.dim $J(z^2+c)$ appear in [Ru2], [Z] and [Mc7]. (Note that $z^{2^n}$ should be $z^{2^{n+1}}$ in Eq. (5) for $a_1(z)$ on [Z, p. 80].)

## 6 Norms and forms on the disk

In this section we introduce a sequence of norms $I_k(f)$ that measure the growth of a holomorphic function $f(z)$ on the unit disk. The quantity $I_0(f)$ introduced in Sect. 4 is included as a special case. We then establish:

**Theorem 6.1** *The deformation of $X_0 = \Delta/\Gamma_0$ represented by the vector field $v$ on $\Delta$ satisfies*

$$\frac{d^2}{dt^2}\mathrm{H.dim}(\Lambda(\Gamma_t))\bigg|_{t=0} = \frac{I_0(v')}{2} = \frac{4I_4(v')}{3} = \frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{3\,\mathrm{area}(X_0)}\,.$$

This result, together with Theorem 2.7, completes the proof of Theorem 1.1. We also discuss norms $J_k(f)$ for use with the Riemann surface lamination $\widehat{X}$.

**Holomorphic forms on $\Delta$.** Let $\Omega^k(\Delta)$ denote the space of (symmetric) holomorphic $k$-forms $\alpha = \alpha(z)\,dz^k$ on the unit disk, $k \geq 0$. We define an operator

$$D : \Omega^k(\Delta) \to \Omega^{k+1}(\Delta)$$

by $D\alpha = \alpha'(z)\,dz^{k+1}$.

Given $\alpha \in \Omega^j(\Delta)$ and $\beta \in \Omega^k(\Delta)$, we measure the asymptotic growth of rate of $\alpha\overline{\beta}$ by the pairing

$$\langle\alpha,\beta\rangle_{j,k} = \lim_{r\to 1}\frac{1}{2\pi}\int_{|z|=r}\rho^{-j-k}z^j\overline{z}^k\alpha\overline{\beta}\,|dz|$$

when $j+k>0$, and by

$$\langle \alpha, \beta\rangle_{0,0} = \lim_{r\to 1} \frac{1}{2\pi|\log(1-r)|}\int_{|z|=r} \alpha\overline{\beta}\,|dz|$$

when $(j,k)=(0,0)$. Here $\rho = 2|dz|/(1-|z|^2)$ denotes the hyperbolic metric. These limits need not exist, but when they do, they obey the usual properties of inner products. Note that we can also write

$$\langle \alpha, \beta\rangle_{0,0} = \lim_{r\to 1} d_\rho(0, S^1(r))^{-1}\frac{1}{2\pi}\int_{|z|=r} \alpha\overline{\beta}\,|dz|,$$

so when this limit exists, the average of $\alpha\overline{\beta}$ over $S^1(r)$ grows linearly with respect to the hyperbolic distance $d_\rho(0, S^1(r))$.

**Power series.** To make these pairings more concrete, let $f(z) = \sum_0^\infty a_n z^n$ be a holomorphic function on the unit disk, and define

$$I_{j+k}(f) = \langle D^j f, D^k f\rangle_{j,k}.$$

It is easy to see that this definition depends only on the sum $\ell = j+k$, and satisfies

$$I_\ell(f) = \lim_{r\to 1}(1-r)^\ell \sum_0^\infty n^\ell |a_n|^2 r^{2n}$$

when $\ell > 0$, and

$$I_0(f) = \lim_{r\to 1}\frac{1}{|\log(1-r)|}\sum_0^\infty |a_n|^2 r^{2n},$$

consistent with (4.1).

**Theorem 6.2** *If $I_k(f)$ exists, then so does $I_j(f)$ for all $j<k$, and*

$$I_k(f) = \frac{(k-1)!}{2^k} I_0(f).$$

*Proof.* If $I_k(f)$ exists, $k>0$, then as $r\to 1$ we have

$$\sum_{n=0}^\infty n^k |a_n|^2 r^{2n} \sim \frac{I_k(f)}{(1-r)^k};$$

multiplying by 2 and integrating with respect to $r$, we obtain

$$\sum_{n=0}^\infty n^{k-1}|a_n|^2 r^{2n} \sim 2I_k(f)\int_0^r \frac{ds}{(1-s)^k}.$$

For $k=1$ the integral on the left grows like $|\log(1-r)|$ and we obtain $2I_0(f) = I_1(f)$; while for $k>1$ it grows like $1/((k-1)(1-r)^{k-1})$, which shows that $I_{k-1}(f)$ exists and agrees with $2I_k(f)/(k-1)$. By induction, we obtain $I_0(f) = 2^k I_k(f)/(k-1)!$. $\square$

*Proof of Theorem 6.1.* The first equality is a restatement of Theorem 1.3, and the second is immediate from Theorem 6.2.

For the final equality, let $\pi : \Delta \to X_0 = \Delta/\Gamma_0$ be the natural covering map, and recall that $\widetilde{\phi} = -2v'''(z)dz^2 \in \Omega^2(\Delta)$ satisfies $\widetilde{\phi} = \pi^*(\phi)$ where

$$\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = \frac{1}{\mathrm{area}(X_0)} \int_{X_0} \rho^{-4}|\phi|^2\, \rho^2(z)|dz|^2.$$

By mixing of the geodesic flow, the projected circle $\pi(S^1(r))$ becomes equidistributed on $X_0$ as $r \to 1$ (see e.g. [EsM]). Thus the average of $|\widetilde{\phi}|^2/\rho^4$ over $|z| = r$ converges to the average of $|\phi|^2/\rho^4$ over $X_0$, and we obtain

$$\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = \langle \widetilde{\phi}, \widetilde{\phi} \rangle_{2,2} = 4\langle D^2(v'), D^2(v') \rangle_{2,2} = 4I_4(v').$$

$\square$

**Césaro norms.** A central difference between the geodesic flows on $\mathrm{T}_1 X$ and $\widehat{X}$ is that, while both are ergodic, the geodesic flow on the Riemann surface lamination $\widehat{X}$ need not be mixing (Sect. 10). Because of this, we will need to replace the norms $I_k(f)$ with norms $J_k(f)$ that involve iterated averages. These are defined for $f(z) = \sum a_n z^n$ by $J_0(f) = I_0(f)$ and

$$J_k(f) = \lim_{r\to 1} \frac{1}{|\log(1-r)|} \int_0^r (1-s)^{k-1} \sum_{n=0}^{\infty} n^k |a_n|^2 s^{2n}\, ds$$

for $k > 0$. It is easy to see that

$$J_{2k}(f) = \lim_{r\to 1} \frac{1}{2\pi|\log(1-r)|} \int_0^r \frac{ds}{1-s} \int_{|z|=s} \rho^{-2k}|D^k f|^2\, |dz| \tag{6.1}$$

for $k > 0$. Thus $J_{2k}(f)$ is obtained by first averaging $S^1(s)$, then averaging over $s \in [0, r] \subset \Delta$ with respect to the hyperbolic metric, and finally taking the limit as $r \to 1$.

**Theorem 6.3** *If $J_k(f)$ exists for some $k \geq 0$ and* $\sup \rho^{-1}|Df| < \infty$, *then $J_k(f)$ exists for all k and satisfies*

$$J_k(f) = \frac{(k-1)!}{2^k} J_0(f).$$

*Proof.* It suffices to show that $J_{k+1}(f) = (k/2)J_k(f)$ whenever one or the other exists. Note that $J_k(f) = \lim_{r\to 1} J_k(f,r)$ where

$$J_0(f,r) = \frac{1}{|\log(1-r)|}\sum |a_n|^2 r^{2n}$$

and

$$J_k(f,r) = \frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty} |a_n|^2 n^k \int_0^r (1-s)^{k-1} s^{2n}\, ds$$

for $k > 0$; thus it suffices to show

$$J_{k+1}(f,r) \sim (k/2)J_k(f,r) \tag{6.2}$$

as $r \to 1$, provided one side or the other has a nonzero limit.

The proof of (6.2) is by induction on $k$. For $k = 0$ we have

$$J_1(f,r) = \frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty} |a_n|^2 \frac{nr^{2n+1}}{2n+1} \sim \frac{1}{2}J_0(f,r),$$

so (6.2) certainly holds. For the inductive step we integrate by parts to obtain

$$J_{k+1}(f,r) = B_k(f,r) + C_k(f,r),$$

where

$$B_k(f,r) = \frac{(1-r)^k}{|\log(1-r)|}\sum_{n=0}^{\infty}\frac{n^{k+1}|a_n|^2 r^{2n+1}}{2n+1} \sim \frac{(1-r)^k}{2|\log(1-r)|}\sum n^k |a_n|^2 r^{2n}$$

and

$$C_k(f,r) = \frac{1}{|\log(1-r)|}\int_0^r k(1-s)^{k-1}\frac{n^{k+1}|a_n|^2 s^{2n+1}}{2n+1}\, ds \sim \frac{k}{2}J_k(f,r).$$

Now the assumption $\sup \rho^{-1}|Df| < \infty$ implies (by Cauchy's integral formula) that $\sup \rho^{-2j}|D^j f|^2 < \infty$ for all $j > 0$, and hence

$$(1-r)^{\ell}\sum_{n=0}^{\infty} n^{\ell}|a_n|^2 r^{2n} \asymp \int_{|z|=r} \rho^{-2j}|f^{(j)}(z)|^2\,|dz| = O(1)$$

when $\ell = 2j$ is even. Integrating with respect to $r$ shows the same $O(1)$ bound holds when $\ell > 0$ is odd. The case $\ell = k$ implies that $B_k(f,r) \to 0$ as $r \to 1$, and thus

$$J_{k+1}(f,r) \sim (k/2)J_k(f,r)$$

as desired. □

*Example.* The function $f(z) = \sum_0^{\infty} z^{2^n}$ satisfies $I_0(f) = 4J_2(f) = \log 2$, but $(1-r)^2\sum n^2|a_n|^2 r^{2n}$ oscillates as $r \to 1$, so $I_2(f)$ does not exist. This function $f(z)$ arises in the study H.dim$(J(z^2+c))$ for small $c$ (Sect. 5).

## 7 The foliated unit tangent bundle

In this section we develop the theory of holomorphic forms on the foliated unit tangent bundle $\mathrm{T}_1 X$. In particular we introduce an inner product on forms and a differential operator $D$, and establish:

**Theorem 7.1** *Any holomorphic 1-form $\alpha$ along the leaves of $\mathcal{F}$ satisfies*

$$2\langle\alpha, \alpha\rangle = \frac{2^{2k-1}}{(2k-1)!}\langle D^{k-1}\alpha, D^{k-1}\alpha\rangle$$

*for all $k > 0$.*

Parallel results for the Riemann surface lamination $\widehat{X}$ will be presented in Sect. 10.

**The unit disk.** Let $\Delta$ be the unit disk in the hyperbolic metric. By identifying each unit tangent vector $\xi \in T_z\Delta$ with the endpoint $p \in S^1$ of the geodesic ray in the direction $\xi$, we obtain an isomorphism

$$\mathrm{T}_1\Delta \cong \Delta \times S^1.$$

We denote the geodesic flow by $g_s : \mathrm{T}_1\Delta \to \mathrm{T}_1\Delta$.

There is a natural foliation $\widetilde{\mathcal{F}}$ of $\mathrm{T}_1\Delta$ whose leaves $L_p \cong \Delta \times \{p\}$ consist of all the geodesics converging to a given point $p \in S^1$. Each leaf admits a natural *affine coordinate*

$$z_p : L_p \to \mathbb{H} = \{z : \mathrm{Im}(z) > 0\},$$

unique up to automorphisms of $\mathbb{H}$ fixing $\infty$, such that $z_p$ sends the geodesics in $L_p$ to vertical lines in $\mathbb{H}$. In terms of $z_p = x + iy$, the geodesic flow on $L_p$ is given by $g_s(x + iy) = x + ie^s y$.

These affine coordinates can be assembled to obtain an isomorphism

$$Z : \mathrm{T}_1\Delta \cong \mathbb{H} \times S^1$$

given by

$$Z(z, p) = \left(i\left(\frac{p+z}{p-z}\right), p\right). \tag{7.1}$$

Here we have normalized so that the unit tangent circle over $z = 0$ maps to $\{i\} \times S^1$.

**Passage to the quotient surface.** Now let $X = \Delta/G$ be a compact quotient of the disk. Then $\widetilde{\mathcal{F}}$ descends to a foliation $\mathcal{F}$ of $\mathrm{T}_1 X$ inheriting the structure above. In particular, in local affine coordinates $z = x + iy$ on any leaf of $\mathcal{F}$, we have the hyperbolic metric $\rho = |dz|/y$ and the hyperbolic area element $dA = dx\,dy/y^2$. Since $X$ is compact, $\mathrm{T}_1 X$ carries a unique smooth probability measure $d\xi$ invariant under the geodesic flow.

**The harmonic current.** Next we describe a natural *harmonic current* $T$ adapted to $\mathcal{F}$. This current takes the role of the fundamental class of $\mathcal{F}$.

Choose local coordinates $(z, p)$ on $\mathrm{T}_1 X$ such that $z$ is affine along each leaf of $\mathcal{F}$ and $p$ ranges in a transversal to $\mathcal{F}$. Then the invariant 3-form for the geodesic flow is given locally by

$$d\xi = \frac{dx\,dy}{y} d\tau(p) = \frac{dx\,dy}{y^2} y\,d\tau(p),$$

where $d\tau$ is a 1-form satisfying $\mathrm{T}\mathcal{F} = \operatorname{Ker} d\tau$. Since the hyperbolic area element $dx\,dy/y^2$ is globally well-defined, so is the 1-form

$$T = y d\tau(p).$$

Note that

$$dT = \frac{dy}{y} \wedge T \quad \text{and} \quad d\xi = \frac{dx\,dy}{y^2} \wedge T.$$

Geometrically, $T$ represents the (non-closed) current of integration along the leaves of $\mathcal{F}$ given locally by $y[L_p]d\tau(p)$; in other words, a compactly supported 2-form $\beta$ satisfies

$$\int_p \Big(\int_{[L_p]} y\beta\Big) d\tau(p) = \int \beta \wedge T.$$

The current $T$ is *harmonic* since $y$ is a harmonic function on each complex leaf $L_p$. Harmonic current arise frequently for foliations like $\mathcal{F}$ which admit no transverse invariant measure; see e.g. [Ga], [FS].

**Holomorphic forms on $\mathcal{F}$.** A (symmetric) *holomorphic $k$-form* on $\mathcal{F}$ is a continuous section of the bundle $(T^*\mathcal{F})^k \to \mathrm{T}_1 X$ given locally in affine coordinates by

$$\alpha = \alpha(z, p)\, dz^k,$$

where $\alpha(z, p)$ is holomorphic in $z$. We denote the space of all such forms by $\Omega^k(\mathcal{F})$.

*Example.* Any holomorphic $k$-form $\beta(z)dz^k$ on $X$ pulls back, under the projection $\mathrm{T}_1 X \to X$, to a holomorphic $k$-form on $\mathcal{F}$.

Note that every $\alpha \in \Omega^k(\mathcal{F})$ is *bounded* in the hyperbolic metric, since $\rho^{-k}|\alpha|$ is continuous and $\mathrm{T}_1 X$ is compact. In particular, we have

$$|\alpha(z)| = O(y^{-k}) \tag{7.2}$$

in affine coordinates on any leaf of $\mathcal{F}$.

**Differentiation of $k$-forms.** Let $D : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$ denote the natural differential operator given in affine coordinates by

$$D\alpha = (d\alpha/dz)\, dz^{k+1}.$$

(This operator is well-defined since any two affine coordinates are related by $z_1 = az_2 + b$). We define the inner product between a $j$-form and a $k$-form by

$$\langle \alpha, \beta \rangle_{j,k} = \int_{T_1 X} y^{j+k} \alpha(z) \overline{\beta}(z)\, d\xi,$$

so in particular

$$\langle \alpha, \alpha \rangle_{k,k} = \int_{T_1 X} y^{2k} |\alpha(z)|^2\, d\xi$$

gives the average value of $\rho^{-2k}|\alpha|^2$. We suppress the subscripts $(j, k)$ when they are clear from the context.

**Theorem 7.2** *The holomorphic forms on $\mathcal{F}$ satisfy*

$$\langle D\alpha, \beta \rangle_{j+1,k} = \frac{i}{2}(j+k)\langle \alpha, \beta \rangle_{j,k}. \tag{7.3}$$

*Proof.* Let $\ell = j + k$, and consider the (0, 1)-form along the leaves of $\mathcal{F}$ given by

$$\gamma = y^{\ell-1} \alpha(z) \overline{\beta}(z)\, d\overline{z}.$$

By Stokes' theorem we have $\int \gamma \wedge dT = \int d\gamma \wedge T$. Since $dT = (dy/y)T$, we have

$$\begin{aligned}\int \gamma \wedge dT &= \int y^{\ell-2} \alpha(z) \overline{\beta}(z)\, d\overline{z}\, dy \wedge T \\ &= \int y^{\ell} \alpha(z) \overline{\beta}(z) \frac{dx\, dy}{y^2} \wedge T = \langle \alpha, \beta \rangle_{j,k},\end{aligned}$$

while differentiation of $\gamma$ yields

$$\begin{aligned}\int d\gamma \wedge T &= \int ((\ell-1) y^{\ell-2} \alpha(z) \overline{\beta}(z) dy\, d\overline{z} + y^{\ell-1} \alpha'(z) \overline{\beta}(z)\, dz\, d\overline{z}) \wedge T \\ &= (1-\ell)\langle \alpha, \beta \rangle_{j,k} - 2i \langle D\alpha, \beta \rangle_{j+1,k}.\end{aligned}$$

Equating these expressions gives the theorem. □

**Theorem 7.3** *The map $D : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$ is an isomorphism for all $k > 0$, and satisfies*

$$\langle D\alpha, D\alpha \rangle = \frac{2k(2k+1)}{4} \langle \alpha, \alpha \rangle.$$

*Proof.* First note that if $\beta(z)$ is a holomorphic function on $\mathbb{H}$ satisfying

$$|\beta(z)| = O(y^{-k-1}), \tag{7.4}$$

then there is a *unique* holomorphic function $\alpha$ on $\mathbb{H}$ satisfying $\alpha'(z) = \beta(z)$ and $|\alpha(z)| = O(y^{-k})$ (as required by (7.2)). Indeed, we can simply take $\alpha(z) = -\int_z^{i\infty} \beta(w)dw$, using the fact that $\int_1^\infty dy/y^{k+1} < \infty$ (since $k > 0$).

Next observe that any $\beta \in \Omega^{k+1}(\mathcal{F})$ satisfies (7.4) in affine coordinates, since it is bounded in the hyperbolic metric. Thus $\beta$ has a canonical primitive on each leaf of $\mathcal{F}$, and these fit together to give the unique form $\alpha \in \Omega^k(\mathcal{F})$ satisfying $D\alpha = \beta$. Therefore $D$ is an isomorphism, and (7.3) implies

$$\langle D\alpha, D\alpha\rangle = \frac{i}{2}(2k+1)\langle \alpha, D\alpha\rangle = \frac{i}{2}(2k+1)\overline{\langle D\alpha, \alpha\rangle} = \frac{2k(2k+1)}{4}\langle \alpha, \alpha\rangle.$$

□

**Corollary 7.4** *The space $\Omega^k(\mathcal{F})$ is infinite-dimensional for all $k > 0$.*

*Proof.* The dimension of $\Omega^k(\mathcal{F})$ is independent of $k$ and bounded below by $\dim \Omega^k(X)$, which tends to infinity as $k \to \infty$ by Riemann–Roch. □

*Proof of Theorem 7.1.* Apply induction, using Theorem 7.3. □

## 8 Growth along leaves

In this section we study holomorphic forms on the foliated unit tangent bundle by passing to the universal cover $\widetilde{L} \cong \Delta$ of an individual leaf. In particular, we use the relation $I_0(f) = 4I_2(f)$ on the unit disk to establish:

**Theorem 8.1** *If $\alpha \in \Omega^1(\mathcal{F})$ is Hölder continuous, then we have*

$$\mathrm{Var}(\mathrm{Re}\,\alpha/\rho) = 2\langle \alpha, \alpha\rangle.$$

We also derive consequences for the power series of automorphic forms.

**Uniformization of leaves.** We begin by relating the $L^2$-norm of a holomorphic form $\alpha \in \Omega^k(\mathcal{F})$ to its growth along leaves. To study this growth, choose a holomorphic covering map

$$\pi : \Delta \to L \subset \mathrm{T}_1 X$$

from the unit disk to a leaf $L$ of $\mathcal{F}$.

**Theorem 8.2** *Let $f(z)$ be a holomorphic function on the unit disk such that $f^{(k)}(z)\, dz^k = \pi^*(\alpha)$. Then $I_\ell(f)$ exists for all $\ell$, and we have*

$$I_{2k}(f) = \int_{\mathrm{T}_1 X} \rho^{-2k}|\alpha|^2\, d\xi.$$

*Proof.* Note there is a $p \in S^1$ such that the map $z_p : \Delta \to \mathbb{H}$ given by $z_p = i(p+z)/(p-z)$ corresponds to an affine coordinate on $L$ (cf. (7.1)).

By mixing of the geodesic flow, the normal vectors of the projection of $S^1(r)$ to $X$ become uniformly distributed in $\mathrm{T}_1 X$ as $r \to 1$ [EsM]. On the other hand, most hyperbolic geodesics from $p$ to $S^1(r)$ are nearly normal to the circle. Thus $\pi(S^1(r))$ also becomes equidistributed in $\mathrm{T}_1 X$, which implies

$$\int_{T_1 X} \rho^{-2k}|\alpha|^2 \, d\xi = \lim_{r\to 1} \frac{1}{2\pi} \int_{|z|=r} \rho^{-2k} |f^{(k)}(z)|^2 \, |dz| = I_{2k}(f).$$

To see that $I_{2k+2}(f)$ also exists, consider a solution to $g^{(k+1)}(z) = \pi^*(D\alpha)$. Then as we have just shown, $I_{2k+2}(g)$ exists. On the other hand, we have

$$\begin{aligned}\pi^*(D\alpha) &= dz_p^k d\big(\pi^*(\alpha)/dz_p^k\big) = \left(f^{(k+1)}(z) + 2k\frac{f^{(k)}(z)}{z-p}\right) dz^{k+1} \\ &= D^{k+1}f + 2k\omega D^k f,\end{aligned}$$

where $\omega = dz/(z-p)$. Since $\alpha$ is bounded, so is $\rho^{-2k}|f^{(k)}|^2$. Consequently

$$\langle \omega D^k f, \omega D^k f\rangle = O\Big(\lim_{r\to 1}\int_{|z|=r} \rho^{-2}(z)|z-p|^{-2}\,|dz|\Big) = 0,$$

since $(1-|z|)/(|z-p|) \to 0$ a.e. as $|z| \to 1$. It follows that $\langle D^{k+1}g, D^{k+1}g\rangle = \langle D^{k+1}f, D^{k+1}f\rangle$, and hence $I_{2k+2}(f)$ also exists.

By similar reasoning, $I_\ell(f)$ exists for all $\ell$. □

**Corollary 8.3** *We have $I_{2j+2k}(f) = \langle D^j\alpha, D^j\alpha\rangle$ for all $j \geq 0$.*

*Proof.* We have just seen this equality for $j = 0$, and by Theorems 6.2 and 7.3 both sides multiply by $(2k+2j)(2k+2j+1)/4$ when we replace $j$ by $j+1$. □

**Variance under the geodesic flow.** We now turn to the proof of Theorem 8.1. Recall that $I_0(f) = \lim_{r\to 1} I_0(f,r)$ where

$$I_0(f,r) = \frac{1}{2\pi|\log(1-r)|}\int_{|z|=r} |f(z)|^2\,|dz| = \frac{1}{|\log(1-r)|}\sum_{n=0}^{\infty}|a_n|^2 r^{2n}.$$

We begin by showing there is some uniformity in the calculation of $I_0(f)$.

**Theorem 8.4** *If $f(0) = 0$ and $1/2 < r < 1$, then we have*

$$I_0(f,r) = O\big(\sup_{\Delta}\rho^{-1}|Df|\big).$$

*Proof.* If $M = \sup_\Delta \rho^{-1}|Df|$ is finite, then $|f'(z)|^2 = O(M(1-r)^{-2})$ on the circle $|z| = r$, and thus we have

$$\frac{1}{2\pi}\int_{|z|=r} |zf'(z)|^2\,|dz| = \sum_{n=0}^{\infty} |a_n|^2 n^2 r^{2n} = O(M(1-r)^{-2}).$$

Integrating twice gives $\sum |a_n|^2 r^{2n} = O(M|\log(1-r)|)$. □

**Lifts of 1-forms.** Now let $\alpha = \alpha(z)\,dz \in \Omega^1(\mathcal{F})$ be a holomorphic 1-form, and let $h : \mathrm{T}_1 X \to \mathbb{R}$ denote the function given by

$$h = \operatorname{Re}\alpha/\rho = \operatorname{Re}\alpha(z)\cdot \operatorname{Im}(z)$$

in local affine coordinates on $\mathcal{F}$. We then have

$$\operatorname{Var}(h) = \frac{1}{\operatorname{area}(X)} \lim_{S\to\infty} \int_X \operatorname{Var}_x(h, S)\,dA, \tag{8.1}$$

where $dA$ is hyperbolic area on $X$, where

$$\operatorname{Var}_x(h, S) = \frac{1}{2\pi S}\int_0^{2\pi} \left|\int_0^S h(g_s\xi(x,\theta))\,ds\right|^2 d\theta,$$

and where $\xi(x, \theta) \in \mathrm{T}_1 X$ parameterizes the unit tangent circle over $x \in X$.

To study $\operatorname{Var}_x(h, S)$ more explicitly, let us normalize the covering map $\Delta \to X = \Delta/G$ so that $0 \in \Delta$ lies over $x \in X$. For each $p \in S^1$ let

$$\pi_p : \Delta \to \mathrm{T}_1 X$$

be the natural map sending $\Delta$ to the leaf of $\mathcal{F}$ consisting of geodesics that converge to $p$ in forward time. Let $\alpha_p = \pi_p^*(\alpha)$, and let $f_p : \Delta \to \mathbb{C}$ be the unique holomorphic function satisfying $f_p(0) = 0$ and $\alpha_p = df_p$.

Choose $r = r(S)$ so $|z| = r$ is a circle of hyperbolic radius $S$. Then for $q = e^{i\theta}$ we have

$$\int_0^S h(g_s\xi(x,\theta))\,ds = \int_0^{rq} \frac{\operatorname{Re}\alpha_q}{\rho}\rho = \operatorname{Re} f_q(rq).$$

Consequently we have

$$\operatorname{Var}_x(\operatorname{Re}\alpha/\rho, S) = \frac{1}{2\pi S}\int_{S^1} |\operatorname{Re} f_q(rq)|^2\,|dq|. \tag{8.2}$$

Similarly, we can write

$$\frac{1}{2} I_0(f_p, r) = \frac{1}{2\pi|\log(1-r)|}\int_{S^1} |\operatorname{Re} f_p(rq)|^2\,|dq| \tag{8.3}$$

(using the fact that $(1/2)\int|f|^2 = \int|\operatorname{Re} f|^2$). To compare these expressions, we observe:

**Theorem 8.5** *If $\alpha$ is Hölder continuous, then we have*

$$|f_p(rq) + f_q(rq)| \leq M(1 + |\log|p - q||)$$

*for all $p \neq q \in S^1$ and $z \in \Delta$, where $M$ is independent of $x$.*

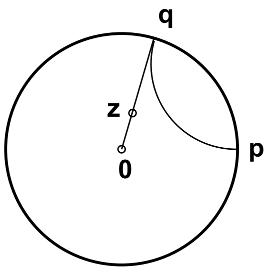


**Fig. 1** Convergent geodesics

*Proof.* Let $[a, b]$ denote the hyperbolic geodesic between two points in $\overline{\Delta}$, and let $\theta(z)$ denote the angle between $[z, 0]$ and $[z, p]$ for $z = rq \in [0, q]$. Assume $\alpha$ is Hölder continuous of exponent $\delta$. Recalling that $\alpha_p$ and $\alpha_q$ are the values of $\alpha$ along geodesics tending to $p$ and $q$ respectively, we find

$$\rho^{-1}|\alpha_p + \alpha_q|(z) = O(|\theta(z)^\delta|).$$

(The $+$ sign comes from the fact that the geodesics $[z, p]$ and $[z, q]$ make an angle of $\pi - \theta(z)$.)

It is easy to see that $\theta(z) \to 0$ exponentially fast with respect to arclength on $[0, q]$, and indeed $\theta(z) = O(d(z, [p, q]))$ where $d(\cdot)$ is the hyperbolic metric. Integrating along $[0, z]$, we obtain

$$|f_p(rq) + f_q(rq)| \leq O(d(0, [p, q])) = O(1 + |\log|p - q||).$$

□

**Theorem 8.6** *If $\alpha$ is Hölder continuous then* $\operatorname{Var}_x(\operatorname{Re}\alpha/\rho, S)$ *is bounded uniformly in $x$, and we have*

$$\lim_{S\to\infty} \operatorname{Var}_x(\operatorname{Re}\alpha/\rho, S) = 2\langle\alpha, \alpha\rangle$$

*for all $x \in X$.*

*Proof.* Fix any $p \in S^1$. In light of the preceding theorem, (8.2) and (8.3) imply

$$\operatorname{Var}_x(\operatorname{Re}\alpha/\rho, S) = \frac{1}{2} I_0(f, r) + O(1/S). \tag{8.4}$$

Here we have used the elementary fact that $S = d(0, r) = |\log(1-r)| + O(1)$ and that $\int_{S^1}(1 + |\log|p - q||)^2\,|dq| = O(1)$. Since

$$\sup_{\Delta} \rho^{-1}|Df_p| = \sup_{\Delta} \rho^{-1}|\alpha_p| = \sup_{X} \rho^{-1}|\alpha| < \infty,$$

Theorem 8.4 provides a bound for $I_0(f_p, r)$ depending only on $\alpha$. Consequently $\mathrm{Var}_x(\mathrm{Re}\,\alpha/\rho, S)$ also has such a bound. Finally (8.4) implies

$$\lim_{S\to\infty} \mathrm{Var}_x(\mathrm{Re}\,\alpha/\rho, S) = \frac{1}{2}\lim_{r\to 1} I_0(f_p, r) = \frac{1}{2} I_0(f_p),$$

and $(1/2)I_0(f_p) = 2I_2(f_p) = 2\langle\alpha, \alpha\rangle$ by Theorems 6.2 and 8.2. □

*Proof of Theorem 8.1.* Since $\mathrm{Var}_x(\mathrm{Re}\,\alpha/\rho, S)$ is bounded, by the dominated convergence theorem we can interchange limits in (8.1) to obtain

$$\mathrm{Var}(\mathrm{Re}\,\alpha/\rho) = \frac{1}{\mathrm{area}(X)} \int_X \lim_{S\to\infty} \mathrm{Var}_x(\mathrm{Re}\,\alpha/\rho, S)\,dA = 2\langle\alpha, \alpha\rangle.$$

□

*Remark (Rankin–Selberg).* For the special case of forms coming from the inclusion $\Omega^k(X) \hookrightarrow \Omega^k(\mathcal{F})$, Theorems 6.2 and 8.2 yield:

**Corollary 8.7** *Let $\alpha(z) = \sum a_n z^n\,dz^k$ be an automorphic form for a cocompact Fuchsian group $G \subset \mathrm{Aut}(\Delta)$. Then the limits*

$$A_\ell = \lim_{r\to 1}(1 - r)^\ell \sum_1^\infty n^{\ell-2k}|a_n|^2 r^{2n}$$

*exist for all $\ell > 0$, and satisfy $A_{\ell+1} = (\ell/2)A_\ell$.*

One can compare this corollary to a classical result of Rankin and Selberg [Sel, (2.16)], which states that any cusp form

$$\alpha = \alpha(z)\,dz^k = \sum_{n=0}^\infty a_n e^{2\pi i z}\,dz^k$$

for a lattice $G \subset \mathrm{Aut}(\mathbb{H})$ containing $g(z) = z + 1$ satisfies

$$\sum_{n\le N} \frac{|a_n|^2}{n^{2k-1}} = AN + o(N) \tag{8.5}$$

for some $A \in \mathbb{R}$. (It is not required that $G$ is arithmetic.) Wolpert showed that (8.5) also holds for the power series $\alpha = \sum a_n z^n\,dz^k$ of an automorphic form for a cocompact Fuchsian group $G \subset \mathrm{Aut}(\Delta)$; the equivalent statement

$\sum_{n \le N} |a_n|^2/n \sim AN^{2k-1}$ appears in [Wol1, Thm. 3]. This result implies Corollary 8.7 by summation by parts.

## 9 Deformations of Fuchsian groups

In this section we show that the derivatives $v^{(k)}(z)\,dz$ of a deformation of a Fuchsian group $G$ determine holomorphic forms on the foliated unit tangent bundle $\mathrm{T}_1 X$, $X = \Delta/G$. We then prove the formulas for the Weil–Petersson metric stated as Theorems 1.5 and 1.4 in the introduction.

**Forms from vector fields.** Let $v$ be a holomorphic vector field on the unit disk, representing a deformation of $G$ as in the introduction. Then the quadratic differential $v'''(z)\,dz^2$ on the disk descends to give a form in $\Omega^2(X) \hookrightarrow \Omega^2(\mathcal{F})$, which we also denote by $v'''$. Let $v'' \in \Omega^1(\mathcal{F})$ be the unique solution to $Dv'' = v'''$ guaranteed by Theorem 7.3, and let $v^{(k+1)} = D^{k-1}v'' \in \Omega^{k+1}(\mathcal{F})$. Then Theorems 7.1 and 8.1 immediately imply:

**Theorem 9.1** *For any $k > 0$, the vector field $v$ satisfies*

$$\frac{2^{2k-1}}{(2k-1)!}\langle v^{(k+1)}, v^{(k+1)}\rangle = \mathrm{Var}(\mathrm{Re}\, v''/\rho).$$

*Proof of Theorem 1.5.* Since $\mu = \rho^{-2}\,\overline{\phi} = -2\overline{v'''}$, we have

$$\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = \frac{\|\mu\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = 4\langle v''', v'''\rangle.$$

The preceding result gives $2\langle v'', v''\rangle = (4/3)\langle v''', v'''\rangle = \mathrm{Var}(\mathrm{Re}\, v''/\rho)$, and the equation

$$\frac{1}{3}\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} = 2\int_{\mathrm{T}_1 X_0} \rho^{-2}|v''|^2\, d\xi = \mathrm{Var}(\mathrm{Re}\, v''/\rho) \qquad (9.1)$$

stated as Theorem 1.5 then follows. □

*Proof of Theorem 1.4.* By Theorem 8.2, $I_k(v')$ exists for all $k > 0$. Since $\widetilde{\phi}(z) = \sum a_n z^n = -2v'''(z)$, we have $I_k(\sum a_n z^n/n^2) = 4I_k(v')$. Thus Theorems 6.1 and 6.2 yield

$$\begin{aligned}\frac{1}{3}\frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} &= \frac{I_0(v')}{2} = \frac{1}{2}\frac{2^k I_k(v')}{(k-1)!} = \frac{1}{8}\frac{2^k I_k\big(\sum a_n z^n/n^2\big)}{(k-1)!} \\ &= \frac{1}{8}\frac{2^k}{(k-1)!}\lim_{r\to 1}(1-r)^k\sum_1^\infty n^{k-4}|a_n|^2 r^{2n},\end{aligned}$$

as stated in Theorem 1.4. □

*Remark (Unitary representations).* One can regard Theorem 9.1 as an instance (for $k = 1$) of the general principle that any two inner products on $\Omega^k(X)$ constructed naturally from hyperbolic geometry must be proportional. To justify this principle, recall that $\Omega^k(X)$ is associated to the irreducible discrete-series representation $V_{2k}$ of $\mathrm{SL}_2(\mathbb{R})$ (see e.g. [Kn, Prop. 2.7]); by Schur's lemma, any two invariant inner products on $V_{2k}$ are proportional, so the same is true of the induced inner products on $\Omega^k(X)$.

## 10 The Riemann surface lamination

In this section we recall the Riemann surface lamination $\widehat{X}$ associated to a Blaschke product $f : \Delta \to \Delta$. We define the geodesic flow on $\widehat{X}$, and prove it is ergodic (but generally not mixing). We then develop the theory of holomorphic forms on $\widehat{X}$, parallel to the theory of holomorphic forms on the foliated unit tangent bundle $\mathrm{T}_1 X$. In particular we establish:

**Theorem 10.1** *Any holomorphic 1-form $\alpha$ along the leaves of $\widehat{X}$ satisfies*

$$2\langle \alpha, \alpha \rangle = \frac{2^{2k-1}}{(2k-1)!} \langle D^{k-1}\alpha, D^{k-1}\alpha \rangle$$

*for all $k > 0$.*

We also show $\mathrm{Var}(\mathrm{Re}\, \alpha/\rho) = 2\langle \alpha, \alpha \rangle$ for suitable $\alpha \in \Omega^1(\widehat{X})$.

**The solenoid.** Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an expanding rational map of degree $d$ with $J(f) = S^1$ and $f(\Delta) = \Delta$. The associated *solenoid* is the space

$$\widehat{S}^1 = \varprojlim (f : S^1 \to S^1) = \left\{ (u_i) \in (S^1)^{\mathbb{Z}} : f(u_i) = u_{i+1} \right\}.$$

A point in $\widehat{S}^1$ is given by a point $u_0 \in S^1$ together with a consistent choice of preimages $u_{-n} \in f^{-n}(u_0)$.

Each connected component or *leaf* of the solenoid has a natural affine coordinate chart $x : L \cong \mathbb{R}$, well-defined up to $x \mapsto ax + b$, inherited from the smooth structure on $S^1$. Topologically we have

$$\widehat{S}^1 \cong (\mathbb{Z}_d \times \mathbb{R})/\mathbb{Z},$$

where $\mathbb{Z}_d = \varprojlim \mathbb{Z}/d^n$ is the group of dyadic integers and $\mathbb{Z} \subset \mathbb{Z}_d \times \mathbb{R}$ is embedded diagonally.

**The Riemann surface lamination.** We can also form the space of backwards orbits of $f$ on the unit disk,

$$\begin{aligned} \widehat{\Delta} &= \varprojlim (f : \Delta \to \Delta) \\ &= \left\{ (z_i) \in \Delta^{\mathbb{Z}} : f(z_i) = z_{i+1} \text{ and } |z_i| \to 1 \text{ as } i \to \infty \right\}. \end{aligned}$$

(The condition $|z_i| \to 1$ excludes the orbit $(\ldots, p, p, p, \ldots)$, where $p$ is the attracting fixed point of $f|\Delta$.) The expanding property of $f|S^1$ implies that every leaf $L$ of $\widehat{\Delta}$ has a natural affine coordinate chart $z : L \cong \mathbb{H}$, well-defined up to $z \mapsto az + b$.

The action of $f$ determines a homeomorphism $\widehat{f}$ on $\widehat{S}^1 \cup \widehat{\Delta}$, given by

$$\widehat{f}((z_i)) = (f(z_i)) = (z_{i+1}).$$

The map $\widehat{f}|\widehat{\Delta}$ is properly discontinuous, yielding as quotient space the *Riemann surface lamination*

$$\widehat{X} = \widehat{\Delta}/\langle \widehat{f} \rangle$$

associated to $f$. See [Sul3], [MeSt, Ch. VI.6] for more details.

**Geodesic flow.** The hyperbolic metric $\rho = |dz|/y$ is well-defined in affine coordinates $z = x + iy$ along each leaf of $\widehat{\Delta}$ and $\widehat{X}$, and varies continuously in the transverse direction. The *geodesic flow* $g_s : \widehat{\Delta} \to \widehat{\Delta}$, given by

$$g_s(x + iy) = x + ie^s y$$

in affine coordinates, is also transversally continuous. It commutes with $\widehat{f}$ and hence descends to a flow on $\widehat{X}$.

To see this flow more readily, we observe there is an isomorphism

$$E : \widehat{S}^1 \times \mathbb{R}_+ \cong \widehat{\Delta}$$

given by the *exponential map*

$$E((u_i), t) = (z_i) = \Big( \lim_{n \to \infty} f^n(u_{i-n} + v_{i-n}) \Big), \qquad (10.1)$$

where $v_0 = -tu_0 = (f^{n-i})'(u_{i-n})v_{i-n}$ for $n \geq i$. The expansion of $f|S^1$ yields the useful estimate

$$z_0 = (1 - t)u_0 + O(t^2). \qquad (10.2)$$

In exponential coordinates, the geodesic flow and the action of $\widehat{f}$ take the simple form

$$g_s((u_i), t) = ((u_i), e^s t)$$

and

$$\widehat{f}((u_i), t) = \big((u_{i+1}), \big|f'(u_0)\big|t\big).$$

This shows:

**Theorem 10.2** *The geodesic flow on $\widehat{X}$ is isomorphic to the suspension of $\widehat{f} : \widehat{S}^1 \to \widehat{S}^1$ under the roof function $r((u_i)) = \log|f'(u_0)|$.*

**Ergodicity.** Let $m$ denote the unique absolutely continuous $f$-invariant probability measure on $S^1$. (If we normalize so that $f(0) = 0$, then $m = |dz|/2\pi$; cf. [Mar].) The corresponding $\widehat{f}$-invariant probability measure $\widehat{m}$ on $\widehat{S}^1$ is characterized by the property that it pushes forward to $m$ under each coordinate function $z_i : \widehat{S}^1 \to S^1$.

Since $m$ is a continuous multiple of $|dz|$, $\widehat{m}$ conditions to a constant multiple of $dx$ in affine coordinates on each leaf of $\widehat{S}^1$. Thus we can locally write

$$\widehat{m} = dx\, d\tau, \tag{10.3}$$

where $d\tau$ is a measure on the transverse Cantor set of $\widehat{S}^1$ and $x$ is affine on each leaf.

From $\widehat{m}$ we obtain an invariant measure $d\xi$ for the geodesic flow on $\widehat{X}$, given upstairs on $\widehat{\Delta} \cong \widehat{S}^1 \times \mathbb{R}_+$ by

$$d\xi = \frac{\widehat{m} \times (dy/y)}{L(f, m)} \cdot$$

The denominator is chosen so that $\int_{\widehat{X}} d\xi = 1$.

**Theorem 10.3** *The geodesic flow on $(\widehat{X}, d\xi)$ is ergodic.*

*Proof.* Since $f|(S^1, m)$ is ergodic, so is $\widehat{f}\,|(\widehat{S}^1, \widehat{m})$ and its suspension. □

**Failure of mixing.** The geodesic flow on $\widehat{X}$ need *not* be mixing; for example, when $f(z) = z^d$, the flow factors over a rotation of the circle because the roof function $\log|f'(u_0)| = \log d$ is constant. See [Ru3] and [Po] for more on the behavior of mixing under suspension.

**The harmonic current.** We now define a harmonic current $T$ on $\widehat{X}$ that plays the role of the fundamental class of $\widehat{X}$ (although it is not closed).

By (10.3) the invariant measure for the geodesic flow can be written locally as

$$d\xi = \frac{dx\, dy}{y} d\tau = dA \wedge T,$$

where $z = x + iy$ is an affine coordinate on each leaf, $dA = dx\, dy/y^2$ is the hyperbolic area form, $d\tau$ is a measure on the transverse Cantor set and

$$T = y d\tau.$$

The current $T$ pairs naturally with a continuous 2-form $\alpha$ along the leaves of $\widehat{X}$ to give

$$\int_{\widehat{X}} \alpha \wedge T = \int_{\widehat{X}} (\alpha/dA)\, d\xi.$$

Now let $\gamma$ be a transversally continuous smooth 1-form on $\widehat{X}$. Stokes' theorem implies that $\int_U d\gamma d\tau = 0$ if $\gamma$ is supported in a coordinate chart $U$. Using a partition of unity, we obtain

$$\int_{\widehat{X}} (d\gamma) \wedge T = \int_{\widehat{X}} \gamma \wedge (dy/y) \wedge T$$

for any $\gamma$, and hence formally

$$dT = (dy/y) \wedge T. \tag{10.4}$$

**Holomorphic forms on $\widehat{X}$.** Let $T^*\widehat{X} \to \widehat{X}$ denote the complex cotangent bundle determined by the complex structure along the leaves of $\widehat{X}$. A *holomorphic $k$-form* on $\widehat{X}$ is a continuous section $\alpha : \widehat{X} \to (T^*\widehat{X})^k$ that is holomorphic along leaves. We denote the space of all such forms by $\Omega^k(\widehat{X})$. Since $\widehat{X}$ is compact, every $\alpha \in \Omega^k(\widehat{X})$ satisfies

$$\sup_{\widehat{X}} \rho^{-k}|\alpha| < \infty.$$

We define the differential operator $D : \Omega^k(\widehat{X}) \to \Omega^{k+1}(\widehat{X})$ in affine coordinates by

$$D\alpha = (d\alpha/dz)\, dz^{k+1},$$

and the inner product between a $j$-form and a $k$-form by

$$\langle \alpha, \beta \rangle_{j,k} = \int_{\widehat{X}} y^{j+k} \alpha(z) \overline{\beta}(z)\, d\xi$$

where $z = x + iy$. The proofs in Sect. 7 for $\mathrm{T}_1 X$ then carry over to yield the following corresponding results for $\widehat{X}$.

**Theorem 10.4** *The holomorphic forms on $\widehat{X}$ satisfy*

$$\langle D\alpha, \beta \rangle_{j+1,k} = \frac{i}{2}(j+k)\langle \alpha, \beta \rangle_{j,k}.$$

**Theorem 10.5** *The map $D : \Omega^k(\widehat{X}) \to \Omega^{k+1}(\widehat{X})$ is an isomorphism for all $k > 0$, and satisfies*

$$\langle D\alpha, D\alpha \rangle = \frac{2k(2k+1)}{4}\langle \alpha, \alpha \rangle.$$

It is known that $\Omega^k(\widehat{X})$ is infinite-dimensional when $k = 2$ [GS2, §7], and thus the same is true for all $k > 0$.

*Proof of Theorem 10.1.* This follows by induction from Theorem 10.5. □

**Almost invariant functions.** A continuous function $h : \Delta \to \mathbb{C}$ is *almost invariant* under $f$ if

$$\lim_{r\to 1} \sup \{|h(f^i(z)) - h(z)| : |f^i(z)| > r\} = 0. \tag{10.5}$$

When this holds, $h$ determines a continuous function $\widehat{h} : \widehat{\Delta} \to \mathbb{C}$ by

$$\widehat{h}((z_i)) = \lim_{i\to\infty} h(z_{-i}).$$

An almost invariant function behaves like the lift of a continuous function on a compact quotient of $\Delta$; in particular $h$ is bounded and uniformly continuous in the hyperbolic metric on $\Delta$.

**Theorem 10.6** *If $h : \Delta \to \mathbb{C}$ is almost invariant under $f$, then we have*

$$\lim_{r\to 1} \frac{1}{|\log(1-r)|} \int_0^r \frac{h(sz)ds}{1-s} = \int_{\widehat{X}} \widehat{h}\, d\xi$$

*for almost every $z \in S^1$.*

*Proof.* Ergodicity of the geodesic flow implies

$$\lim_{T\to 0} \frac{1}{|\log T|} \int_T^1 \widehat{h}(E((u_i), t))\, \frac{dt}{t} = \int_{\widehat{X}} \widehat{h}\, d\xi$$

for almost every $(u_i) \in \widehat{S}^1$. Let $(z_i(t)) = E((u_i), t)$. Then as $t \to 0$, we have $|z_0(t)| \to 1$ and thus

$$|\widehat{h}((u_i), t) - h(z_0(t))| \to 0$$

by (10.5), and

$$|h(z_0(t)) - h((1-t)u_0)| \to 0$$

by uniform continuity of $h$ in the hyperbolic metric and (10.2). Consequently we have

$$\int_{\widehat{X}} \widehat{h}\, d\xi = \lim_{T\to 0} \frac{1}{|\log T|} \int_T^1 h((1-t)u_0)\, \frac{dt}{t}$$

for almost every $(u_i) \in \widehat{S}^1$, or equivalently for almost every $u_0 \in S^1$ (since $m$ is absolutely continuous). The proof is completed by the change of variables $(1-t)u_0 = sz$ and $T = 1-r$. □

**Almost invariant forms.** A holomorphic form $\alpha \in \Omega^k(\Delta)$ is *almost invariant* under $f$ if

$$\lim_{r\to 1} \sup \{\rho^{-k}|(f^i)^*\alpha - \alpha|(z) : |f^i(z)| > r\} = 0.$$

In this case we can use the coordinate functions $z_i : \widehat{\Delta} \to \Delta$ to define an $\widehat{f}$-invariant form $\widehat{\alpha} \in \Omega(\widehat{\Delta})$ by

$$\widehat{\alpha} = \lim_{i\to\infty} (z_{-i})^* \alpha.$$

**Theorem 10.7** *Let $g(z)$ be a holomorphic function on the disk such that $\alpha = g^{(k)}(z)\, dz^k$ is almost invariant. Then we have*

$$\langle \widehat{\alpha}, \widehat{\alpha} \rangle = J_{2k}(g).$$

*Proof.* Since the hyperbolic metric on the disk is almost invariant, so is the function

$$h(z) = \rho^{-2k} |\alpha|^2 = \rho^{-2k}(z) |f^k(z)|^2,$$

and we have $\widehat{h} = \rho^{-2k} |\widehat{\alpha}|^2$, so

$$\int_{\widehat{X}} \widehat{h}\, d\xi = \langle \widehat{\alpha}, \widehat{\alpha} \rangle.$$

By Theorem 10.6, we also have

$$h_r(z) = \frac{1}{|\log(1-r)|} \int_0^r \frac{ds}{1-s} h(sz)\, ds \to \langle \widehat{\alpha}, \widehat{\alpha} \rangle$$

as $r \to 1$ for almost every $z \in S^1$. Since $h$ is bounded, so is $h_r$, and thus dominated convergence implies

$$\lim_{r\to 1} \int_{S^1} h_r(z)\, |dz| = \langle \widehat{\alpha}, \widehat{\alpha} \rangle.$$

Reversing the order of integration over $z$ and $r$, we obtain

$$\langle \widehat{\alpha}, \widehat{\alpha} \rangle = \lim_{r\to 1} \frac{1}{2\pi |\log(1-r)|} \int_0^r \frac{ds}{1-s} \int_{|z|=s} \rho^{-2k} |g^{(k)}(z)|\, |dz| = J_{2k}(g).$$

□

**Almost invariance with Hölder bounds.** Given $\delta > 0$, we say a continuous function $h : \Delta \to \mathbb{R}$ is *$\delta$-almost invariant* if

$$\sup_{|z|=r} |h(f(z)) - h(z)| = O((1-r)^\delta).$$

This condition implies $h$ is almost invariant. Using the expansion property of $f$, it also implies the comparison

$$|\widehat{h}((z_i)) - h(z_0)| = O((1 - |z_0|)^\delta), \qquad (10.6)$$

which shows that $\widehat{h}$ is Hölder continuous on $\widehat{X}$. For $k$-forms, the condition of $\delta$-almost invariance becomes

$$\sup_{|z|=r} \rho^{-k}|f^*\alpha - \alpha| = O((1-r)^\delta).$$

**Theorem 10.8** *Let $\alpha = g'(z)\,dz \in \Omega^1(\Delta)$ be a $\delta$-almost invariant form on the disk. Then we have*

$$\mathrm{Var}(\mathrm{Re}\,\widehat{\alpha}/\rho) = 2\langle\widehat{\alpha},\widehat{\alpha}\rangle = (1/2)I_0(g).$$

*Proof.* First note that $\rho^{-1}|\alpha| = \rho^{-1}|Dg|$ is bounded by almost invariance of $\alpha$; thus $J_0(g) = I_0(g)$ exists and satisfies

$$(1/2)J_0(g) = 2J_2(g) = 2\langle\widehat{\alpha},\widehat{\alpha}\rangle$$

by Theorems 6.3 and 10.7.

Let $h(z) = \mathrm{Re}\,\alpha/\rho$. Since $\alpha$ is $\delta$-almost invariant, so is $h$. We will regard $\widehat{\alpha}$ and $\widehat{h}$ as $\widehat{f}$-invariant forms and functions on $\widehat{\Delta}$ in exponential coordinates $((u_i), t) \in \widehat{S}^1 \times \mathbb{R}_+$.

Given $0 < t < 1$ and $(u_i) \in \widehat{S}^1$, let $(z_i) = E((u_i), t)$. We then have $1 - |z_0| \asymp t$, and thus

$$\widehat{h}((u_i), t) = h(z_0) + O(t^\delta)$$

by (10.6). Since $\rho^{-1}|\alpha|$ is bounded, $h$ is Lipschitz in the hyperbolic metric on the disk, so the estimate $z_0 = (1-t)u_0 + O(t^2)$ given in (10.2) implies

$$\widehat{h}((u_i), t) = h((1-t)u_0) + O(t^\delta). \tag{10.7}$$

Thus if we define $S_T : \widehat{S}^1 \to \mathbb{R}$ for $0 < T < 1$ by

$$S_T((u_i)) = \int_T^1 \widehat{h}((u_i), t)\,\frac{dt}{t},$$

the estimate (10.7) gives

$$S_T((u_i)) = \int_T^1 h((1-t)u_0)\,\frac{dt}{t} + O(1). \tag{10.8}$$

Now recall that the $f$-invariant measure $m|S^1$ has a smooth density, i.e. we can write

$$|dz|/(2\pi) = \delta(z)\,dm(z)$$

for a suitable smooth function $\delta(z)$ satisfying $\int \delta\,dm = 1$. Since the geodesic flow on $\widehat{X}$ can be presented as a suspension of $\widehat{f} : \widehat{S}^1 \to \widehat{S}^1$, Theorem 3.5

implies

$$\operatorname{Var}(\widehat{h}) = \lim_{T\to 0} \frac{1}{|\log T|} \int_{\widehat{S}^1} \delta(u_0) S_T((u_i))^2 d\widehat{m}.$$

Together with (10.8), this yields

$$\operatorname{Var}(\widehat{h}) = \lim_{T\to 0} \frac{1}{2\pi |\log T|} \int_{S^1} \left( \int_T^1 h((1-t)u_0) \frac{dt}{t} \right)^2 |du_0|.$$

Since $\rho((1-t)u_0) \sim 1/t$, we have

$$h((1-t)u_0)dt/t = (\operatorname{Re}\alpha/\rho)dt/t \sim \operatorname{Re}\alpha$$

exponentially fast, and thus

$$\int_T^1 h((1-t)u_0) \frac{dt}{t} = \operatorname{Re} g((1-T)u_0) + O(1).$$

A change of variables then gives

$$\operatorname{Var}(\operatorname{Re}\widehat{\alpha}/\rho) = \operatorname{Var}(\widehat{h}) = I_0(\operatorname{Re} g) = (1/2) I_0(g),$$

completing the proof. □

## 11 Deformations of Blaschke products

In this section we return to the setting of the introduction, and consider a deformation $v$ of an expanding Blaschke product $f(z)$ with $J(f) = S^1$. We show that $v$ determines holomorphic forms $v^{(k)}$ on the Riemann surface lamination $\widehat{X}$ determined by $f$, and then establish:

**Theorem 11.1** *Any deformation $v$ of a Blaschke product $f \in \mathcal{B}_d$ satisfies*

$$\frac{d^2}{dt^2} \operatorname{H.dim}(J(F_t))\Big|_{t=0} = \operatorname{Var}(\operatorname{Re} v''/\rho) = \frac{2^{2k-1}}{(2k-1)!} \int_{\widehat{X}} \rho^{-2k} |v^{(k+1)})|^2 \, d\xi$$

*for all $k > 0$.*

The cases $k = 1, 2$ yield Theorem 1.9 of the introduction.

**Virtual cocycles and holomorphic forms.** Let $f \in \mathcal{B}_d$ be an expanding Blaschke product with $J(f) = S^1$ and $f(\Delta) = \Delta$. Let $F_t(z)$ be a family of rational maps such that $f(z) = F_0(z)$, and such that there exists a smooth family of conformal maps $H_t : \Delta \to \mathbb{C}$ satisfying $H_0(z) = z$ and $F_t H_t(z) = H_t f(z)$. The holomorphic vector field $v = dH_t/dt|_{t=0}$ records the associated deformation of $f$.

We begin by constructing the forms $v^{(k+1)} \in \Omega^k(\widehat{X})$.

**Theorem 11.2** *The 1-form $\alpha = v''(z)\,dz$ on the unit disk is $(1-\epsilon)$-almost invariant for every $\epsilon > 0$.*

*Proof.* By Theorems 4.4 and 4.5, the holomorphic function

$$h(z) = \frac{d}{dt} \log F_t'(H_t(z))\Big|_{t=0}$$

extends to $C^{1-\epsilon}(\overline{\Delta})$ and satisfies

$$h(z) = v'(f(z)) - v'(z)$$

for $z \in \Delta$. Consequently we have

$$f^*\alpha - \alpha = (v''(f(z))f'(z) - v'(z))\,dz = h'(z)\,dz.$$

By Hölder continuity of $h$ and Cauchy's integral formula, we have the estimate

$$|h'(z)| = O((1-|z|)^{-\epsilon});$$

and since $1/\rho(z) \sim 1 - |z|$, this implies

$$\sup_{|z|=r} \rho^{-1}|f^*\alpha - \alpha| = O((1-|z|)^{1-\epsilon})$$

as desired. □

Since $\alpha = v''(z)\,dz$ is almost invariant, it determines a form $\widehat{\alpha} \in \Omega^1(\widehat{X})$ which we also denote by $v''$. Let $v^{(k+1)} = D^{k-1}(v'') \in \Omega^k(\widehat{X})$. With this notation in place, we can now give the:

*Proof of Theorem 11.1.* We have

$$\frac{d^2}{dt^2} \mathrm{H.dim}(J(F_t))\Big|_{t=0} = (1/2)I_0(v')$$

by the results of Sect. 4,

$$(1/2)I_0(v') = \mathrm{Var}(\mathrm{Re}\, v''/\rho) = \langle v'', v''\rangle,$$

by Theorem 10.8, and

$$\langle v'', v''\rangle = \frac{2^{2k-1}}{(2k-1)!}\langle v^{(k+1)}, v^{(k+1)}\rangle$$

by Theorem 10.1 . □

## 12 Random geodesics

We conclude by presenting a new proof of:

**Theorem 12.1 (Wolpert)** *The length on $X_t$ of a random geodesic on $X_0$ satisfies*

$$\frac{d^2}{dt^2} \log \ell(X_t, g_0)\bigg|_{t=0} = \frac{4}{3} \frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} \cdot$$

**Random geodesics.** Let $X_t = \Delta/G_t$ be a smooth family of compact Riemann surfaces of genus $g$. Any closed, oriented geodesic $\gamma \subset X_0$ determines a unique probability measure $m(\gamma)$ supported on its lift to $\mathrm{T}_1 X_0$ and invariant under the geodesic flow. It also determines a smooth function $\ell(Y, \gamma)$ on Teichmüller space, giving the hyperbolic length on $Y$ of the unique geodesic in the same homotopy class as $\gamma$ on $X$.

The smooth invariant probability measure $d\xi$ on $\mathrm{T}_1 X$ can be regarded as the lift of a 'random geodesic' $g_0$ on $X_0$. To make this more precise, choose a sequence of closed geodesics $\gamma_n$ such that

$$m(\gamma_n) \to d\xi$$

in the weak topology on measures. (Such a sequence exists by the ergodic theorem and the closing lemma.) We can then define the length on $Y$ of a random geodesic on $X_0$ by

$$\ell(Y, g_0) = \lim \frac{\ell(Y, \gamma_n)}{\ell(X_0, \gamma_n)} \cdot \tag{12.1}$$

The limit $\ell(Y, g_0)$ is independent of the choice of the sequence $\langle \gamma_n \rangle$, and convergence to the limit is uniform on compact subsets of $\mathcal{T}_g$. Indeed, one can formalize the statement $\gamma/\ell(\gamma, X_0) \to g_0$ using the space of geodesic currents $\mathcal{C}_g$, and interpret $\ell(Y, \cdot)$ in terms of the continuous intersection pairing $i : \mathcal{C}_g \times \mathcal{C}_g \to \mathbb{R}$ [Bon].

**Random orbits.** Now let $f_0 : S^1 \to S^1$ be an expanding Markov map associated to the action of $G_0|S^1$ as in Sect. 2. Let $h_t : S^1 \to S^1$ be the unique isotopy conjugating $G_0$ to $G_t$ and satisfying $h_0(z) = z$, and let $f_t = h_t f_0 h_t^{-1}$ be the corresponding Markov map for $G_t$. Let $m_0$ be the unique absolutely continuous invariant probability measure for $f_0|S^1$, and let $m_t = (h_t)_*(m_0)$.

Let $\langle \gamma_n \rangle$ be a sequence of oriented closed geodesics on $X_0$ satisfying (12.1). We will construct a corresponding sequence of periodic cycles for $f_0$ that become equidistributed for the invariant measure $m_0$.

Let $[a, b] \subset \Delta$ denote the hyperbolic geodesic joining $a, b \in S^1$. Choose a compact convex fundamental domain $D \subset \Delta$ for the action of $G_0$, and choose lifts

$$\widetilde{\gamma_n} = [a_n, b_n]$$

of $\gamma_n$ meeting $D$. Excluding finitely many $n$ if necessary, we can assume that $G_0 \cdot a_n$ avoids the endpoints of the tiles defining $f_0|S^1$, and hence

$$a_n(i) = f_0^i(a_n)$$

is well-defined for all $i \geq 0$.

There are unique points $b_n(i) \in S^1$ such that $[a_n(i), b_n(i)]$ is also a lift of $\gamma_n$. By the expanding property of $f_0$ and compactness of $D$, there is an $\epsilon > 0$ such that

$$|a_n(i) - b_n(i)| > \epsilon \tag{12.2}$$

for all $i$ and $n$. But the orbit of $G_0 \cdot (a_n, b_n) \subset S^1 \times S^1$ accumulates only on the diagonal, so the sequence $[a_n(i), b_n(i)]$, $i = 1, 2, 3, \ldots$, takes on only finitely many distinct values. Consequently there is an integer $p_n > 0$ such that

$$[a_n(i + p_n), b_n(i + p_n)] = [a_n(i), b_n(i)]$$

for all $i \gg 0$.

Replacing $\widetilde{\gamma_n}$ with $[a_n(i), b_n(i)]$ for $i \gg 0$, we can assume $a_n$ is actually a periodic point for $f_0$, and $p_n$ is its period. Let $\mu_n$ be the measure with a $\delta$-mass on each point of the periodic cycle $\{a_n(1), \ldots, a_n(p_n)\} \subset S^1$.

**Theorem 12.2** *The probability measures $\mu_n/p_n$ on $S^1$ converge to the absolutely continuous measure $m_0$ as $n \to \infty$, and satisfy*

$$\frac{L(f_t, \mu_n)}{L(f_0, \mu_n)} = \frac{\ell(X_t, \gamma_n)}{\ell(X_0, \gamma_n)} \cdot \tag{12.3}$$

*Proof.* By the definition of $f_0$, for every $n$ there is a $g_n \in G_0$ such that $f_0^{p_n} = g_n$ on a neighborhood of $a_n$. Since $g_n(a_n) = a_n$, the Möbius transformation $g_n$ stabilizes $[a_n, b_n] = \widetilde{\gamma_n}$. Hence there is an integer $C_n$ such that

$$L(f_0, \mu_n) = \log\left|\left(f_0^{p_n}\right)'(a_n)\right| = \log\left|g_n'(a_n)\right| = C_n \ell(X_0, \gamma_n). \tag{12.4}$$

Letting the parameter $t$ vary, we obtain (12.3).

We claim the period $p_n$ of $a_n$ is comparable to length $\ell_n = \ell(X_0, \gamma_n)$. Indeed, since $|f_0'|$ is bounded and $C_n \geq 1$, (12.4) implies $\ell_n = O(p_n)$. On the other hand, the length of $\gamma_n$ is comparable to the number of its intersections with the image of $\partial D$ on $X$, which in turn bounds the number of distinct lifts meeting $D$ and hence the number of distinct lifts whose endpoints satisfy (12.2). The latter quantity is an upper bound for the period of $a_n$, and hence $p_n = O(\ell_n)$.

To see that $\mu_n/p_n \to m_0$, consider the lift of $\mu_n$ to a measure $\nu_n$ on $S^1 \times S^1$ with $\delta$-masses on the pairs $(a_n(i), b_n(i))$, $i = 1, \ldots, p_n$. Similarly, let $\xi_n$ be the measure with a $\delta$-mass at *every* point in the orbit $G_0 \cdot (a_n, b_n) \subset S^1 \times S^1$.

The assumption $\gamma_n/\ell_n \to g_0$ implies the measures $\xi_n/\ell_n$ converge to an absolutely continuous measure on $S^1 \times S^1$. Since $\nu_n \le \xi_n$ and $p_n \asymp \ell_n$, any accumulation point of the probability measures $\nu_n/p_n$ is also absolutely continuous. Consequently the same is true for $\mu_n$. But $f_0|S^1$ admits a unique absolutely continuous invariant probability measure, and hence $\mu_n \to m_0$. $\square$

Taking the limit as $n \to \infty$ in (12.3), we obtain:

**Theorem 12.3** *The Lyapunov exponent of $f_t$ and the length of a random geodesic on $X_t$ are related by*

$$\frac{L(f_t, m_t)}{L(f_0, m_0)} = \ell(X_t, g_0).$$

*Proof of Theorem 12.1.* At $t = 0$ we have

$$\frac{d^2}{dt^2} \log \ell(X_t, g_0) = \frac{d^2}{dt^2} \log L(f_t, m_t)$$

by the preceding result,

$$\frac{d^2}{dt^2} \log L(f_t, m_t) = -\frac{d^2}{dt^2} \log \mathrm{H.dim}(m_t)$$

by the entropy relation $\mathrm{H.dim}(m_t) L(f_t, m_t) = h(f_t, m_t) = h(f_0, m_0)$, and

$$-\frac{d^2}{dt^2} \log \mathrm{H.dim}(m_t) = \frac{4}{3} \frac{\|\dot{X}_0\|^2_{\mathrm{WP}}}{\mathrm{area}(X_0)} \tag{12.5}$$

by Theorem 1.1. $\square$

*Remark.* Conversely, one can use Theorem 12.3 and the entropy relation to deduce (12.5) from Wolpert's theorem.

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